Out-of-Time-Ordered Correlator's growth rate and Lyapunov Exponent to inspect classical and quantum chaos

Quantum Information and Computing Course

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29/07/2021

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Introduction

The project

The aim of this project is to study how chaos arises in quantum systems.

"Lyapunov exponent and out-of-time-ordered correlator's growth rate in a chaotic system" by E. B. Rozenbaum, S. Ganeshan and V. Galitski.
Other reference articles can be found in the last slide.

All the material can be found at `https://github.com/CleliaCorridori/QuantumInformation_project`.

Quantum chaos

- Quantum chaos studies the properties of quantum systems having a chaotic classical analogous.
- Impossible to use classical arguments such as trajectories in the phase space.
- **Correspondence principle**
  $\longrightarrow$ Classical system as limit of quantum ones.

Ehrenfest theorem

A **quantum wave-packet** evolution initially **follows** the **classical trajectory** having initial momentum and position in the center of the packet.

- Until t_E the wave-packet center follows a classical trajectory.
- After t_E the wave-packet loses coherence.

Quantum chaos - OTOC

Out-of-Time-Ordered four-point Correlator (OTOC) $\mathrm{C(t)}$:

$$C(t) = -\langle [P(t), P(0)]^2 \rangle$$

- The OTOC grows exponentially until t_E.
- OTOC's exponential growth rate (CGR) is a promising parameter to characterize quantum chaos.

Lyapunov Exponent (LE) λ:

$\rightarrow$ rate of exponential separation of initially close trajectories in the phase space

$$\lambda = \left\langle\!\!\left\langle \lambda(x,p) \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \lim_{t\to\infty} \lim_{d(0)\to 0} \frac{1}{t} \ln \frac{d(t)}{d(0)} \right\rangle\!\!\right\rangle$$

where $d(t)$ is the distance between two trajectories

$$d(t) \approx d(0) e^{\lambda(x,p)t}$$

Key difference

OTOC CGR

States are wave-packets:

- OTOC is sensitive to neighborhood properties
- CGR computed using the mean OTOC
- sensitive to **local** chaos

LE

Computed on single trajectories:

- Not sensitive to neighborhood
- λ computed as mean of trajectory separation rates
- sensitive to **global** chaos

Hamiltonian for a particle of unitary mass:

$$H(X,P,t) = \frac{P^2}{2} + K\cos(X)\sum_{n=-\infty}^{+\infty}\delta(t-nT)$$

- K = kicking strength
- T = kicking period
- X = space coordinate (operator) for CKR (QKR)
- P = momentum coordinate (operator) for CKR (QKR)

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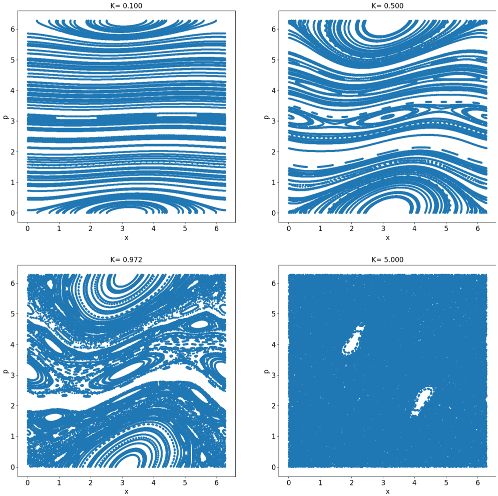
Transition from regular to chaotic dynamics:

- $K \sim 0$: both regular and chaotic trajectories in the phase space
- $K \sim K_c \sim 0.972$: mostly chaotic trajectories with some regular islands in the phase space
- $K \gg K_c$: only chaotic trajectories in the phase space



We expect that:

- If $K < K_c$, $\lambda \ll 1$
- If $K > K_c$, λ will grow

Using the Chirikov's formula the LE can be studied analytically. For $K \gg K_c$ the LE grows as:

$$\lambda \approx \ln\left(\frac{K}{2}\right)$$

Quantum Kicked Rotor

Key parameters:

- Kicking strength K
- Dimensionless effective Planck's constant $\hbar_{\text{eff}}$

QKR action $I = PT$:

$$[X, I] = T\,[X, P] = i\hbar T = i\hbar_{\text{eff}}$$
$$\Downarrow$$
$$\hbar_{\text{eff}} = \hbar T$$

Semi-classical limit: $\hbar_{\text{eff}} \to 0$

- OTOC is sensitive to chaotic trajectories nearby the center of the wave-packet
- OTOC's growth rate greater than the LE
- Particularly for low K where classically there are both regular and chaotic trajectories

OTOC for CKR $\mathbf{C^{cl}(t)}$:

$$C(t) = \hbar_{\text{eff}}^2 \left\langle \frac{\delta p(t)^2}{\delta x(0)^2} \right\rangle \approx \hbar_{\text{eff}}^2 \left\langle\!\!\left\langle \frac{\Delta p(t)^2}{\Delta x(0)^2} \right\rangle\!\!\right\rangle = C^{cl}(t).$$

OTOC's growth rate for CKR $\tilde{\lambda}$:

$$C^{cl}(t) = C^{cl}(1)e^{2\tilde{\lambda}(t-1)}$$

$$\Downarrow$$

$$\tilde{\lambda} = \lim_{t\to\infty} \lim_{\Delta x(0)\to 0} \frac{1}{2t} \ln \frac{C^{cl}(t+1)}{C^{cl}(1)}.$$

Code Development

Code Development

Here we will consider the following computations:

1. OTOC for the QKR for different values of K and for different values of $\hbar_{\text{eff}}$.
2. The growth rate of $C(t)$, CGR, for the QKR;
3. Trajectories with Chirikov standard map;
4. The Lyapunov exponent for the CKR;
5. The growth rate of $C^{cl}(t)$, CGR, for the CKR.

OTOC: the parameters

The used parameters are:

- N, the number of points for the discretization of the spatial coordinate;
- T, period selected considering the relation $2\hbar_{\text{eff}}N \in [2^7; 2^{16}]$ with $\hbar_{\text{eff}} = \hbar T = T$, with $\hbar = 1$;
- N_{kicks}, the number of kicks;
- K, the kicking strength;
- Number of trials.

The OTOC expression is:

$$\boxed{C(t) = -\langle [P(t), P(0)]^2 \rangle}$$

$\rightarrow$ We use **normalized Gaussian wave-packets** in the momentum representation as initial state $|\Psi(0)\rangle$:

$$|\Psi(0)\rangle = \sum_{p=-\infty}^{\infty} a_p^{(0)} |p\rangle \quad \text{with} \quad a_p^{(0)} \sim \exp\left[\frac{-(p-p_0)^2}{2\sigma^2}\right],$$

where $\sigma = 4$ and p_0 is randomly selected form a uniform distribution in the range $[-\pi; \pi]$.

OTOC: commutator's explicit expression

The OTOC expression is:

$$\boxed{C(t) = -\langle [P(t), P(0)]^2 \rangle}$$

$\rightarrow$ Making explicit the **commutator** we get:

$$\begin{aligned} -\left[P(t), P(0)\right]^2 = &+ P(0)P(t)P(t)P(0) + P(t)P(0)P(0)P(t) \\ &- P(t)P(0)P(t)P(0) - P(0)P(t)P(0)P(t), \end{aligned}$$

with $P(t) = U^\dagger(t)P(0)U(t)$, $U(t) = e^{-\frac{i}{\hbar}\hat{H}(t)}$ time evolution operator.

OTOC: the Floquet operator

The **Floquet operator** for the QKR is:

$$F(T) = U_V U_0(T) = e^{-ik\cos X} e^{-i\frac{P^2}{2}T}$$

with $k = K/\hbar_{eff}$.

$\rightarrow$ $U(t)$ for the QKR after n periods is:

$$U(t = nT) = F(T)^n$$

$\rightarrow$ The time evolution for $P(t)$ is:

$$P(t) = P(nT) = \left[U_0^\dagger U_V^\dagger\right]^n P(0) \left[U_V U_0\right]^n$$

with $n \in [0; N_{kicks}]$.

$\rightarrow$ **The time evolution of the OTOC is discretized**, $C(nT)$

CGR for the QKR (I)

- For $t < t_E$ we can approximate the quantum OTOC:

$$C(t) \approx C^{cl}(t) = \hbar_{\text{eff}}^2 \left\langle\!\!\left\langle \frac{\Delta p(t)^2}{\Delta x(0)^2} \right\rangle\!\!\right\rangle = C^{cl}(1) e^{2\tilde{\lambda}(t-1)}$$

  Recalling that $C(0) = 0$ we considered also $t > 0$.

- Thanks to this equation we compute the **CGR,** $\tilde{\lambda}$, as:

$$\tilde{\lambda}(t) = \frac{1}{2} \ln \frac{C(t)}{C(t-1)}$$

  Then we fit $\tilde{\lambda}(t)$ with $y = costant$ in $[t_{min}, t_E)$.

CGR for the QKR (II)

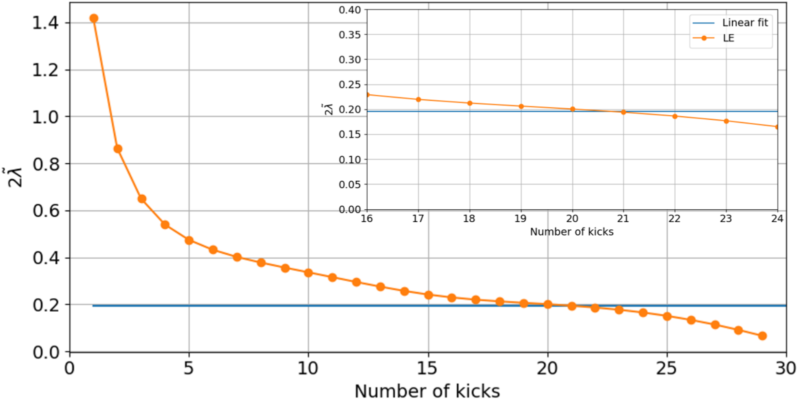


Figure: $2\tilde{\lambda}(t)$ as a function of the number of kicks, t, for $K = 0.045$ and fit $y = constant$ in the region $t \in [16, 24]$.

Chirikov standard map (I)

GOAL: Time evolution of the distance at time t, $d(t)$, between two initially close trajectories, $d(0) \ll 1$, in the phase space.

The trajectories are described by the **Chirikov standard map** [4]:

$$\begin{cases} p_{n+1} \overset{\text{mod } 2\pi}{=} p_n + K \sin x_n \\ x_{n+1} = x_n + p_{n+1} \end{cases}$$

$$\begin{cases} p'_{n+1} \overset{\text{mod } 2\pi}{=} p'_n + K \sin x'_n \\ x'_{n+1} = x'_n + p'_{n+1} \end{cases}$$

with $n \in [0, N_{kicks}]$.

Tangent map (II)

Define ξ_n and η_n as relative coordinates, s.t. $\boldsymbol{d}(n) = \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix}$.

1. The standard map for ξ_n and η_n is:

$$\begin{cases} \eta_{n+1} = \eta_n + K(\sin x'_n - \sin x_n) \\ \xi_{n+1} = \xi_n + \eta_{n+1} \end{cases}$$

2. By using a trigonometric identity:

$$\sin x'_n - \sin x_n = \sin x_n(\cos \xi_n - 1) + \sin \xi_n \cos x_n = \xi_n \cos x_n.$$

   Obtained considering the linear order of ξ_n, with $\xi_n \ll 1$.

3. The **tangent map** is:

$$\begin{cases} \eta_{n+1} = \eta_n + (K \cos x_n)\xi_n \\ \xi_{n+1} = \xi_n + \eta_{n+1} \end{cases}$$

Benettin algorithm (III)

Due to the **overflow** of $d(t)$ caused by the exponential stretching, consider the Benettin algorithm [1]:

1. Set $\boldsymbol{d}(0)$ with unitary norm and random initial position (x_0, p_0) in phase space;
2. Propagate the *standard map* and the *tangent map* for N_{kicks};
3. Repeat for M times: $\forall i = 1, ..., M$ save $d(iN_{kicks}) = \|\boldsymbol{d}(iN_{kicks})\|$, the final normalized $\boldsymbol{d}(iN_{kicks})$ and the final coordinates, $x(iN_{kicks}), p(iN_{kicks})$;

This way we obtained a set of distances $\{d(iN_{kicks})\}$.

Lyapunov Exponent for CKR

To compute the **Lyapunov Exponent**:

- Consider the set of initial positions $\{(x_{0,i}, p_{0,i})\}_{i=1}^{Ntrials}$ uniformly distributed in the interval $[0, 2\pi]$.
- Compute $\lambda(x_0, p_0)$ for each initial condition:

$$\lambda(x_0, p_0) = \frac{1}{MN_{kicks}} \sum_{t=1}^{N_{kicks}} \ln d(t)$$

  N_{kicks} is selected as large as possible *avoiding the overflow*.

- λ is obtained averaging $\lambda(x_0, p_0)$ over the phase space.

CGR for the CKR

The **growth rate of** $C^{cl}(t)$ for the CKR is give by:

$$\widetilde{\lambda} = \frac{1}{2(t_c - 1)} \sum_{t=2}^{t_c} \ln \frac{C^{cl}(t)}{C^{cl}(t-1)}$$

It is not possible to re-scale $\boldsymbol{d}(t) \Rightarrow t_c$ selected to *prevent the overflow*.

$\rightarrow$ Substituting the expression for $C^{cl}(t)$ in the equation above and considering $\Delta x(0) \approx costant$, we get the **CGR** for the CKR:

$$\boxed{\widetilde{\lambda} = \frac{1}{2(t_c - 1)} \sum_{t=2}^{t_c} \ln \frac{\left\langle\!\!\left\langle [\Delta p(t)]^2 \right\rangle\!\!\right\rangle}{\left\langle\!\!\left\langle [\Delta p(t-1)]^2 \right\rangle\!\!\right\rangle}}$$

where $\Delta p(t) = \eta(t)$ is computed using the tangent map.

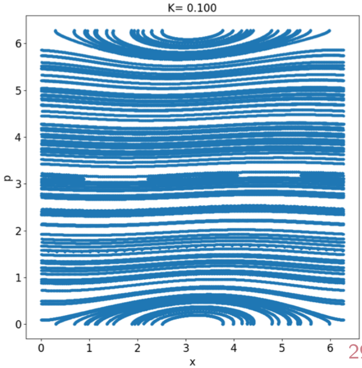
Results

The transition to chaos is observed when we introduce a perturbation in the system as:

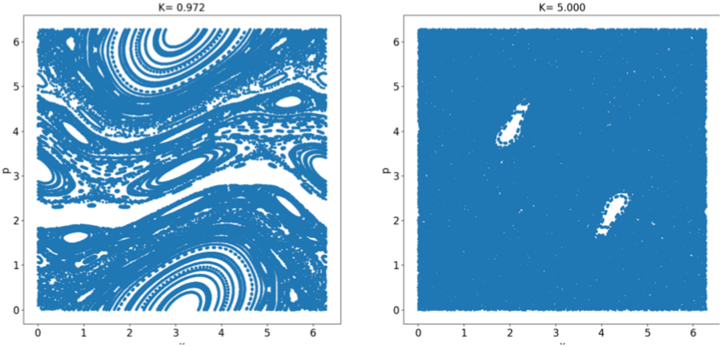
$$H'(X,t) = K\cos(X)\sum_{n=-\infty}^{+\infty}\delta(t-nT), \tag{1}$$

For small values of the kicking strength K we observe stable trajectories.

- Chirikov standard map
- Phase space $[0, 2\pi]$
- number of trajectories=100
- $N_{kicks} = 1000$
- K=0.1



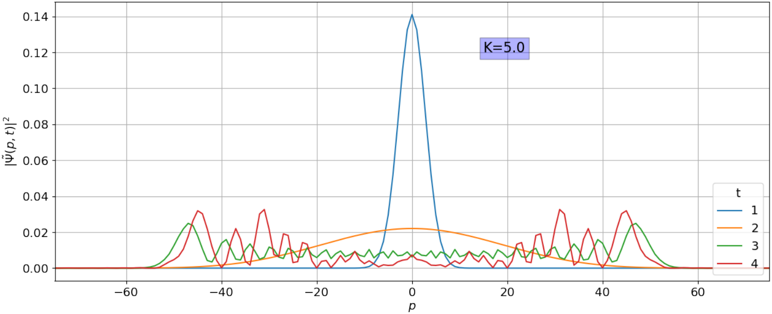
Classical Kicked Rotor (II)

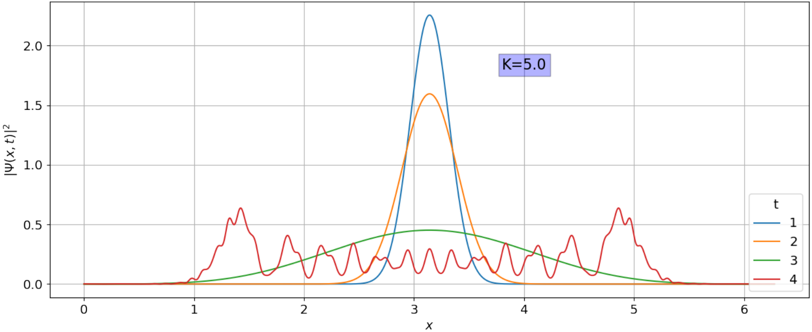


- $K = K_c = 0.972 \rightarrow$ Destruction of regular trajectories
- $K > K_c \rightarrow$ Chaos. The particle visits all points in the phase space

What changes in QKR?
Observing the evolution in time of the probability distribution associated to the Gaussian wave-function.

- $N = 2^{10} \quad T = 2^{-4} \rightarrow 2TN \in [2^7, 2^{16}]$
- Kicking strength $K = 5$

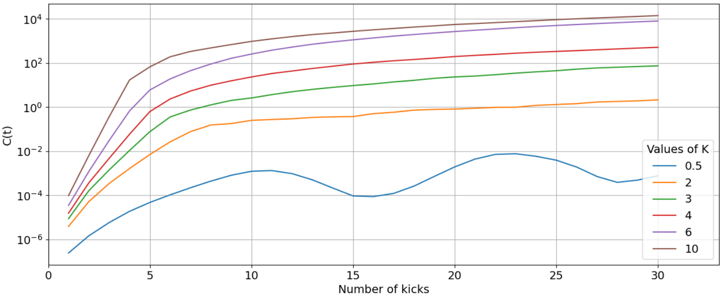




Complete spreading of the wave packet after only $N_{kicks} = 4$

OTOC: Different Values of K (I)

- $K = [0.5, 2, 3, 4, 6, 10]$
- $N = 2^{14}$, $\hbar_{\text{eff}} = T = 2^{-7} \to 2TN \in [2^7, 2^{16}]$
- $N_{kicks} = 100$, $N_{trials} = 50$



C(t) in *y* log-scale shows a linear growth in time until t_E.

Presence of chaotic islands $\rightarrow$ Oscillations for $K < 1$

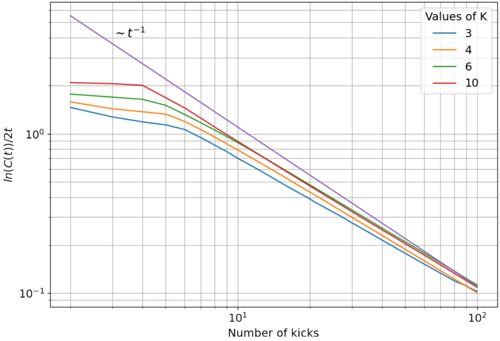
OTOC: Different Values of K (II)

Computing $\ln[C(t)]/2t$, normalizing $C(t)$ to $C(1)$ and plotting in log-log scale, we can observe:

$\rightarrow$ Constant behavior before t_E.

$\rightarrow$ Power-growth behavior with decreasing power

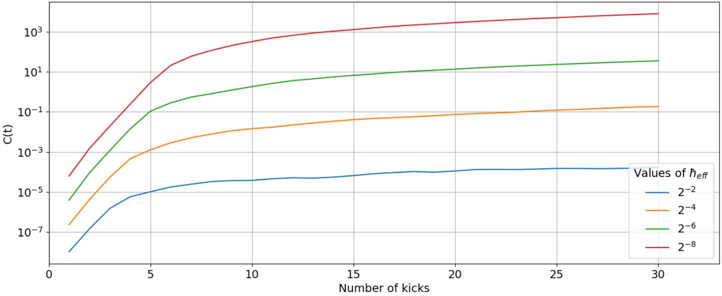
$\rightarrow$ t_E increases decreasing K



- $t_E \sim \frac{|\ln(\hbar_{\text{eff}})|}{\ln(K/2)}$:
  $t_E(K=10) \sim 4$,
  $t_E(K=6) \sim 5$

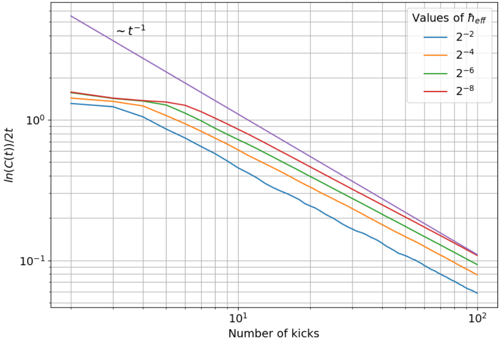
OTOC: Different Values of $\hbar_{\text{eff}}$ (I)

- $\hbar_{\text{eff}} = [2^{-2}, 2^{-4}, 2^{-6}, 2^{-8}]$
- $K = 4$
- $N_{kicks} = 100,\ N_{trials} = 50$



Similar exponential growth rate for each value of $\hbar_{\text{eff}}$

OTOC: Different Values of $\hbar_{\text{eff}}$ (II)



$$t_E(\hbar_{\text{eff}} = 2^{-2}) \sim 3$$
$$t_E(\hbar_{\text{eff}} = 2^{-4}) \sim 4$$
$$t_E(\hbar_{\text{eff}} = 2^{-6}) \sim 5$$
$$t_E(\hbar_{\text{eff}} = 2^{-8}) \sim 6$$

The exponential growth rate is almost the same of each $\hbar_{\text{eff}}$

$\rightarrow$ **The CGR is independent on $\hbar_{\text{eff}}$**

Comparison between CKR and QKR (I)

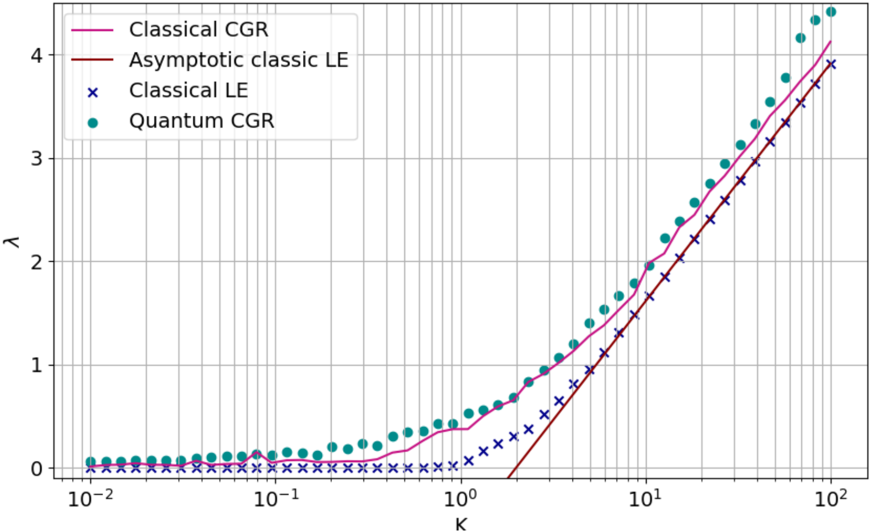
We have computed the OTOC's Growth Rate for CKR and QKR and the Lyapunov Exponent for CKR with:

CGR for QKR			
$\hbar_{\text{eff}} = 2^{-7}$	$N = 2^{14}$	$N_{kicks} = 30$	$N_{trials} = 10$

LE for CKR	
$N_{kicks} = 50$	$N_{trials} = 100$

CGR for CKR	
$N_{kicks} = t_c = 50$	$N_{trials} = 1000$

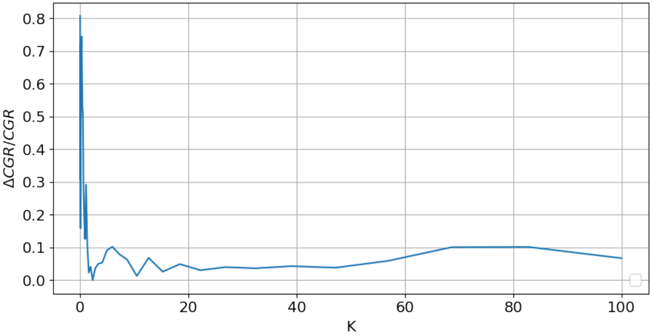
Comparison between CKR and QKR (II)



Comparison between CKR and QKR (III)

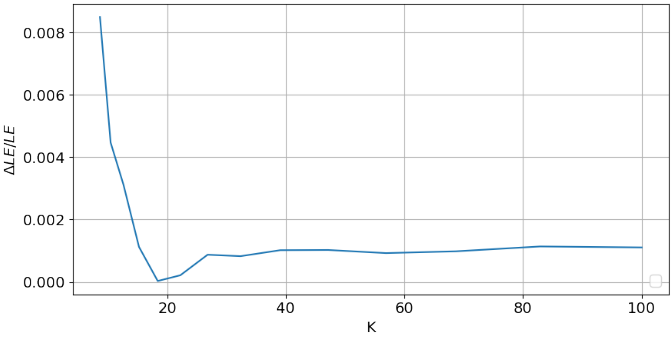
- Fluctuations due to the low number of trials
- Low K, LE quickly decreases while OTOC's Growth Rate decreases slower for the presence of chaotic islands.

The behavior of the classic and quantum CGR is similar



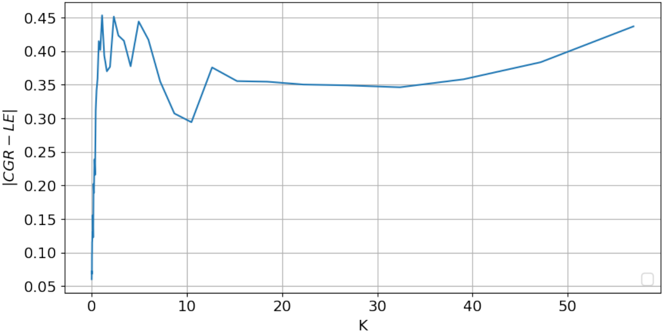
Comparison between CKR and QKR (IV)

We observed that the Lyapunov Exponent agrees with its asymptotic value computed as $\ln(K/2)$ for high values of K.



Comparison between CKR and QKR (V)

Computing the difference between LE and CGR we observe that it stabilize for high values of K, $K \gtrsim 15$.



For $K > 60$ it is not possible compare the values due to the fit procedure on QKR.

Conclusions

$\rightarrow$ CGR and the LE are in general distinct quantities: the order of the logarithm and average operations are exchanged in the CGR and in the LE;

$\rightarrow$ CGR is more sensitive than the LE to the presence of chaotic islands in the phase space;

$\rightarrow$ **LE** could be used to detect *global chaos*;

$\rightarrow$ **CGR** could be used to detect *local chaos*.

Thanks for your attention

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[1] Giancarlo Benettin et al. “Lyapunov Characteristic Exponents for smooth dynamical systems and for Hamiltonian systems; A method for computing all of them. Part 2: Numerical application”. In: *Meccanica* 15 (Mar. 1980), pp. 21–30. DOI: 10.1007/BF02128237.

[2] Gennady P Berman and George M Zaslavsky. “Condition of stochasticity in quantum nonlinear systems”. In: *Physica A: Statistical Mechanics and its Applications* 91.3-4 (1978), pp. 450–460.

[3] Giulio Casati et al. “Stochastic behavior of a quantum pendulum under a periodic perturbation”. In: *Stochastic behavior in classical and quantum Hamiltonian systems*. Springer, 1979, pp. 334–352.

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[5] Clelia Corridori Cristina Cicali and Anna Steffinlongo. *Out-of-Time-Ordered Correlator's growth rate and Lyapunov Exponent to inspect classical and quantum chaos*. 2021. URL: %5Curl%7Bhttps://github.com/CleliaCorridori/QuantumInformation_project%7D.

[6] P.M. Koch and K.A.H. van Leeuwen. “The importance of resonances in microwave “ionization” of excited hydrogen atoms”. In: *Physics Reports* 255.5 (1995), pp. 289–403. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(94)00093-I. URL: https://www.sciencedirect.com/science/article/pii/037015739400093I.

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Chirikov's formula

The Chirikov's formula can be used to compute analytically the behaviour of the LE:

$$\lambda \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln L(x)\, dx$$

where

$$L(x) = \left| 1 + \frac{K \cos(x)}{2} + \mathsf{sgn}[K \cos(x)] \sqrt{K \cos(x) \left(1 + \frac{K \cos(x)}{4} \right)} \right|$$

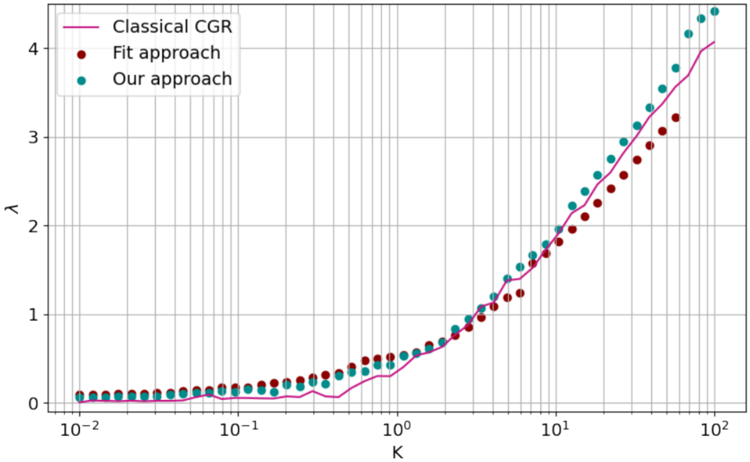

Figure: Fit approach presented in the article compared to the fit approach used in this work.