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1. Inverse Matrix

- Recall that the inverse of A is equal to: $\frac{1}{|A|} \times \text{Adjugate of A}$
- Further recall that the adjugate is equal to the transpose of the cofactor matrix.
- Find the inverse of the following matrix:

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$$

determinant:

$$\begin{aligned} & 4[(3\cdot 7)-(0\cdot 2)] \\ & -1[(0\cdot 7)-(3\cdot 2)] \\ & +(-1)\cdot[(0\cdot 0)-(3\cdot 3)] \\ & = 84+6+9=99 \end{aligned}$$

1. Inverse Matrix

- The cofactor matrix of $A=\begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$ is:

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & +7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & +\begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

1. Inverse Matrix

- Cofactor matrix

$$\begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

- Transpose of cofactors:

$$\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

- Inverse of A:

$$\frac{1}{|\mathrm{A}|} adj\,\mathrm{A} = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

2. SVAR Identification: restrictions

- Imposing the restriction makes the number of unknown parameters in the structural model equal to the number of parameters known from the standard VAR estimation:

1. $g_{10} = \beta_{10}$
2. $g_{20} = -a_{21}\beta_{10} + \beta_{20}$
3. $g_{11} = \beta_{11}$
4. $g_{21} = -a_{21}\beta_{11} + \beta_{12}$
5. $g_{12} = \beta_{12}$
6. $g_{22} = -a_{21}\beta_{12} + \beta_{22}$

Since $e_{yt} = u_{yt}$ then,

7. $\text{var}(e_{yt}) = \sigma_{uy}^2$
8. $\text{var}(e_{rt}) = \sigma_{ur}^2 + a_{21}^2\sigma_{uy}^2$
9. $cov(e_{yt}e_{rt}) = E(u_{yt})(u_{rt} - a_{21}u_{yt}) = -a_{21}\sigma_{uy}^2$

- Substitute the estimated values for these 9 parameters $g_{10}, g_{11}, g_{12}, g_{20}, g_{21}, g_{22}, \sigma^2{}_{ey}, \sigma^2{}_{er}$ and $\sigma_{ey\text{-}er}$ in the 9 equations, to solve for: $a_{21}, \beta_{10}, \beta_{20}, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}, \sigma^2{}_{\upsilon y}$, and $\sigma^2{}_{\upsilon r}$.

3. Impulse-responses

- In a first order autoregressive model:

$$y_t = g_0 + g_1 y_{t-1} + \varepsilon_t$$

- The stability condition is that $|g_1| < 1$
- In the case of a VAR, an equivalent condition is required. It becomes clear when the standard VAR is iterated backwards. For a simple VAR(1):

$$X_t = G_0 + G_1 X_{t-1} + e_t \quad \longrightarrow \quad X_{t-1} = G_0 + G_1 X_{t-2} + e_{t-1}$$

$$X_t = G_0 + G_1(G_0 + G_1 X_{t-2} + e_{t-1}) + e_t$$

3. Impulse-responses

- Successive iterations generate:

$$X_t = G_0 + G_1 \overbrace{(G_0 + G_1 \underbrace{(G_0 + G_1 X_{t-3} + e_{t-2})}_{X_{t-2}} + e_{t-1})}^{X_{t-1}} + e_t$$

$$X_t = (I + G_1 + G_1^2 + \cdots + G_1^{\infty})G_0 + e_t + \sum_{i=1}^{\infty} G_1^i e_{t-i} + G_1^i X_{t-n-i} \rightarrow 0$$

As i → +∞ **X** reduces to a sum of errors or Wold representation of **X**:

$$X_t = \mu + \sum_{i=1}^{+\infty} G_1^i e_{t-i} + e_t \text{ assume } \psi_i = G_1^i \text{ then } X_t = \mu + \sum_{i=1}^{+\infty} \psi_i e_{t-i} + e_t$$

If **X** has a Wold representation then **X** is **stable**. The condition that guarantees it is that **G has eigenvalues smaller than 1 in modulus**. It is a general result, not only valid for a VAR (1) but valid for any VAR(p).

3. Impulse-responses

- Since the VAR is stationary, the estimated reduced-form VAR ha a moving average:

$$X_t = \mu + e_t + \sum_{i=1}^{\infty} \psi_i e_{t-i}$$

- Using the same old relation between forecast errors and structural shocks $e_t = A^{-1} u_t$ we find: $X_t = \mu + A^{-1} u_t + \sum_{i=1}^{\infty} \psi_i A^{-1} u_{t-i}$ or more compactly $X_t = \mu + \sum_{i=0}^{\infty} C_i u_{t-i}$

$c_{11,i}$ and $c_{12,i}$ are the responses of ***y*** and ***r*** to a change u_{yt} and u_{rt} respectively.

- Note that $c_{11,0}$ is the effect at impact, $c_{11,1}$ is the effect of u_{yt} on ***y*** on t+1, and so on in succession: $\frac{\partial y_{t+k}}{\partial u_{yt}} = C_{11,k}$

- Also, the cumulative effect is $\sum_{i=0}^{\infty} c_{11,i}$.

4. Variance Decomposition

- Knowledge of the prediction errors can be extremely valuable in examining the relationships among the variables.
- Assume that we know the coefficients G_0 and G_1 and wish to project the values of X_{t+1} conditional on the observed values of X_t.
- If the equation $X_t = G_0 + G_1 X_{t-1} + e_t$ is advanced one period, we obtain $X_{t+1} = G_0 + G_1 X_t + e_{t+1}$ and the prediction error will be $X_{t+1} - EX_{t+1} = e_{t+1}$ for innovation in t+1.

$$X_{t+3} = G_0 + G_1 \overbrace{(G_0 + G_1 \underbrace{(G_0 + G_1 X_t + e_{t+1})}_{X_{t+1}} + e_{t+2})}^{X_{t+2}} + e_{t+3}$$

$$X_{t+3} - EX_{t+3} = G_1^2 e_{t+1} + G_1 e_{t+2} + e_{t+3}$$

4. Variance Decomposition

$$\mathrm{EX}_{t+n} = (\mathrm{I} + \mathrm{G}_1 + \mathrm{G}_1^2 + \ldots + \mathrm{G}_1^{n-1})\mathrm{G}_0 + \mathrm{G}_1^n \mathrm{X}_{t+n}$$

- This may also be expressed in terms of structural errors:

$$\mathrm{X}_{t+n} - \mathrm{EX}_{t+n} = e_{t+n} + \mathrm{G}_1 e_{t+n-1} + \mathrm{G}_1^2 e_{t+n-2} + \cdots + \mathrm{G}_1^{n-1} e_{t+1}$$

- The prediction error for only GDP gap ***n*** steps forward will be:

$$\mathrm{X}_{t+n} = \mu + \sum_{i=0}^{n} \mathrm{G}_1^i \mathrm{A}^{-1} \mathrm{u}_{t+n-i} = \quad \mathrm{X}_{t+n} = \mu + \sum_{i=0}^{n} \psi_i \mathrm{A}^{-1} \mathrm{u}_{t+n-i} = \quad \mathrm{X}_{t+n} = \mu + \sum_{i=0}^{n} C_i \mathrm{u}_{t+n-i}$$

- The variance of this prediction error is:

$$\mathrm{y}_{t+n} - \mathrm{Ey}_{t+n} = c_{11,0} u_{yt+n} + c_{11,1} u_{yt+n-1} + \cdots + c_{11,n-1} u_{yt+1} +$$

$$c_{12,0} u_{rt+n} + c_{12,1} u_{rt+n-1} + \cdots + c_{12,n-1} u_{rt+n}$$

$$\sigma_{y,n}^2 = \sigma_y^2 [c_{11,0}^2 + c_{11,1}^2 + \cdots + c_{11,n-1}^2] +$$

$$\sigma_z^2 [c_{12,0}^2 + c_{12,1}^2 + \cdots + c_{12,n-1}^2]$$

4. Variance Decomposition

- As the values of $c_{11,0}^2$ are necessarily positive, the variance of the errors increases with the projection horizon.
- It is possible to decompose the prediction error n periods forward by the contribution each of the two shocks in our example.
- The proportions of $\sigma_{y,n}^2$ attributable to each structural shock are:

$$\frac{\sigma_y^2[c_{11,0}^2 + c_{11,1}^2 + \cdots + c_{11,n-1}^2]}{\sigma_{y,n}^2} \qquad \frac{\sigma_r^2[c_{12,0}^2 + c_{12,1}^2 + \cdots + c_{12,n-1}^2]}{\sigma_{y,n}^2}$$

- This is the proportion of the changes in one variable attributable to shocks to it and to shocks in another variable.
- If u_{rt-i} fails to explain any changes in ***y*** the latter is exogenous.
- The restriction imposed above requires that the entire variance in the prediction error for ***y*** one period forward be attributable to u_{yt-i}