Coupled System Oscillators

Connor Adams

May 14, 2021

Abstract

In this paper We will be looking at a coupled system of oscillators and comparing it to a single oscillator. We begin by setting up the single oscillator using the difference equation:

$$x(n+1) = f(x(n)), \text{ with } f(x) = 1 - \alpha x^2$$

Alpha will be a constant that is a positive real number. As the value of alpha is altered, we will notice that the oscillations are effected in an interesting way. As alpha increases we introduce a stable two cycle at around $\alpha = .75$, then a stable four cycle at around $\alpha = 1.25$, next is a stable eight cycle at around $\alpha = 1.37$, stable six cycle at around $\alpha = 1.48$, lastly, we notice a stable three cycle around $\alpha = 1.75$. Between some of the cycles there seems to be some chaos. After exploring the effects of alpha with single oscillators we move on to more complicated systems.

Now, it is time to introduce a coupled system of oscillators. We look at the system where:

$$x(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_p(n) \end{bmatrix}$$

and the p first order nonlinear equations are:

$$x_1(n+1) = bf(x_2(n)) + af(x_1(n)) + bf(x_p(n))$$

$$x_i(n+1) = bf(x_{i-1}(n)) + af(x_i(n)) + bf(x_{i+1}(n))$$

$$x_p(n+1) = bf(x_{p-1}(n)) + af(x_p(n)) + bf(x_1(n))$$

Note that a and b are real constants and $f(x) = 1 - \alpha x^2$. when letting $p = 32$, $a = .6$, and $b = .2$ we notice that coupled system experiences almost the same stable cycles as the single oscillator at the same alphas.

---

1 Introduction

Consider a single oscillator using the difference equation:

$$x(n+1) = f(x(n)), \text{ with } f(x) = 1 - \alpha x^2, \qquad (1)$$

where alpha is a constant positive real number. $x(n+1)$ relates to the position of the object after each iteration. We are essentially predicting the future. Since this is a single oscillator, we would be mapping something like a pendulum in motion. We will be examining where the stable cycles are and which alpha values give us those stable cycles.

Now, consider a coupled system oscillator such as:

$$x(n) = \begin{bmatrix} x_1(n) \\ \vdots \\ x_p(n) \end{bmatrix}$$

and the p first order nonlinear equations are:

$$x_1(n+1) = bf(x_2(n)) + af(x_1(n)) + bf(x_p(n))$$
$$x_i(n+1) = bf(x_{i-1}(n)) + af(x_i(n)) + bf(x_{i+1}(n))$$
$$x_p(n+1) = bf(x_{p-1}(n)) + af(x_p(n)) + bf(x_1(n))$$

I believe a good way to think about this is that there are p amount of masses connected together by springs. If there is an initial displacement on those masses, $x(0)$, then the nonlinear equations will help us predict where the masses' next position will be, $x(n+1)$. Just like in the single oscillators, we will be establishing where the stable cycles are and which alpha values give us those stable cycles.

---

2 Discussion of The Results and Project

2.1 Single Oscillator

To begin I would like to analyze the bifurcation plot of f for $0 \leq \alpha \leq 2$.

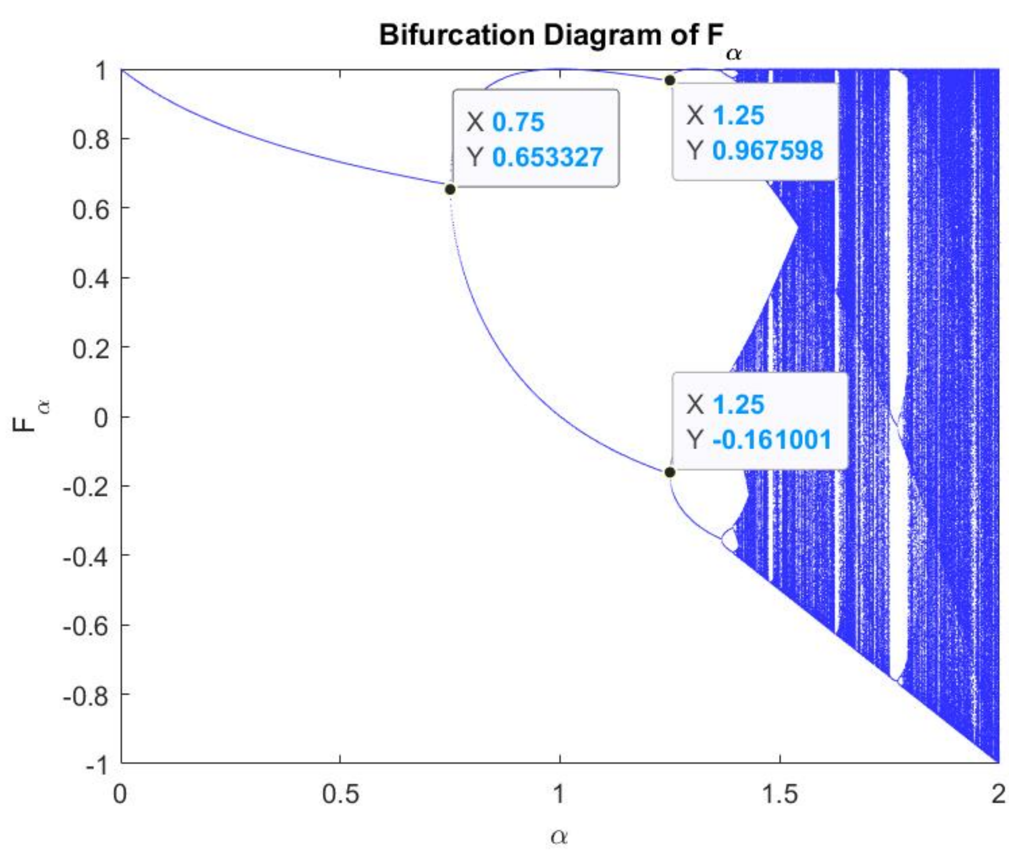

Figure 1: Bifurcation Diagram of f.

Notice that there is a stable two cycle beginning at around $\alpha = .75$ and continues to about $\alpha = 1.25$, but not at $\alpha = 1.25$, this will be explained later. When looking at α before .75 we just get single line, which should represent all points converging together. We can test this by plotting multiple iterations of $x(n)$ with respect to the iterations, n:

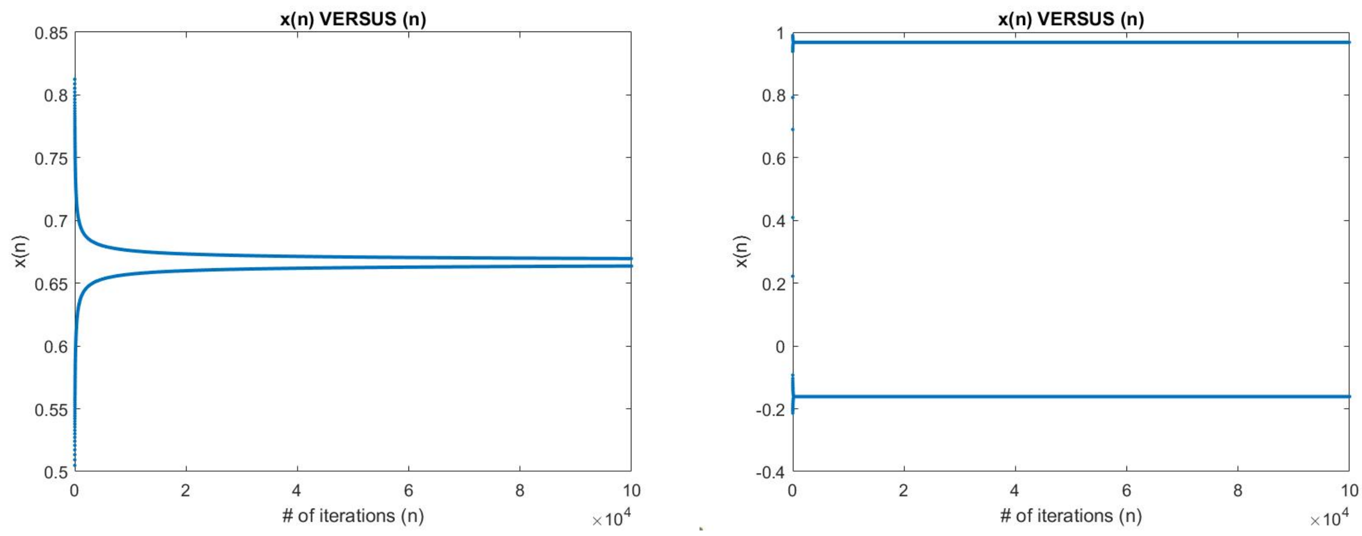

Figure 2: The plot on the left illustrates the case when $\alpha = .75$ and the one on the right illustrates the case when $\alpha = 1.24$.

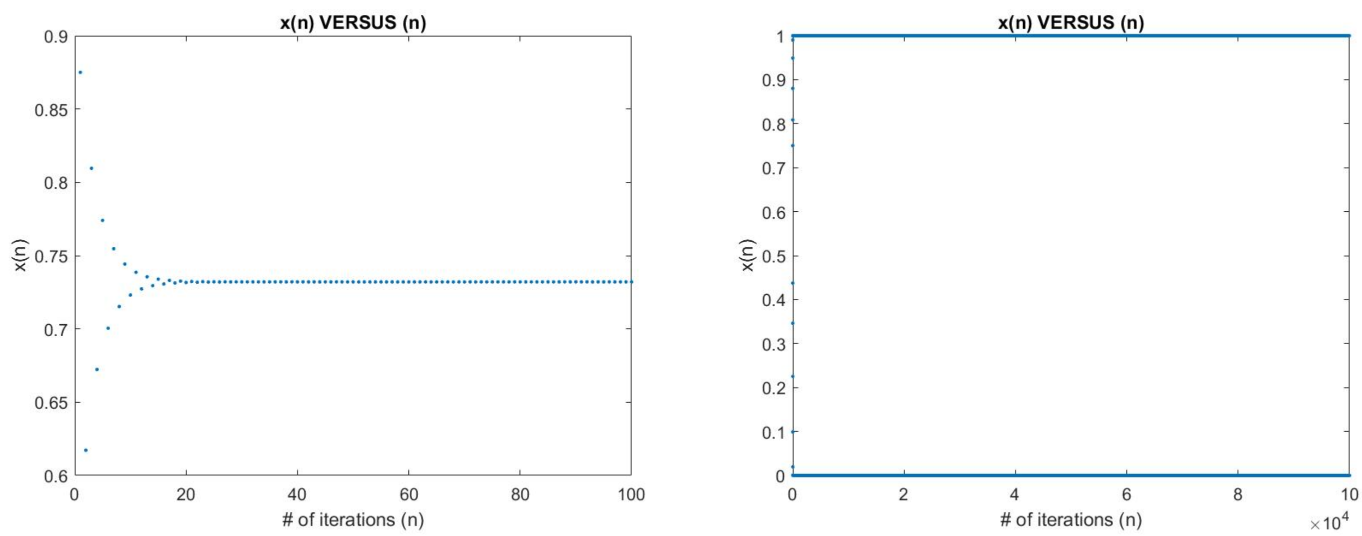

Figure 3: $\alpha = .5$ on the left and $\alpha = 1$ on the right. The plot on the left shows that all the points converge to the same $x(n)$ over a safe number of iterations, n. The plot on the right illustrates what happens when α is between .75 and 1.25.

The iterations for the 3 figures in which $\alpha \geq .75$ are large enough that we can safely determine that the two dotted lines will not converge into one, hence the conclusion that the stable two cycle beginning at around $\alpha = .75$ and continues to about $\alpha = 1.25$. So, what happens at $\alpha = 1.25$ is this:

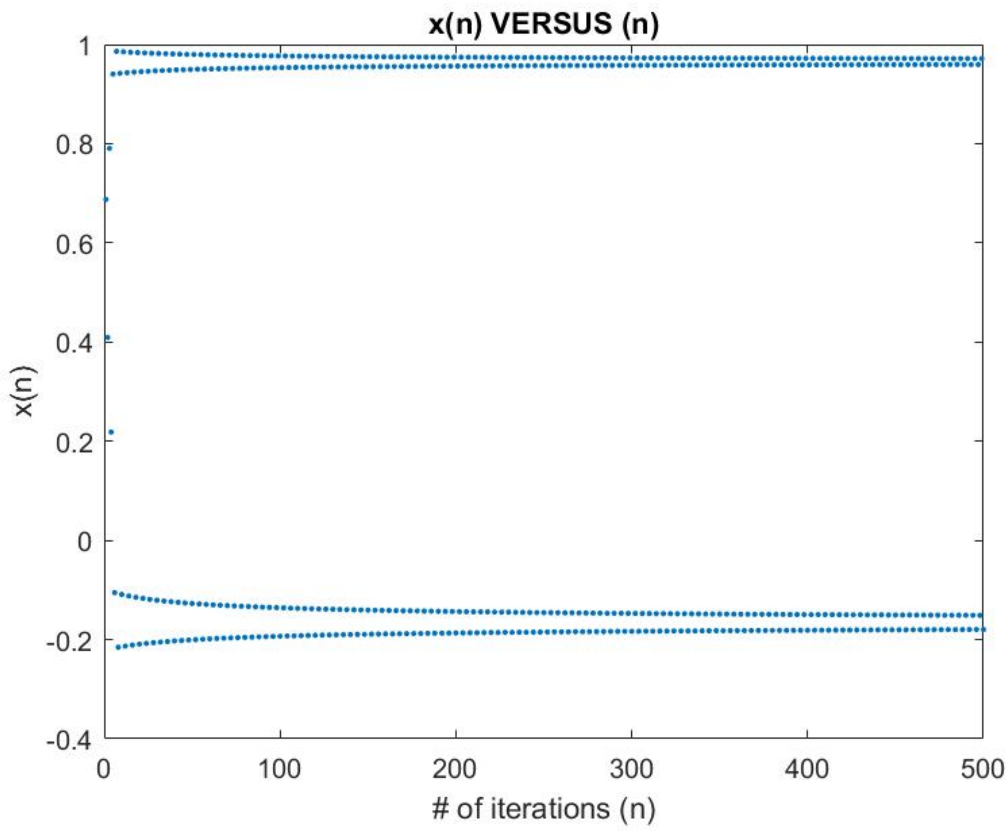

Figure 4: Illustrating the points when $\alpha = 1.25$.

When α hits 1.25 the stable cycles go from 2 to 4. Looking back at the bifurcation diagram we can visually see this happening. We can keep following α on the bifurcation diagram and some new points will occur where the cycle is different.

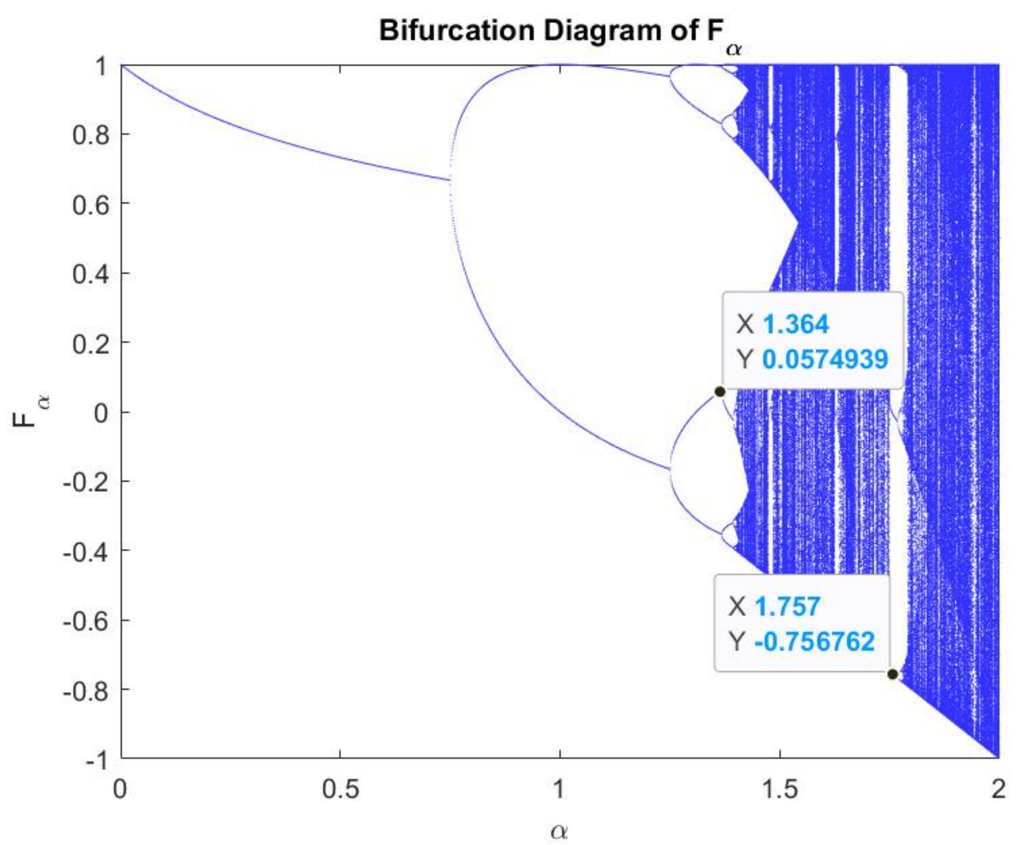

Figure 5: Some new points to look at for the bifurcation diagram.

I selected some new points to evaluate. When $\alpha = 1.364$ it seems that f will be transforming into a stable 8 cycle and when $\alpha = 1.75$, f will reduce back down to a stable 3 cycle.

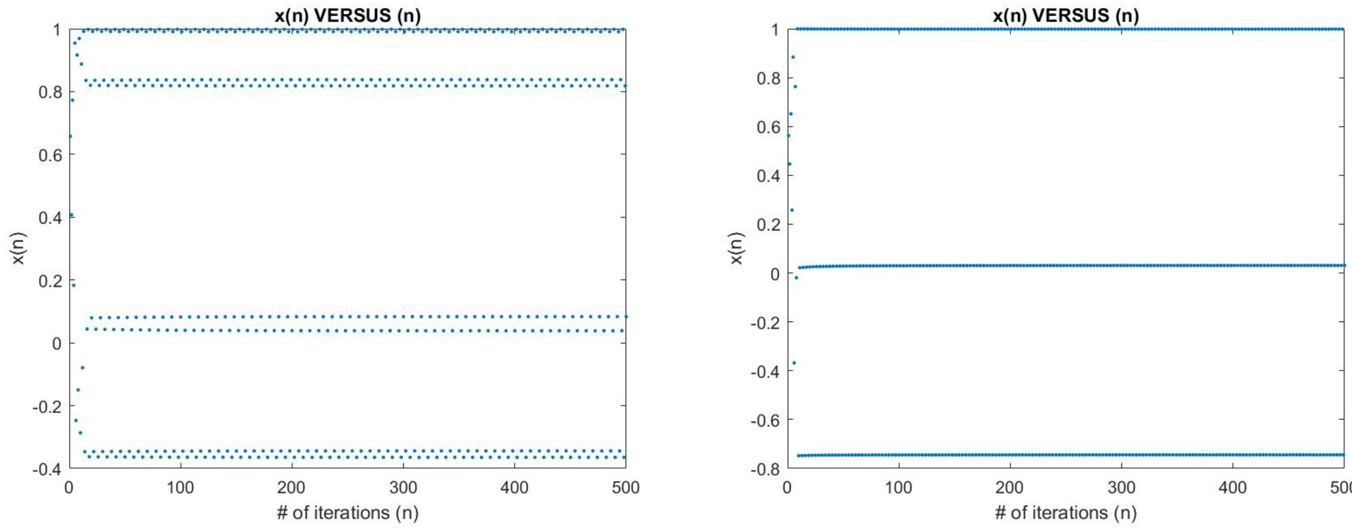

Figure 6: The plot on the left is created when $\alpha = 1.37$ *and the other one is when* $\alpha = 1.75$.

The reason I chose to $\alpha = 1.37$ for the left plot is for visual reasons, so if you evaluate α at 1.364 you will see an 8 cycle, this plot is more clear. Now, lets look at value for α that doesn't fall between these cycles, like $\alpha = 1.9$. When we let $\alpha = 1.9$ we get a plot that looks like:

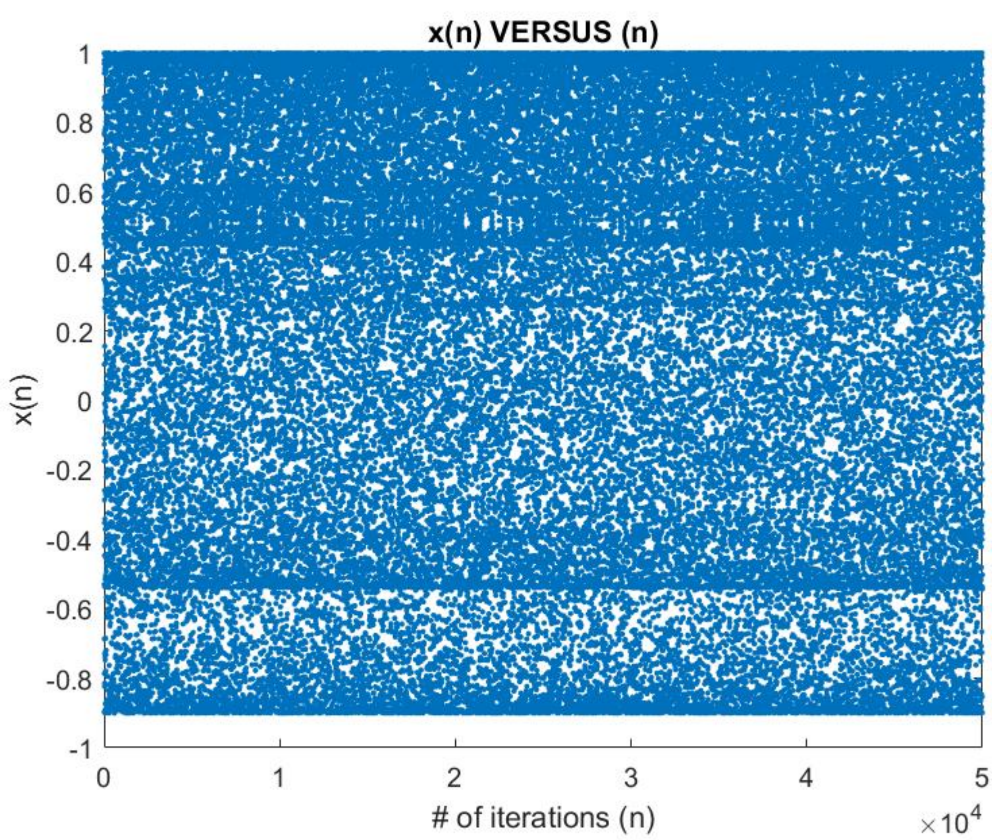

Figure 7: This is what the plot looks like when alpha $= 1.9$.

For this illustrations I let the iterations be really high because I wanted to really demonstrate it's chaos. It's almost like every point leads to an entirely different point that has never been hit before on the grid..

2.2 Coupled System of Oscillators and Markov Chains

In order to make life a whole lot easier I am going to convert the p first order nonlinear equations into matrix-vector form. This will allow iterating the system of difference equations a lot easier. So, my goal is to get something like this:

$$x(n+1) = Ag(x(n))$$

The A matrix is going to be p x p and look like this:

$$A = \begin{bmatrix} a & b & 0 & 0 & 0 & \dots & 0 & b \\ b & a & b & 0 & 0 & \dots & 0 & 0 \\ 0 & b & a & b & 0 & \dots & 0 & 0 \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & b & a & b \\ b & 0 & 0 & 0 & 0 & \dots & b & a \end{bmatrix}$$

This honestly reminds me of a set up for solving Markov chains just more complicated in my opinion because we are dealing with nonlinear difference equations instead of linear equations. Remember when I gave the example of a coupled system of oscillators. The idea is that knowing the initial displacement of the masses we could predict where they masses will be during any point in time. Well the idea is sort of the same with Markov chains, but there are key differences. Below is an image of a Markov chain and our coupled oscillator system. Note the similarities and differences:

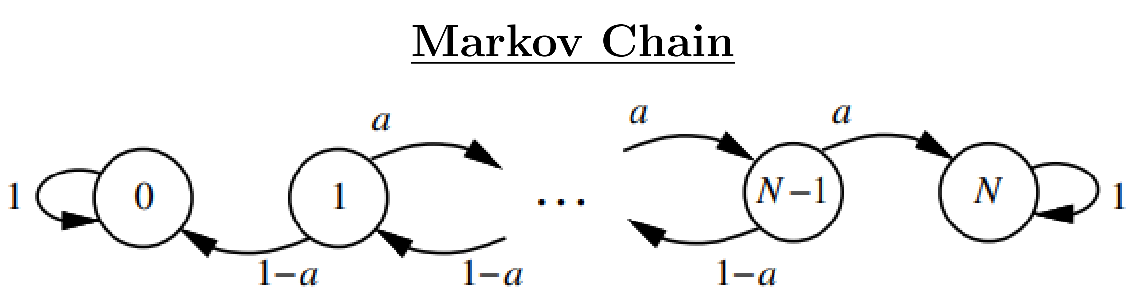

Figure 8: This is a diagram of a Markov Chain.

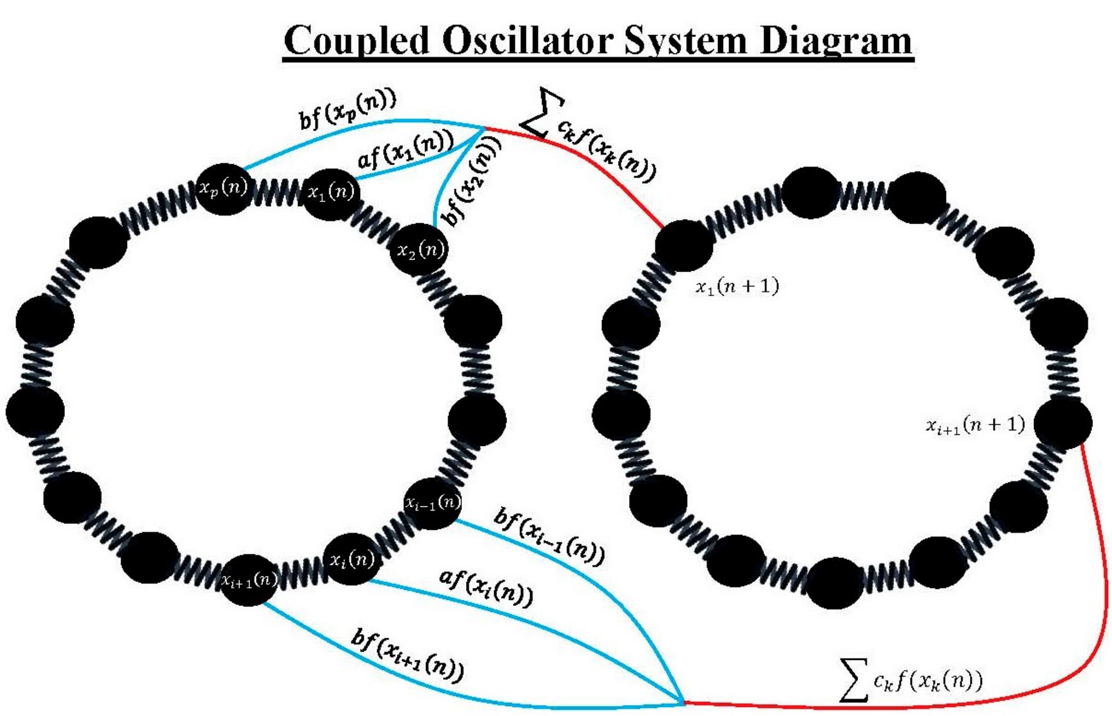

Figure 9: This is diagram of our Couples System Oscillator.

Now, a transition matrix for a Markov chain can look exactly the same as our A matrix if set up correctly. The difference between the meaning of the matrices is important. The similarities are that both matrices are meant to determine the next state, or destination, in that iteration. A key difference is that the transition matrix of a Markov chain is using b and a as probabilities of moving from state to state and using a high number of iterations to determine which state the system will converge to, while our A matrix is using b and a as values that determine how much a mass effects the destination of it's neighbour and the high number of iterations determines where those masses will be location wise. It is very interesting to see this matrix set up appearing to two different fields of mathematics.

2.3 Iterate a Coupled System of Oscillators

Now that we have the A matrix, we can set the vector, $g(x(n))$, to:

$$g(x(n)) = \begin{bmatrix} f(x_1(n)) \\ \vdots \\ f(x_p(n)) \end{bmatrix}$$

Remember $f(x) = 1 - \alpha x^2$, so by letting $p = 32$, $b = .6$, $a = .2$, and $n = 10^6$, where n is the number of iterations we can study what is happening to $x_1(n)$ over that iteration period. By letting α be a vector we can get multiple plots for $x_1(n)$ instead of just one. So, the vector I created for alpha is:

$$g(x(n)) = \begin{bmatrix} .5 \\ .75 \\ 1 \\ 1.25 \\ 1.37 \\ 1.75 \\ 1.9 \end{bmatrix}$$

These are the values that we used earlier to assess what would happen to the single oscillator over a given number of iterations. When putting it all together we get something along the lines of:

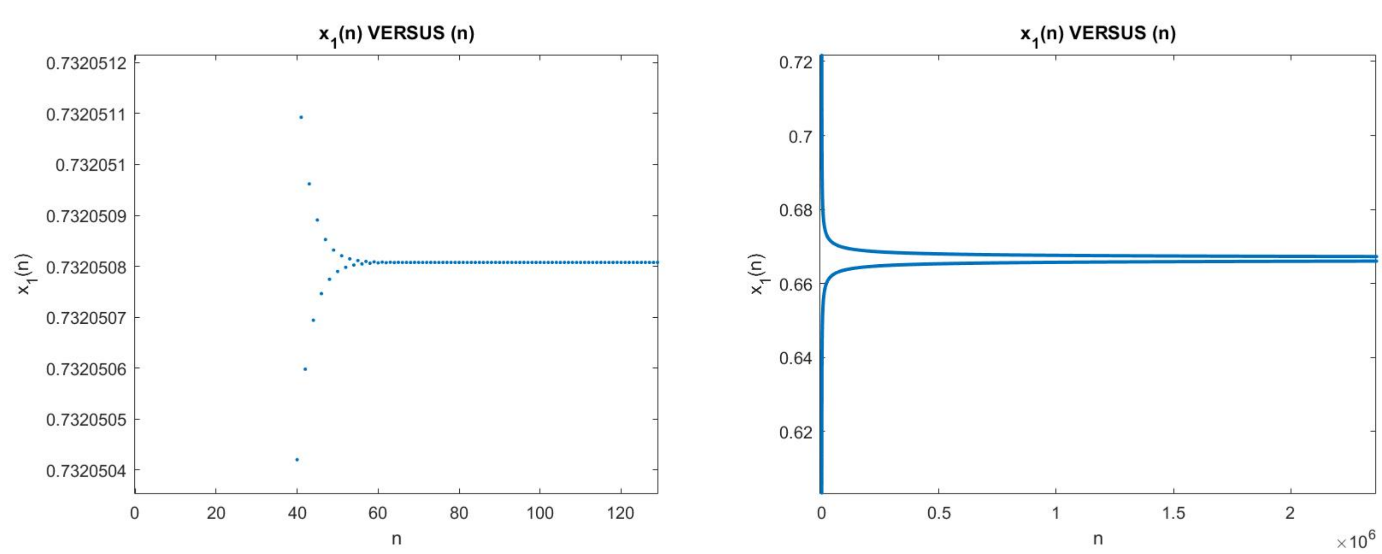

Figure 10: The left side of the figure represents the plot when $\alpha = .5$ and the

right side of the figure represents the plot when $\alpha = .75$.

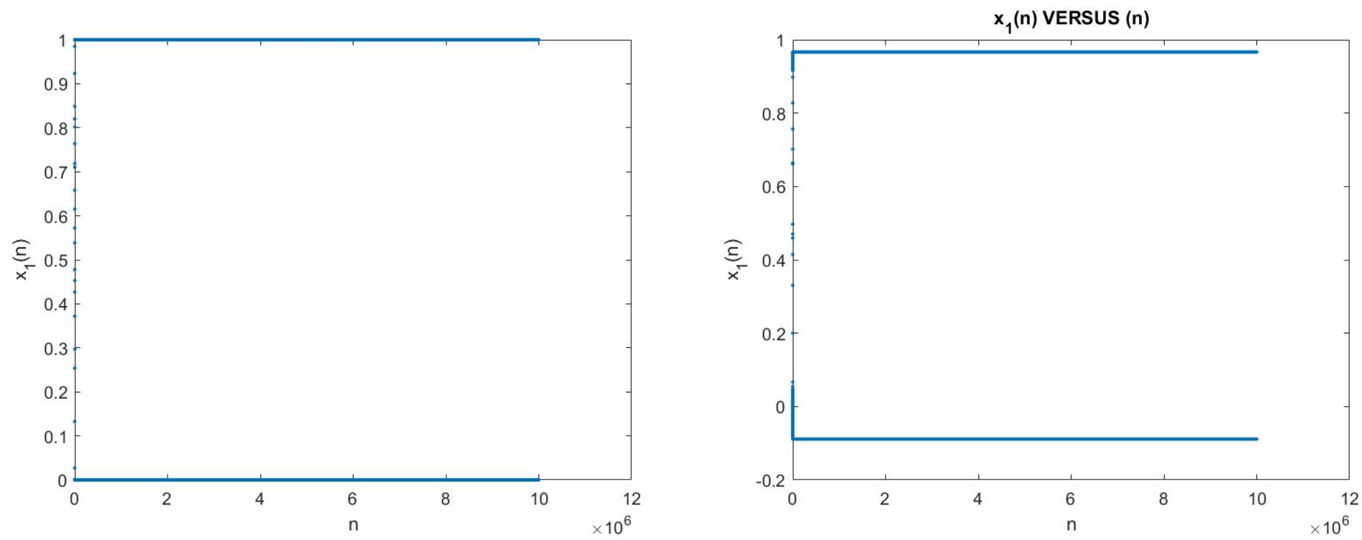

Figure 11: The left side of the figure represents the plot when $\alpha = 1$ and the other side of the figure represents the plot when $\alpha = 1.25$.

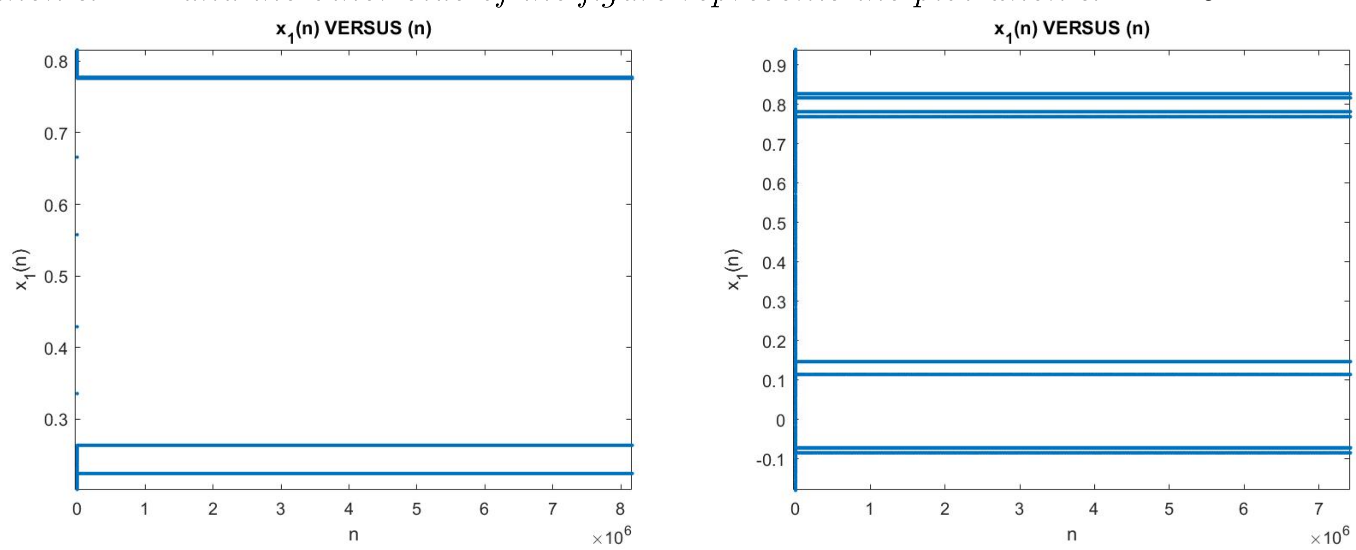

Figure 12: The left side of the figure represents the plot when $\alpha = 1.37$ and the right side of the figure represents the plot when $\alpha = 1.75$.

When looking at figure 10, and the left side for figure 11 we notice something very similar to the figures for the single oscillator. For example, when $\alpha = .5$ the sequence converges to a single $x_1(n)$ value just like in figure 3. When $\alpha = 1.25$ we start noticing some difference such as the plot is no longer a 4-cycle when $\alpha = 1.25$, instead it is a 2-cycle. When $\alpha = 1.37$ we are receiving a 4-cycle and when $\alpha = 1.75$ we get a plot that has an 8-cycle. The last difference I have come to find is when $\alpha = 1.9$.

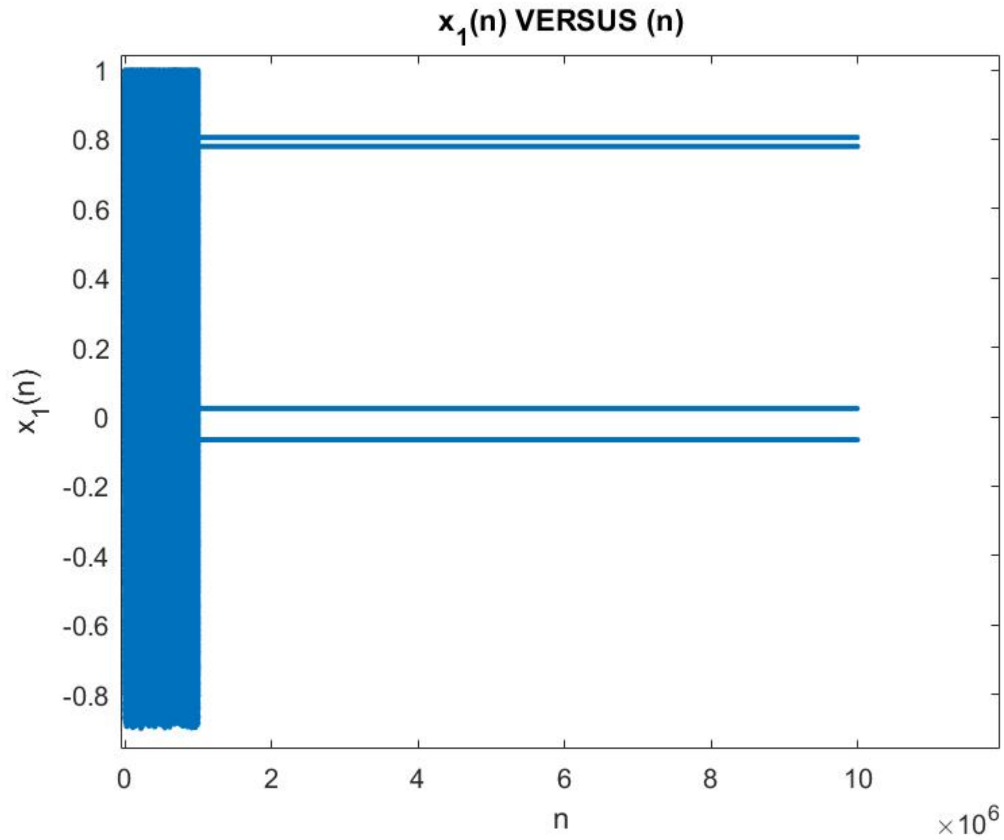

Figure 10: The wild story of $x_1(n)$ over n iterations when $\alpha = 1.9$

I find this plot to be quite interesting because of how chaotic it looks off the start. With a small or medium amount of iteration this plot may seem to just continue to be chaotic, but in actuality it comes to a stable 4-cycle.

3 References

References

[1] Alex Blumenthal: NSF Mathematics Postdoc specializing in Dynamical Systems,
`https://www.quora.com/Whats-the-difference-between-a-linear-system-and-a`

[2] Jia Wu: Determining the Coupling Source on a Set of Oscillators from Experimental Data,
`https://www.hindawi.com/journals/complexity/2017/8017138/`

[3] Vassili N. Kolokoltsov: Nonlinear Markov processes and kinetic equations,
`https://warwick.ac.uk/fac/sci/statistics/staff/academic-research/kolokol`

[4] Alexander F. Vakakis, Richard H. Rand:Non-linear dynamics of a system of coupled oscillators with essential stiness non-linearities,
`http://pi.math.cornell.edu/ rand/randpdf/alex1.pdf`

[5] G. Goldin, Strongly Nonlinear Stochastic Processes in Physics and the Life Sciences,
`https://www.hindawi.com/journals/isrn/2013/149169/`

[6] Yen-Jie Lee, MIT 8.03SC Physics III: Vibrations and Waves, Fall 2016,
`https://www.youtube.com/watch?v=BX4QPdP7fT8`

4 Source Code

```
%Sinle Oscillator Bifurcation
clear all;
close all;
clc;

xvals = [];

for alpha = 0:.001:2
    x_n = .5;
    for i = 1:5000
        f_x = 1 - ( alpha * x_n ^ 2);
        x_n = f_x;
    end
    x = f_x;
    for i = 1:10000
        f_x = 1 - ( alpha * x_n ^ 2);
        x_n = f_x;
        xvals(1,length(xvals)+1) = alpha;
        xvals(2,length(xvals)) = f_x;
        if (abs(f_x-x)<.001)
            break;
        end
    end
end
plot(xvals(1,:),xvals(2,:),'.','LineWidth',.1,'MarkerSize',1.1,'Color',[.2,.2,1]);
ylabel('F_\alpha');
xlabel('\alpha');
title('Bifurcation Diagram of F_\alpha')
```

```
%Sinle Oscillator Exploration
clear all;
close all;
clc;

alpha = .5;
x_n = .5;

for i = 1:100

   f_x = 1 - ( alpha * x_n ^ 2 );
   y(i) = f_x;
   x_n = f_x;


end

plot(y,'.')
ylabel('x(n)');
xlabel('# of iterations (n)');
title('x(n) VERSUS (n)')
```

```
%Couples System Oscillator x_1(n) values
clear all;
close all;
clc;

p = 32;
a = .6;
b = .2;
alpha_vector = [.5; .75; 1; 1.25; 1.37; 1.75; 1.9];
number_of_alphas = length(alpha_vector);
 for t = 1:number_of_alphas

alpha = alpha_vector(t);
x_n = rand(p, 1);

a_diag = a*eye(p);
b_1 = b * diag(ones(1,p-1),1);
b_2 = b * diag(ones(1,p-1),-1);
A = a_diag + b_1 + b_2;
A(1,p) = b;
A(p,1) = b;

n = 10^7;

for m = 1:n+1

for i = 1:p
    f_x = 1 - ( alpha * x_n(i)^2 );

    g_x(i) = f_x;

end

x_n_plus_m = A * g_x';

x_n_plus_m_matrix(:, m) = x_n_plus_m;

x_n = x_n_plus_m;


end

y = x_n_plus_m_matrix;

figure(t)
plot(y(1,:),'.')
ylabel('x_1(n)');
xlabel('n');
title('x_1(n) VERSUS (n)')
hold on
 end
```