偏导数计算

多元函数的偏导数

- 双变量函数 $f(x,y)$
- 求高阶偏导数

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n} f(x,y)$$

- MATLAB语法

$$f = \texttt{diff}(\texttt{diff}(\mathit{fun}, x, m), y, n)$$

- 或

$$f = \texttt{diff}(\texttt{diff}(\mathit{fun}, y, n), x, m)$$

例3-17 一阶偏导数

- 二元函数的偏导数图形表示

$$z = f(x,y) = (x^2 - 2x)\mathrm{e}^{-x^2-y^2-xy}$$

- 两个一阶偏导数 $\partial z/\partial x,\ \partial z/\partial y$

```
>> syms x y;
   z=(x^2-2*x)*exp(-x^2-y^2-x*y);
   zx=simplify(diff(z,x)),
   zy=diff(z,y)
```

梯度函数图形表示

➢绘制三维曲面

```
>> [x0,y0]=meshgrid(-3:.2:2,-2:.2:2);
z0=double(subs(z,{x,y},{x0,y0}));
surf(x0,y0,z0), zlim([-0.7 1.5])
```

➢引力线（负梯度）

```
>> contour(x0,y0,z0,30), hold on;
zx0=subs(zx,{x,y},{x0,y0});
zy0=subs(zy,{x,y},{x0,y0});
quiver(x0,y0,-zx0,-zy0)
```

例3-18 三元函数求偏导

- 三元函数 $f(x,y,z)=\sin(x^2y)\mathrm{e}^{-x^2y-z^2}$
- 求偏导数 $\dfrac{\partial^4 f(x,y,z)}{(\partial x^2\partial y\partial z)}$
- MATLAB求解

```
>> syms x y z;
   f=sin(x^2*y)*exp(-x^2*y-z^2);
   df=diff(diff(diff(f,x,2),y),z);
   df=simplify(df)
```

隐函数的偏导数

- 隐函数 $f(x_1, x_2, \cdots, x_n) = 0$
- 一阶偏导数

$$\frac{\partial x_i}{\partial x_j} = -\frac{\dfrac{\partial}{\partial x_j} f(x_1, x_2, \cdots, x_n)}{\dfrac{\partial}{\partial x_i} f(x_1, x_2, \cdots, x_n)}$$

- MATLAB代码

$$F = -\texttt{diff}(f, x_j)/\texttt{diff}(f, x_i)$$

二元隐函数的高阶偏导数

- 二元函数 $f(x,y)$
- 一阶偏导数 $\partial y/\partial x = F_1(x,y)$
- 二阶偏导数

$$F_2(x,y) = \frac{\partial^2 y}{\partial x^2} = \frac{\partial F_1(x,y)}{\partial x} + \frac{\partial F_1(x,y)}{\partial y} F_1(x,y)$$

- 高阶偏导数

$$F_n(x,y) = \frac{\partial^n y}{\partial x^n} = \frac{\partial F_{n-1}(x,y)}{\partial x} + \frac{\partial F_{n-1}(x,y)}{\partial y} F_1(x,y)$$

高阶偏导数MATLAB求解

➢递推公式

$$F_n(x,y)=\frac{\partial^n y}{\partial x^n}=\frac{\partial F_{n-1}(x,y)}{\partial x}+\frac{\partial F_{n-1}(x,y)}{\partial y}F_1(x,y)$$

➢MATLAB求解 $f_1=\texttt{impldiff}(f,x,y,n)$

```
function dy=impldiff(f,x,y,n)
if mod(n,1)~=0,
   error('n should positive integer, please correct')
else, F1=-simplify(diff(f,x)/diff(f,y)); dy=F1;
   for i=2:n, dy=simplify(diff(dy,x)+diff(dy,y)*F1);
end, end
```

例3-19 隐函数求导

- 二元隐函数

$$z = f(x,y) = (x^2 - 2x)\mathrm{e}^{-x^2-y^2-xy} = 0$$

- 一阶偏导数

```
>> syms x y; f=(x^2-2*x)*exp(-x^2-y^2-x*y);
   F1=impldiff(f,x,y,1)
```

- 二阶、三阶偏导数

```
>> F2=impldiff(f,x,y,2),
   F3=impldiff(f,x,y,3),
   [n,d]=numden(F3), simplify(n)
```

例3-20 隐函数求导与化简

- 隐函数 $x^2+xy+y^2=3$
- 各阶导数

```
>> f=x^2+x*y+y^2-3; F1=impldiff(f,x,y,1)
   f2=impldiff(f,x,y,2); F2=subs(f2,x^2+x*y+y^2,3)
   f3=impldiff(f,x,y,3); F3=subs(f3,x^2+x*y+y^2,3)
   f4=impldiff(f,x,y,4); F4=subs(f4,x^2+x*y+y^2,3)
```

- 有时需要手工化简

$$F_4=-\frac{648\left(4x^2+xy+y^2\right)}{(x+2y)^7}\quad\rightarrow\quad F_4=-\frac{1944(x^2+1)}{(x+2y)^7}$$

多元函数的Jacobi矩阵

- 多元函数

$$\begin{cases} y_1 = f_1(x_1, x_2, \cdots, x_n) \\ y_2 = f_2(x_1, x_2, \cdots, x_n) \\ \quad\vdots \qquad\qquad\qquad \vdots \\ y_m = f_m(x_1, x_2, \cdots, x_n) \end{cases}$$

- Jacobi矩阵

$$\boldsymbol{J} = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \cdots & \partial y_1/\partial x_n \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \cdots & \partial y_2/\partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial y_m/\partial x_1 & \partial y_m/\partial x_2 & \cdots & \partial y_m/\partial x_n \end{bmatrix}$$

- MATLAB求解 $\boldsymbol{J} = \texttt{jacobian}(\boldsymbol{y}, \boldsymbol{x})$

例3-21 Jacobi矩阵

- 直角坐标和极坐标变换公式

$$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi,\ \ z = r\cos\theta$$

- 求Jacobi矩阵

```
>> syms r theta phi;
   x=r*sin(theta)*cos(phi);
   y=r*sin(theta)*sin(phi);
   z=r*cos(theta);
   J=jacobian([x; y; z],[r theta phi])
```

Hessian偏导数矩阵

- 给定的n元函数 $f(x_1, x_2, \cdots, x_n)$
- Hessian矩阵

$$\boldsymbol{H} = \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 & \cdots & \partial^2 f/\partial x_1 \partial x_n \\ \partial^2 f/\partial x_2 \partial x_1 & \partial^2 f/\partial x_2^2 & \cdots & \partial^2 f/\partial x_2 \partial x_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial^2 f/\partial x_n x_1 & \partial^2 f/\partial x_n \partial x_2 & \cdots & \partial^2 f/\partial x_n^2 \end{bmatrix}$$

- MATLAB $\boldsymbol{H} = \texttt{hessian}(f, \boldsymbol{x})$
- 早期版本 $\boldsymbol{H} = \texttt{jacobian}(\texttt{jacobian}(f, \boldsymbol{x}), \boldsymbol{x})$

例3-22 Hessian矩阵

- 试求出下列二元函数的Hessian矩阵

$$z = f(x,y) = (x^2 - 2x)\mathrm{e}^{-x^2-y^2-xy}$$

- MATLAB代码

```
>> syms x y;
f=(x^2-2*x)*exp(-x^2-y^2-x*y);
H=simplify(hessian(f,[x,y]))
```

- 另一种求法

```
>> X=[x,y];
H1=simplify(jacobian(jacobian(f,X),X))
```