ALASKAN BROWN BEARS AND PACIFIC SALMON FACE THE EFFECTS OF GLOBAL WARMING

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ABSTRACT

Climate change has been a popular topic since James Hansen gave his testimony to Congress in 1988, expressing the disasters that would come from global warming. Many researchers are studying climate change in hopes of predicting its effects. If we can anticipate the outcomes of climate change, we can take measures to minimize or eliminate the catastrophes that will follow. In this thesis, we compare two models that determine the long-term outcome of two interactive species, pacific salmon *Oncorhynchus* and Alaskan brown bears *Ursus arctos*. The first model predicts the outcome of the species when temperature is constant, and the other when temperature is a function of time. We conclude that the effects of global warming could cause the pacific salmon to either die off or migrate to an area that is more suitable for their environmental needs, resulting in the brown bear population decreasing in size to accommodate for the elimination of a food source.

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Chapter 1

Introduction

The topic of climate change has been debated for many years, with some people believing that global warming is a myth and misusing scientific research to support their claims [1]. Many researchers have concluded that the earth's temperature has increased significantly since the early 20$^{\text{th}}$ century and will continue to increase until at least the mid to late 21$^{\text{st}}$ century [2, 3, 4]. A group of researchers implemented a joint Bayesian hierarchical model, which determined that global temperatures will likely increase by an average of 3.2°C entering the 22$^{\text{nd}}$ century [3]. When the industrial revolution began, a significant amount of greenhouse gases, such as carbon dioxide, methane, nitrous oxide, and fluorinated gases, filled our atmosphere, acting as a canopy that prevents heat from escaping the atmosphere, thus causing an escalation in climate temperature [5, 4, 6]. In 2019, the EPA reported that 80% of the earth's greenhouse gases are carbon dioxide [6]. Because of the high demand for transportation, electricity, industrial production, and residential/commercial use of fossil fuels, carbon dioxide will continue to be the primary cause of greenhouse gas emissions [6]. Now realizing the damage we have done to the earth, we are trying to resolve the problem without sacrificing our comfort. Unfortunately, this realization is too late for some species, such as the polar and koala bears [7, 8, 9]. In this thesis, we investigate the effects of climate change on a pair of species known to interact with each other, pacific salmon *Oncorhynchus* and Alaskan brown bears *Ursus arctos*.

Pacific salmon are sensitive to their environment and rely on sufficient river temperatures to survive spawning migration [10]. Adult salmon live in the ocean, but when the time comes to reproduce, they swim up river streams to lay their eggs and usually die shortly after; this is referred to as spawning. Salmon like to begin their journey from salt water to fresh water between late spring and early summer, but this depends on the species and location of pacific salmon [10]. Specifically in Alaska, salmon can be seen spawning in river streams between the middle of July through late October [11]. As river temperatures rise, the months in which they spawn and where they spawn may change, respectively [12]. Thus, global warming could affect the population of pacific salmon as well as any species that interact with them.

Alaskan brown bears feed on salmon as they migrate upstream, and if the population of salmon is susceptible to changes in temperature, then the brown bears could also be affected. Bears hibernate during winter and emerge during spring. Once emerged, they consume an enormous amount of food, such as berries, roots of plants, squirrels, moose, caribou, and fish [10]. Alaskan brown bears have various food sources, but salmon is an essential part of their diet, consuming an average of 1099 kg per year [13, 14]. Pacific salmon are already migrating further north, where temperatures are more suitable for them [12]. Since salmon is an essential food source for brown bears, it is probable that the species will either migrate with the salmon or look for another abundant resource to feed on [14].

In chapter 2, we use the logistic growth equation to model the population growth of pacific salmon and Alaskan brown bears. We begin by estimating the growth rate parameter for salmon using the reproduction rate from the Western Fisheries Research Center (WFRC) and the proportion of escapement from the National Park Service (NPS). Then, using data from the 2021 Bristol Bay annual management report, we calculate the carrying capacity by determining the maximum volume of salmon for any given run. Next, we compare growth rates from 3 different articles and calculate their average to approximate our growth rate for the brown bear model. Lastly, we pick a carrying capacity for the brown bears using information published by the Alaskan Department of Fish and Game (ADFG).

In Chapter 3, we propose a salmon growth rate function dependent on time, which replaces the growth rate parameter in the salmon logistic model. We use articles by Dr. Phyllis Weber Scannell and Katherine Carter to model the proportion of salmon that survive spawning migration at different temperatures. Applying linear regression to 30 years of recorded Alaskan river data, we create a function to represent the change in Alaskan river temperature over time. With this temperature function, we redesign the survival proportion model as a function of time. Now, we construct the proposed salmon growth rate function by combining the growth rate parameter with the survival proportion function. Finally, the growth rate function, $G(t)$, replaces the growth rate parameter in the salmon logistic model, creating a non-autonomous model for the salmon species.

In Chapter 4, we begin by constructing a variation of Theodore Modis' model to introduce interaction between Alaskan brown bears and pacific salmon when neither species is affected by climate change. We then solve for the critical points of our model and determine their stability by finding the eigenvalues of the Jacobian matrix. We explore different interaction parameters and visualize their effect by comparing the plots of our model's solutions for each pair of parameters. Next, we fix the interaction parameters to $c_{xy} = 0.0627$, and $c_{yx} = 0.0313$, then plot each species population with respect to time. We determine that the brown bear population will surpass the salmon population before both species oscillate toward their critical point, $(x = 0.79,\ y = 7.1)$. Following this section, we introduce climate change into our model, letting the function, $G(t)$, represent the growth rate of salmon. We then plot the solutions of the model using different initial conditions. To compare the effects of climate change, we plot the solutions to this model with respect to time on top of the previous model's plot and denote the differences. Lastly, we find that global warming causes the salmon species in Alaska to become regionally extinct, and in response to a diminished food resource, the brown bear species decrease in size relative to their population without the influence of climate change.

Chapter 2

Logistic Models For The Species

In this chapter, we introduce simple logistic growth models for the salmon and brown bear species. Using information from the Alaskan Department of Fish and Game, we estimate a growth rate parameter for the salmon population. Then, we choose the carrying capacity parameter by calculating the maximum volume of salmon for any given inshore run[1] in Bristol Bay, Alaska. We find 3 growth rates from 3 articles for the Alaskan brown bears and calculate their mean, which we use to represent their growth rate parameter [15, 16, 17]. Also, for this model, we estimate the carrying capacity based on information from the Alaskan Department of Fish and Game (ADFG) [10].

2.1 Logistic Model For Pacific Salmon

The Alaskan rivers and streams are comprised of 5 species of salmon; sockeye *O. nerka*, chinook *O. tshawytscha*, coho *O. kisutch*, chum *O. keta*, and pink *O. gorbuscha*. Of the 5 species, sockeye salmon is the most common food source for

[1] Inshore runs are when salmon migrate back from the sea to spawn.

Alaskan brown bears [18]. Sockeye salmon begin their journey hatching in streams and making their way down to the ocean. At this point, they spend a year to 2 years out at sea before migrating back to the streams where they originated. According to the National Park Service (NPS), only $25-40\%$ of returning salmon in Bristol Bay, Alaska, escape harvesting from commercial fisheries [19]. They will then travel several miles upstream, where they lay and fertilize their eggs, called spawning. Salmon will then lay between 1,500 to 10,000 eggs when spawning, but only 0 to 10 of these eggs will reach adulthood [20]. A significant proportion of energy for spawning salmon is spent reaching an optimal place to lay and fertilize their eggs, so much so that once they finish this process, they usually die shortly after [18].

Based on the information above, the rate at which the salmon population grows appears to be exponential. On average, for every salmon that lays eggs, 5 of their offspring will survive long enough to be adults ready to migrate back to their birthplace. Then, according to the NPS, approximately 32% of those 5 offspring will make it past escapement to reproduce. These conditions allow us to model the salmon population, shown below,

$$x(t) = x_0(0.32 * 5)^t, \tag{2.1}$$

where x_0 is the initial number of salmon that laid eggs and t represents time in years. We can then rewrite this exponential function with base e, as shown below,

$$x(t) = x_0 e^{r_x t}, \tag{2.2}$$

where $r_x = \ln(0.32 * 5)$ represents the growth rate. Taking the derivative of $x(t)$ give us the salmon population's instantaneous rate of change,

$$\frac{dx}{dt} = r_x x_0 e^{r_x t} = r_x x. \tag{2.3}$$

To examine the solution of the above model, we plot the function with an initial starting point of 20.

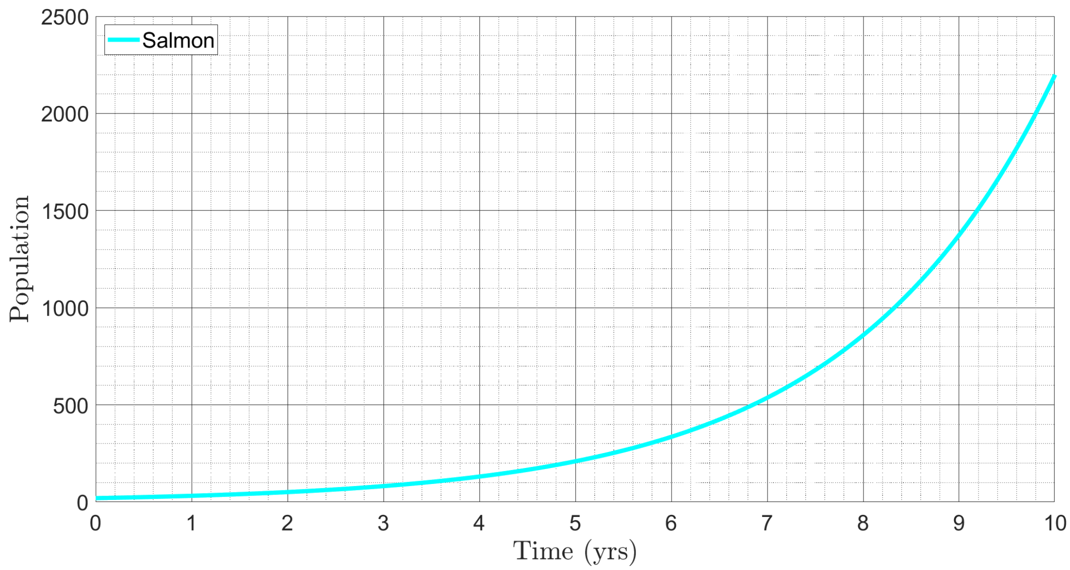


Figure 2.1: Plot of the exponential growth model for the salmon population with respect to time.

This figure illustrates that the population of salmon increases quickly in a short period. In just 9 years, the population of salmon increases from 20 to approximately 1,400, and 1 year later, the population grows to about 2,200. The issue with this model is that the population gets extremely large in a short period of time, eventually reaching values that would fall well outside physical possibility. Now that we have established a growth rate, a carrying capacity can be estimated using data from the ADFG in Bristol Bay, Alaska.

Bristol Bay is located on the easternmost side of the Bering Sea and is where many salmon migrate when returning home to reproduce. There was a dramatic increase in sockeye salmon returning to Bristol Bay in 2021 compared to previous years. However the average weight of sockeye salmon this year decreased by a pound compared to the average for the past 20 years [21]. The sockeye species make up a large majority of the inshore runs, harvests[2], and escapements[3] in Bristol Bay each year, which explains why the brown bear population mainly feeds on sockeye salmon [21].

Table 2.1: Sockeye Comparison Between Weight and Run Size in Bristol Bay

Year	**Weight (lbs)**	**Run (mil)**
2001	6.7	22.3
2002	6.1	16.9
2003	6.3	24.9
2004	5.8	41.9
⋮	⋮	⋮
2017	5.5	57.6
2018	5.1	63.0
2019	5.1	56.4
2020	5.1	58.3
2021	4.7	67.7

*This table gives a brief look at the relationship between run size and the average annual weight of sockeye salmon in Bristol Bay. See Table A.1 in Appendix A for the complete data.

In the first 3 years of this table, the weight seems to be dramatically higher than in the last 3 years, but the opposite effect appears in run size. This proposes the question that there might be a correlation between the run size and the average weight of salmon each year. When taking a closer look at Table A.1 in Appendix A, the trend becomes more apparent when comparing the sockeye's run size and the

[2]Harvests is defined as the number of fish gathered by commercial fisheries.

[3]Escapements are salmon that escape the fisheries and continue upstream to spawn.

average annual weight. Before calculating the correlation between these two events, we must create a plot to see if the trend is linear.

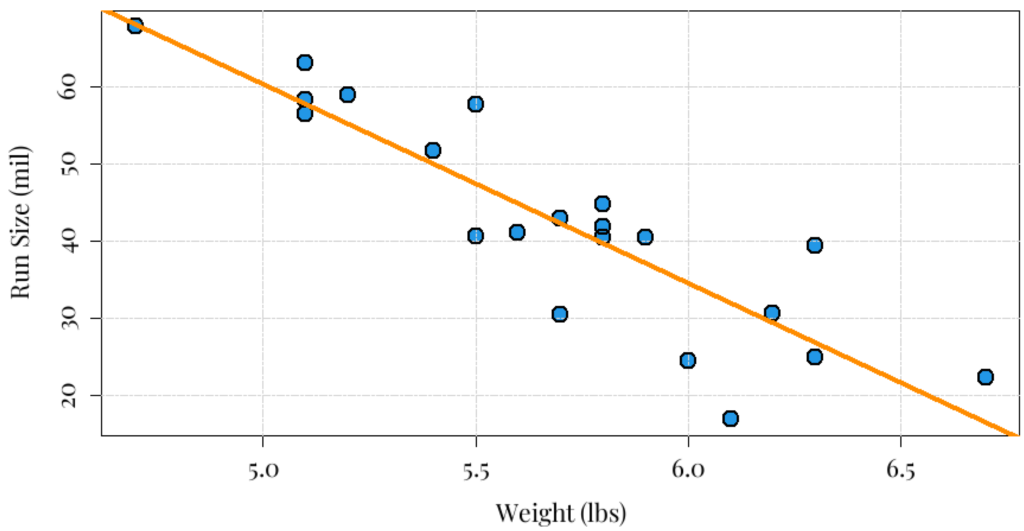


Figure 2.2: Scatter plot of the variables; inshore run size and the average weight of salmon during that year's run, with the line of best fit.

Based on the figure above, there is a linear correlation between sockeye run size and their average weight. The run size of the sockeye salmon decreases as their average weight increases. Since multiple variables make up the size of a salmon run each year, this helps explain the variance seen in the plot. So, calculating the correlation of these two events gives a value of -0.88. Since the correlation of these events is strong, an environmental limit of salmon can be estimated based on the maximum annual volume of sockeye salmon for the past 21 years. When looking at Table A.2 in Appendix A, the maximum volume for any given run in the last 21 years was 7.34 million cubic feet (MMCF) in 2018. The average weight of sockeye salmon during this year was 5.1 lbs, which is 0.4 lbs more than the lowest average weight of 4.7 lbs in 2021. Now, the carrying capacity for sockeye salmon can be estimated using the maximum volume and the lowest average weight, producing a value of 68.4 million sockeye salmon. Sockeye are not the only salmon in the river streams, but according to Table A.3 in Appendix A, they make up approximately 94% of the average annual commercial harvest in Bristol Bay. Assuming that the run proportions are the same as the average annual commercial harvest, 72.8 million becomes the carrying capacity for inshore salmon runs in Bristol Bay. As stated earlier, approximately 32% of adult salmon will escape commercial harvesting, which changes the carrying capacity to 29.1 million salmon each year [19].

Now that we have the carrying capacity of the salmon population, we can construct their logistic growth model to be

$$\frac{dx}{dt} = r_x x \left(1 - \frac{x}{K_x}\right), \tag{2.4}$$

where $K_x = 29,100,000$ is the carrying capacity, and $r_x = \ln(0.32 * 5)$ is the growth rate. If we start with an initial population of 20 million, the population should have the below trend.

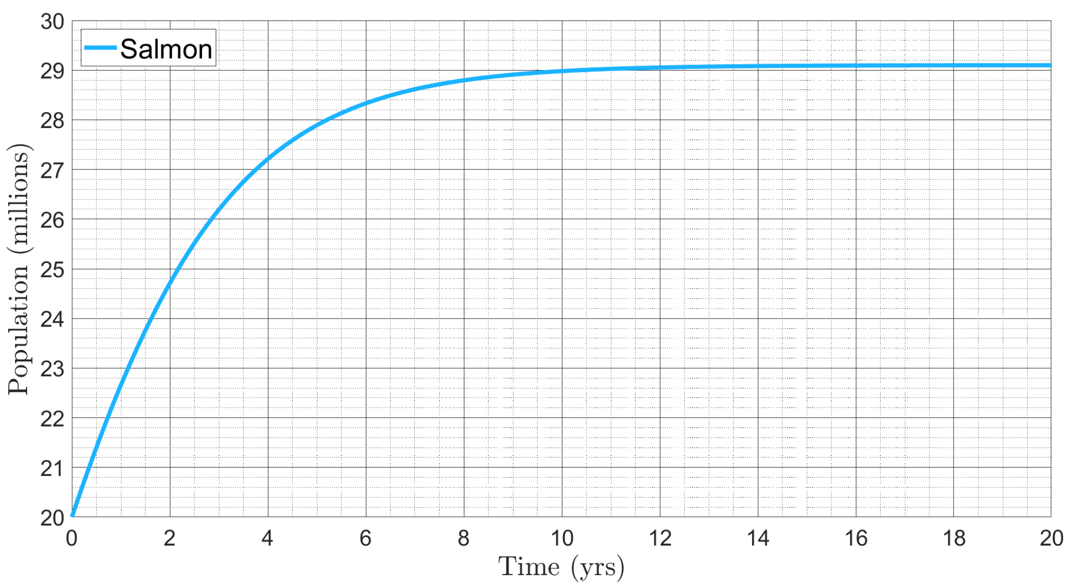


Figure 2.3: Plot of the logistic growth model for the salmon population with an initial population of 20 million.

From the graph above, the population of salmon grows rapidly for about 14 years before approximately reaching the carrying capacity. In the next chapter, we will examine if the changes in temperature affect the growth rate of salmon, and if so, incorporate climate change into the model and evaluate the results.

2.2 Logistic Model For Alaskan Brown Bears

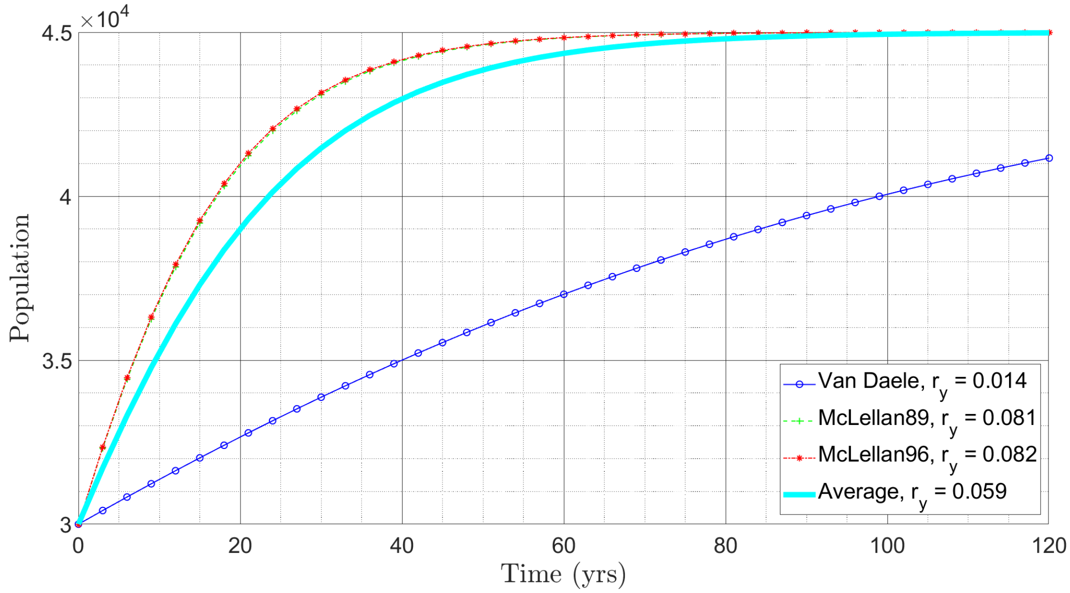
Throughout this thesis, the logistic model for the Alaskan brown bear population will be of the form,

$$\frac{dy}{dt} = r_y y \left(1 - \frac{y}{K_y}\right). \tag{2.5}$$

Researchers Lawrence J. Van Daele and Victor G. Barnes Jr. wrote a report "MANAGEMENT OF BROWN BEAR HUNTING ON KODIAK ISLAND, ALASKA" in 2010 that discusses the growth of the brown bear population and harvest strategies to maintain a healthy species [15]. They collected data using aerial surveys in Kodiak Archipelago and developed a deterministic population model using RISKMAN, a RISK MANagement Decision Tool [15, 22]. From the model, they approximated a growth rate of $\lambda = 1.014$ for Kodiak bears. So, we estimate $r_y = \ln(1.014) \approx 0.014$ to be the stochastic growth rate for Barnes' and Van Daele's deterministic model.

Van Daele and Barnes compare their results to Bruce N. McLellan's research on the dynamics of grizzly bears. McLellan's article, "Dynamics of a grizzly bear population during a period of industrial resource extraction. III. Natality and rate of increase," used the Lotka equation to estimate the growth rate of a grizzly bear population in Flathead Valley, British Columbia from 1979 to 1989 [16]. In this article, McLellan achieves an estimated growth rate of $r_y = 0.081$. In another article, "Estimating population growth of grizzly bears from the Flathead River drainage using computer simulations of reproduction and survival rates," Frederick W. Hovey and Bruce N. McLellan have continued the research, now from 1979 to 1994, in Flathead Valley and chose a different method of estimating the growth rate on the extended data [17]. Hovey and McLellan used the bootstrap method to improve the accuracy of estimating bias and standard error compared to the method used in McLellan's 1989 article. With the new method of calculating the growth rate and the increase in data size, Mclellan and Hovey have found similar results to Mclellan's 1989 article, where the newfound growth rate is $r_y = 0.082$.

Before comparing these growth rates, a carrying capacity for the brown bear population needs to be estimated. According to the Alaska Fish & Game Department (ADFG), the current recorded population of brown bears is estimated to be 30,000 [10]. From all the articles above, the common consensus is that the ADFG would like to maintain the size of the Alaskan bear population and keep it from climbing much higher than it is currently [16, 17, 15]. For this reason, a carrying capacity of $K_y = 45,000$ would be an appropriate estimation of the brown bears' environmental size limit. Now, for comparison, the graphs below display the solutions of Equation (2.5) for each growth rate.



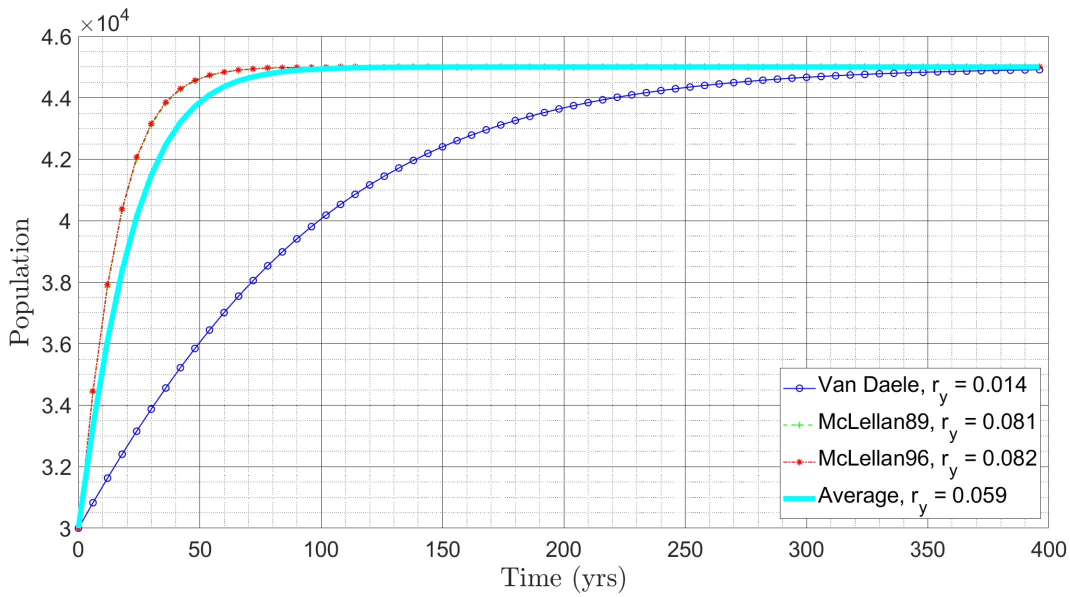


Figure 2.4: Plots of the Alaskan brown bear logistic growth equation, Equation (2.5), for each of the growth rate parameter values, r_y, discussed in the articles above, as well as the average of all the growth rates. The first graph represents a time span of 120 years, and the second represents 400 years.

In the plots above, McLellan's growth rates illustrate that the brown bear population will reach its environmental capacity within 80 years from an initial population of 30,000, while Van Daele's and Barnes' show that it would take the brown bears approximately 400 years. For this thesis, we estimate the growth rate parameter for the brown bear population by calculating the mean of all the growth rates discussed earlier, resulting in an estimated value of $r_y = 0.059$. Figure 2.4 shows that using this growth rate would result in the brown bears reaching their environmental limit in approximately 100 years.

2.3 Conclusion

In this chapter, we found the growth rate and carrying capacity for the salmon and brown bear species to be: $r_x = \ln(0.32 * 5) \approx 0.47$, $K_x = 29.1 * 10^6$, $r_y = 0.059$, and $K_y = 4.5 * 10^4$. A logistic model was used to represent both species, as shown below,

$$\frac{dx}{dt} = r_x x \left(1 - \frac{x}{K_x}\right),$$

and

$$\frac{dy}{dt} = r_y y \left(1 - \frac{y}{K_y}\right).$$

The solutions to these models show that the salmon species reaches its environmental limit in 14 years and the brown bears reach theirs in 100 years. Now that each species has a foundational model for their growth behavior, in the next chapter, we will create a function dependent on time that replaces the growth rate parameter for salmon.

Chapter 3

Salmon Growth Rate Function

In this chapter, we propose a salmon growth rate function that depends on time. First, we use research articles to obtain river temperatures during spawning migration that indicate when salmon survival is optimal versus minimal. Then, we use Katherine Carter's article, "The effects of temperature on steelhead trout, coho salmon, and chinook salmon biology and function by life stage" to extrapolate the proportion of salmon that would survive spawning migration in each of the obtained temperatures. Next, we design a function that estimates the survival proportion of salmon during spawning migration with respect to temperature. After that, we sample data from the United States Geological Survey (USGS) to construct a temperature growth model dependent on time. Then, we replace the temperature parameter in the survival proportion function with the temperature model, resulting in the function being dependent on time. Lastly, we combine this function with the current growth rate parameter, which produces the proposed salmon growth rate function.

3.1 Growth Rate Function Dependent on Temperature

Salmon have an optimal temperature range for the rate at which they grow, migrate, and reproduce. If the temperature of their environment goes outside of that range, then salmon may change their spawning location or time frame of migration [23]. If they do not take either of those options, then they may fatigue and die before reaching their spawning location [23, 24]. So, when temperatures reach a critical point, mortality rates increase significantly, which consequently decreases their growth rate [23]. The purpose of this section is to use salmon's mortality rate during spawning migration at different temperatures to approximate a growth rate function, $R(T)$, that is dependent upon temperature. While there is little research that scientifically describes the effects of salmon population growth at each temperature, there are reports that estimate the optimum temperature range for maximum survival and critical points where survival becomes unlikely. Dr. Phyllis Weber Scannell wrote an article in 1992 for the Alaskan Department of Fish and Game (ADFG) about the optimal temperature ranges for cold water fish. In this article, she highlights the optimal range as well as the critical high temperature of sockeye salmon in Alaska [23]. Also, Katherine Carter has published an article that suggests temperatures below 2°C will result in a high mortality rate [25].

Table 3.1: Optimal Temperature Range For Pacific Sockeye Salmon

Species	**Optimal (°C)**	**Low (°C)**	**High (°C)**
Sockeye	$11 - 14$	< 2	> 22.2

*The optimal, critical high and low temperature range for the sockeye salmon species in Alaska [23, 25].

As Dr. Scannel's report states, each researcher estimates slightly different temperature ranges due to a multitude of variables such as acclimation, age, size, genetic strain, and physiological conditions of the fish [23]. That being said, we use Table 3.1 to help fit a curve that best illustrates the impact of temperature on the proportion of salmon that survive their spawning migration. Now, Katherine Carter's article explains that at these critical points the population could have a mortality rate of 50% [25]. So, we estimate that under ideal conditions and optimal temperatures, 100% of the salmon population would survive the spawning migration to reproduce, and at critical temperatures, only 50% would survive. From Table 3.1 the optimal temperature range is $11 - 14$°C and the critical temperature points are at 2°C and 22.2°C, which can be observed on the graph below.

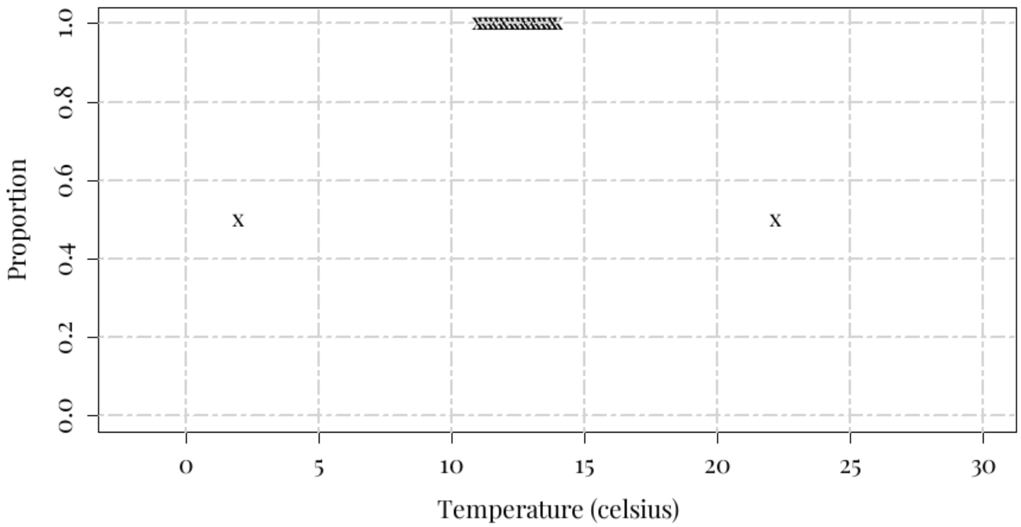


Figure 3.1: Scatter plot of the survival rates in the optimal temperature range and at the critical temperature points.

Given that the width of the optimal range is rather large, developing a function to approximate these data points will be rather difficult. The proportion of salmon surviving spawning migration cannot drop below 0, which implies that we should be looking at a function similar to the one displayed below,

$$P(T) = \frac{1}{1 + c(T - T_{opt})^p}, \quad p \,\epsilon\, 2\mathbb{N}, \tag{3.1}$$

where $P(T)$ estimates the proportion of salmon that will survive spawning migration at a given temperature, T, in Celsius, and T_{opt} is the average of the optimal temperature range, 12.5°C. The power of the binomial, p, controls the average rate of change of the proportions, and c is a constant that will be calculated to stretch or compress the function horizontally. Now, we get the graph below by setting the parameters $p = 2$ and $c = 1$ as a starting point for the function above.

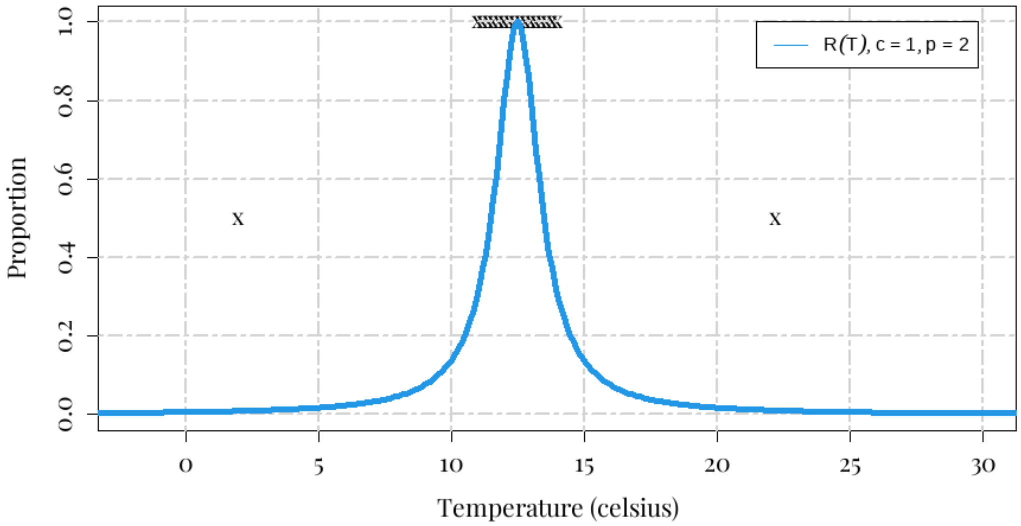


Figure 3.2: Plot of the proportion function, where $c = 1$ and $p = 2$.

The main issue with these parameters is that the peak of the curve is too narrow to represent the optimal range well, and the curve is too far away from the critical temperature points, $(2, 0.5)$ and $(22.2, 0.5)$. From the graph above, the survival proportion of migrating salmon at the limits of their optimal temperature range is $P(T = 11) = P(T = 14) \approx 0.31$, which is a major deviation from the proposed survival proportion. So, by adjusting the parameter c, we can stretch the function to better fit the survival proportions for the critical and optimal temperatures. This can be done by taking the average of the distances from T_{opt} to the critical temperatures, as shown below,

$$Avg = \frac{|T_{opt} - 2| + |T_{opt} - 22.2|}{2} = 10.1.$$

From here we can set $P(T) = 0.5$ and $T - T_{opt} = Avg = 10.1$ and solve for c,

$$c = \frac{1 - P(T)}{P(T)(T - T_{opt})^2} = \frac{1 - 0.5}{0.5(10.1)^2} = \frac{1}{10.1^2} = \frac{1}{102.1} \approx 0.01.$$

Now, plotting $P(T)$ with parameters $p = 2$ and $c = 0.01$ produces the plot below.

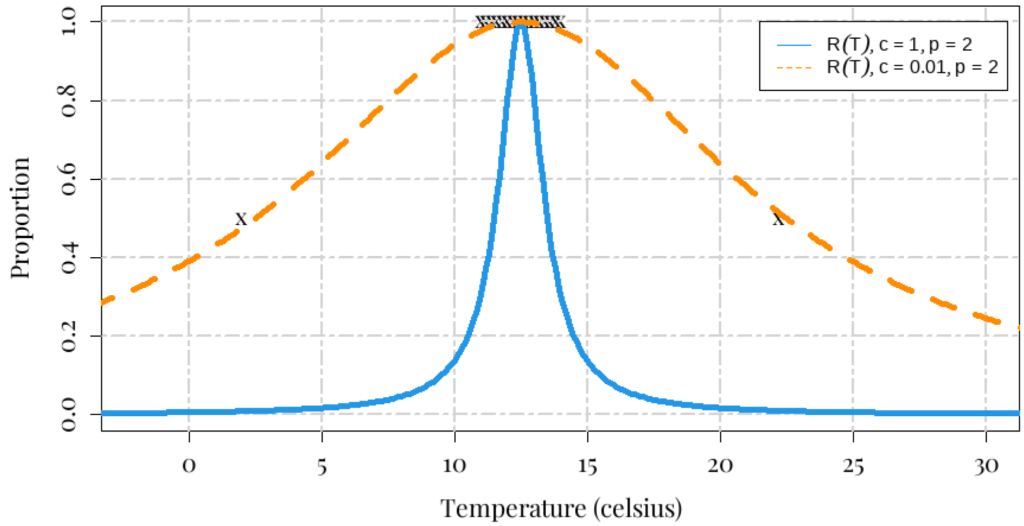


Figure 3.3: Compares the plots of the proportion function, where $c = 1$ and $c = 0.01$, but $p = 2$ remains the same.

From the figure above, the low and high critical temperature points are better represented with the new parameter, $c = 0.01$. However, at the limits of the optimal temperature range, the survival proportion is 0.978, which should be closer to 1. This can be resolved by changing the power of the binomial, $p = 2$, in Equation (3.1) to $p = 4$, which will widen the curve while maintaining a steep descent as the temperature escapes the optimal region. So, substituting the new parameters, $c = 0.01$, and $p = 4$, the graph below is produced.

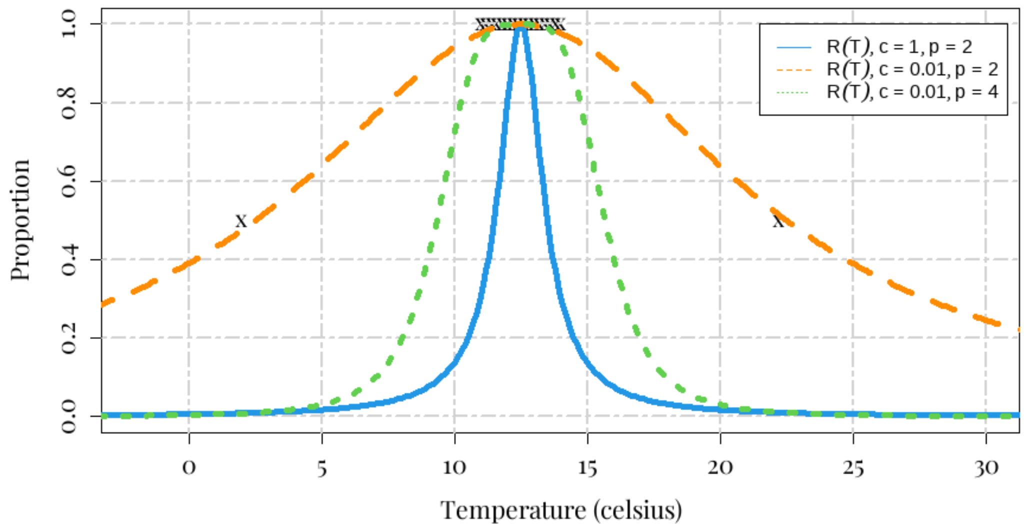


Figure 3.4: Compares the plots of Figure 3.3 with the plot of the proportion function, where $c = 0.01$ and $p = 4$.

With this figure, the representation of the optimal range is better, but the proportions of salmon survival decrease significantly as T approaches the limits of the optimal range. Also, the survival proportions at the critical temperatures are far from the points, $(2, 0.5)$ and $(22.2, 0.5)$. To resolve this issue, we can repeat the same process as earlier to select a new c value that accurately reflects the proportions at the optimal and critical temperatures. As a result, the graph below is produced by substituting the new parameter $c = 10^{-4}$ into the proportion function.

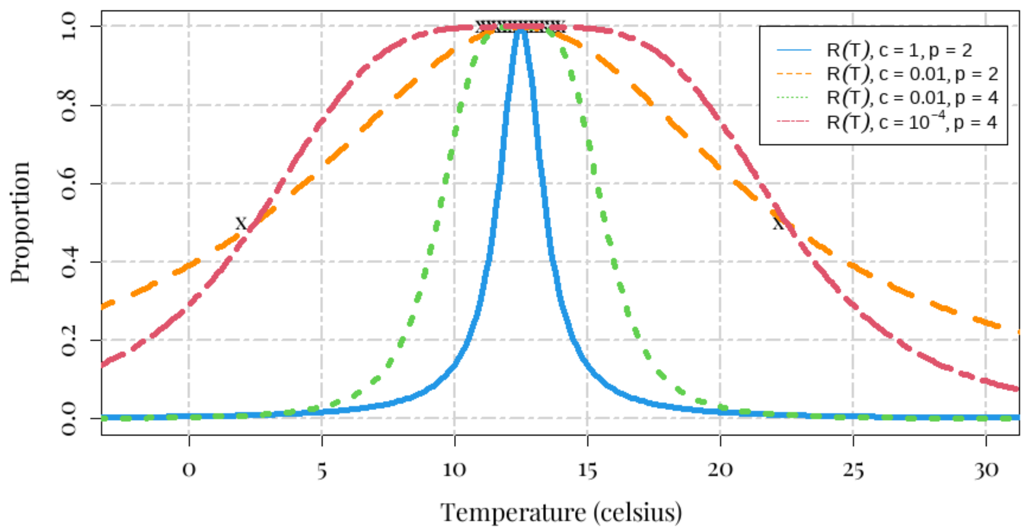


Figure 3.5: Compares the plots of Figure 3.4 with the plot of the proportion function, where $c = 10^{-4}$ and $p = 4$.

The parameters, $c = 10^{-4}$ and $p = 4$, offer a better fit, with the survival proportion being 0.9995 at the optimal temperature limits and $P(2) \approx 0.45$ and $P(22.2) \approx 0.53$. By fixing $c = 10^{-4}$, $p = 4$ and T_{opt} in Equation (3.1), we get

$$P(T) = \frac{1}{1 + c(T - T_{opt})^p} = \frac{1}{1 + 10^{-4}(T - 12.5)^4}, \tag{3.2}$$

where P(T) represents the proportion of salmon that survive spawning migration with respect to temperature. During the salmon migration of 2004, Weaver Creek sockeye salmon experienced a drastic rise in water temperature, which resulted in a higher than usual mortality rate [26]. According to Anthony P. Farrell, temperatures were around 20.4°C and 30% of the salmon population did not make it to the spawning location due to the excessive heat [26]. Using Equation (3.2), we get $P(20.4) = 0.7197$. Therefore, we estimate a 72% survival rate, or a mortality rate of approximately 28%, for the salmon migrating to their spawning locations, which is close to Anthony P. Farrell's estimation of 30%.

Looking back, the growth rate, $r_x = \ln(0.32 * 5)$, was estimated when temperatures were ideal, or in the optimal range, so combining the proportion function with the current growth rate, we get the function below,

$$R(T) = \ln(0.32 * 5 * P(T)) = \ln\left(\frac{0.32 * 5}{1 + c(T - T_{opt})^4}\right), \tag{3.3}$$

where $c = 10^{-4}$, $T_{opt} = 12.5°\text{C}$, and T is temperature. Lastly, we will replace the growth rate, r_x, with the growth rate function, $R(T)$, in Equation (2.4) to get the below equation,

$$\frac{dx}{dt} = R(T)x\left(1 - \frac{x}{K_x}\right). \tag{3.4}$$

To see the effect of temperature on the salmon population, we will compare Equation (3.4) at different temperatures.

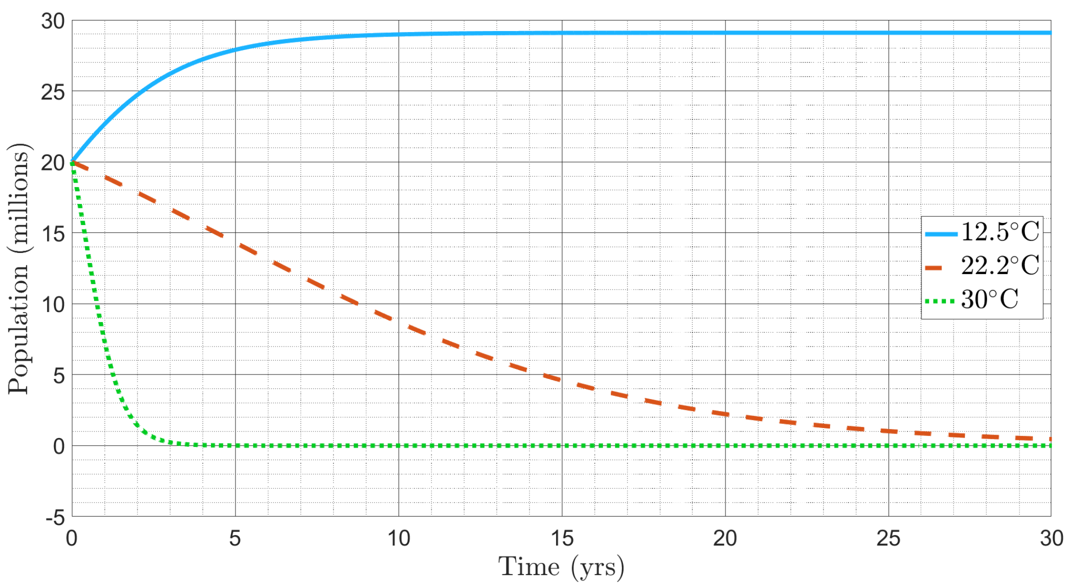


Figure 3.6: Plot of the salmon logistic growth model using the growth rate function, Equation (3.4), at 3 different temperatures.

Notice, at the optimal temperature, $T = 12.5°\text{C}$, the curve is the same as in Figure 2.3 because $R(12.5) = r_x = 0.47$. As the temperature moves further away from the optimal temperature, the reproduction of salmon is negatively affected, resulting in a decay rate, which can be observed in the middle and bottom curves. When $T = 22.2°\text{C}$ the growth rate is $R(22.2) = -0.1641$, which is explains why the population is decreasing over time. Notice, as the temperature moves drastically far away from the optimal temperature, $T = 30°\text{C}$, the growth rate changes to $R(30) = -1.8698$, causing the population to die off in about 5 years. By replacing the growth rate of salmon with a function dependent upon temperature, we can see the drastic effects on the salmon population as temperature changes.

3.2 Temperature Function Dependent on Time

The global temperature of the earth has been increasing exponentially over the past 100 years [5]. Temperatures are expected to keep increasing for at least the next 30 years unless changes are made now to the emissions of greenhouse gas [27, 28]. Below is a graph illustrating the annual deviation of global surface temperature from the 20^{th} century average of 13.9°C.

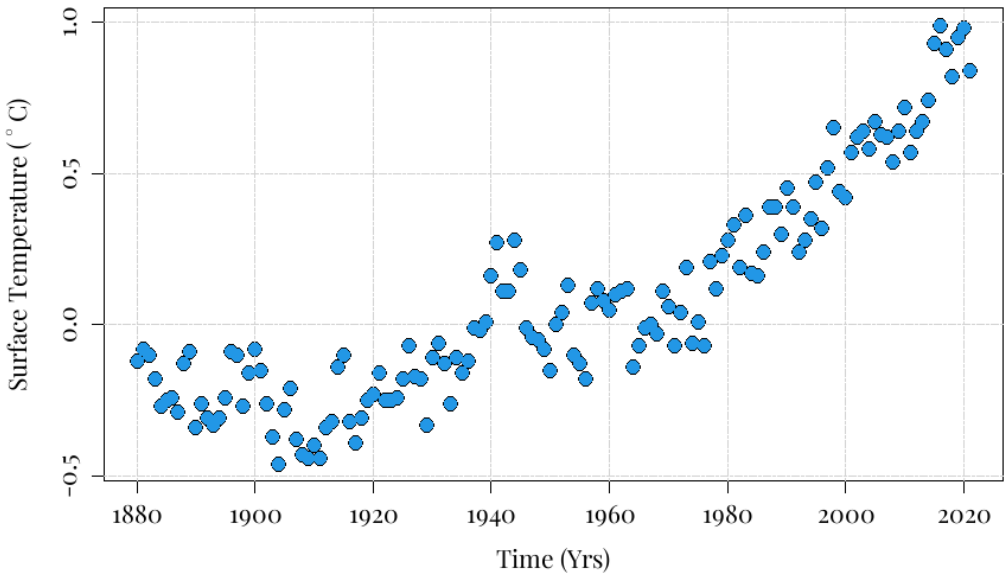


Figure 3.7: Scatter plot of the average annual global temperatures compared to the 20^{th} century average.

The data for Figure 3.7 comes from the National Oceanic and Atmospheric Administration (NOAA) [29]. Any points below 0°C represent the years when temperatures were less than 13.9°C, and the points above 0°C represent the years when temperatures were greater than 13.9°C. The earth's surface temperature appears to decrease from 1880 to 1910, then increases exponentially to a little after 1940. From here, the temperature decreases by approximately 0.05°C entering 1965 before increasing again to the present. Starting around 1970 to the present, global surface temperatures appear to increase linearly. While an exponential regression model can be used to fit the data, a quadratic model would seem to work better because of the initial decrease from 1880 to 1910. The quadratic model would look like

$$T(t) = at^2 + bt + c, \tag{3.5}$$

where $a = 7.95 * 10^{-4}$, $b = -30.25 * 10^{-2}$, and $c = 287.57$. The response variable, $T(t)$, represents temperature with the units, °C, that is dependent on time, t, expressed in years. This function seems to fit the data well with the graph below.

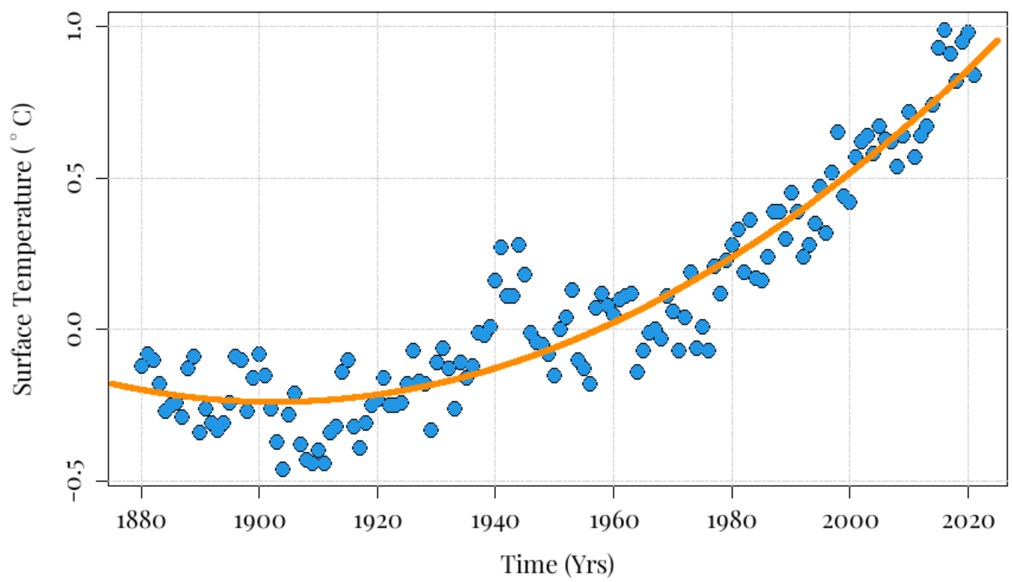


Figure 3.8: Plot of the quadratic function, $T(t)$, on top of the scatter plot given in Figure 3.7.

There is a possible issue that should be explored before continuing. This model projects the change in global surface temperatures of the earth, but salmon live in the ocean. So, designing a model to fit the earth's change in surface temperature over time might not accurately reflect the environmental temperatures of this species. The National Oceanic and Atmospheric Administration also collect data on the global sea surface temperature over the same time period [29]. Below is a graph looking at the global sea surface temperature anomalies with respect to the 20$^{\text{th}}$ century average of 13.9°C.

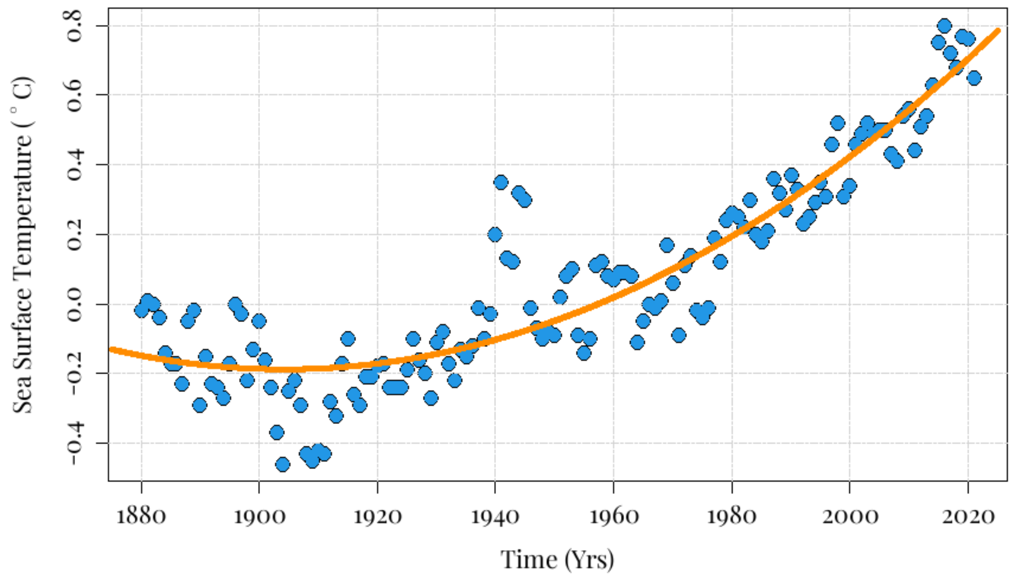


Figure 3.9: Scatter plot of the average annual sea surface temperatures compared to the 20th century average fitted with the quadratic function, $T(t)$, with new coefficients.

Since the graph above has a similar trend to Figure 3.7, a quadratic model seems to fit this data well. The new parameters for the quadratic model are, $a = 6.67 * 10^{-5}$, $b = -0.25$, and $c = 241.53$. Looking at Figure 3.9 after 1970, the trend appears linear, which means the quadratic equation may not be the best choice for predicting temperature. Because of this, we will look at sea surface temperatures (SST) after 1970. Also, Alaska is ranked 40th in the nation with total greenhouse gas emissions, which may affect that region's SST trend differently than other regions [30].

Alaska has a large number of river streams, but salmon can be seen predominately in the southern parts of Alaska, such as Anchorage, the Kenai Peninsula, near Juneau, and Alaska Peninsula [10]. According to the ADFG, salmon swim in these streams from June to September [10]. Therefore, we sampled water temperature data in these regions during these months to model the change in temperature over time. The data was provided by the United States Geological Survey (USGS) [31]. When using the USGS database, there are plenty of streams where the Alaskan government was collecting data, but there are a few issues when looking at the data sets for some of the streams [31]. First, most data sets are a small duration of a few years, which is not enough time to model a trend. Second, some data sets are missing data for a couple of months every year or even just had big gaps for several years. We set criteria for the rivers we wanted to sample; each river needed consistent data for at least 15 years during the months when salmon swim upstream. In the end, only 5 data sets are usable for analyzing trends over time. The 5 streams we use for this analysis are Cooper Creek on the Kenai Peninsula, Kenai River at Cooper Landing, Russell Creek on the Alaska Peninsula, Terror River, just south of the Alaska Peninsula, and Staney Creek which is south of Juneau. We initially combined the data for each river and calculated the average water temperature per year.

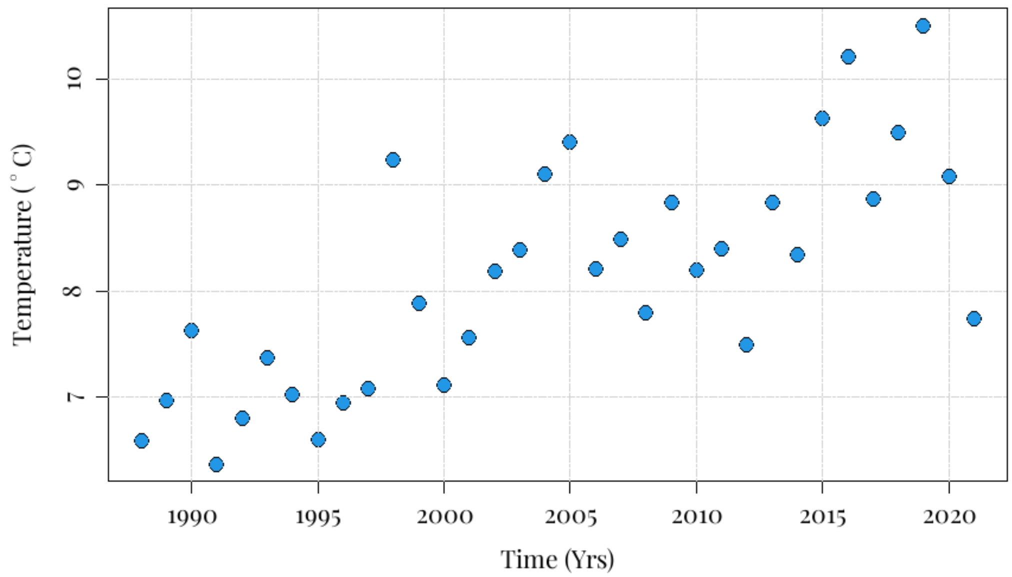


Figure 3.10: Scatter plot of the average annual water temperatures during the months of June to September for the combined data of the rivers; Cooper Creek, Kenai River, Russell Creek, Terror River, and Staney Creek.

Figure 3.10 shows that the trend for the average water temperature per year is fluctuating between increasing and decreasing from 1988 to 2021. Overall, the data appears to be increasing for the past 33 years. From Figure 3.10, we can see that, on average, the water temperature is consistently increasing. This may imply that a linear regression model would fit the data well. Before fitting the combined river data with a linear model, we have to make sure that each river's change in water temperature over time appears to be increasing linearly. Below are the plots of each stream fitted with a linear model that closely represents their trend.

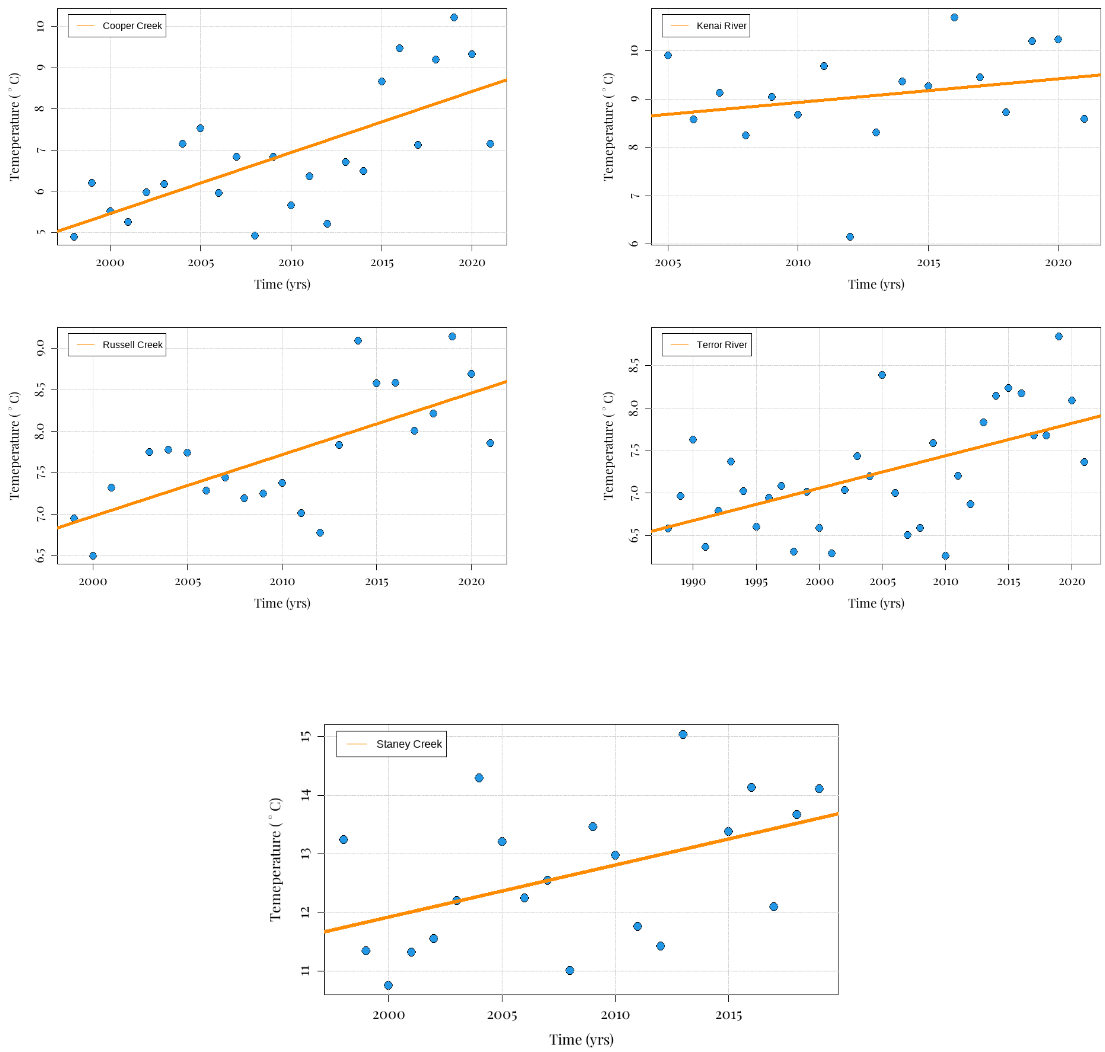


Figure 3.11: Plots of each river's average annual water temperature trend, fitted with a linear model.

Notice in Figure 3.11, the time span of the recorded water temperatures for some of the rivers are different. This could cause the combined data used in Figure 3.10 to be biased towards rivers with a longer time span. We can compare the average increase in water temperature for Figure 3.10 to the average of the slopes for the rivers in Figure 3.11. If the data is biased, we should be able to see it in the comparison. Each river has a similar trend to the combined data in Figure 3.10

with the mean of their slopes representing an average increase in annual water temperature of 0.0797 °C per year. When fitting a linear model to Figure 3.10 the average increase in water temperature is 0.0803 °C per year.

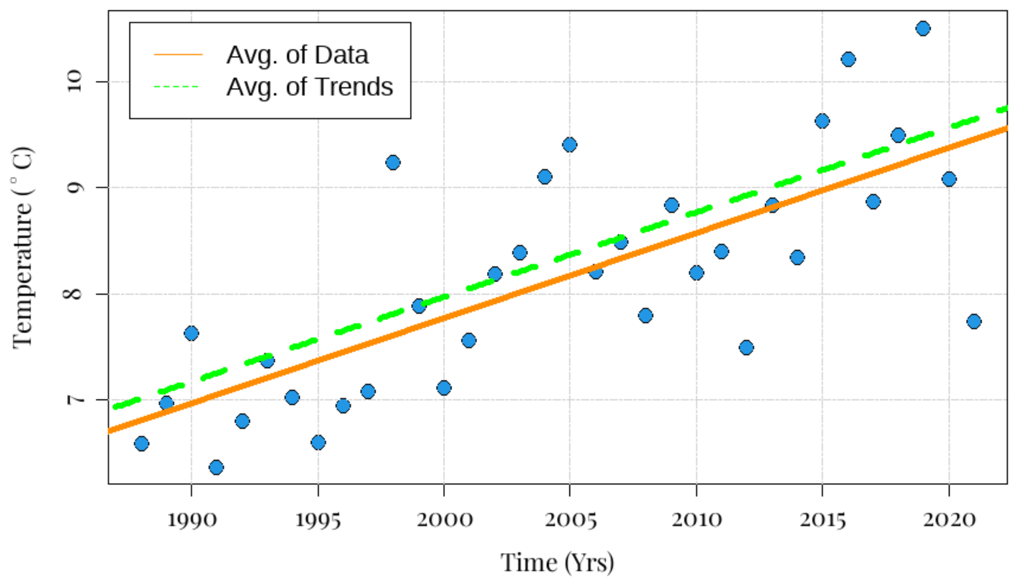


Figure 3.12: The solid line represents the average change in annual water temperature of the combined data for the 5 rivers seen in Figure 3.10. The dashed line represents the average of the slopes from each river's linear model seen in Figure 3.11.

The figure above illustrates that for the past 33 years the average change in Alaskan river temperature during the months of salmon spawning migration has a linear growth of approximately 0.08 °C per year. The model for the change in water temperature in Alaska can now be represented as

$$T(t) = a * t + b, \tag{3.6}$$

with $a = 0.08$ and $b = 9.54$. The coefficient, a, represents the average increase in water temperature over time, and the intercept, b, represents the average water temperature for the year 2022. We will use this model to predict water temperatures in Alaska during the months when salmon migrate to spawning locations. Now, substituting the function $T(t)$ for the parameter T in Equation (3.3) gives us

$$\begin{aligned} R(T(t)) = \ln\big[0.32 * 5 * P(T(t))\big] &= \ln\left(\frac{0.32 * 5}{1 + c(T(t) - T_{opt})^4}\right) \\ &= \ln\left(\frac{0.32 * 5}{1 + c(at + b - T_{opt})^4}\right). \end{aligned} \tag{3.7}$$

Then, substituting in all the parameter values gives the following equation,

$$R(t) = \ln\left(\frac{0.32 * 5}{1 + 10^{-4}(0.08t - 2.96)^4}\right). \tag{3.8}$$

In Equation (2.3) the growth rate, r_x, was revealed after taking the derivative of Equation (2.2). So, by replacing r_x in Equation (2.2) with $R(t)$ we get

$$x(t) = x_0 e^{R(t)t}.$$

Then, taking the derivative produces the following,

$$\frac{dx}{dt} = [R(t) + R'(t)t]x_0 e^{R(t)t} = [R(t) + R'(t)t]x,$$

where

$$R'(t) = \frac{d}{dt}\ln(0.32 * 5 * P(t)) = \frac{P'(t)}{P(t)}. \tag{3.9}$$

As briefly shown in Equation (3.7), $P(T)$ from Equation (3.2) can be rewritten as a function of time using Equation (3.5),

$$P(t) = \frac{1}{1+10^{-4}(0.08t-2.96)^4}, \tag{3.10}$$

where

$$P'(t) = \frac{-4*10^{-4}*0.08(0.08t-2.96)^3}{(1+10^{-4}(0.08t-2.96)^4)^2}. \tag{3.11}$$

So, the growth rate function can now be written as

$$G(t) = \ln\left(\frac{0.32 * 5}{1 + 10^{-4}(0.08t - 2.96)^4}\right) - \frac{4*10^{-4}*0.08t(0.08t-2.96)^3}{1+10^{-4}(0.08t-2.96)^4}, \tag{3.12}$$

or in a more simplified form,

$$G(t) = R(t) + \frac{P'(t)t}{P(t)}. \tag{3.13}$$

To better understand what the growth rate function is doing, we graph the function below.

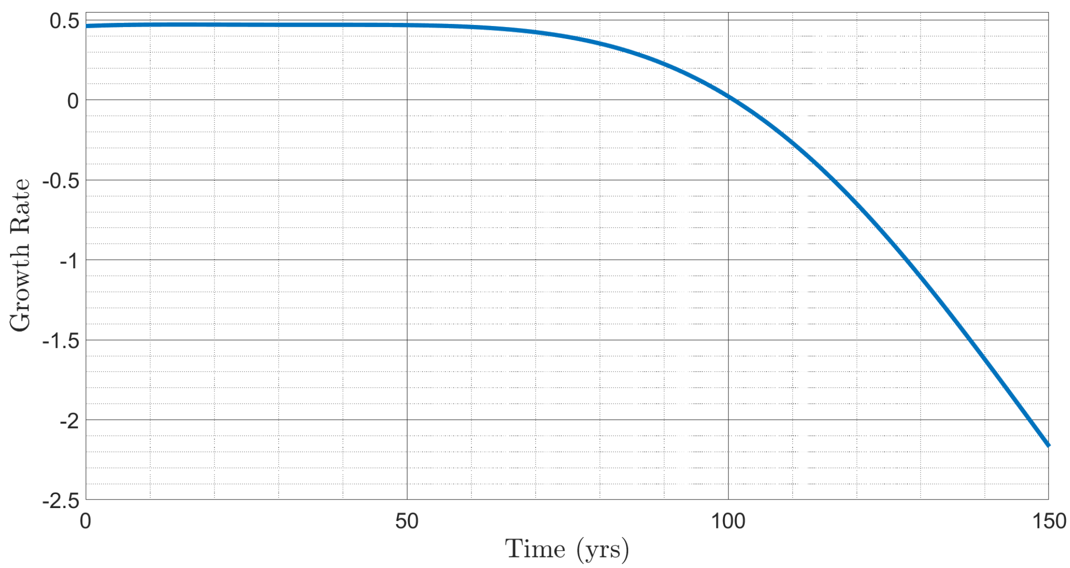


Figure 3.13: Plot of the growth rate function, Equation (3.13), over a time span of 150 years.

We can see in the figure above that the growth rate remains positive for approximately the first 100 years. After this, the growth rate becomes negative indefinitely. Now, as shown below, we can substitute the above growth rate function into Equation (3.4).

$$\frac{dx}{dt} = G(t)x\left(1 - \frac{x}{K_x}\right). \tag{3.14}$$

Since the model for the salmon population is now dependent on time, it becomes a non-autonomous ordinary differential equation. When comparing this model to Equation (3.4), the below figure is produced.

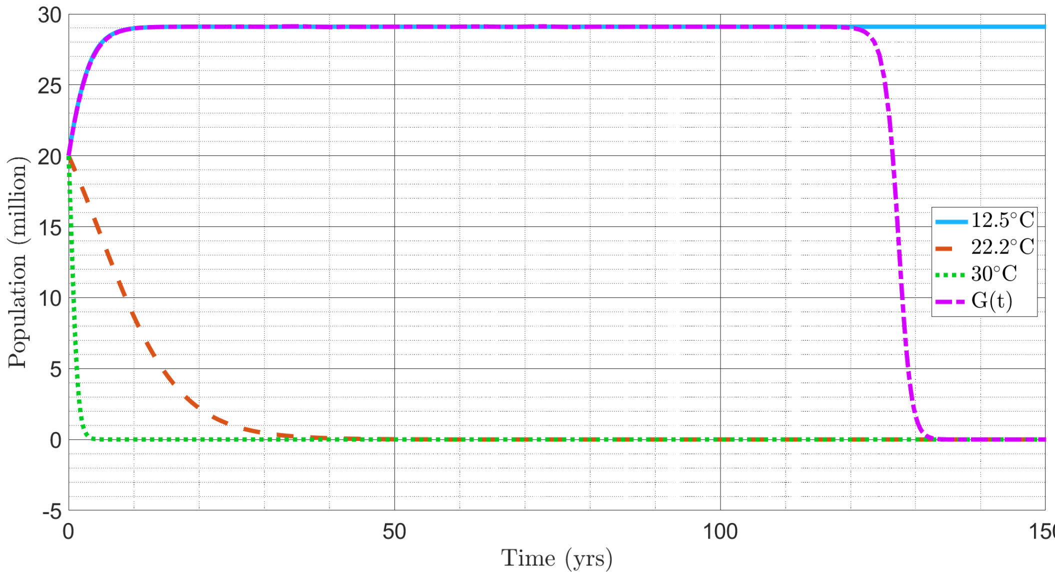


Figure 3.14: Solutions to the autonomous system for some values of water temperature T, and the non-autonomous system as a function of time.

Using the **vpasolve** function in MATLAB, we found that in approximately 101 years, the growth rate will change from positive to negative, which is the beginning of the population's descent. When substituting $t = 101$ into Equation (3.6), we get $T(101) = 17.62$°C. This temperature can be denoted as our inflection temperature for the salmon population. Figure 3.14 suggests that the water temperature will be too hot for the salmon population in the future, resulting in their death or regional extinction.

3.3 Conclusion

In this chapter, we proposed a salmon growth rate function that depends on water temperature. We used river temperature data from the United States Geological Survey (USGS) to design a function that models the increase in Alaskan river temperature over time, which can be seen below,

$$T(t) = a * t + b,$$

where $a = 0.08$, and $b = 9.54$. Also, $t = 0$ represents the current year, 2022. We then make the growth rate function dependent on time by replacing the temperature parameter T with the temperature function, as shown bellow,

$$\begin{aligned} G(t) &= R(t) + \frac{P'(t)t}{P(t)} \\ &= \ln\left(\frac{0.32*5}{1+10^{-4}(0.08t-2.96)^4}\right) - \frac{4*10^{-4}*0.08t(0.08t-2.96)^3}{1+10^{-4}(0.08t-2.96)^4}. \end{aligned}$$

Lastly, we replaced this new growth rate function with the growth rate parameter in the original logistic model, Equation (2.4), and compare its results. We found that after approximately 100 years, the salmon population will begin to decline and eventually die off or migrate elsewhere. In the next chapter, we will examine the effect of interaction between the brown bear and salmon species. We will compare the results of the interaction with and without the growth rate function.

Chapter 4

Interaction Between Species

We begin this chapter by introducing the Lotka-Volterra equations, which describe the interactive relationship between two species using a system of first-order nonlinear differential equations. The Lotka-Volterra equations, also called the predator-prey model, demonstrate that without interaction, the prey population will grow exponentially while the predator population will decay exponentially. We then introduce Theodore Modis' competitive predator-prey model, a variation of the Lotka-Volterra equations where the logistic growth equation is the base model for both species. In the following section, we propose a model similar to Theodore Modis' that portrays the interactive relationship between brown bears and salmon when both species are unaffected by climate change. In the third section, we solve for the critical points of our model and determine the stability of the populations near their critical points. We then evaluate the solutions of the model with different pairs of interaction parameters to determine the behavior of both species over time. We fix the interaction parameters to a pair of values that create a noticeable oscillation of solutions for both species. In the next section, we introduce climate change to our model by replacing the growth rate parameter with the growth rate function, $G(t)$. Using different initial conditions for the model, we visualize the species' behavior over time. Lastly, in this chapter, we compare both models, concluding that global warming will cause the salmon population in Alaska to become regionally extinct and the Alaskan brown bears to compensate for the loss of a food resource by decreasing their population size.

4.1 Lotka-Volterra Equations

In nature, most animals share their environment, which sometimes causes species to interact, like salmon and brown bears. This relationship can be portrayed by incorporating interaction terms into each species' population equation. The Lotka-Volterra model, also referred to as the predator-prey model, is a simple system of two nonlinear ordinary differential equations that utilize interaction terms to imitate the relationship between two species [32],

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= \delta xy - \gamma y. \end{aligned} \tag{4.1}$$

Consider x as the prey, y as the predator, and α, β, δ, γ are positive real parameters that describe the interaction of the two species. Also, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent each species' instantaneous population rate of change. The interaction term for species x is subtracted from the exponential growth component, αx, to describe that the instantaneous growth rate of species x will decrease as y increases. The opposite effect happens to species y because its interaction term is added instead of subtracted. The author of "US Nobel laureates: Logistic growth versus Volterra–Lotka", Theodore Modis, developed a competitive predator-prey model that implements logistic growth into the Lotka-Volterra equations,

$$\begin{aligned} \frac{dx}{dt} &= a_x x - b_x x^2 + c_{xy} xy, \\ \frac{dy}{dt} &= a_y y - b_y y^2 + c_{yx} xy, \end{aligned} \tag{4.2}$$

where a_x, a_y, b_x, b_y, c_{xy} and c_{yx} are real parameters that describe the interaction of the two species [33].

4.2 Introducing Interaction

Currently, we have constructed models for both species that represent their populations without the influence of each other. By including interaction terms for both models, we can simulate a trade-off of environmental resources, as we would see in the real world. First, we use a variation of Theodore Modis' model, Equation (4.2), to introduce interaction between the brown bears and salmon when both species are unaffected by climate change,

$$\begin{aligned} \frac{dx}{dt} &= r_x x \left(1 - \frac{x}{K_x}\right) - c_{xy} xy, \\ \frac{dy}{dt} &= r_y y \left(1 - \frac{y}{K_y}\right) + c_{yx} xy, \end{aligned} \tag{4.3}$$

where $r_x = \ln(0.32 * 5)$, $r_y = 0.059$, and $c_{xy},\ c_{yx} > 0$. Equations (2.4) and (2.5) represent the base models of the above system of differential equations where both species are unaffected by climate change. Notice, for the salmon ODE, we subtract its interaction term, but for the brown bears, we add its interaction term. We do this because the salmon population should have a negative consequence when there is an increase in brown bears. In contrast, brown bears should be rewarded when their food source increases. The interaction parameters alter the effect of the carrying capacity, so we change $K_x = 15$ and $K_y = 5$ as a vague measure of their environmental limits. We can rewrite the system of equations in a similar form to Theodore Modis' model, as shown below,

$$
\begin{aligned}
\frac{dx}{dt} &= r_x x - \frac{r_x x^2}{K_x} - c_{xy}xy, \\
\frac{dy}{dt} &= r_y y - \frac{r_y y^2}{K_y} + c_{yx}xy.
\end{aligned}
\tag{4.4}
$$

Then, substituting $a_x = r_x$, $b_x = \frac{r_x}{K_x}$, $a_y = r_y$, and $b_y = \frac{r_y}{K_y}$ we get

$$
\begin{aligned}
\frac{dx}{dt} &= a_x x - b_x x^2 - c_{xy}xy, \\
\frac{dy}{dt} &= a_y y - b_y y^2 + c_{yx}xy.
\end{aligned}
\tag{4.5}
$$

Condensing the model reduces the number of parameters in each equation, which makes the model and any operations done to the model more readable and interpretable. In the next section, we determine how different values for the parameters, c_{xy} and c_{yx}, control the stability of the populations near their critical points.

4.3 Critical Points and Their Stability

We find the critical points by setting the rate functions in Equation (4.5) to 0,

$$
\begin{aligned}
0 &= x(a_x - b_x x - c_{xy}y), \\
0 &= y(a_y - b_y y + c_{yx}x).
\end{aligned}
$$

Then, we solve for x and y to get the following critical points;

$$
\begin{aligned}
x_1^* &= 0, & y_1^* &= 0, \\
x_2^* &= \tfrac{a_x}{b_x} = K_x, & y_2^* &= 0, \\
x_3^* &= 0, & y_3^* &= \tfrac{a_y}{b_y} = K_y, \\
x_4^* &= \tfrac{a_x b_y - c_{xy} a_y}{c_{xy} c_{yx} + b_x b_y}, & y_4^* &= \tfrac{a_x c_{yx} + b_x a_y}{c_{xy} c_{yx} + b_x b_y}.
\end{aligned}
$$

We can determine the stability at each of the above critical points by finding the eigenvalues of our model, Equation (4.5) [34]. We begin by constructing the Jacobian matrix,

$$J_{(x,y)} = \begin{pmatrix} a_x - 2b_x x - c_{xy} y & -c_{xy} x \\ c_{yx} y & a_y - 2b_y y + c_{yx} x \end{pmatrix}. \tag{4.6}$$

Then, we derive the characteristic polynomial,

$$\begin{aligned} \det(J_{(x,y)} - \lambda I) = \lambda^2 - \big[\, (a_x - 2b_x x - c_{xy} y) + (a_y - 2b_y y + c_{yx}) \,\big]\lambda \\ + \big[\, (a_x - 2b_x x - c_{xy} y)(a_y - 2b_y y + c_{yx}) + c_{xy} x c_{yx} y \,\big]. \end{aligned} \tag{4.7}$$

Note that the trace and determinant of the Jacobian matrix are

$$\begin{aligned} \mathrm{T} &= \mathrm{tr}(J_{(x,y)}) = (a_x - 2b_x x - c_{xy} y) + (a_y - 2b_y y + c_{yx}), \\ \mathrm{D} &= \det(J_{(x,y)}) = (a_x - 2b_x x - c_{xy} y)(a_y - 2b_y y + c_{yx}) + c_{xy} x c_{yx} y. \end{aligned}$$

So, substituting the above variables in Equation (4.7) produces

$$\det(J_{(x,y)} - \lambda I) = \lambda^2 - \mathrm{T}\lambda + \mathrm{D}.$$

Now, solving for our eigenvalues, λ, gives the below equation,

$$\lambda = \frac{\mathrm{T} \pm \sqrt{\mathrm{T}^2 - 4\mathrm{D}}}{2}.$$

Using the equation above, we can determine the signs and potentially the numeric values of the real and imaginary parts of the eigenvalues for each critical point. Starting with the first critical point, $(0,\ 0)$, we get

$$\begin{aligned} &\mathrm{T} = a_x + a_y, \\ &\mathrm{T}^2 - 4\mathrm{D} = (a_x - a_y)^2. \end{aligned}$$

So, substituting in the values for a_x and a_y, we get

$$\begin{aligned} &\mathrm{T} = \ln(0.32 * 5) + 0.059 \approx 0.529, \\ &\mathrm{T}^2 - 4\mathrm{D} = (\ln(0.32 * 5) - 0.059)^2 \approx 0.169. \end{aligned}$$

Since $\mathrm{T} \approx 0.529 > \sqrt{\mathrm{T}^2 - 4\mathrm{D}} \approx 0.411$, both eigenvalues are positive real values, which implies that the first critical point is an unstable node. Now, for the second critical point, $\left(\frac{a_x}{b_x},\ 0\right)$, we get

$$\mathrm{T} = a_x\left(\frac{c_{yx}}{b_x} - 1\right) + a_y,$$
$$\mathrm{T}^2 - 4\mathrm{D} = \frac{(a_x(b_x+c_{yx})+a_y b_x)^2}{b_x^2},$$

where the stability is dependent on c_{yx}. Because the value for the discriminant is squared, it is always positive, which implies that the eigenvalues are real. When we substitute the parameters, a_x, b_x, and a_y for their numeric values, we get the following criterion;

$$\mathrm{T} + \sqrt{\mathrm{T}^2 - 4\mathrm{D}} > 0,$$
$$\mathrm{T} - \sqrt{\mathrm{T}^2 - 4\mathrm{D}} < 0.$$

This implies that one eigenvalue is positive while the other is negative, which makes $\left(\frac{a_x}{b_x},\ 0\right)$ a saddle point. Now, looking at the third critical point, $\left(0,\ \frac{a_y}{b_y}\right)$, the trace and discriminant are;

$$\mathrm{T} = a_x - \frac{a_y(b_y+c_{xy})}{b_y},$$
$$\mathrm{T}^2 - 4\mathrm{D} = \frac{(a_x b_y+a_y(b_y-c_{xy}))^2}{b_y^2},$$

where the stability is now dependent on c_{xy}. Just like for the last critical point, the discriminant is always positive, which implies the eigenvalues are also real. Then, we substitute the values for, a_x, a_y, and b_y, into the above equations to get the criterion below;

$$\mathrm{T} + \sqrt{\mathrm{T}^2 - 4\mathrm{D}} > 0 \quad \text{when} \quad c_{xy} < 0.094,$$
$$\mathrm{T} - \sqrt{\mathrm{T}^2 - 4\mathrm{D}} < 0.$$

From the signs of the eigenvalues, we determine that $\left(0,\ \frac{a_y}{b_y}\right)$ is a saddle point when $c_{xy} < 0.094$, and a stable node when $c_{xy} > 0.094$. Lastly, with the critical point,

$$(x_4^*,\ y_4^*) = \left(\frac{a_x b_y - c_{xy} a_y}{c_{xy} c_{yx} + b_x b_y},\ \frac{a_x c_{yx} + b_x a_y}{c_{xy} c_{yx} + b_x b_y}\right),$$

we get the following trace and discriminant equations;

$$\begin{aligned}
\mathrm{T} &= \frac{a_y b_x (c_{xy} - b_y) - a_x b_y (b_x + c_{yx})}{b_x b_y + c_{xy} c_{yx}}, \\
\mathrm{T}^2 - 4\mathrm{D} &= \frac{(a_y b_x (c_{xy} - b_y) - a_x b_y (b_x + c_{yx}))^2}{(b_x b_y + c_{xy} c_{yx})^2} \\
&\quad - \frac{4(a_x c_{yx} + a_y b_x)(a_x b_y - a_y c_{xy})(b_x b_y + c_{xy} c_{yx})}{(b_x b_y + c_{xy} c_{yx})^2},
\end{aligned}$$

where the stability is dependent on parameters, c_{xy} and c_{yx}. Since the other critical points contain one or more species with a population size of 0, we construct the following constraints for the critical point;

$$x_4^* = \frac{a_x b_y - c_{xy} a_y}{c_{xy} c_{yx} + b_x b_y} > 0, \quad \text{and} \quad y_4^* = \frac{a_x c_{yx} + b_x a_y}{c_{xy} c_{yx} + b_x b_y} > 0,$$

which eliminates the chance of repeating any of the first 3 critical points. From the above constraint, we create the below criterion for c_{xy} and c_{yx};

$$\begin{aligned}
0 < c_{xy} &< \frac{a_x b_y}{a_y}, \\
c_{yx} &> -\frac{b_x a_y}{a_x}.
\end{aligned}$$

Since $a_x,\ b_x,\ a_y > 0$, the criterion changes to

$$\begin{aligned}
0 < c_{xy} &< \frac{a_x b_y}{a_y} \approx 0.094, \\
c_{yx} &> 0.
\end{aligned}$$

The values for the discriminant, $\mathrm{T}^2 - 4\mathrm{D}$, and trace, T, can be plotted by designing a meshgrid for the parameters c_{xy} and c_{yx} based on their constraints as shown below.

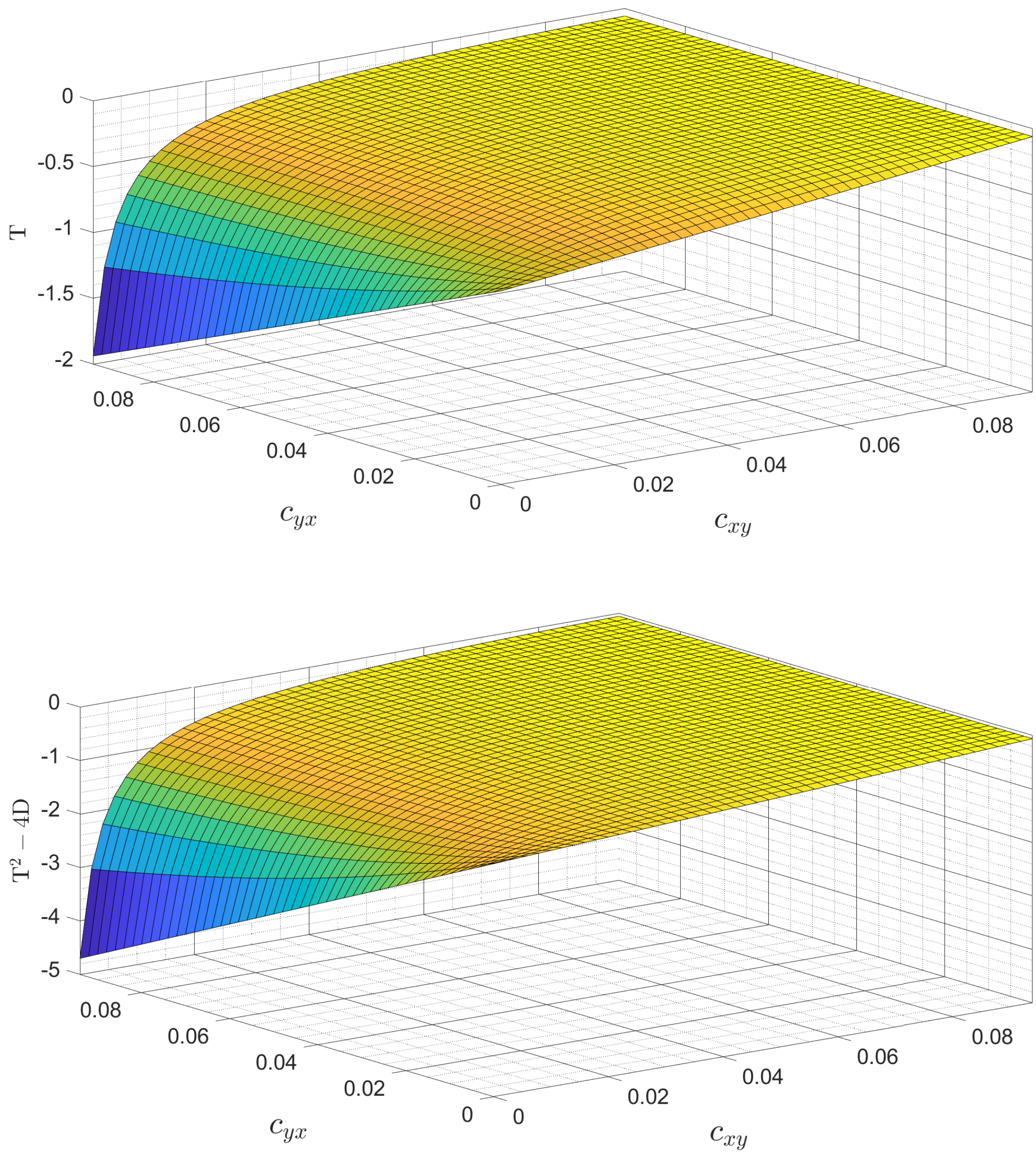


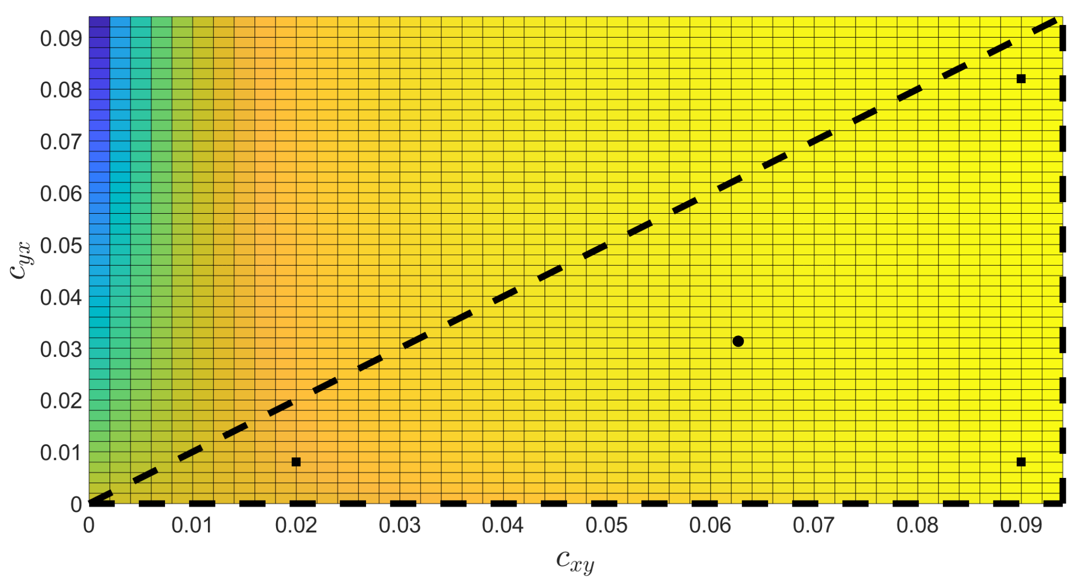
Figure 4.1: The graphs above are the trace and discriminant of $J_{(x_4^*,y_4^*)}$ for different values of the parameters c_{xy} and c_{yx}.

Notice that the trace and discriminant values are always negative for all c_{xy} and c_{yx} that belong in their constraints. If we assume that $c_{yx} > c_{xy}$, then we are stating that the impact of salmon on brown bears is greater than brown bears on salmon. However, brown bears eat a large quantity of salmon yearly, which means to support a brown bear's diet, there would need to be an abundance of salmon, whereas one brown bear kills thousands of salmon yearly [13, 14]. So, the brown bear population should have a higher effect on the salmon population. Therefore, the constraints for the parameters c_{xy} and c_{yx} are

$$0 < c_{xy} < 0.094,$$

$$0 < c_{yx} < c_{xy}.$$

Looking at Figure 4.1 from a top-down view, we can see which values for c_{xy} and c_{yx} satisfy the new constraints.



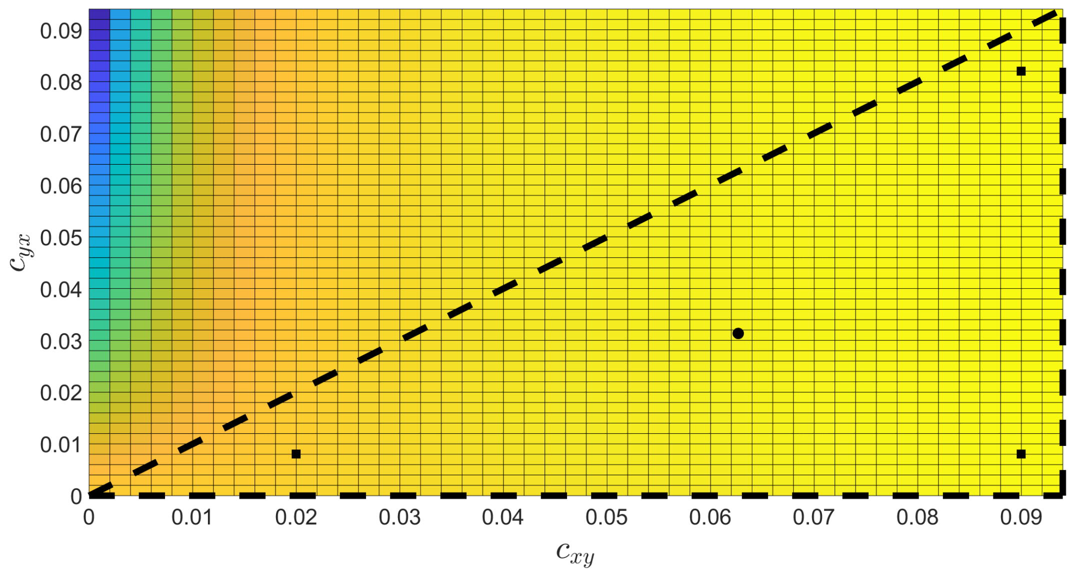


Figure 4.2: Top-down view of Figure 4.1. The points inside the right triangle are all values that satisfy the constraints of parameters c_{xy} and c_{yx}. The right triangle's center of mass is marked with a black dot at the coordinate point $(0.0627, 0.0313)$.

Now, we begin testing different c_{xy} and c_{yx} values to illustrate how different interaction rates affect the outcome of each species' population. The values chosen for $(c_{xy},\ c_{yx})$ are $(0.02,\ 0.008)$, $(0.09,\ 0.008)$, $(0.09,\ 0.082)$, and $(0.0627,\ 0.0313)$. These pair of interaction parameters can be seen plotted in the figure above. To compare each of the parameters, we plot the solutions to Equation (4.3), where the brown bear population is along the y-axis, and the salmon population is along the x-axis as shown below for each pair of parameters.

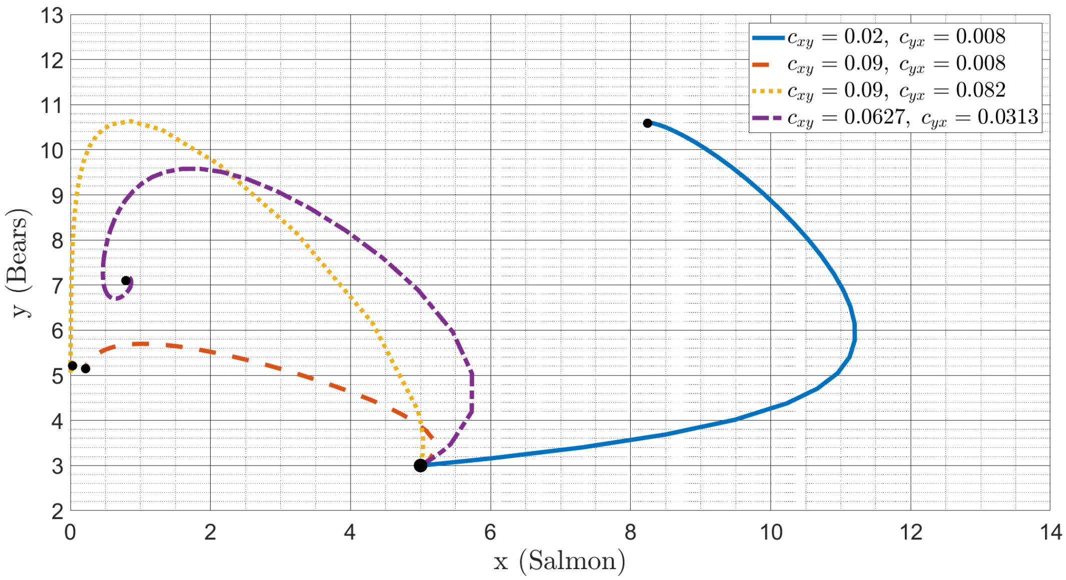


Figure 4.3: Compares the effect of different interaction rates, $(c_{xy},\ c_{yx})$, for the autonomous model, Equation (4.3), where the initial conditions are $x_0 = 5$ and $y_0 = 3$.

The graph above shows that each of these parameters affects the location of the critical point, (x_4^*, y_4^*), as well as the oscillations of the populations. We chose the initial conditions, $x_0 = 5$ and $y_0 = 3$, to illustrate the dramatic difference in interaction parameters. When c_{xy} is large, the salmon population dies off, and when c_{yx} is large, the brown bear population increases faster before converging near its carrying capacity. Lastly, when the pair of parameters is equal to the right triangle's center of mass in Figure 4.2, the population oscillates and converges to its critical point $(0.79, 7.1)$. We use $c_{xy} = 0.0627$ and $c_{yx} = 0.0313$ to represent the interaction rates of the two species for Equation (4.3) because with these parameters, the populations oscillate around their critical point much longer than the other parameters we tested, which will provide a more visible comparison for when we incorporate climate change. So, with all the parameters selected, the solutions to the autonomous model, Equation (4.3), with respect to time, are shown below.

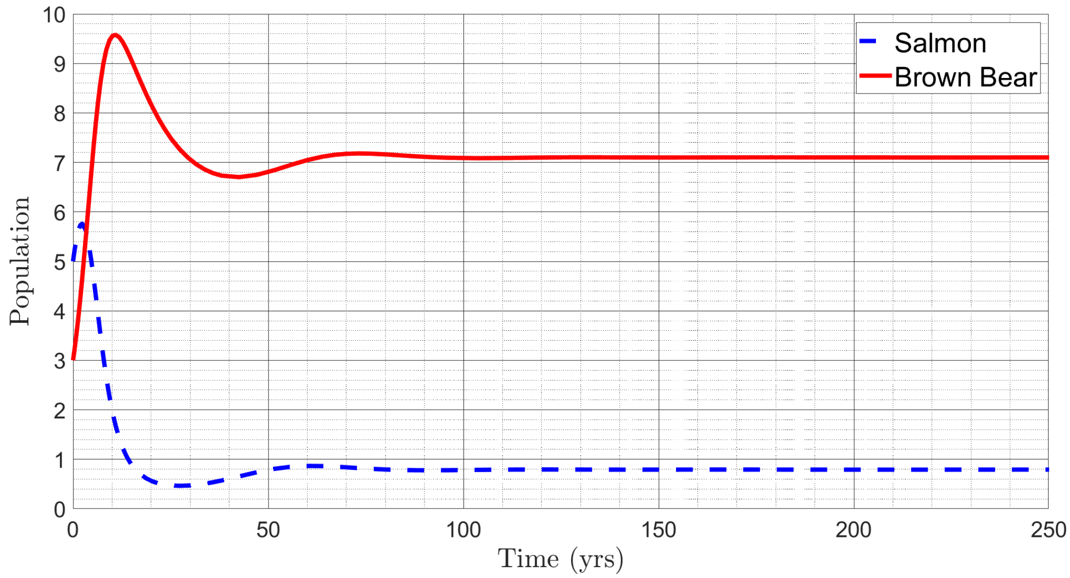


Figure 4.4: Plot of the solutions to the autonomous model, Equation (4.3), with respect to time.

In the figure above, both populations briefly increase before changing directions and oscillating toward their equilibrium points. Based on this figure, the brown bear population quickly overtook the salmon population, forcing the salmon near regional extinction.

4.4 Non-Autonomous System of ODEs

Now, we can compare the results from the previous section to a non-autonomous system of ODEs, where the growth rate is a function of time, shown below.

$$\begin{aligned}\frac{dx}{dt} &= G(t)x\left(1-\frac{x}{K_x}\right)-c_{xy}xy,\\ \frac{dy}{dt} &= r_y y\left(1-\frac{y}{K_y}\right)+c_{yx}xy.\end{aligned} \tag{4.8}$$

Now, using the same parameters for the autonomous model, Equation (4.3), we consider the model with different initial conditions.

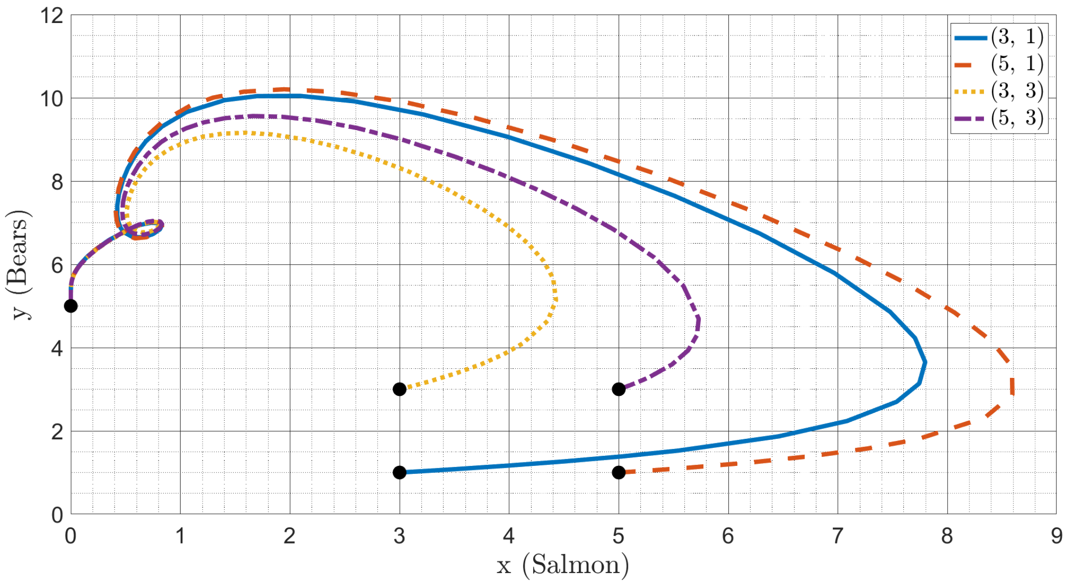


Figure 4.5: Compares the solutions to the non-autonomous model, Equation (4.8), with different initial conditions, $(x_0,\ y_0)$.

As expected, the salmon population converges to zero as seen in Figure 3.14, resulting in the brown bear population converging to their carry capacity. When the salmon population dies off, the interaction terms in the model will approach zero, and eventually, the behavior of the brown bear species will be represented by its logistic equation, Equation (2.5). Lastly, in the graph below, we compare the results to the autonomous and non-autonomous models.

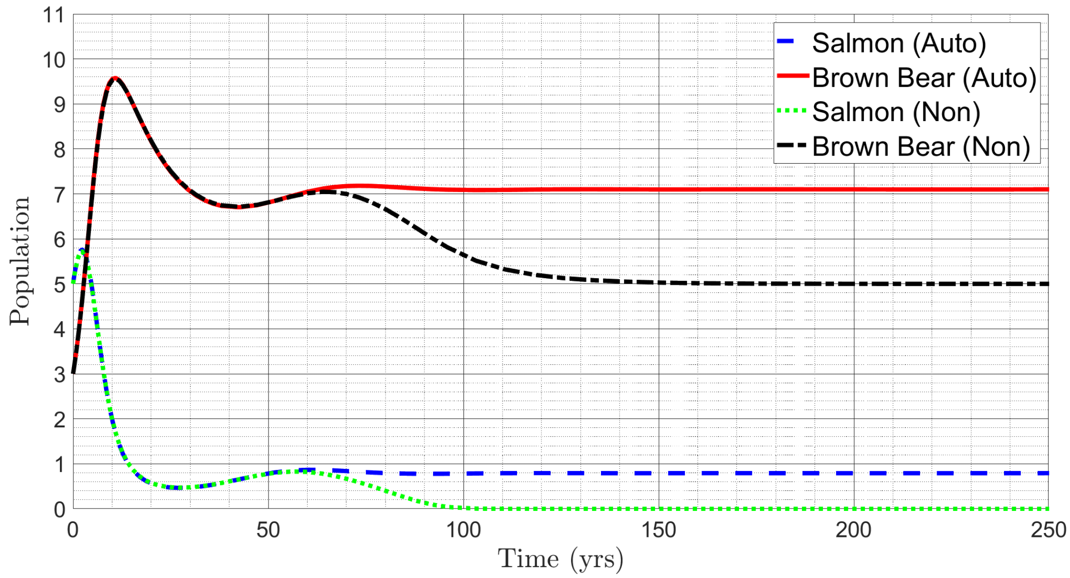


Figure 4.6: Plot of the solutions to the autonomous and non-autonomous model with respect to time.

Initially, the two models follow the same path, but after approximately 60 years, the curves begin to deviate. According to our temperature function, Equation (3.6), the projected Alaskan river temperature in 60 years is $T(60) \approx 14.34°\text{C}$. Therefore, the graph illustrates that soon after the river temperature leaves the optimal range, the difference in the outcomes of the species' populations becomes prominent. The non-autonomous model shows similar trends to the autonomous model, but ultimately resulting in the salmon population dying off and the Alaskan brown bear population converging to its carrying capacity.

4.5 Conclusion

In this chapter, we introduce a variation of Theodore Modis' model, Equation (4.2), to simulate the interaction between the brown bear and salmon species. We start with the autonomous system of ODEs, Equation (4.3), and analyze its stability near its critical points. We established that for the critical point where neither population is extinct, both eigenvalues of the Jacobian matrix contain imaginary parts and negative reals parts, which implies the fixed point is a stable spiral. We then substituted $c_{xy} = 0.0627$ and $c_{yx} = 0.0313$, into the autonomous model, Equation (4.5), which created an oscillation around the critical point $(0.79,\ 7.1)$. When plotting the solutions for the autonomous model, it was apparent that the brown bear species will overtake the salmon species, bringing the salmon near regional extinction. In the next section, we applied the above interaction parameters to the non-autonomous system of ODEs, Equation (4.8), which includes the growth rate function and compared its solutions to the autonomous model. The results of the non-autonomous model were similar to the autonomous version for the first 65 years. As the temperature began to leave the optimal range, the solutions to the non-autonomous model separated from those of the autonomous version, resulting in the entire salmon population becoming regional extinct from Alaska, and the brown bear population converging to its carrying capacity. We conclude that global warming could eventually cause the salmon population to go extinct and the brown bear population to decrease in size to accommodate for the lack of a resource, assuming the brown bear population is not affected in any other way.

Chapter 5

Conclusion

In chapter 2, we use the logistic growth model to simulate the population of both species, Alaskan brown bears and pacific salmon. In the first section, we estimated the growth rate parameter, $r_x = \ln(0.32 * 5)$, for salmon using reports from the Alaskan Department of Fish and Game. We estimated the salmon's carrying capacity, $K_x = 29.1 * 10^6$ by calculating the maximum volume of salmon in Bristol Bay, Alaska, for any given inshore run. In the second section, we approximate the growth rate parameter, $r_y = 0.059$, for brown bears by taking the average of three growth rates reported in three articles. Using information from the Alaskan Department of Fish and Game, we estimated a parameter value, $K_y = 4.5 * 10^4$, for the carrying capacity of the brown bear species. Ultimately, we found that after 14 years, the salmon population should reach its environmental capacity if they start with an initial population of 20 million, and the brown bears will reach their environmental capacity in approximately 100 years if they begin with an initial population of 30 thousand.

In the first section of chapter 3, we use data from Dr. Phyllis Weber Scannel and Katherine Carter to approximate the proportion of salmon who survive spawning migration at different temperatures. We develop a survival proportion function dependent on temperature for migrating salmon by fitting a curve to our approximated survival proportions. Combining this function with the growth rate parameter of salmon found in chapter 2, we propose a growth rate function dependent on temperature. In the next section of chapter 3, we used river temperature data from the United States Geological Survey to design a function that models the increase in Alaskan river temperature over time. Substituting the temperature equation for the temperature parameter in the growth rate function proposed in the first section changes the growth rate for salmon to a function of time. We found that after 100 years, the water temperature for salmon will become too hot to survive spawning migration, resulting in their regional extinction.

In chapter 4, we introduce interaction terms to both species' logistic models using a variation of Theodore Modis' system of ordinary differential equations. In the second section of chapter 4, we analyze different interaction parameters for the model when neither species is affected by climate change. After testing different parameters, we chose $c_{xy} = 0.0627$ and $c_{yx} = 0.0313$ to represent the interaction between the species and evaluated the solutions to the autonomous system. The solutions to the autonomous model showed that when neither species is affected by climate change, the brown bears will overtake the salmon, and both species will oscillate toward their equilibrium point, $(0.79,\ 7.1)$. In the third section, we incorporate climate change into the autonomous system by making the growth rate a function of time, creating a non-autonomous model. Using the interaction parameters in the previous section, we analyzed the behavior of both species by plotting the solutions to the non-autonomous model with different initial conditions. When comparing the two models proposed in this chapter, we found that as the temperature begins to leave the optimal range, the solutions to the non-autonomous model separate from those of the autonomous one, forcing the entire salmon population to become regional extinct from Alaska, and the brown bear population converging to its carrying capacity.

Plenty of variations of our non-autonomous model can be implemented to improve its accuracy. While the logistic growth equation is useful for modeling population growth, another is age-structure models. With brown bears and salmon having different survival and reproduction rates at different ages, an age-structure model might be more effective for predicting their behaviors [15, 35]. Another aspect of our model worth exploring is growth rate functions. A limitation we faced in developing a growth rate function for the salmon species was the need for more research in determining survival rates for migrating salmon at different temperatures. Lastly, the Alaskan Department of Fish and Game reports that they adjust harvest rates periodically for brown bears to maintain their population size, so including a term to represent the change in harvest rates would significantly improve the accuracy of our model [15]. We hope our model is used in further research to protect salmon and brown bears from the extreme weather that global warming will bring.

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Appendix A

TABLES

Table A.1: Sockeye Comparison Between Weight and Run Size in Bristol Bay

Year	**Weight (lbs)**	**Run (mil)**
2001	6.7	22.3
2002	6.1	16.9
2003	6.3	24.9
2004	5.8	41.9
2005	6.3	39.3
2006	5.7	42.9
2007	5.8	44.8
2008	5.8	40.4
2009	5.9	40.4
2010	5.5	40.6
2011	6.2	30.6
2012	5.7	30.4
2013	6.0	24.4
2014	5.6	41.1
2015	5.2	58.8
2016	5.4	51.7
2017	5.5	57.6
2018	5.1	63.0
2019	5.1	56.4
2020	5.1	58.3
2021	4.7	67.7

*Comparing the average weight of sockeye salmon to their run size in Bristol Bay each year. The data used to make this table was taken from the "*2021 BRISTOL BAY AREA ANNUAL MANAGEMENT REPORT*" [21].

Table A.2: Volume of Sockeye Salmon Runs Each Year in Bristol Bay

Year	**Volume (MMCF)**
2001	3.41
2002	2.35
2003	3.58
2004	5.55
2005	5.65
2006	5.58
2007	5.93
2008	5.35
2009	5.44
2010	5.1
2011	4.33
2012	3.96
2013	3.34
2014	5.25
2015	6.98
2016	6.37

Continued on next page

Table A.2 – continued from previous page	
Year	**Volume (MMCF)**
2017	7.23
2018	7.34
2019	6.57
2020	6.79
2021	7.26

*Using Table A.1 to calculate the volume for each year.

Table A.3: Average Annual Harvest For Salmon in Bristol Bay

Species	Harvest
Sockeye	28,100,000
Chinook	39,571
Chum	1,100,000
Coho	95,583
Pink	510,000
Total	**29,845,154**

*Average annual commercial harvest for each salmon species from (2001 – 2020) [21]. Pink Salmon are reported in even years because of their two-year life cycle pattern.

Appendix B

R Code

B.1 Salmon Run Size Vs Their Average Weight

```
# Comparing run versus weight of sockeye salmon in Bristol Bay

setwd('C:/Users/Connor/OneDrive/Desktop/GitHub/Alaskan-Brown-Bears/
    R/Salmon Escapement/Data')

rm(list = ls())

# Average Weight of Salmon During Annual Run
weight <- c(6.7,6.1,6.3,5.8,6.3,5.7,5.8,5.8,5.9,5.5,6.2,5.7,
            6,5.6,5.2,5.4,5.5,5.1,5.1,5.1,4.7)

# Run Size During Annual Run
run <- c(22.3,16.9,24.9,41.9,39.3,42.9,44.8,40.4,
         40.4,40.6,30.6,30.4,24.4,41.1,58.8,51.7,
         57.6,63,56.4,58.3,67.7)

```

```r
# Constructs Dataframe
df <- data.frame(weight,run)

colnames(df) <- c('Weight','Run')

# Calculates The Correlation Between The Two Variables
cor(df$Weight,df$Run)

# Scatter Plot of The Two Variables
plot(df$Weight,df$Run,
     main = 'Sockeye Run Vs. Average Weight',
     xlab = 'Weight (lbs)',
     ylab = 'Run Size (mil)',
     cex = 2,
     col = 4,
     pch = 19,
     weight = 2,
     cex.main = 1.5,
     cex.lab = 1.5
     )

grid(NULL,NULL,col='lightgrey',lty=6)

points(df$Weight,df$Run,
     cex = 2,
     lwd = 2
     )

# Linear Regression Model
slr <- lm(Run~.,
```

```
           data = df
           )

# Graphs the Line on the Plot
abline(slr,
       lwd = 4,
       col = 'darkorange'
       )

# Calculates the Volume of Salmon Per Run
df$Volume <- df$Weight*df$Run

df$'V (MMCF)' <- round(df$Volume / 43.8,2)

df

# Constructs a Table with All Three Varaibles
dfsub <- data.frame(seq(2001,2021),df$`V (MMCF)`)

write.csv(dfsub,'salmonvolume.csv',row.names = FALSE)
```

B.2 Proportion Function

```
# Reproduction Function for Salmon

rm(list = ls())

#Libraries

library(latex2exp)
```

```r
# Collect some data about sockeye salmon mortality at different
    temperatures

temps <- c(seq(11,14,.2),2,22.2)
survival <- c(rep(1,16),.5,.5)


# Scatter Plot of Data
plot(temps,survival,
     pch='x',
     cex=1.25,
     ylab='Proportion',
     xlab='Temperature (celsius)',
     main='Proportion of Salmon Survival',
     cex.lab=1.5,
     cex.main=2,
     xlim=c(-2,30),
     ylim=c(0,1))
grid(NULL,NULL,col='lightgrey',lty=6,lwd=2)


temp <- seq(-7,37,.1)

t_opt <- 12.5


# Function 1: squared with c = 1
```

```
c <- 1
c

f1 <- 1 / (1 + c*(temp - t_opt)^2)

# Graphs the Line of Function 1
lines(temp,f1,lwd=5,col=4)


# Function 2: squared with c = 0.01
c <- 1/(2.5-t_opt)^2
c

f2 <- 1 / (1 + c*(temp - t_opt)^2)

# Graphs the Line of Function 2
lines(temp,f2,lwd=5,col='darkorange',lty=2)


# Function 3: 4th power with c = 0.01
c <- 1/(2.5-t_opt)^4
c <- .01
c

f3 <- 1 / (1 + c*(temp - t_opt)^4)

# Graphs the Line of Function 3
lines(temp,f3,lwd=5,col=3,lty=3)

# Function 4: 4th power with c = 0.0001
```

```
c <- 1/10000
c

f4 <- 1 / (1 + c*(temp - t_opt)^4)

# Graphs the Line of Function 4
lines(temp,f4,lwd=5,col=2,lty=6)

legend(22,1,
       legend=TeX(c('$R(T), c=1.00, p=2$',
                    '$R(T), c=0.01, p=2$',
                    '$R(T), c=0.01, p=4$',
                    '$R(T), c=10^{-5}, p=4$')
                  ),
       col=c(4,'darkorange',3,2),
       lty=c(1,2,3,6),
       cex=1
       )
```

B.3 Water Temperature Dependent on Time

B.3.1 Polynomial Fit of Surface Temperature

```
# EPA air temperature from 1880 - 2020

# Clear list
rm(list=ls())

# Directory
setwd("C:/Users/Connor/OneDrive/Desktop/GitHub/Alaskan-Brown-Bears/
```

```
    R/Water Temp Data/")

# Libraries
library(latex2exp)

tab <- read.csv("NOAA air.csv")

# Organize column names
colnames(tab) <- c('Year','Temp')

# Plot data
attach(tab)

plot(Year,Temp,ylab=TeX(r'(Surface Temperature ( $\degree$ C))'),
                         xlab='Time (Yrs)',
                         main='Temperature Anomalies From 1880-2021'
    )
grid(NULL,NULL,col='lightgrey',lty=6)
points(Year,Temp,pch=19,cex=.7,col=4)

# ELR Model
elr <- lm(log(Temp+.4601)~Year)
timevalues <- seq(1875,2025,.1)
exp.temp <- exp(predict(elr,list(Year=timevalues)))
lines(timevalues,exp.temp-.4601,lwd=3,col='green')

# Poly Model
plr <- lm(Temp~poly(Year,2,raw=TRUE))
poly.temp <- predict(plr,newdata=data.frame(Year=timevalues))
lines(timevalues,poly.temp,lwd=3,col='darkorange')
```

```
A = plr$coef['(Intercept)']
B1 = plr$coef['poly(Year, 2, raw = TRUE)1']
B2 = plr$coef['poly(Year, 2, raw = TRUE)2']
A
B1
B2
```

B.3.2 Polynomial Fit of Sea Surface Temperature

```
# NOAA SST from 1880 - 2020

# Clear list
rm(list=ls())

# Directory
setwd("C:/Users/Connor/OneDrive/Desktop/GitHub/Alaskan-Brown-Bears/
    R/Water Temp Data/")

# Libraries
library(latex2exp)

tab <- read.csv("NOAA SST.csv")

# Organize column names
colnames(tab) <- c('Year','Temp')

# Plot data
attach(tab)

plot(Year,Temp,ylab=TeX(r'(Sea Surface Temperature ( $\degree$ C))'
```

```
    ),
      xlab='Time (Yrs)',
      main='SST Anomalies From 1880-2021')
grid(NULL,NULL,col='lightgrey',lty=6)
points(Year,Temp,pch=19,cex=.7,col=4)

# ELR Model
elr <- lm(log(Temp+.4601)~Year)
timevalues <- seq(1875,2025,.1)
exp.temp <- exp(predict(elr,list(Year=timevalues)))
# lines(timevalues,exp.temp-.4601,lwd=3,col='darkorange')

# Poly Model
plr <- lm(Temp~poly(Year,2,raw=TRUE))
poly.temp <- predict(plr,newdata=data.frame(Year=timevalues))
lines(timevalues,poly.temp,lwd=3,col='darkorange')

A = plr$coef['(Intercept)']
B1 = plr$coef['poly(Year, 2, raw = TRUE)1']
B2 = plr$coef['poly(Year, 2, raw = TRUE)2']
A
B1
B2
slr <- lm(Temp[Year>=1960]~Year[Year>=1960])
abline(slr,lwd=2,col='green')
plot(Year[Year>=1960],residuals(slr))
```

B.3.3 Linear Fit

```
# USGS Rivers

```

```
# clear variable
rm(list=ls())

# Libraries
library(zoo)
library(latex2exp)

# Directory
setwd("C:/Users/Connor/OneDrive/Desktop/GitHub/Alaskan-Brown-Bears/
    R/")

# Load data frames
tab1 <- read.delim("USGS River Temp/Battle")
tab2 <- read.delim("USGS River Temp/Bradley")
tab3 <- read.delim("USGS River Temp/Cooper")
tab4 <- read.delim("USGS River Temp/Iliamna")
tab5 <- read.delim("USGS River Temp/Kenai R - Cooper")
tab6 <- read.delim("USGS River Temp/Kenai R - Soldotna")
tab7 <- read.delim("USGS River Temp/Koktuli")
tab8 <- read.delim("USGS River Temp/Kokwok")
tab9 <- read.delim("USGS River Temp/Kroto Creek")
tab10 <- read.delim("USGS River Temp/Matanuska")
tab11 <- read.delim("USGS River Temp/Moose")
tab12 <- read.delim("USGS River Temp/Nuyakuk")
tab13 <- read.delim("USGS River Temp/Palmer")
tab14 <- read.delim("USGS River Temp/Red Cloud")
tab15 <- read.delim("USGS River Temp/Russell")
tab16 <- read.delim("USGS River Temp/Ship")
tab17 <- read.delim("USGS River Temp/Susitna")
tab18 <- read.delim("USGS River Temp/talkeetna")
```

```r
tab19 <- read.delim("USGS River Temp/Terror")
tab20 <- read.delim("USGS River Temp/Willow")
tab21 <- read.delim("USGS River Temp/Wolverine")
tab22 <- read.delim("USGS River Temp/Taiya")
tab23 <- read.delim("USGS River Temp/Indian R")
tab24 <- read.delim("Usgs River Temp/Staney")


# Remove first row of each data frame
tab1 <- tab1[-c(1), ,drop=FALSE]
tab2 <- tab2[-c(1), ,drop=FALSE]
tab3 <- tab3[-c(1), ,drop=FALSE]
tab4 <- tab4[-c(1), ,drop=FALSE]
tab5 <- tab5[-c(1), ,drop=FALSE]
tab6 <- tab6[-c(1), ,drop=FALSE]
tab7 <- tab7[-c(1), ,drop=FALSE]
tab8 <- tab8[-c(1), ,drop=FALSE]
tab9 <- tab9[-c(1), ,drop=FALSE]
tab10 <- tab10[-c(1), ,drop=FALSE]
tab11 <- tab11[-c(1), ,drop=FALSE]
tab12 <- tab12[-c(1), ,drop=FALSE]
tab13 <- tab13[-c(1), ,drop=FALSE]
tab14 <- tab14[-c(1), ,drop=FALSE]
tab15 <- tab15[-c(1), ,drop=FALSE]
tab16 <- tab16[-c(1), ,drop=FALSE]
tab17 <- tab17[-c(1), ,drop=FALSE]
tab18 <- tab18[-c(1), ,drop=FALSE]
tab19 <- tab19[-c(1), ,drop=FALSE]
tab20 <- tab20[-c(1), ,drop=FALSE]
tab21 <- tab21[-c(1), ,drop=FALSE]
```

```r
tab22 <- tab22[-c(1), ,drop=FALSE]
tab23 <- tab23[-c(1), ,drop=FALSE]
tab24 <- tab24[-c(1), ,drop=FALSE]

# Can't use
  # tab23 - too short
  # tab22 - too short
  # tab21 - too short
  # tab20 - too short
  # tab17 - too short
  # tab16 - too short
  # tab14 - too short
  # tab13 - too short
  # tab12 - too short
  # tab10 - too short
  # tab9 - too short
  # tab8 - too short
  # tab7 - too many missing data values in each year
  # tab6 - too short
  # tab4 - too short
  # tab2 - too much of a gap between years
  # tab1 - too much of a gap between years


# Adjust duration of data because of gaps in years
tab24 <- tab24[which(tab24$year_nu<2020 & tab24$year_nu>=1998),]
tab5 <- tab5[which(tab5$year_nu>=2005),]

# Combine all data frames into a single data frame
tab <- rbind(tab3,tab5,tab15,tab19,tab24)
```

```r

# List of site number
site <- as.numeric(as.matrix(unique(tab$site_no),ncol=1))

# tab <- rbind(tab3,tab5,tab11,tab15,tab18,tab19,tab22)
  # Tab3 -> Cooper -> Consistent data for months 6-10 starting 1999
  # Tab5 -> Kenai R - Cooper -> Consistent starting 2005
  # Tab11 -> Moose -> Consistent data for months 7-10 starting 2013
  # Tab15 -> Russell -> Consistent data for months 6-10 starting
    1999
  # Tab18 -> Talkeetna -> Consistent data for months 6-10 starting
    2012
  # Tab19 -> Terror -> Consistent data starting 1988
  # Tab22 -> Taiya -> Consistent data form months 7-9 starting 2004

# data frames that look good to sample
  # Characterisitcs I am looking for are
    # data longer than 10 years.
    # no gaps between years
# tab 3,5,11,15,18,19
# Maybe 2,4,7

# Reorganize data frame
df <- data.frame(tab$year_nu,tab$month_nu,tab$mean_va,tab$site_no)
colnames(df) <- c('year','month','SST','site #')

# Change character variable to numeric
df$year <- as.numeric(df$year)
df$month <- as.numeric(df$month)
df$SST <- as.numeric(df$SST)
```

```
df$‘site #‘ <- as.numeric(df$‘site #‘)
df$date <- as.yearmon(df$year+df$month)

# Setting up plot frame
par(mfrow=c(1,1))
plot(df$date,df$SST)

# Label the sites by name of location
site.name <- c(’Cooper Creek’, ’Kenai River’, ’Russell Creek’,
               ’Terror River’, ’Staney Creek’)

B <- data.frame(1:length(site))
A <- data.frame(1:length(site))
# Collect average point estimate for each data frame

for (i in site) {
  r <- which(site==i,arr.ind = TRUE)
avg.monthly.tab<- aggregate(SST[month>=6 & month<=9 & year>=1950 & ‘site #‘ == i]~
                            year[month>=6 & month<=9 & year>=1950 & ‘site #‘==i],
                            data=df,FUN=mean)
colnames(avg.monthly.tab) <- c(’year’,’SST’)
slr <- lm(avg.monthly.tab$SST~avg.monthly.tab$year)
B[r,1] <- slr$coef[’avg.monthly.tab$year’]
A[r,1] <- slr$coef[’(Intercept)’]
plot(avg.monthly.tab$year,avg.monthly.tab$SST,
     ylab=TeX(r’(Temeperature ( $\degree$ C))’),
     xlab=’Time (yrs)’,
     main=substitute(paste(’Annual Water Temperature for ’,
```

```
                                  a),
                          list(a=site.name[r]))
      )
grid(NULL,NULL,col='lightgrey',lty=6)
points(avg.monthly.tab$year,avg.monthly.tab$SST,
        pch=19,
        cex=.7,
        col=4)
abline(slr,lwd=2,col='darkorange')
}
glog <- median(B[1:nrow(B),]); glog
goog <- mean(B[1:nrow(B),]); goog
A.avg <- mean(A[1:nrow(A),]); A.avg

# Average of all the SST for each month
avg.monthly<- aggregate(SST[month>=6 & month<=9 & year>=1950]~
                                year[month>=6 & month<=9 & year>=1950],
                             data=df,FUN=mean)
colnames(avg.monthly) <- c('year','SST')

# Attach data frame for ease of use
attach(avg.monthly)

# Focus on spawning months
mon.begin <- 7
mon.end <- 9

# Plot data
slr <- lm(SST~year)
plot(year,SST,
```

```
      cex=1.8,
      ylab=TeX(r'(Temperature ( $\degree$ C))'),
      xlab='Time (Yrs)',
      main='Average Annual Water Temperatures')
grid(NULL,NULL,col='lightgrey',lty=6,lwd=2)
points(year,SST,pch=19,cex=1.5,col=4)
abline(slr,lwd=5,col='darkorange')
B_T <- slr$coef['year']; B_T
A_T <- slr$coef['(Intercept)']; A_T
timevalues <- seq(1980,2040,.1)
lines(timevalues,goog*timevalues+A.avg,lwd=5,col='green',lty=2)
legend(1987,10.6,legend=c('Avg. of Data','Avg. of Trends'),
       col=c('darkorange','green'),
       lty=c(1,2),
       cex=1.4
       )
# dynamic harmonic regression - fable library
goog
x <- seq(1988,2021,1)
y.sin <- 10^(-39.45)*exp(.045*x)*sin(10^(-4)*x^2-31)+B_T*x+A_T
lines(x,y.sin,
      col='red',
      lwd=2
      )
```

Appendix C

MATLAB Code

C.1 Ordinary Differential Equation

C.1.1 Salmon Exponential Equation

```
% Exponential Growth Model for Salmon.

close all
clear
clc

    % dx/dt = r*x_0

% Parameters
x = 20;
r = log(.32*5);

t = 0:.01:10;

```

```
% Exponential ODE set up
h = @(t,y)(r.*y(1));

% Solutions to the eponential ODE
[t, za] = ode45(h,t,x);

% Plot of the exponential growth model
figure(1)
plot(t,za(:,1),'c','LineWidth',4)
xlabel("Time (yrs)", 'FontSize', 25)
ylabel("Population", 'FontSize', 25)
title("The Population of Salmon Over Time", 'FontSize', 25)
legend('Exponential', 'FontSize', 20, 'Location', 'NorthWest')
grid on
grid minor
ax = gca;
ax.GridAlpha = 1;
ax.MinorGridAlpha = 1;
set(gca,'FontSize',20)
```

C.1.2 Salmon Logistic Equation

```
% Salmon Logistic Growth

clear
close all
clc

% Parameters
x_0 = 20;
k = 29.1;
```

```
r = log(0.32*5);
time_max = 20;

t = 0:.01:time_max;

% Logistic Function
dx =@(x) r*x*(1-x/k);

salmon = @(t,x)(dx(x));

% Solutions to the Function
[t,pops] = ode45(salmon,t,x_0);


% Plotting the population over the time duration
figure(1)
p1 = plot(t,pops,'Color',[0.10,0.70,1.00],'LineWidth',4);
xlabel("Time (yrs)", 'FontSize', 25)
ylabel("Population of Salmon (millions)", 'FontSize', 25)
str = "The Population of Alaskan Salmon Over Time";
title(str, 'FontSize', 35)
legend('Salmon', 'FontSize', 25, 'Location', 'NorthWest')
ax = gca;
grid on
grid minor
ax.MinorGridAlpha = 1;

ax.GridAlpha = 1;
set(gca,"FontSize",20)
```

C.1.3 Brown Bear Logistic Equation

```
 % Brown Bear Logstic Equation

clear
close all
clc

% Initial Population
y_0 = 30000;

% r by Van Daele
r_VD = 0.014;
% r by McLellan (Table 2)
r_FV = 0.081;
% r by McLellan
r_M = 0.082;
% r by the average
r_AVG = mean([r_VD,r_FV,r_M]);

% Vector of Growth Rates
r = [r_VD,r_FV,r_M,r_AVG];

% Carrying Capacity
K = 45000;

% Time Intervals
% t = 0:3:120;
t = 0:6:400;


```

```
legendNames{1} = 'Van Daele, r_y = 0.014';
legendNames{2} = 'McLellan89, r_y = 0.081';
legendNames{3} = 'McLellan96, r_y = 0.082';
legendNames{4} = 'Average, r_y = 0.059';

% line properties
% List a bunch of markers; they will be selected in
% order and then the selection will start again if
% there are more lines than markers.
markers = {'o','+','*','s','d','v','>','h'};
% List a bunch of colors; like the markers, they
% will be selected circularly.
colors = {'b','g','r','k','c','m'};
% Same with line styles
linestyle = {'-','--','-.',':'};
% This function will do the circular selection
% Example:  getprop(colors, 7) = 'b'
getFirst = @(v)v{1};
getprop = @(options, idx)getFirst(circshift(options,-idx+1));

% For loop for the different growth rates.
for i = 1:length(r)-1

    % Logistic Function
    h = @(t,y) r(i).*y(1).*(1-(y(1)/K));

    % Solutions to the Function
    [t za] = ode45(h,t,y_0);

    % Plots the Function with the chosedn growth rate
```

```
    figure(1)
    plot(t,za(:,1),...
        'Marker',getprop(markers,i),...
        'color',getprop(colors,i),...
        'linestyle',getprop(linestyle,i),...
        'LineWidth',1)
    hold on
end
% Plots the Average Growth Rate with a bold line
h = @(t,y) r(4).*y(1).*(1-(y(1)/K));
[t za] = ode45(h,t,y_0);
plot(t,za(:,1),"Color",'c','LineWidth',5)
set(gca,"FontSize",20)
xlabel("Time (yrs)",'Interpreter','latex', 'FontSize', 25)
ylabel("Population",'Interpreter','latex', ...
    'FontSize', 25)
title("The Population of Alaskan Brown Bears Over Time", ...
    'Interpreter','latex','FontSize', 25)
legend(legendNames, 'FontSize', 20,'Location','SouthEast')
grid on
ax = gca;
ax.GridAlpha = 1;
grid minor
ax.MinorGridAlpha = 1;
```

C.2 Growth Rate Function

C.2.1 Growth Rate Function Dependent on Time

```
% Growth Rate Function
```

```
clc
clear
close all

% Parameters
r = 5;
d = .32;
c = 1e-4;
a = .08;
% Temp Intercept - T_opt
b = 2.96;
x_o = 10;

% Allows for symbolic function to be created
syms t

% Symbolic Proportion Function with Respect to Time
R = d*r / (1 + c*(a*t-b)^4);

% Derivative of the Proportion Function with Respect to Time
R_prime = diff(R,t);

% Growth Rate Function Dependent on Time
f = log(R) + R_prime*t/R;


% Plotting the Growth Rate Function
t = 0:1:150;
plot(t, subs(f,t),'LineWidth',4)
```

```
ax = gca;
grid on
grid minor
ax.MinorGridAlpha = 1;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
xlabel("Time (yrs)", 'FontSize', 25)
ylabel("Growth Rate", 'FontSize', 25)
title('Growth Rate Function', "FontSize",30)
ylim([-2.5,.55])
```

C.2.2 Salmon Model with Growth Rate Function

```
% Salmon Logistic Equation with Growth Function

clc
clear
close all
format long

    % Growth Rate Function as a Function of Temperature

% Paramters
x_0 = 20;
k = 29.1;
time_max = 150;
r = 5;
d = .32;
c = 1e-4;
T_opt = 12.5;
t = 0:.1:time_max;
```

```matlab

% Vector of Different Temperatures
T = [12.5, 22.2, 30];

% Empty Matrix to Store Solutions
pop = zeros([time_max/.1+1,3]);

% For loop for testing different temperatures
for i = 1:3
    % Growth Rate Function of Temperature
    R = log( d*r / (1 + c*(T(i)-T_opt)^4) );

    % Salmon Logistic Model
    dx =@(x) R*x*(1-x/k);

    salmon = @(t,x)(dx(x));

    % Solutions to the Function
    [t,pops] = ode15s(salmon,t,x_0);

    % Stores the Solutions in the empty matrix
    pop(:,i) = pops;
end


% Plotting the population over the time duration
figure(1)
p1 = plot(t,pop(:,1),'-','Color',[0.10,0.70,1.00], ...
    'LineWidth',4);
hold on
```

```
p2 = plot(t,pop(:,2),'--','Color',[0.8500 0.3250 0.0980], ...
    'LineWidth',4);
hold on
p3 = plot(t,pop(:,3),':','Color',[0.0000 0.80 0.130], ...
    'LineWidth',4);
hold on
set(gca,"FontSize",20)
xlabel("Time (yrs)",'Interpreter', 'Latex', 'FontSize', 25)
ylabel("Population (million)",'Interpreter', 'Latex', 'FontSize',
   25)
% str = "The Population of Alaskan Salmon Over Time";
% title(str,"Interpreter","Latex", 'FontSize', 25)
ax = gca;
grid on
grid minor
ax.MinorGridAlpha = 1;
ax.GridAlpha = 1;



    % Growth Rate Function as a Function of Time

% New Parameters
a = .08;
b = 9.54;

% Allows for symbolic function to be created
syms t

% Symbolic Proportion Function with Respect to Time
```

```
R = d*r / (1 + c*(a*t + b-T_opt)^4);

% Derivative of the Proportion Function with Respect to Time
R_prime = diff(R);

% Growth Rate Function Dependent on Time
f = log(R) + R_prime*t/R;

% Non-autonomous ODE for Salmon
dx =@(t,x) (log(8/(5*(((2*t)/25 - 74/25)^4/10000 + 1))) - ...
    (4*t*((2*t)/25 - 74/25)^3*(((2*t)/25 - 74/25)^4/16000 + ...
    5/8)) / (78125*(((2*t)/25 - 74/25)^4/10000 + 1)^2)) .*...
    x .* (1-x/k);

salmon = @(t,x) [dx(t,x)];

% Creates Time Interval for Model
t = [0 time_max];

% Solutions to the Non-autonomous model
[t,pops] = ode45(salmon,t,x_0);


% Plotting the solutions on the figure 1
figure(1)
p4 = plot(t,pops,'-.','Color',[0.83,0.00,1.00],'LineWidth',4);
legend('$12.5^{\circ}$C', '$22.2^{\circ}$C', '$30^{\circ}$C', ...
    'G(t)', 'Interpreter', 'Latex', 'FontSize', 20, ...
    'Location', 'East')
ax.GridAlpha = 1;
```

```
set(gca,"FontSize",20)
```

C.3 Interaction Term Parameters

C.3.1 The Jacobian Matrix

```
% Jacobian Matrix of The Autonomous Model

clear
clc
format rational

% Parameters
r_S = 5;
c = .0001;
T_opt = 12.5;

% Growth Rate Dependent on Temperature
R =@(T) log( .32*r_S / ( 1 + c*(T - T_opt)^4 ) );

% Chosed Temp
T = 12.5;

% Theodore Modis Form of Parameters
a1 = R(T);
b1 = a1 / 15;
c1 = .0627;
a2 = .059;
b2 = a2 / 5;
c2 = .0313;
```

```
fprintf(['The water temperature will remain constant at' ...
     ' T=%.1f\n\n'],T)
% Jacobian for critical point (0,0)
fprintf('The Jacobian for the critical point (0,0) is:')
J_1 = [a1, 0; 0, a2];
display(J_1)

    % Eigenvalues of J_1
fprintf('The eigenvalues of the Jacobian above are:\n')
disp(eig(J_1))

% Jacobian for critical point (0,K_2)
fprintf('The Jacobian for the critical point (0,%d) is:',a2/b2)
J_2 = [a1-c1*a2/b2, 0; c2*a2/b2, -a2];
display(J_2)

    % Eigenvalues of J_2
fprintf('The eigenvalues of the Jacobian above are:\n')
disp(eig(J_2))

% Jacobian for critical point (K_1,0)
fprintf('The Jacobian for the critical point (%.0f,0) is:',a1/b1)
J_3 = [-a1, -c1*a1/b1; 0, a2+c2*a1/b1];
display(J_3)

    % Eigenvalues of J_3
fprintf('The eigenvalues of the Jacobian above are:\n')
disp(eig(J_3))

```

```
% Jacobian for critical point (x*,y*)
x_star = (a1*b2 - c1*a2) / (c1*c2 + b1*b2);
y_star = (a1*c2 + b1*a2) / (c1*c2 + b1*b2);
fprintf('The Jacobian for the critical point (%f,%f) is:',[x_star,
    y_star])
J_4 = [(-a1*b1*b2+2*b1*c1*a2-b1*c1*a2) / (c1*c2 + b1*b2),...
     (-a1*c1*b2 + c1^2*a2) / (c1*c2 + b1*b2);...
     (a1*c2^2 + b1*a2*c2) / (c1*c2 + b1*b2),...
     (-a2*b1*b2 - 2*a1*b2*c2 + a1*b2*c2) / (c1*c2 + b1*b2)];
display(J_4)

    % Eigenvalues of J_4
fprintf('The eigenvalues of the Jacobian above are:\n')
disp(eig(J_4))
```

C.3.2 Trace and Discriminant

```
% Determine the Stability of the Autonomous Model

clear
clc
close all

format long

% Growth Rate Dependent on Temperature
R =@(T) log( .32*5 / ( 1 + .0001*(T - 12.5)^4 ) );

% Parameters
T = 12.5;
k1 = 15;
```

```
k2 = 5;

a1 = R(T);
b1 = a1 / k1;

a2 = .059;
b2 = a2 / k2;

% Symbolic Values for the Interaction Parameters and Populations
syms c1 c2 x y

% Autonomous System of ODEs
f1 = a1*x - b1*x^2 - c1*x*y;
f2 = a2*y - b2*y^2 + c2*x*y;

    % Critical Values

x_crit = (a1*b2 - c1*a2) / (c1*c2 + b1*b2);
% To keep x_crit within constraints: c1 < a1*b2/a2 = a1/k2
y_crit = (a1*c2 + b1*a2) / (c1*c2 + b1*b2);
% To keep y_crit within constraints: c2 > -b1*a2/a1 = -a2/k1

% By setting Tau = 0, we can solve for c2 with any c1 value
c_1 = 0:.001:a1*b2/a2;
c_2 = ( b1*c_1*a2 - a1*b1*b2 - b1*b2*a2 ) / (a1*b2);

% Jacobian Matrix
A = [diff(f1,x), diff(f1,y);...
    diff(f2,x), diff(f2,y)];

```

```
% The Trace of the Jacobian
Tau = trace(A);

% Trace at the Critical Point Chosen
Tau = subs(Tau,[x,y],[x_crit,y_crit]);

% Create Meshgrid of Interaction Parameters
[C1, C2] = meshgrid(0:.002:a1*b2/a2,0:.002:a1*b2/a2);

% Sub in the Interaction Parameters
Tau = subs(Tau,[c1,c2],{C1,C2});

% Plotting the Trace with a Meshgrid of c_xy and c_yx values
figure(2)
surf(C1,C2,double(Tau))
hold on
% Plot of the Triangle
zz = zeros(length(c_1),1);
vv = a1*b2/a2 * ones([length(c_1),1]);
plot3(c_1,c_1,zz,'LineWidth',6,'Color','k','LineStyle','--')
hold on
plot3(vv,c_1,zz,'LineWidth',6,'Color','k','LineStyle','--')
hold on
plot3(c_1,zz,zz,'LineWidth',6,'Color','k','LineStyle','--')
hold on
plot3((2/3)*a1*b2/a2,(1/3)*a1*b2/a2,0,'ko','MarkerSize',10, ...
    'MarkerFaceColor','k')
hold on
plot3(.02,.008,0,'k.','MarkerSize',10,'MarkerFaceColor','k', ...
    'Marker','square')
```

```
hold on
plot3(.09,.008,0,'k.','MarkerSize',10,'MarkerFaceColor','k', ...
    'Marker','square')
hold on
plot3(.09,.082,0,'k.','MarkerSize',10,'MarkerFaceColor','k', ...
    'Marker','square')
xlim([0,a1*b2/a2])
ylim([0,a1*b2/a2])
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
grid minor
ax.MinorGridAlpha = 1;
title('Trace: $\mathrm{T}$','Interpreter','Latex', ...
    'FontSize',25)
xlabel('$c_{xy}$','Interpreter','Latex','FontSize',30)
ylabel('$c_{yx}$','Interpreter','Latex','FontSize',30)
zlabel('$\textrm{T}$','Interpreter','Latex','FontSize',22)

% The Determinate of the Jacobian
D = det(A);

% Determinate at the Critical Point Chosen
D = subs(D,[x,y],[x_crit,y_crit]);

% Sub in the Interaction Parameters
D = subs(D,[c1,c2], {C1,C2});

% Discriminant
```

```
F = double(Tau) - 4*double(D);

% Plotting the Discriminant with a Meshgrid of c_xy and c_yx values
figure(3)
surf(C1,C2,F)
hold on
% Plot of the triangle
plot3(c_1,c_1,zz,'LineWidth',6,'Color','k','LineStyle','--')
hold on
plot3(vv,c_1,zz,'LineWidth',6,'Color','k','LineStyle','--')
hold on
plot3(c_1,zz,zz,'LineWidth',6,'Color','k','LineStyle','--')
hold on
plot3((2/3)*a1*b2/a2,(1/3)*a1*b2/a2,0,'ko','MarkerSize',10, ...
    'MarkerFaceColor','k')
hold on
plot3(.02,.008,0,'k.','MarkerSize',10,'MarkerFaceColor','k', ...
    'Marker','square')
hold on
plot3(.09,.008,0,'k.','MarkerSize',10,'MarkerFaceColor','k', ...
    'Marker','square')
hold on
plot3(.09,.082,0,'k.','MarkerSize',10,'MarkerFaceColor','k', ...
    'Marker','square')
xlim([0,a1*b2/a2])
ylim([0,a1*b2/a2])
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
```

```
grid minor
ax.MinorGridAlpha = 1;
xlabel('$c_{xy}$','Interpreter','Latex','FontSize',30)
ylabel('$c_{yx}$','Interpreter','Latex','FontSize',30)
zlabel('$\mathrm{T}^2-4\mathrm{D}$','Interpreter','Latex', ...
    'FontSize',22)
title('Discriminant: $\mathrm{T}^2-4\mathrm{D}$','Interpreter','
    Latex', ...
    'FontSize',25)
```

C.3.3 Autonomous Model with Different Parameters

```
% Solutions to the Autonomous Model with Different
% Interaction Parameters

clc
clear
close all

    % Parameters
% Growth Rates
r_y = .059;
r_x = 5;

% Carry capacity
K_y = 5;
K_x = 15;

% Initial populations
x_o = 5;
y_o = 3;
```

```
% Time duration starting at 1990.
t=[0 250];

% Reproduction function parameters
c = .0001;
T_opt = 12.5; % Celsius

    % Functions
% Temperature function
T = 12.5;

% Growth Rate function
R =@(T) log( .32*r_x / ( 1 + c*(T - T_opt)^4 ) );
r = R(T);

% Interaction parameters
cMat = [0.02, .008; 0.09, .008; 0.09, .082; 0.0627, .0313];
styles = ['-',"--",':','-.'];

for i = 1:4
    % Bear ODE
    dy =@(y) r_y.*y(2).*(1 - ( y(2)./K_y ) ) +...
        cMat(i,2).*y(2).*y(1);

    % Salmon ODE
    dx =@(t,y) r .* y(1)*( 1 - ( y(1)./K_x ) ) -...
        cMat(i,1).*y(2).*y(1);

    % Autonomous System of ODE function
```

```
    MODEL = @(t,y) [dx(t,y); dy(y)];

    [t,Y] = ode45(MODEL, t, [x_o, y_o]);

    % Creates vector field
    figure(2)
    plot(Y(:,1),Y(:,2),'LineWidth',4,'LineStyle',styles(i))
    hold on

end
for i = 1:4
    % Critical Points
    x_crit = (r*r_y/K_y - cMat(i,1)*r_y) /...
        (cMat(i,1)*cMat(i,2) + (r/K_x)*(r_y/K_y));
    y_crit = (r*cMat(i,2) + r/K_x*r_y) /...
        (cMat(i,1)*cMat(i,2) + (r/K_x)*(r_y/K_y));

    % Plots Critical Points
    plot(x_crit,y_crit,'ko','MarkerSize',8,'MarkerFaceColor','k')
    hold on
end
% Plots the Initial Populations
plot(x_o,y_o,'ko','MarkerSize',12,'MarkerFaceColor','k')
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
grid minor
ax.MinorGridAlpha = 1;
xlim([0,14]);
```

```
ylim([2,13]);
xlabel('x (Salmon)','Interpreter', 'Latex', 'FontSize', 25)
ylabel('y (Bears)','Interpreter', 'Latex', 'FontSize', 25)
title('Solutions For The Autonomous Model','Interpreter', ...
    'Latex', 'FontSize', 25)
legend('$c_{xy}=0.02,\;c_{yx}=0.008$', ...
    '$c_{xy}=0.09,\;c_{yx}=0.008$', ...
    '$c_{xy}=0.09,\;c_{yx}=0.082$', ...
    '$c_{xy}=0.0627,\;c_{yx}=0.0313$',...
    'Interpreter', 'Latex', 'FontSize', 20, ...
    'Location', 'NorthEast')
```

C.4 The System of ODEs Model

C.4.1 The Autonomous Model

```
% Solutions to the Autonomous Model

clc
clear
close all

    % Parameters
% Growth Rates
r_y = .059;
r_x = 5;

% Carry Capacity
K_y = 5;
K_x = 15;
```

```
% Initial Populations
x_o = 5;
y_o = 3;

% Interaction Terms
c_x = 0.0627;
c_y = 0.0313;

    % Try these Parameters to see something a bit more realistic
% c_x = .02;
% c_y = .0008;

% Time Duraction.
t=[0 250];

% Growth Rate Function Parameters
c = .0001;
T_opt = 12.5; % Celsius

% Temperature
T = 12.5;

% Growth Rate function
R =@(T) log( .32*r_x / ( 1 + c*(T - T_opt)^4 ) );
r = R(T);

% Bear ODE
dy =@(y) r_y.*y(2).*(1 - ( y(2)./K_y ) ) + c_y.*y(2).*y(1);

```

```
% Salmon ODE
dx =@(t,y) r .* y(1)*( 1 - ( y(1)./K_x ) ) - c_x.*y(2).*y(1);

% System of ODE function
MODEL = @(t,y) [dx(t,y); dy(y)];

% System of ODE solver
[t,pop] = ode15s(MODEL, t, [x_o, y_o]);

% Plotting the Autonomous System Over the Time
figure(1)
plot(t, pop(:,1), 'b--', 'LineWidth', 4)
hold on
plot(t, pop(:,2), 'r', 'LineWidth', 4)
hold off
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
grid minor
ax.MinorGridAlpha = 1;
xlabel("Time (yrs)","Interpreter","Latex", ...
    'FontSize', 25)
ylabel("Population","Interpreter","Latex", ...
    'FontSize', 25)
str = "The Autonomous Model";
title(str,"Interpreter","Latex", 'FontSize', 25)
legend('Salmon', 'Brown Bear', 'FontSize', 25, ...
    'Location', 'NorthEast')

```

```
% ==============================================================

% ==============================================================

% Plotting the Autonomous Model with Different Initial Populations
for y = 1:2:3
    for x = 3:2:5
        % Solutions to the Autonomous System
        [t,Y] = ode45(MODEL, t, [x, y]);

        % Creates vector field
        figure(2)
        plot(Y(:,1),Y(:,2),'LineWidth',4)
        hold on
    end
end
for y = 1:2:3
    for x = 3:2:5
        % Plots the Initial Populations
        figure(2)
        plot(x,y,'ko','MarkerSize',8,'MarkerFaceColor','k')
        hold on
    end
end
% Plots the Critical Point
plot(0.79,7.1,'ko','MarkerSize',12,'MarkerFaceColor','k')
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
```

```
grid minor
ax.MinorGridAlpha = 1;
xlabel('x (Salmon)',"Interpreter","Latex", ...
    'FontSize', 25)
ylabel('y (Bears)',"Interpreter","Latex", ...
    'FontSize', 25)
title('Solutions For The Autonomous Model', ...
    'FontSize', 25)
legend('$(3,\;1)$', '$(5,\;11)$', '$(3,\;3)$','$(5,\;3)$', ...
    'Interpreter', 'Latex', 'FontSize', 20,...
    'Location', 'NorthEast')
```

C.4.2 The Non-Autonomous Model

```
% Solutions to the Non-autonomous Model

clc
clear
close all

    % Parameters
% Growth Rates
r_y = .059;
r_x = 5;

% Carry Capacity
K_y = 5;
K_x = 15;

% Initial Populations
x_o = 5;
```

```
y_o = 3;

% Interaction Terms
c_x = 0.0627;
c_y = 0.0313;

    % Try these Parameters to see something a bit more realistic
% c_x = .02;
% c_y = .0008;

% Time Duraction.
t=[0 250];

% Bear ODE
dy =@(y) r_y.*y(2).*(1 - ( y(2)./K_y ) ) + c_y.*y(2).*y(1);

% Salmon ODE
dx =@(t,y) (log(8/(5*(((2*t)/25 - 74/25)^4/10000 + 1))) -...
    (4*t*((2*t)/25 - 74/25)^3*(((2*t)/25 - 74/25)^4/16000 +...
    5/8))/(78125*(((2*t)/25 - 74/25)^4/10000 + 1)^2)).*...
    y(1)*( 1 - ( y(1)./K_x ) ) - c_x.*y(2).*y(1);

% Non-Autonomous System of ODE Function
MODEL = @(t,y) [dx(t,y); dy(y)];

% Solutions to the Non-Autonomous System of ODE
[t,pop] = ode15s(MODEL, t, [x_o, y_o]);

% Plotting the Non-Autonomous System Over the Time
figure(1)
```

```
plot(t, pop(:,1),'b--', 'LineWidth', 4)
hold on
plot(t, pop(:,2),'r', 'LineWidth', 4)
hold off
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
grid minor
ax.MinorGridAlpha = 1;
xlabel("Time (yrs)","Interpreter","Latex", 'FontSize', 25)
ylabel("Population","Interpreter","Latex", 'FontSize', 25)
str = "The Non-Autonomous Model";
title(str,"Interpreter","Latex", 'FontSize', 25)
legend('Salmon', 'Brown Bear', 'FontSize', 25, 'Location', '
    NorthEast')

% ==============================================================

% ==============================================================

% List for Cycling Through Different Line Styles
styles = ['-',"--",':','-.'];
i = 1;
% Plotting the Non-Autonomous Model with Different Initial
% Populations
for y = 1:2:3
    for x = 3:2:5
        % Solutions to the Non-Autonomous System
        [t,Y] = ode45(MODEL, t, [x, y]);
```

```
        % Creates vector field
        figure(2)
        plot(Y(:,1),Y(:,2),'LineWidth',4,'LineStyle',styles(i))
        hold on
        i = i + 1;
    end
end
for y = 1:2:3
    for x = 3:2:5
        % Plots the Initial Populations
        plot(x,y,'ko','MarkerSize',12,'MarkerFaceColor','k')
        hold on
    end
end
% Plots the Critical Point
plot(0,K_y,'ko','MarkerSize',12,'MarkerFaceColor','k')
grid on
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
grid minor
ax.MinorGridAlpha = 1;
xlabel('x (Salmon)',"Interpreter","Latex", 'FontSize', 25)
ylabel('y (Bears)',"Interpreter","Latex", 'FontSize', 25)
title('Solutions For The Non-Autonomous Model', ...
    "Interpreter","Latex", 'FontSize', 25)
legend('$(3,\;1)$', '$(5,\;1)$', '$(3,\;3)$',...
    '$(5,\;3)$',...
    'Interpreter', 'Latex', 'FontSize', 20, 'Location', 'NorthEast'
```

```
    )
xlim([0,9])
ylim([0,12])
```

C.4.3 Comparing Autonomous Vs Non-Autonomous

```
% Autonomous Vs Non-Autonomous

clc
clear
close all

    % Parameters
% Growth Rates
r_y = .059;
r_x = 5;

% Carry Capacity
K_y = 5;
K_x = 15;

% Initial Populations
x_o = 5;
y_o = 3;

% Interaction Terms
c_x = 0.0627;
c_y = 0.0313;

    % Try these Parameters to see something a bit more realistic
% c_x = .02;
```

```
% c_y = .0008;

% Time Duraction.
t=[0 250];

% Growth Rate Function Parameters
c = .0001;
T_opt = 12.5; % Celsius

% Temperature
T = 12.5;

% Growth Rate Function
R =@(T) log( .32*r_x / ( 1 + c*(T - T_opt)^4 ) );
r = R(T);

% Bear ODE
dy =@(y) r_y.*y(2).*(1 - ( y(2)./K_y ) ) + c_y.*y(2).*y(1);

% Salmon ODE
dx =@(t,y) r .* y(1)*( 1 - ( y(1)./K_x ) ) - c_x.*y(2).*y(1);

% Autonomous System of ODE Function
MODEL = @(t,y) [dx(t,y); dy(y)];

% Solutions to the Autonomous System of ODE
[t,pop] = ode15s(MODEL, t, [x_o, y_o]);

% Plotting the Autonomous System Over the Time
figure(1)
```

```
plot(t, pop(:,1), 'b--', 'LineWidth', 4)
hold on
plot(t, pop(:,2), 'r', 'LineWidth', 4)
grid on
ylim([0,11])
ax = gca;
ax.GridAlpha = 1;
set(gca,"FontSize",20)
grid minor
ax.MinorGridAlpha = 1;
xlabel("Time (yrs)","Interpreter","Latex", 'FontSize', 25)
ylabel("Population","Interpreter","Latex", 'FontSize', 25)
str = "Autonomous and Non-Autonomous Model";
title(str,"Interpreter","Latex", 'FontSize', 25)

% =============================================================

% =============================================================

    % Non-autonomous

% Bear ODE
dy =@(y) r_y.*y(2).*(1 - ( y(2)./K_y ) ) + c_y.*y(2).*y(1);

% Salmon ODE
dx =@(t,y) (log(8/(5*(((2*t)/25 - 74/25)^4/10000 + 1))) -...
    (4*t*((2*t)/25 - 74/25)^3*(((2*t)/25 - 74/25)^4/16000 +...
    5/8))/(78125*(((2*t)/25 - 74/25)^4/10000 + 1)^2)).* y(1)*( 1
   -...
    ( y(1)./K_x ) ) - c_x.*y(2).*y(1);
```

```
% Non-Autonomous System of ODE Function
MODEL = @(t,y) [dx(t,y); dy(y)];

% Solutions to the Non-Autonomous System of ODE
[t,pop] = ode15s(MODEL, t, [x_o, y_o]);

% Plotting the Non-Autonomous Model on top of Figure 1
figure(1)
plot(t, pop(:,1),'g:', 'LineWidth', 4)
hold on
plot(t, pop(:,2),'k-.', 'LineWidth', 4)
legend('Salmon (Auto)', 'Brown Bear (Auto)', 'Salmon (Non)', ...
    'Brown Bear (Non)', 'FontSize', 25, 'Location', 'NorthEast')
```