Notes on State Abstractions

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【强化学习理论 62】StatisticalRL 6

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这是UIUC姜楠老师开设的CS598统计强化学习（理论）课程的第四讲的第二部分，主要讲的内容是state abstraction。

原文传送门

CS598 Note4
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回顾

对于如下的定义

Definition 3 (Approximate abstractions). Given MDP $M = (\mathcal{S}, \mathcal{A}, P, R, \gamma)$ and state abstraction ϕ that operates on $\mathcal{S}$, define the following types of abstractions:

1. ϕ is an $\epsilon_{\pi^\star}$-approximate $\pi^\star$-irrelevant abstraction, if there exists an abstract policy $\pi : \phi(\mathcal{S}) \to \mathcal{A}$, such that $\|V_M^\star - V_M^{[\pi]_M}\|_\infty \le \epsilon_{\pi^\star}$.
2. ϕ is an $\epsilon_{Q^\star}$-approximate $Q^\star$-irrelevant abstraction if there exists an abstract Q-value function $f : \phi(\mathcal{S}) \times \mathcal{A} \to \mathbb{R}$, such that $\|[f]_M - Q_M^\star\|_\infty \le \epsilon_{Q^\star}$.
3. ϕ is an (ϵ_R, ϵ_P)-approximate model-irrelevant abstraction if for any $s^{(1)}$ and $s^{(2)}$ where $\phi(s^{(1)}) = \phi(s^{(2)})$, $\forall a \in \mathcal{A}$,
$$|R(s^{(1)}, a) - R(s^{(2)}, a)| \le \epsilon_R, \quad \left\|\Phi P(s^{(1)}, a) - \Phi P(s^{(2)}, a)\right\|_1 \le \epsilon_P. \tag{3}$$

Note that Definition 1 is recovered when all approximation errors are set to 0.

Definition 2 (*lifting*). For any function f that operates on $\phi(\mathcal{S})$, let $[f]_M$ denote its *lifted* version, which is a function over $\mathcal{S}$, defined as $[f]_M(s) := f(\phi(s))$. Similarly we can also lift a state-action value function. Lifting a real-valued function f over states can also be expressed in vector form: $[f]_M = \Phi^\top f$.

有以下性质

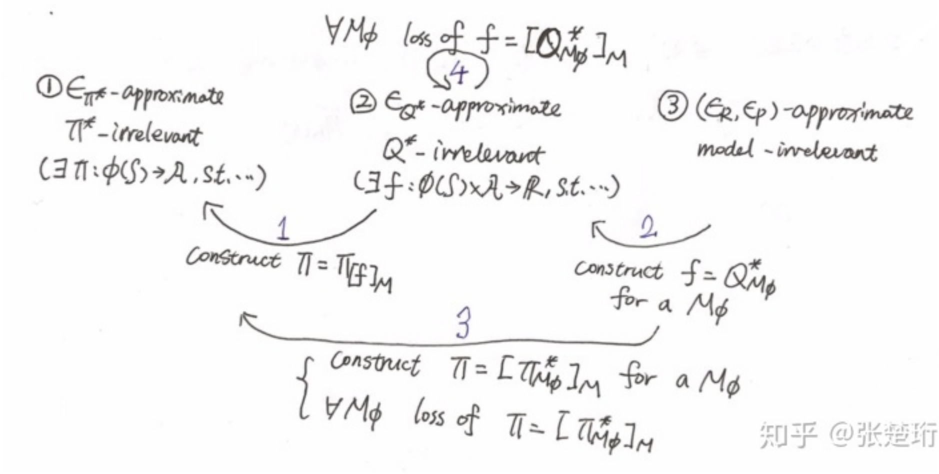
Theorem 2. *(1) If ϕ is an (ϵ_R, ϵ_P)-approximate model-irrelevant abstraction, then ϕ is also an approximate $Q^\star$-irrelevant abstraction with approximation error $\epsilon_{Q^\star} = \frac{\epsilon_R}{1-\gamma} + \frac{\gamma \epsilon_P R_{\max}}{2(1-\gamma)^2}$.*
(2) If ϕ is an $\epsilon_{Q^\star}$-approximate $Q^\star$-irrelevant abstraction, then ϕ is also an approximate $\pi^\star$-irrelevant abstraction with approximation error $\epsilon_{\pi^\star} = 2\epsilon_{Q^\star}/(1-\gamma)$.

Theorem 4. *Let ϕ be an (ϵ_R, ϵ_P)-approximate model-irrelevant abstraction of M, and M_ϕ be an abstract model defined as in Lemma 3 with arbitrary distributions $\{p_x\}$, then*

$$\left\| V_M^\star - V_M^{[\pi^\star_{M_\phi}]_M} \right\|_\infty \le \frac{2\epsilon_R}{1-\gamma} + \frac{\gamma \epsilon_P R_{\max}}{(1-\gamma)^2}.$$

Theorem 5. *Let ϕ be an $\epsilon_{Q^\star}$-approximate $Q^\star$-irrelevant abstraction for M. Then, for M_ϕ constructed as in Lemma 3 with arbitrary distributions $\{p_x\}$, we have $\|[Q^\star_{M_\phi}]_M - Q^\star_M\|_\infty \le 2\epsilon_{Q^\star}/(1-\gamma)$.*

这几个定理的关系如下



1. 要用第二种近似bound第一种近似，即找到一个 π 满足第一种近似中的条件即可；
2. 要用第三种近似bound第二种近似，即找到一个 f 满足第二种近似中的条件即可；为了找到这个 f ，我们将新定义一个与 ϕ 有关的MDP M_ϕ ，并说明这个新MDP下的最优价值函数满足第二种近似中的条件；
3. 可以连用1和2来用第三种近似bound第一种近似；为了得到更紧的bound，可以直接找到一个 π 满足第一种近似中的条件，同样，我们找到的这个 π 就是 M_ϕ ；这个证明又告诉我们了另外一件事情：第三种近似下，直接在 M_ϕ 上找到的最优策略的损失有多大，其中我们使用 M_ϕ 上最优策略和真实最优策略之间价值函数的差来衡量“损失”。
4. 这里证明的是，在第二种近似下，直接在 M_ϕ 上找到的最优策略的损失有多大。

在证明这四件事情之前，我们先看看如何定义一个与 ϕ 有关的MDP M_ϕ ，并且如果 ϕ 满足第三种近似，那么这个MDP M_ϕ 有什么样的性质。

Construct a MDP with respect to the approximate model-irrelevant

Lemma 3. *Let ϕ be an (ϵ_R, ϵ_P)-approximate model-irrelevant abstraction of M. Given any distributions $\{p_x : x \in \phi(\mathcal{S})\}$ where each p_x is supported on $\phi^{-1}(x)$, define $M_\phi = (\phi(\mathcal{S}), \mathcal{A}, P_\phi, R_\phi, \gamma)$, where $R_\phi(x,a) = \mathbb{E}_{s\sim p_x}[R(s,a)]$, and $P_\phi(x'|x,a) = \mathbb{E}_{s\sim p_x}[P(x'|s,a)]$. Then for any $s \in \mathcal{S}, a \in \mathcal{A}$,*

$$|R_\phi(\phi(s),a) - R(s,a)| \le \epsilon_R, \quad \|P_\phi(x,a) - \Phi P(s,a)\|_1 \le \epsilon_P.$$

即对于满足第三种近似的 ϕ 来说，这个关于它的新MDP的reward和dynamics都距离真实MDP相差不远。这样构造的原因是当原MDP中很多状态都抽象到 M_ϕ 中的一个状态之后，原MDP中的状态就无法区分了，我们需要对于每一个抽象出来的状态，都规定一个条件概率分布，即给定x它对应s的概率是多少。这样才能把原MDP中的reward/dynamics联系起来并且得到一个完全定义在x-space上的reward/dynamics。

Proof. We only prove for the transition part; the reward part follows from a similar (and easier) argument. Consider any fixed x and a. Let $q_s := \Phi P(s,a)$. By the definition of approximate bisimulation we have $\|q_{s^{(1)}} - q_{s^{(2)}}\|_1 \le \epsilon_P$ for any $\phi(s^{(1)}) = \phi(s^{(2)})$. The LHS of the claim on transition function is (let $x := \phi(s)$)

$$\begin{aligned}
&\|P_\phi(x,a) - \Phi P(s,a)\|_1 \\
=& \Big\| \sum_{\tilde{s}\in\phi^{-1}(x)} p_x(\tilde{s}) q_{\tilde{s}} - q_s \Big\|_1 = \Big\| \sum_{\tilde{s}\in\phi^{-1}(x)} p_x(\tilde{s})(q_{\tilde{s}} - q_s) \Big\|_1 \\
\le& \sum_{\tilde{s}\in\phi^{-1}(x)} \Big\| p_x(\tilde{s})(q_{\tilde{s}} - q_s) \Big\|_1 \le \sum_{\tilde{s}\in\phi^{-1}(x)} p_x(\tilde{s})\epsilon_P = \epsilon_P.
\end{aligned}$$

证明很容易，大体上就是平均值小于等于最大值。

下面证明第一件事情：

Approximate Q-star-irrelevant abstraction bounds approximate pi-star-irrelevant abstraction

第二种近似里面提到存在一个价值函数 f 满足某个性质，第一种近似要求存在一个策略 π 满足某个性质。这样我们就规定策略 π 是相对于价值函数 f 的最优策略，并且证明它满足第一种近似的要求。

这件事情在第一讲中已经证明了，可以拿来直接使用。

Lemma 4 ([8]). $\|V^\star - V^{\pi_f}\|_\infty \le \dfrac{2\|f - Q^\star\|_\infty}{1-\gamma}$.

下面证明第二件事情：

Approximate model-irrelevant abstraction bounds approximate Q-star-irrelevant abstraction

Define M_ϕ to be an abstract model as in Lemma 3 w.r.t. arbitrary distributions $\{p_x\}$. We will use $Q^\star_{M_\phi}$ as the f function in the definition of approximate $Q^\star$-irrelevance, and upper bound $\|[Q^\star_{M_\phi}]_M - Q^\star_M\|_\infty$ as:

$$\Big\|[Q^\star_{M_\phi}]_M - Q^\star_M\Big\|_\infty \le \frac{1}{1-\gamma}\Big\|[Q^\star_{M_\phi}]_M - \mathcal{T}[Q^\star_{M_\phi}]_M\Big\|_\infty = \frac{1}{1-\gamma}\Big\|[\mathcal{T}_{M_\phi}Q^\star_{M_\phi}]_M - \mathcal{T}[Q^\star_{M_\phi}]_M\Big\|_\infty.$$

For any (s,a),

$$\begin{aligned}
&|([\mathcal{T}_{M_\phi}Q^\star_{M_\phi}]_M)(s,a) - (\mathcal{T}[Q^\star_{M_\phi}]_M)(s,a)| \\
=& |(\mathcal{T}_{M_\phi}Q^\star_{M_\phi})(\phi(s),a) - (\mathcal{T}[Q^\star_{M_\phi}]_M)(s,a)| \\
=& |R_\phi(\phi(s),a) + \gamma\langle P_\phi(\phi(s),a), V^\star_{M_\phi}\rangle - R(s,a) - \gamma\langle P(s,a), [V^\star_{M_\phi}]_M\rangle| \\
\le& \epsilon_R + \gamma\left|\langle P_\phi(\phi(s),a), V^\star_{M_\phi}\rangle - \langle P(s,a), \Phi^\top V^\star_{M_\phi}\rangle\right| \\
=& \epsilon_R + \gamma\left|\langle P_\phi(\phi(s),a), V^\star_{M_\phi}\rangle - \langle \Phi P(s,a), V^\star_{M_\phi}\rangle\right| \qquad (*) \\
\le& \epsilon_R + \gamma\epsilon_P\|V^\star_{M_\phi} - \tfrac{R_{\max}}{2(1-\gamma)}\mathbf{1}\|_\infty \\
\le& \epsilon_R + \gamma\epsilon_P R_{\max}/(2(1-\gamma)).
\end{aligned}$$

In step (*), we notice that $[V^\star_{M_\phi}]_M$ is piece-wise constant, so when we take its dot-product with $P(s,a)$, we essentially first collapse $P(s,a)$ onto $\phi(\mathcal{S})$ (which is done by the Φ operator) and then take its dot-product with $V^\star_{M_\phi}$. The rest of the proof is similar to that of the simulation lemma. □

其中，第一个式子中的第一个不等式的推导和下面的推导方法类似，第二个等号是由于Bellman算子的不动点性质。

$$\|Q^{\star}_{\widehat{M}} - Q^{\star}_{M}\|_{\infty} \leq \frac{1}{1-\gamma}\|Q^{\star}_{M} - \mathcal{T}_{\widehat{M}} Q^{\star}_{M}\|_{\infty}. \tag{8}$$

This is because

$$\begin{aligned}
\|Q^{\star}_{\widehat{M}} - Q^{\star}_{M}\|_{\infty} &= \|\mathcal{T}_{\widehat{M}} Q^{\star}_{\widehat{M}} - \mathcal{T}_{\widehat{M}} Q^{\star}_{M} + \mathcal{T}_{\widehat{M}} Q^{\star}_{M} - Q^{\star}_{M}\|_{\infty} \\
&\leq \gamma \|Q^{\star}_{\widehat{M}} - Q^{\star}_{M}\|_{\infty} + \|\mathcal{T}_{\widehat{M}} Q^{\star}_{M} - Q^{\star}_{M}\|_{\infty}. \qquad (\mathcal{T}_{\widehat{M}} \text{ is a } \gamma\text{-contraction})
\end{aligned}$$

第二串式子的推导难点在于分清楚每个符号表示的含义；倒数第二个不等号要考虑到P不是普通的向量，它的每一项都为正数并且加起来为1，因此它与任何单位向量的内积都为1，因此在倒数第二行里面减去一个常向量，这样能够让bound缩小一半；最后一个不等式用到 $u^T v \leq \|u\|_1 \|v\|_\infty$ 。

下面证明第三件事情：

Approximate model-irrelevant abstraction bounds approximate pi-star-irrelevant abstraction (How lossy is the optimal policy for M_ϕ)

为了证明 M_ϕ 下的最优策略的损失，先证明：对于任意 M_ϕ 下的策略 π ，策略 π 在 M_ϕ 下的性能和策略 $[\pi]_M$ 在 M 下的性能的差距上界。注意到这个上界刚好是我们要证明损失的一半。

Proof. We first prove that for any *abstract* policy $\pi : \phi(\mathcal{S}) \to \mathcal{A}$,

$$\left\| [V^{\pi}_{M_\phi}]_M - V^{[\pi]_M}_M \right\|_\infty \leq \frac{\epsilon_R}{1-\gamma} + \frac{\gamma \epsilon_P R_{\max}}{2(1-\gamma)^2}. \tag{4}$$

To prove this, first recall the contraction property of policy-specific Bellman update operator for state-value functions, which implies that

$$\left\| [V^{\pi}_{M_\phi}]_M - V^{[\pi]_M}_M \right\|_\infty \leq \frac{1}{1-\gamma} \left\| [V^{\pi}_{M_\phi}]_M - \mathcal{T}^{[\pi]_M} [V^{\pi}_{M_\phi}]_M \right\|_\infty = \frac{1}{1-\gamma} \left\| [\mathcal{T}^{\pi}_{M_\phi} V^{\pi}_{M_\phi}]_M - \mathcal{T}^{[\pi]_M} [V^{\pi}_{M_\phi}]_M \right\|_\infty .$$

For notation simplicity let $R^{\pi'}(s) := R(s, \pi'(s))$ and $P^{\pi'}(s) := P(s, \pi'(s))$. For any $s \in \mathcal{S}$,

$$\begin{aligned}
&|[\mathcal{T}^{\pi}_{M_\phi} V^{\pi}_{M_\phi}]_M(s) - \mathcal{T}^{[\pi]_M} [V^{\pi}_{M_\phi}]_M(s)| \\
&= |(\mathcal{T}^{\pi}_{M_\phi} V^{\pi}_{M_\phi})(\phi(s)) - \mathcal{T}^{[\pi]_M} [V^{\pi}_{M_\phi}]_M(s)| \\
&= |R^{\pi}_{\phi}(\phi(s)) + \gamma \langle P^{\pi}_{\phi}(\phi(s)), V^{\pi}_{M_\phi} \rangle - R^{[\pi]_M}(s) - \gamma \langle P^{[\pi]_M}(s), V^{[\pi]_M}_M \rangle| \\
&\leq \epsilon_R + \gamma |\langle P^{\pi}_{\phi}(\phi(s)), V^{\pi}_{M_\phi} \rangle - \langle P^{[\pi]_M}(s), [V^{\pi}_{M_\phi}]_M \rangle| \\
&= \epsilon_R + \gamma \left| \langle P^{\pi}_{\phi}(\phi(s)), V^{\pi}_{M_\phi} \rangle - \langle \Phi P^{[\pi]_M}(s), V^{\pi}_{M_\phi} \rangle \right| \\
&\leq \epsilon_R + \frac{\gamma \epsilon_P R_{\max}}{2(1-\gamma)}.
\end{aligned}$$

Now that we have a uniform upper bound on evaluation error, it might be attempting to argue that we under-estimate $\pi^\star_M$ and over-estimate $\pi^\star_{M_\phi}$ at most this much, hence the decision loss is twice the evaluation error. This argument does not apply here because $\pi^\star_M$ cannot be necessarily expressed as a lifted abstract policy when ϕ is not an exact bisimulation!

接下来我们要证明的损失可以拆开，并且使用前面的结论

Instead we can use the following argument: for any $s \in \mathcal{S}$,

$$\begin{aligned}
V^\star_M(s) - V^{[\pi^\star_{M_\phi}]_M}_M(s) &= V^\star_M(s) - V^\star_{M_\phi}(\phi(s)) + V^\star_{M_\phi}(\phi(s)) - V^{[\pi^\star_{M_\phi}]_M}_M(s) \\
&\leq \left\| Q^\star_M - [Q^\star_{M_\phi}]_M \right\|_\infty + \left\| [V^{\pi^\star_{M_\phi}}_{M_\phi}]_M - V^{[\pi^\star_{M_\phi}]_M}_M \right\|_\infty .
\end{aligned}$$

Here both terms can be bounded by $\frac{\epsilon_R}{1-\gamma} + \frac{\gamma \epsilon_P R_{\max}}{2(1-\gamma)^2}$ but for different reasons: the bound applies to the first term due to Claim (1) of Theorem 2, and applies to the second term through Eq.(4) as $\pi^\star_{M_\phi}$ is an abstract policy. □

下面证明第四件事情

Under approximate Q-star-irrelevant, how lossy is the optimal policy for M_ϕ

这件事情看起来不是很自然，因为第二种近似下，有些状态可能本质上不一样，但是它们价值函数差不多，我们现在把它们当做一样的状态来看待。但是这里后面证明了在这种高度抽象的MDP下找最优策略，性能相比于最优策略相差也不多。

为了理解这件事情，我们考虑exact Q-star-irrelevant的情况。考虑我们通过Bellman operator来找最优策略，如果两个状态 $s^{(1)}$ 和 $s^{(2)}$ ，如果它们Q值相同，那么它们在Bellman operator作用下产生的效果类似。进一步来说，可以说明原本MDP中的最优价值函数，也是 M_ϕ 下Bellman operator的不动点。

Exact $Q^\star$-irrelevance To develop intuition, let's see what happens when ϕ is an exact $Q^\star$-irrelevant abstraction: we can prove that $[Q^\star_{M_\phi}]_M = Q^\star_M$, despite that the dynamics and rewards in M_ϕ "do not make sense". In particular, we know that for any $s^{(1)}$ and $s^{(2)}$ aggregated by ϕ, for any $a \in \mathcal{A}$,

$$R(s^{(1)}, a) + \gamma\langle P(s^{(1)}, a), V^\star_M\rangle = Q^\star(s^{(1)}, a) = Q^\star(s^{(2)}, a) = R(s^{(2)}, a) + \gamma\langle P(s^{(2)}, a), V^\star_M\rangle.$$

This equation tells us that, although ϕ aggregates states that can have very different rewards and dynamics, they at least share one thing: the Bellman operator updates $Q^\star_M$ in exactly the same way at $s^{(1)}$ and $s^{(2)}$ (for any action).

Let $[Q^\star_M]_\phi(x, a) = Q^\star_M(s, a)$ for any $s \in \phi^{-1}(x)$; note that the notation $[\cdot]_\phi$ can only be applied to functions that are piece-wise constant under ϕ. We now show that $[Q^\star_M]_\phi$ is the fixed point of $\mathcal{T}_{M_\phi}$, which proves the claim. This is because, for any $x \in \phi(\mathcal{S})$, $a \in \mathcal{A}$, let s be any state in $\phi^{-1}(x)$:

$$\begin{aligned}
(\mathcal{T}_{M_\phi}[Q^\star_M]_\phi)(x, a) &= R_\phi(x, a) + \gamma\langle P_\phi(x, a), [V^\star_M]_\phi\rangle \\
&= \sum_{s \in \phi^{-1}(x)} p_x(s)\,(R(s, a) + \gamma\langle \Phi P(s, a), [V^\star_M]_\phi\rangle) \\
&= \sum_{s \in \phi^{-1}(x)} p_x(s)\,(R(s, a) + \gamma\langle P(s, a), V^\star_M\rangle) \\
&= \sum_{s \in \phi^{-1}(x)} p_x(s)\,[Q^\star_M]_\phi(x, a) = [Q^\star_M]_\phi(x, a).
\end{aligned}$$

下面的证明也类似，大致上（用语言太难说清楚了，放弃说明了。。）：

M 下最优价值函数相比于 M_ϕ 下最优价值函数

M 下最优价值函数相比于它被某个与 M_ϕ 有关Bellman operator作用过的函数

核心就是：exact情形证明容易是由于 M 下的最优价值函数可以投影到x-space上，然后在x-space上分析都很容易；approximate情形下，不可以这样做，分析都需要在s-space上进行。因此会定义另外的在s-space上但又与 ϕ 有关的一个MDP M_ϕ 来绕一下。其实想法挺直接，证明挺繁琐。

The approximate case The more general case is much trickier, as $Q^\star_M$ is not piece-wise constant when ϕ is not exactly $Q^\star$-irrelevant, so we cannot apply $\mathcal{T}_{M_\phi}$ to it.

To get around this issue, define a new MDP $M'_\phi = (\mathcal{S}, \mathcal{A}, P'_\phi, R'_\phi, \gamma)$, with

$$R'_\phi(s,a) = \mathbb{E}_{\tilde{s}\sim p_{\phi(s)}}[R(\tilde{s},a)], \qquad P'_\phi(s'|s,a) = \mathbb{E}_{\tilde{s}\sim p_{\phi(s)}}[P(s'|\tilde{s},a)].$$

Recall that $\{p_x\}$ are a set of arbitrary distributions and we use them as weights for defining M_ϕ. The model here, $M_{\phi'}$, also combines parameters from aggregated states, but is defined over the *primitive* state space. This seemingly crazy model has two important properties: (1) Its optimal Q-value function coincides with that of M_ϕ (after lifting), and (2) It's defined over $\mathcal{S}$ so we can apply its Bellman operator to $Q^\star_M$.

We first prove that $[Q^\star_{M_\phi}]_M = Q^\star_{M'_\phi}$, by showing that $\mathcal{T}_{M'_\phi}[Q^\star_{M_\phi}]_M = [Q^\star_{M_\phi}]_M$:

$$\begin{aligned}
(\mathcal{T}_{M'_\phi}[Q^\star_{M_\phi}]_M)(s,a) &= R'_\phi(s,a) + \gamma\langle P'_\phi(s,a), [V^\star_{M_\phi}]_M\rangle \\
&= \sum_{\tilde{s}:\phi(\tilde{s})=\phi(s)} p_x(\tilde{s}) \left(R(\tilde{s},a) + \gamma\langle P(\tilde{s},a), [V^\star_{M_\phi}]_M\rangle\right) \\
&= \sum_{\tilde{s}:\phi(\tilde{s})=\phi(s)} p_x(\tilde{s}) R(\tilde{s},a) + \sum_{\tilde{s}:\phi(\tilde{s})=\phi(s)} p_x(\tilde{s})\gamma\langle \Phi P(\tilde{s},a), V^\star_{M_\phi}\rangle \\
&= R_\phi(\phi(s),a) + \gamma\langle P_\phi(\phi(s),a), V^\star_{M_\phi}\rangle \\
&= Q^\star_{M_\phi}(\phi(s),a) = [Q^\star_{M_\phi}]_M(s,a).
\end{aligned}$$

With this result, we have

$$\left\|[Q^\star_{M_\phi}]_M - Q^\star_M\right\|_\infty = \left\|Q^\star_{M'_\phi} - Q^\star_M\right\|_\infty \le \frac{1}{1-\gamma}\left\|\mathcal{T}_{M'_\phi}Q^\star_M - Q^\star_M\right\|_\infty.$$

And

$$\begin{aligned}
&|(\mathcal{T}_{M'_\phi}Q^\star_M)(s,a) - Q^\star_M(s,a)| \\
&= |R'_\phi(s,a) + \gamma\langle P'_\phi(s,a), V^\star_M\rangle - Q^\star_M(s,a)| \\
&= \left|\left(\sum_{\tilde{s}:\phi(\tilde{s})=\phi(s)} p_x(\tilde{s})\left(R(\tilde{s},a) + \gamma\langle P(\tilde{s},a), V^\star_M\rangle\right)\right) - Q^\star_M(s,a)\right| \\
&= \left|\sum_{\tilde{s}:\phi(\tilde{s})=\phi(s)} p_x(\tilde{s})\left(Q^\star_M(\tilde{s},a) - Q^\star_M(s,a)\right)\right| \\
&\le \left|\sum_{\tilde{s}:\phi(\tilde{s})=\phi(s)} p_x(\tilde{s})(2\epsilon_{Q^\star})\right| = 2\epsilon_{Q^\star}.
\end{aligned}$$

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