Math Notes

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0.1 Variance, the arbitrariness thereof

For some time I have wondered about the arbitrariness around the mean and variance. For example why the arithmetic mean instead of the geometric or root-mean-squared? For the why square root the variance to give the standard deviation?

Well the strictness of Markov's and Chebyshev's might provide a reason. Both rely of the conditional expected value, so just to reiterate:

$$E[X|X \geq a] \geq a$$

Since everything X can be is greater then a it's expected value must be greater than a. Notice the strictness of the inequality, this will be used to make the following inequalities much stricter.

Markov

$$\begin{aligned} \mu =& E[X] \\ =& P(X \leq a)E[X|X \leq a] + P(X \geq a)E[X|X \geq a] \\ \geq& 0 \cdot E[X|X \leq a] + P(X \geq a)a \\ \frac{\mu}{a} \geq& P(X \geq a) \end{aligned}$$

Chebyshev

$$\begin{aligned} &E[(X-a)^2] \\ =& P(|X-a| \leq b)E[(X-a)^2||X-a| \leq b] + P(|X-a| > b)E[(X-a)^2||X-a| > b] \end{aligned}$$

A General Relation

Assume:

$$f(S') \geq 0, \quad g(S) \geq 0$$

Then through:

$$E[f(X)] = P(X \in S)E[f(X)|X \in S] + P(X \in S')E[f(X)|X \in S']$$

We have:

$$1 - \frac{E[g(X)]}{E[g(X)|X \in S']} \leq P[X \in S] \leq \frac{E[f(X)]}{E[f(X)|X \in S]}$$

With dual equality if:

$$f = 1_S, \quad g = 1_{S'}$$

Covariance

Lets try to find the lest squares regression between X and Y such that:

$$E[X] = E[Y] = 0, E[X^2] = E[Y^2] = 1$$

Since the expected values are both zero the line is through the origin

$$\begin{aligned}
\sum_n (mx_n + c - y_n)^2 =& nE[(mX + c - Y)^2] \\
=& n\Big(E[m^2X^2] + E[c^2] + E[Y^2] + E[2cmX] + E[-2mXY] + E[-2cY] \Big) \\
=& n(m^2 + c^2 + 1 - 2mE[XY])
\end{aligned}$$

Trying to minimize this value by our selection of trivially gets:

$$c = 0, \quad m = E[XY]$$

Just expanding the definitions gives:

$$COV[X, Y] = E[(X - E[X])(Y - E[Y])] = E[XY] = m$$

Hence the covariance can 'naturally' be interpreted and the first order function between the valuables.

$$E[f(X)] \approx E[f_0 + f_1X + f_2X^2/2] = f_0 + \mu f_1 + \sigma^2 f_2/2$$

$$E[f(X)] \approx E[f(\mu) + (X - \mu)f'(\mu) + (X - \mu)^2/2f''(\mu)] = f(\mu) + \frac{f''(\mu)}{2}\sigma^2$$