Supplementary Handout
Stability and Stationary Conditions

Important characteristicc of a *VAR(p)*-process are its stability and stationarity. A *VAR(p)* is weakly or covariance stationary if it has finite means and variances and the covariances depend only on the time elapsed, and not on the reference period. A *VAR(p)* process is considered *stable* if that will not diverge to infinity ("blow up"). Stability implies stationarity – thus it is sufficient to test for stability to ensure that a *VAR(p)* process is both stable and stationary.

Let's assume the following reduced form *VAR(p)*:

$$\mathbf{y_t} = \mathbf{G_0} + \mathbf{G_1 y_{t-1}} + \mathbf{G_2 y_{t-2}} + \ldots + \mathbf{G_p y_{t-p}} + \mathbf{e_t}$$

Using the lag operator, this can be further re-written as:

$$(\mathbf{I_n} - \mathbf{G_1 L} - \mathbf{G_2 L^2} - \ldots - \mathbf{G_p L^p})\mathbf{y_t} = \mathbf{G_0} + \mathbf{e_t}$$

Let's denote the lag polynomial as:

$$\mathbf{G(L)} = \mathbf{I_n} - \mathbf{G_1 L} - \mathbf{G_2 L^2} - \ldots - \mathbf{G_p L^p}$$

One can assess stability by evaluating the roots of the ***characteristic polynomial***, which is obtained by replacing the lag operator $\mathbf{L}$ by a variable (call it $\mathbf{z}$) and set the resulting polynomial equal to 0, i.e.:

$$\mathbf{I_n} - \mathbf{G_1 z} - \mathbf{G_2 z^2} - \ldots - \mathbf{G_p z^p} = \mathbf{0}$$

The characteristic roots are the values of $\mathbf{z}$ that solve the ***characteristic equation:***

$$\det(\mathbf{I_n} - \mathbf{G_1 z} - \mathbf{G_2 z^2} - \ldots - \mathbf{G_p z^p}) = \mathbf{0}$$

Note that there are *kp* roots, where *k* is the number of variables in the VAR and *p* is the number of lags.

A *VAR(p)* process is stable (thus stationary) if all the roots of the characteristic polynomial are (in modulus) *outside* the unit imaginary circle (i.e. are greater than 1 in absolute value if real, or in modulus, if complex).

If a root equals one or minus one, it is called a unit root. If there is at least one unit root, or if any root lies between plus and minus one, then either some or all variables in the *VAR(p)* process are integrated of order 1. It might be the case that cointegration between the variables exists (this instance can be better analyzed in the context of a VECM).

Note that EViews calculates the *inverse* roots of the characteristic polynomial ($\lambda = \frac{1}{\mathbf{z}}$), i.e. based on the equation:

$$\mathbf{I_n}\lambda^{\mathbf{p}} - \mathbf{G_1}\lambda^{\mathbf{p-1}} - \mathbf{G_2}\lambda^{\mathbf{p-2}} - \ldots - \mathbf{G_p} = \mathbf{0}$$

which should then lie *within* the unit imaginary circle.

An equivalent definition is that a *VAR(p)* process is stationarity when the eigenvalues of the companion matrix:

$$\mathbf{F} = \begin{bmatrix} \mathbf{G_1} & \mathbf{G_2} & \cdots & \mathbf{G_k} \\ \mathbf{I_k} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{I_k} & \mathbf{0} \end{bmatrix}$$

are smaller than 1 in modulus.

The matrix $\mathbf{F}$ comes from re-writing the *VAR(p)* model as a *VAR(1)* , i.e.:

$$\boldsymbol{\xi}_{\mathbf{t}} = \mathbf{F}\,\boldsymbol{\xi}_{\mathbf{t}} + \boldsymbol{\varepsilon}_{\mathbf{t}}$$

where:

$$\boldsymbol{\xi}_{\mathbf{t}} = \begin{bmatrix} \mathbf{y_t} \\ \mathbf{y_{t-1}} \\ \cdot \\ \mathbf{y_{t-p}} \end{bmatrix} \text{ and } \boldsymbol{\varepsilon}_{\mathbf{t}} = \begin{bmatrix} \mathbf{e_t} \\ \mathbf{0} \\ \cdot \\ \mathbf{0} \end{bmatrix}$$

Eigenvalues, Eigenvectors and the Characteristic Polynomial

For a general *kxk* square matrix **A**, a scalar λ is an ***eigenvalue*** of **A** if and only if there is an ***eigenvector*** $\nu \neq 0$ such that:

$$\mathbf{A}\nu = \lambda\nu$$

Or, written more explicitly:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_k \end{bmatrix} = \lambda \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_k \end{bmatrix}$$

This is equivalent to:

$$(\lambda\mathbf{I} - \mathbf{A})\nu = 0$$

where $\mathbf{I}$ is *kxk* the identity matrix.

Since v is non-zero, this holds only when the matrix $\lambda\,\mathbf{I} - \mathbf{A}$ is singular (non-invertible), which in turn means that its determinant has to be 0, i.e.:

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

The ***characteristic polynomial*** of a square matrix **A** is the polynomial defined by:

$$p_A(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A})$$

Thus, the roots of the characteristic polynomial are the eigenvalues of **A**.