Lecture Notes in Finance 2 (MiQE/F, MSc course at UNISG)

Paul Söderlind[1]

7 January 2017

[1]University of St. Gallen. *Address:* s/bf-HSG, Rosenbergstrasse 52, CH-9000 St. Gallen, Switzerland. *E-mail:* Paul.Soderlind@unisg.ch. Document name: Fin2MiQEFAll.TeX.

Contents

Warning: a few of the tables and figures are reused in later chapters. This can mess up the references, so that the text refers to a table/figure in another chapter. No worries: it is really the same table/figure. I promise to fix this some day...

Chapter 15

Forwards and Futures

Main References: Elton, Gruber, Brown, and Goetzmann (2014) 24 and Hull (2009) 5 and 8–9
Additional references: McDonald (2014) 6–8

15.1 Derivatives

Derivatives are assets whose payoff depend on some underlying asset (for instance, the stock of a company). The most common derivatives are futures contracts (or similarly, forward contracts) and options. Sometimes, options depend not directly on the underlying, but on the price of a futures contract on the underlying. See Figure 15.1.

Derivatives are in zero net supply, so a contract must be issued (a short position) by someone for an investor to be able to buy it (long position). For that reason, gains and losses on derivatives markets sum to zero.

15.2 Forward and Futures

15.2.1 Present Value

Forward prices play an important role in simplifying option analysis, so we first discuss the forward-spot parity.

The present value of Z units paid m periods (years) into the future is

$$\mathrm{PV}_m(Z) = [1 + Y(m)]^{-m} Z, \text{ or} \tag{15.1}$$

$$= e^{-my(m)} Z, \tag{15.2}$$

Underlying and Derivatives

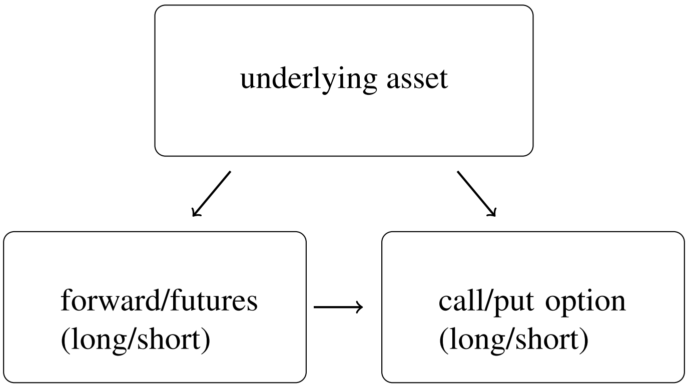


Figure 15.1: Derivatives on an underlying asset

where $Y(m)$ is effective spot interest rate for a loan until m periods ahead, and $y(m)$ is the continuously compounded interest rate ($y(m) = \ln[1 + Y(m)]$). As usual, the interest rates are expressed as the rate per year, so m should be also expressed in terms of years. On notation: trade time subscripts are mostly suppressed in these notes, except when strictly needed for clarity. It should be noticed, however, that interest rates change over time.

Example 15.1 *(Present value) With* $y(m) = 0.05$ *and* $m = 3/4$ *we have the present value* $e^{-0.05\times 3/4}Z \approx 0.963Z$.

15.2.2 Definition of a Forward Contract

A forward contract specifies (among other things) which asset should be delivered at expiration and how much that should be paid for it: the forward price, F. See Figure 15.2 for an illustration. The forward (and also a futures, see below) are zero sum games: the profit of the buyer is the loss of the seller (or vice versa).

The profit (payoff) of a forward contract at expiration is very straightforward. Let S_{t+m} be the price (on the spot market) of the underlying asset at expiration (in $t + m$). Then, for the *buyer* of a forward contract

$$\text{payoff of a forward contract} = S_{t+m} - F. \tag{15.3}$$

The owner of the forward contract pays F to get the asset, sells it immediately on spot

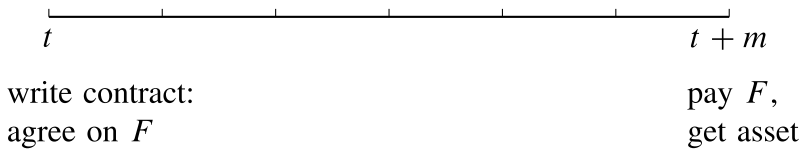


Figure 15.2: Timing convention of forward contract

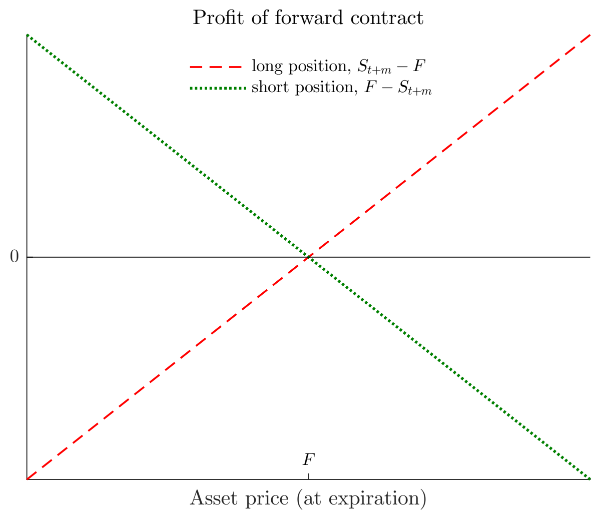


Figure 15.3: Profit (payoff) of forward contract at expiration

market for S_{t+m}. See Figure 15.3. Similarly, the payoff for the *seller* of a forward contract is $F - S_{t+m}$ (she buys the asset on spot market for S_{t+m}, gets F for asset according to the contract). This sums to *zero*.

15.2.3 Forward-Spot Parity

Proposition 15.2 *(Forward-spot parity, no dividends) The present value of the forward price, $F(m)$, contracted in t (but to be paid in $t+m$) on an asset without dividends equals the spot price:*

$$e^{-my(m)}F(m) = S, \text{ so} \tag{15.4}$$

$$F(m) = e^{my(m)}S, \tag{15.5}$$

where S is the spot price today (when the forward contract is written).

The intuition is that the forward contract is like buying the underlying asset on credit—

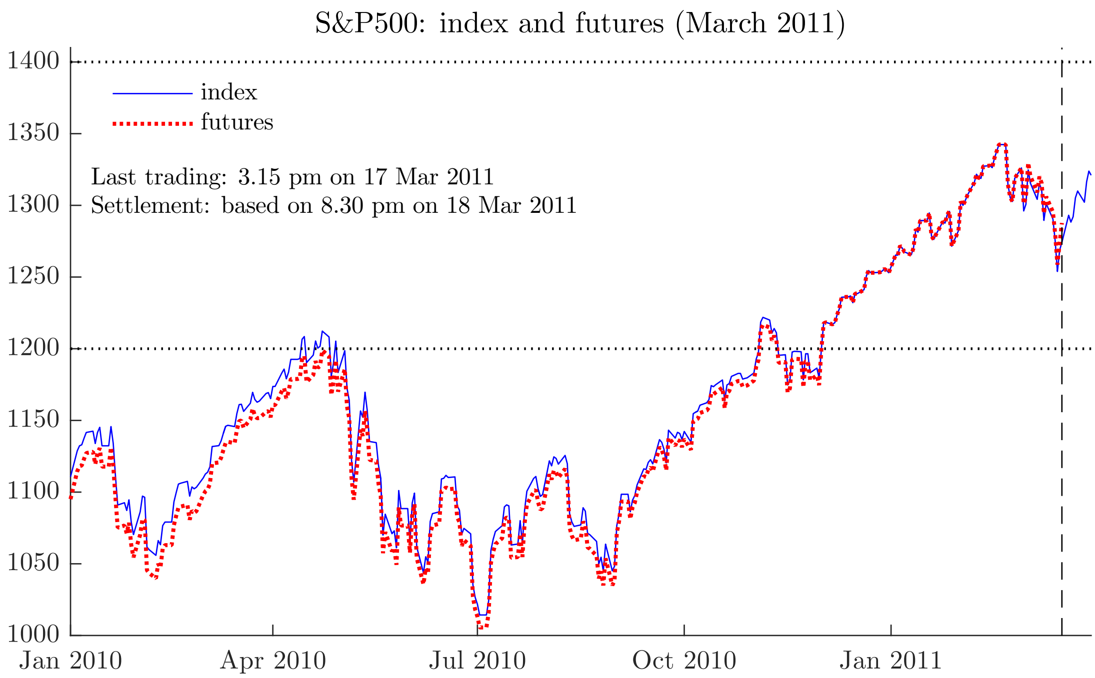


Figure 15.4: S&P 500 index level and futures

$e^{-my(m)}F(m)$ can be thought of as a prepaid forward contract. If you prefer effective interest rates, then the expression reads $F(m) = [1 + Y(m)]^m S$.

Proof. (of Proposition 15.2) Portfolio A: enter a forward contract, with a present value of $e^{-my}F$. Portfolio B: buy one unit of the asset at the price S. Both portfolios give one asset at expiration, so they must have the same costs today. ■

The essence of the forward-spot parity is that *the value of a new forward contract is zero*, that is, if you try to sell off the forward contract a split second after you entered it, you will get nothing for it. A forward contract entails both a right (to get the underlying asset at expiration) and an obligation (to pay the forward price at expiration), so it is perhaps not obvious what the total value is. However, the no-arbitrage argument in the proof gives a simple answer: if you are long a forward contract, then you can cancel all risk by going short the underlying asset today (and put the money on a bank account). At expiration, you have the safe profit of $e^{my_t(m)}S_t$ (at your bank account) minus the forward price F_t. Since you have not invested anything and you have no risk, your profit must be zero (or else there is an arbitrage opportunity)—which requires that (15.5) holds.

Proposition 15.3 *(Forward-spot parity, discrete dividends) Suppose the underlying asset pays the dividend d_i at m_i $(i = 1, ..., n)$ periods into the future (but before the expiration* *date of the forward contract). The dividends must be known already in t. The forward price then satisfies*

$$e^{-my(m)}F(m) = S - \sum\nolimits_{i=1}^{n} e^{-m_i y(m_i)} d_i, \text{ so} \tag{15.6}$$

$$F(m) = e^{my(m)}S - e^{my(m)} \sum\nolimits_{i=1}^{n} e^{-m_i y(m_i)} d_i. \tag{15.7}$$

The last term of (15.6) is the sum of the present values of the dividend payments. The intuition is that the forward contract does not give the right to these dividends so its value is the underlying asset value stripped of the present value of the dividends. Dividends decrease the forward price.

Proof. (of Proposition 15.3) Portfolio A: enter a forward contract, with a present value of $e^{-my}F$. Portfolio B: buy one unit of the asset at the price S and sell the rights to the known dividends at the present value of the dividends. Both portfolios give one asset at expiration, so they must have the same costs today. ■

Proposition 15.4 *(Forward-spot parity, continuous dividends) When the dividend is paid continuously as the rate δ (of the price of the underlying asset), then*

$$e^{-my(m)}F(m) = Se^{-\delta m}, \text{ so} \tag{15.8}$$

$$F(m) = Se^{m[y(m)-\delta]} \tag{15.9}$$

Proof. (of Proposition 15.4) Portfolio A: enter a forward contract, with a present value of $e^{-my}F$. Portfolio B: buy $e^{-\delta m}$ units of the asset at the price $e^{-\delta m}S$, and then collect dividends and reinvest them in the asset. Both portfolios give one asset at expiration, so they must have the same costs today. ■

Example 15.5 *(Forward-spot parity) With $y(m) = 0.05$, $m = 0.75$ and $S = 100$ we have the forward price $F = e^{0.75\times0.05}100 \approx 103.82$. Instead with a continuous dividend rate of $\delta = 0.01$, we get $F = e^{0.75\times(0.05-0.01)}100 \approx 103.04$.*

Remark 15.6 *Figure 15.4 provides an example of how the futures price (on S&P 500), the intrinsic value of the option and the option price developed over a year. Notice how the futures prices converges to the index level at expiration of the futures. Before it can deviate because of delayed payment (+) and no part in dividend payments (−).*

15.2.4 The Value of an Old Forward Contract

Consider a forward contract that expires in $t+m$, although the contract was written at some earlier point in time ($\tau < t$) and specified a forward price of F_τ (time subscripts are needed for the analysis here). The value of this contract in t is

$$\text{Value of old forward contract} = e^{-ym}(F_t - F_\tau), \tag{15.10}$$

where F_t is today's forward price on the same underlying asset (and same expiration date). This is what someone would pay in order to buy that old forward contract. The intuition is that an owner of an old (τ) forward contract can short sell a new forward contract (t) and thereby cancel all risk—and stand to win $F_t - F_\tau$ at expiration. The present value of this is (15.10). Clearly, for a new contract ($t = \tau$), the value is zero—as discussed before.

Proof. (15.10) An investor sells (issues) a forward contract in t. At expiration, this will give $F_t - S_{t+m}$, where S_{t+m} is the price of the underlying asset at expiration. If she buys an old forward contract for the price V_t, the payoff of that is $S_{t+m} - F_\tau$ at expiration. Hence, the total portfolio has the payoff $F_t - F_\tau$, which is riskfree. There is an arbitrage opportunity unless the price of the old contract is $V_t = e^{-ym}(F_t - F_\tau)$. ■

Remark 15.7 *("Return" on a forward contract*) In a traditional forward contract there is no up-front payment, so it is tricky to define a return. However, we can define a kind of return in the following way. Suppose that when you enter the forward contract in period* τ*, you put* $e^{-y(m+t-\tau)}F_t$ *on a bank account to be sure to cover the forward price at expiration. Consider this as your investment (this is just like a prepaid forward contract). You are also promised to get the underlying asset at expiration of the contract in* $t+m$*. In period* t $(> \tau)$ *you shorten the forward contract, which requires you to deliver the underlying asset in* $t+m$*, but it also promises the payment of* F_t *which has a present value of* $e^{-ym}F_t$*. The combination of these two transactions is that you do not deliver/receive any of the underlying asset at expiration. You also "paid"* $e^{-y(m+t-\tau)}F_\tau$ *in period* τ *and "received"* $e^{-ym}F_t$ *in period* t*. The gross return (received in t/paid in* τ*) of* $e^{y(t-\tau)}F_t/F_\tau$*. (Subtract one to get the net return.) For an asset without dividends, the forward-spot parity (15.5) then shows that the gross return is just* S_t/S_τ*.*

15.2.5 Application of the Forward-Spot Parity: Forward Price of a Bond

Consider a forward contract (expiring in $t+m$) on a discount (zero coupon) bond that matures in $t+n$ (assuming $n > m$). See Figure 15.5 for an illustration.

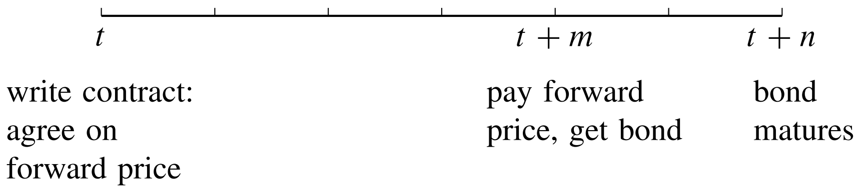


Figure 15.5: Timing convention of forward contract on a bond

By the forward spot parity (15.5) and the definition of a present value (15.3), today's forward price is

$$F(m) = e^{my(m)}B(n)$$
$$= B(n)/B(m), \tag{15.11}$$

where $B(n)$ is the price of an n-period bond today and $B_t(m)$ is the price of an m-period bond.

Example 15.8 *(Forward price of a bond) Let* $(m, n, B(m), B(n)) = (5, 7, 0.779, 0.657)$. *Then,* $F = 0.657/0.779 \approx 0.843$.

15.2.6 Application of the Forward-Spot Parity: Forward Price of Foreign Currency

Let S be the price (measured in domestic currency) of foreign currency. (Watch out: sometime the exchange rate quotation is the inverse of this.) Investing in foreign currency effectively means investing in a foreign interest bearing instrument which earns the continuous interest rate ("dividend") $y^*(m)$. Use $\delta = y^*(m)$ in (15.9)

$$F(m) = Se^{m[y(m)-y^*(m)]}. \tag{15.12}$$

This is called the *covered interest rate parity* (CIP). The price is quoted at the forward price F, or as the forward premium $F - S$. The premium is sometimes multiplied by 10,000 to give the premium in "pips." For instance, with $F = 1.22$ and $S = 1.20$, we have 200 pips.

Notice that $F > S$ (a positive premium) means that $y(m) > y^*(m)$. That is, if the domestic interest rate is higher than the foreign interest rate, then the forward price (of foreign currency) is higher than the spot price. In this way, the extra yield from the domestic interest rate is exactly matched by the "forward appreciation" of the foreign currency—to make the return the same. Conversely, $F < S$ (a negative premium) means that the domestic interest rate is lower than the foreign interest rate.

To be more precise, notice that buying one unit of foreign currency now costs S. At expiration we have $e^{my^*(m)}$ units for foreign currency. Converted back into the domestic currency at the (predetermined forward price) we have $e^{my^*(m)}F = Se^{my(m)}$. Since we invested S_t, the return on this investment is the same as on the domestic money market.

Example 15.9 *(CIP) With $S = 1.20, m = 1, y = 0.0665$ and $y^* = 0.05$ we have*

$$F = 1.20e^{0.0165} = 1.22.$$

Buying one unit of foreign currency costs 1.20 and after one year we have $e^{0.05} = 1.0513$ units of foreign currency, which are (when converted with $F = 1.22$) worth $1.0513 \times 1.22 = 1.2826$ in domestic currency. Since we invested 1.20, the gross return is $1.2826/1.20 = 1.0688$, which equals $\exp(0.0665)$.

Remark 15.10 *(CIP, alternative version*) If $\tilde{S}$ is the price of domestic currency ($\tilde{S} = 1/S$) and $\tilde{F}$ is analogous, then (15.12) becomes*

$$\tilde{F}(m) = \tilde{S}e^{m[y^*(m)-y(m)]},$$

which is just the reciprocal.

15.2.7 Forwards versus Futures

A forward contract is typically a private contract between two investors—and can therefore be tailor made. A futures contract is similar to a forward contract (write contract, get something later), but is typically traded on an exchange—and is therefore standardized (amount, maturity, settlement process). The settlement is either cash settlement or physical settlement. The latter does not work for synthetic assets like equity indices.

Another important difference is that a forward contract is settled at expiration, whereas a futures contract is settled daily (*marking-to-market*). This essentially means that gains and losses (because of price changes) are transferred between issuer and owner daily—but kept at an interest bearing account at the exchange. The counter parties have to post an initial margin—and the marking-to-market then adds to/subtracts from this margin. If the amount decreases below a certain level (maintenance margin), then a margin call is issued to the investor—informing him/her to add cash to the margin account. If interest rates change randomly over time (and they do), the rate at which the money on the margin account is invested at (refinanced) will be different from the rate when the futures was issued. This risk of this happening is reflected in the futures price.

The proposition below shows that, if the interest rate path was non-stochastic (provided there is no counter party risk), then the forward and futures prices would be the same. In practice, the difference between forward and futures prices is typically small.

Proposition 15.11 *(Forward vs. futures prices, non-stochastic interest rates) The forward and futures prices would be the same (a) if there were no counter party risk; (b) and if the interest rate only changed in a non-stochastic way.*

Proof. (of Proposition 15.11) To simplify the notation, let $t = 0$ and $m = 2$. Also, let r_s be the continuously compounded one-day interest rate and f_s be the futures price. Strategy A: have e^{r_0} long futures contracts on (the end of) day 0, increase it to $e^{r_0+r_1}$ on day 1. Provided we reinvest the settlements in one-day bills, we have

Day (s)	Position	Settlement	End-value of reinvested settlement
0	e^{r_0}	0	0
1	$e^{r_0+r_1}$	$e^{r_0}(f_1 - f_0)$	$e^{r_0}(f_1 - f_0)e^{r_1}$
2	0	$e^{r_0+r_1}(f_2 - f_1)$	$e^{r_0+r_1}(f_2 - f_1)$

The end-value of strategy A is therefore $e^{r_0+r_1}(f_2 - f_0)$, which equals $e^{r_0+r_1}(S_2 - f_0)$ since the value at expiration is the value of the underlying asset. Strategy B: be long $e^{r_0+r_1}$ forward contracts, which gives a payoff on day 2 of $e^{r_0+r_1}(S_2 - F_0)$. Both strategies take on exactly the same risk, so the prices must be the same: $f_0 = F_0$. (The proof relies on knowing r_1 already on day 0.) ■

Example 15.12 *(Margin account) Margin account of a buyer (holder) of a futures contract (maintenance margin = 0.75×initial margin) could be as follows (assuming a zero interest rate):*

Day	*Futures price*	*Daily gain*	*Posting of margin*	*Margin account*
0	100		4	4
1	99	−1		3
2	97	−2	2	3
3	99	2		5

On day 2, the investor received a margin call to add cash to the account—to make sure that the maintenance margin (here 3) is kept. Notice that the overall profit is the difference of what has been put into the margin account $(4+2)$ *and the final balance* (5)*, that is,* -1*. This is also the* cumulative *daily gain* $(-1-2+2=-1)$*. With marking to market this is all that happens: no payment of the futures price and no delivery of the underlying asset. However, it is equivalent to what happen without marking to market, since at expiration, the gain is* $99-100=-1$ *(futures = underlying at expiration).*

15.3 Appendix: Data Sources*

The data used in these lecture notes are from the following sources:

1. website of Kenneth French,
   `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`
2. Datastream
3. Federal Reserve Bank of St. Louis (FRED), `http://research.stlouisfed.org/fred2/`
4. website of Robert Shiller, `http://www.econ.yale.edu/~shiller/data.htm`
5. yahoo! finance, `http://finance.yahoo.com/`
6. OlsenData, `http://www.olsendata.com`

Chapter 16

Interest Rate Calculations

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 21–22 and Hull (2009) 4
Additional references: McDonald (2014) 9; Fabozzi (2004); Blake (1990) 3–5; and Campbell, Lo, and MacKinlay (1997) 10

16.1 Zero Coupon (discount or bullet) Bonds

16.1.1 Zero Coupon Bond Basics

Consider a zero coupon bond which costs $B(m)$ in t and gives one unit of account in $t+m$ (the trade time index t is suppressed to simplify notation—in case of potential confusion, we can write $B_t(m)$). See Figure 16.1 for an illustration.

The gross return (payoff divided by price) from investing in this bond is $1/B(m)$, since the face value is normalized to unity. The relation between the *bond price* $B(m)$

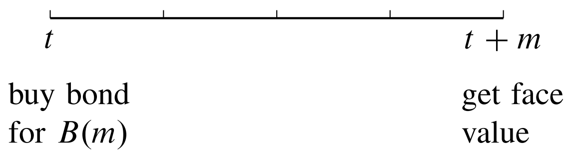


Figure 16.1: Timing convention of zero coupon bond

and the *effective (spot) interest rate* $Y(m)$ is

$$\frac{1}{B(m)} = [1 + Y(m)]^m, \tag{16.1}$$

$$B(m) = [1 + Y(m)]^{-m}, \tag{16.2}$$

$$Y(m) = B(m)^{-1/m} - 1. \tag{16.3}$$

The interest rate is therefore an annualized rate of return from investing $B(m)$ and receiving the face value (here normalized to 1) m periods later. Another way to think about this is that if we invest the amount $B(m)$ by buying one bond, then after m periods we get $B(m)$ times the interest rate factor, that is, $B(m)[1 + Y(m)]^m = 1$. In practice, bond quotes are typically expressed in percentages (like 97) of the face value, whereas the discussion here effectively uses the fraction of the face value (like 0.97). Notice that you can calculate the *present value* (of getting Z in $t + m$) as $B(m) Z$.

The relation between the rate and the price is clearly non-linear—and depends on the time to maturity (m): short rates are more sensitive to bond price movements than long rates. Conversely, prices on short bonds are less sensitive to interest rate changes than prices on long bonds. See Figure 16.2 for an illustration.

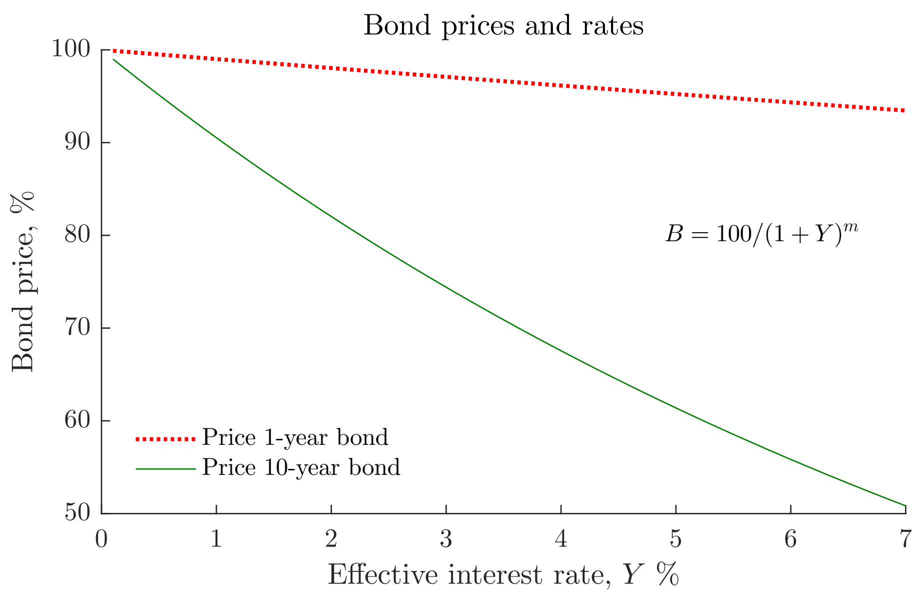


Figure 16.2: Interest rate vs. bond price

We also have the following relation between the bond price and the *continuously com-*

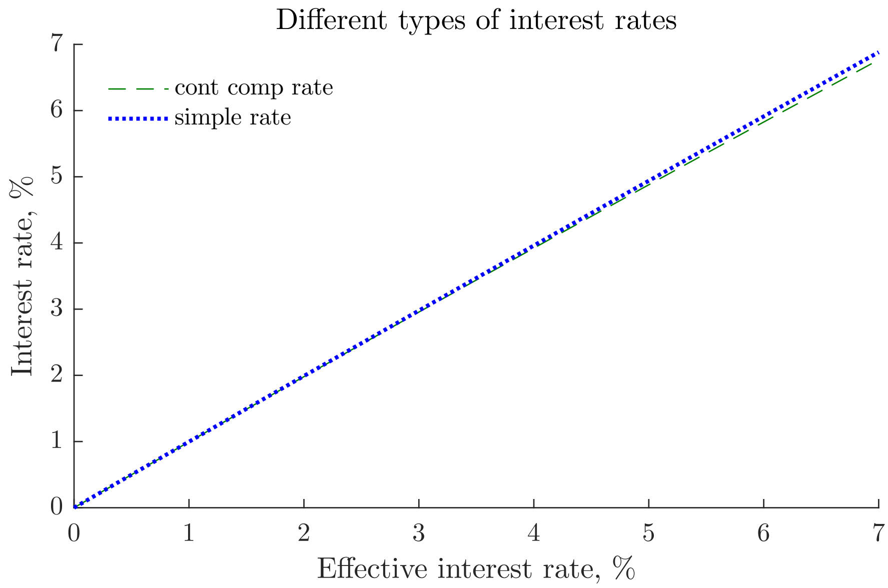


Figure 16.3: Different types of interest rates

pounded interest rate

$$\frac{1}{B(m)} = \exp\left[my(m)\right], \tag{16.4}$$

$$B(m) = \exp\left[-my(m)\right], \tag{16.5}$$

$$y(m) = -\ln B(m)/m. \tag{16.6}$$

Example 16.1 *(Effective and continuously compounded rates) Let the period length be a year (which is the most common convention for interest rates). Consider a six-month bill so $m = 0.5$. Suppose $B(m) = 0.95$. From (16.1) we then have that*

$$\frac{1}{0.95} = [1 + Y(0.5)]^{0.5}, \text{ so } Y(0.5) \approx 0.108, \text{ and } y(0.5) \approx 0.103.$$

Some fixed income instruments (in particular inter bank loans, LIBOR/EURIBOR) are quoted in terms of a *simple interest rate*, $\tilde{Y}$. The "price" of a deposit that gives unity at maturity is then related to the simple interest rate according to

$$B(m) = \frac{1}{1 + m\tilde{Y}(m)}, \text{ or} \tag{16.7}$$

$$\tilde{Y}(m) = \frac{1/B(m) - 1}{m}. \tag{16.8}$$

Example 16.2 *(Simple rates) Consider a six-month bill so* $m = 0.5$*. Suppose* $B(m) = 0.95$*. From (16.7) we then have that*

$$0.95 = \frac{1}{1 + 0.5 \times \tilde{Y}(0.5)}, \text{ so } \tilde{Y}(0.5) \approx 0.105.$$

Remark 16.3 *(The transformation from one type of rate to the other*) We have*

$$\begin{aligned}
&y(m) = \ln[1 + Y(m)] \text{ and } y(m) = \ln\left[1 + m\tilde{Y}(m)\right]/m, \\
&Y(m) = \exp[y(m)] - 1 \text{ and } Y(m) = [1 + m\tilde{Y}(m)]^{1/m} - 1 \\
&\tilde{Y}(m) = \{[1 + Y(m)]^m - 1\}/m \text{ and } \tilde{Y}(m) = \{\exp[y(m)] - 1\}/m.
\end{aligned}$$

The different interest rates (effective, continuously compounded and simple) are typically very similar, except for very high rates. See Figure 16.3 for an illustration.

Example 16.4 *(Different interest rates) With* $m = 1/2$*,* $Y = 0.108$*,* $y = 0.103$ *and* $\tilde{Y} = 0.106$

$$1.053 \approx (1 + 0.108)^{0.5} \approx \exp(0.5 \times 0.103) \approx 1 + 0.5 \times 0.105.$$

16.1.2 The Return from Holding a Zero Coupon Bond

The log *return from holding a zero coupon bond until maturity* is $my(m)$. This follows directly from the definition of the log interest rate (see (16.4)).

The log *return from holding a zero coupon bond* from t to $t + s$ is clearly the relative change of the bond price

$$\ln(1 + R_{t+s}) = \ln \frac{B_{t+s}(m - s)}{B_t(m)}, \tag{16.9}$$

where the subscripts indicate the trading date (previously suppressed). Notice that the bond's maturity decreases with time: in this case from m to $m - s$. (This is a return over s periods and it is not rewritten on a "per period" basis as interest rates are.)

Example 16.5 *(Bond return) If the bond price decreases from* 0.95 *to* 0.86*, then (16.9) gives the log return*

$$\ln \frac{0.86}{0.95} = -0.1.$$

Using the relation between the continuously compounded interest rate and the bond price (16.4) gives the log return as

$$\begin{aligned}\ln(1+R_{t+s}) &= my_t(m) - (m-s)y_{t+s}(m-s) \\ &= -m[y_{t+s}(m-s) - y_t(m)] + sy_{t+s}(m-s)\,. \end{aligned} \tag{16.10}$$

This expression is useful for looking at some special cases—to highlight the key properties of zero coupon bond returns.

The first special case is *a short holding period* (s is very small). The second term in (16.10) is then virtually zero, so we can write

$$\ln(1+R_{t+s}) = -m\Delta y_{t+s}(m)\,, \tag{16.11}$$

where $\Delta y_{t+s}(m)$ is the change in the interest rate (the term in brackets in (16.10)). This is clearly negative if the interest rate change is positive—and more so if the maturity (m) is long.

Example 16.6 *(Bond returns vs interest rate changes) Suppose that, over a split second (so the time to maturity is virtually unchanged), the interest rates for all maturities increase from 0.5% to 1.5%. Using (16.4) gives the following bond prices*

	at 0.5%	*at 1.5%*	*Change in logs (%)*
1-year bond	$e^{-1\times0.005}=0.995$	$e^{-1\times0.015}=0.985$	-1
10-year bond	$e^{-10\times0.005}=0.951$	$e^{-10\times0.015}=0.861$	-10

Using (16.11) directly gives the same: $-1\times0.01=0.01$ *and* $-10\times0.01=-0.1$.

The second special case if a long *a short holding period, but with an unchanged flat yield curve*. In this case, all interest rates in (16.10) are the same (denoted y), so we get

$$\ln(1+R_{t+s}) = sy, \tag{16.12}$$

which is just the holding period times the interest rate. The reason is simply that the bond starts out as the m-maturity bond, but becomes an $(m-s)$-maturity bond—and the latter has a higher price. See Figure 16.4.

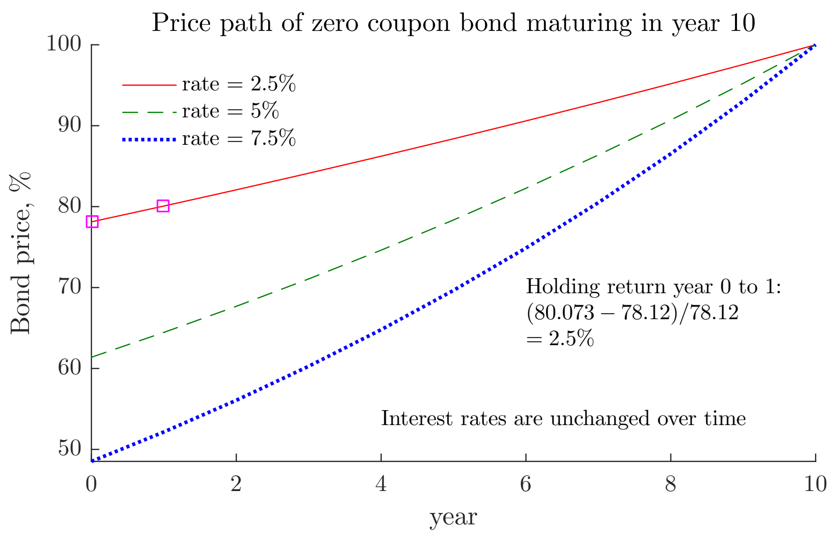


Figure 16.4: The price of a zero coupon bond maturing in year 10

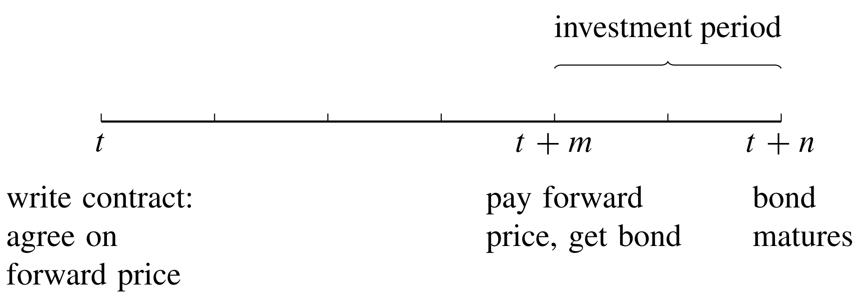


Figure 16.5: Timing convention of forward contract

16.2 Forward Rates

16.2.1 Definition of Forward Rates

A forward contract on a bond can be used to lock in an interest rate for an investment over a future period. Consider "buying" a forward contract in t: it stipulates what you have to pay in $t+m$ (the forward price) and that you then get a discount bond that pays the face value (here normalized to 1) at time $t+n$. See Figure 16.5 for an illustration.

16.2.2 Implied Forward Rates

The forward-spot parity implies that the forward price is

$$F = [1 + Y(m)]^m B(n) = B(n)/B(m). \tag{16.13}$$

Buying a forward contract is effectively an investment from $t + m$ to $t + n$, that is, over $n - m$ periods. The gross return (which is known already in t) is 1/forward price. We define a per period effective rate of return, a *forward rate*, $\Gamma(m, n)$, analogous with an interest rate as

$$\frac{1}{F} = \frac{B(m)}{B(n)} = [1 + \Gamma(m, n)]^{n-m}. \tag{16.14}$$

Notice that $\Gamma(m, n)$ here denotes a forward rate, not a forward price. This is the rate of return over $t + m$ to $t + n$ that can be guaranteed in t. In many cases, it is convenient to use $B(m)/B(n)$ in calculations involving the gross return on the forward contract. However, it is sometimes more instructive to rewrite in terms of the forward rate. By using the relation between bond prices and yields (16.1), the gross forward rate can be written

$$1 + \Gamma(m, n) = F^{-1/(n-m)} \tag{16.15}$$

$$= \frac{[1 + Y(n)]^{n/(n-m)}}{[1 + Y(m)]^{m/(n-m)}}. \tag{16.16}$$

This shows that the forward rate depend on both interest rates, and thus, the general shape of the yield curve. As discussed later, the forward rate can be seen as the "marginal cost" of making a loan longer. See Figure 16.6 for an illustration.

Example 16.7 *(Forward rate) Let $m = 0.5$ (six months) and $n = 0.75$ (nine months), and suppose that $Y(0.5) = 0.04$ and $Y(0.75) = 0.05$. Then (16.16) gives*

$$[1 + \Gamma(0.5, 0.75)]^{0.75-0.5} = \frac{(1 + 0.05)^{0.75}}{(1 + 0.04)^{0.5}},$$

which gives $\Gamma(0.5, 0.75) \approx 0.07$. See Figure 16.6 for an illustration.

Example 16.8 *(Forward rate) Let the period length be a year. Let $m = 1$ (one year) and $n = 2$ (two years), and suppose that $Y(1) = 0.04$ and $Y(2) = 0.05$. Then (16.16) gives*

$$1 + \Gamma(1, 2) = \frac{(1 + 0.05)^2}{(1 + 0.04)^1} \approx 1.06,$$

so the forward rate is approximately 6%.

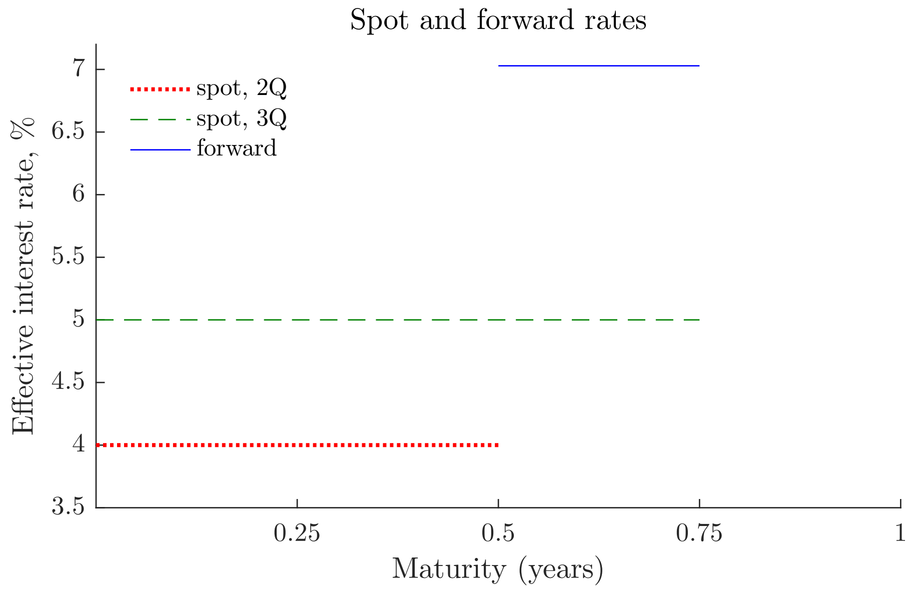


Figure 16.6: Spot and forward rates

Remark 16.9 *(Forward Rate Agreement) An* FRA *is an over-the-counter contract that guarantees an interest rate during a future period. The FRA does not involve any lending/borrowing—only compensation for the deviation of the future interest rate (typically LIBOR) from the agreed forward rate. An FRA can be emulated by a portfolio of zero-coupon bonds, similarly to a forward contract.*

Remark 16.10 *(Alternative way of deriving the forward rate*) Rearrange (16.16) as*

$$[1 + Y(m)]^m \, [1 + \Gamma(m,n)]^{n-m} = [1 + Y(n)]^n \, .$$

This says that compounding $1 + Y(m)$ *over* m *periods and then* $1 + \Gamma(m,n)$ *for* $n - m$ *periods should give the same amount as compounding the long rate,* $1 + Y(n)$*, over* n *periods.*

16.3 Coupon Bonds

16.3.1 Coupon Bond Basics

Consider a bond which pays coupons, c, for K periods (at $t + m_1, t + m_2, .., t + m_K$), and also the “face” (or “par” value, here normalized to 1) at maturity, $t + m_K$. See Figure 16.7 for an illustration.

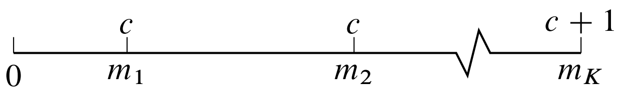


Figure 16.7: Timing convention of coupon bond

The coupon bond is, in fact, a portfolio of zero coupon bonds: c maturing in $t+m_1$, c in $t+m_2$,..., and $c+1$ in $t+m_K$. The price of the coupon bond, P, must therefore equal the price of the portfolio

$$P = \sum_{k=1}^{K} B(m_k)c + B(m_K) \tag{16.17}$$

where $B(m_k)$ is the price of a zero coupon bond maturing m_k periods later. Using the relation between (zero coupon) bond prices and yields in (16.1), this can also be written

$$P = \sum_{k=1}^{K} \frac{c}{[1+Y(m_k)]^{m_k}} + \frac{1}{[1+Y(m_K)]^{m_K}}. \tag{16.18}$$

This shows that coupon bond price is just the present value of the cash flow (from coupons and payment of the face value), but where the discounting is made by the different spot interest rates. In these calculations, P is the full (invoice) price of the bond—which can differ from quoted prices (also called "clean prices") by an accrued interest rate term. See below for details. Sometimes it will be convenient to let $P(m)$ denote the price of a coupon bond that matures in m periods ahead.

Example 16.11 *(Coupon bond price) Suppose $B(1) = 0.95$ and $B(2) = 0.90$. The price of a bond with a 6% annual coupon with two years to maturity is then*

$$1.01 \approx 0.95 \times 0.06 + 0.90 \times 0.06 + 0.90 \times 1.$$

Equivalently, the bond prices imply that $Y(1) \approx 5.3\%$ and $Y(2) \approx 5.4\%$ so

$$1.01 \approx \frac{0.06}{1.053} + \frac{0.06+1}{1.054^2}.$$

Example 16.12 *(Coupon bond price at par) A 9% (annual coupons) Suppose $B(1) = 1/1.06$ and $B(2) = 1/1.091^2$. The price of a bond with a 9% annual coupon with two years to maturity is then*

$$\frac{0.09}{1.06} + \frac{0.09}{1.091^2} + \frac{1}{1.091^2} \approx 1.$$

This bond is (approximately) sold "at par", that is, the bond price equals the face (or par) value (which is 1 in this case).

Remark 16.13 *(STRIPS, Separate Trading of Registered Interest and Principal of Securities*) A coupon bond can be split up into its embedded zero coupon bonds—and traded separately. STRIPS are therefore zero coupon bonds.*

Remark 16.14 *(Bond price quotes*) Bond prices are typically quoted as percentage of face (par) value, e.g. a quote of 97 on a bond with face value is 1000 means that you pay 970. On the U.S. Treasury bond market, the quotes are often not in a decimal form. Instead, the quoted prices use fractions of 4, 8, 26, 32 and 62 as in*

$$\begin{aligned}
&\textit{91-21 means } 91 + 21/32 \approx 91.6562 \\
&\textit{91-21+ means } 91 + 21/32 + 1/64 \approx 91.6719 \\
&\textit{91-21}\tfrac{3}{4}\textit{ means } 91 + (21 + 3/4)/32 \approx 91.6797 \\
&\textit{91-213 means } 91 + (21 + 3/8)/32 \approx 91.6680.
\end{aligned}$$

16.3.2 Coupon Bond Pricing with a Flat Yield Curve*

If we knew all the spot interest rates, then it would be easy to calculate the correct price of the coupon bond. The special (admittedly unrealistic) case when all spot rates are the same (flat yield curve) is interesting since it provides good intuition for how coupon bond prices are determined. In particular, if the next coupon payment is one period ahead ($m_k = k$), then (16.18) becomes

$$P = 1 + \frac{c - Y}{Y}[1 - (1 + Y)^{-K}], \tag{16.19}$$

where Y is the (common) spot rate and K is the maturity. The term in square brackets is positive (assuming $Y > 0$ and $K > 0$), so when the interest rate (which then equals the yield to maturity, see below) is below the coupon rate, then the bond price is above the face value (since $c - Y > 0$), and vice versa. When $c = Y >$ the bond trades at par, that is, the bond price equals the face value (here normalized to unity).

Example 16.15 *(of (16.19)) With $c = 5\%$, $Y = 2.5\%$ and $K = 10$ we get $P = 121.88\%$. Instead, with $c = 1\%$ we get 86.87%*

Proof. (of (16.19)) Write (16.18) as

$$P = \textstyle\sum_{k=1}^{K} \frac{c}{(1 + Y)^k} + \frac{1}{(1 + Y)^K}.$$

Consider the first term. We know that

$$D = \sum_{k=1}^{\infty} \frac{c}{(1+Y)^k} = \frac{c}{Y} \text{ and that}$$
$$E = \sum_{k=K+1}^{\infty} \frac{c}{(1+Y)^k} = (1+Y)^{-K} \frac{c}{Y}.$$

Clearly, the first term ($\Sigma_{k=1}^{K} c/(1+Y)^k$) equals $D - E = \frac{c}{Y}[1-(1+Y)^{-K}]$. Add the last term ($1/(1+Y)^K$) to get the bond price $P = \frac{c}{Y}[1-(1+Y)^{-K}] + (1+Y)^{-K}$, which can be written as (16.19). ■

16.3.3 Yield to Maturity

The effective *yield to maturity* (also called redemption yield), θ, on a coupon bond is the internal rate of return which solves

$$P = \sum_{k=1}^{K} \frac{c}{(1+\theta)^{m_k}} + \frac{1}{(1+\theta)^{m_K}}, \tag{16.20}$$

where the bond pays coupons, c, at $m_1, m_2, ..., m_K$ periods ahead. This equation can be solved (numerically, see Appendix) for θ. Bonds are quoted in terms of the yield to maturity (instead of the price). For a *par bond* (the bond price equals the face value, here 1), the yield to maturity equals the coupon rate. For a zero coupon bond, the yield to maturity equals the spot interest rate.

Example 16.16 *(Yield to maturity) A 4% (annual coupon) bond with 2 years to maturity. Suppose the price is 1.019. The yield to maturity is 3% since it solves*

$$1.019 \approx \frac{0.04}{1+0.03} + \frac{0.04}{(1+0.03)^2} + \frac{1}{(1+0.03)^2}.$$

Example 16.17 *(Yield to maturity of a par bond) A 4% (annual coupon) par bond (price of 1)with 2 years to maturity. The yield to maturity is 4% since*

$$\frac{0.04}{1+0.04} + \frac{0.04}{(1+0.04)^2} + \frac{1}{(1+0.04)^2} = 1.$$

Example 16.18 *(Yield to maturity of a portfolio) A 1-year discount bond with a ytm (effective interest rate) of 7% has the price* $1/1.07$ *and a 3-year discount bond with a ytm of 10% has the price* $1/1.1^3$*. A portfolio with one of each bond has a ytm*

$$\frac{1}{1.07} + \frac{1}{1.1^3} = \frac{1}{1+\theta} + \frac{1}{(1+\theta)^3}, \text{ with } \theta \approx 0.091.$$

This is clearly not the average ytm of the two bonds. It would be, however, if the yield curve was flat.

16.3.4 The Return from Holding a Coupon Bond until Maturity

To calculate the *return from holding a coupon bond until maturity* we need to specify how the coupons are reinvested. For instance, the yield to maturity is the return of holding the bond until maturity if all the coupons are reinvested in assets that generate returns equal to the bond's ytm.

Proposition 16.19 *(Return from holding a coupon bond until maturity, a special case) If all coupons are reinvested in assets that generate returns equal to the bond's yield to maturity θ, then the grossreturn of holding the bond until maturity is $(1+\theta)^{mK}$. This means that the annualized rate of return is θ.*

Proof. (of Proposition 16.19) Consider a 2-period coupon bond with ytm θ. From (16.20), the price of the bond is

$$P = \frac{c}{1+\theta} + \frac{c+1}{(1+\theta)^2}.$$

If we can reinvest the first coupon payment to give the return θ, it is worth $c(1+\theta)$ are maturity—and we also receive $c+1$ at maturity. Divide the end value with the initial investment (the bond price P)

$$\frac{c(1+\theta)+c+1}{c/(1+\theta)+(c+1)/(1+\theta)^2} = (1+\theta)^2.$$

■

Another useful assumption is that the coupons are reinvested via forward contracts (agreed on at the time the bond is bought). This means that the investor buys the bond now and receives nothing until maturity—and he/she knows already now much will be received at maturity. This is just like he/she had bought a zero-coupon bond. Indeed, no-arbitrage arguments show that the return (from now to maturity) is indeed the spot interest on a zero-coupon bond. In short, we have

Proposition 16.20 *(Return from holding a coupon bond until maturity, another special case) If the the coupons are reinvested by forward contracts, then the yield on holding the bond until maturity is the current spot rate (on a zero coupon bond with the same maturity).*

Notice that this holds irrespective of the coupon rate. For this reason, it can well be said that coupons do not really matter for returns. With other assumptions about how the coupons are reinvested, the result is different (but typically not very much so).

Proof. (of Proposition 16.20) Consider a 2-period coupon bond. From (16.18), the price of the bond is

$$P_t = B_t(1)c + B_t(2)(c + 1).$$

From (16.14), we know that the forward contract for the first coupon has the gross return (until maturity) $B_t(1)/B_t(2)$. The value of the reinvested coupon and the face value at maturity is then

$$\frac{B_t(1)}{B_t(2)}c + c + 1.$$

Dividing by the first equation (the investment) gives $1/B_t(2)$ so the return on buying and holding (and reinvesting the coupons) this coupon bond is the same as the 2-period spot interest rate. (The extension to more periods is straightforward.) ■

Example 16.21 *(Holding a coupon bond until maturity) Suppose that the spot (zero coupon) interest rates are 4% for one year to maturity and 5% for 2 years to maturity (the zero coupon bond prices are $B(1) = 0.962$ and $B(2) = 0.907$). A 3% coupon bond with 2 years to maturity must have the current price*

$$\frac{0.03}{1.04} + \frac{0.03 + 1}{1.05^2} \approx 0.963.$$

However, the value of the bond at maturity, if the coupon is reinvested by a forward contract, is

$$0.03 \times \frac{0.962}{0.907} + 0.03 + 1 \approx 1.062,$$

so the gross return over two years is approximately $1.062/0.963 \approx 1.102$. Compare that to $(1 + 0.05)$^2, which is approximately the same (some small rounding differences).

16.3.5 The Return from Selling a Coupon Bond before Maturity

The gross *return from holding a coupon bond from t to $t + s$* depends on both the price development on the bond and the value in $t + s$ of the (reinvested) coupon payments received between t and $t + s$

$$1 + R_{t+s} = \frac{P_{t+s} + \text{value}_{t+s}(\text{coupon payments})}{P_t}, \tag{16.21}$$

where the subscripts indicate the trading date. The price change follows the same general pattern as for zero coupon bonds: if interest rates increase, then that is the same as if the bond prices decrease (leading to a capital loss for the bond holder). The second term depends, as usual, on the timing, size and reinvestment return of the coupons. We can disentangle this in case we are explicit about how we reinvest the coupons.

Proposition 16.22 *(Bond holding return, a special case) Suppose we reinvest the coupons with forward contracts—as if we were going to hold the bond until maturity m_K (see Proposition 16.20). Holding the bond until $t+s$ ($s \leq m_K$) gives the total gross return (between t and $t+s$) $1+R_{t+s} = B_{t+s}(m_K - s)/B_t(m_K)$, where $B_t(m)$ denotes the price of an m-period zero coupon bond in t. This implies that the portfolio has the same return as an m_K-period zero coupon bond bought in t, which becomes an $m_K - s$ zero coupon bond in $t+s$.*

For instance, with $m_K = 3$, the gross return on holding the bond for one period is $B_{t+1}(2)/B_t(3)$, while the gross return from holding it for two periods is $B_{t+2}(1)/B_t(3)$. Clearly, the strategy to reinvest the coupons with forward contracts essentially turns this into an m_K-period zero coupon bond (where you invest in t but do not receive any payoffs until $t+m_K$). The return of the strategy is thus the same as on holding this zero coupon bond for s periods. Once again, with other assumptions about how the coupons are reinvested, the result is different. The general insight is correct however: a main part of the holding return is due to interest rate changes.

Proof. (of Proposition 16.22) Consider a 2-period coupon bond which we hold for 1 period. Enter forward contracts like in the proof of Proposition 16.20. The value in $t+1$ must then be the present value of the value at maturity, that is,

$$B_{t+1}(1)\left[\frac{B_t(1)}{B_t(2)}c + c + 1\right],$$

where $B_{t+1}(1)$ denotes the price in $t+1$ of a one-period zero coupon bond. Dividing by the bond price (see proof of Proposition 16.20) gives the gross return

$$1 + R_{t+1} = B_{t+1}(1)/B_t(2)$$

■

Proof. (of Proposition 16.22, more general*) Consider a 3-period coupon bond which we hold for 1 period. Enter forward contracts like in the proof of Proposition 16.20. The value of this portfolio in $t+1$ must be the present value of the value at maturity, that is,

$$B_{t+1}(2)\left[\frac{B_t(1)}{B_t(3)}c+\frac{B_t(2)}{B_t(3)}c+c+1\right],$$

where $B_{t+1}(2)$ denotes the price in $t+1$ of a two-period zero coupon bond. Dividing by the bond price in t

$$P_t = B_t(1)c + B_t(2)c + B_t(3)(c+1)$$

gives the gross return

$$1+R_{t+1} = B_{t+1}(2)/B_t(3).$$

■

Example 16.23 *(Holding a coupon bond for one period) Use the same numbers as in Example 16.21 and assume that the interst rates are unchanged. The present value in $t+1$ of the value at maturity is*

$$0.962 \times 1.062 = 1.022.$$

Dividing by the bond price P_t, the gross return is

$$\frac{1.022}{0.963} \approx 1.06.$$

Using Proposition 16.22 directly gives $B_{t+1}(1)/B_t(2)$, which is approximately the same. Instead, if the interest rates change so $B_{t+1}(1) = 0.957$, then the return is $0.957 \times 1.062/0.963 \approx 1.055$, which is the same as $B_{t+1}(1)/B_t(2)$.

Example 16.24 *(Playing the yield curve) (a) On 1 Oct 2015:1y LIBOR is 0.86% and 5y T-bond rate is 1.37%; (b) you believe the 1y rate will not change much over the next 5 years; (c) buy 5y T-bond ("receiving 1.37%") and finance it by 1y borrowing (paying 0.86% at most); (d) next year you roll over your debt; (e) works well unless the 1y rates increase. Orange County got bankrupt by doing something similar in 1994 (using a very leveraged position in inverse floaters, whose coupons are inversely related to the short interest rate). Short rates went up.*

Notice that in the special case of holding the bond until maturity ($s = m_K$), then Proposition 16.22) shows that $1 + R_{t+s} = 1/B(m_K)$ (since $B_{t+s}(0) = 1$), which is the same result as in Proposition 16.20). In this case, the bond earns the spot interest rate $Y(m_K)$ per period.

Also, notice that in the very special case of a flat and unchanged yield curve (with the interest rate Y for all maturities), then then Proposition 16.22) shows that the return is

$$1 + R_{t+s} = (1 + Y)^s. \tag{16.22}$$

See Figure 16.8 for an illustration.

However, when there are changes in the interest rate level and we sell the bond before maturity, then the capital gains/losses can dominate: lower interest rates mean capital gains and vice versa (just like for zero coupon bonds). For long-maturity bonds, the effects can be considerable. See Figure 16.9 for an illustration and Figure 16.10 for an empirical example. Later sections of these notes will focus on this topic, by analysing the price changes over very short holding periods ("overnight") when the entire yield curve shifts.

Remark 16.25 *(Realized forwards*) Sometimes another set of assumptions (labelled "realized forwards") is used to analyse the return on holding a coupon bond. In this case, we do not write forward contracts on the coupons. Rather, the coupons are reinvested at the spot rates prevailing at the time of the coupon payment. However, it is assumed that those future spot rates actually are equal to today's forward rates (hence "realized"). This is clearly unrealistic, but can be used to gauge the expected return on holding the bond, at least if today's forwards are close approximations of the expected future spot rates. The result is similar to Proposition 16.22, except that it effectively assumes* $B_{t+s}(m_K - s) = B_t(m_K - s)$*, that is, no surprise changes in interest rates.*

16.3.6 Par Yield*

A par yield is the coupon rate at which a bond would trade at par (that is, have a price equal to the face value). Setting $P = 1$ in (16.17) and solving for the implied coupon rate gives

$$c = \frac{1}{\sum_{k=1}^{K} B(m_k)} \left[1 - B(m_K)\right], \text{ or} \tag{16.23}$$

$$= \frac{1}{\sum_{k=1}^{K} \frac{1}{[1+Y(m_k)]^{m_k}}} \left[1 - \frac{1}{[1 + Y(m_K)]^{m_K}}\right]. \tag{16.24}$$

Typically, this is very similar to the zero coupon rates: see Figure 16.11.

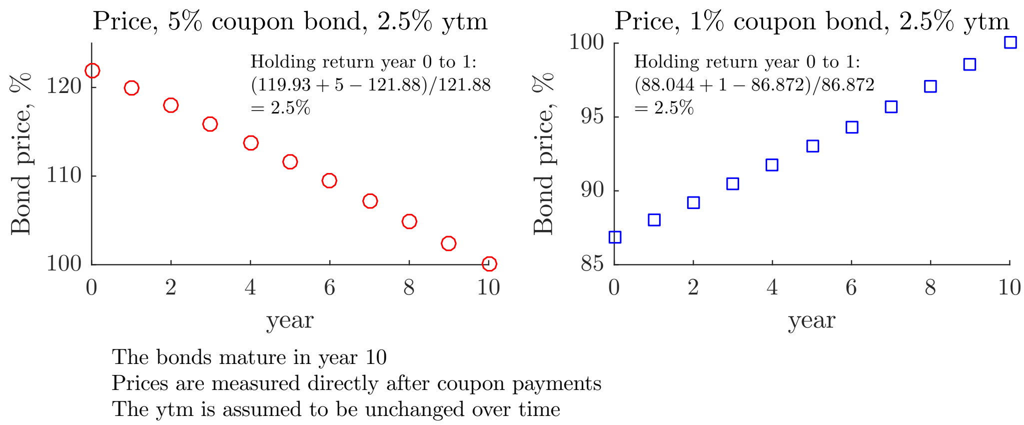


Figure 16.8: Bond price and yield to maturity

Example 16.26 *Suppose* $B(1) = 0.95$ *and* $B(2) = 0.90$*. We then have*

$$1 = (0.95 + 0.9)c + 0.9, \text{ so } c = \frac{1}{0.95 + 0.9}(1 - 0.9) \approx 0.054.$$

16.4 Price and Yield to Maturity of Bond Portfolios

Bond portfolios have a more complicated cash flow process than a traditional bond, but the pricing and yield to maturity follow the same general principles.

To handle this case, suppose the bond portfolio pays cf_k at m_k periods from now. This cash flow includes both coupon payments and face values. The bond pricing formula (16.18) becomes

$$P = \sum_{k=1}^{K} \frac{cf_k}{[1 + Y(m_k)]^{m_k}}, \tag{16.25}$$

and formula (16.20) for the yield to maturity, θ, becomes

$$P = \sum_{k=1}^{K} \frac{cf_k}{(1 + \theta)^{m_k}}. \tag{16.26}$$

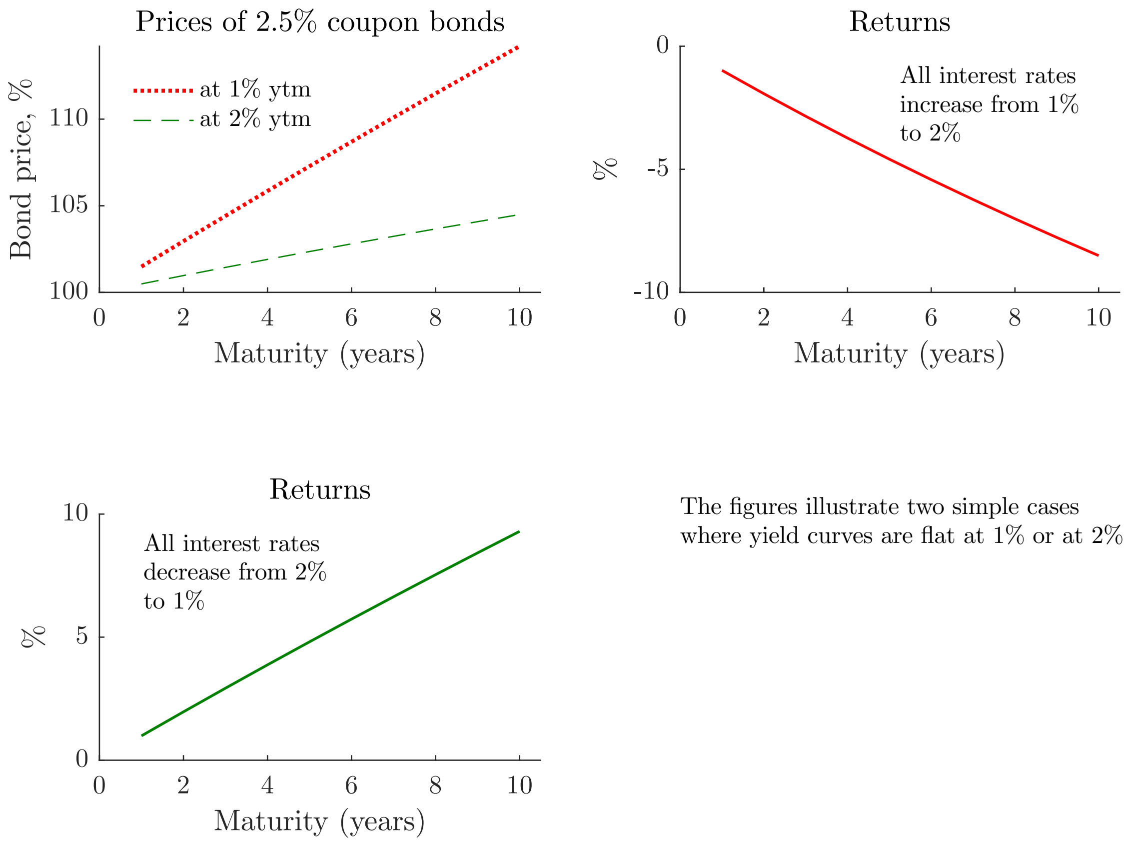


Figure 16.9: Gains and losses from interest rate changes

16.5 Swap and Repo

16.5.1 Swap

A swap contract involves a sequence of payment over the life time (maturity) of the contract: for each tenor (that is, sub period, for instance a quarter) it pays the floating market rate (say, the 3-month Libor) in return for a fixed *swap rate*. Split up the time until maturity n into n/h intervals of length h—see Figures 16.12–16.13. In period sh, the swap contract pays

$$h[\tilde{Y}_{(s-1)h}(h) - R] \tag{16.27}$$

where $\tilde{Y}_{(s-1)h}(h)$ is the short (floating) simple h-period interest rate in $(s-1)h$ and R is the (fixed) swap rate determined in t (as part of the swap contract).

The issuer can lock in the floating rate payments by a sequence of forward rate agreements that pay the floating rate in return for the forward rate. In this way the swap contract

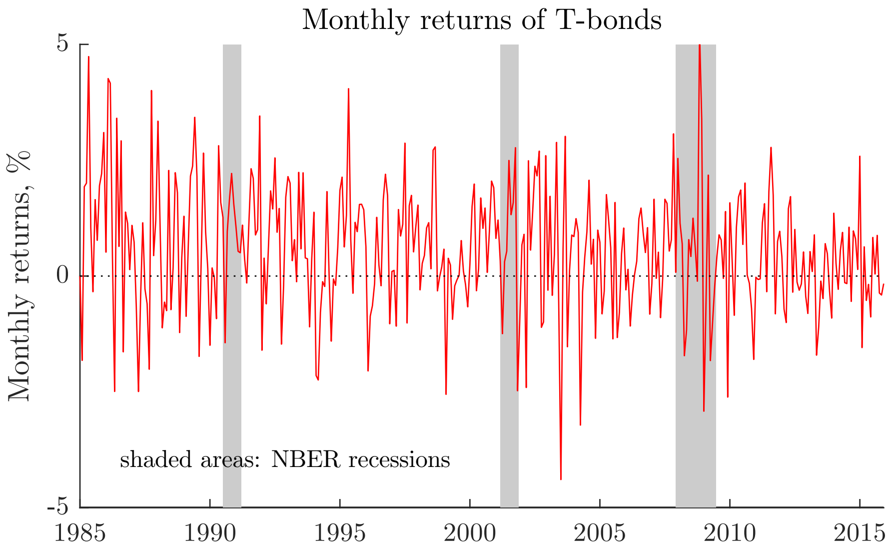


Figure 16.10: Returns on an index of U.S. Treasury bonds

becomes riskfree so its present value must be zero. This implies that the swap rate must therefore be (assuming no default or liquidity premia)

$$R = \frac{1}{h} \frac{1 - B(n)}{\sum_{s=1}^{n/h} B(sh)}, \tag{16.28}$$

which is proportional to the par yield in (16.23).

Example 16.27 *(Swap rate) Consider a one-year swap contract with quarterly periods ($n = 1, h = 1/4$). (16.28) is then*

$$R = 4 \frac{1 - B(1)}{B(1/4) + B(1/2) + B(3/4) + B(1)}.$$

With the bond prices (0.99,0.98,0.97,0.96) we have

$$R = 4 \frac{1 - 0.96}{0.99 + 0.98 + 0.97 + 0.96} \approx 4.1\%.$$

An *Overnight Indexed Swap* (OIS) is a swap contract where the floating rate is tied to an index of floating rates (for instance, federal funds rates in the U.S., EONIA in Europe—which is a weighted average of all overnight unsecured interbank lending transactions). Since the OIS has very little risk (as the face value or notional never changes hands—only the interest payment is risked in case of default), it is little affected by interbank risk

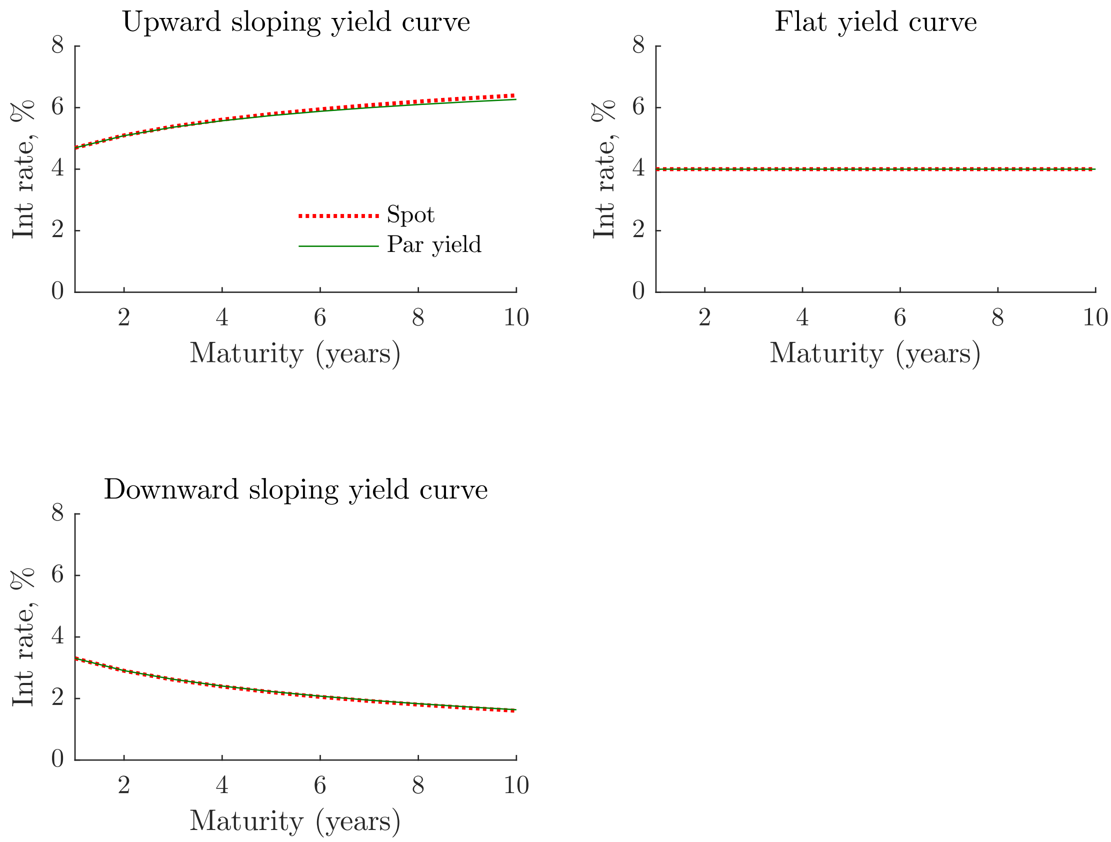


Figure 16.11: Spot and par yield curve

premia. The quote is in terms of the fixed rate (called the swap rate, quoted a simple interest rate)—which typically stays close to secured lending rates like repo rates.

Proof. (of (16.28)) Notice that a simple forward rate for an investment from sh to $(s+1)h$ is

$$\tilde{\Gamma}[sh,(s+1)h] = \frac{1}{h}\left[\frac{B(sh)}{B[(s+1)h]} - 1\right].$$

We can therefore write the present value of (16.27) as

$$\text{PV} = \sum\nolimits_{s=1}^{n/h} B(sh)\left\{\left[\frac{B[(s-1)h]}{B(sh)} - 1\right] - hR\right\}.$$

Since it is riskfree (assuming no default and liquidity premia) the PV should be zero (or

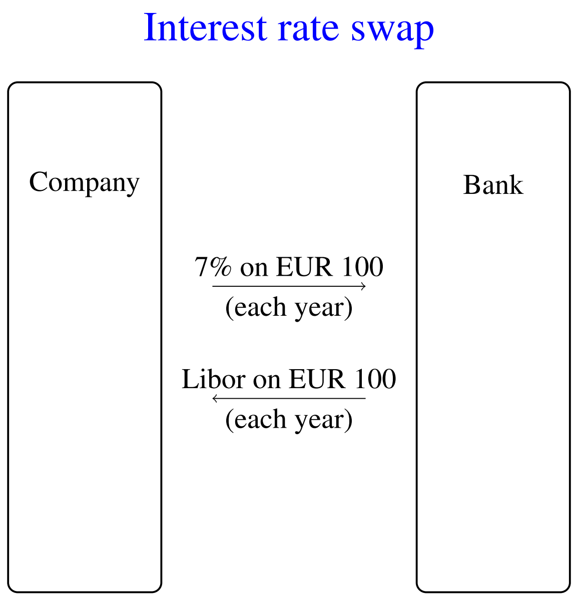


Figure 16.12: Interest rate swap

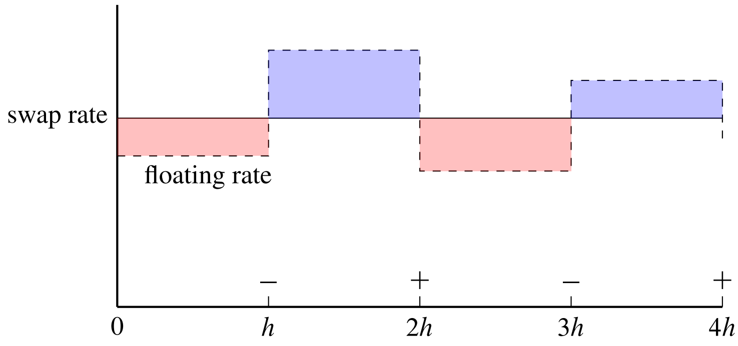
else there are arbitrage opportunities), which we rearrange as

$$hR\sum_{s=1}^{n/h} B(sh) = \sum_{s=1}^{n/h} B(sh)\left[\frac{B[(s-1)h]}{B(sh)} - 1\right]$$
$$hR\sum_{s=1}^{n/h} B(sh) = 1 - B(n),$$

where we have used the fact that $B(0) = 1$. Finally, solve for hR to get (16.28). ■

16.5.2 Repo

A *Repo* (Repurchase agreement) is a way of borrowing against a collateral. Suppose bank A sells a security to bank B, but there is an agreement that bank A will buy back the security at some fixed point in time (the next day, after a week, etc.)—at a price that is predetermined (or decided according to some predetermined formula). This means that bank A gets a loan against a collateral (the asset)—and pays an interest rate (final buy price/initial sell price minus one). See Figure 16.14. Bank B is said to have made a reverse repo. Another way to think about the repo is that bank A has made a sale of the security, but also acquired a forward contract on it (the position of bank B is just the reverse). The repo clearly means that bank B has "borrowed" the security—which can then be sold to someone else. This is a way of shortening the security, so the repo rate is



(The party receiving the floating rate pays the fixed swap rate)
(The net payments are marked by + or −)

Figure 16.13: Timing convention of interest rate swap

low if there is a demand for shortening the security. A *haircut* (of 3%, say) means that the collateral (security) has market value that is 3% higher than the price agreed in the repo. This provides a safety margin to the lender—since the market price of the security could decrease over the life span of the repo.

Example 16.28 *(Long-short bond portfolio). First, buy bond X and use it as collateral in a repo (the repo borrowing finances the purchase of the bond). Second, enter a reverse repo where bond Y is used as collateral and sell the bond (selling provides cash for the repo lending).*

16.6 Estimating the Yield Curve

The (zero coupon) spot rate curve is of particular interest: it helps us price any bond or portfolio of bonds—and it has a clear economic meaning ("the price of time").

In some cases, the spot rate curve is actually observable—for instance from swaps and STRIPS. In other cases, the instruments traded on the market include some zero coupon instruments (bills) for short maturities (up to a year or so), but perhaps only coupon bonds for longer maturities. This means that the spot rate curve needs to be calculated (or estimated). This section describes different methods for doing that.

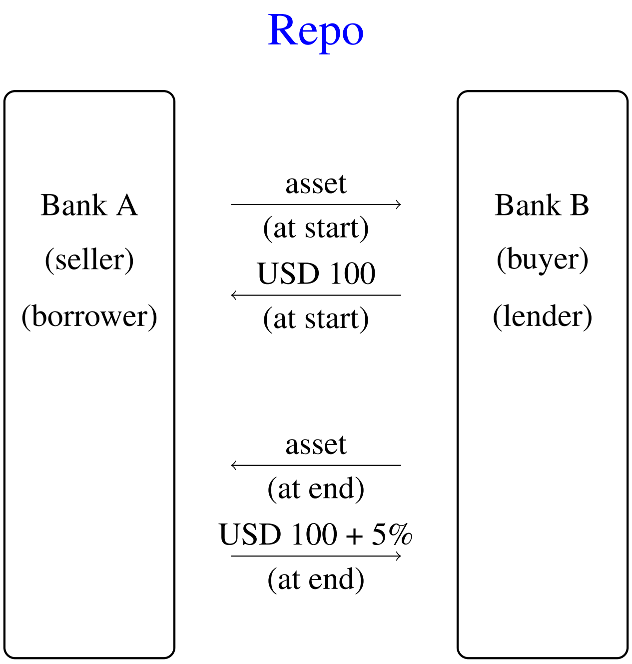


Figure 16.14: Repo

16.6.1 Direct Calculation of the Yield Curve ("Bootstrapping")

We can sometimes calculate large portions of the yield curve directly from bond prices by a method called "bootstrapping."

The basic idea is to recursively use the fact that a coupon bond is a portfolio of discount (zero coupon) bonds. For instance, suppose we have a one-period coupon bond, here denoted $P(1)$, which by (16.17) must have the price

$$P(1) = B(1)[c(1) + 1], \tag{16.29}$$

where we use $c(1)$ to indicate the coupon value of this particular bond. The equation immediately gives the price of a one-period discount bond, $B(1)$. In this setting the discount bond prices, $B(m)$, are also called a *discount function* (considered as a function of m).

Suppose we also have a two-period coupon bond, which pays the coupon $c(2)$ in $t+1$ and $t+2$ as well as the principal in $t+2$, with the price (see (16.17))

$$P(2) = B(1)c(2) + B(2)[c(2) + 1]. \tag{16.30}$$

The two period discount function, $B(2)$, can be calculated from this equation since it is the only unknown. We can then move on to the three-period bond,

$$P(3) = B(1)c(3) + B(2)c(3) + B(3)[c(3) + 1] \tag{16.31}$$

to calculate $B(3)$, and so forth. Finally, we can use (16.1) to transform these zero coupon bond prices to spot interest rates.

Remark 16.29 *(Numerical calculation of the bootstrap) Equations (16.29)–(16.31) can clearly be written*

$$\begin{bmatrix} P(1) \\ P(2) \\ P(3) \end{bmatrix} = \begin{bmatrix} c(1)+1 & 0 & 0 \\ c(2) & c(2)+1 & 0 \\ c(3) & c(3) & c(3)+1 \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \\ B(3) \end{bmatrix},$$

which is a recursive (triangular) system of equations.

Example 16.30 *(Bootstrapping) Suppose we know that $B(1) = 0.95$ and that the price of a bond with a 6% annual coupon with two years to maturity is 1.01. Since the coupon bond must be priced as*

$$0.95 \times 0.06 + B(2) \times 0.06 + B(2) = 1.01,$$

we can solve for the price of a two-period zero coupon bond as $B(2) \approx 0.90$. The spot interest rates are then $Y(1) \approx 0.053$ and $Y(2) \approx 0.054$. In this case the system of equations is

$$\begin{bmatrix} 0.95 \\ 1.01 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0.06 & 1.06 \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \end{bmatrix}.$$

Unfortunately, the bootstrap approach is tricky to use. First, there are typically gaps between the available maturities. One way around that is to interpolate. Second (and quite the opposite), there may be several bonds with the same maturity but with different coupons/prices, so it is hard to calculate a unique yield curve. This could be solved by forming an average across the different bonds or by simply excluding some data.

16.6.2 Estimating the Yield Curve with Regression Analysis

Recall equation (16.17) which expresses the coupon bond price in terms of a series of discount bond prices. It is reproduced here

$$P = \textstyle\sum_{k=1}^{K} B(m_k)c + B(m_K). \tag{16.32}$$

If we attach some random error to the bond prices, then this looks very similar to regression equation: the coupon bond price is the dependent variable; the coupons are the regressors, and the discount function (discount bond prices) are the coefficients to estimate—perhaps with OLS. This is a way of overcoming the second problem discussed above since multiple bonds with the same maturity, but different coupons, are just additional data points in the estimation.

The first problem mentioned above, gaps in the term structure of available bonds, is harder to deal with. If there are more coupon dates than bonds, then we cannot estimate all the necessary zero coupon bond prices from data (fewer data points than coefficients). The way around this is to decrease the number of coefficients by assuming that the discount function, $B(m)$, is a linear combination of some J predefined functions of maturity, $g_1(m)$,..., $g_J(m)$,

$$B(m) = 1 + \textstyle\sum_{j=1}^{J} a_j g_j(m), \tag{16.33}$$

where $g_j(0) = 0$ since $B(0) = 1$ (the price of a bond maturing today is one).

Once the $g_j(m)$ functions are specified, (16.33) is substituted into (16.17) and the j coefficients a_1,..., a_j are estimated by minimizing the squared pricing error (see, for instance, Campbell, Lo, and MacKinlay (1997) 10).

One possible choice of $g_j(m)$ functions is a polynomial, $g_j(m) = m^j$. Another common choice is to make the discount bond price a spline (see McCulloch (1975)).

Example 16.31 *(Quadratic discount function) With a quadratic discount function*

$$B(m) = a_0 + a_1 m + a_2 m^2,$$

we get from (16.17)

$$\begin{aligned} P(m_K) &= \textstyle\sum_{k=1}^{K} B(m_k)c + B(m_K) \\ &= \textstyle\sum_{k=1}^{K} \left(a_0 + a_1 m_k + a_2 m_k^2\right) c + \left(a_0 + a_1 m_K + a_2 m_K^2\right). \end{aligned}$$

Collect all constants (that does not depend on m) into a first regressor, then all terms that are linear in m into a second regressor and finally all terms that are quadratic in m into a third regressor

$$P(m_K) = a_0 \underbrace{(Kc + 1)}_{\text{term 0}} + a_1 \underbrace{(c \textstyle\sum_{k=1}^{K} m_k + m_K)}_{\text{term 1}} + a_2 \underbrace{(c \textstyle\sum_{k=1}^{K} m_k^2 + m_K^2)}_{\text{term 2}}.$$

For a 1-year bonds that pays no coupons and a 2-year bond that pays a 6% coupons *at $m_1 = 1$ and $m_2 = 2$, we have the following matrix of regressors (the bonds are on different rows)*

Bond ↓	term 0	term 1	term 2
1-year, 0%	1	1	1
2-year, 6%	$2 \times 0.06 + 1 = 1.12$	$0.06 \times (1+2) + 2 = 2.18$	$0.06 \times (1^2 + 2^2) + 2^2 = 4.30.$

The a_0, a_1, and a_2 can be estimated by OLS if we have data on at least two bonds. This method can, however, lead to large errors in the fitted yields (if not the prices). See Figure 16.15 for an example.

Example 16.32 *(Cubic discount function*) With a cubic discount function*

$$B(m) = a_0 + a_1 m + a_2 m^2 + a_3 m^3,$$

we get

$$P(m_K) = a_0 (Kc + 1) + a_1 \left(c \textstyle\sum_{k=1}^{K} m_k + m_K \right) + a_2 \left(c \textstyle\sum_{k=1}^{K} m_k^2 + m_K^2 \right) + a_3 \left(c \textstyle\sum_{k=1}^{K} m_k^3 + m_K^3 \right).$$

16.6.3 Estimating a Parametric Forward Rate Curve*

Yet another approach to estimating the yield curve is to start by specifying a function for the instantaneous forward rate curve, and then calculate what this implies for the discount bond prices (discount function). (These will typically be complicated and not satisfy the simple linear structure in (16.33).)

Let $f(m)$ denote the instantaneous forward rate with time to settlement m. The *extended Nelson and Siegel forward rate function* (Svensson (1995)) is

$$f(m; b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right), \quad (16.34)$$

where $b = (\beta_0, \beta_1, \beta_2, \tau_1, \beta_3, \tau_2)$ is a vector of parameters (β_0, τ_1 and τ_2 must be positive, and $\beta_0 + \beta_1$ must also be positive—see below). The original Nelson and Siegel function sets $\beta_3 = 0$. Note that in either case

$$\lim_{m \to 0} f(m; b) = \beta_0 + \beta_1, \text{ and}$$
$$\lim_{m \to \infty} f(m; b) = \beta_0,$$

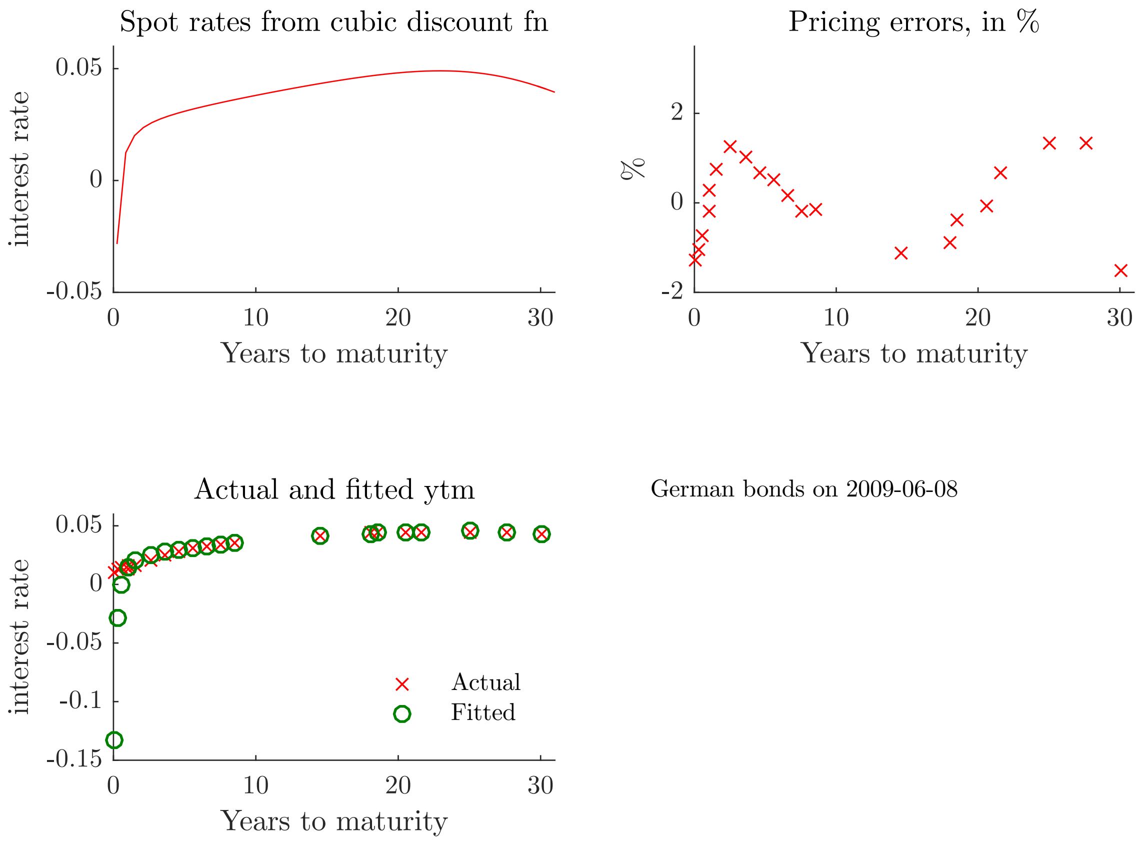


Figure 16.15: Estimated yield curves

so $\beta_0 + \beta_1$ corresponds to the current very short spot interest rate (an overnight rate, say) and β_0 to the forward rate with settlement very far in the future (the asymptote).

The spot rate implied by (16.34) is (integrate as in (16.53) to see that)

$$y(m;b) = \beta_0 + \beta_1 \frac{1 - \exp(-m/\tau_1)}{m/\tau_1} + \beta_2 \left[\frac{1 - \exp(-m/\tau_1)}{m/\tau_1} - \exp\left(-\frac{m}{\tau_1}\right) \right]$$
$$+ \beta_3 \left[\frac{1 - \exp(-m/\tau_2)}{m/\tau_2} - \exp\left(-\frac{m}{\tau_2}\right) \right]. \tag{16.35}$$

One way of estimating the parameters in (16.34) is to substitute (16.35) for the spot rate in (16.4), and then minimize the sum of the squared price errors (differences between actual and fitted prices), perhaps with 1/maturity (or 1/modified duration) as the weight for the squared error (a practice used by many central banks). Alternatively, one could minimize the sum of the squared yield errors (differences between actual and fitted yield to maturity). See Figure 16.16 for an illustration.

Figure 16.16: Estimated yield curves

16.6.4 Par Yield Curve

When many bonds are traded at (approximately) par, the par yield curve (16.23) can be obtained by just plotting the coupon rates. In practice, the yield to maturity is used instead (to partly compensate for the fact that the bonds are only approximately at par)—and the gaps (across maturities) are filled by interpolation. (Recall that for a par bond, the yield to maturity equals the coupon rate.) This is basically the way the Constant Maturity Treasury yield curve, published by the US Treasury, is constructed.

16.6.5 Swap Rate Curve

The swap rates for different maturities can also be used to construct a yield curve.

16.7 Conventions on Important Markets*

16.7.1 Compounding Frequency

Suppose the interest rate r is compounded 2 times per year. This means that the amount B invested at the beginning of the year gives $B \times (1 + r/2)$ after six months—which is reinvested and therefore gives $B \times (1 + r/2)(1 + r/2)$ after another six months (at the end of the year). To make this payoff equal to unity (as we have used as our convention) it must be the case that the bond price $B = 1/(1+r/2)^2$. By comparing with the definition of the effective interest rate (with annual compounding) in (16.1) we have

$$\frac{1}{B} = \left(1 + \frac{r}{2}\right)^2 = 1 + Y, \tag{16.36}$$

where Y is the annual effective interest rate.

This shows how we can transform from semi-annual compounding to annual compounding (and vice versa).

More generally, with compounding n times per year, we have

$$\frac{1}{B} = \left(1 + \frac{r}{n}\right)^n = 1 + Y. \tag{16.37}$$

Clearly, as $n \to \infty$, the expression in (16.37) goes to e^r, where r is the continuously compounded rate.

16.7.2 US Treasury Notes and Bonds

The convention for *US Treasury notes and bonds* (issued with maturities longer than one year) is that coupons are paid semi-annually (as half the quoted coupon rate), and that yields are semi-annual effective yields. (This applies also to most US corporate bonds and UK Treasury bonds.)

However, both are quoted on an annual basis by multiplying by two. The quoted *yield to maturity*, ϕ, solves

$$P = \sum_{k=1}^{K} \frac{c/2}{(1 + \phi/2)^{n_k}} + \frac{1}{(1 + \phi/2)^{n_K}}, \tag{16.38}$$

where the bond pays coupons $c/2$, at $n_1, n_2, ..., n_K$ half-years ahead. By using (16.36), the yield quoted, ϕ, can be expressed in terms of an annual effective interest rate.

Example 16.33 *A 9% US Treasury bond (the coupon rate is 9%, paid out as 4.5% semi-*

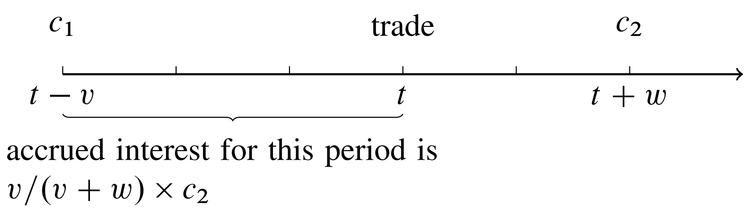


Figure 16.17: Accrued interest

annually) with a yield to maturity of 7%, and one year to maturity has the price

$$\frac{0.09/2}{1+0.07/2} + \frac{0.09/2}{(1+0.07/2)^2} + \frac{1}{(1+0.07/2)^2} = 1.019.$$

From (16.36), we get that the yield to maturity rate expressed as an annual effective interest is $(1+0.035)^2 - 1 \approx 0.071$.

16.7.3 Accrued Interest on Bonds

The quotes of bond prices (as opposed to yields) are not the full price (also called the dirty price, invoice price, or cash price) the investor pays. Instead, the full price is

full price = quoted price + accrued interest.

The buyer of the bond (buying in t) will typically get the next coupon (trading is "cum-dividend"). The accrued interest is the faction of that next coupon that has been accrued during the period the seller owned the bond. It is calculated as

accrued interest = next coupon $\times$ days since last coupon/days between coupons.

For instance, for most US bonds, the next coupon is half the coupon rate and the days between coupons are 182.5. See Figures 16.17–16.18.

16.7.4 US Treasury Bills

Discount Yield

US Treasury bills have no coupons and are issued in 3, 6, 9, and 12 months maturities—but the time to maturity does of course change over time. They are quoted in terms of the

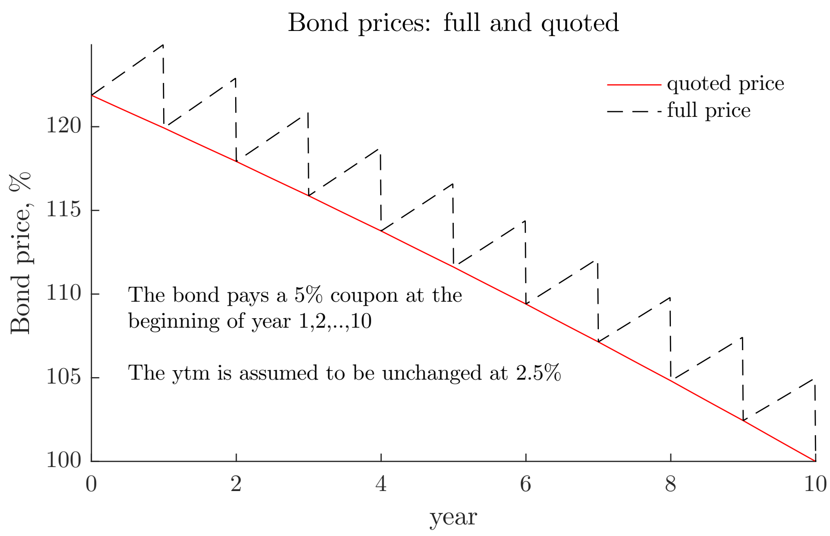


Figure 16.18: Full and quoted bond prices

(banker's) *discount yield*, $Y_{db}(m)$, which satisfies

$$B(m) = 1 - mY_{db}(m), \text{ where } m = \text{days}/360, \text{ so} \tag{16.39}$$

$$Y_{db}(m) = [1 - B(m)]/m. \tag{16.40}$$

Notice the convention of m =days/360. (If the face value is different from one, then we have $Y_{db}(m) = [\text{face} - B(m)]/(\text{face} \times m)$.)

From (16.1) and (16.39) it is the clear that the effective interest rate and the continuously compounded interest rates can be solved as

$$Y(m) = [1 - mY_{db}(m)]^{-1/m} - 1 \tag{16.41}$$

$$y(m) = -\ln[1 - mY_{db}(m)]/m. \tag{16.42}$$

Example 16.34 *A T-bill with 44 days to maturity and a quoted discount yield of 6.21% has the price* $1-(44/360)\times 0.0621 \approx 0.992$. *The effective interest rate is* $[1-(44/360)\times 0.0621]^{-360/44} - 1 \approx 6.43\%$.

16.7.5 LIBOR and EURIBOR

The LIBOR (London Interbank Offer Rate) and the EURIBOR (Euro Interbank Offered Rate) are the simple interest rate on a short term loan without coupons. It is quoted as a simple annual interest rate, using a "actual/360" day count—with the exception of pounds which are quoted "actual/365." This means that borrowing one dollar for 150 days at a 6% LIBOR requires the payment of $0.06 \times 150/360$ dollars in interest at maturity. Rescaling to make the payment at maturity equal to unity (which is the convention used in these lecture notes), the loan must be $1/(1 + 0.06 \times 150/360)$—which is the "price" of a deposit that gives unity in 150 days.

16.7.6 European Bond Markets

The major continental European bond markets (in particular, France and Germany) typically have annual coupons and the accrued interest is calculated according to the "actual/actual" convention, that is, as

$$\text{accrued interest} = \text{next coupon} \times \text{days since last coupon}/365 \text{ (or 366)}.$$

(The computation is slightly more complicated for the UK and the Scandinavian countries, since they have ex-dividend periods.)

16.8 Other Instruments

16.8.1 Collateralized Debt Obligations

CDO is a repackaging of a set of assets ("collaterals," typically bonds) where the claims (payouts) are tranched (have different priorities). See Figure 16.19 for an example.

CDOs are created for two main reasons. First, it is a way for the issuer (typically a bank), to "package and sell off,":that, it is a way to shrink the balance sheet for the bank (securitisation) but still earn a fee. Second, a CDO transforms risky bonds to (a) some safe bonds and (b) some very risky ones. This opens up new possibilities for investors. For instance, it may allow risk averse investors (including pension funds) to invest into the safe tranches, while they would otherwise not dare (or be allowed to) invest into the original bonds.

It is clear that the correlation of the defaults of the bonds in the CDO is important. The idea of tranching (in particular, to regard the senior tranche as safe) depends on the assumption that not all underlying debts default at the same time. Underestimating the correlation can lead to serious overpricing of the senior tranches—as was often the case just before the financial crisis 2008–9.

Another important aspect of the CDO is whether the originator (bank) holds the junior trance or not. If it does, then it has the incentives to screen the borrowers/monitor the

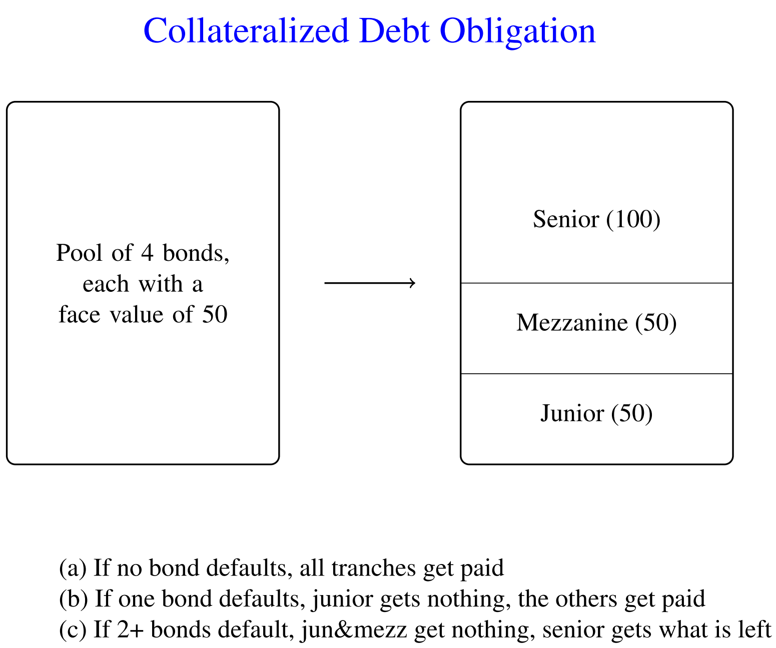


Figure 16.19: Collateralized Debt Obligation

loans, otherwise not.

16.8.2 Credit Default Swaps

A credit default swap is an insurance against default on a bond (eg. , Greece). See Figure 16.20 for an example.

If you hold a portfolio of one risky bond and a CDS on it, then you effectively own a riskfree bond. The other way around is to buy one riskfree bond and issue a CDS, which gives effectively the same as owning the risky bond. This simple observation is the key to understanding how the CSD ("insurance") premium is determined.

16.8.3 Inflation-Indexed Bonds*

Reference: Deacon and Derry (1998)

Consider an inflation-indexed coupon bond issued in t, which has both coupons and principal adjusted for inflation up to the period of payment (this is called "capital indexed," which is the most common type). Let Q_t be the value of the relevant price index (typically a CPI) in period t. The coupon payments are cQ_{t+m_1-l}/Q_{t-l} at $t+m_1$, cQ_{t+m_2-l}/Q_{t-l} at $t+m_2$, and so forth—and also the principal is paid as Q_{t+m_K-l}/Q_{t-l}

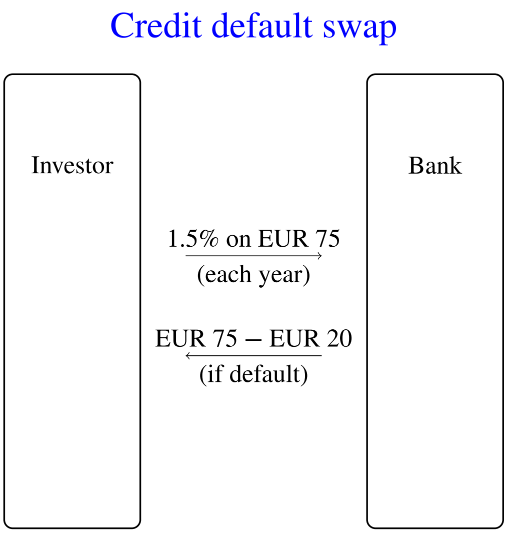


Figure 16.20: Credit default swap

year	Probability of survival through year t	Probability of default in year t	Expected spread payment	Expected payment from insurance	Expected PV of net payment
1	0.98	0.02	$0.98s$	0.02×0.6	$0.98s - 0.012$
2	0.95	0.03	$0.95s$	0.03×0.6	$0.95s - 0.018$
Sum					$1.93s - 0.03$

Table 16.1: Example of the payment flows of a 2-year CDS with an assumed recovery rate of 0.4 and a riskfree interest rate of zero. The CDS spread is denoted s.

in $t + m_K$.

The lag factor l is the *indexation lag*. There are two reasons for this lag. First, the convention on many markets is that the bond price is quoted disregarding accrued interest (clean price). The typical case is as follows. The next coupon payment is m_1 periods ahead. The buyer of the bond in t will get this coupon (trading is "cum-dividend"). The full price the buyer pays to the seller in t is therefore

$$\text{full price} = \text{quoted price} + \text{accrued interest},$$

where the accrued interest is typically the coupon payment times the fraction of this coupon period that has already passed. To pay this accrued interest, we have to know the next coupon payment, that is, cQ_{t+m_1-l}/Q_{t-l}; in t we must know the price level in

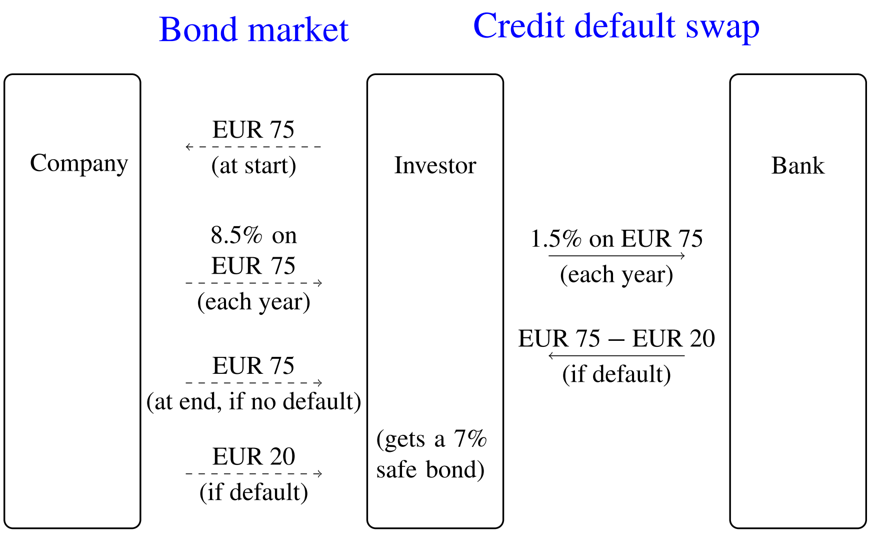


Figure 16.21: Credit default swap

$t + m_1 - \dot{l}$. This mean that $l \geq m_1$ must always hold: the indexation lag must be at least as long as the time between coupon payments (six months in the UK).

Second, it takes time to calculate and publish price indices. Suppose we learn to know Q_s in $s + k$. This means that the indexation lag must be an additional k periods, $l \geq m_1 + k$, so it uses a known price level. For instance, in the UK, the indexation lag is 8 months.

To simplify matters in the rest of this section, suppose the indexation lag is zero. Use (16.17), modified to allow for different coupons, to price the inflation-indexed bond. To further simplify, suppose that bonds do not have any risk premia (clearly a strong assumption), so that the bond price equals the discounted expected payoffs

$$P = \sum_{k=1}^{K} \frac{c \, \mathrm{E}_t \, Q_{t+m_k}/Q_t}{[1+Y(m_k)]^{m_k}} + \frac{\mathrm{E}_t \, Q_{t+m_K}/Q_t}{[1+Y(m_K)]^{m_K}}. \tag{16.43}$$

The Fisher equation is

$$[1+Y(m)]^m = [1+R(m)]^m \frac{\mathrm{E}_t \, Q_{t+m}}{Q_t}, \tag{16.44}$$

where R is the real interest rate. It splits up the gross nominal return in the bond into a gross real return and gross inflation rate. Notice that the Fisher equation assumes that there is no risk premia, which is a strong assumption.

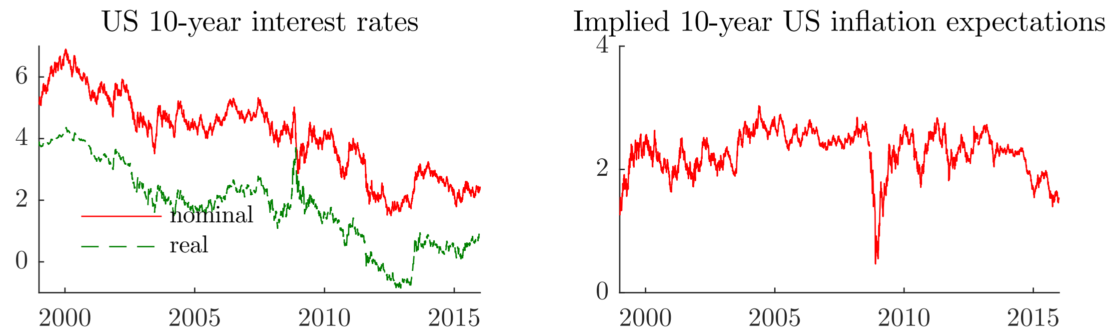


Figure 16.22: US nominal and real interest rates

Use (16.44) to rewrite (16.43) as

$$\begin{aligned}
P &= \sum_{k=1}^{K} \frac{c \, \mathrm{E}_t \, Q_{t+m_k}/Q_t}{[1+R(m_k)]^{m_k} \, \mathrm{E}_t \, Q_{t+m_k}/Q_t} + \frac{\mathrm{E}_t \, Q_{t+m_K}/Q_t}{[1+R(m_K)]^{m_K} \, \mathrm{E}_t \, Q_{t+m_K}/Q_t} \\
&= \sum_{k=1}^{K} \frac{c}{[1+R(m_k)]^{m_k}} + \frac{1}{[1+R(m_K)]^{m_K}} \qquad (16.45)
\end{aligned}$$

With a set of inflation-indexed bonds, we could therefore estimate a *real yield curve*, that is, how $R(m)$ depends on m. If the Fisher equation indeed holds, then the difference between a nominal interest rate and a real interest rate can be interpreted as a measure of the market's inflation expectations (often called the "break-even inflation rate").

16.9 Appendix: More on Forward Rates*

16.9.1 Forward Rate as a Marginal Cost*

Split up the time until n into n/h intervals of length h (see Figure 16.23). Then, the n-period spot rate equals the geometric average of the h-period forward rates over t to $t+ns$

$$\begin{aligned}
1+Y(n) &= [1+\Gamma(0,h)]^{h/n} \times [1+\Gamma(h,2h)]^{h/n} \times \ldots \times [1+\Gamma(n-h,n)]^{h/n} \\
&= \prod\nolimits_{s=0}^{n/m-1} \{1+\Gamma[sh,(s+1)h]\}^{h/n}. \qquad (16.46)
\end{aligned}$$

This means that the forward rate can be seen as the "marginal cost" of making a loan longer. See Figure 16.24 for an illustration.

Proof. (of (16.46)) Let $n = 2m$ and use (16.14) for forward contracts between 0 to m

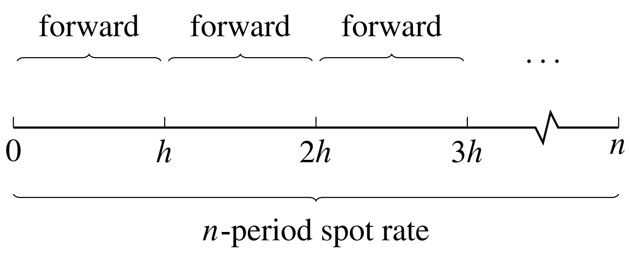


Figure 16.23: Forward contracts for several future periods

and m to $2m$

$$\frac{1}{B(m)/B(0)} = [1 + \Gamma(0,m)]^m \text{ and } \frac{1}{B(2m)/B(m)} = [1 + \Gamma(m,2m)]^m .$$

Multiply and simplify to get

$$\frac{1}{B(n)} = [1 + \Gamma(0,m)]^m \times [1 + \Gamma(m,2m)]^m .$$

Raise to the power of $1/n$ to get the interest rate

$$1 + Y(n) = [1 + \Gamma(0,m)]^{m/n} \times [1 + \Gamma(m,2m)]^{m/n} .$$

■

Example 16.35 *(Spot as average forward rate) In the previous example, (16.46) gives, using $\Gamma(0,1) = Y(1)$,*

$$1.04^{1/2}1.06^{1/2} \approx 1.05,$$

which indeed equals $1 + Y(2)$.

16.9.2 Continuously Compounded and Simple Forward Rates*

Taking logs of $1 + \Gamma(m,n)$ in (16.16) we get the continuously compounded forward rate

$$f(m,n) = \frac{1}{n-m} \ln \frac{B(m)}{B(n)} = \frac{ny(n) - my(m)}{n-m}. \tag{16.47}$$

Conversely, the n-period (continuously compounded) spot rate equals the average (continuously compounded) forward rate (take logs of 16.46)

$$y(m) = \frac{h}{n} \sum\nolimits_{s=0}^{n/h-1} f[sh,(s+1)h]. \tag{16.48}$$

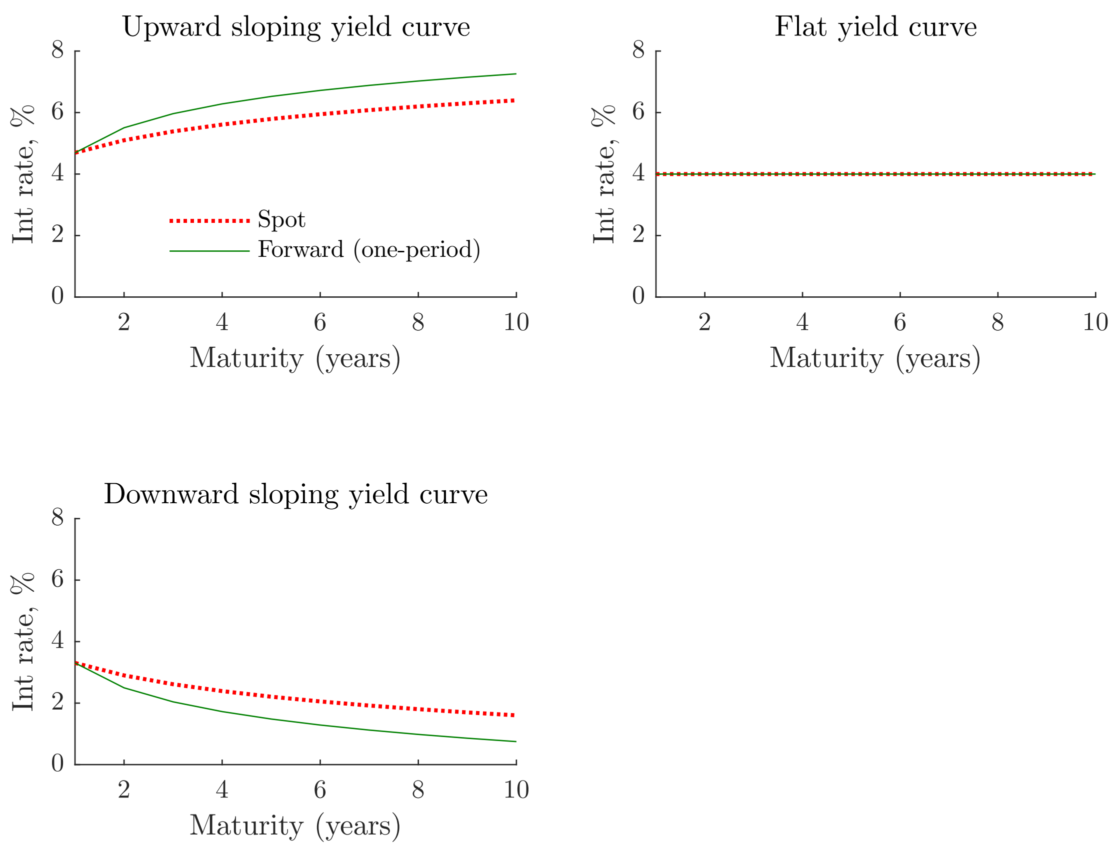


Figure 16.24: Spot and forward rates

A simple forward rate (used on interbank markets) is defined as

$$\frac{1}{B(n)/B(m)} = 1 + (n-m)\tilde{\Gamma}(m,n), \text{ so} \tag{16.49}$$

$$\tilde{\Gamma}(m,n) = \frac{1}{n-m}\left[\frac{B(m)}{B(n)} - 1\right] = \frac{n\tilde{Y}(n) - m\tilde{Y}(m)}{(n-m)[1+m\tilde{Y}(m)]}. \tag{16.50}$$

16.9.3 Instantaneous Forward Rates*

The instantaneous forward rate, $f(m)$, is defined as the limit when the maturity date of the bond approaches the settlement date of the forward contract, $n \to m$. This can be thought of as a forward "overnight" rate m periods ahead in time. From (16.47) it is

$$
\begin{aligned}
f(m) &= \lim_{n\to m} f(m,n) && (16.51)\\
&= \lim_{n\to m} \frac{n-m}{n-m} y(n) - \lim_{n\to m} \frac{m\,[y(m)-y(n)]}{n-m}\\
&= y(m) + m\frac{dy(m)}{dm}. && (16.52)
\end{aligned}
$$

Conversely, the average of the forward rates over t to $t+n$ is the spot rate, which we see by integrating (16.52) to get

$$
y(n) = \frac{1}{n}\int_0^n f(s)ds. \tag{16.53}
$$

Equations (16.52) and (16.53) show that the difference between the forward and spot rates, $f(n) - y(n)$, is proportional to the slope of the yield curve.

Proof. (of (16.53)) Integrating the first term on the right hand side of (16.52) over $[0,n]$ gives $\int_0^n y(s)ds$. Integrating (by parts) the second term on the right hand side of (16.52) over $[0,n]$, $\int_0^n s\frac{dy(s)}{ds}ds$, gives $ny(n) - \int_0^n y(s)ds$. Adding the two terms gives $ny(n)$. ■

16.10 Appendix: More Details on Bond Conventions*

16.10.1 Bond Equivalent Yields on US Bonds

The financial press typically quotes a bond equivalent yield for T-bills—in an attempt to make the yields comparable. The bond equivalent yield is the coupon (and yield to maturity) of a par bond that would give the same yield as the T-bill. For a T-bill with at most half a year to maturity, this gives a simple interest rate, but for longer T-bills the expression is more complicated.

We first analyse *a T-bill with more than half a year to maturity*. Consider a coupon bond with face value B (which equals the current price of the T-bill), semi-annual coupon $c/2$ and the same yield to maturity. Since the coupon and the yield to maturity are the same, the "clean price" of the bond (the price to pay if the seller gets to keep the accrued interest on the first coupon payment) equals the face value (here B): it is traded at par. Notice that the latter means that the buyer gets the following fraction of the next coupon payment (which is $B \times c/2$): the fraction of a half year until the next coupon payment (or (days to next coupon)$\times 2/365$).

When the T-bill has more than half a year to maturity, then the bond has two coupon payments left (including the maturity). At maturity, the owner will have the following: *(i)* the principal plus final coupon, $B \times (1+c/2)$; *(ii)* the part of the first coupon that belongs to the current owner, $d = B \times 2n \times c/2$, where n =(days to next coupon)/365; and *(iii)* the interest on d when reinvested at the semi-annual rate $c/2$ for half a year, $d \times c/2$.

To get the same return as on the T-bill, the owner of the coupon bond must get a value of one at maturity (the return is then $1/B$), or

$$1 = B \times [1 + c/2 + 2n \times c/2 \times (1 + c/2)]. \tag{16.54}$$

Solving for c gives the bond equivalent yield

$$c = \frac{\sqrt{2n/B + 1/4 - n + n^2} - n - 1/2}{n}. \tag{16.55}$$

Example 16.36 *A T-bill with 212 days to maturity and a quoted discount yield of 5.9% has the price* $1-(212/360)\times 0.059 \approx 0.965$. *There must be* $212-182 = 30$ *days to the next coupon payment, so* $n = 30/365$. *The bond equivalent yield is the* c *such that*

$$c = \frac{\sqrt{2\,(30/365)\,/0.965 + 1/4 - (30/365) + (30/365)^2} - (30/365) - 1/2}{(30/365)} \approx 6.2\%$$

Remark 16.37 *If we define* $h =$ *(days to maturity)*$/365$, *then* $n = h - 1/2$ *and we can rearrange (16.55) as*

$$c = \frac{2\sqrt{h^2 + (2h-1)\,(1/B - 1)} - 2h}{2h - 1}.$$

This is the expression in McDonald (2014) Appendix 9.A and Blake (1990) 4.2.

We now apply the same logic to a *T-bill with at most half a year to maturity*. The bond then only has the final coupon left (which is split with the previous owner), and the face value (which is not split). In particular, there is no reinvestment. In this case, (16.54) simplifies to

$$1 = B \times (1 + 2n \times c/2). \tag{16.56}$$

Solving for c (and using the fact that $n = h =$ (days to maturity)$/365$) gives

$$c = \frac{1/B - 1}{h} \text{ or} \tag{16.57}$$

$$B = \frac{1}{1 + h \times c}. \tag{16.58}$$

Example 16.38 *A T-bill with 44 days to maturity and a quoted discount yield of 6.21% has the price* $1 - (44/360) \times 0.0621 \approx 0.992$*. The bond equivalent yield is the* c *such that*

$$0.992 = \frac{1}{1 + \frac{44}{360}c} \text{ or } c = 6.6\%.$$

Remark 16.39 *There are two other, but equivalent, expressions for the bond equivalent yield for maturities of at most half a year (see, for instance, McDonald (2014) Appendix 9.A). The first is*

$$c_1 = \frac{1 - B}{B}\frac{1}{m}.$$

Substituting for B *using (16.58) shows that* $c_1 = c$*. The second is*

$$c_2 = \frac{365 \times Y_{db}}{360 - Y_{db} \times days}.$$

Substituting for Y_{db} *using (16.40) shows that* $c_2 = c_1 = c$*.*

16.10.2 Calculating the Yield to Maturity

Example 16.40 *(Par bond) A 9% (annual coupons) 2-year bond with a yield to maturity of 9%, and exactly two years to maturity has the price*

$$\frac{0.09}{1 + 0.09} + \frac{0.09}{(1 + 0.09)^2} + \frac{1}{(1 + 0.09)^2} = 1.$$

Remark 16.41 *(Newton-Raphson algorithm for solving (16.20)) It is straightforward to use a Newton-Raphson algorithm to solve (16.20). It is then useful to note that the derivative is*

$$\frac{dP(\theta)}{d\theta} = -\textstyle\sum_{k=1}^{K} \frac{m_k c}{(1+\theta)^{m_k+1}} - \frac{m_K}{(1+\theta)^{m_K+1}}.$$

Notice that $P(\theta)$ *is just a way of indicating that the bond price depends on the ytm* θ*. The Newton-Raphson algorithm is based on a first order Taylor expansion of the bond price equation*

$$P(\theta_1) = P(\theta_0) + \frac{dP(\theta_0)}{d\theta}(\theta_1 - \theta_0).$$

Set the left hand side equal to the observed price, P*, guess a value of* θ *and call it* θ_0*; then solve for* θ_1 *as* $\theta_1 = \theta_0 + [P - P(\theta_0)]/\frac{dP(\theta_0)}{d\theta}$*.* θ_1 *is probably a better guess of* θ *than* θ_0*. Improve by repeating this updating as* $\theta_2 = \theta_1 + [P - P(\theta_1)]/\frac{dP(\theta_1)}{d\theta}$*, and so* *forth until θ_n converges. A good starting value is*

$$\theta_0 = \frac{c + (\text{face value} - P)/m_K}{(\text{face value} + P)/2}.$$

Remark 16.42 *(Bisection method for solving (16.20)) The bisection method is a very simple (no derivatives are needed) and robust way to solve for the yield to maturity. First, start with a lower (θ_L) and higher (θ_H) guess of the yield which are known to bracket the true value, that is, $P(\theta_H) \leq B \leq P(\theta_L)$ where P is the observed bond price and $P(\theta)$ is the value according to (16.20). Recall that $P(\theta)$ is decreasing in θ. Second, calculate the bond price at the average of the two guesses: $P[(\theta_L + \theta_H)/2]$. Third, replace either θ_L or θ_H according to: if $P[(\theta_L + \theta_H)/2] \geq P$ (so the midpoint $(\theta_L + \theta_H)/2$ is below the true yield) then replace θ_L by $(\theta_L + \theta_H)/2$ (a higher value), but if $P[(\theta_L + \theta_H)/2] < P$ then replace θ_H by $(\theta_L + \theta_H)/2$ (a lower value). Fourth, iterate until $\theta_L \approx \theta_H$.*

Example 16.43 *(Bisection method). The first couple of iterations for a 2-year bond with a 4% coupon and a price of 1.019 are (see also Figure 16.25)*

Iteration	θ_L	θ_H	$(\theta_L + \theta_H)/2$	$B[(\theta_L + \theta_H)/2]$
1	0	0.05	0.0250	1.0289
2	0.025	0.05	0.0375	1.0047
3	0.025	0.0375	0.03125	1.0167
4	0.025	0.03125	0.028125	1.0228
5	0.028125	0.03125	0.029687	1.0197

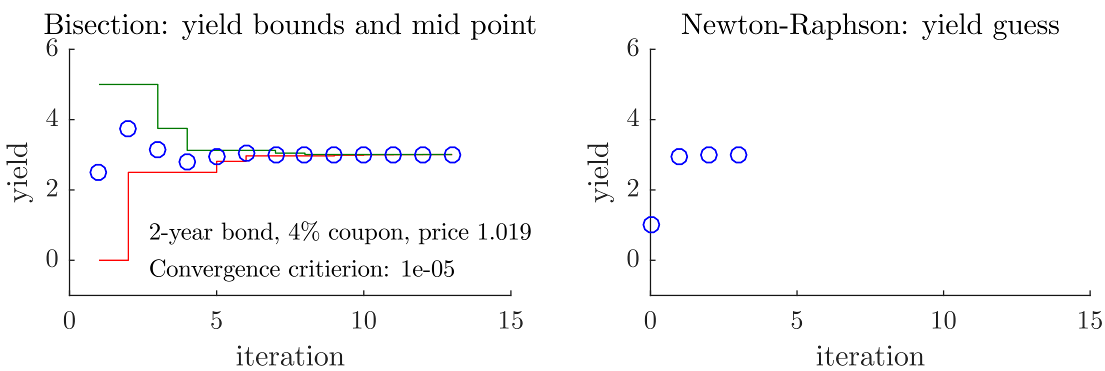


Figure 16.25: Bisection method to calculate yield to maturity

Remark 16.44 *(Calculating θ in a simple special case) If m_k in (16.20) is the integer k, then subtracting B from both sides of (16.20) gives a K^{th} order polynomial in $\Theta = 1/(1+\theta)$,*

$$0 = -P + \sum_{k=1}^{K-1} c\Theta^k + (c+1)\,\Theta^K,$$

where all coefficients except one are positive. There is then only one positive real root, Θ_1. Many software packages contain routines for finding roots of polynomials. Once that is done, pick the only positive real root, Θ_1, and calculate the yield as $\theta = (1-\Theta_1)/\Theta_1$.

Remark 16.45 *(Calculating θ in the simplest special case) If the bond price, P, is unity, then the bond is sold "at par." If also m_k in (16.20) is the integer k (as in the previous remark), then $\theta = c$.*

Chapter 17

Bond Portfolios and Hedging

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 21–22 and Hull (2009) 4
Additional references: McDonald (2014) 9

17.1 Bond Hedging

Suppose we want to hedge against price movements of a bond portfolio or a liability stream. (This is also called immunization.) See Figure 17.1 for a motivation.

The basic idea is to form a total portfolio of both that bond portfolio/liability stream and some other bonds that help us to make the overall portfolio "immune" to changes in the overall interest rate level. To simplify, the analysis is focused on changes over a short time period (so the holding return only depends on changes of the bond prices, not how coupons are reinvested) and we often make strong assumptions about how the yield curve changes (for instance, only parallel movements).

Example 17.1 *(Why a liability is not hedged by putting its present value on a bank account) Suppose your liability stream is that you owe 150 next year and 250 the year after that. Suppose all interest rates are 5%. The present value of your liability stream is*

$$\frac{150}{1.05} + \frac{250}{1.05^2} = 369.615.$$

Put 369.615 in a bank account. Next day, the interest rate decreases to 4%. The present value (market value) of the liability stream is

$$\frac{150}{1.04} + \frac{250}{1.04^2} = 375.37,$$

which is more than you put away in the bank (369.615).

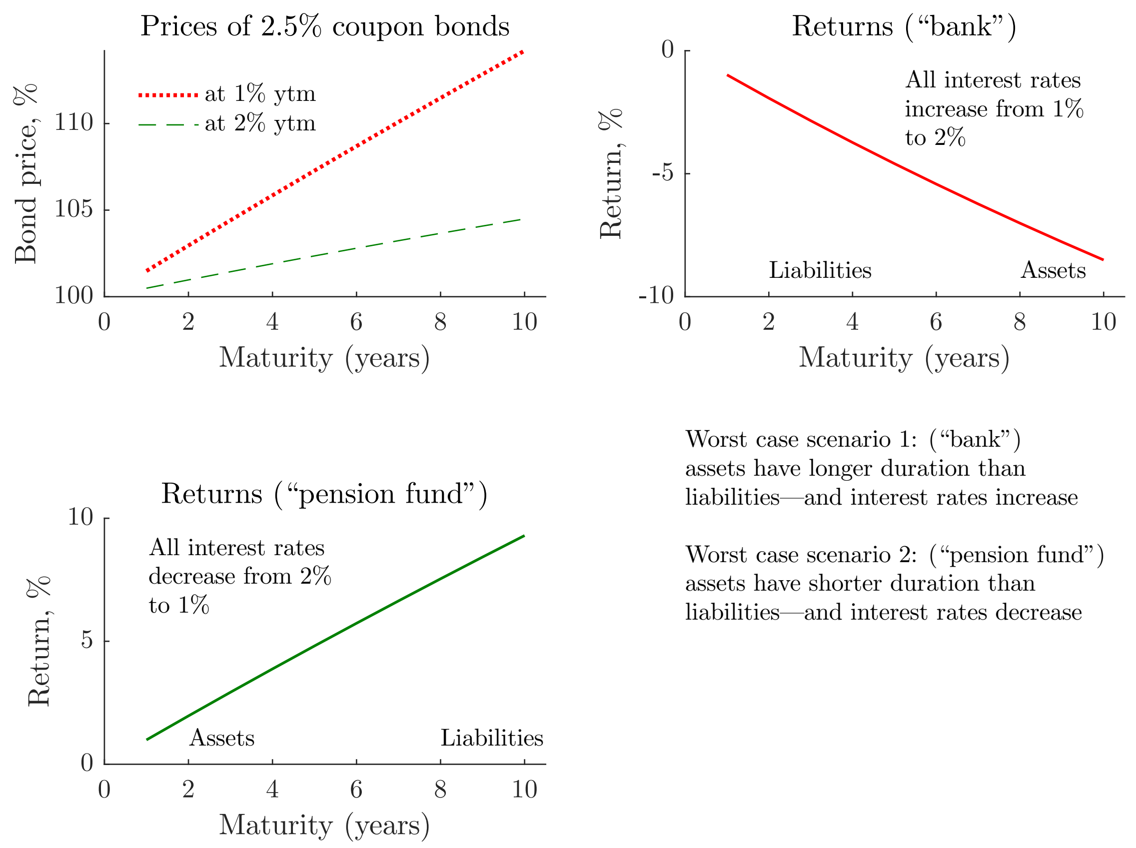


Figure 17.1: Gains and losses at interest rate changes

Example 17.2 *(continued) Another perspective on the same problem. With* 369.615 *in the bank and 4% interest rate, you have*

$$369.615 \times 1.04 = 384.4 \text{ at end of year 1.}$$

Use 150 to cover the first payment—leaving 234.4. After another year (still 4% rate) you have

$$234.4 \times 1.04 = 243.776 \text{ at end of year 2.}$$

Not enough to cover the 250. See Table 17.1.

17.2 Duration: Definitions

The "duration" of a coupon bond is used to analyse how the bond price will change in response to changes in the yield curve. This section gives the definitions of the most

	5% interest rate		5% → 4% interest rate on day 1			
	bank account		bank account		3.9962 1.6-year bonds	
	Flow	Balance	Flow	Balance	Flow	Balance
Day 0	369.615	369.615	369.615	369.615	369.615	369.615
Day 1		369.615		369.615		375.313
Year 1	18.481 − 150	238.096	14.785 − 150	234.400	15.012 − 150	240.326
Year 2	11.905 − 250	0.001	9.376 − 250	−6.224	9.613 − 250	−0.061

Table 17.1: Effect of interest rate decrease on the possibility to finance a liability of 150 after one year and 250 after two years. With the 1.6-year bonds, it is assumed that we sell them after the interest rate decrease and put the balance on a bank account.

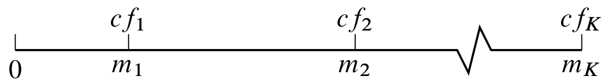


Figure 17.2: Timing convention of bond portfolio

commonly used duration measures.

Consider a bond portfolio with the cash flows cf_k as illustrated in Figure 17.2. Recall that the price P and the yield to maturity θ are related according to

$$P = \sum_{k=1}^{K} \frac{cf_k}{[1+Y(m_k)]^{m_k}} = \sum_{k=1}^{K} \frac{cf_k}{\exp[m_k y(m_k)]} \tag{17.1}$$

$$= \sum_{k=1}^{K} \frac{cf_k}{(1+\theta)^{m_k}}. \tag{17.2}$$

The derivative of the price with respect to its yield to maturity is

$$\frac{dP(\theta)}{d\theta} = -\frac{1}{1+\theta} \sum_{k=1}^{K} m_k \frac{cf_k}{(1+\theta)^{m_k}}. \tag{17.3}$$

This measures the sensitivity of the price to a small change in the yield to maturity. The *dollar duration*, $D^{\$}$, is typically defined as this derivative times minus one

$$D^{\$} = -\frac{dP(\theta)}{d\theta} \tag{17.4}$$

$$= \frac{1}{1+\theta} \sum_{k=1}^{K} m_k \frac{cf_k}{(1+\theta)^{m_k}}. \tag{17.5}$$

The change of the price, ΔP, due to a small change in the yield, $\Delta\theta$, is approximately

$$\Delta P \approx \frac{dP(\theta)}{d\theta} \times \Delta\theta \tag{17.6}$$

$$= -D^{\$} \times \Delta\theta \tag{17.7}$$

This says that an increase in the interest rate (more precisely, the yield to maturity, θ) translates into a decrease in the bond price—and more so if the duration ($D^{\$}$) is long.

Example 17.3 *(Price of discount bond: exact and approximation) Consider a 5-year zero coupon bond with an effective interest rate of 2% or 3%. Its price at the two different interest rates would be*

$$\frac{1}{(1+0.02)^5} = 0.9057$$
$$\frac{1}{(1+0.03)^5} = 0.8626.$$

If the interest rate changes from 2% to 3%, the price change is $0.8626 - 0.9057 = -0.043$. *The dollar duration at 2% is*

$$D^{\$} = \frac{5}{1+0.02} \times 0.9057 \approx 4.44,$$

so the approximate price change according to (17.7) is

$$-4.44 \times (0.03 - 0.02) = -0.044,$$

which is close to the exact number.

It is common to divide the dollar duration by the price, P, to get the *adjusted (or modified) duration*, D^a,

$$D^a = D^{\$}/P. \tag{17.8}$$

By dividing both sides of (17.7) by the bond price and using the definition of the adjusted duration we see that the relative (percentage) change of the bond price due to a small change in the yield is approximately

$$\frac{\Delta P}{P} \approx -D^a \times \Delta\theta \tag{17.9}$$

It is also common to multiply the dollar duration by $(1+\theta)/P$ to get *Macaulay's duration*, D^M,

$$\begin{aligned} D^M &= D^{\$}(1+\theta)/P & (17.10)\\ &= \sum_{k=1}^{K} m_k \frac{cf_k}{(1+\theta)^{m_k} P}. & (17.11) \end{aligned}$$

By multiplying both sides of (17.7) by $(1+\theta)/P$ and using the definition of Macaulay's duration we see that the relative (percentage) change of the bond price due to a small relative (percentage) change in the yield is approximately

$$\frac{\Delta P}{P} \approx -D^M \times \frac{\Delta\theta}{1+\theta}. \tag{17.12}$$

The term last term, $\Delta\theta/(1+\theta)$, is the relative change in the gross yield—since $\Delta\theta = \Delta(1+\theta)$.

Example 17.4 *(Relative price change of discount bond) Using the same figures as in Example 17.3, Macaulay's duration is* 5 *(same as the time to maturity). The approximate relative price change according to (17.12) is*

$$-5 \times \frac{0.03-0.02}{1+0.02} = -0.049.$$

In contrast, the exact relative change is $(0.8626-0.9057)/0.9057 = -0.048$.

Notice that Macaulay's duration is a weighted average of the time to the coupon (and face) payments $(m_1, m_2, ..., m_K)$. The weight of m_k is $cf_k/[(1+\theta)^{m_k} P]$, so the weights sum to unity and they are clearly the percentage of the bond price accounted for by the respective coupon (or principal) payments. Macaulay's duration is therefore an average "time to payment" of the bond. For instance, for a zero coupon bond, Macaulay's duration is the time to maturity (set $cf = 0$ in (17.11)). For bonds with coupons, Macaulay's duration is less than the time to maturity—and this effect is more pronounced at high coupon rates and at high yields to maturity. This is illustrated in Figure 17.3.

In any case, the different duration measures are fairly similar, so we can typically think of the dollar duration to represent "average time to payment."

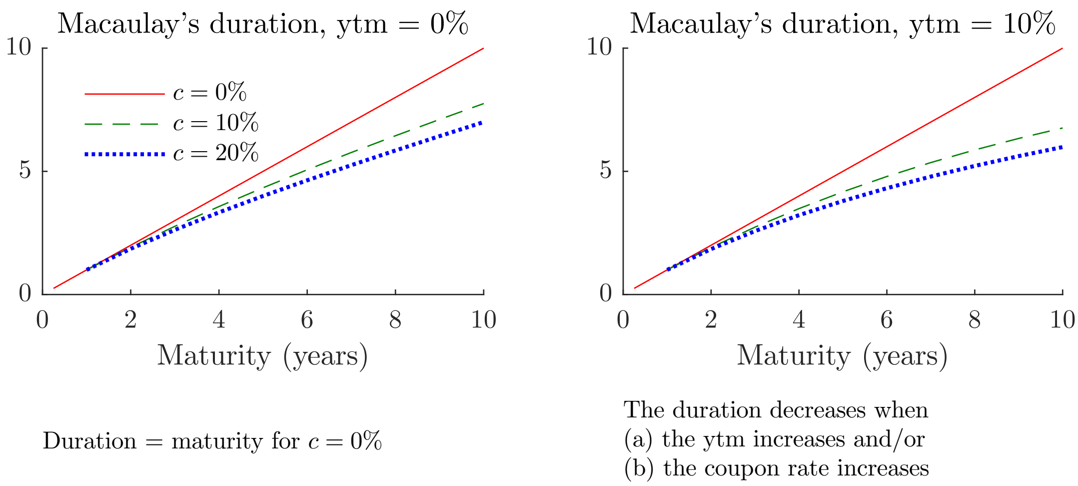


Figure 17.3: Macaulay's duration

Remark 17.5 *(Duration of a zero coupon bond) For a zero-coupon bond with a face value of unity and maturity of K, the price is $B = 1/(1+y)^K$, where y is the yield to maturity. The duration measures are*

$$D^{\$} = \frac{K}{1+y} B$$
$$D^a = \frac{K}{1+y}, \text{ and}$$
$$D^M = K.$$

Example 17.6 *(Duration) Consider a 4% (annual) coupon bond with 2 years to maturity. Suppose the price is 1.019. The the yield to maturity is 3% since it solves*

$$1.019 \approx \frac{0.04}{1+0.03} + \frac{1.04}{(1+0.03)^2}.$$

The dollar duration is

$$D^{\$} = \frac{1}{1.03}\left[\frac{0.04}{1.03} + 2\frac{1.04}{1.03^2}\right] \approx 1.94,$$

so the adjusted duration and Macaulay's duration are

$$D^a = 1.94\frac{1}{1.019} \approx 1.90$$
$$D^M = 1.94\frac{1.03}{1.019} \approx 1.96.$$

Example 17.7 *(Duration of a zero coupon bond) A two-period zero coupon bond with price 0.94 has a ytm equal to 0.03, since*

$$0.94 \approx \frac{1}{1.03^2}.$$

The dollar duration is

$$\frac{1}{1.03} 2 \frac{1}{1.03^2} \approx 1.83,$$

and Macaulay's duration is

$$\frac{1.03}{1/1.03^2} \times \frac{1}{1.03} 2 \frac{1}{1.03^2} = 2.$$

The duration of a portfolio of bonds is easily calculated from the durations of the individual bonds—at least if the bonds have the same ytm. This is summarised in the next proposition.

Proposition 17.8 *(Duration of a portfolio) If the yield to maturities of bond i and j (with prices denoted by P_i and P_j) are the same, then a portfolio of both bonds has the dollar duration $D_i^\$ + D_j^\$$ and the Macaulay's duration $P_i/(P_i + P_j)D_i^M + P_j/(P_i + P_j)D_j^M$ (the value weighted average of the different Macaulay's durations). If the ytms are different, this does not hold.*

Proof. (Duration of a portfolio*) The first part of the proposition is intuitive since the dollar duration of a coupon bond is considered "correct"—and it uses the same ytm for all the coupons. For the second part of the proposition, multiply the dollar duration $D_i^\$ + D_i^\$$ by the ytm and divide by the portfolio value $(P_i + P_j)$. This is Macaulay's duration of the portfolio. Now, rewrite by using $D^\$ = PD^M/(1 + \theta)$ to get the result in the proposition. ■

Example 17.9 *(Duration of a portfolio, same ytm*) A 1-year discount bond with a ytm (effective interest rate) of 10% has the price* $1/1.1$ *and a 3-year discount bond with a ytm of 10% has the price* $1/1.1^3$*.The dollar duration and Macaulay's durations are*

$$\text{1-year bond: } D^\$ = \frac{1}{1.1^2} \approx 0.83 \text{ and } D^M = 1$$
$$\text{3-year bond: } D^\$ = \frac{3}{1.1^4} \approx 2.05 \text{ and } D^M = 3.$$

A portfolio with one of each bond has a price $P = 1/1.1 + 1/1.1^3 \approx 1.66$ *and a ytm*

$$P = \frac{1}{1+\theta} + \frac{1}{(1+\theta)^3}, \text{ with } \theta = 0.1.$$

The duration and Macaulay's duration of the portfolio are then

$$\begin{aligned} D^{\$} &= \frac{1}{1.1}\left[\frac{1}{1.1} + 3\frac{1}{1.1^3}\right] \approx 2.88, \\ D^{M} &= D^{\$}\frac{1.1}{P} \approx 1.90. \end{aligned}$$

Compare with

$$\begin{aligned} &0.83 + 2.05 \approx 2.88 \text{ and} \\ &\frac{1/1.1}{P}1 + \frac{1/1.1^3}{P}3 \approx 1.90, \end{aligned}$$

which are the same.

Example 17.10 *(Duration of a portfolio, different ytm*) A 1-year discount bond with a ytm (effective interest rate) of 7% has the price* $1/1.07$ *and a 3-year discount bond with a ytm of 10% has the price* $1/1.1^3$*.The dollar duration and Macaulay's durations are*

$$\begin{aligned} \text{1-year bond: } D^{\$} &= \frac{1}{1.07^2} \approx 0.87 \text{ and } D^{M} = 1 \\ \text{3-year bond: } D^{\$} &= \frac{3}{1.1^4} \approx 2.05 \text{ and } D^{M} = 3. \end{aligned}$$

A portfolio with one of each bond has a price $P = 1/1.07 + 1/1.1^3 \approx 1.69$ *and a ytm*

$$P = \frac{1}{1+\theta} + \frac{1}{(1+\theta)^3}, \text{ with } \theta \approx 0.091.$$

The duration and Macaulay's duration of the portfolio are then

$$\begin{aligned} D^{\$} &= \frac{1}{1.091}\left[\frac{1}{1.091} + 3\frac{1}{1.091^3}\right] \approx 2.96, \\ D^{M} &= D^{\$}\frac{1.091}{P} \approx 1.91. \end{aligned}$$

Compare with

$$\begin{aligned} &0.87 + 2.05 \approx 2.92 \text{ and} \\ &\frac{1/1.07}{P}1 + \frac{1/1.1^3}{P}3 \approx 1.89, \end{aligned}$$

which are slightly different.

17.3 Using Duration to Improve the Hedging of a Bond Portfolio

17.3.1 Basic Setup

Suppose we want to hedge against price movements of a bond portfolio or a liability stream. (This is also called immunisation.) The portfolio can be thought of as a coupon bond (with a possibly complicated set of coupons), so the previous formulas apply.

One way of hedging is to use a (potentially large) set of forward contracts (or a swap)—to *match every cash flow* of the bond, but that may well be both difficult and costly (transaction costs). Another way, *duration matching* is the other extreme: finding a single instrument to use in the hedging.

A liability is the same as being short one unit of a bond (portfolio) with price P_L and Macaulay's duration D_L^M. We will hedge this portfolio by buying h units of a bond portfolio (denoted H) with price P_H and Macaulay's duration D_H^M. The value of the overall position is then

$$V = hP_H + M - P_L, \tag{17.13}$$

where M is a short-term money market account which has an almost zero duration. In a first step, we choose which bond portfolio to use as bond H. Choosing a bond portfolio with a duration similar to the liability is typically a good idea. In a second step, we find h so that hP_H and P_L are equally sensitive to the main risk: changes in the general interest rate level.

The choice of M is typically such that the initial (on the first day of the hedging) value of V is zero. The subsequent value of the money market account will change as the interest rates are collected/paid and the gains and losses of the bonds accumulate.

Remark 17.11 *(Overall portfolio value over several subperiods*) Start by creating a portfolio with a zero initial value*

$$0 = h_t P_{H,t} + M_t - P_{L,t}, \text{ so } M_t = 0 - h_t P_{H,t} + P_{L,t},$$

where M_t is the amount held in a money market account (almost zero duration) with an interest rate y_t. In $t+s$ (say, after a day so $s = 1/365$), this portfolio is worth

$$V_{t+s} = h_t(P_{H,t+s} + cf_{H,t+s}) + M_t(1+y_t)^s - (P_{L,t+s} + cf_{L,t+s}),$$

where $cf_{H,t+s}$ and $cf_{L,t+s}$ are the coupon payments (or any other cash flows) and the bond prices are measured after coupons. After rebalancing in $t+s$, we need h_{t+s} units of *bond H and we are still short one bond L, so the balance on the money market account is*

$$M_{t+s} = V_{t+s} - h_{t+s} P_{H,t+s} + P_{L,t+s}.$$

This is similar to the expression for M_t in the first equation, except that V_{t+s} may be non-zero. The value of the portfolio in $t+2s$ is computed as in the second equation, but with subscripts advanced one period.

Using the approximate relation of the bond price change (17.7) we have that the change of value (due to a sudden change in the interest rates) of the overall position is

$$\Delta V = h\Delta P_H - \Delta P_L \tag{17.14}$$

$$\approx -hD_H^M P_H \times \frac{\Delta\theta_H}{1+\theta_H} + D_L^M P_L \times \frac{\Delta\theta_L}{1+\theta_L}, \tag{17.15}$$

where the durations are Macaulay's duration.

Several of the hedging approaches (discussed below) assume that $\Delta\theta_H = \Delta\theta_L$, that is, a parallel shift of the yield curve. The weakness of that assumption is also discussed below.

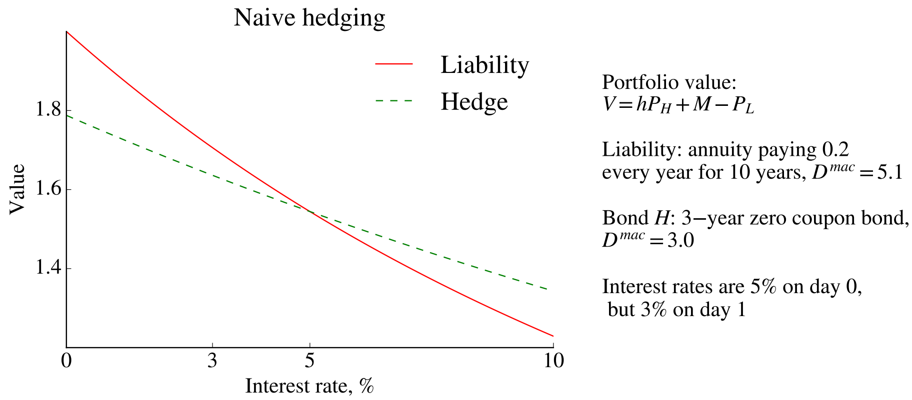
17.3.2 Naive Hedging (Flat Yield Curve)

Suppose we just set $h = P_L/P_H$, so the amount invested in bond H equals the value of our liability. Using (17.15) and assuming a flat yield curve that shifts up/down ($\theta_H = \theta_L$ both before and after the change) gives the change in hedge portfolio value, expressed as a fraction of the value of the liability, as

$$\frac{\Delta V}{P_L} \approx \left(D_L^M - D_H^M\right) \times \frac{\Delta\theta}{1+\theta}. \tag{17.16}$$

For instance, suppose interest rates decrease ($\Delta\theta < 0$) and the duration of the liability is longer than of the hedge bond ($D_L^M > D_H^M$). Then, the portfolio will lose money. See Figure 17.4 for an example. In that figure, "m-to-m" stands for the marking-to-market stage and "rebalancing" for the stage after rebalancing the portfolio.

The reason is that the value of the liability goes up more than the value of the hedge bond—as longer bonds are more sensitive to interest rate changes than short bonds. To overcome this problem, we need to invest more than proportionally into the hedge bond. See Figure 17.1 (lower-left).



	y	P_H	h	hP_H	M	$-P_L$	V_0	hP_H/P_L
Day 0	0.05	0.86	1.79	1.54	0.00	-1.54	0.00	1.00
Day 1 (m-to-m)	0.03	0.92	1.79	1.64	0.00	-1.71	-0.07	0.96
Day 1 (rebalance)	0.03	0.92	1.86	1.71	-0.07	-1.71	-0.07	1.00

Figure 17.4: Example of naive hedging

Example 17.12 *(Naive hedging) The liability in Example 17.1 has* $D_L^M = 1.6$*. If we use a hedge bond with* $D_H^M = 0.5$*, then (17.16) gives*

$$\frac{\Delta V}{P_L} \approx (1.6 - 0.5) \times \frac{-0.01}{1 + 0.05} = -0.01,$$

so the loss is around 1% of the value of the liability. The problem is that the hedging bond has a short duration—so it does not increase much in value when the interest rate decreases.

Remark 17.13 *(Effect of yield curve shift on a bank) A bank typically has liabilities with short duration (deposits, inter-bank lending) and assets with long duration (loans to companies and households), so* $D_L^M < D_H^M$*. Equation (17.16) shows that an increase in the interest rate level will hurt the bank: the assets decrease more than the liabilities. This can also be phrased as follows: the bank has fixed incomes from the loans it has made, but it now needs to refinance itself (deposits and inter-bank loans) at a higher cost.*

17.3.3 Duration Hedging (Parallel Shift of the Yield Curve)

Suppose we instead choose

$$h = \frac{D_L^M}{D_H^M} \times \frac{P_L}{P_H}, \text{ so} \tag{17.17}$$

$$\frac{hP_H}{P_L} = \frac{D_L^M}{D_H^M}, \tag{17.18}$$

where the second equation shows that the *amount* (dollars) invested into the hedge bond (hP_H) exceeds the value of the liability if the liability has a longer duration than the hedge bond. In this way, we compensate for the hedge bonds lower interest rate sensitivity by having a larger exposure to the hedge bond (than to the liability).

Combine (17.15) and the hedge ratio (17.17) to get that the relative change in the portfolio value is approximately

$$\frac{\Delta V}{P_L} \approx D_L^M \times \left(\frac{\Delta\theta_L}{1+\theta_L} - \frac{\Delta\theta_H}{1+\theta_H} \right). \tag{17.19}$$

Suppose the yield curve shifts up in a parallel fashion (so $\Delta\theta_L/(1+\theta_L) = \Delta\theta_H/(1+\theta_H)$) as in Figure 17.5. Then, (17.19) shows that the overall portfolio value will not change ($\Delta V/P_L \approx 0$). See Figure 17.6 for an example how the duration hedging works.

The intuition is that the price of a bond with long duration is more sensitive to a yield curve shift than the price of a short bond. Therefore, to hedge a bond with a long duration (as bond L) we need to buy more of the bond with a short duration (bond H). See Figure 17.7 to see how prices on long bonds are sensitive to interest rate changes.

See Figures 17.8–17.9 for an empirical illustration.

Remark 17.14 *(Using the dollar duration instead*) Recall that $D^M = D^\$(1+\theta)/P$), so (17.15) can be rewritten as*

$$\Delta V \approx -hD_H^\$ \times \Delta\theta_H + D_L^\$ \times \Delta\theta_L.$$

Set $\Delta V = 0$ to get the hedge ratio

$$h = \frac{D_L^\$}{D_H^\$} \times \frac{\Delta\theta_L}{\Delta\theta_H}.$$

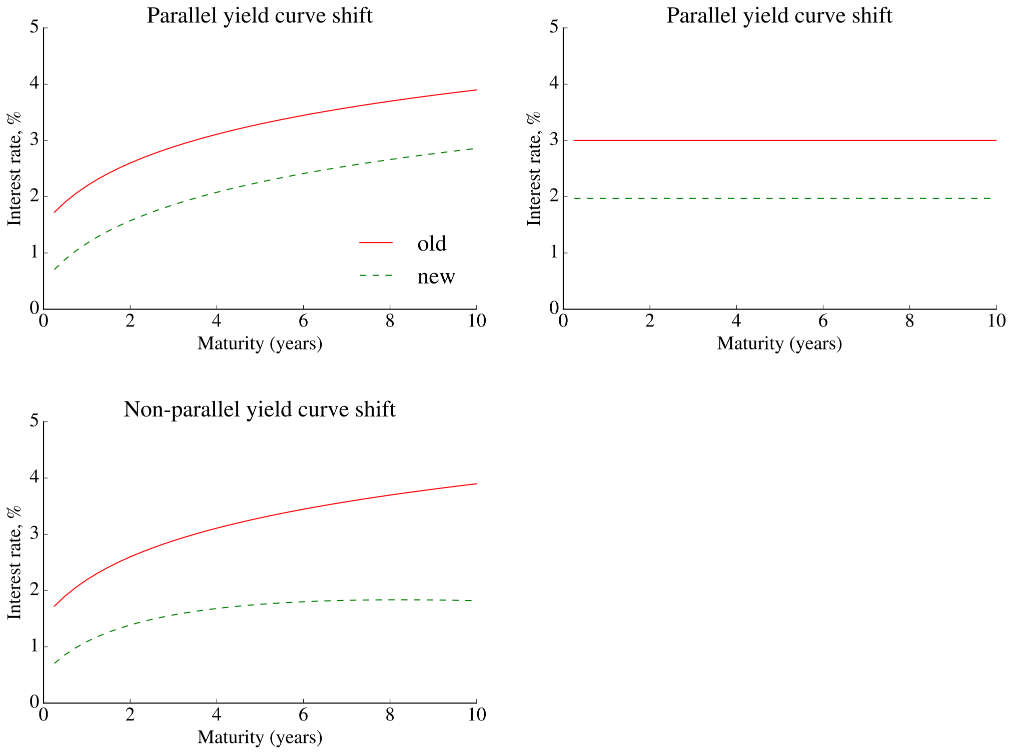


Figure 17.5: Yield curve shifts

If we assume that both yields change equally much, then

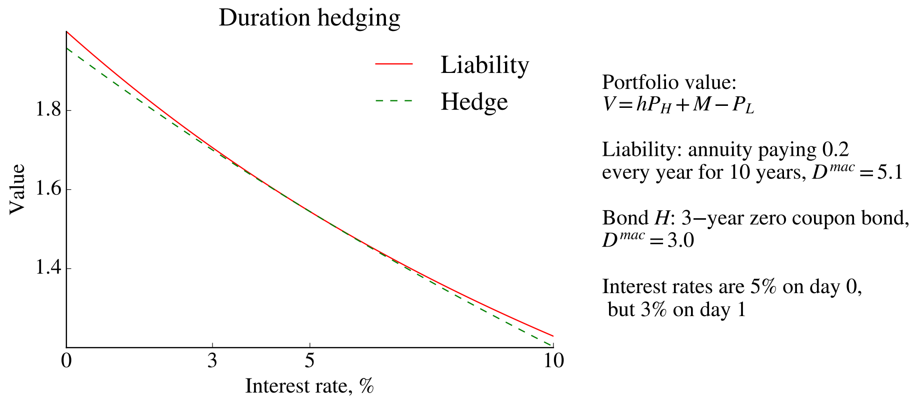
$$h = \frac{D_L^{\$}}{D_H^{\$}}.$$

Example 17.15 *(Duration hedging) The liability in Example 17.1 has* $D_L^M = 1.6$*. If we use a hedge bond with* $D_H^M = 1.6$*, and assume that all interest rates are the same (flat yield curve) then (17.19) gives*

$$\frac{\Delta V}{P_L} \approx 1.61 \times \left(\frac{-0.01}{1+0.05} - \frac{-0.01}{1+0.05} \right) = 0,$$

so the loss is approximately zero.

Example 17.16 *(Duration hedging of the liability in Example 17.1) Instead of putting* 369.615 *on a bank account, buy 1.6-year bonds for this amount—since Macaulay's duration of the liability is 1.6 years. According to (17.18) we should then choose the number of*



	y	P_H	h	hP_H	M	$-P_L$	V_0	hP_H/P_L
Day 0	0.05	0.86	3.04	2.62	-1.08	-1.54	0.00	1.70
Day 1 (m-to-m)	0.03	0.92	3.04	2.78	-1.08	-1.71	-0.01	1.63
Day 1 (rebalance)	0.03	0.92	3.27	2.99	-1.29	-1.71	-0.01	1.75

Figure 17.6: Example of duration hedging

1.6 year bonds bought (h) so that the investment equals the value of the liability (369.615). The price of each bond (with face value 100) is $100/1.05^{1.6}$, so you buy h of them

$$h\frac{100}{1.05^{1.6}} = 369.615 \text{ or } h = 3.9962.$$

The value of this position after the interest rate has changed to 4% is

$$3.9962 \times \frac{100}{1.04^{1.6}} = 375.313,$$

which is almost the same as the PV of the liability stream after the interest rate change. You could therefore sell your bond and put the money in a bank account. It would be enough to pay the liabilities—if there were no further interest changes...

17.4 Problems with Duration Hedging

17.4.1 Problem 1: Approximation Error

The formula for the price change (17.7) is only exact for infinitesimal yield changes—and the approximation error is likely to be large when the yield changes are substantial.

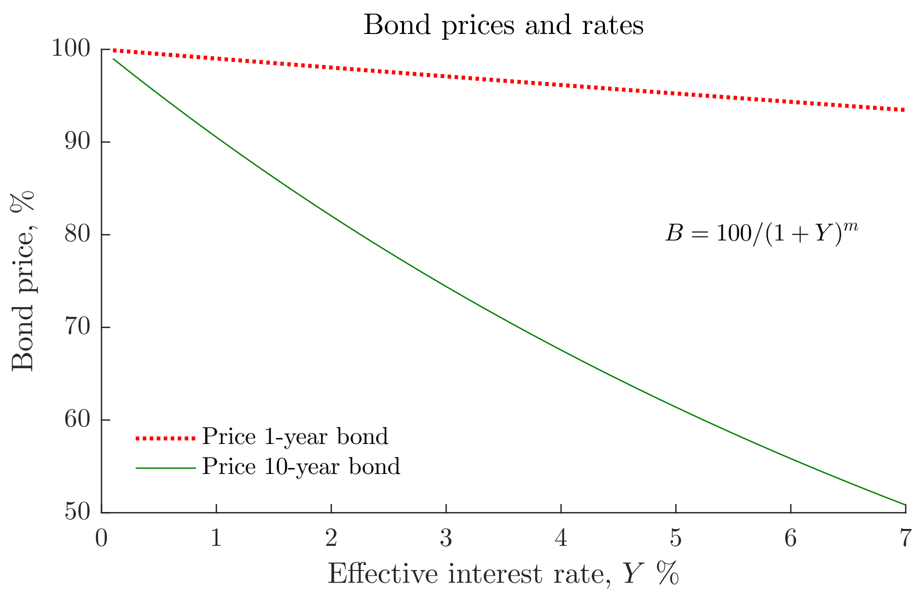


Figure 17.7: Interest rate vs bond prices

The formula is really a first-order Taylor approximation of the form

$$\Delta P \approx \frac{dP}{d\theta} \times \Delta\theta. \tag{17.20}$$

Obviously, a second-order Taylor approximation is more precise. It would be

$$\Delta P \approx \frac{dP}{d\theta} \times \Delta\theta + \frac{1}{2}\frac{d^2 P}{d\theta^2} \times (\Delta\theta)^2. \tag{17.21}$$

where the last term includes the second derivative of the bond price with respect to the yield to maturity. The second derivative is easily calculated to be

$$\frac{d^2 P}{d\theta^2} = \textstyle\sum_{k=1}^{K} m_k(m_k+1)\frac{c_{fk}}{(1+\theta)^{m_k+2}}. \tag{17.22}$$

Dividing (17.21) by the bond price and using (17.9) gives

$$\frac{\Delta P}{P} \approx -D^a \times \Delta\theta + \frac{1}{2}C \times (\Delta\theta)^2, \tag{17.23}$$

where C (often called “convexity”) is the second derivative in (17.21) divided by the bond price. It can be shown that the convexity is positive, but decreasing in the coupon rate — for a given ytm and maturity. (The convexity is actually increasing in the coupon rate for a given ytm and modified duration.) See Figure 17.7 for an illustration of the non-linear

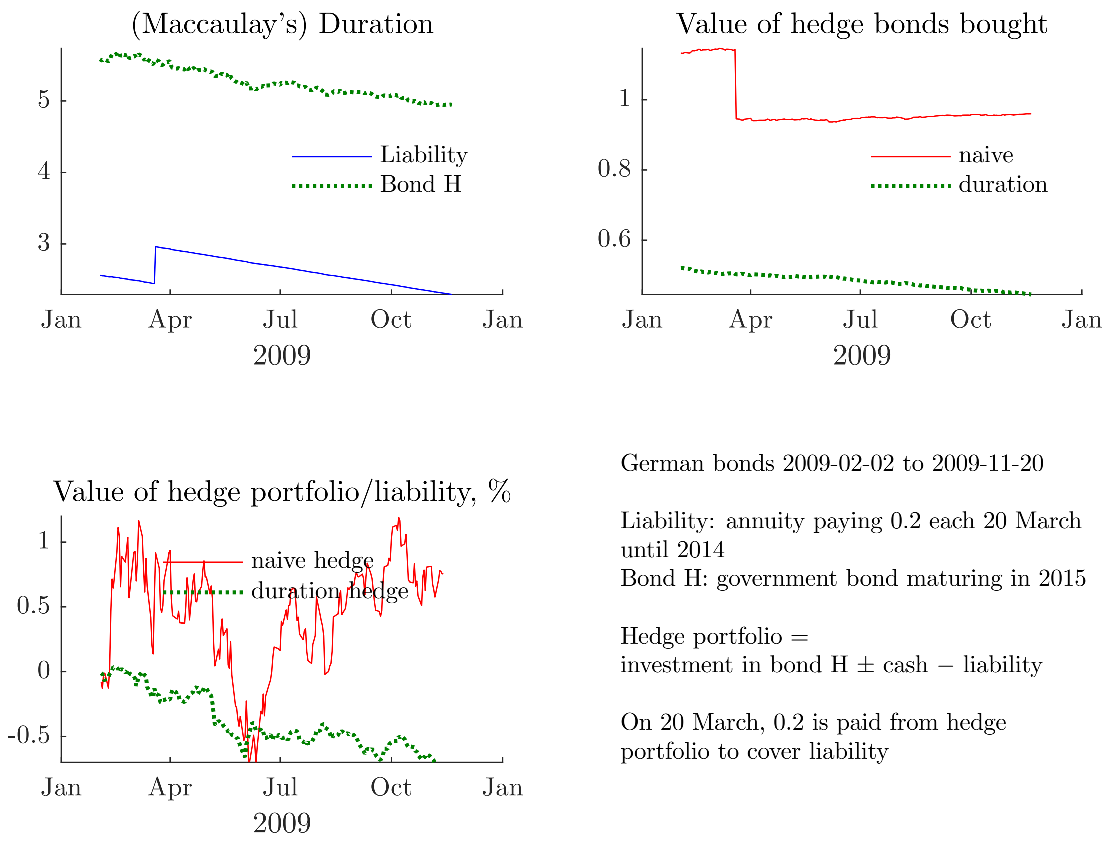


Figure 17.8: Duration hedging

effect.

Choosing the hedging bond (portfolio) so that it has a similar convexity to the bond to be hedged may make the hedge more precise.

Example 17.17 *(Convexity) The convexity of the bond in Example 17.6 is*

$$C = \frac{1}{1.019}\left[1(1+1)\frac{0.04}{1.03^3} + 2(2+1)\frac{1.04}{1.03^4}\right] \approx 5.51.$$

For a zero coupon bond in Example 17.7 (which has the same ytm and maturity), the convexity is

$$C = \frac{1}{1/1.03^2}\left[2(2+1)\frac{1}{1.03^4}\right] = \frac{6}{1.03^2} \approx 5.66.$$

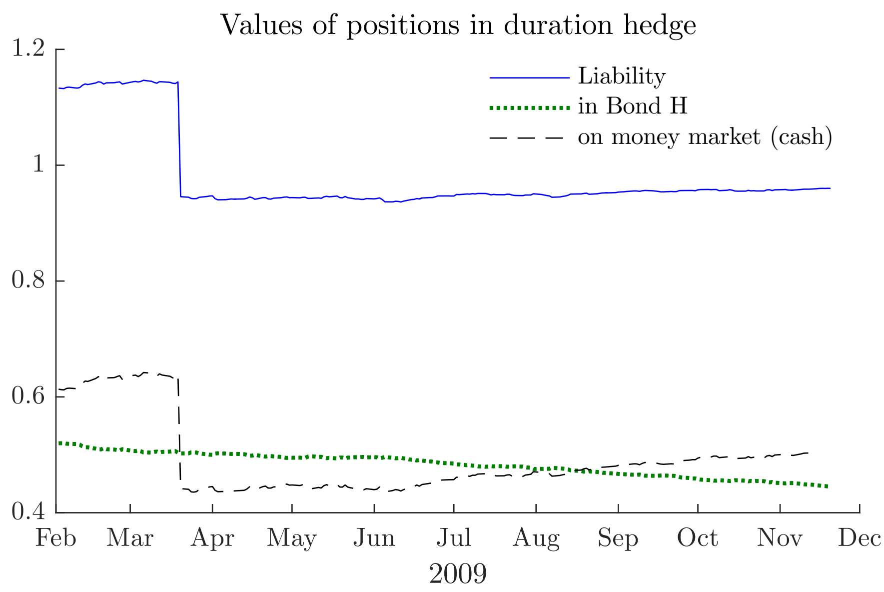


Figure 17.9: Duration hedging

17.4.2 Problem 2: Changing Cash Flows

The duration measures assume that the times when the coupons and the face value are paid are unaffected by the yield change. That is true for many instruments (like most government bonds), but not for callable bonds—and effectively not for bonds whose risk premium depends on the interest rate level as most corporate bonds do (as the interest rate level affects the default risk).

17.4.3 Problem 3: Yield Curve Changes vs. Changes in Yields to Maturity

The probably most important problem with using duration for hedging is that the hedge ratio in (17.24) depends on how the yields change—and that is not known when we construct the hedging portfolio.

The ideal case for duration hedging is when the yields (to maturity) move in parallel. This will be the case, for instance, if the yield curve is flat (across maturities)—and the only movements are parallel shifts up and down. In reality, most movements in the yield curve are parallel, but changes in slope and curvature are not uncommon either. Often the short interest rates move more (in response to news) than long rates.

Equation (17.19) shows how the value of the overall portfolio depends on the yields of the liability and the hedge bond. For instance, suppose the yield curve changes from being flat to being downward sloping and the hedging bond has shorter duration than the liability. In this case, the overall portfolio loses value. The reason is that the value of the hedging portfolio increases less (the yield decreases less) in price than the liability. See Figure 17.5 for an illustration.

To overcome this problem, the hedge ratio should be (set $\Delta V = 0$ in (17.15))

$$h = \frac{D_L^M}{D_H^M} \times \frac{P_L}{P_H} \times \frac{\Delta\theta_L/(1+\theta_L)}{\Delta\theta_H/(1+\theta_H)}. \tag{17.24}$$

This is indeed the same as (17.17) if all changes of the yield curve are parallel shifts (last term in (17.24) is unity). However, the relative frequencies of the yield curve movements (level, slope, curvature) seem to change over time (according to business cycle conditions and monetary policy regime). This suggests that the ability of a simple duration matching to provide a hedge is different in different time periods and different markets.

Explicit models of how the entire yield curve moves in response to a small number of factors have implications for how the two yields will change—which may vary across instruments and time. This would allow us to also model how the last term in (17.24) would react to the drivers of the yield curve—and this provide a more precise hedge ratio.

Chapter 18

Interest Rate Models

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 21–22 and Hull (2009) 4
Additional references: McDonald (2014) 9

18.1 Empirical Properties of Yield Curves

Yield curves (in the US and most other developed countries) tend to have the following features (see Figure 18.1 for some examples).

First, most of the time, the yield curve is upward sloping. This is only consistent expectations hypothesis if short rates are expected to be higher in the future. This means that short rates should (most of the time) be increasing over time—which contradicts empirical evidence. It is more likely that long rates tend to be high because of risk premia. See Figure 18.3 for an illustration.

Second, the yield curve changes over time. It is common to describe the movements in terms of three "factors": level, slope, and curvature. One way of measuring these factors is by defining

$$\begin{aligned}
\text{Level}_t &= y_{10y} \\
\text{Slope}_t &= y_{10y} - y_{3m} \\
\text{Curvature}_t &= (y_{2y} - y_{3m}) - (y_{10y} - y_{2y}).
\end{aligned} \tag{18.1}$$

This means that we measure the level by a long rate, the slope by the difference between a long and a short rate—and the curvature (or rather, concavity) by how much the medium/short spread exceeds the long/medium spread. For instance, if the yield curve is hump shaped (so y_{2y} is higher than both y_{3m} and y_{10y}), then the curvature measure is positive. In contrast, when the yield curve is U-shaped (so y_{2y} is lower than both y_{3m} and

y_{10y}), then the curvature measure is negative. See Figure 18.4 for an example.

Most evidence on US data suggest (see, for instance, Cochrane (2001) 19) that changes in the level dominate—perhaps accounting for 80–90% of the total variation in yields. The slope comes second (perhaps accounting for 10%), and hump third (accounting for a few percent). Similar results are found by principal component analysis.

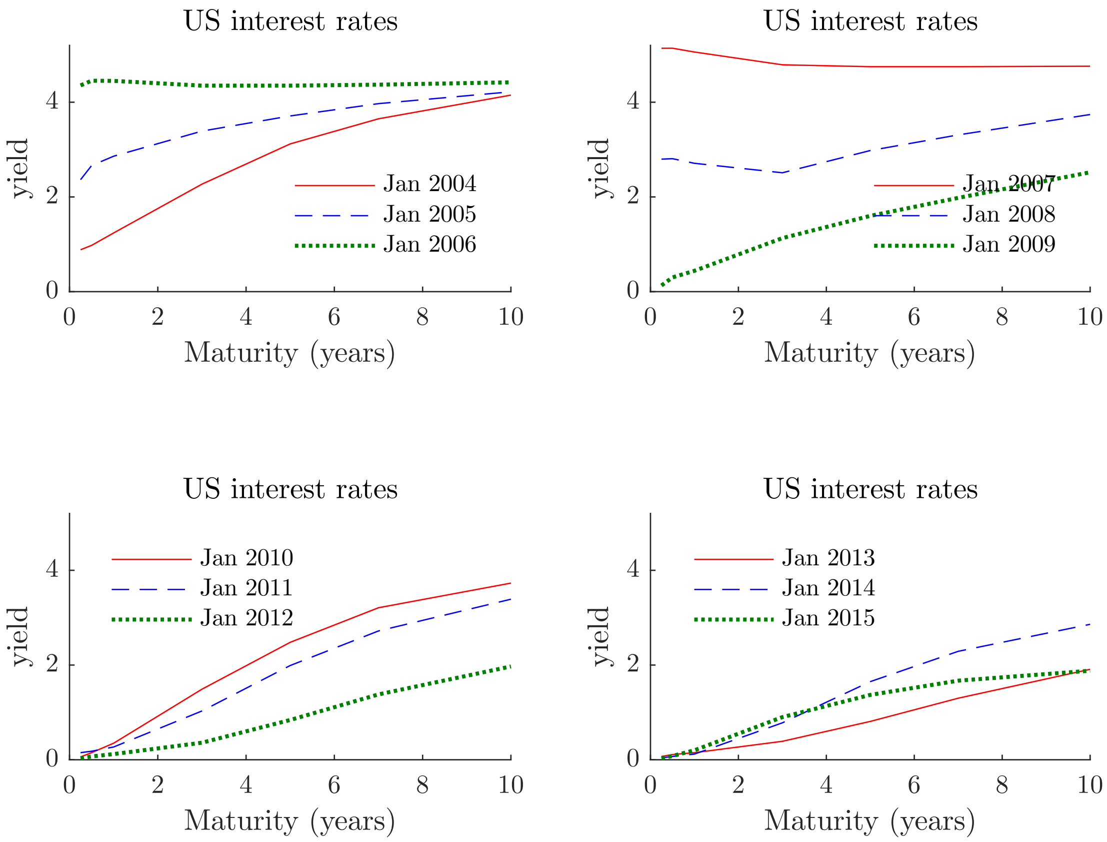


Figure 18.1: Estimated yield curve, Nelson-Siegel method

Interest rates are strongly related to business cycle conditions, so it often makes sense to include macro economic data in the modelling. See Figure 18.5 for how the term spreads are related to recessions: term spreads typically increase towards the end of recessions. The main reason is that long rates increase before short rates. There is also a tendency for the term spreads to be very small (or even negative) at the beginning of recessions.

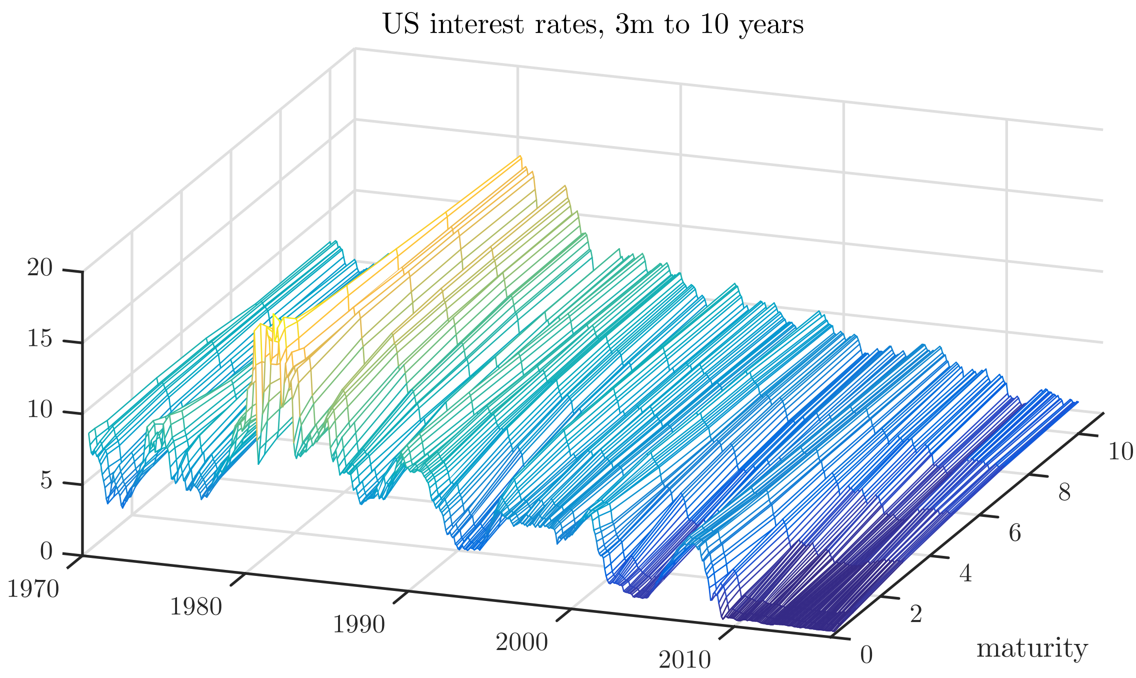


Figure 18.2: US yield curves

18.2 Yield Curve Models

18.2.1 The Expectations Hypothesis of Interest Rates

The expectations hypothesis of interest rates says that long bonds have no, or possibly constant, risk premia. In that case, forward interest rates (or their changes) can be interpreted as expected future short interest rates (or their changes). The evidence on the expectations hypothesis is mixed, so it can only be thought of as a rough, although convenient, approximation.

To illustrate how the expectations hypothesis works, it is easiest to work with continuously compounded interest rates. Recall that a continuously compounded interest rate for a loan between now and m period later, $y_t(m)$, satisfies

$$\frac{1}{B_t\left(m\right)} = \exp\left[my_t\left(m\right)\right], \text{ or } y_t(m) = -\ln B_t(m)/m, \tag{18.2}$$

where $B_t(m)$ is the price (in t) of a zero-coupon bond which matures in $t+m$.

The expectations hypothesis says that the current long rate equals the expected future average short interest rate (the “short rate” is the interest rate for the shortest maturity). To

Figure 18.3: Average US yield curve

formalize that, split up the time until m into m intervals of length 1. Then, the expectations hypothesis says that the m-period spot rate equals the average of the 1-period (the shortest maturity) rates over t to $t+m$

$$y_t(m) = \lambda(m) + \frac{1}{m}\left(r_t + \mathrm{E}_t\, r_{t+1} + \mathrm{E}_t\, r_{t+2} + \ldots + \mathrm{E}_t\, r_{t+m-1}\right), \tag{18.3}$$

where r_t is short hand notation for the 1-period interest rates ($y_t(1)$).

The expectations hypothesis allows for constant risk premia ($\lambda(m) \neq 0$), which may differ across maturities (m). If $\lambda(m) = 0$, then the *pure* expectations hypothesis is said to hold. See Figure 18.6 for an illustration.

18.2.2 Risk Premia

There are several reasons for why bonds should have risk premia. First, long bonds are risky for investors who do not intend to keep them until maturity—and will therefore have term premia. Second, some bonds are not traded much (for instance, off-the-run bonds and many index-linked bonds)—so they are likely to have liquidity premia. Third, the real

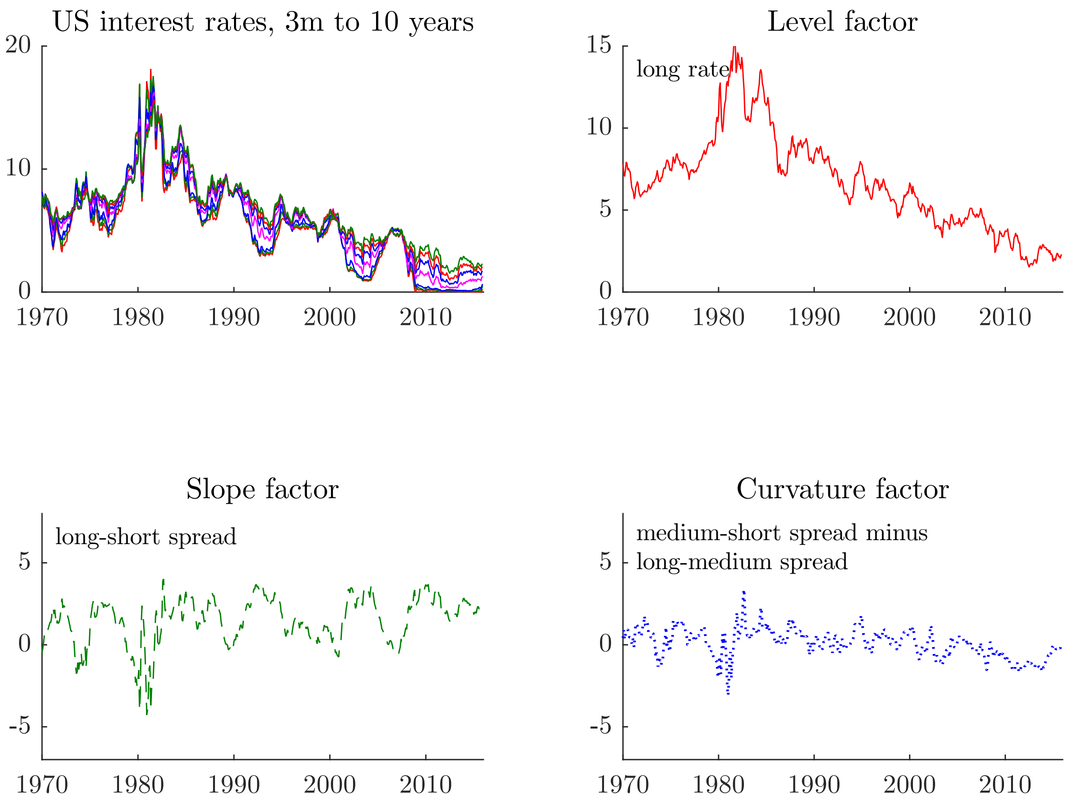


Figure 18.4: US yield curves: level, slope and curvature

return of a long bond is very sensitive to inflation changes—probably more than equity. Bonds are therefore likely to have inflation risk premia.

18.2.3 A Simple One-Factor Model: The Vasicek Model

The Vasicek model assumes that the state variable (the short interest rate) is an AR(1). The specification typically involves shifting the mean of the process to allow for a risk (term) premia. To simplify, I will crudely assume that there are some unspecified constant risk premia (the expectations hypothesis). (The more general formulation derives the risk premia in terms of the mean reversion and volatility of the short rate.)

To simplify the notation, let the short rate, r_t, be the state variable. It follows an AR(1)

$$r_{t+1} - \mu = \rho (r_t - \mu) + \varepsilon_{t+1}, \tag{18.4}$$

where μ is the mean. See 18.7 for an illustration.

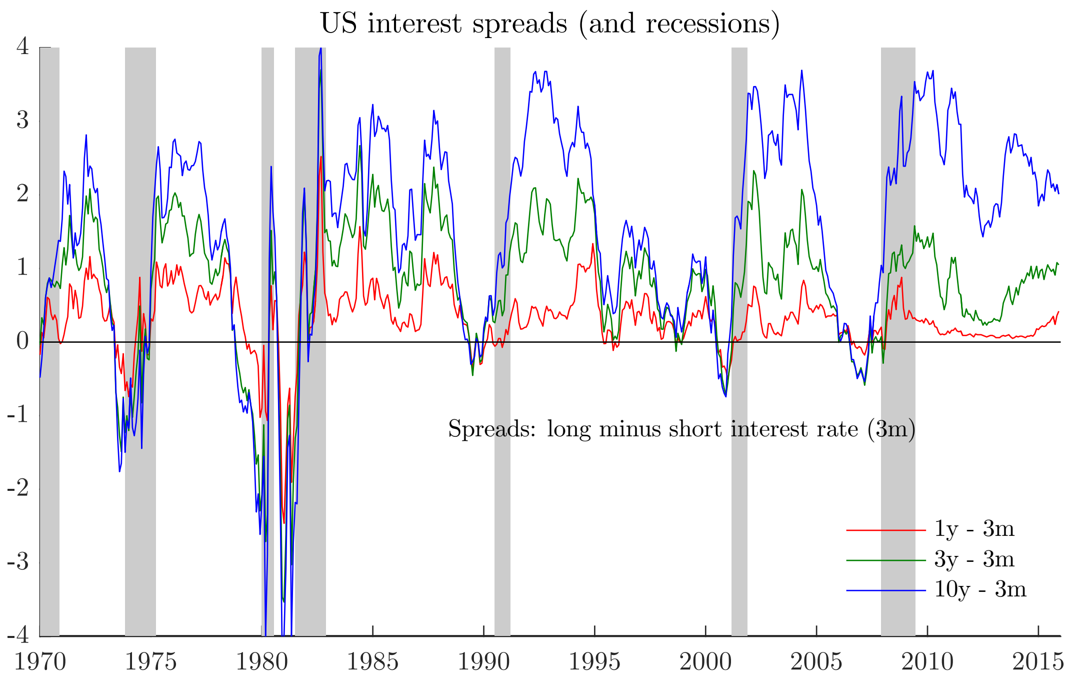


Figure 18.5: US term spreads (over a 3m T-bill)

Remark 18.1 *(Alternative formulation of (18.4)*) The process is sometimes specified in terms of changes as*

$$r_{t+1} - r_t = a(\mu - r_t) + \varepsilon_{t+1}.$$

Clearly, this can be written

$$r_{t+1} - \mu = (1 - a)(r_t - \mu) + \varepsilon_{t+1},$$

where $1 - a$ *corresponds to* ρ*. With* $0 < a < 1$ *(that is, with* $0 < \rho < 1$*) the process is mean reverting.*

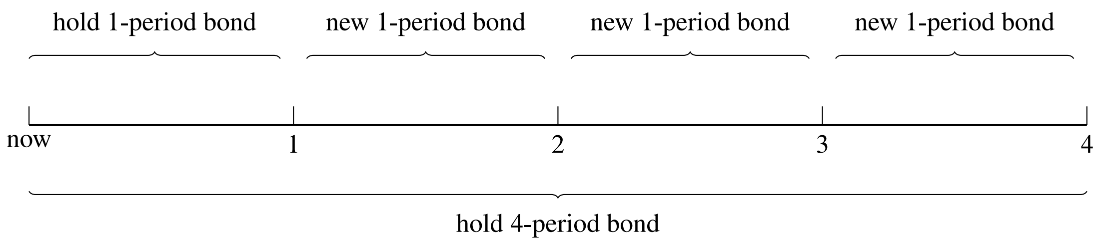


Figure 18.6: Timing for expectations hypothesis

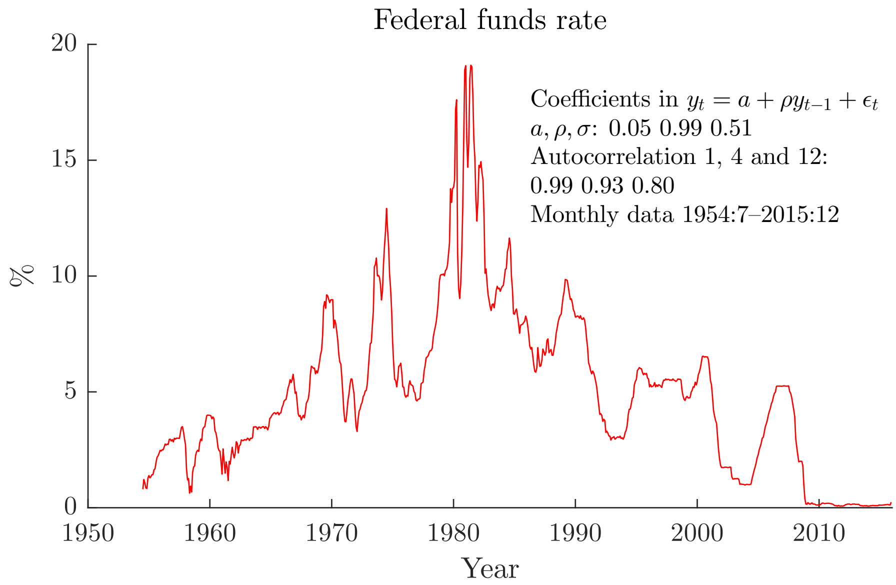


Figure 18.7: Federal funds rate, monthly data

The forecast for $t + s$ is

$$\mathrm{E}_t\, r_{t+s} = (1 - \rho^s)\, \mu + \rho^s r_t. \tag{18.5}$$

See Figure 18.8.

Remark 18.2 *(Calibrating the AR(1) to data) Notice that (18.4) implies that* $\mathrm{Corr}(r_t, r_{t-s}) = \rho^s$*, so we could thus estimate* ρ *by* $\mathrm{Corr}(r_t, r_{t-s})^{1/s}$*. If the AR(1) is a very good fit to data, then it should not matter (much) if you use* $s = 1$ *or* $s = 12$ *(say). In practice, the results may well differ. For instance, suppose monthly data gives* $\mathrm{Corr}(r_t, r_{t-1}) = 0.99$ *but* $\mathrm{Corr}(r_t, r_{t-12}) = 0.80$*, which imply* $\rho = 0.99$ *and* $\rho = 0.982$ *respectively. This matters for the pricing of long-maturity bonds: with 120 months (10 years) we get* $0.99^{120} = 0.3$ *while* $0.98^{120} = 0.09$*. Which value we choose to use depends on whether we are most interested in the short maturities (use* $\rho = 0.99$*) or the long maturities (use* $\rho = 0.982$*).*

We now assume that the expectations hypothesis of interest rates holds. Using this in (18.3) gives the long interest rate. For instance, the $m = 2$ rate is

$$\begin{aligned} y_t(2) &= \lambda(2) + \frac{1}{2}\left[r_t + (1-\rho)\,\mu + \rho r_t\right] \\ &= \lambda(2) + \mu\,(1-\rho)/2 + r_t\,(1+\rho)/2. \end{aligned} \tag{18.6}$$

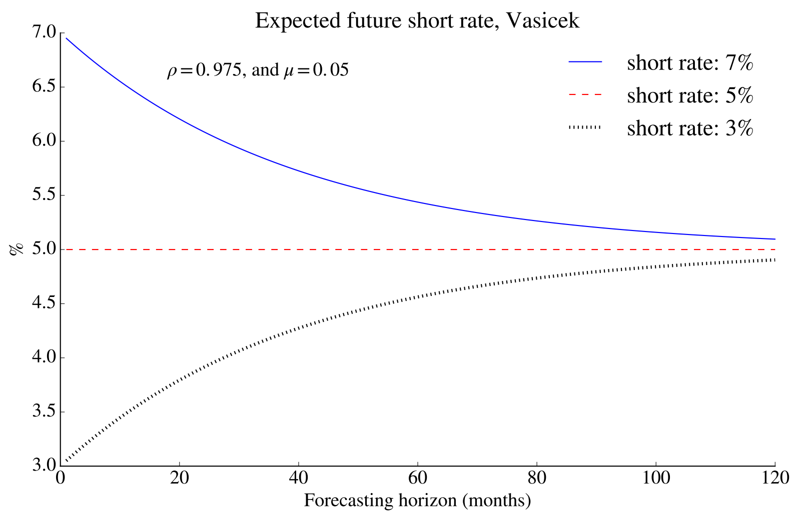


Figure 18.8: Expected future short rate in Vasicek model, for different intial short rates

The general expression for a maturity of m is

$$y_t(m) = a(m) + b(m)r_t, \text{ where} \tag{18.7}$$
$$a(m) = \lambda(m) + \mu\left[1 - b(m)\right] \text{ and}$$
$$b(m) = (1 + \rho + \ldots + \rho^{m-1})/m = (1 - \rho^m)/[(1 - \rho)m].$$

In this model, all movements of the yield curve are driven by the state variable (the short rate), so it is a *one-factor model*. The shifts of the yield curve are parallel if $\rho = 1$ (the random walk model) since then $b(m) = 1$ in (18.7), so we get

$$y_t(m) = \lambda(m) + r_t, \text{ if } \rho = 1. \tag{18.8}$$

For lower values of ρ, the short rate process is mean-reverting, so the expected future short rates (and therefore the current long rates) are always closer the the mean than the current short rate. See Figures 18.9–18.10 for an illustration. In particular, notice that the driver of the long rates is the average expected future short rates.

Example 18.3 *(Vasicek model) For $\rho = 0.9$ and $\mu = 0.05$, (18.7) gives (assuming no*

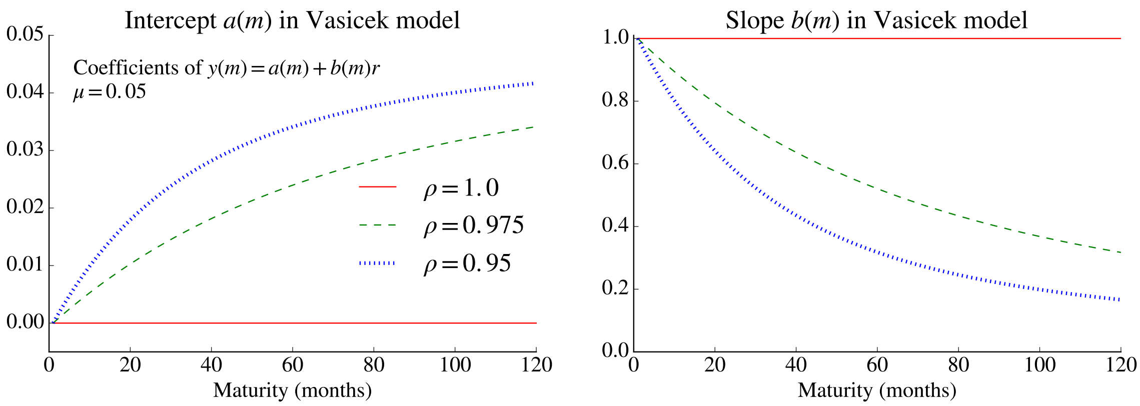


Figure 18.9: Intercept and slope in the Vasicek model

risk premia)

$$\begin{bmatrix} y_t(1) \\ y_t(2) \\ y_t(3) \\ y_t(4) \end{bmatrix} \approx \begin{bmatrix} 0 \\ 0.0025 \\ 0.0048 \\ 0.007 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.9/2 \\ 2.71/3 \\ 3.44/4 \end{bmatrix} r_t.$$

18.2.4 The Vasicek Model: Hedging a Bond

The Vasicek model allows us to calculate (or rather estimate) the proper way of *hedging a bond*.

The change of the hedge portfolio is

$$\Delta V = h\Delta P_H - \Delta P_L, \tag{18.9}$$

and a bond price can be calculated as

$$P = \sum_{k=1}^{K} \frac{cf_k}{\exp[m_k y(m_k)]} \tag{18.10}$$

Once we know the parameters of the Vasicek model, it is straightforward to calculate (at least numerically) what ΔP_H and ΔP_L are—as a function of the change in the short rate interest rate (Δr_t). In practice, this can be done by the following steps.

1. For an initial value of the short log interest rate r, use (18.7) for calculate all log spot rates, $y(m)$ for m starting at 1 and going until the maturity of the liability or hedge (whichever is longest).

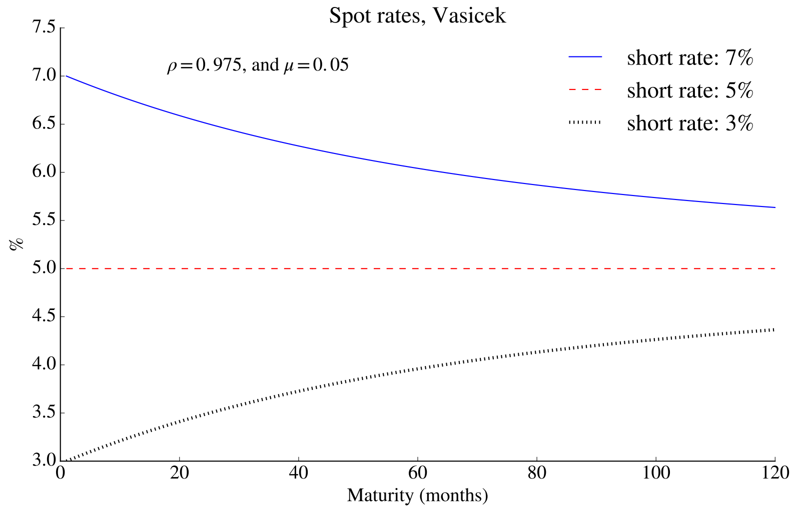


Figure 18.10: Vasicek model, spot rates for different initial short rates

2. Use the spot interest rates $y(m)$ to calculate the prices of the hedge bond the liability according to (18.10).

3. Redo points 1 and 2, but starting from another short log rate.

4. Calculate $(\Delta P_H, \Delta P_L)$ as a response to Δr. We then set h so that $\Delta V = 0$, that is, $h = \Delta P_L / \Delta P_H$.

Remark 18.4 *(Duration hedging with the Vasicek model*) The Vasicek model can also be used to calculate the yield changes in a duration hedge. Recall that the following value (dollars) invested into a hedge bond is (H) relative to the value of the liability (L) should provide a good hedge:*

$$hP_H/P_L = \frac{D_L^M}{D_H^M} \times \frac{\Delta\theta_L/(1+\theta_L)}{\Delta\theta_H/(1+\theta_H)}$$

where D_i^M is Macaulay's duration, θ_i the yield to maturity and P_i the price of bond i. In the typical duration hedge we assume that all yield curve moments are parallel, so the last term in this expression equals one. Follow the same steps as above, but also calculate the durations (only at the initial short interest rate) and the yield to maturities. Then calculate hP_H/P_L according to the equation above. The results are very similar to the easier approach discussed above,

Figure 18.11 gives an illustration. The hedge ratio h converges to the duration hedge

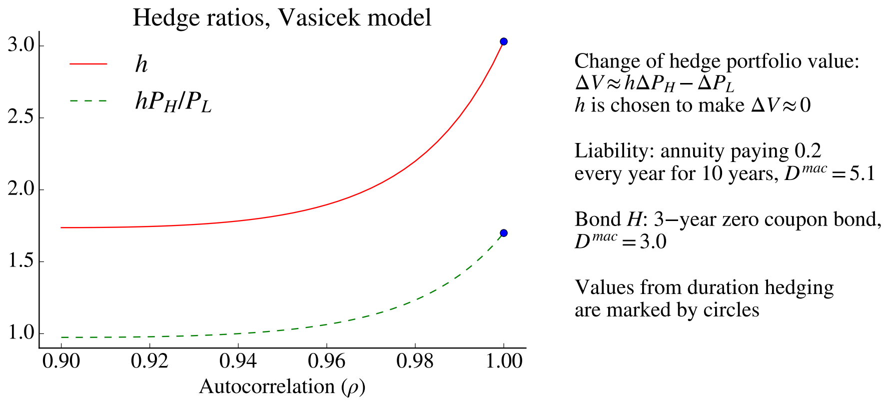


Figure 18.11: Hedge ratios in the Vasicek model

ratio as the autocorrelation (ρ) in the short rate process (18.4) increases towards unity: in that limiting case all yield curve movements are indeed parallel. For lower values of the autocorrelation, the hedge ratio is lower. The main reason is that mean-reversion, that is, low autocorrelation makes interest rates on long maturity bonds (here, the liability) move less than interest rates on short bonds. This means that the value of the liability is not that sensitive to the short rate (when ρ is low)—and hence we need not invest so much into the (shorter maturity) hedge bond.

Notice, however, that all one-factor models (not least the Vasicek model) imply that all yields are perfectly correlated (there is a common single driving force) and only fairly limited yield curve movements are possible. *Multi-factor models* overcome most of those limitations. For instance, the model in Nelson and Siegel (1987) is a two-factor model.

18.3 Interest Rates and Macroeconomics

This section outlines several (not mutually exclusive) macroeconomic approaches to modelling the yield curve.

18.3.1 The Fisher Equation and Index-Linked Bonds

Let π_{t+n} be the one period inflation rate over t to $t+n$ and $y_t^r(n)$ the n-period continuously compounded real interest rate (an interest rate measured in purchasing power terms).

The Fisher equation (here in the form of continuously compounded rates) says that

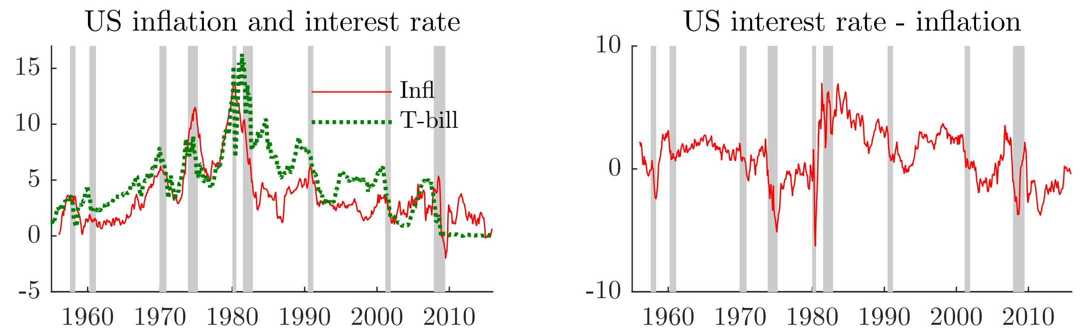


Figure 18.12: US inflation and 3-month interest rate

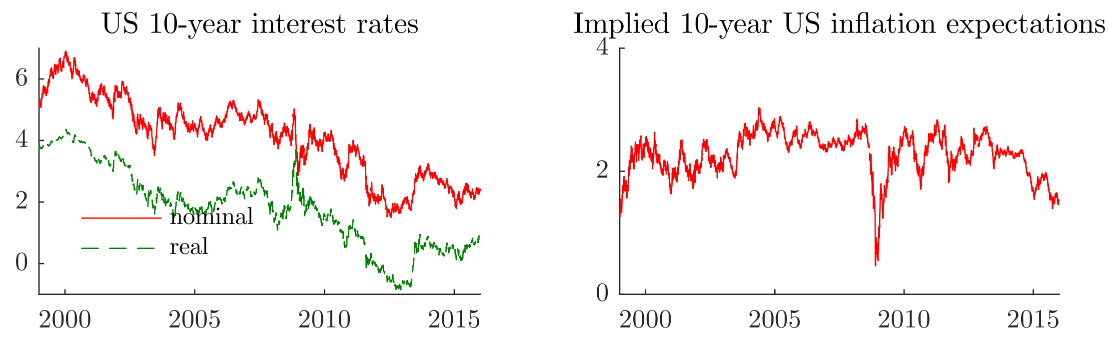


Figure 18.13: US nominal and real interest rates

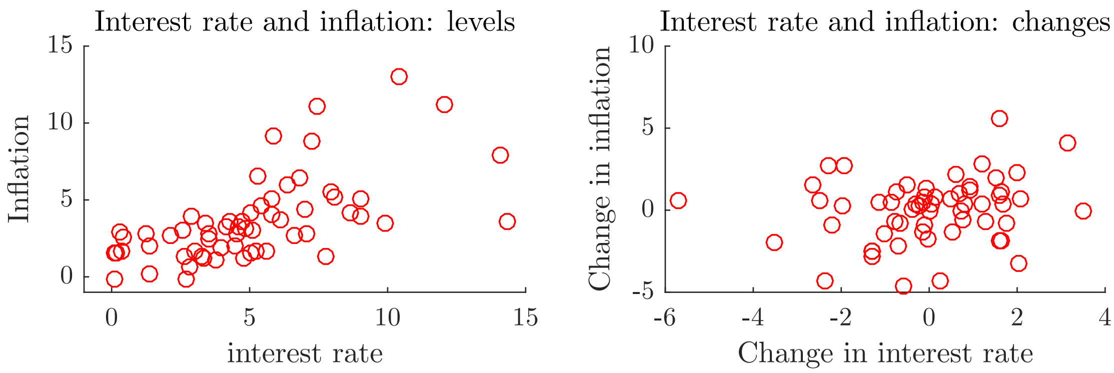
the nominal interest rate includes compensation both for inflation expectations, $\mathrm{E}_t\, \pi_{t+n}$, the real interest rate, $y_t^r(n)$, and possibly a constant (across time) risk premium, $\psi(n)$,

$$y_t(n) = \mathrm{E}_t\, \pi_{t+n} + y_t^r(n) + \psi(n). \tag{18.11}$$

Example 18.5 *(Fisher equation) Suppose the nominal interest rate is* $y(n) = 0.07$*, the real interest rate is* $y^r(n) = 0.03$*, and the nominal bond has no risk premium (*$\psi = 0$*), then the expected inflation is* $\mathrm{E}_t\, \pi_{t+n} = 0.04$.

The same type of relation holds for forward rates. The Fisher equation suggests a framework for analysing nominal interest rates in terms of real interest rates and inflation expectations. This is commonly used for long rates. Information about real interest rates can be elicited from *index-linked bonds*, that is, bonds which give automatic compensation for actual inflation.

Empirical results typically indicate that there are non-trivial movements in the real interest rate and/or risk premia—especially for short forecasting horizons. This holds



Sample: US 12-month interest rates and next-year inflation 1955:1–2015:12

Figure 18.14: US nominal interest rates and subsequent inflation

also when inflation expectations, as measured by surveys, are used as the dependent variable. Inflation expectations seem to vary by less than the interest rate. It is therefore not straightforward to extract inflation expectations from nominal interest rates.

The Fisher equation could also be embedded in a macro model to construct a sophisticated (and complicated) model of the yield curve. This involves using macro theory/empirics to model how real interest rates and inflation expectations (for different maturities) depend on the state of the economy.

18.3.2 The Expectations Hypothesis of Interest Rates

The expectations hypothesis of interest rates says that long interest rates equal an average of expected future short rates, possibly with a constant (across time, not maturities) risk premium as in (18.3). Alternatively, that forward rates equal expected future spot rates.

The expectations hypothesis is often used to calculate implied "forecasts" of future short interest rates. For instance, suppose the central bank increases its policy rate (typically a very short rate, at most a week or two). This is likely to affect also longer interest rates, but how is another matter. Let us consider a few different cases. For simplicity we assume that risk premia are unaffected by this move in the policy rate.

Example 18.6 *(Macroeconomics and the Pricing of Long-maturity Bonds I) An example of a macro based approach:*

- *Suppose the macroeconomic outlook becomes worse, inflation expectations down*

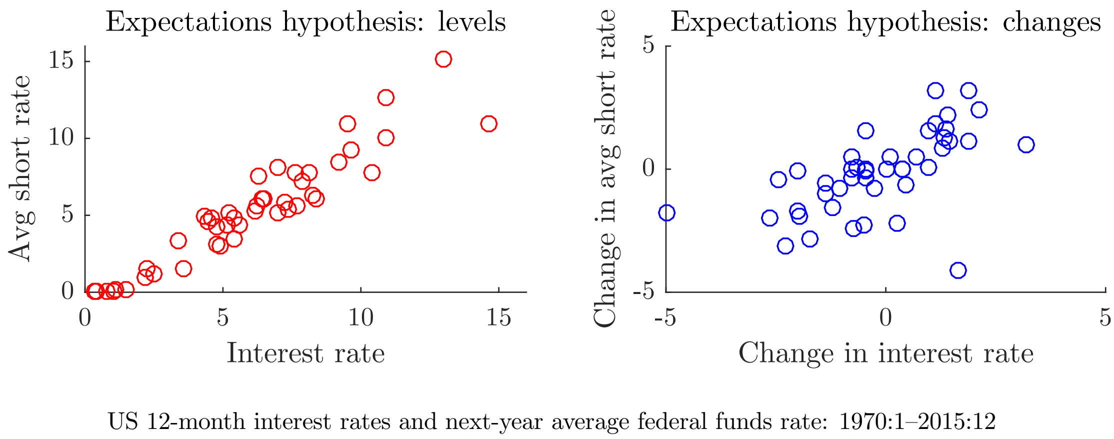


Figure 18.15: US 12-month interest and average federal funds rate (next 12 months)

- *the central bank is expected to keep short interest rates low for an extended period*
- *the expectation hypothesis⇒interest rates on long bonds decrease today (⇔bond prices up)*

Example 18.7 *Macroeconomics and the Pricing of Long-maturity Bonds II) an alternative, more asset market focused approach, but with the same result:*

- *Suppose the macroeconomic outlook becomes worse, inflation expectations down*
- *(a) inflation expectations down: nominal bonds more attractive; (b) growth down: bonds more attractive than stocks*
- *demand for long bonds up⇒ prices of long bonds up (⇔long interest rates down)*

First, one possibility is that only the very short interest rates change, and that all longer interest rates stay unchanged. This would happen if the policy move was well anticipated.

Second, another possibility is that most long interest rates increase. Under the expectations hypothesis of interest rates the interpretation is that the market now expects high short interest rates also in the future. That is, that the central bank will not reverse its policy action in the foreseeable future. If we are willing to assume that the real interest rate was not affected by the policy move, then one possible interpretation is that the central bank has received information about a long-lasting inflation pressure.

Third, and finally, short rates may increase, but really long interest rates decrease. A common interpretation of this scenario is that the central bank has become more inflation averse. It therefore raises the policy rate to bring down inflation. If the market believes that it will succeed, then it follows that it will eventually be possible to lower interest rates (when inflation and inflation expectations are lower).

The expectations hypothesis has been tested many times, typically by an ex post linear regression (realized interest rates regressed on lagged forward rates). The results often reject the expectations hypothesis, but the results depend on how the test is done. It is not clear, however, if the rejection is due to systematic risk premia or to fairly small samples (compared to the long swings in interest rates). The expectations hypothesis gets more support when survey data on interest rate expectations is used instead on realized interest rates.

18.3.3 Uncovered Interest Rate Parity

Uncovered interest rate parity says that the difference between a domestic and foreign interest rate equals the expected depreciation plus a constant (across time, not maturities) risk premium

$$y_t(m) - y_t^*(m) = \mathrm{E}_t\, s_{t+m} - s_t + \varphi_m, \tag{18.12}$$

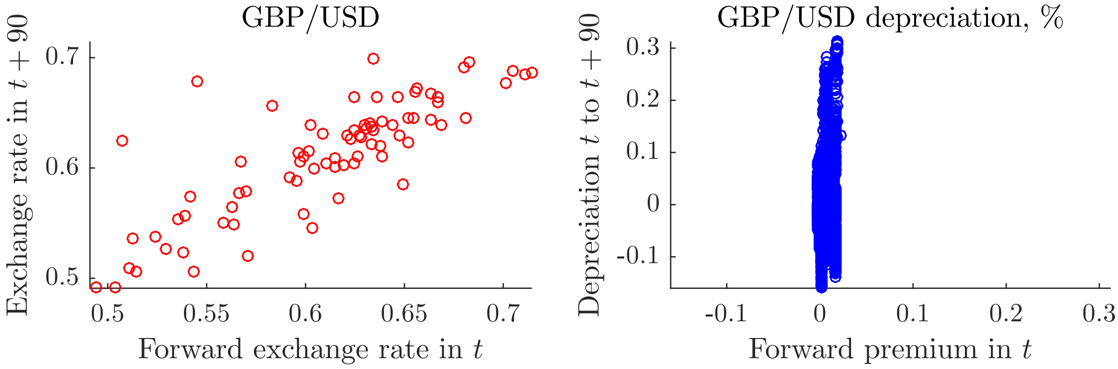
where $y_t^*(m)$ is a (continuously compounded) foreign interest rate, and s_t is the logarithm of the exchange rate (number of domestic currency units per foreign currency unit). If this condition holds with a zero risk premium, then the expected return from investing in foreign bonds and then buying domestic currency equals the known return from investing in domestic bonds.

Empirical evidence suggests that there might be large movements in the risk premia over time (or that there have been systematic surprises in historical samples).

18.3.4 A New-Keynesian Model of Monetary Policy

Monetary policy is a crucial part of the macroeconomic picture these days, so it is important to understand how monetary policy is formed. It has not always been this way: there are long periods when many countries adopted a very simple (or so it seemed) monetary policy by pegging the currency to another currency. Macroeconomic policy was then synonymous with fiscal policy. Recently, the roles have changed.

Modern macro models are often smaller than the older macroeconometric models and they pay more attention to both the supply side of the economy and the role of expectations. These models try to capture the key elements in the way central banks (and most other observers) reason about the interaction between inflation, output, and monetary policy.



Sample: 1988:7–2015:12

To predict exchange rate level S_{t+90}, use:
(a) regression: $a + bF_t$ from OLS
(b) random walk: S_t
(c) forward rate: F_t

To predict depreciation, $S_{t+90}/S_t - 1$, use:
(a) regression: $a + bF_t/S_t$ from OLS
(b) random walk: 0
(c) forward premium: $F_t/S_t - 1$

	Slope (LS)	R^2 (LS)	MSE (LS)	MSE (RW)	MSE (forward)
Levels	0.82	0.67	8.91	8.83	8.90
Depreciation	0.95	0.01	25.95	26.20	25.99

Figure 18.16: GBP/USD spot and forward exchange rates

In these models, inflation depends on expected future inflation (some prices are set today for a long period and will therefore be affected by expectations about future costs and competitors' prices), lagged inflation, and a "Phillips effect" where an *output gap* (output less trend output) affects price setting via demand pressure. For instance, inflation (π_t) is often modelled as

$$\pi_t = \alpha \operatorname{E}_t \pi_{t+1} + \beta \pi_{t-1} + \phi x_t + \varepsilon_{\pi t}, \tag{18.13}$$

where x_t is the output gap and $\varepsilon_{\pi t}$ can be interpreted as "cost push" shocks (wage demands, oil price shocks). This equation can be said to represent the supply side of the economy and it is typically derived from a model where firms with some market power want to equate marginal revenues and marginal costs, but choose to change prices only gradually.

The demand side of the economy is modelled from consumers' savings decision, where the trade off between consumption today and tomorrow depends on the real interest rates. Simplifying by setting consumption equal to output we get something like the following equation for the output gap

$$x_t = x_{t-1} - \gamma(i_t - \mathrm{E}_t\pi_{t+1}) + u_t, \tag{18.14}$$

where i_t is the nominal interest rate (set by the central bank) and u_t is a shock to demand. Note that the expected *real* interest rate affects demand (negatively).

In some cases, the real exchange rate is added to both (18.13) and (18.14), capturing price increases on imported goods and foreign demand for exports, respectively. The exchange rate is then linked to the rest of the model via an assumption of uncovered interest rate parity (that is, expected exchange rate depreciation equals the interest rate differential).

Some of the important features of this simple model are: *(i)* inflation expectations matter for today's inflation (think about wage inflation), *(ii)* the instrument for monetary policy, the short interest rate i_t, can ultimately affect inflation only via the output gap; *(iii)* it is the real, not the nominal, interest rate that matters for demand.

To make the model operational, two more things must be added: the monetary policy (the way the interest rate is set) and the expectations in (18.13)–(18.14) must be specified.

It is common to assume that the central bank has some instrument rule like the famous "Taylor rule"

$$i_t = \theta_0 + 0.5x_t + 1.5\pi_t + v_t. \tag{18.15}$$

The residual v_t is a "monetary policy shock," which picks up factors left out of the model (for instance, the central bank's concern for the banking sector or simply changes in the central bank's objectives). This simple reaction function has been able to track US monetary policy fairly well over the last decade or so. Another approach to find a policy rule is to assume that the central bank has some loss function that it minimizes by choosing a policy rule. This loss function is often a weighted average of the variance of inflation and the variance of the output gap. The policy rule is the solution of the minimization problem, and can often look more complicated than the Taylor rule. However, there is one interesting special case. Suppose the central bank wants to minimize the (unconditional) variance of inflation. The formal optimization problem is then

$$\min_{i_t} \mathrm{Var}(\pi_t), \text{ subject to (18.13) and (18.14).} \tag{18.16}$$

The solution is then that the interest rate should be set so that actual inflation is zero (here the mean) in every period. If the model is changed so there is a time lag between the interest rate decision and its effect on inflation (for instance, by letting inflation in (18.13) react to x_{t-1} instead of x_t), then the interest rate should be set so that the conditional expectation of next period's inflation is zero (the mean), $\mathrm{E}_t\, \pi_{t+1} = 0$. This type of "rule" is used in much of the monetary policy debate.

The expectations in (18.13)–(18.14) can be handled in many ways. The perhaps most straightforward way is to assume that the expectations about the future equal the current value of the same variable (a "random walk"). A more satisfactory way is to use survey data on inflation expectations. Finally, many model builders assume that expectations are "rational" (or "model consistent") in the sense that the expectation equals the best guess we could do under the assumption that the model is correct. This latter approach typically requires a sophisticated way of solving the model (as the model both generates the best guesses and depends on them).

18.4 Forecasting Interest Rates

The expectations hypothesis of interest suggests that current long rates can help predict future short rates). Empirically, this has some (moderately strong) support. However, there are also a number of other forecasting approaches.

18.4.1 Forecasting Monetary Policy or Inflation?

There is a two-way causality: inflation and the real economy (which depend on the real interest rate) affect monetary policy, and monetary policy can surely affect inflation and the real economy. This makes it difficult to analyse and forecast interest rates. However, for short term forecasting, the emphasis is typically on forecasting the next monetary policy move. Long run forecasting relies more on understanding the determinants of real interest rates and inflation, which depends on the general business cycle prospects, but also on the long run stance of monetary policy ("tough on inflation or not?").

18.4.2 Interest Rate Forecasts by Analysts

Kolb and Stekler (1996) use a semi-annual survey of (12 to 40) professional analysts' interest rate forecasts published in Wall Street Journal. The (6 months ahead) forecasts are for the 6-month T-bill rate and the yield on 30-year government bonds. The paper studies four questions, and I summarize the findings below.

1. Q. Is the distribution of the forecasts (across forecasters) at any point in time symmetric? (Analysed by first testing if the sample distribution could be drawn from

a normal distribution; if not, then checking asymmetry (skewness).) A. Yes, in most periods. (The authors argue why this makes the median forecast a meaningful representation of a "consensus forecast.")

2. Q. Are all forecasters equally good (in terms of ranking of (absolute?) forecast error)? A. Yes for the 90-day T-bill rate; No for the long bond yield.

3. Q. Are some forecasters systematically better (in terms of absolute forecast error)? (Analysed by checking if the absolute forecast error is below the median more than 50% of the time) A. Yes.

4. Q. Do the forecasts predict the direction of change of the interest rate? (Analysed by checking if the forecast gets the sign of the change right more than 50% of the time.) A. No.

18.4.3 Market Positions as Interest Rate Forecasts

Hartzmark (1991) has data on daily futures positions of large traders on eight different markets, including futures on 90-day T-bills and on government bonds. He uses this data to see if the traders changed their position in the right direction compared to realized prices (in the future) and if they did so consistently over time.

The results indicate that these large investors in T-bills and bond futures did no better than an uninformed guess of the direction of change of the bill and bond prices. He gets essentially the same results if the size of the change in the position and in the price are also taken into account.

There is of course a distribution of how well the different investors do, but it looks much like one generated from random guesses (uninformed forecasts). The investors change places in this distribution over time: there is very little evidence that successful investors continue to be successful over long periods.

18.5 Risk Premia on Fixed Income Markets

There are many different types of risk premia on fixed income markets.

Nominal bonds are risky in real terms, and are therefore likely to carry *inflation risk premia*. Long bonds are risky because their market values fluctuate over time, so they probably have *term premia*. Corporate bonds and some government bonds (in particular, from developing countries) have *default risk premia*, depending on the risk for default.

Interbank rates may be higher than T-bill of the same maturity for the same reason (see the TED spread, the spread between 3-month Libor and T-bill rates) and illiquid bonds may carry *liquidity premia* (see the spread between off-the run and on-the-run bonds).

Figures 18.17–18.20 provide some examples.

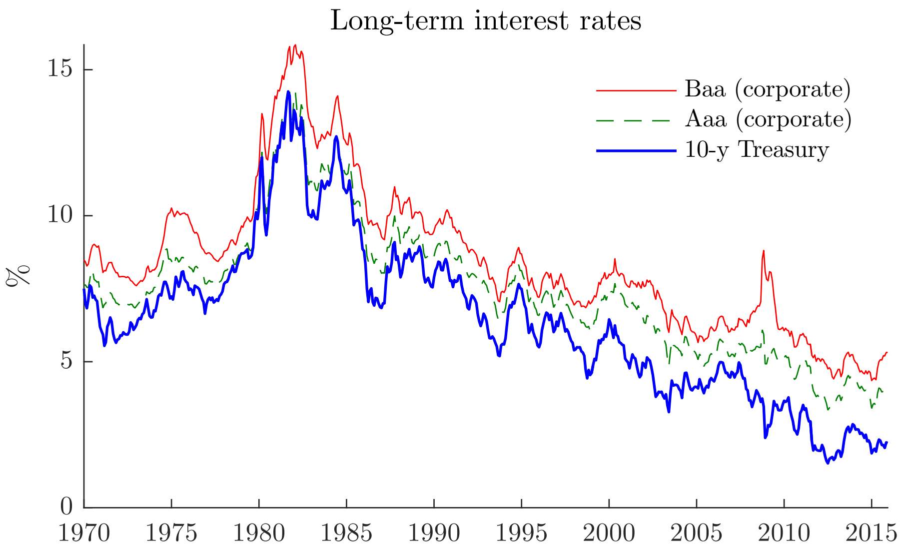


Figure 18.17: US interest rates

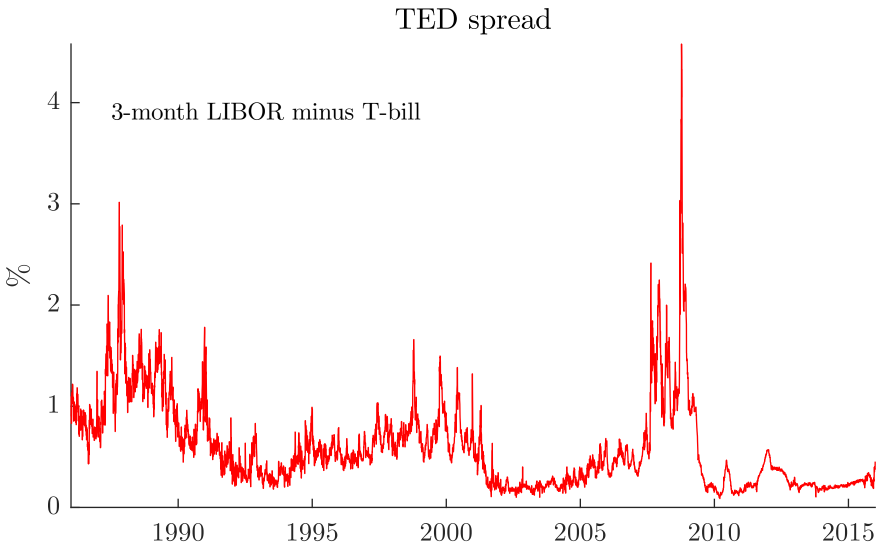


Figure 18.18: TED spread

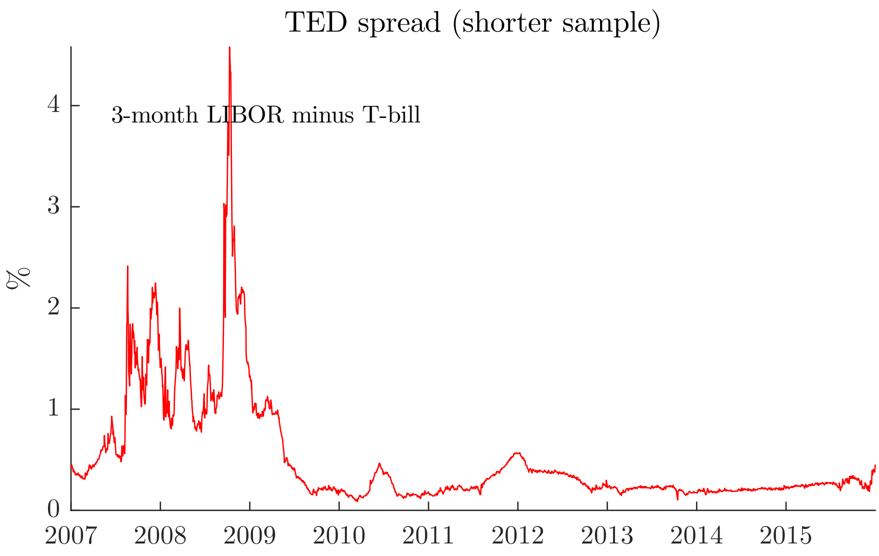


Figure 18.19: TED spread recently

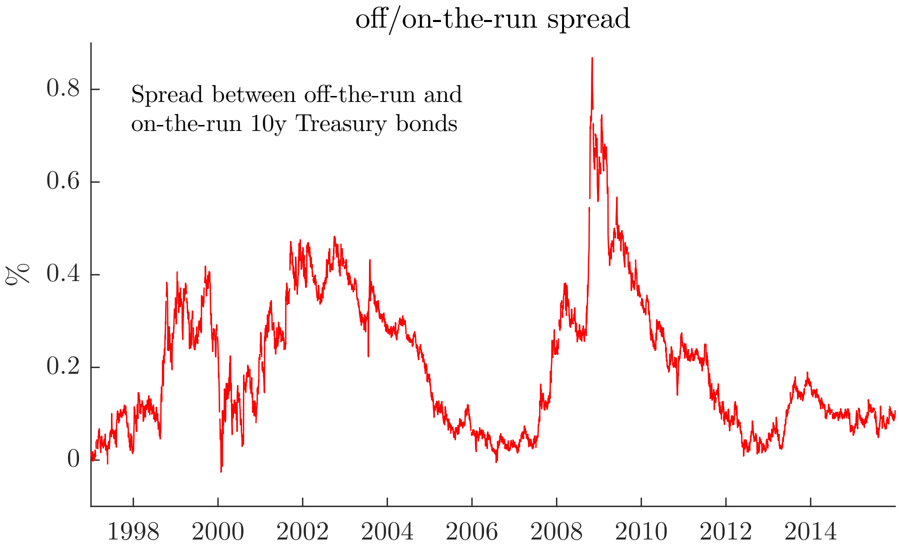


Figure 18.20: Off-the-run liquidity premium

Chapter 19

Basic Properties of Options

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23
Additional references: McDonald (2014) 11–12

19.1 Derivatives

Derivatives are assets whose payoff depend on some underlying asset (for instance, the stock of a company). The most common derivatives are futures contracts (or similarly, forward contracts) and options. Sometimes, options depend not directly on the underlying, but on the price of a futures contract on the underlying. See Figure 19.1.

Derivatives are in zero net supply, so a contract must be issued (a short position) by someone for an investor to be able to buy it (long position). For that reason, gains and losses on derivatives markets sum to zero.

19.2 Introduction to Options

19.2.1 Definition of European Calls and Puts

A European *call* option contract traded in t may stipulate that the buyer of the contract has the right to buy one unit of the underlying asset from the issuer of the option on the expiration date $t+m$ at the strike price K. See Figure 19.2 for the timing convention.

The payoff at exercise is zero or, if larger, the price of the underlying asset, S_{t+m}, minus the strike price

$$\text{call payoff}_{t+m} = \max\left(0, S_{t+m} - K\right). \tag{19.1}$$

Clearly, an owner of a call option benefits from an increase in the price of the underlying

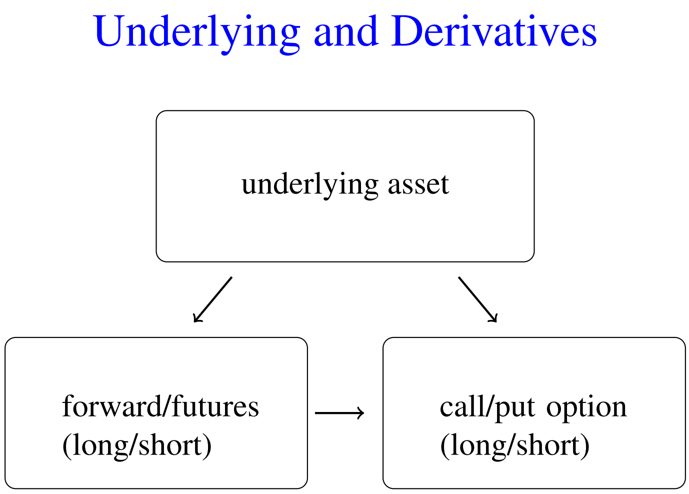


Figure 19.1: Derivatives on an underlying asset

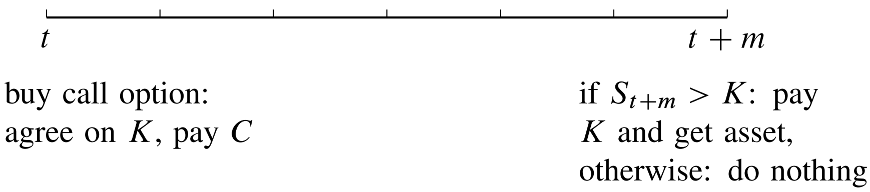


Figure 19.2: Timing convention of a European call option contract

asset (exercise the right to buy for K and sell asset at a higher price). The payoff of the original seller of the option (the option writer who has a short option position) is the mirror image of the buyer's payoff: the buyer's gain is the writer's loss: a *zero sum game*. See Figures 19.3 for an illustration.

The zero sum game property is true both for the payoff at exercise (19.1) as well as the profit

$$\text{call profit}_{t+m} = \text{call payoff}_{t+m} - C, \tag{19.2}$$

where C is the call price (typically paid in t). See Figure 19.4 for an illustrations. Notice that the price of the option (C) is always paid, irrespective of whether the option in exercised or not.

A buyer of an option does not have to post any margin, but a seller does. The reason is that a default of the seller would create a loss for the option buyer if the option is worth exercising. In contrast, the default of the buyer cannot create a loss for the seller.

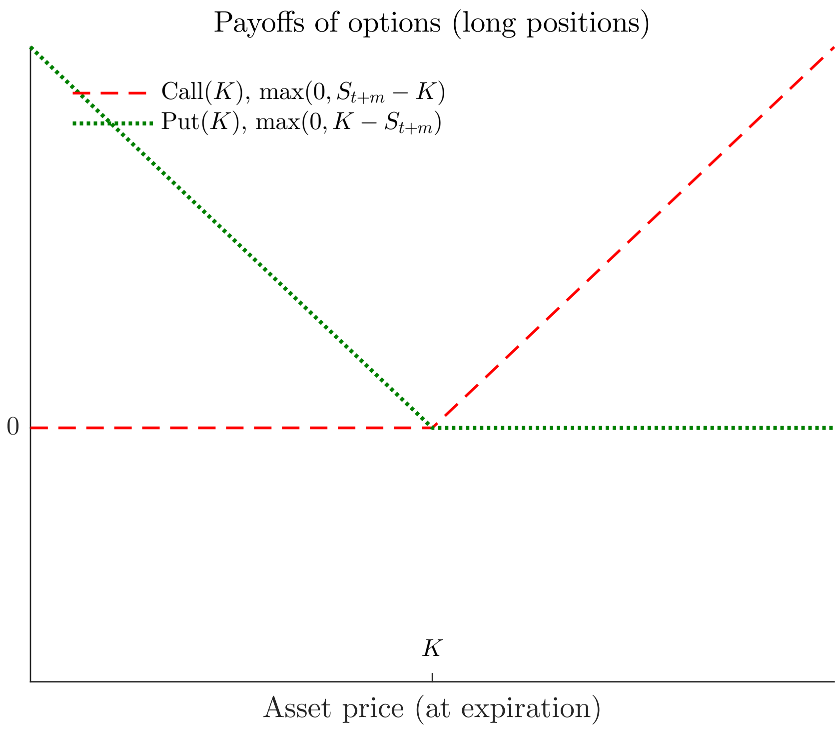


Figure 19.3: Payoffs of options, long positions

A *put* option instead gives the buyer of the contract the right to sell one unit of the underlying asset. The put price is here denoted by P. An owner of a put option benefits from a decrease in the price of the underlying asset (buy the asset cheaply and exercise the right to sell for K). The payoff is

$$\text{put payoff}_{t+m} = \max\left(0, K - S_{t+m}\right). \tag{19.3}$$

An option that would be profitable to exercise now is called *in-the-money*; an option that would be unprofitable to exercise is called *out-of-the-money*—and an option that would just break even is called at-the-money.

Figures 19.6–19.7 illustrate the trade intensity of options with different strike prices (but same expiration and underlying asset). It seems clear that most of the trade is in out-of-the-money options (high strike prices for the calls and low strike prices for the puts). Figure 19.8 shows that most of the trade is close to expiration, and there is a seasonality pattern related to rolling over the investment from other (expired) options. Figure 19.9 shows how the trading volume at CBOE has developed over time. The volume grew up

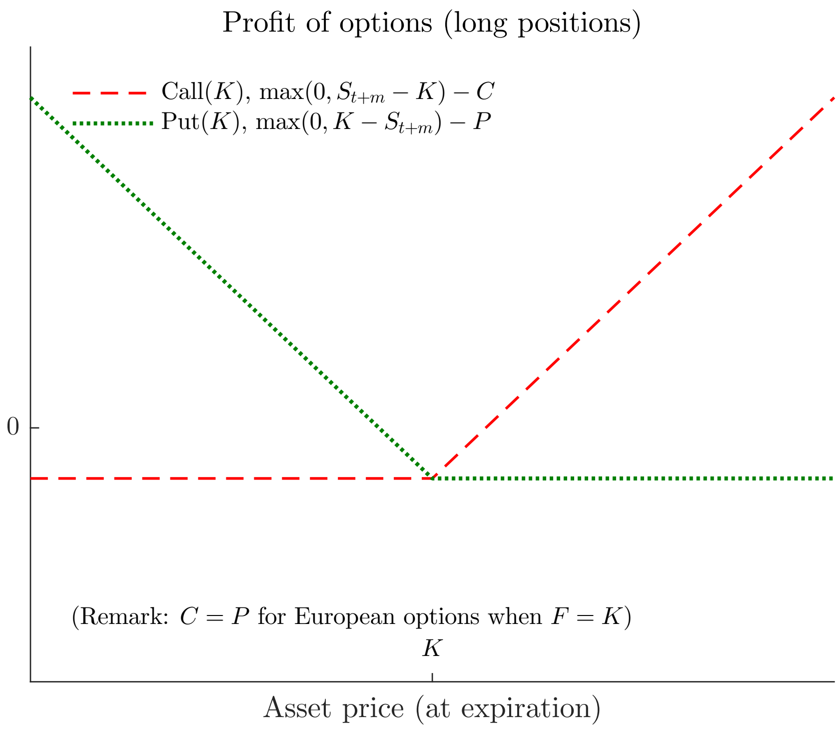


Figure 19.4: Profit of options, long positions

to the financial crisis, decreased somewhat during the crisis and has stabilized since. The ratio of traded put contracts to traded call contracts in Figure 19.10 is sometimes used to gauge market nervousness. The idea is that investors will demand put contracts if they want to insure against a stock market decline.

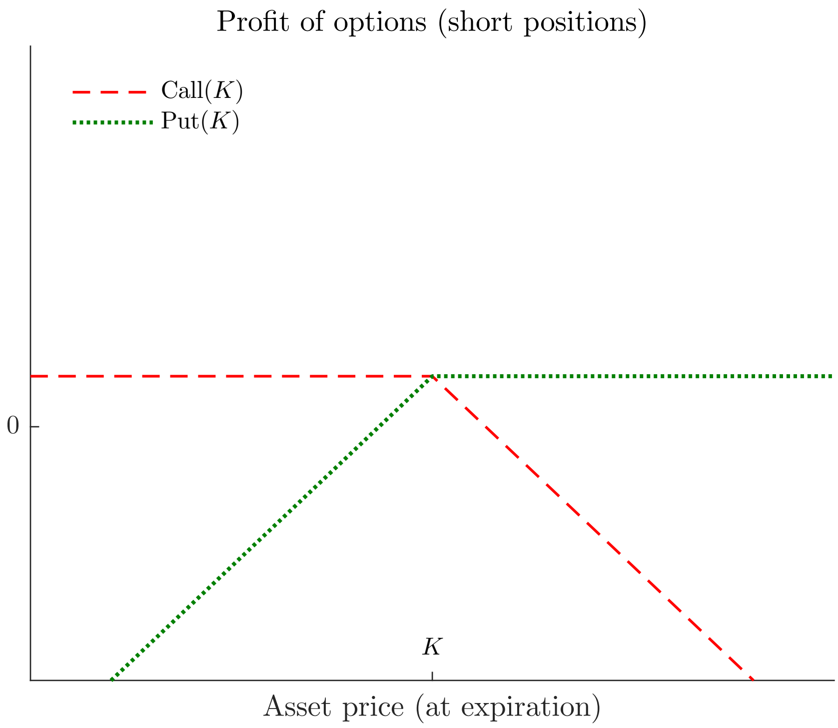


Figure 19.5: Profit of options, short positions

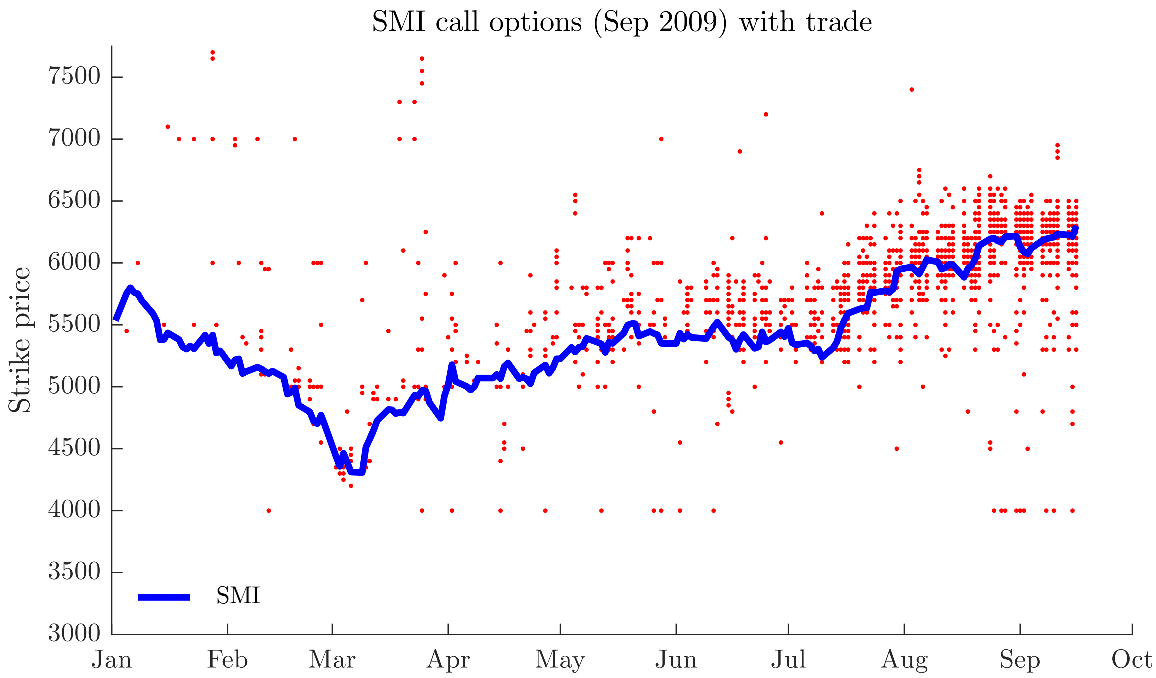


Figure 19.6: Traded options

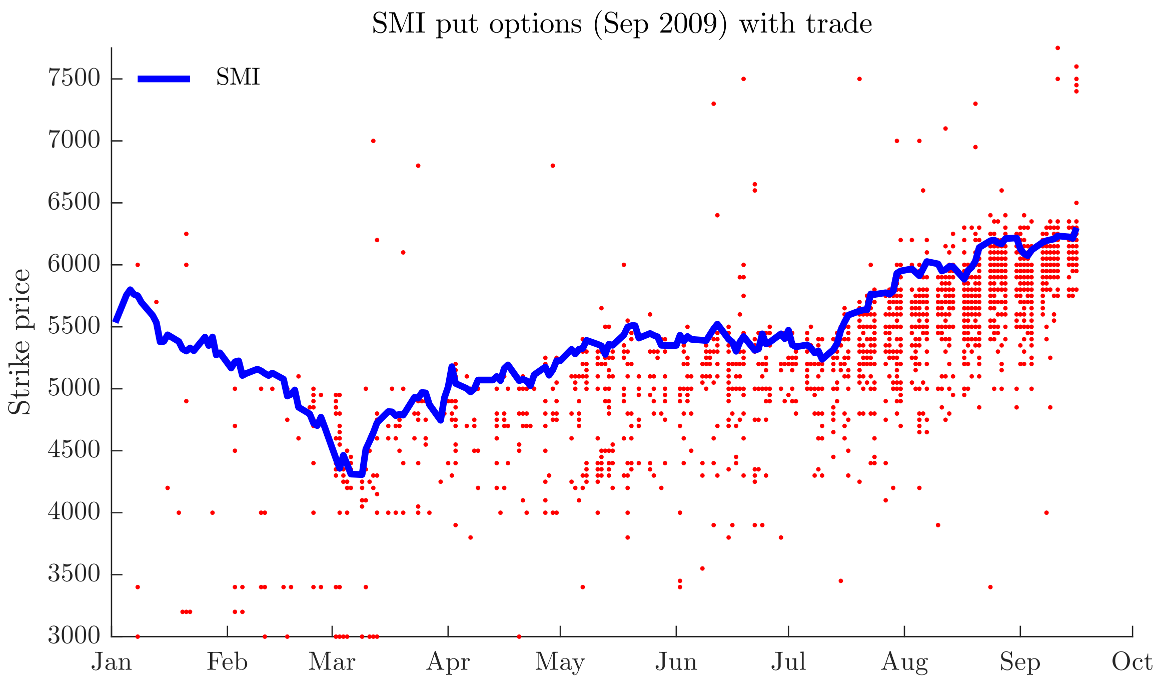


Figure 19.7: Traded options

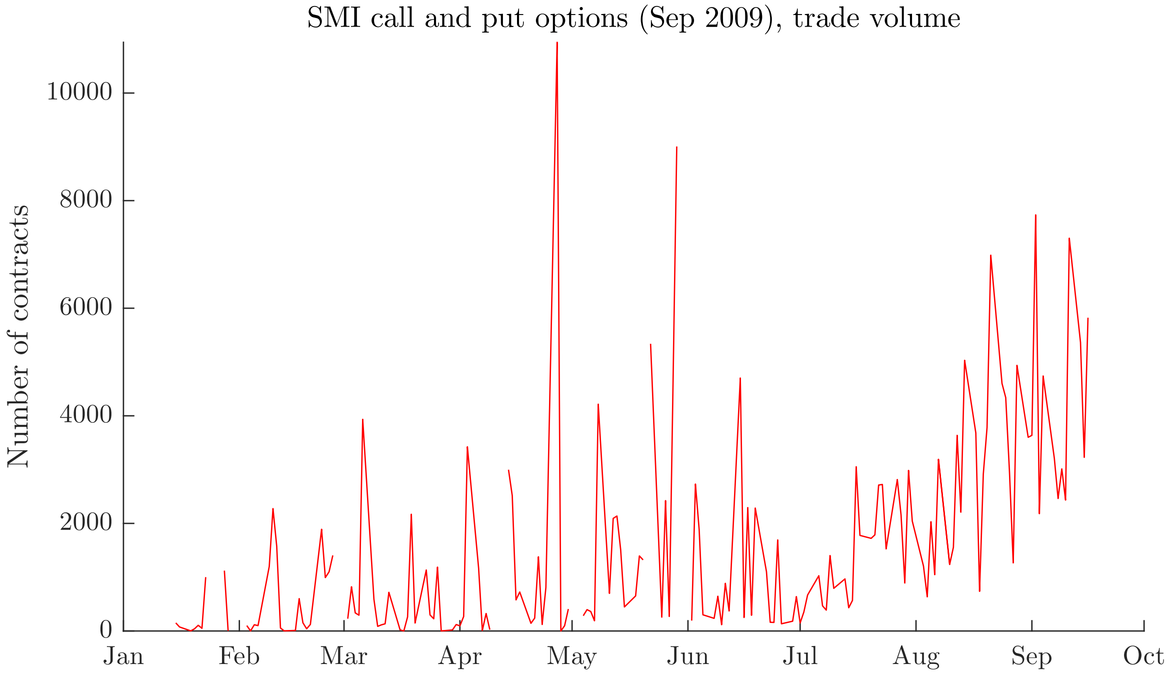


Figure 19.8: Option trade volume

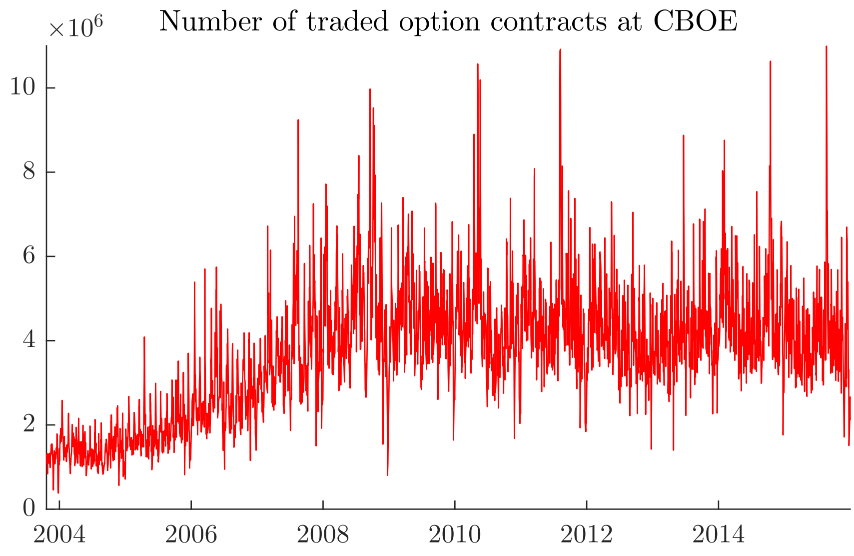


Figure 19.9: Option trade volume

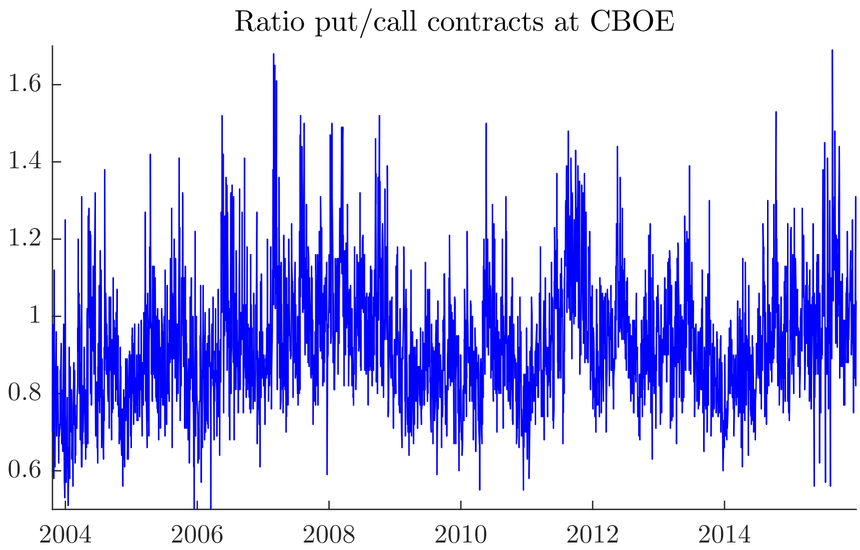


Figure 19.10: Option trade volume

19.2.2 Financial Engineering

Hedging a Future Purchase of the Underlying

Suppose your firm needs to buy oil next year, since you use oil to run the factory. The oil price fluctuates, so it is a source of risk. One way of hedging this risk is to enter a forward contract: this provides a complete insurance. Alternatively, could buy a call option: in this case you know that you will be able to buy oil for the strike price or less: it hedges against the risk up the oil price increasing, but keeps the chance of gaining from a lower price.

Replicating a Forward

Options markets are often very liquid—and are therefore useful for constructing replicating portfolios. The portfolio Call(K) - Put(K) for $K = F$ (the forward price) replicates a forward contract, so it is a synthetic forward. (Actually, Call(K) - Put(K) for any K will replicate a forward–but with an initial payment. If $K = F$, then there is no initial payment, just like for a forward contract. See the "put-call parity" below.) Clearly, we can then replicate a short position in a forward contract by selling such a portfolio. See Figure 19.11.

Portfolio Insurance

A *protective put* is a combination of a put and a position in the underlying asset. This allows the owner to capture the upside of the price movement (of the underlying), at the same time as insuring against the downside. This is indeed very similar to just buying a call option. See Figure 19.12.

Betting on Large Changes

An option is a bet on a change in a specific direction. Option portfolios can be constructed to instead make a bet on a large change in either direction (that is, high volatility): a *straddle* is Call(K) + Put(K), and a *strangle* is Call(K_2) + Put(K_1) where $K_1 < K_2$. See Figure 19.13.

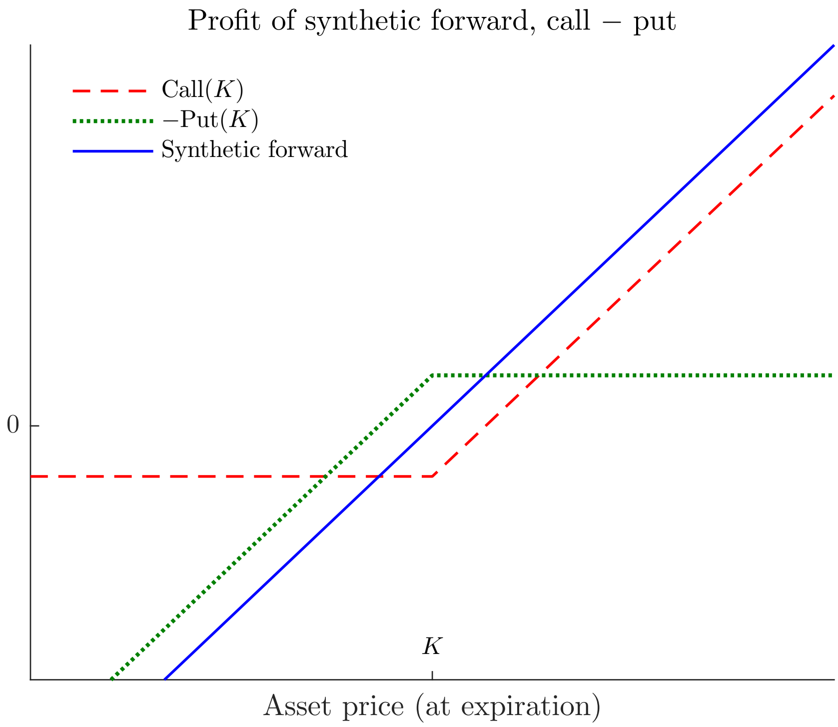


Figure 19.11: Profit of an option portfolio that replicates a forward contract

Putting a Collar on Losses and Gains

A *collared stock* is a combination of the underlying asset, a put with a low strike price (K_1) and a short call with a high strike price (K_2). This portfolio has a profit that increases one-for-one with the underlying asset as long as it is between K_1 and K_2. The losses for values of the underlying below K_1 are limited (by the put), and the gains for values above K_2 are also capped (by the short call). See Figure 19.14.

Betting on a Large Price Decrease

A variation on the synthetic short forward is the *collar*: $-\text{Call}(K_2) + \text{Put}(K_1)$ where $K_1 < K_2$. It also looks like a short position in a forward contract, except that the payoff is flat between the strike prices. Clearly, this is betting on a large price decrease. Selling a collar (or *reversal*) is instead a bet on a large price increase.

A collar (reversal) can be used to hedge a long (short) position in the underlying asset, except that there is no hedge between the strike prices. It provides insurance outside the strike prices. See Figure 19.15.

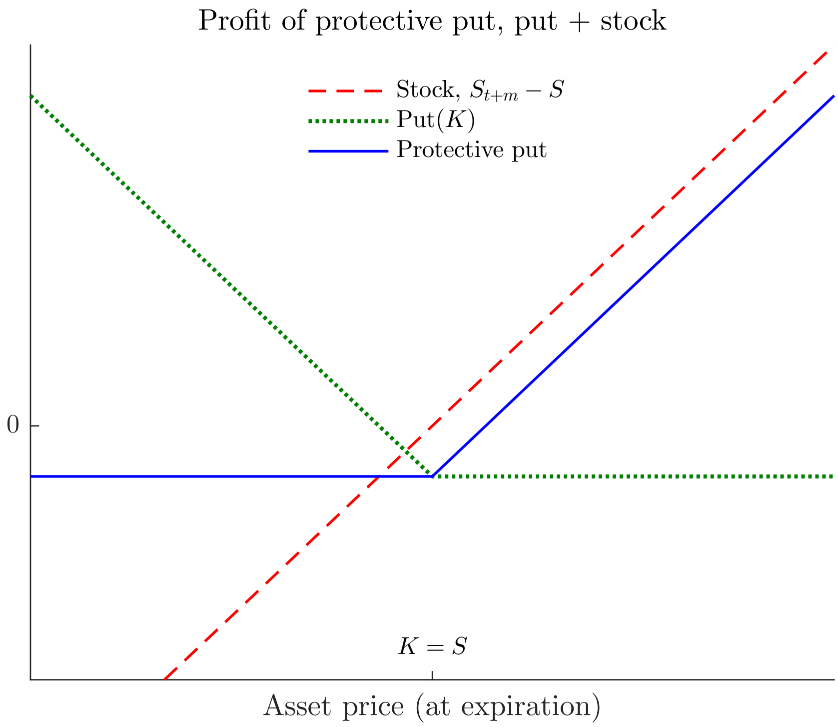


Figure 19.12: Profit of an option portfolio that insures a stock

Betting On a Small Price Increase

To bet on a small increase in the price of the underlying asset we can use a *bull spread*: Call(K_1) - Call(K_2) where $K_1 < K_2$. This portfolio has flat payoffs outside the strike prices, but a payoff that increases with the underlying asset between them. Selling a bull spread creates a *bear spread*, which is a bet on a small decrease of the underlying price. (These spreads can also be constructed by combing puts.) See Figure 19.15.

19.2.3 Options Are Risky Assets

The buyer always stands the risk of getting a zero payoff, that is, a return of -100%. For instance, the net return on a European call option is

$$\text{return on call}_{t+m} = \frac{\max\left(0, S_{t+m} - K\right)}{C} - 1, \tag{19.4}$$

where C is the call option price. Whenever the option isn't exercised, the whole investment is lost (and the return is -100%). However, for the owner of an option the risk is limited to the initial option price. In contrast, the option issuer can lose much more than

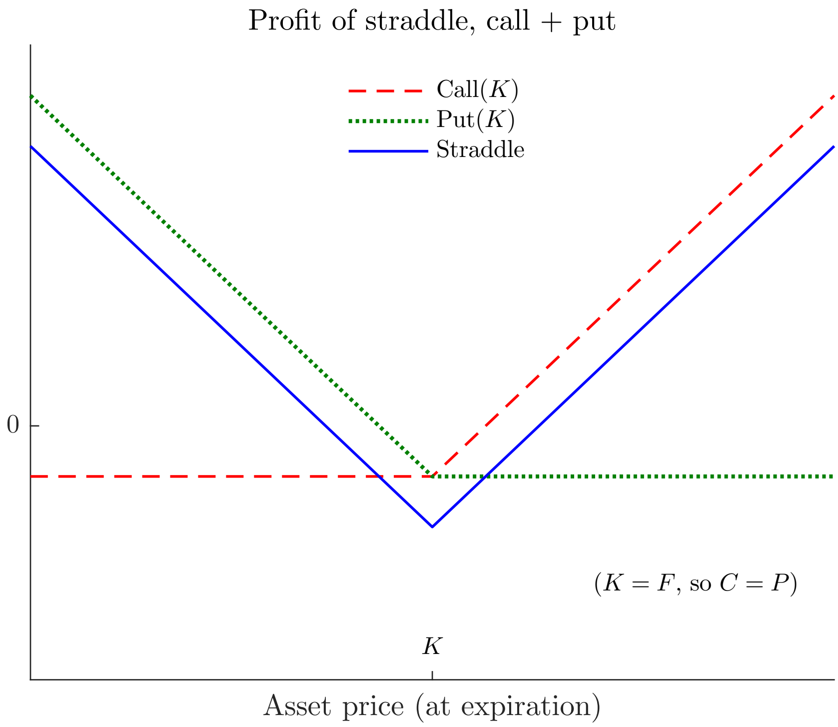


Figure 19.13: Profit of an option portfolio than bets on volatility

that. For these reasons, options issuers need to post margin, while option buyers do not.

It is clear that option returns cannot be normally (or even lognormally) distributed: the density function has a spike at -100% (whose probability is the same as the probability of $S_{t+m} \leq K$). This means, that we cannot motivate "mean-variance" pricing of options by referring to a normal distribution of the return. (This does not rule out mean-variance pricing, which could be motivated by, for instance, mean-variance preferences.)

19.2.4 Basic Properties of Option Prices

Options prices depend on many things, but there are some fairly general results

First, *call option prices are decreasing in the strike price*, while put options prices are increasing in the strike price, see Figure 19.16. The intuition is illustrated in Figure 19.17 which shows the perceived (by the market) distribution of the asset price at expiration. Notice that a higher strike price means that an owner of a call option will have to pay more in case of exercise—and there is also a lower chance of exercise. (Later sections provide a more formal proof on the same property.)

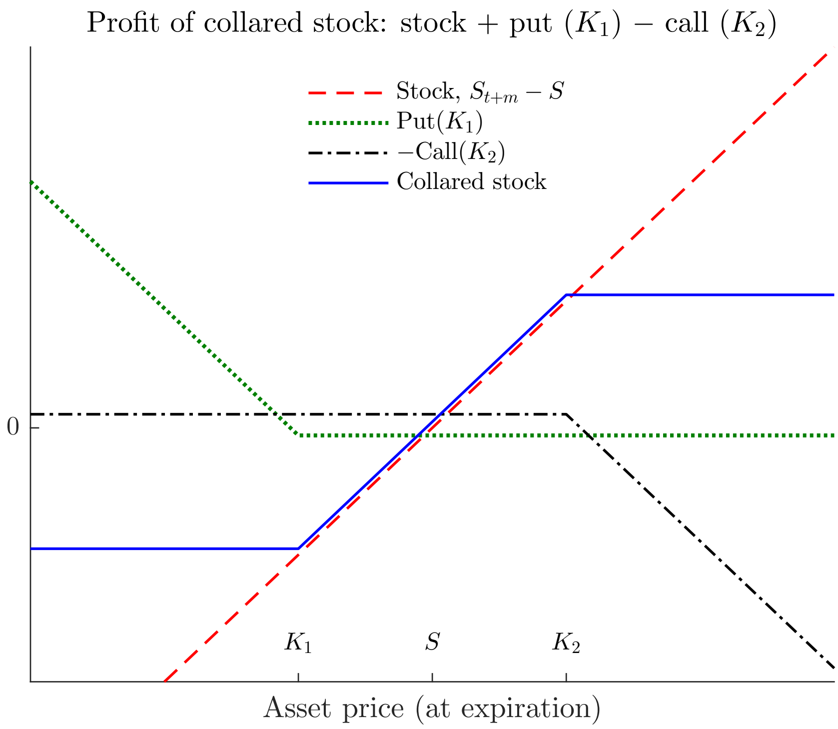


Figure 19.14: Profit of a collared stock

Second, both *call and put option prices are typically increasing in the (perceived) uncertainty* of the future price of the underlying asset, see Figure 19.18. The intuition is illustrated in Figure 19.19, which shows that a wider dispersion of the distribution increases the probability of a really high price of the underlying asset (although the figure is constructed to have the same probability of exercise in the two cases). Of course, it also increases the probability of a really low asset price, but that is of less concern since the call option payoff is bounded below at zero.

19.2.5 Definition of American Calls and Puts

An American option is like a European option, except that it *can be exercised on any day* before or on the expiration date. This means that an American option has more rights than a European option and is therefore worth at least as much

$$C_A \geq C_E \text{ and } P_A \geq P_E. \tag{19.5}$$

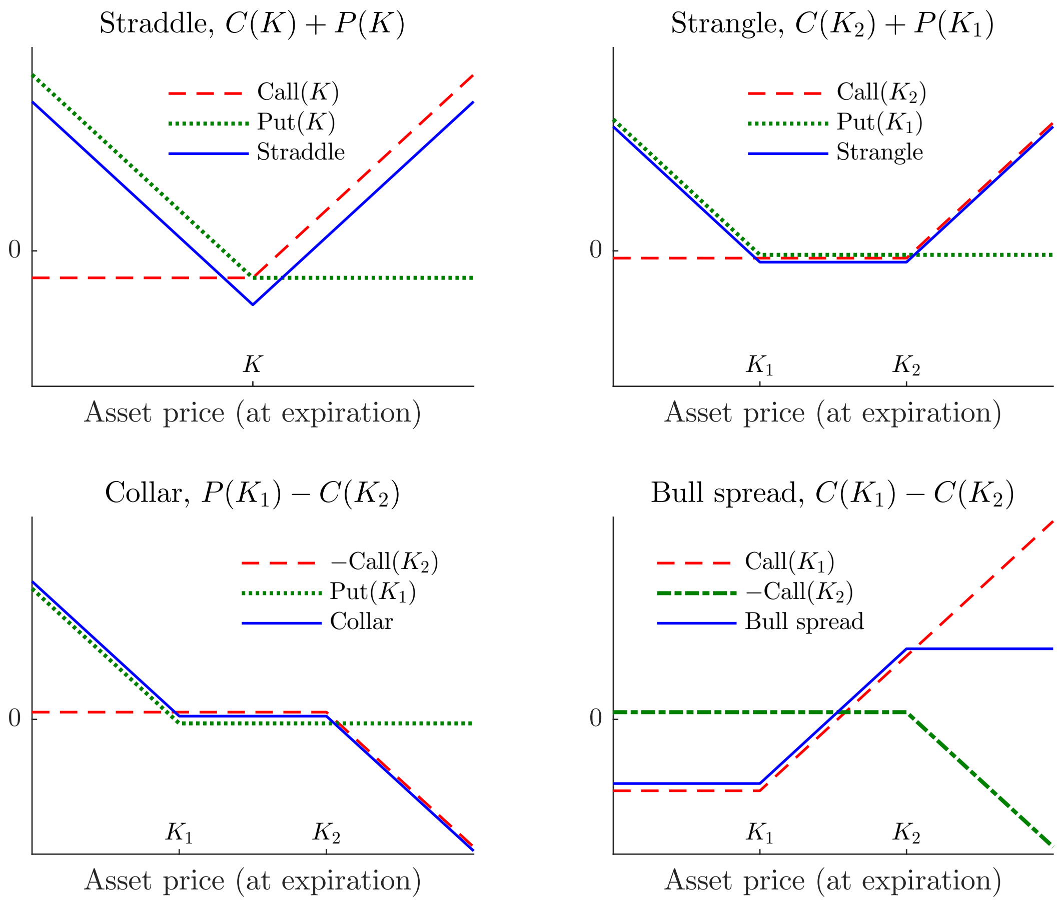


Figure 19.15: Profits of option portfolios

If the (American) option is exercised, then the immediate payoff is $\max(0, S-K)$ for a call and $\max(0, K-S)$ for a put. This means, that the options prices must (at any point in time) obey

$$
\begin{aligned}
C_A &\geq \max(0, S-K) \\
P_A &\geq \max(0, K-S),
\end{aligned}
\tag{19.6}
$$

where S is the current price of the underlying. The right hand sides are what you get by exercising the option now, and are called the "intrinsic values." (It can be shown that the first of these expressions is true also for European style options, but the second need not be.)

We will later demonstrate the following results. First, if there are no dividends, then it is never optimal to exercise an American call option early (such a call option will have the

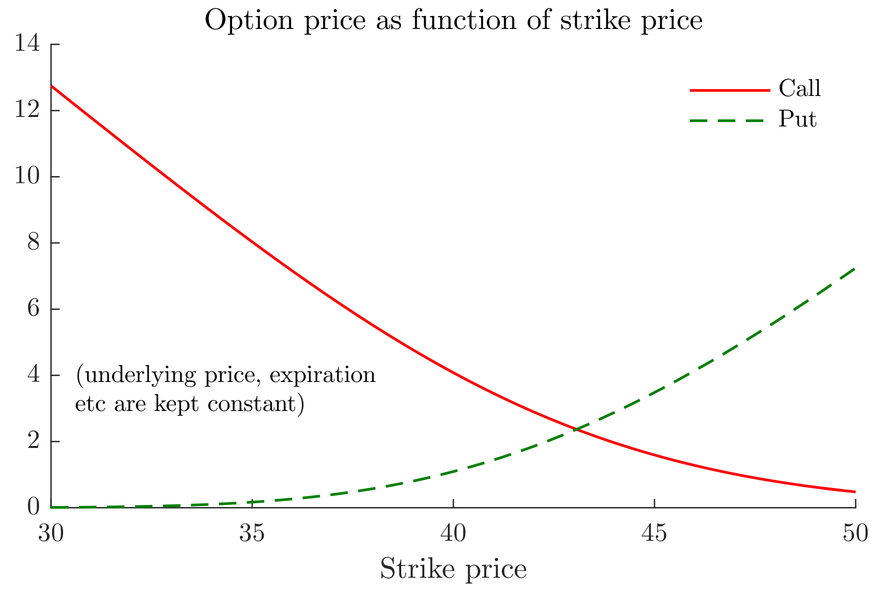


Figure 19.16: Option price as a function of the strike price

same price as a European call option), but it can still be optimal to exercise an American put option early. Second, if there are dividends, then the American call option should only be exercised just prior to the dividend payments, while an American put should perhaps also be exercised also at other times. See Figure 19.21.

Remark 19.1 *Figures 19.20 and 19.21 provide an example of how the futures price (on S&P 500), the intrinsic value of the option and the option price developed over a year. Notice how the futures prices converges to the index level at expiration of the futures. Be-*

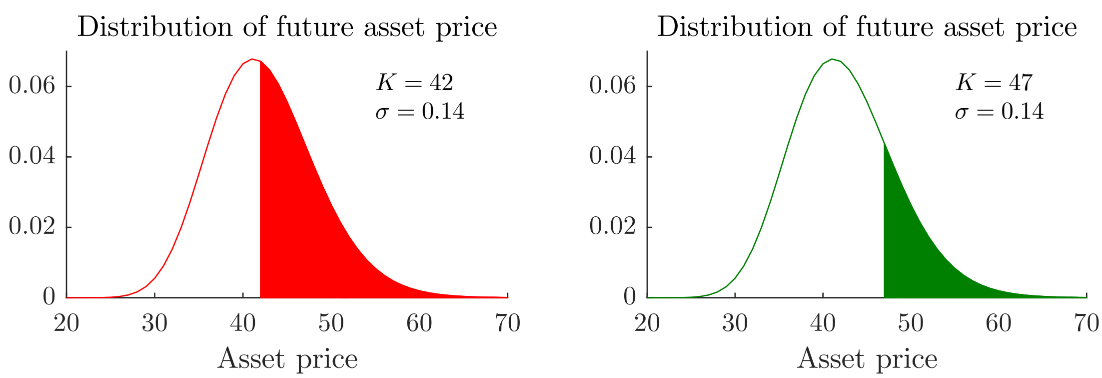


Figure 19.17: Distribution of future stock price

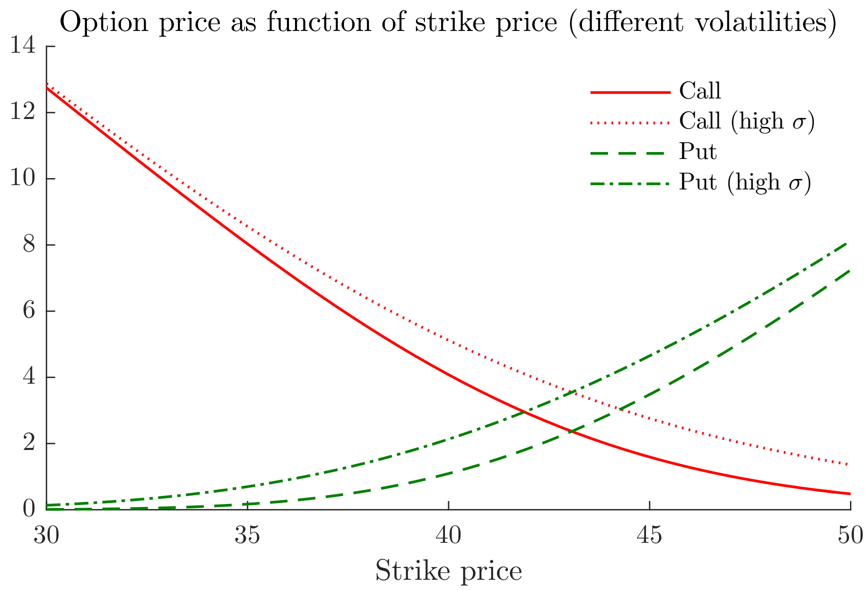


Figure 19.18: Option price as a function of the strike price

fore it can deviate because of delayed payment (+) and no part in dividend payments (−). Also notice that even options with zero intrinsic value (zero payoff if exercised now) can have fairly high option prices—at least if the time to expiration is long, but it converges to zero the expiration date gets closer.

19.3 Put-Call Parity for European Options

There is a tight link between European call and put prices. If you know one of them (and the forward price), then you can easily calculate what the other must be. The following proposition is more precise.

Proposition 19.2 *(Put-call parity for European options) The put-call parity for European options is*

$$C - P = e^{-my}(F - K), \tag{19.7}$$

where $e^{-my}(F - K)$ is the present value of the forward price minus the strike price.

Time subscripts and indicators of maturity have been suppressed to make the notation a bit easier. The parity holds irrespective of whether the underlying asset has dividends or not (since the expression uses the forward price). Its practical importance is that it allows

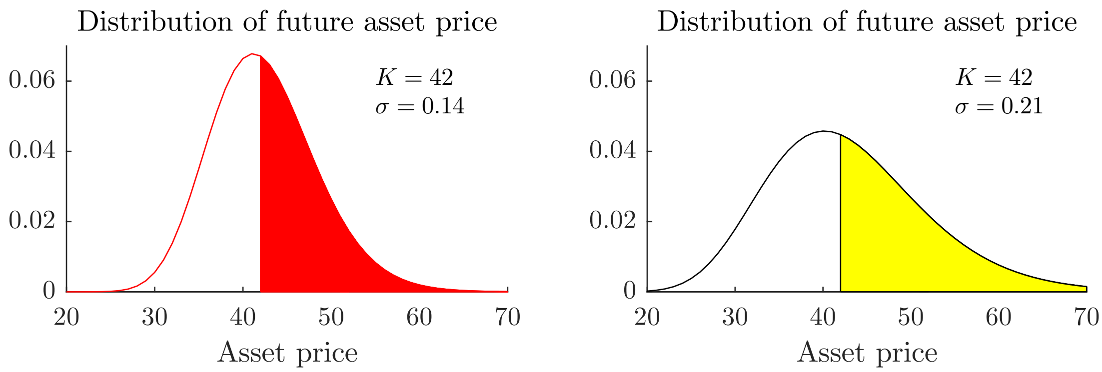


Figure 19.19: Distribution of future stock price

us to use two of the assets to replicate the third asset. For instance, we can combine a call option (with strike price $K = F$) and a forward contract to replicate a put option, or buy a call and sell a put (with strike price $K = F$) to replicate a forward contract. Transaction costs can cause (relatively small) deviations from the parity condition. See Figure 19.22 for an illustration.

Proof. (of Proposition 19.2) Buy one call option and sell one put option, both with the strike price K. This will with certainty give one asset at maturity at the price K (since the class *or* the put will be exercised). The present value of the cost is $C - P + e^{-my}K$. The same is achieved by entering a forward contract—the present value of the cost is $e^{-my}F$.
■

This formula is very general, but a few special cases are of particular interest. First, when the underlying asset pays no dividends, then (19.7) together with the forward-spot parity give

$$C - P = S - e^{-my}K \text{ if no dividends,} \tag{19.8}$$

$$C - P = S - \sum\nolimits_{i=1}^{n} e^{-m_i y_t(m_i)} D_i - e^{-my}K \text{ if dividends,} \tag{19.9}$$

$$C - P = Se^{-\delta m} - e^{-my}K \text{ if continuous dividend rate } \delta. \tag{19.10}$$

Example 19.3 *(Put-call parity) $S = 42, m = 1/2, y = 5\%, K = 38$. If $C = 5.5$ for an underlying asset without dividends, then (19.8) gives*

$$5.5 - P = 42 - e^{-0.5 \times 0.05} 38 \text{ or } P \approx 0.56.$$

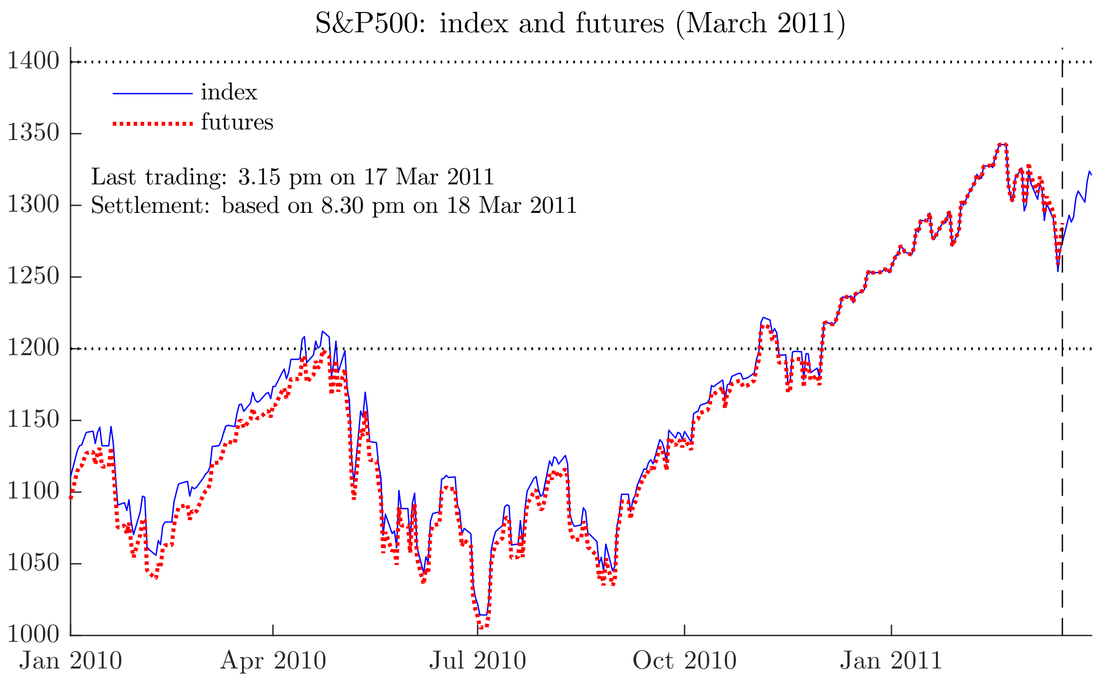


Figure 19.20: S&P 500 index level and futures

19.3.1 Put-Call Parity and Synthetic Replications*

The following remarks provides details on how two assets can be used to replicate a third—since they are all tied together by the put-call parity.

Remark 19.4 *(Synthetic forward) Buy one call and sell one put at a strike price that equals the forward price prevailing in t* ($K = F_t$)*. By (19.7), the cost of this portfolio is zero. At expiration, it will give one unit of the underlying, at the cost* K*. Just like a forward contract. See Figure 19.22.*

Remark 19.5 *(Synthetic call option) Buy one forward and one put with strike price* $K = F$*. By (19.7), this has the price* C*. If, at expiration,* $S_{t+m} < K$*, then the forward pays off* $S_{t+m} - F$ *and the put option* $K - S_{t+m}$*. Since* $K = F$*, the sum is zero. Instead, if* $S_{t+m} > K$*, then the forward pays off* $S_{t+m} - F$ *and the put nothing. In either case, this is just like a call option with strike price* K*. See Figure 19.22.*

Remark 19.6 *(Synthetic put option) Buy one call with strike price* $K = F$ *and sell one forward. By (19.7), this has the price* P*. If, at expiration,* $S_{t+m} < K$*, then the call pays off nothing and the short forward* $-(S_{t+m} - F)$*. Since* $K = F$*, the sum is* $K - S_{t+m}$*.*

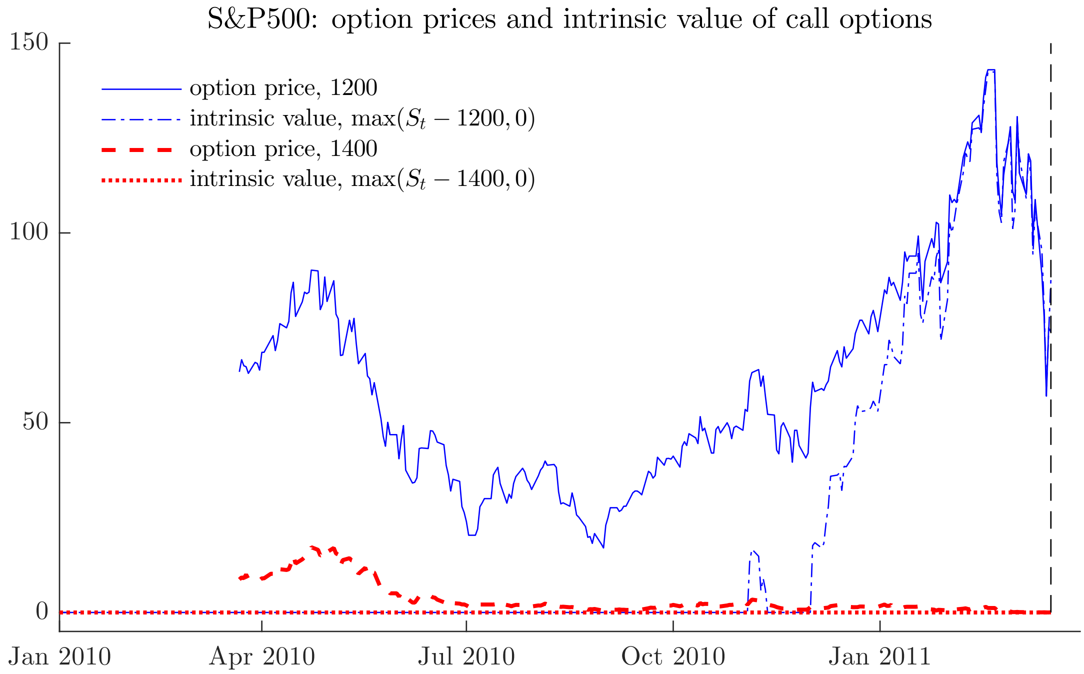


Figure 19.21: S&P 500 options

Instead, if $S_{t+m} > K$, then the call pays off $S_{t+m}-K$ and the short forward $-(S_{t+m}-F)$, which sums to zero. In either case, this is just like a put option with strike price K. See Figure 19.22.

19.4 Pricing Bounds and Convexity of Pricing Functions

19.4.1 Pricing Bounds for (European and American) Call Options

The prices of American or European call options must satisfy the following restrictions

$$C \leq e^{-my} F \leq S \tag{19.11}$$

$$0 \leq C \tag{19.12}$$

$$e^{-my}(F-K) \leq C \tag{19.13}$$

The motivations are basically as follows (the intuition based on European options, but the results extend to American options as well). First, a call option with a zero strike price ($K = 0$) would be the same as owning a prepaid forward contract (which is worth as much or less than the underlying asset). Whenever the strike price is higher, the call

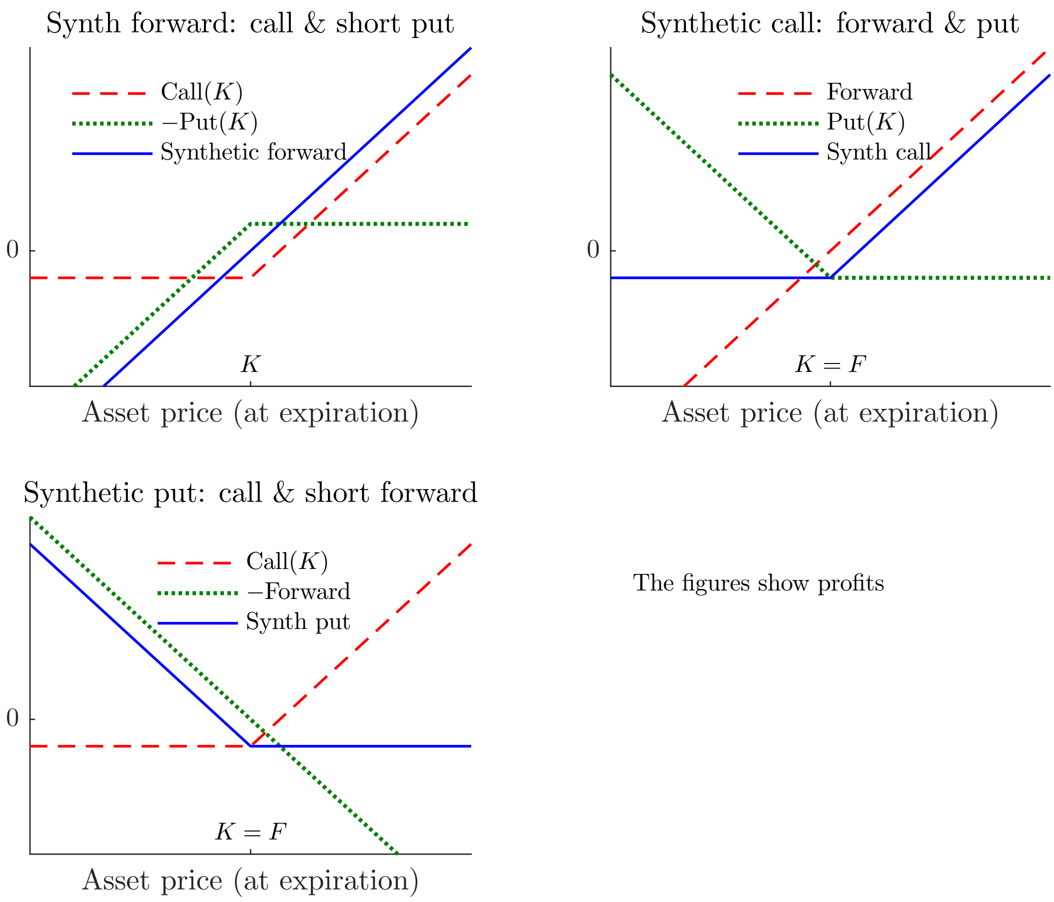


Figure 19.22: Put-call parity

price is lower. Second, the call option gives rights, not obligations: its price value cannot be negative. Third, the lowest possible value of a put option is zero, so the put-call parity (19.7) immediately gives that the call price must exceed the present value of $F - K$. (See below for an alternative proof.) Transaction costs can cause (relatively small) failures of the bounds.

Combining the bounds, we get

$$C \leq e^{-my} F \leq S \tag{19.14}$$

$$C \geq \max[0, e^{-my}(F - K)]. \tag{19.15}$$

In particular, for a financial asset without dividends (until expiration of the option), we have $C \leq S$ and $C \geq \max(0, S - e^{-my} K)$. See Figures 19.23, 19.24 and 19.28 for illustrations.

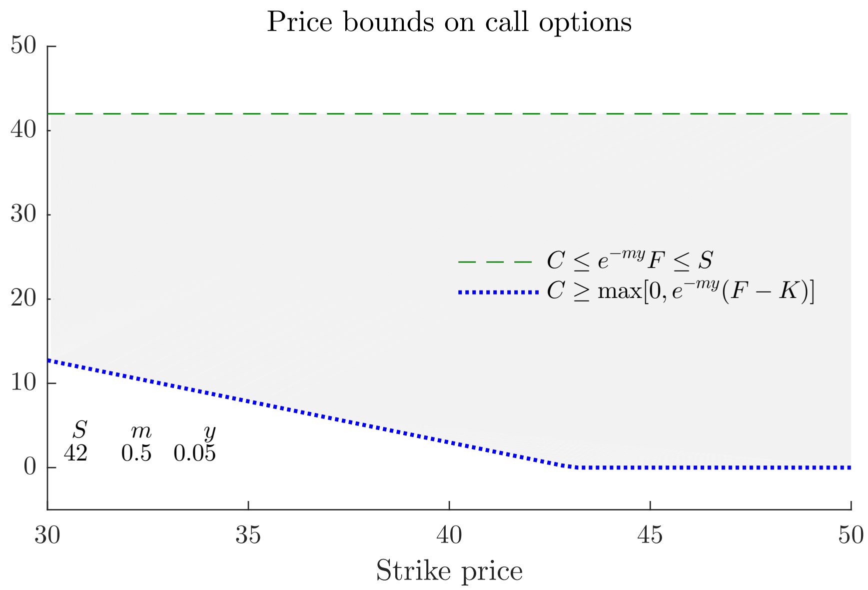


Figure 19.23: Call option price bounds as a function of the strike price

Remark 19.7 *(The put price bounds in Figure 19.24) At very low strike prices, it is almost certain that the option will be exercised at expiration. Therefore, the present value of the cost, $C + e^{-ym}K$, must be almost equal to the present value of a forward contract, $e^{-ym}F$. Combining gives $C = e^{-my}(F - K)$. In contrast, at very high strike prices, the probability of exercise is almost zero—so the option price is too.*

Example 19.8 *(Pricing bounds for call option) Using the same parameters as in Example 19.3, we get the following bounds*

$$4 \leq C \leq 42.$$

In this case, the second bound ($0 \leq C$) is superfluous.

Proof. (of (19.13)) Portfolio A: one European call option and $e^{-my}K$ on a bank account. At expiration, this portfolio is worth S_{t+m} if the option is exercised, and K otherwise: $\max(S_{t+m}, K)$. Portfolio B: one prepaid forward contract, which is worth S_{t+m} at expiration. (Since you pay $e^{-my}F$ now, there is no payment at expiration.) Clearly, portfolio A is always worth more at expiration, so it must also be worth more right now: $C_E + e^{-my}K \geq e^{-my}F$. Rearrange to get (19.13). Since $C_A \geq C_E$, the bound holds also for an American call option. ■

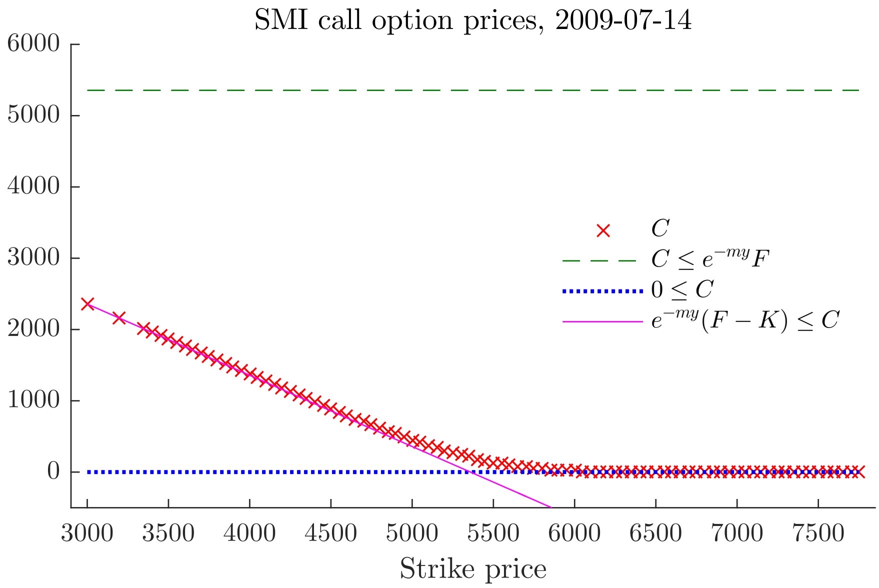


Figure 19.24: Prices and bounds for SMI options

19.4.2 Pricing Bounds for (European and American) Put Options

The prices of American and European put options must satisfy the following restrictions

$$P_E \leq e^{-my}K \text{ and } P_A \leq K \tag{19.16}$$

$$0 \leq P_E \text{ and } 0 \leq P_A \tag{19.17}$$

$$e^{-my}(K-F) \leq P_E \text{ and } K-S \leq P_A. \tag{19.18}$$

See Figure 19.25.

The motivations are as follows. First, the payoff from a put option is $\max(K-S, 0)$, so the maximum value is the strike price (when $S = 0$). For the European put, this payoff is received only at expiration, so the maximum value today is the present value of the strike price. Second, the put option gives rights, not obligations: its price value cannot be negative. Third, the lowest possible value of a call option is zero, so the put-call parity (19.7) immediately gives that the European put price must exceed the present value of $K-F$. (See below for an alternative proof.) In contrast, the American put can be exercised now so its value must be at least as high as the intrinsic value.

Proof. (of (19.18)) Portfolio A: one European put option and a prepaid forward contract. At expiration, this portfolio is worth K if the option is exercised, and S_{t+m} otherwise: $\max(K, S_{t+m})$. Portfolio B: $e^{-my}K$ on a bank account, which is worth K at expi-

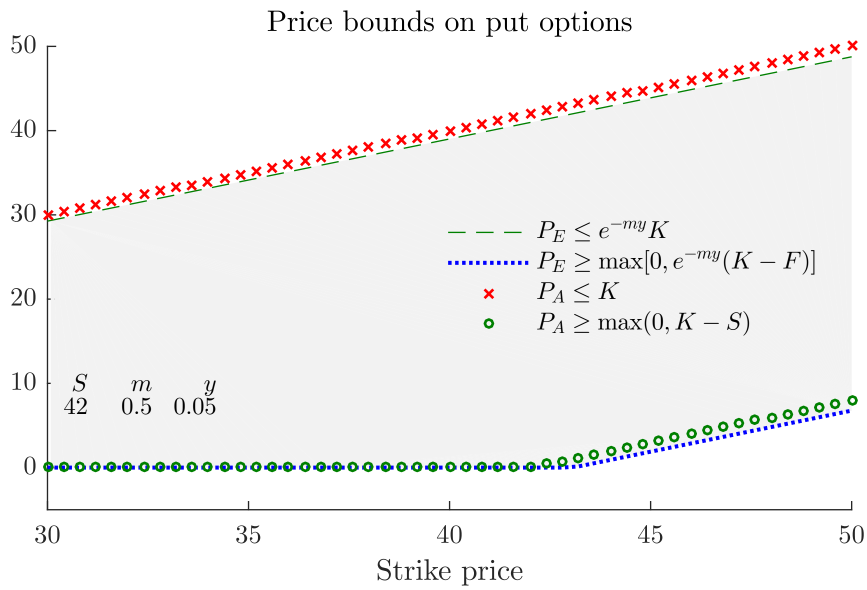


Figure 19.25: Put option price bounds as a function of the strike price

ration. Clearly, portfolio A is always worth more at expiration, so it must also be worth more right now: $P_E + e^{-my}F \geq e^{-my}K$. Rearrange to get (19.18). Since $P_A \geq P_E$, the bound holds also for an American put option. ■

19.4.3 Prices of Call Options for Different Strike Prices*

Suppose we have American or European call options with different strike prices, $K_1 < K_2$. We the have the following price relations

$$C(K_2) - C(K_1) \leq 0 \tag{19.19}$$

$$\frac{C(K_2) - C(K_1)}{K_2 - K_1} \geq -1 \tag{19.20}$$

$$C\left[\lambda K_1 + (1-\lambda)K_2\right] \leq \lambda C(K_1) + (1-\lambda)C(K_2), \text{ for } 0 \leq \lambda \leq 1. \tag{19.21}$$

The first relation says that the call option price is decreasing in the strike price. The intuition is that a higher strike price means that an owner of a call option will have to pay more in case of exercise—and there is also a lower chance of exercise. The second relation says that change is smaller than the change in the strike price. The third relation says that the relation is convex. If these relations do not hold, then there are arbitrage opportunities (see the proofs below).

In other words, these three conditions say that we have the following partial derivatives (if they exist) of the call option price function

$$-1 \leq dC(K)/dK \leq 0 \text{ and } dC^2(K)/dK^2 \geq 0. \tag{19.22}$$

This means that the call option price is decreasing in the strike price, but slower than the strike price itself, but that the curve flattens out at high strike prices. See Figure 19.24 for an illustration.

Proof. (of (19.19)) If (19.19) was not true, so $C(K_2) > C(K_1)$, then a bull spread (buy $C(K_1)$ and sell $C(K_2)$), would have a negative price $(C(K_1) - C(K_2) < 0)$. However, the payoff of a bull spread is

$$\max(0, S - K_1) - \max(0, S - K_2) = \begin{cases} 0 & \text{if } S \leq K_1 \\ S - K_1 & \text{if } K_1 < S \leq K_2 \\ K_2 - K_1 & \text{if } K_2 < S. \end{cases}$$

This would give a non-negative payoff for a negative asset price, which creates arbitrage opportunities. ∎

Proof. (of (19.20)) If (19.20) was not true, so $C(K_1) - C(K_2) \geq K_2 - K_1$, then we can sell a bull spread (sell $C(K_1)$ and buy $C(K_2)$) and invest the proceeds in a T-bill (zero investment). The payoff at expiration (m period later) is then

$$\max(0, S-K_2) - \max(0, S-K_1) = \underbrace{[C(K_1) - C(K_2)]e^{rm}}_{>K_2-K_1} + \begin{cases} 0 & \text{if } S \leq K_1 \\ -(S - K_1) & \text{if } K_1 < S \leq K_2 \\ -(K_2 - K_1) & \text{if } K_2 < S. \end{cases}$$

In either case, there is a positive profit (recall that the initial investment is zero), which creates arbitrage opportunities. ∎

Proof. (of (19.21)) Let $\bar{K} = \lambda K_1 + (1 - \lambda)K_2$. If (19.21) was not true, so $C(\bar{K}) > \lambda C(K_1) + (1 - \lambda)C(K_2)$, then we can sell $C(\bar{K})$ and buy $\lambda C(K_1) + (1 - \lambda)C(K_2)$ (zero investment). The payoff at expiration (m period later) is then

$$\lambda \max(0, S - K_1) - \max(0, S - \bar{K}) + (1 - \lambda) \max(0, S - K_2)$$
$$= \begin{cases} 0 & = 0 & \text{if } S \leq K_1 \\ \lambda(S - K_1) & = \lambda(S - K_1) & \text{if } K_1 < S \leq \bar{K} \\ \lambda(S - K_1) - (S - \bar{K}) & = (1 - \lambda)(S - K_2) & \text{if } \bar{K} < S \leq K_2 \\ \lambda(S - K_1) - (S - \bar{K}) + (1 - \lambda)(S - K_1) & 0 & \text{if } K_2 < S, \end{cases}$$

where the second column uses the definition of $\bar{K}$. All payoffs are non-negative, and some are positive. Since the initial investment is zero, this creates arbitrage opportunities. ■

19.5 Early Exercise of American Options

This section discusses early exercise of American options. There are some cases where we can exclude early exercise, so the American option is priced as a European option. In other cases, we cannot exclude early exercise—but we may still be able to say something about when early exercise is likely. More precise answers will require building a model for the pricing. Clearly, the answer is then model dependent.

The key results are as follows (assuming interest rates are positive):

	without dividends	with dividends
Call	no early exercise	early exercise (at high S)
Put	early exercise (at low S)	early exercise

Proofs and details are found in the Appendix. (Negative interest rates means that you could plausibly have early exercise for all four types. In general, negative interest rates makes it more attractive to exercise call options earlier—to get rid of the cash, and exercise put options later—to not receive cash early.)

An American put may well be exercised early if the current price of the underlying is low. The bankruptcy case ($S = 0$) is an extreme, but illustrating case.

Example 19.9 *(Bankruptcy, American put, no dividends) Suppose the underlying asset goes bankrupt, then* $S = 0$ *and it is known that it will stay at* $S = 0$. *Exercising the American put option now gives* K*, whereas waiting until expiration has a present value of* $e^{-my}K$ *(which is lower): early exercise is optimal.*

See Figures 19.26–19.27 for an illustration, based on a numerical solution (of a specific model, so the precise results are not general) for the price on an American put option. The first figure shows in which nodes early exercise is optimal: at low asset prices. The second picture illustrates that the American put price is close to the European put price when the asset price is low, that is, when early exercise is unlikely to happen in the near future. However, the American put price starts to increase above the European put price when the asset price is lower.

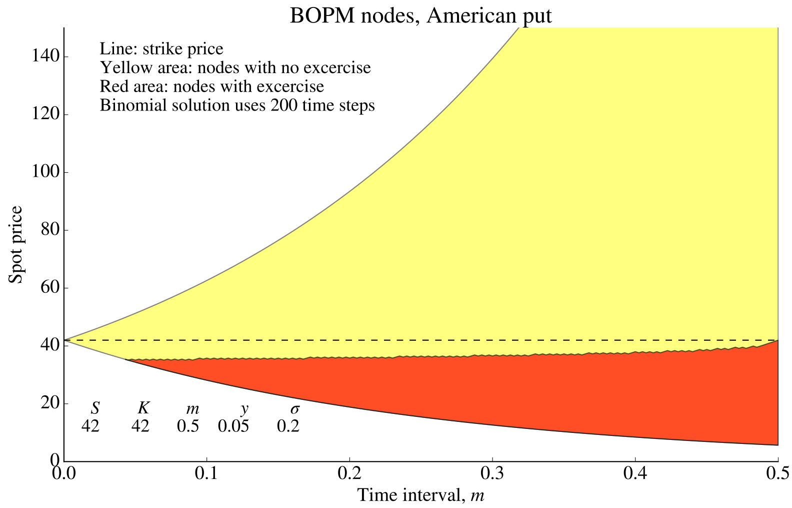


Figure 19.26: Numerical solution of an American put price (no dividends)

19.6 Appendix: Details on Early Exercise of American Options*

19.6.1 Early Exercise of American Call Options (No Dividends)

American call options on an asset without dividends (until expiration of the option) are not exercised early. The following proposition is more precise.

Proposition 19.10 *(No early exercise, American call, no dividends) An American call option on an asset without dividends should never be exercised early (if the interest rate is positive). It therefore has the same price as a European call option.*

See Figure 19.28 for an illustration of the fact that early exercise is not profitable since $C_A \geq C_E > \max(0, S - K)$.

Suppose that you are pretty sure that price of the underlying will drop tomorrow. The above proposition suggests that you should still not exercise the call option, but it might be sensible to sell the option today. If we exercised early, then we would effectively throw away the downside protection inherent in the call option and be left with the underlying asset and also pay the strike price now instead of later—neither of which is good (and which a potential buyer of the call option would be willing to pay for).

One way to think about this situation is as follows. If you think the underlying asset will drop in price soon, then you might be tempted to exercise now. If you do so and immediately sell the underlying, then you get $S_t - K$, which is worth $e^{y365}(S_t - K)$

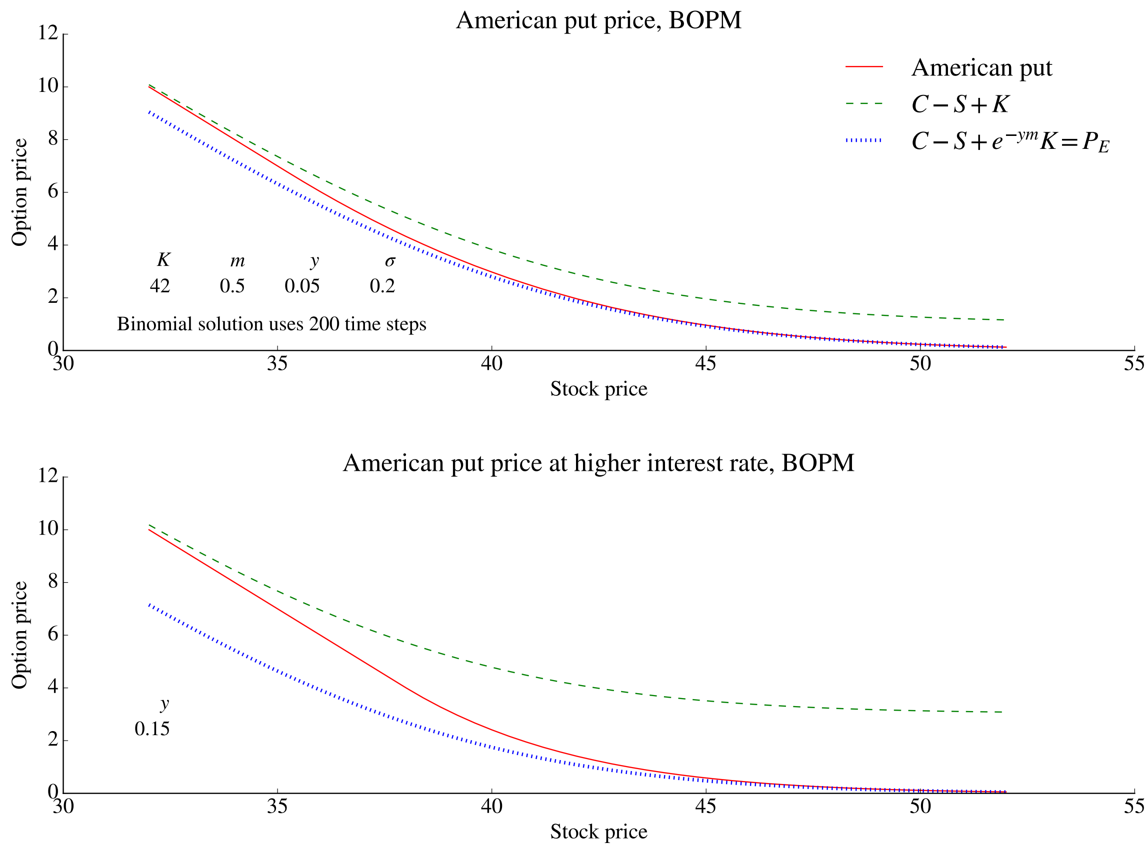


Figure 19.27: Numerical solution of an American put price (no dividends)

tomorrow. Rather, by keeping the option and shortening the underlying asset you will be better off. By shortening today you also get S_t, which is worth $e^{y365}S_t$ tomorrow. Tomorrow, you need to close the short position: buy the asset for S_{t+1} or exercise the option (whichever is cheaper), so you pay $\min(K, S_{t+1})$, giving you a total value of at least $e^{y365}S_t - \min(K, S_{t+1})$ tomorrow. (In case you do not exercise the option you also keep the value of it.) This is more than if you had exercised today. This shows that you should wait with exercising (and the same type of argument applies tomorrow).

Proof. (of Proposition 19.10) First, consider the case when $C_A > 0$. Then, (19.15) shows that, as long as the interest rate is positive, $C_A > \max(0, S - K)$ for an underlying asset without dividends (since $S = e^{-my}F$ and $K > e^{-my}K$). This is higher than what exercising now gives, $S - K$, so exercising now is suboptimal. Second, if $C_A = 0$, then we know from (19.6) that $S \leq K$: (a) with $S < K$ exercise brings a loss; (b) while with $S = K$ and $C = 0$ we are indifferent (early exercise could thus happen, but it would be

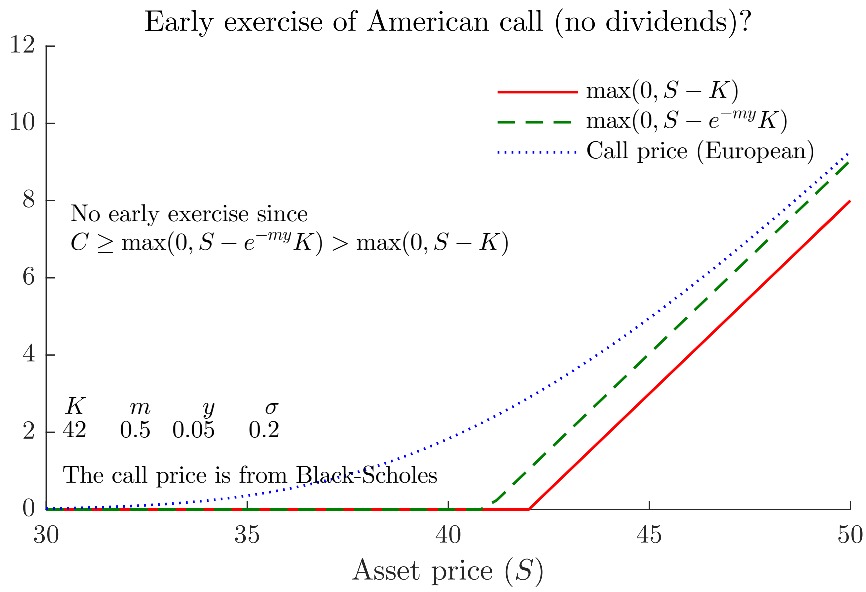


Figure 19.28: Early exercise of American call option (no dividends)

pointless). ■

Proof. (of Proposition 19.10, alternative*) First, consider the case when $C_A > 0$. Consider an investor who is willing to keep the underlying asset until tomorrow (at least). Clearly, such investors must exist, or else the underlying asset would not be worth its current price. Portfolio A: one American call option and $e^{-y/365}K$ on a bank account. Tomorrow, this portfolio is worth S_{t+1} if the option is exercised, and $K + C_{A,t+1}$ otherwise: $\max(S_{t+1}, K + C_{A,t+1})$. Portfolio B: one unit of the underlying asset, which is worth S_{t+1} tomorrow. Clearly, portfolio A is worth at least as much as B tomorrow, so it must also be worth at least as much right now: $C_A + e^{-y/365}K \geq S$. Rearranging gives $C_A \geq S - e^{-y/365}K$, and we also know that $C_A > 0$, so $C_A > \max(0, S - K)$ as long as the interest rate is positive. If you are not an investor who is willing to keep the underlying asset until tomorrow, then you should sell the option to such an investor. Second, if $C_A = 0$, then we know from (19.6) that $S \leq K$: (a) with $S < K$ exercise brings a loss; (b) while with $S = K$ and $C = 0$ we are indifferent (early exercise could thus happen, but it would be pointless). ■

19.6.2 Early Exercise of American Put Options (No Dividends)*

American put options on an asset without dividends (until expiration of the option) may be exercised early. The following proposition is more precise.

Proposition 19.11 *(Early exercise, American put, no dividends) An American put option on an asset without dividends could be exercised early. However, we can rule out early exercise when* $P_E > \max(0, K - S)$*, since* $P_A \geq P_E$ *then implies that selling the option is better than exercising. From the put-call parity for European options, we notice that* $P_E > \max(0, K - S)$ *happens when* $C_E > (1 - e^{-my})K$*. For instance, this is always the case if the interest rate is zero—so there is no early exercise. This holds when* $K < S$ *(the put is out of the money), when* K *is slightly above* S *(the put is in the money, but not much), but not when the put is deep in the money. Hence, early exercise is only possible when the asset price is very low compared to the strike price.*

See Figure 19.29 for an illustration of the fact that early exercise is not profitable (since $P_E > \max(0, K - S)$) for high asset prices, but might be so for low asset prices (since $P_E < \max(0, K - S)$ means that $P_A < \max(0, K - S)$ is possible). Clearly, the proposition relies on having information about a European put price—or a good model of what the price should be. If we do not have that information, the proposition is not very useful—except in telling us that early exercise is more likely if the asset price is low and the interest rate high.

Example 19.12 *(Early exercise of American put option?) Using the same parameters as in Example 19.3, we have that* $C_E > (1 - e^{-my})K$ *is satisfied since*

$$5.5 > (1 - e^{-1/2 \times 0.05})38 = 0.94,$$

so there is no early exercise of the American put option. The reason is the put-call parity for European options (19.8) and the fact $P_A \geq P_E$ *give*

$$P_A \geq P_E = \underbrace{C_E}_{5.5} + K - S - \underbrace{(1 - e^{-my})K}_{0.94},$$

so selling the put option (getting P_A*) gives the same as exercising* $(K - S)$ *plus at least* $5.5 - 0.94 = 4.56$*, so selling gives more than exercising. If, for some reason, we instead have* $y = 35\%$ *(so* $(1 - e^{-my})K = (1 - e^{-1/2 \times 0.35})38 = 6.1$*) but the same prices, then*

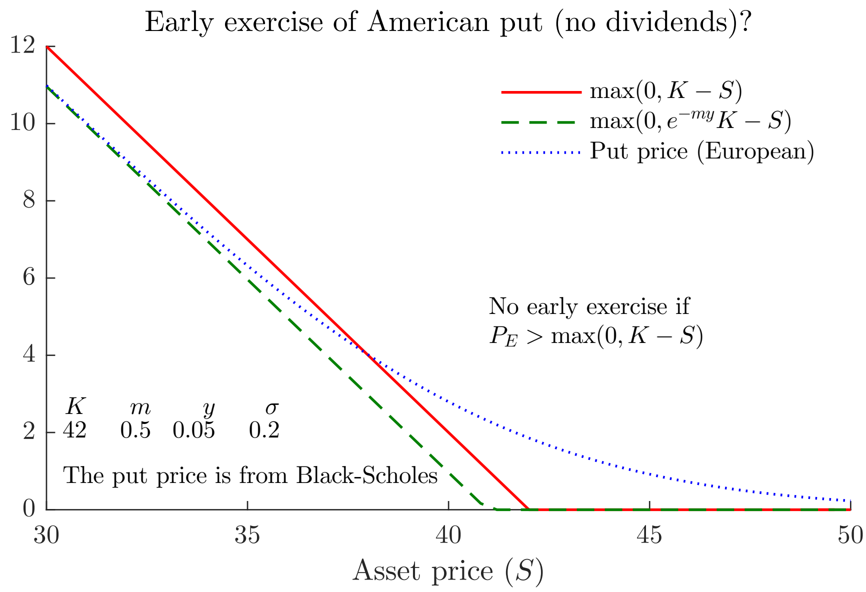


Figure 19.29: Early exercise of American put option (no dividends)

we would perhaps get early exercise. In particular, the expression above would say

$$P_A \geq P_E = \underbrace{C_E}_{5.5} + K - S - \underbrace{(1 - e^{-my})K}_{6.1},$$

so selling the put gives the same as exercising $(K - S)$ *plus at least* $5.5 - 6.1 = -0.6$*, so it's not sure that selling is better than exercising (could be or not).*

Proof. (of Proposition 19.11) To avoid early exercise, selling (getting P_A) should be more profitable than exercising (getting $K - S$), $P_A > K - S$. Put-call parity for European options (19.8) says

$$P_E = C_E + K - S - (1 - e^{-my})K.$$

If

$$C_E > (1 - e^{-my})K,$$

then $P_A \geq P_E > K - S$ so selling is better than exercising. This means that there is no early exercise if the European call price is high (high asset price compared to strike price), the strike price is low, or if the discounting until expiration is low (low interest rate or small time to expiration). For instance, with a zero interest rate, $P_A \geq C_E + K - S$, so there is never early exercise as long as $C_E > 0$. If these conditions are not satisfied, we cannot rule out early exercise. ■

19.6.3 Early Exercise of American Call and Put Options (Dividends)*

American call and put options on an asset with dividends (until expiration of the option) may be exercised early. The following propositions are more precise.

Proposition 19.13 *(Early exercise, American call, dividends) An American call option on an asset with dividends could be exercised early, especially just before a dividend payment and when the option is deep in-the-money (low strike price/high asset price). Conversely, there is no early exercise if* $(1-e^{-my})K > \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i$*, that is, with a high strike price and low present value of the dividends.*

Example 19.14 *(Early exercise, American call, dividends?) Suppose there is one dividend payment one month ahead:* $D_1 = 0.95$ *at* $m_1 = 4/12$*. If we use the same parameters as in Example 19.3, we then have*

$$(1-e^{-1/2\times 0.05})38 = 0.94 > e^{-4/12\times 0.05}0.95 = 0.93,$$

so we can rule out early exercise. However, if the dividend payment is at $m_1 = 1/12$*, then we cannot.*

Proof. (of Proposition 19.13) To avoid early exercise, selling (getting C_A) should be more profitable than exercising (getting $S-K$), $C_A > S-K$. Put-call parity for European options (19.9) says

$$C_E = S - K - \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i + (1-e^{-my})K + P_E.$$

If

$$(1-e^{-my})K > \sum_{i=1}^{n} e^{-m_i y_t(m_i)} D_i,$$

and $P_E \geq 0$ (always true), then $C_A \geq C_E > S-K$: selling is better than early exercise. Hence, there is no early exercise if the present value of dividends is low, the strike price is high or if the discounting until expiration is large (high interest rate or long time to expiration). In the opposite case, we cannot rule out early exercise. ■

Proposition 19.15 *(Early exercise, American put, dividends) Early exercise is possible...*

19.7 Appendix: Put-Call Relation for American Options*

There is no put-call parity for American options. However, pricing bounds (based on the values of European options) can be derived.

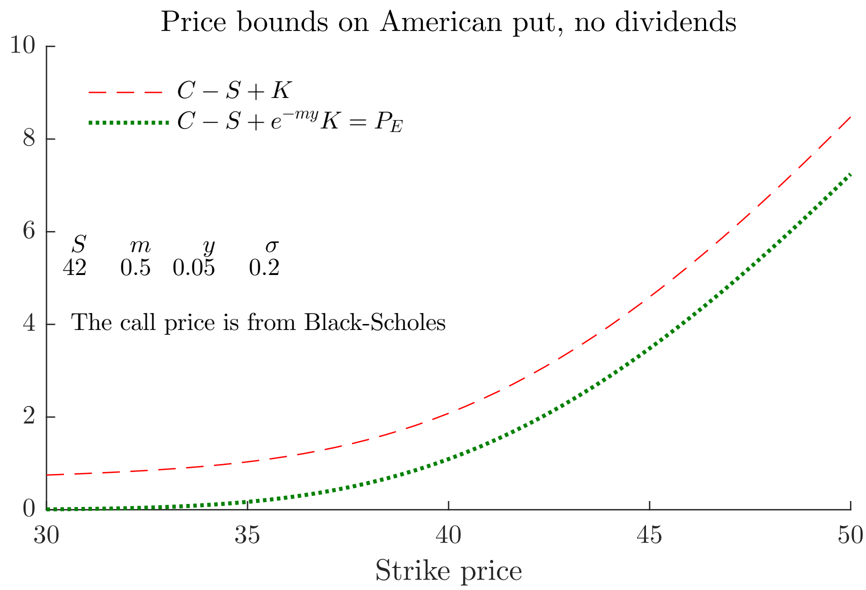


Figure 19.30: Option price as a function of the strike price

Proposition 19.16 *(Put-call, American option, no dividend) For an American option on an asset without dividends, the put price must be inside the interval*

$$\underbrace{C_A - S + e^{-my}K}_{P_E} \leq P_A \leq \underbrace{C_A}_{C_E} - S + K. \tag{19.23}$$

The lower boundary is the European put price from (19.8). The reason is that the American and European call options have the same prices (the American call option on an asset without dividends is never exercised early—see Section 19.5). The upper bound is very similar, except that it involves the strike price, not its present value. Clearly, when the interest rate is low, then the interval is narrow—and with a zero interest rate it collapses to the put-call parity of European options. (The latter corresponds to the fact that an American put option on an asset without dividends is never exercised early if the interest rate is zero, see Section 19.5).

See Figures 19.27 and 19.30 for illustrations.

Example 19.17 *(Bounds for an American put option) Using the same parameters as in Example 19.3, we get the following bounds for an American put option (no dividends)*

$$0.56 \leq P_A \leq 5.5 - 42 + 38 = 1.5.$$

Proof. (of Proposition 19.16) The lower boundary is the European put price (since $C_A = C_E$ when there are no dividends) and it is always true that $P_A \geq P_E$.

The upper boundary follows from the following argument where we compare two portfolios. Portfolio A: one call option with strike price K plus a deposit of K. Portfolio B: one put option plus one underlying asset. If the put option is held until expiration (the call is not exercised early), then portfolio A will be worth $\max(0, S_m - K) + e^{my}K$ in period m (where m is date of expiration), and portfolio B will be worth $\max(0, K - S_m) + S_m$, so portfolio A is worth (weakly) more. If, instead, the put is exercised earlier ($l < m$), then portfolio A will be worth $C_{A,l} + e^{ly}K$ in period l, and portfolio B will be worth $K - S_l + S_l = K$, so portfolio A is worth (weakly) more. In period 0 ($0 \leq l < m$) we don't know when/if the early exercise of the put will happen—but we know that in either case A portfolio will then be worth more than a portfolio B: portfolio A must therefore be worth (weakly) more than B already in 0: $C_{A,0} + K \geq P_{A,0} + S_0$, which is the upper bound in (19.23). ■

Proposition 19.18 *(Put-call, American option, dividends) With dividends, the upper boundary in (19.23) is changed by adding the present value of the dividend stream*

$$C_A - S + e^{-my}K \leq P_A \leq C_A - S + K + \sum\nolimits_{i=1}^{n} e^{-m_i y_t(m_i)} D_i. \tag{19.24}$$

Notice that the lower boundary is not equal to the European put price anymore (since $C_A \geq C_E$ and the present value of the dividends is not added). Together this means that the interval is wider with dividends than without dividends.

Proof. (of Proposition 19.18) The lower boundary follows from the following argument. Buy one call option, lend $e^{-my}K$, and sell one asset—the total value is $C_A + e^{-my}K - S$, which is the left hand side of (19.24). If the call is exercised prior to expiry, the payoff is $S - K + e^{-my}K - S = (e^{-my} - 1)K < 0$ which must be less than the value of the put whose value is nonnegative. If no early exercise, then the payoff at expiration is $\max(0, S - K) + K - S = \max(0, K - S)$ which is the same as the put payoff.

The upper boundary is a bit trickier, so we leave it for now. ■

Chapter 20

The Binomial Option Pricing Model

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23 and Hull (2009) 11
Additional references: McDonald (2014) 13–14

20.1 Overview of Option Pricing

There are basically two ways to model option prices: by some sort of factor model (like CAPM) or by a no-arbitrage argument. The latter is clearly much more precise, so it is typically preferred—when it works. These notes focus on a particularly simple case: when the underlying asset follows a binomial process.

20.2 The Basic Binomial Model

The binomial model, where the change of the price of the underlying asset can take only two values, is very stylized, but it is useful for establishing the key ideas of option pricing. It can also be transformed into a realistic model by cumulating many (short) subperiods. In the limit (as the subperiods became very many/very short) this binomial option pricing model (BOPM) converges to the well-known Black-Scholes model.

20.2.1 Binomial Process for the Stock Price

The binomial tree for the underlying asset starts at the price S and has probability q of moving to Su in the next period and a probability of $1-q$ of moving to Sd. This is illustrated in Figure 20.1. These probabilities are the true ("natural") probabilities. If we

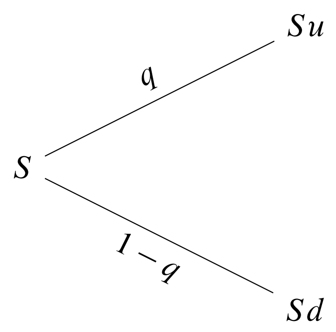


Figure 20.1: Natural binomial process for S

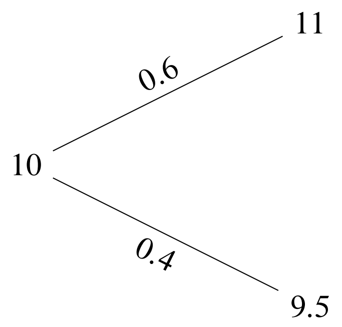


Figure 20.2: Natural binomial process for S

denote the price today by S_t and in the next period by S_{t+h}, then we have

$$S_{t+h}/S_t = \begin{cases} u & \text{with probability } q \\ d & \text{with probability } 1-q. \end{cases} \tag{20.1}$$

Remark 20.1 *(Mean and variance of a binomial process) The mean of a (shifted) binomial process like (20.1) is $qu + (1-q)d$ and the variance is $q(1-q)(u-d)^2$.*

Example 20.2 *(Binomial process) Suppose $S = 10, u = 1.1, d = 0.95$, and $q = 0.6$. Then, the process has a 60% probability of increasing from 10 to 11 and a 40% probability of decreasing to 9.5. See Figure 20.2. This gives an expected relative change of* $0.6 \times 1.1 + 0.4 \times 0.95 = 1.04$ *and a variance of the relative change of* $0.6 \times 0.4 \times (1.1 - 0.95)^2 = 0.0054$.

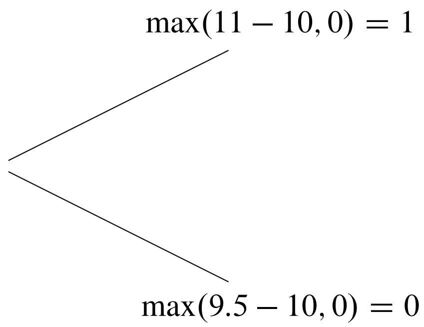


Figure 20.3: Numerical example of call option payoff

20.2.2 No-Arbitrage Pricing of a Derivative

Consider a derivative asset that will be worth f_u in case we end up at Su and f_d if we end up at Sd—see Figure 20.1. Notice that f_u is the notation for the value (price) of the derivative in the up state (it should *not* be read as f times u). Clearly, since we know the value of the underlying in the up (down) state, we can calculate the value of the derivative in that state (since it is the expiration date). As an example, the derivative could be a call option with strike price K, so if the next time period is the time expiration, then

$$f_u = \max(Su - K, 0) \text{ and } f_d = \max(Sd - K, 0). \tag{20.2}$$

Example 20.3 *(European call option) With the parameters in Example 20.2, equation (20.2) shows that a European call option with strike price of 10 has*

$$f_u = \max(11 - 10, 0) = 1 \textit{ and } f_d = \max(9.5 - 10, 0) = 0,$$

while a strike price of 9 gives

$$f_u = \max(11 - 9, 0) = 2 \textit{ and } f_d = \max(9.5 - 9, 0) = 0.5.$$

See Figure 20.3.

Alternatively, it could be a forward contract (and the next time period is the time of expiration), so

$$f_u = Su - F \text{ and } f_d = Sd - F. \tag{20.3}$$

We next use a no-arbitrage argument to derive what today's price of the derivative (denoted f) must be. In doing so, we take it for granted that

$$u > e^{yh} > d. \tag{20.4}$$

If this condition is not satisfied, then there are a trivial arbitrage opportunities. For instance, if $e^{yh} > u$, then we could short the stock and buy bonds: this would guarantee a positive payoff for a zero investment (an arbitrage possibility).

Step 1: Construct a Riskfree Portfolio

We now construct the following portfolio

$$\begin{aligned} &\Delta \text{ of the underlying asset, and} \\ &-1 \text{ of the derivative,} \end{aligned} \tag{20.5}$$

where will pick the value of Δ to make the portfolio riskfree.

For a given value of Δ (not yet determined), the payoff of the portfolio at expiry is $\Delta Su - f_u$ in the "up" state and $\Delta Sd - f_d$ in the "down" state. To make the portfolio riskfree, Δ must be such that the payoff is the same in both cases

$$\begin{aligned} \Delta Su - f_u &= \Delta Sd - f_d, \text{ so} \\ \Delta &= \frac{f_u - f_d}{S(u-d)}. \end{aligned} \tag{20.6}$$

With this choice of Δ (also called the "delta hedge") the portfolio is riskfree and must therefore have the same return as the riskfree rate.

Example 20.4 *(European call option) Continuing Example 20.3 we get*

$$\Delta = \frac{1-0}{10(1.1-0.95)} = \frac{2}{3} \textit{ for } K = 10, \textit{ and } \Delta = \frac{2-0.5}{10(1.1-0.95)} = 1 \textit{ for } K = 9.$$

The payoff of this portfolio is indeed safe. For instance, for the $K = 10$ *option the value in the up state is* $\frac{2}{3}11 - 1 = 19/3$ *and in the down state* $\frac{2}{3}9.5 - 0 = 19/3$.

Step 2: Make the Return of the Portfolio Equal to the Riskfree Rate

Since the choice of Δ in (20.6) makes the portfolio safe, it must have same return as the riskfree asset. This will help us determine what today's price of the derivative (f) is. The gross return on the riskfree from now until expiration (h periods later) is e^{yh} for the

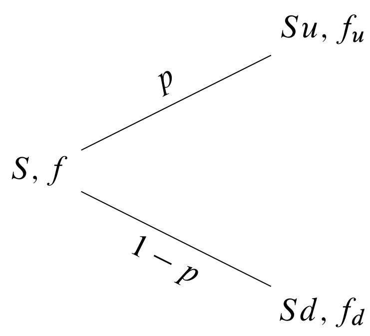


Figure 20.4: Risk neutral binomial process for S and f

riskfree asset (y is the continuously compounded interest rate). For our portfolio, it is the portfolio value at expiration (same in the up and down states) divided by the price of the portfolio today ($\Delta S - f$). Equating gives

$$\frac{\Delta Su - f_u}{\Delta S - f} = e^{yh}, \tag{20.7}$$

where we still keep the Δ notation (to save ink), but assume that Δ is determined as in (20.6). Taking logs shows that the (safe) log return on the portfolio equals yh.

Solve for the (current) price of the derivative, f, and use the value of Δ from (20.6) that ensures that the portfolio is riskfree

$$f = \Delta S(1 - e^{-yh}u) + e^{-yh} f_u \tag{20.8}$$

$$= \frac{f_u - f_d}{u - d}(1 - e^{-yh}u) + e^{-yh} f_u \tag{20.9}$$

$$= e^{-yh}\left[p f_u + (1-p) f_d\right] \text{ with } p = \frac{e^{yh} - d}{u - d} \tag{20.10}$$

$$= e^{-yh}\,\mathrm{E}^*(\text{future payoff of derivative}) \tag{20.11}$$

Equation (20.9) shows what the price of the derivative must be—and is written in terms of the possible outcomes and the interest rate. Notice that neither probabilities (of the different outcomes), nor risk preferences enter this expression—since we have used a no-arbitrage argument to price this derivative. This works (that is, we can construct a riskfree portfolio) because we have as many (relevant) assets (riskfree and underlying risky asset) as there are possible outcomes (up or down).

Equations (20.9)–(20.11) are alternative ways to write the price of the derivative. Equation (20.10) shows that the current price of the derivative is the discounted value

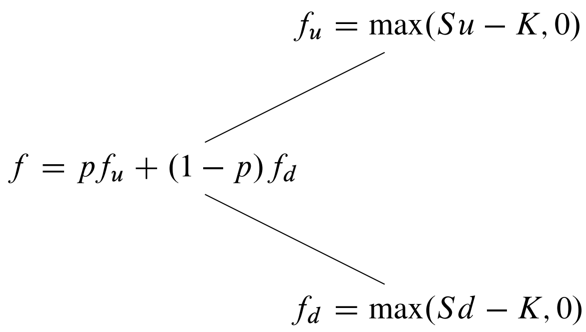


Figure 20.5: Solving for a call option price, zero interest rate

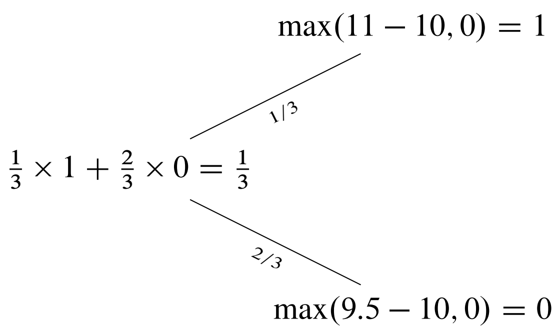


Figure 20.6: Numerical example of call option price, zero interest rate

(e^{-yh}) times what seems as an expectation of the payoff of the derivative. This expression is quite useful since we can think of p as a "risk neutral probability"—although it is not a probability in the usual sense: it is just a convenient construction. Notice that p does not depend on which derivative (with the same underlying asset) we consider: p depends on the underlying asset (and the interest rate), not the derivative. Under the restrictions in (20.4), $0 < p < 1$, as any "probability" should be. This interpretation is highlighted in (20.11), where E^* stands for the expectations according to the *risk neutral distribution* (more about that later). See Figure 20.4 for an illustration of the risk neutral process. Compared with the natural process in Figure 20.1, only the probabilities are different.

The computation in (20.10) are illustrated in Figure 20.5 (assuming a zero interest rate to make the expression simpler).

Example 20.5 *(European call option) Continuing Example 20.3 and assuming that $y =$*

0, *equation (20.10) gives the price of a call option with strike price 10 as*

$$f = e^{-0}\left[p1 + (1-p)\,0\right] \text{ with } p = \frac{1-0.95}{1.1-0.95} = 1/3$$
$$= 1/3.$$

See Figure 20.6. For the call option with a strike price of 9, we get

$$f = e^{-0}\left[(1/3)\times 2 + (2/3)\times(1/2)\right] = 1.$$

20.2.3 Applying the No-Arbitrage Pricing on Different Derivatives

This section discusses how we apply (20.10) on some special derivatives.

Consider *the underlying asset itself*. It is clearly a (trivial) derivative with $f_u = Su$ and $f_d = Sd$. According to (20.10) the current price of the underlying asset should be

$$S = e^{-yh}\left[pSu + (1-p)\,Sd\right]. \tag{20.12}$$

This looks (again) like a discounted expected future payoff.

Example 20.6 *(The underlying asset itself) Continuing Example 20.5, equation (20.12) gives*

$$S = e^{-0}\left[(1/3)\times 11 + (2/3)\times 9.5\right] = 10.$$

A *forward contract* has a zero current price (nothing is paid until expiry), and the payoff at expiry is $f_u = Su - F$ in the up state (the value of the underlying asset minus the forward price) and $f_d = Sd - F$ in the down state. Using this in (20.10) gives

$$0 = e^{-yh}\left[p\,(Su-F) + (1-p)\,(Sd-F)\right], \text{ so} \tag{20.13}$$
$$F = pSu + (1-p)\,Sd. \tag{20.14}$$

This shows that the mean of the risk neutral distribution equals the forward price. Combining (20.12) and (20.14) clearly gives the spot-forward parity, $F = e^{yh}S$.

Example 20.7 *(A forward contract) Continuing Example 20.5, we get* $F = 10$ *(same as in Example 20.6) since the interest rate is zero).*

A *riskfree asset* can also be priced by this method. The only way an asset can be riskfree in this setting is if $f_u = f_d$. We then get a zero hedge ratio (Δ) and (20.10) gives

$$f = e^{-yh} f_u, \tag{20.15}$$

which is the discounted value of the (sure) payoff.

An "*Arrow-Debreu asset*" (a sort of theoretical derivative often used in asset pricing models) for the "up" pays off one unit in the up state and zero otherwise ($f_u = 1$ and $f_d = 0$). This is also a so-called "cash-or-nothing" call option provided the up state means that the option is in the money ($Su > K$). From (20.10) we have

$$f = e^{-yh} p. \tag{20.16}$$

20.2.4 Replicating (and Hedging) a Derivative

The no-arbitrage argument in (20.6) was based on the fact that a portfolio of Δ of the underlying asset and of -1 of the derivative replicated a bond.

This argument can be turned around to replicate the derivative by holding the following portfolio

$$\begin{aligned} &\Delta \text{ of the underlying asset, and} \\ &-e^{-yh}(\Delta Su - f_u) \text{ bills.} \end{aligned} \tag{20.17}$$

The payoff of this portfolio in the up state is $\Delta Su - (\Delta Su - f_u) = f_u$ and in the down state it is $\Delta Sd - (\Delta Sd - f_d) = f_d$ (since $\Delta Su - f_u = \Delta Sd - f_d$). This replicates the derivative's payoff. We can therefore hedge a short position in the derivative by holding Δ of the underlying asset ("delta hedging").

Example 20.8 *(Replicating the call option) For the call option with a strike price of 10 and with a zero interest rate, we have (see Example 20.4) $\Delta = 2/3$ and*

$$-e^{-yh}(\Delta Su - f_u) = -1(\frac{2}{3} \times 10 \times 1.1 - 1) = -6\frac{1}{3}$$

20.2.5 Where is the Risk Premium?

We have used a no-arbitrage method to price the derivative. It works since the derivative is a redundant asset: it can be replicated by a portfolio of the underlying asset and a riskfree asset—and therefore must have the same price as this portfolio. This does not mean, however, that the option is in itself riskfree. In fact, options are typically very risky and therefore carry large risk premia. It may seem as if the pricing formula (20.10) is free from the preference parameters that would determine the risk premium. Not correct. The pricing formula contains the current asset price (through f_u and f_d) which is indeed affected by preference parameters.

The easiest way to see this is perhaps to recall that we can replicate the portfolio by holding a portfolio of the underlying asset and bills, see (20.17). Clearly, this portfolio will incorporate a risk premium—and so must the derivative.

The only case without a risk premium is when the derivative payoffs are unrelated to the asset price—so the derivative is actually a safe asset as in (20.15).

20.3 Interpretation of the Risk Neutral Probabilities

The relation between the true probabilities (q) and the risk neutral probabilities (p) depends whether the underlying asset has a risk premium or not.

When the underlying asset has an expected return (over t to $t+h$) higher than the risk free rate, that is, a positive risk premium, then

$$\frac{\mathrm{E}_t\, S_{t+h}}{S_t} > e^{yh} \text{ (with positive risk premium).} \tag{20.18}$$

From the spot-forward parity for an asset without dividends, we know that $F = e^{yh} S_t$. Combining gives

$$\mathrm{E}_t\, S_{t+h} > F \text{ (with positive risk premium).} \tag{20.19}$$

The binomial process implies that the expected value of the future asset price is

$$\mathrm{E}_t\, S_{t+h} = qSu + (1-q)\, Sd, \tag{20.20}$$

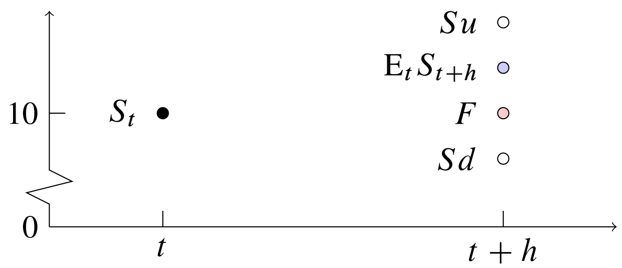
where q is the natural probability of the up state. At the same time, the risk neutral expected value equals the forward price (see (20.14))

$$F = pSu + (1-p)\, Sd. \tag{20.21}$$

For (20.19) to hold, we must have

$$q > p \text{ (with positive risk premium).} \tag{20.22}$$

This result is intuitively clear: both $\mathrm{E}_t\, S_{t+h}$ and F are averages of the same values (Sd and Su) and they only differ with respect to the probabilities. Clearly, for $\mathrm{E}_t\, S_{t+h}$ to exceed F, the former must have a higher probability for the high value (Su). See Figure 20.7 for an illustration. One interpretation is that a risk neutral investor would be happy with a lower probability of the up state (and thus a lower expected return), than a risk averse investor.



Calculations:
$S_t = 10$, $Sd = 9.5$, $Su = 11$
$E_t S_{t+h} = 0.6 \times 11 + 0.4 \times 9.5 = 10.4$
$F = 1/3 \times 11 + 2/3 \times 9.5 = 10$

Figure 20.7: Risk premium and risk neutral probabilities

Example 20.9 *(Natural versus risk neutral probability) With the parameters in Example 20.2*

$$E_t\, S_{t+h} = 0.6 \times 11 + (1 - 0.6) \times 9.5 = 10.4.$$

With $y = 0$, $F = S = 10$. In this case, the underlying asset indeed has a positive risk premium (see (20.19)), and $q = 0.6$ while $p = 1/3$. See Figure 20.7 for an illustration.

Proof. (of (20.22)) For (20.19) to hold, we need $qSu + (1-q)\,Sd > pSu + (1-p)\,Sd$. Subtract Sd from both sides to get $qS(u-d) > pS(u-d)$ and notice that $S(u-d) > 0$ to conclude that $q > p$ is required. ■

In contrast, when the risk premium on the underlying asset is zero (so $E_t\, S_{t+h} = F$), then

$$q = p \text{ (with no risk premium).} \tag{20.23}$$

This means that if the underlying asset is priced by *"risk neutral investors,"* then p equals the true probabilities—suggesting the name "risk neutral probability."

20.4 Numerical Applications of the Binomial Model

20.4.1 How to Construct a Tree for the Asset Price

We now discuss how to construct a binomial tree with many small time steps—so that it mimics the behaviour of the asset price process.

The binomial distribution converges to a normal distribution as we chop up a given time to expiration into smaller and smaller time steps—and the normal distribution is fully described by the mean and variance. It is therefore common practice to construct the binomial tree to match the mean and variance of the underlying series.

Suppose the log price of the underlying asset is a random walk with drift

$$\begin{aligned} \ln S_{t+k} &= \ln S_t + \mu k + \varepsilon_{t+k}, \text{ or} \qquad (20.24)\\ \ln S_{t+k} - \ln S_t &= \mu k + \varepsilon_{t+k}, \text{ with} \\ \varepsilon_{t+k} \text{ being iid, } \mathrm{E}\,\varepsilon_{t+k} &= 0 \text{ and } \mathrm{Var}(\varepsilon_{t+k}) = \sigma^2 k. \end{aligned}$$

For instance, when we measure periods in years, then $k = 1/252$ corresponds to daily data (only counting the trading days). In practice, we estimate μk as the sample mean of $\ln S_{t+k} - \ln S_t$ and $\sigma^2 k$ as the sample variance (or $\sigma\sqrt{k}$ as the sample standard deviation). Expressing the moments in terms of annualised numbers (μ and σ^2) helps relating to the binomial model—and to compare results across different sampling intervals.

Example 20.10 *(Variance for daily return) If the data is daily ($k = 1/250$, assuming 250 trading days per year) and the standard deviation is estimated to be 0.0126, then the annualised variance is $\sigma^2 = 0.0126^2 \times 250 = 0.2^2$ and the annualized standard deviation is $\sigma = 0.0126 \times \sqrt{250} = 0.2$.*

Equation (20.24) implies that data sampled as $t, t+k, t+2k, \ldots$ has the mean and variance

$$\mathrm{E}(\ln S_{t+k} - \ln S_t) = \mu k \qquad (20.25)$$

$$\mathrm{Var}(\ln S_{t+k} - \ln S_t) = \sigma^2 k. \qquad (20.26)$$

Example 20.11 *(Variance for intervals of five days) With $h = 5/250 = 1/50 = 0.02$. With $\sigma = 0.2$, we have $\sigma\sqrt{h} = 0.2 \times \sqrt{0.02} \approx 0.028$.*

To approximate the price process for such h sampling with the binomial model (20.1), we use

$$\ln S_{t+h} - \ln S_t = \begin{cases} \ln u & \text{with probability } q \\ \ln d & \text{with probability } 1-q. \end{cases} \qquad (20.27)$$

Notice that (20.1) says that $S_{t+h}/S_t = u$ with probability q. Just take logs to get the results here. Adding $\ln S_t$ to both sides of (20.27) gives a tree for the log price instead of

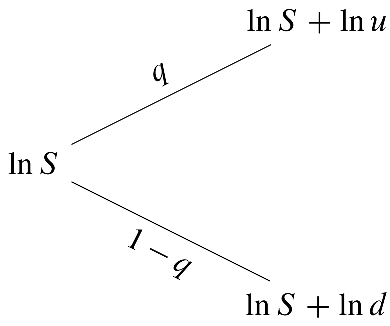


Figure 20.8: Natural binomial process for $\ln S$

the changes, see Figure 20.8. The binomial process implies that the mean and variance of the asset price change are therefore (see Remark 20.1)

$$\mathrm{E}(\ln S_{t+h} - \ln S_t) = q \ln u + (1-q) \ln d, \tag{20.28}$$

$$\mathrm{Var}(\ln S_{t+h} - \ln S_t) = q(1-q)(\ln u - \ln d)^2. \tag{20.29}$$

There are three parameters (u, d, and q) which can be chosen to match the two moments (mean and variance), that is, to make (20.28)–(20.29) equal to (20.25)–(20.26). We can therefore make one arbitrary choice. The following is a common approach.

First, for any u and d (not yet decided), pick q to match the mean drift over a time step of size h (which is μh, see (20.24)), that is,

$$q \ln u + (1-q) \ln d = \mu h, \text{ so} \tag{20.30}$$

$$q = \frac{\mu h - \ln d}{\ln u - \ln d}. \tag{20.31}$$

Strictly speaking, we do not need the physical probability q in order to price derivatives (the risk neutral probability is enough), but it is still useful to understand the logic of calibrating d and u (below).

Second, we try to pick u and d to match the variance over a time step of size h (which is $\sigma^2 h$, see (20.24)), that is, we try to match

$$q(1-q)(\ln u - \ln d)^2 = \sigma^2 h. \tag{20.32}$$

Use (20.31) to substitute for q and simplify

$$(\mu h - \ln d)(\ln u - \mu h) = \sigma^2 h. \tag{20.33}$$

This is one equation with two unknowns (d and u), so we must impose further restrictions. There are several ways to proceed, but the most common is the approach of Cox, Ross, and Rubinstein (1979) where

$$u = e^{\sigma\sqrt{h}} \text{ and } d = e^{-\sigma\sqrt{h}}. \tag{20.34}$$

Example 20.12 *(Parameters to binomial tree) With $h = 1/50$ and $\sigma = 0.2$, (20.34) gives $u \approx 1.029$ and $d \approx 0.972$.*

Using (20.34) on the left hand side of (20.33) gives the variance of the binomial process as

$$\text{Var}\,(\text{binomial process}) = \sigma^2 h - \mu^2 h^2. \tag{20.35}$$

This does not fit the volatility exactly because of the $\mu^2 h^2$ term (compare with the right hand side of (20.33)), but the approximation improves quickly as h decreases (the second order term h^2 vanishes fast). There are other ways to construct the binomial tree, but they have similar properties.

Notice that once we have the values of u and d, the pricing of derivatives does not use the natural probability of the up state (q).

20.4.2 Multiperiod Trees

The binomial model is very useful for numerical calculations of the option price. In such numerical applications, the time to expiry is divided into many small time steps, and it is assumed that the price of the underlying asset can make an up or down movement in each subinterval—and that the no-arbitrage portfolio is rebalanced every time step. The logic is that the binomial pricing model applies to each up/down branching, so we can combine many such branchings into a tree. Of course, the size of the up and down movements (u and d in the previous analysis), as well as the discounting, is scaled by the number of subintervals.

Let m be the time to expiration of the derivative. With n short time intervals, the length of each interval is $h = m/n$. Clearly, if we use shorter time steps, then we need more of them to reach the same time to expiration, see Figure 20.9.

Example 20.13 *(Parameters to binomial tree) If the time to maturity of the option is 6 months, then $m = 0.5$. If we want $n = 25$ short time intervals in the binomial tree, then $h = m/n = 0.5/25 = 1/50$, which corresponds to 5 days if there are 250 trading*

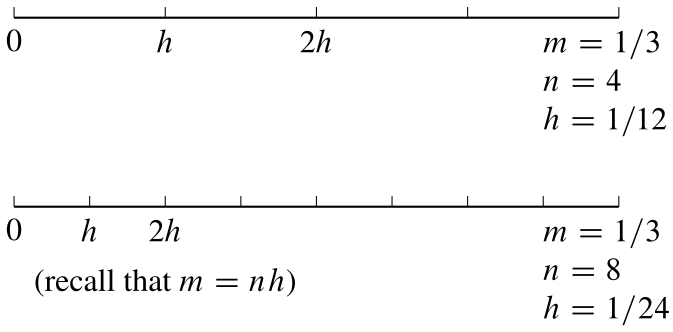


Figure 20.9: Two different time steps with same time to expiration m

days during a year. Instead, if we want $n = 250$*, then* $h = 0.5/250 = 1/500$ *which corresponds to half a day.*

The Cox, Ross, and Rubinstein (1979) approach implies (from using (20.10) and (20.34))

$$u = e^{\sigma\sqrt{h}}, d = e^{-\sigma\sqrt{h}}, \tag{20.36}$$
$$p = (e^{yh} - d)/(u - d), \text{ and}$$
$$\text{and discounting by } e^{-yh}.$$

Notice that we must keep h small enough so (20.4) holds ($u > e^{yh} > d$, to rule arbitrage opportunities), that is,

$$e^{\sigma\sqrt{h}} > e^{yh} > e^{-\sigma\sqrt{h}}, \tag{20.37}$$

which requires $\sigma > y\sqrt{h} > -\sigma$.

Example 20.14 *(Checking parameters of binomial tree) With the parameters in Example 20.12 and assuming* $y = 0.05$*, we notice that* $e^{yh} = e^{0.05/50} \approx 1.001$*, so the requirement is fulfilled*

$$1.029 > 1.001 > 0.972.$$

As we start to change the interval length (h), it may not correspond to the sampling frequency of data used in estimating the mean and variance of data in (20.25)–(20.26). That is not a problem. Rather, use data to extract the σ^2 (and perhaps μ) and then use that in the binomial tree (20.36). In other words, the k in the sampling frequency and the h in the binomial tree need not be the same.

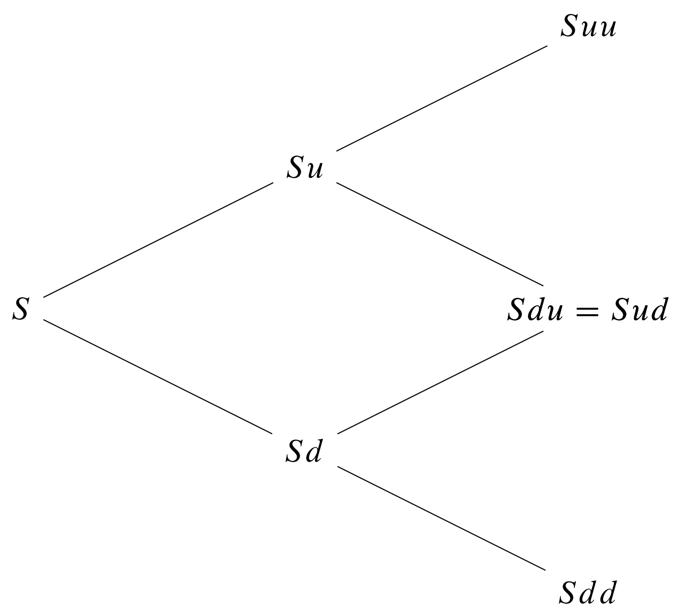


Figure 20.10: Binomial tree for underlying asset ($n = 2$)

Figure 20.10 is an illustration of a binomial tree with two subintervals. This tree has only three final nodes, since $Sud = Sdu$—it is "recombining," which is very useful to keep the number of nodes manageable (when we have many time steps). This would not be the case if the up and down moves were different for different periods (non-iid price process).

Remark 20.15 *(Coding the binomial tree*) Let S_{i1} denote the highest asset price at time node i $(0, 1, ...n)$ and S_{i2} the second highest and so forth. Clearly, $S_{ij} = uS_{i-1,j}$ for all j, except that the lowest value is d times the lowest value of S_{i-1}, that is, $S_{i,i+1} = dS_{i-1,i}$. In this case, this is the same as $S_{ij} = S_t u^{i+1-j} d^{j-1}$.*

Remark 20.16 *(Size of the binomial tree*) With n time steps, there are $n + 1$ different prices at the end nodes. Also, there are a total of $(n + 1)(n + 2)/2$ nodes. There are $n!/[(n - s)!s!]$ different ways to reach the sth node below the top node (where $x! = x \times (x - 1) \times \ldots \times 1$). Summing across the nodes shows that the tree contains 2^n different*

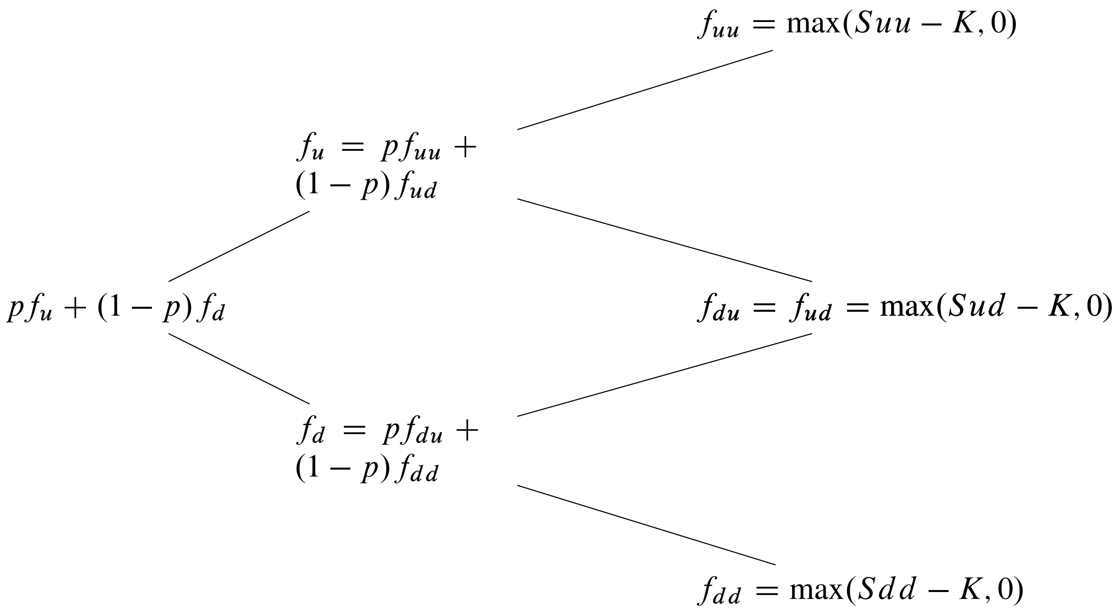


Figure 20.11: Binomial tree for European call option ($n = 2$), zero interest rate

paths. For instance, our (recombining) three has

n	*no. end nodes*	*no. total nodes*	*no. paths*
2	3	6	4
25	26	351	33, 554, 432
200	201	20, 301	1.6×10^{60}

In contrast, a non-recombining tree has 2^n end nodes, that is, as many as there are paths in the recombining tree.

Example 20.17 *(European call option) For a European call option with strike price K and two steps ($n = 2$), in Figure 20.11 are*

$$\textit{step 0: } f = e^{-yh}[pf_u + (1-p)f_d],$$

$$\textit{step 1: } \begin{bmatrix} f_u = e^{-yh}[pf_{uu} + (1-p)f_{ud}] \\ f_d = e^{-yh}[pf_{du} + (1-p)f_{dd}] \end{bmatrix}$$

$$\textit{step 2: } \begin{bmatrix} f_{uu} = \max(Suu - K, 0) \\ f_{ud} = \max(Sud - K, 0) \\ f_{dd} = \max(Sdd - K, 0) \end{bmatrix}$$

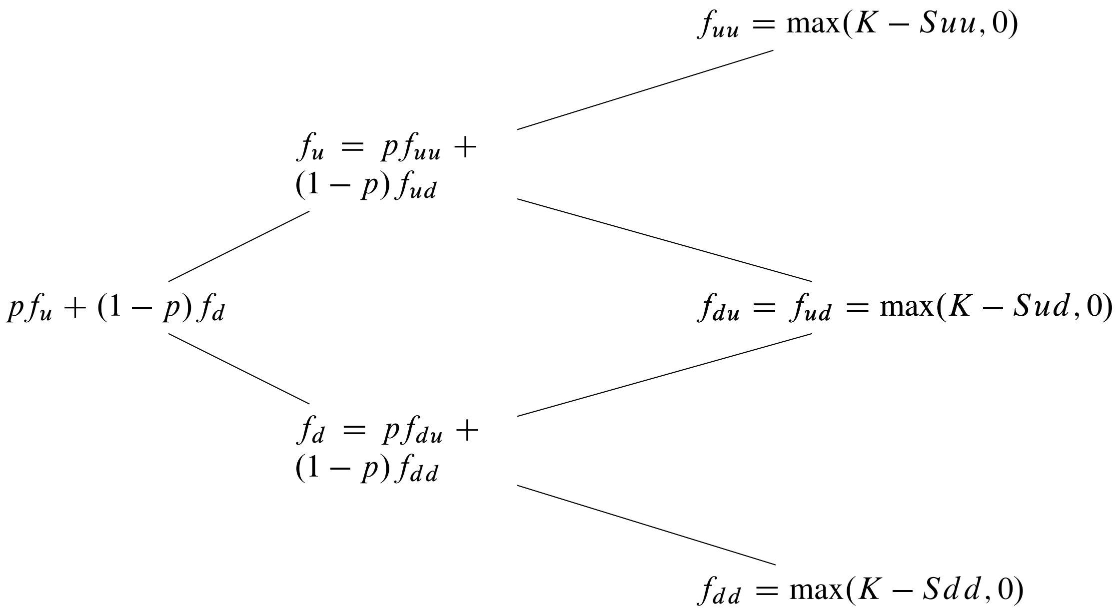


Figure 20.12: Binomial tree for a European put option ($n = 2$), zero interest rate

where $p = (e^{yh} - d)/(u - d)$. *Notice that the calculation begins at the end (step 2) and works backwards towards the start of the tree (step 0). With* $y = 0$, *all discounting terms are 1. In contrast, with* $m = 1/4$ *(three months) and* $n = 2$, $h = 1/8$, *the discounting terms are* $e^{-y/8}$.

Example 20.18 *(European put option) The tree for a European put option is the same as for a European call option, except for the end nodes. With two steps, as in Figure 20.12, we have*

$$\begin{bmatrix} f_{uu} = \max(K - Suu, 0) \\ f_{ud} = \max(K - Sud, 0) \\ f_{dd} = \max(K - Sdd, 0) \end{bmatrix}$$

See Figures 20.13–20.14 for an illustration of how the parameters (p, u, d) and the resulting option price converge as the number of time steps increases (keeping the time to expiration fixed).

20.4.3 Using a Binomial Tree for Pricing American Options

The binomial tree we have used so far assumes that the derivative is "alive" until the end of the period. This is not necessarily the case for American options, so the approach needs to be modified to handle the possibility of early exercise.

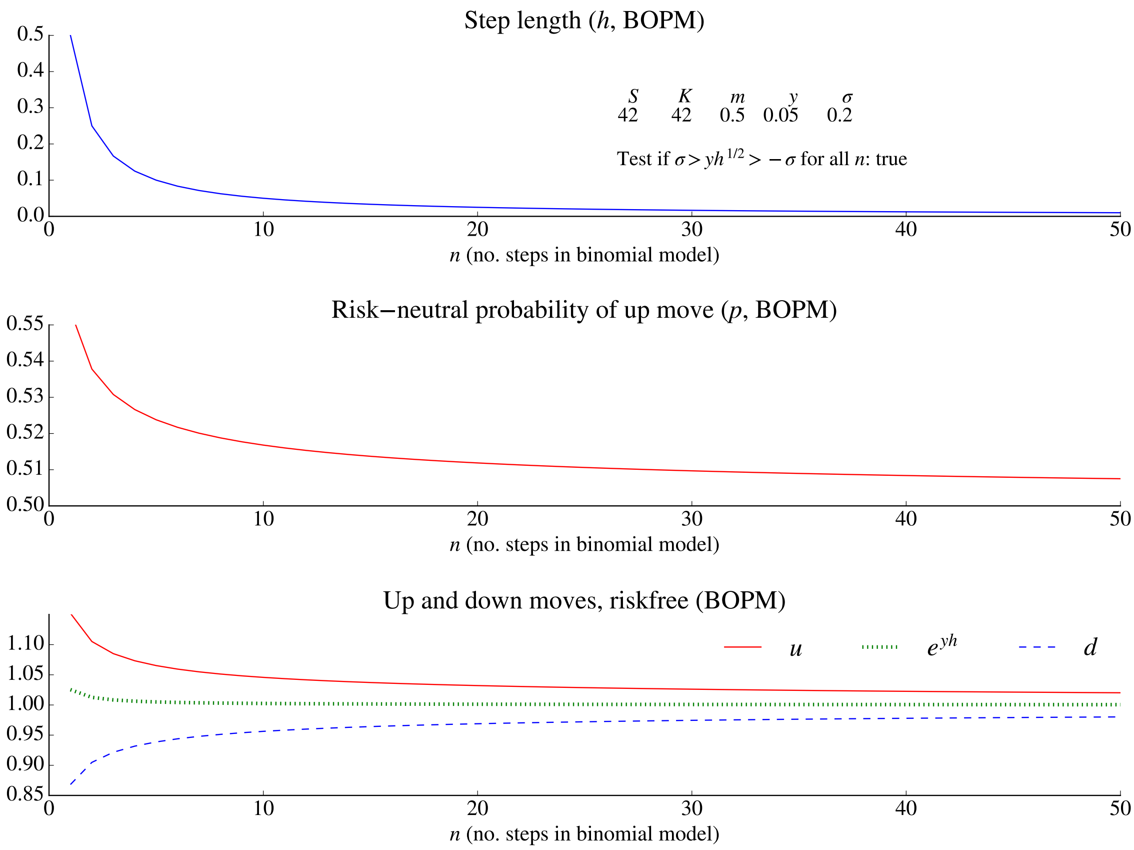


Figure 20.13: Convergence of the parameters in a binomial model

The option value is then the maximum of the exercise value and the value if keeping the option "alive." The latter is defined in the same way as in (20.10). Together this gives the price of the derivative as

$$f = \max(\text{value if exercised now}, e^{-yh}\left[p f_u + (1-p)\, f_d\right]), \tag{20.38}$$

where p is defined as before (in (20.10)). For instance, a two-step tree for an American put option would have

$$f = \max(K - S, e^{-yh}\left[p f_u + (1-p)\, f_d\right]), \text{ where} \tag{20.39}$$

$$f_u = \max(K - Su, 0) \text{ and } f_d = \max(K - Sd, 0). \tag{20.40}$$

Example 20.19 *(An American put option) With an American put option we must account for the possibility of an early exercise. At each node, the option value is the maximum of*

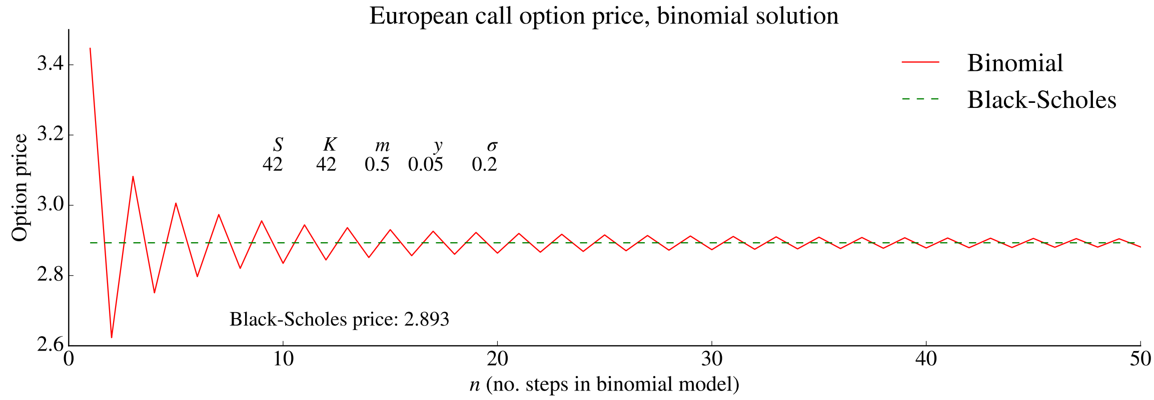


Figure 20.14: Convergence of the binomial price

the value if exercised (K minus the asset price) and the value if kept "alive" (denoted f^a below), see Figure 20.15. The latter is the discounted risk neutral expected value of the option value next period—just like for a European option. We therefore have

$$\textit{step 0: } f = \max(K - S, f^a), \textit{ where } f^a = e^{-yh}[pf_u + (1-p)f_d]$$

$$\textit{step 1: } \begin{bmatrix} f_u = \max(K - Su, f_u^a), \textit{ where } f_u^a = e^{-yh}[pf_{uu} + (1-p)f_{ud}] \\ f_d = \max(K - Sd, f_d^a), \textit{ where } f_d^a = e^{-yh}[pf_{du} + (1-p)f_{dd}] \end{bmatrix}$$

$$\textit{step 2: } \begin{bmatrix} f_{uu} = \max(K - Suu, 0) \\ f_{ud} = \max(K - Sud, 0) \\ f_{dd} = \max(K - Sdd, 0), \end{bmatrix}$$

where $p = (e^{yh} - d)/(u - d)$. As always, the calculation begins at the end (step 2) and works backwards.

Figure 20.16 illustrates the solution for an American put option on an asset without dividends. Notice that the American put price exceeds the European put price—and more so at low asset prices ($S < K$ is necessary, but not sufficient for early exercise) and high interest rates, that is, when it is likely that the option will be exercised early. The lower and upper limits on the put price are from the put-call "parity" (two inequalities) for American options. Notice that the call price C used in the figure is the same for European and American options (since there is no early exercise in thisc case).

Figure 20.17 illustrates the calculations of the American put price for one current value of the underlying asset. The shaded areas show the location of the nodes (future prices of the underlying asset) that are used in the calculation—and at which nodes that

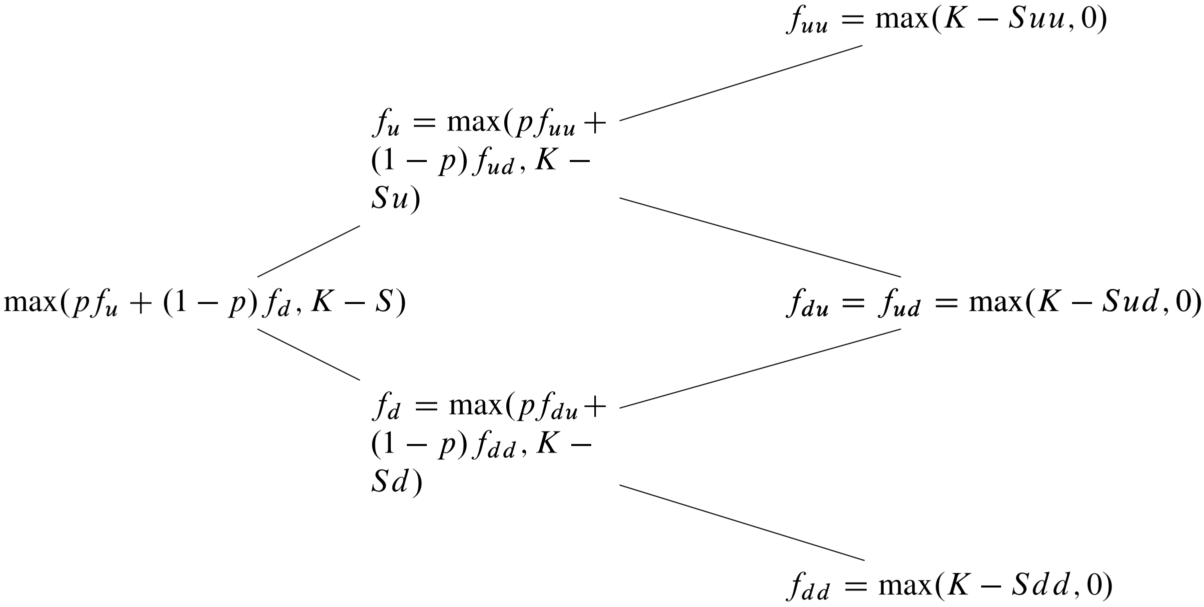


Figure 20.15: Binomial tree for an American put option ($n = 2$), zero interest rate

early exercise will happen.

20.4.4 A Binomial Tree with Continuous Dividends*

It is straightforward to construct another tree that allows for continuous dividends.

Suppose dividends are paid at the continuous *rate* δ. Let the up and down movements in the asset price reflect the ex-dividend price, and assume that any dividends are reinvested in the stock.

First, to construct a riskfree portfolio, hold Δ of the underlying asset and -1 of the derivative. The payoff of the portfolio at expiry is $\Delta S e^{\delta h} u - f_u$ in the "up" state and $\Delta S e^{\delta h} d - f_d$ in the "down" state. The $e^{\delta h}$ factor comes from reinvestment. To make the portfolio riskfree the delta must be

$$\Delta = \frac{f_u - f_d}{S e^{\delta h} (u - d)}. \tag{20.41}$$

Second, to make the return of the portfolio equal to the riskfree rate, we set the present value of our riskfree portfolio equal to the cost of the portfolio

$$e^{-yh} \left(\Delta S e^{\delta h} u - f_u \right) = \Delta S - f. \tag{20.42}$$

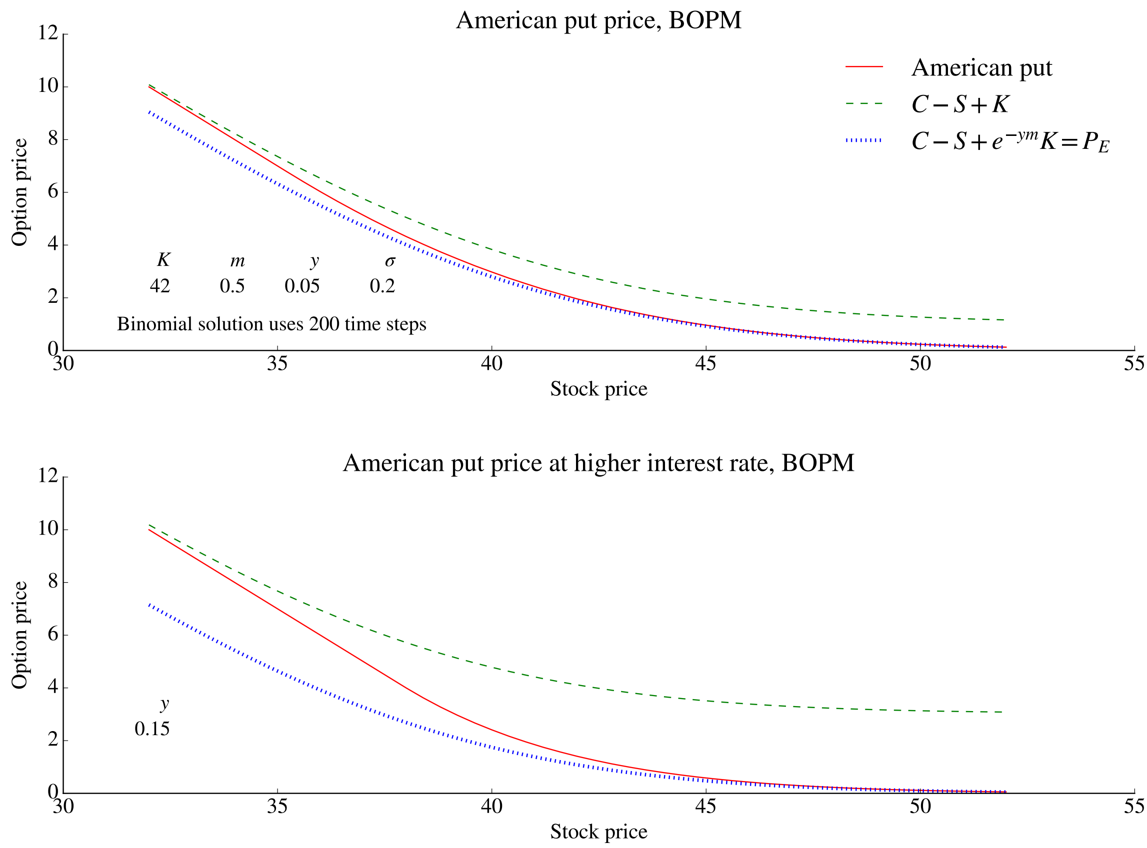


Figure 20.16: Numerical solution of an American put price

Use (20.41) and rearrange as

$$f = \Delta S \left(1 - e^{(\delta - y)h} u\right) + e^{-yh} f_u \tag{20.43}$$

$$= \frac{f_u - f_d}{e^{\delta h}(u-d)} \left(1 - e^{(\delta - y)h} u\right) + e^{-yh} f_u \tag{20.44}$$

$$= e^{-yh} \left[p f_u + (1-p) f_d\right] \text{ with } p = \frac{e^{(y-\delta)h} - d}{u - d}. \tag{20.45}$$

With this new definition of p, the rest of the computations are as in the case without dividends. In particular, the drift of the asset price does not matter, so u and d can be chosen as before, for instance, as in (20.34).

Remark 20.20 *(Risk neutral drift with continuous dividends) With continuous dividends, the risk neutral expected value is* $\mathrm{E}_t^p\, S_{t+h}/S_t = e^{(y-\delta)h}$*, so the drift is* $(y-\delta)h$ *over the short time interval* h*.*

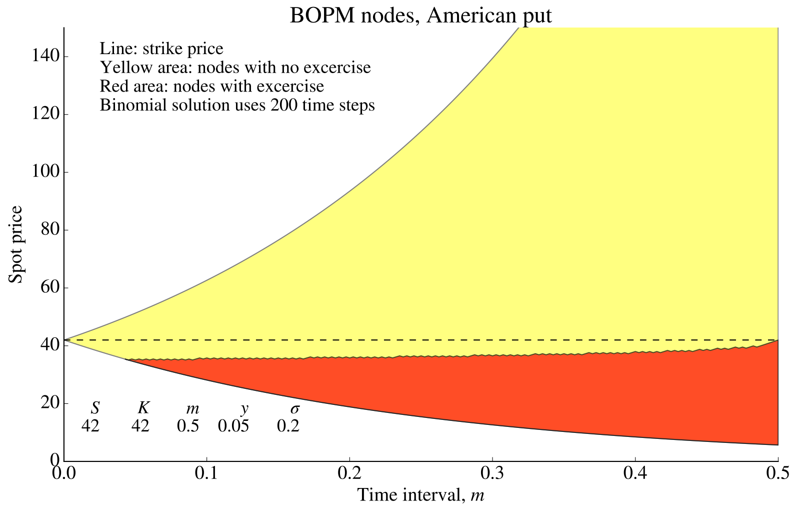


Figure 20.17: Numerical solution of an American put price

Chapter 21

The Black-Scholes Model

Main references: Elton, Gruber, Brown, and Goetzmann (2014) 23 and Hull (2009) 13 and 17
Additional references: McDonald (2014) 15–16 and Cox, Ross, and Rubinstein (1979)

21.1 The Black-Scholes Model

21.1.1 The Basic Black-Scholes Model without Dividends

Assume that the change over a short interval (between t and $t + h$) in the log asset price is an iid process

$$\ln S_{t+h} - \ln S_t = \mu h + \varepsilon_{t+h}, \text{ with } \varepsilon_{t+h} \sim iid\ N(0, \sigma^2 h). \tag{21.1}$$

This clearly means that $\ln S_{t+h}$ is a random walk with drift: just add $\ln S_t$ to both sides of the equation. (Also, notice that $\ln S_{t+h} - \ln S_t = \ln S_{t+h}/S_t$, which is also used below.) This implies that, based on the information in period 0, the logarithm of the stock price in period m, S_m, is normally distributed

$$\ln S_m \sim N(\ln S + \mu m, \sigma^2 m), \tag{21.2}$$

where S is the current ($t = 0$) asset price. (The subscript is dropped in most of these note, except when strictly needed.) See Figure 21.1 for an illustration.

Proof. (of (21.2)) Notice that (21.1) implies that

$$\begin{aligned}
\ln S_1 &= \ln S_0 + \mu + \varepsilon_1 \text{ and} \\
\ln S_2 &= \ln S_1 + \mu + \varepsilon_2 = (\ln S_0 + \mu + \varepsilon_1) + \mu + \varepsilon_2.
\end{aligned}$$

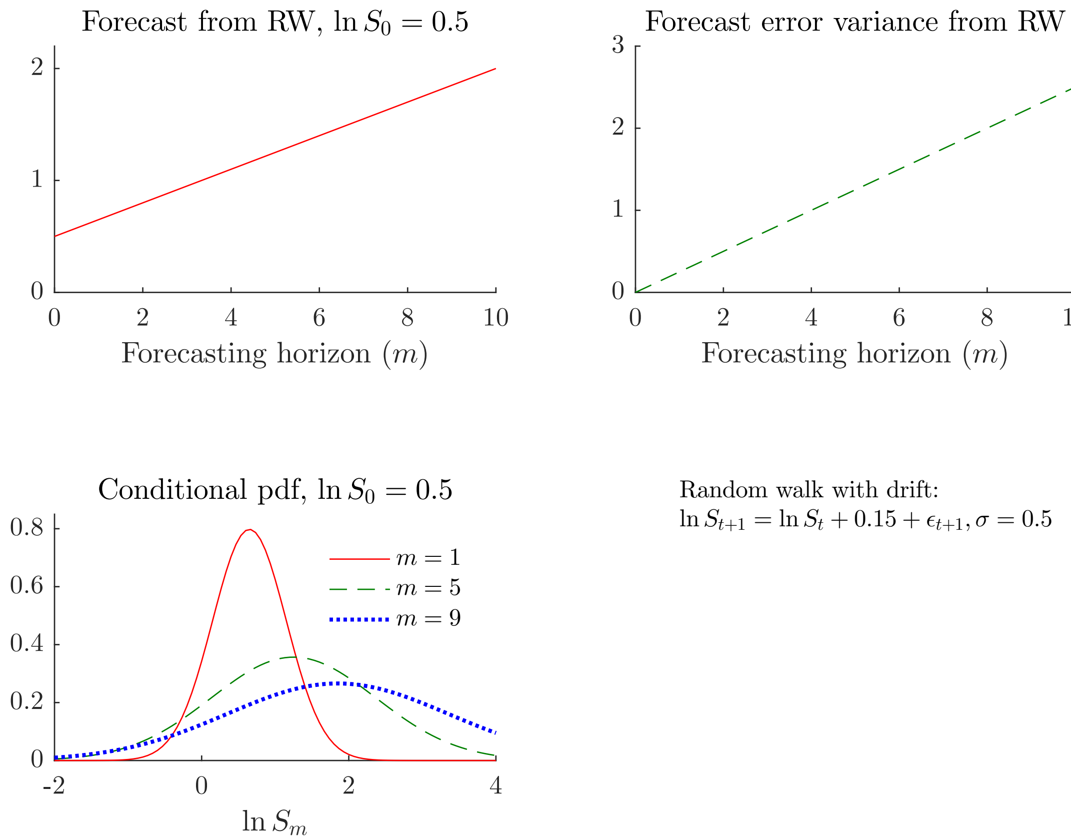


Figure 21.1: Conditional distribution from random walk with drift

Since $\mathrm{E}_0\, \varepsilon_1 = 0$ and $\mathrm{E}_0\, \varepsilon_2 = 0$, the conditional means are $\mathrm{E}_0 \ln S_1 = \ln S_0 + \mu$ and $\mathrm{E}_0 \ln S_2 = \ln S_0 + 2\mu$. The conditional variances are just the variances of the forecast errors (the ε part), so $\mathrm{Var}_0(\ln S_1) = \mathrm{Var}(\varepsilon_1) = \sigma^2$ and $\mathrm{Var}_0(\ln S_2) = \mathrm{Var}(\varepsilon_1) + \mathrm{Var}(\varepsilon_2) = 2\sigma^2$. Finally, $\ln S_1$ and $\ln S_2$ are linear functions of the (independent) normally distributed $(\varepsilon_1, \varepsilon_2)$, and therefore also normally distributed. ■

If we take the proper limit as the time interval h goes towards zero, then we have a Brownian motion for the log asset price ($d \ln S_t = \mu dt + \sigma d W_t$, where $d W_t$ are the increments to a Wiener process).

A hedging/no arbitrage argument similar to the binomial model then leads to the Black-Scholes formula for the price of a European call option(on an asset without dividends)

$$C = S\Phi(d_1) - e^{-ym} K\Phi(d_2), \text{ where} \tag{21.3}$$

$$d_1 = \frac{\ln(S/K) + (y + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{21.4}$$

In this formula, $\Phi(d)$ denotes the probability of $x \le d$ when x has an $N(0,1)$ distribution (that is, the distribution function value at d).

The call option price is increasing in the asset price, volatility, time to maturity and the interest rate, but decreasing in the strike price. See Figure 21.2.

It is straightforward to show (see Appendix) that when $\sigma = 0$ then $C = \max(S - e^{-ym}K, 0)$, and when $m = 0$ then $C = \max(S - K, 0)$.

Example 21.1 *(Call option price) With* $(S, K, y, m, \sigma) = (42, 42, 0.05, 0.5, 0.2)$*, (21.3)–(21.4) give* $C = 2.893$.

21.1.2 The Black-Scholes Model with Dividends

Consider a European option on an underlying asset that pays (continuous or discrete) dividends before expiration. Then, the Black-Scholes formula is not correct. It may seem as if dividends would just affect the mean drift in (21.1), and therefore not affect the option price—but this is wrong. The basic reason is that buying the underlying asset now is different from knowing that you will get the asset at the expiration of the option, since you get the dividends if you hold the asset.

To handle this, we could apply the BS formula to a forward contract (expiring on the same day as the option) instead. Let a prepaid forward contract (present value of forward price), worth $e^{-ym}F$, play the role of the underlying asset in (21.1). This gives the BS formula (21.3)–(21.4) but with $e^{-ym}F$ substituted for S

$$C = e^{-ym} F\Phi(d_1) - e^{-ym} K\Phi(d_2), \text{ where} \tag{21.5}$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{21.6}$$

This is *Black's model* which has many applications.

For instance, for an asset with a continuous dividend rate of δ, the forward-spot parity

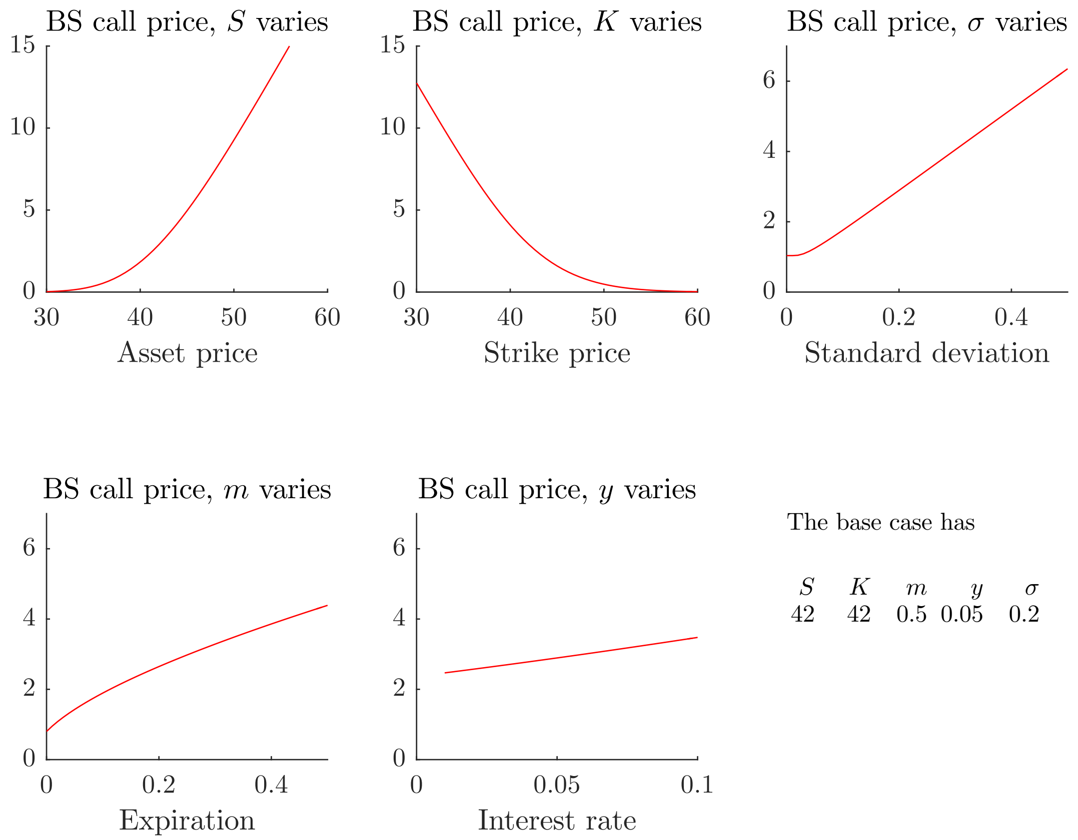


Figure 21.2: Call option price, Black-Scholes model

says $F = Se^{(y-\delta)m}$. In this case (21.5)–(21.6) can also be written

$$C = e^{-\delta m} S\Phi(d_1) - e^{-ym} K\Phi(d_2), \text{ where} \tag{21.7}$$

$$d_1 = \frac{\ln(S/K) + (y - \delta + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{21.8}$$

When the asset is a currency (read: foreign money market account) and δ is the foreign interest rate, then this is the "Garman-Kolhagen" formula. The Appendix contains some hints about how to use computer code written for the Black-Scholes formula to calculate Black's model.

Using the put-call parity formula for an asset with continuous dividends ($C - P = Se^{-\delta m} - e^{-my}K$), the pricing formula for a put option can be written $C - Se^{-\delta m} + e^{-my}K = P$)

$$P = e^{-\delta m} S[\Phi(d_1) - 1] - e^{-ym} K[\Phi(d_2) - 1], \tag{21.9}$$

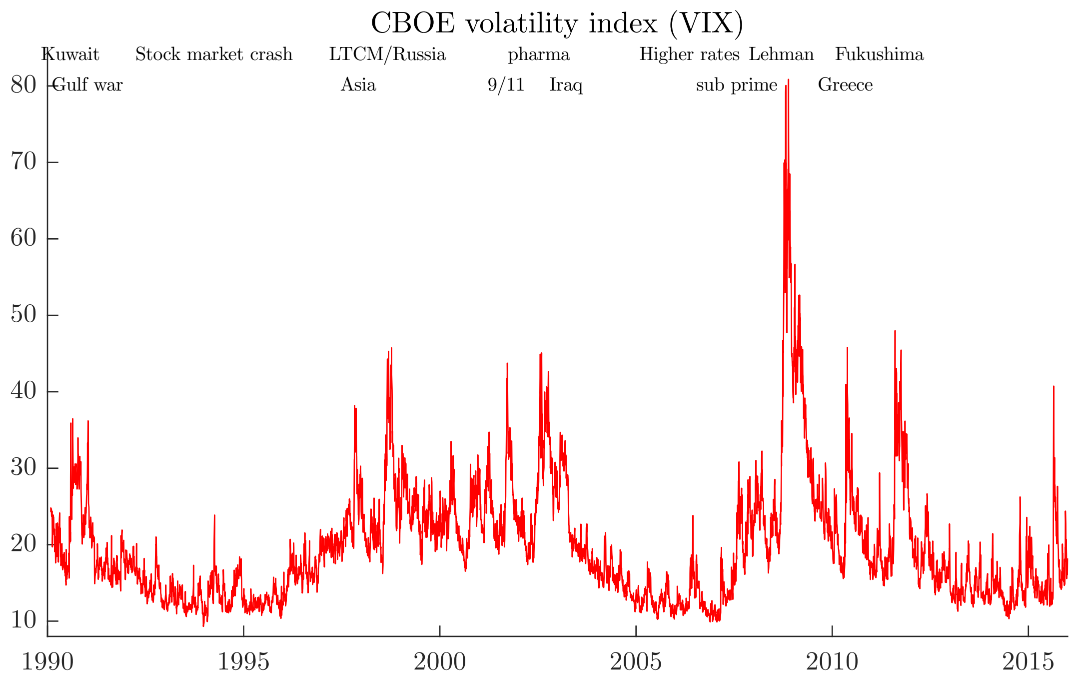


Figure 21.3: CBOE VIX, summary measure of implied volatilities (30 days) on US stock markets

where d_1 and d_2 are defined in (21.8).

Example 21.2 *(Put price) Using the same parameters as in Example 21.1 and* $\delta = 0$*, we get* $P = 1.856$*. Instead, with* $\delta = 0.05$*, we get* $P = 2.309$*.*

Remark 21.3 *(Alternative expression for the put price*) Since* $\Phi(d) + \Phi(-d) = 1$*, (21.9) can also be written*

$$P = e^{-ym} K\Phi(-d_2) - e^{-\delta m} S\Phi(-d_1).$$

21.1.3 Implied Volatility: A Measure of Market Uncertainty

The Black-Scholes formula ((21.3)–(21.4) or (21.5)–(21.6)) contains only one unknown parameter: the variance $\sigma^2 m$ in the distribution of $\ln S_m$ (see 21.2). With data on the option price, spot and forward prices, the interest rate, and the strike price, we can solve for the standard deviation σ (see from Figure 21.2 that the option price and the volatility have a monotonic relation). (You can also calculate σ from a put, since the put-call parity shows that a call and a put with the same strike price have the same implied volatility.)

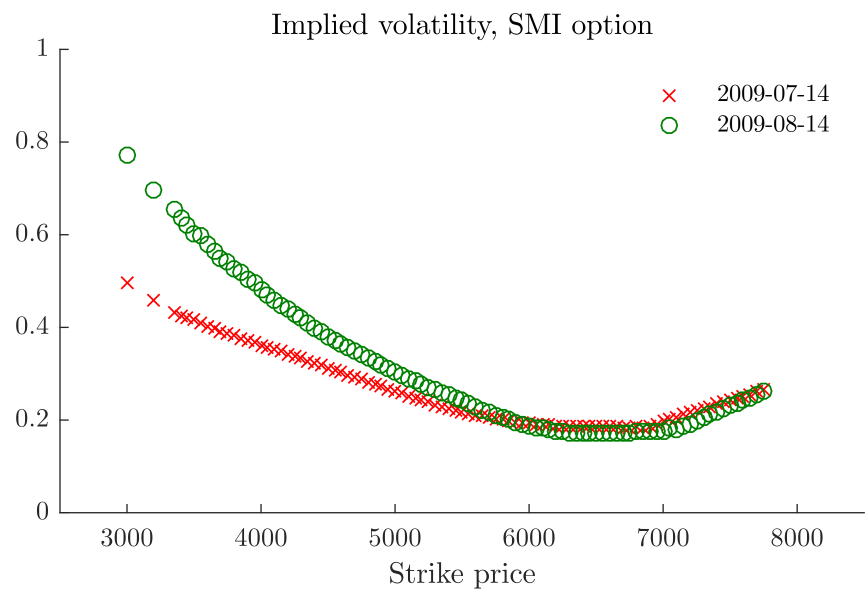


Figure 21.4: Implied volatilities of SMI options, selected dates

The term σ is often called the *implied volatility*—and it is often used as an indicator of market uncertainty about the future asset price, S_m. It can be thought of as an annualized (provided a period is defined as a year) standard deviation. See Figure 21.3 for an empirical example.

Note that we can solve for one implied volatility for each available strike price. If the Black-Scholes formula is correct, that is, if the assumption in (21.1) is correct, then these volatilities should be the same across strike prices. On currency markets, we often find

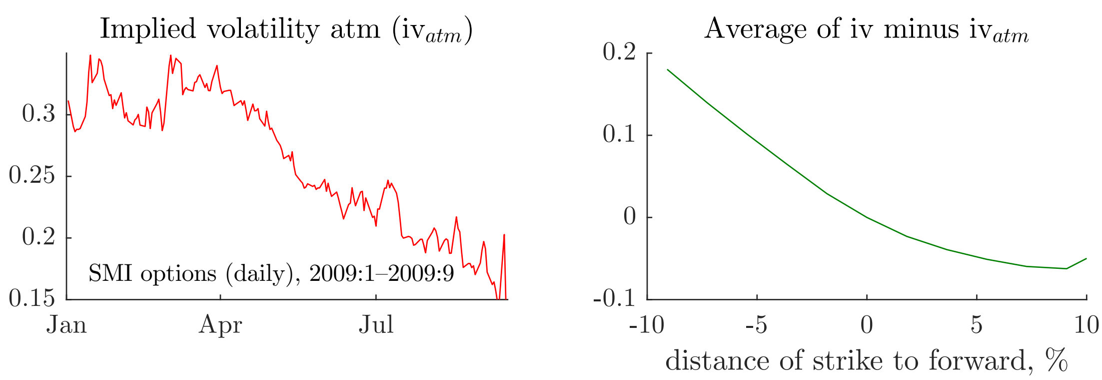


Figure 21.5: Implied volatilities over nine months

a volatility "smile" (volatility is a U-shaped function of the strike price). One possible explanation is that the (perceived) distribution of the future asset price has relatively more probability mass in the tails ("fat tails") than a normal distribution has. On equity markets, we often find a volatility "smirk" instead, where the volatility is very high for very low strike prices. This is often interpreted as that investors are willing to pay a lot for put options that protect them from a dramatic fall in the stock price. One possible explanation is thus that the distribution has more probability mass than a normal distribution at very low stock prices (negative skewness). See Figures 21.4–21.5 for empirical examples.

Remark 21.4 *(Practical hint: starting value for finding σ^*) A good starting guess is $\sigma \approx C/[e^{-ym} F\sqrt{m/(2\pi)}]$. Alternatively, it is often recommended to use the starting value $\sigma = \sqrt{|\ln(F/K)|\, 2/m}$. See Appendix for more details.*

21.2 Convergence of the BOPM to Black-Scholes

21.2.1 The Main Result

This section demonstrates that the option prices from the binomial option pricing model (BOPM) converge to the prices from the Black-Scholes model. See Figures 21.6–21.7 for an illustration of how the parameters (p, u, d) from the CRR approach and the resulting option price converge as the number of time steps increases (keeping the time to expiration fixed).

21.2.2 The Risk Neutral Distribution in Black-Scholes

We know that the risk neutral pricing of a European call option is

$$C = e^{-ym}\,\mathrm{E}^* \max(0, S_m - K), \tag{21.10}$$

where E^* denotes the expectation according to the risk neutral distribution. This can be written

$$C = e^{-ym} \int_K^\infty (S_m - K)\, f^*(S_m) dS_m, \tag{21.11}$$

where $f^*(S_m)$ is the risk neutral density function of the asset price at expiration (S_m). (Below K the value of the integrand is zero.)

For the Black-Scholes model, the (physical) normal distribution for the log asset price

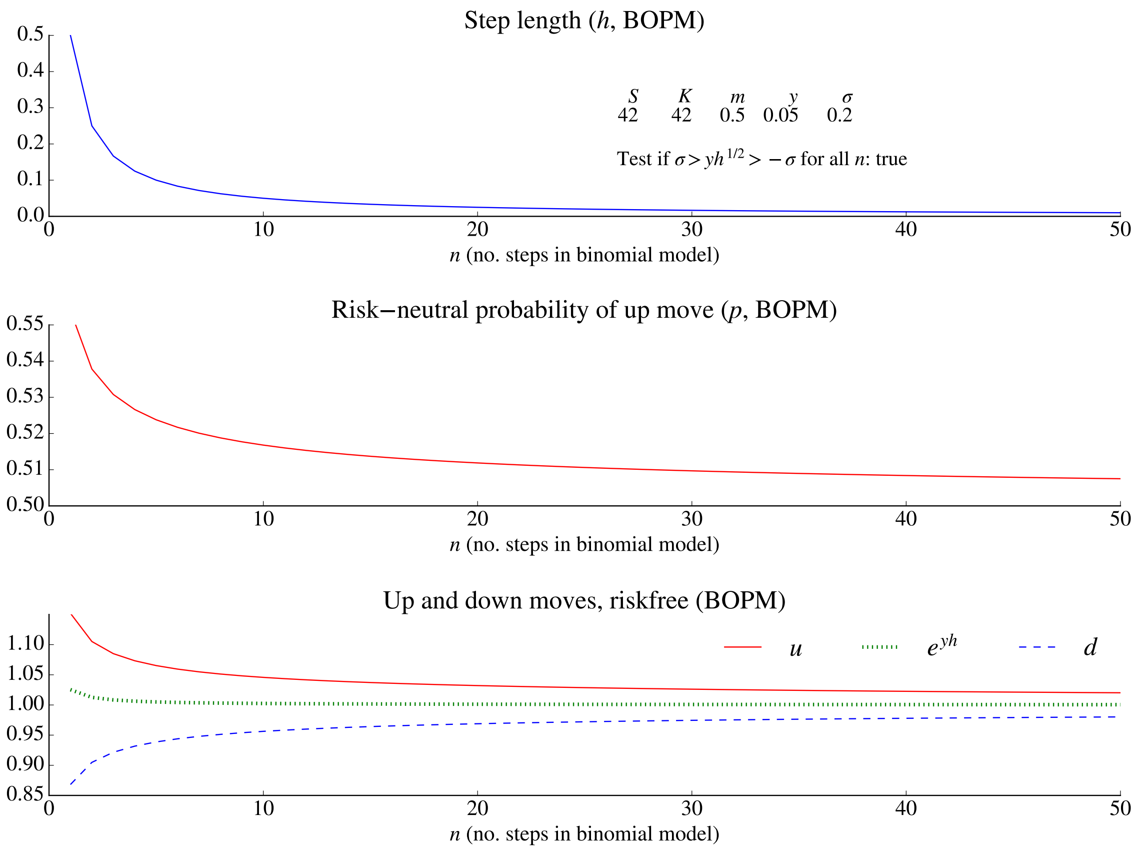


Figure 21.6: Convergence of the parameters in a binomial model

(21.2) implies that the risk neutral distribution of $\ln S_m$ is

$$\ln S_m \sim^* N(\ln S + ym - \sigma^2 m/2, \sigma^2 m), \tag{21.12}$$

where S is the current asset price. This risk neutral distribution typically has a lower mean than the physical distribution (certainly if the expected return is higher than the risk free rate), but the same variance. Calculating the expectation in (21.10) by using (21.12) gives the Black-Scholes formula (21.3). (Proving this is just a matter of calculating the integral. Not difficult, once a squared expression is handled. Alternatively, we can calculate (21.11) by numerical integration to verify that we get the same value as from the Black-Scholes formula.)

Recall that the risk neutral distribution is for the underlying, and that all derivatives on the underlying can be priced using the same risk neutral distribution. For instance, a forward contract has a zero price and the payoff $S_m - F$, so by using (21.12) we can

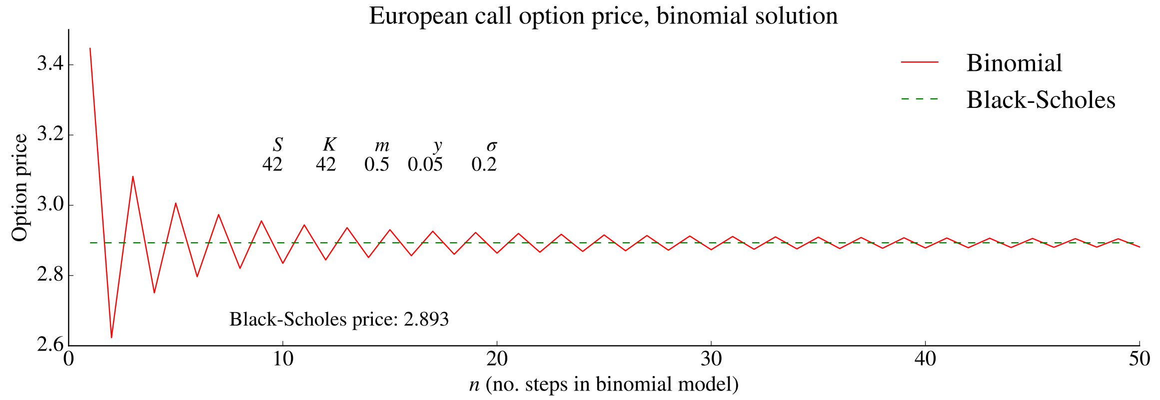


Figure 21.7: Convergence of the binomial price to the Black-Scholes price

evaluate

$$\begin{aligned} 0 &= \mathrm{E}^*(S_m - F) \\ &= \int_{-\infty}^{\infty} (S_m - F) f^*(S_m) dS_m \\ &= Se^{ym} - F, \end{aligned} \tag{21.13}$$

which indeed fits with the forward-spot parity. The expression also shows that the risk neutral expectation equals the forward price.

Proof. (that (21.13) has a mean equal to the forward rate) Recall that $\mathrm{E}[\exp(x)] = \exp(\mu+s^2/2)$, if $x \sim N(\mu, s^2)$. Applying on the distribution in (21.12) gives $\mathrm{E}^*[\exp(\ln S_m)] = \exp(\ln S + ym) = Se^{ym}$, which equals the forward price (by the forward-spot parity). ∎

21.2.3 The Risk Neutral Binomial Distribution

In the binomial option pricing model (BOPM), the risk neutral binomial process for the asset price gives the following binomial process for the *log asset price*

$$\ln S_{t+h} - \ln S_t = \begin{cases} \ln u & \text{with probability } p \\ \ln d & \text{with probability } 1-p. \end{cases} \tag{21.14}$$

(In the risk neutral binomial tree $S_{t+h} = S_t u$ with probability p and $S_t d$ with probability $1-p$. This implies (21.14) for the logs.) The parameters u, d and p all depend on the time step length h in such a way that we match the mean and variance of the price series. In fact, they are chosen so that the mean and variance of $\ln S_{t+h} - \ln S_t$ are (at least in the limit) proportional to h.

Clearly, the binomial tree means that we reach $\ln S_m$ by adding n steps of the kind in (21.14). To save clutter, assume the current time period is zero. Then,

$$\ln S_m = \ln S_0 + \sum\nolimits_{i=1}^{n} [\ln S_{ih} - \ln S_{(i-1)h}]. \tag{21.15}$$

I demonstrate the convergence in two steps: first, that the binomial distribution converges to a normal distribution; and second that both distributions have the same mean and variance in the limit.

21.2.4 The Central Limit Theorem at Work

If we can show that the risk neutral distribution implied by the binomial model converges (as the number of time steps increase, keeping time to expiration constant) to a normal distribution, then it is plausible that the Black-Scholes model can be thought of as the limit of the binomial model.

The Black-Scholes model is based on normally distributed changes of log prices. In the binomial model, the log price changes can only take two values, but the sum of many such changes will converge to a normally distributed variable as the number of time steps increases—providing the step size decreases. This may seem counter intuitive since central limit theorems apply to samples averages (times the square root of the sample size), not to sums. However, the rescaling of the log price changes as the number of time steps increases, means that the sum is effectively a (scaled) sample average—so a CLT indeed applies.

See Figure 21.8 for an example of how the distribution converges.

Proposition 21.5 *If u, d and p in the binomial process (21.14) are such that the mean and variance of $\ln S_{t+h} - \ln S_t$ are proportional to h, then the distribution of $\ln S_m - \ln S_0$ converges to a normal distribution as the number of time steps n increases, keeping the maturity m constant (so $h = m/n$).*

Remark 21.6 *(The Lindeberg-Lévy central limit theorem) If x_i is independently and identically distributed with $\mathrm{E}\, x = 0$ and $\mathrm{Var}(x_i) = \sigma < \infty$, then,*

$$\sqrt{n}\frac{1}{n}\sum\nolimits_{i=1}^{n} x_i = \sum\nolimits_{i=1}^{n} x_i/\sqrt{n} \xrightarrow{d} N(0, \sigma^2).$$

Proof. (of Proposition 21.5) The binomial model (21.14)–(21.15) means that we can write the log price change over a time step of length as $\varepsilon_i \sqrt{h} + \mu h$, where ε_i is an iid

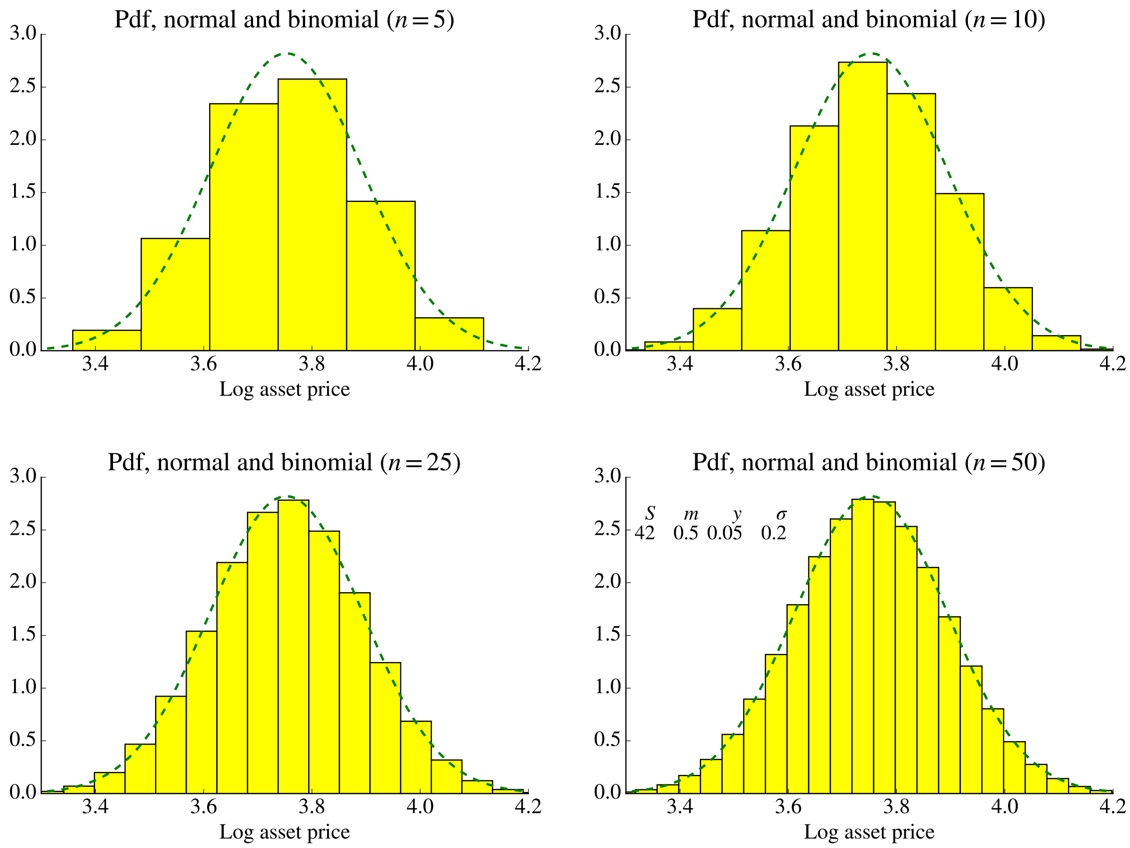


Figure 21.8: Convergence of the binomial model to the Black-Scholes model

zero mean random variable with variance σ^2

$$\varepsilon_i \sqrt{h} + \mu h = \ln S_{ih} - S_{(i-1)h}, \text{ where } \mathrm{E}\, \varepsilon_i = 0 \text{ and } \mathrm{Var}(\varepsilon_i) = \sigma^2.$$

Clearly, $\mathrm{Var}(\varepsilon_i \sqrt{h})$ is proportional to h and the means are all zero, as required. For instance, for the first time interval we have $\varepsilon_1 \sqrt{h} + \mu h = \ln S_h - \ln S_0$. Since we take n steps (of length $h = m/n$) to get from from period 0 to m, we can write the change in the log price (from 0 to m) as the sum of $\sqrt{h}\varepsilon_i + \mu h$ from $i = 1$ to n

$$\begin{aligned} \ln S_m - \ln S_0 &= \sqrt{h} \textstyle\sum_{i=1}^{n} \varepsilon_i + n h \mu \\ &= \sqrt{m} \textstyle\sum_{i=1}^{n} \varepsilon_i / \sqrt{n} + \mu m, \end{aligned}$$

where we have used $h = m/n$. The first term on the right hand (except the constant m) is of the same form as in the central limit theorem in Remark 21.6. The second term is just a constant. ■

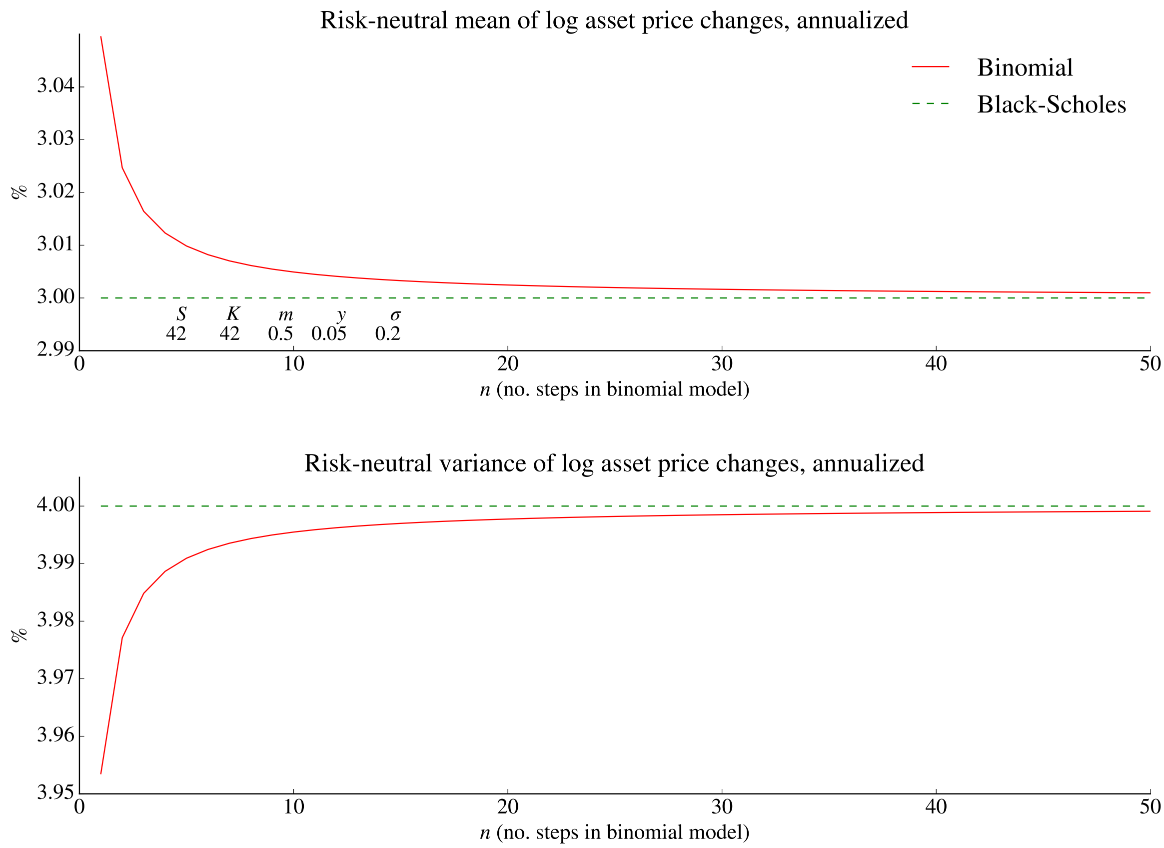


Figure 21.9: Convergence of the binomial mean and variance

21.2.5 Convergence of the Mean and Variance

This section demonstrates that the mean and variance of the binomial distribution converge to the same values as in the risk neutral distribution of the Black-Scholes model (21.12). See Figure 21.9 for an illustration.

Proposition 21.7 *(Moments of CRR steps) In the Cox, Ross, and Rubinstein (1979) tree, the parameters in (21.14) are*

$$\ln u = \sigma\sqrt{h}, \ln d = -\sigma\sqrt{h} \text{ and } p = (e^{yh} - d)/(u - d).$$

As $n \to \infty$, but $h = m/n$ we have (since the price changes are independent) the following results for the sum of them

$$\mathrm{E} \ln S_m - \ln S_0 = m(y - \sigma^2/2) \text{ and } \mathrm{Var}(\ln S_m - \ln S_0) = m\sigma^2.$$

This is the same as in the risk neutral distribution of the Black-Scholes model.

Proof. (of Proposition 21.7) Recall that the mean and variance of $\ln S_{t+h} - \ln S_t$ are $p \ln u + (1-p) \ln d$ and $p(1-p)(\ln u - \ln d)^2$ respectively. Since the terms in (21.15) are uncorrelated, the mean and the variance of the sum are $n\,\mathrm{E}(\ln S_{t+h} - \ln S_t)$ and $n\,\mathrm{Var}(\ln S_{t+h} - \ln S_t)$. Substitute for u, d and p and take the limits of as $n \to \infty$, but $h = m/n$. (This is straightforward, but slightly messy, calculus.) ∎

21.3 Hedging an Option

This section discusses how we can hedge a European call option. The setting is that we have written (sold) such an option, but we do not want to carry the risk.

The derivatives in the following remark will be useful.

Remark 21.8 *(The "Greeks") The derivatives of the Black-Scholes formula for an asset with continuous dividends (21.7)–(21.8) are*

$$
\begin{aligned}
\Delta &= \frac{\partial C}{\partial S} = e^{-\delta m} \Phi\left(d_1\right) \\
\Gamma &= \frac{\partial^2 C}{\partial S^2} = \frac{e^{-\delta m} \phi\left(d_1\right)}{S \sigma \sqrt{m}} \\
\theta &= \frac{\partial C}{\partial t} = -\frac{\partial C}{\partial m} = \delta S e^{-\delta m} \Phi(d_1) - y K e^{-y m} \Phi(d_2) - \frac{1}{2\sqrt{m}} e^{-\delta m} S \phi(d_1) \sigma \\
vega &= \frac{\partial C}{\partial \sigma} = S e^{-\delta m} \phi\left(d_1\right) \sqrt{m} \\
\rho &= \frac{\partial C}{\partial y} = m K e^{-y m} \Phi(d_2),
\end{aligned}
$$

where $\phi()$ is the standard normal probability density function (the derivative of $\Phi()$). See Figures 21.10–21.11.

21.3.1 Delta Hedging

Consider a portfolio with Δ_t of the underlying asset (the hedging portfolio) and short one call option. (Warning: Δ_t does not represent a change, it is the number of assets bought,) The value of the overall position is

$$V_t = \Delta_t S_t - C_t. \tag{21.16}$$

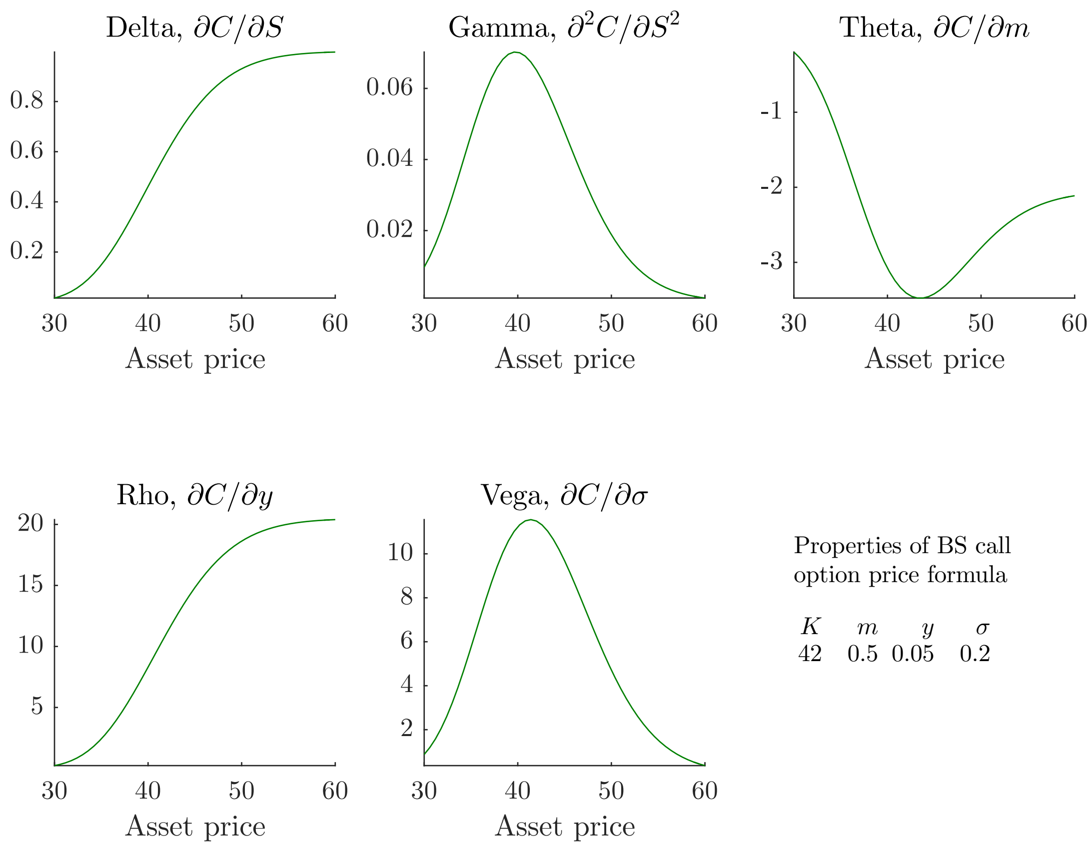


Figure 21.10: The Greeks in the Black-Scholes model as a function of asset price

The idea is now to find Δ_t so that $\Delta_t S_t$ and C_t are equally sensitive to the main risk: changes in S_t.

Assume that of all the drivers of the option price, only the price of the underlying asset can change (clearly not true, but at least a starting point for the analysis). A first-order Taylor approximation of the call option price is

$$C_{t+h} - C_t \approx \frac{\partial C_t}{\partial S} (S_{t+h} - S_t) . \tag{21.17}$$

Use (21.17) to approximate the change of the value of the overall portfolio as

$$\begin{aligned} V_{t+h} - V_t &= \Delta_t (S_{t+h} - S_t) - (C_{t+h} - C_t) \\ &\approx \Delta_t (S_{t+h} - S_t) - \frac{\partial C_t}{\partial S} (S_{t+h} - S_t) \\ &\approx 0 \text{ if } \Delta_t = \frac{\partial C_t}{\partial S} . \end{aligned} \tag{21.18}$$

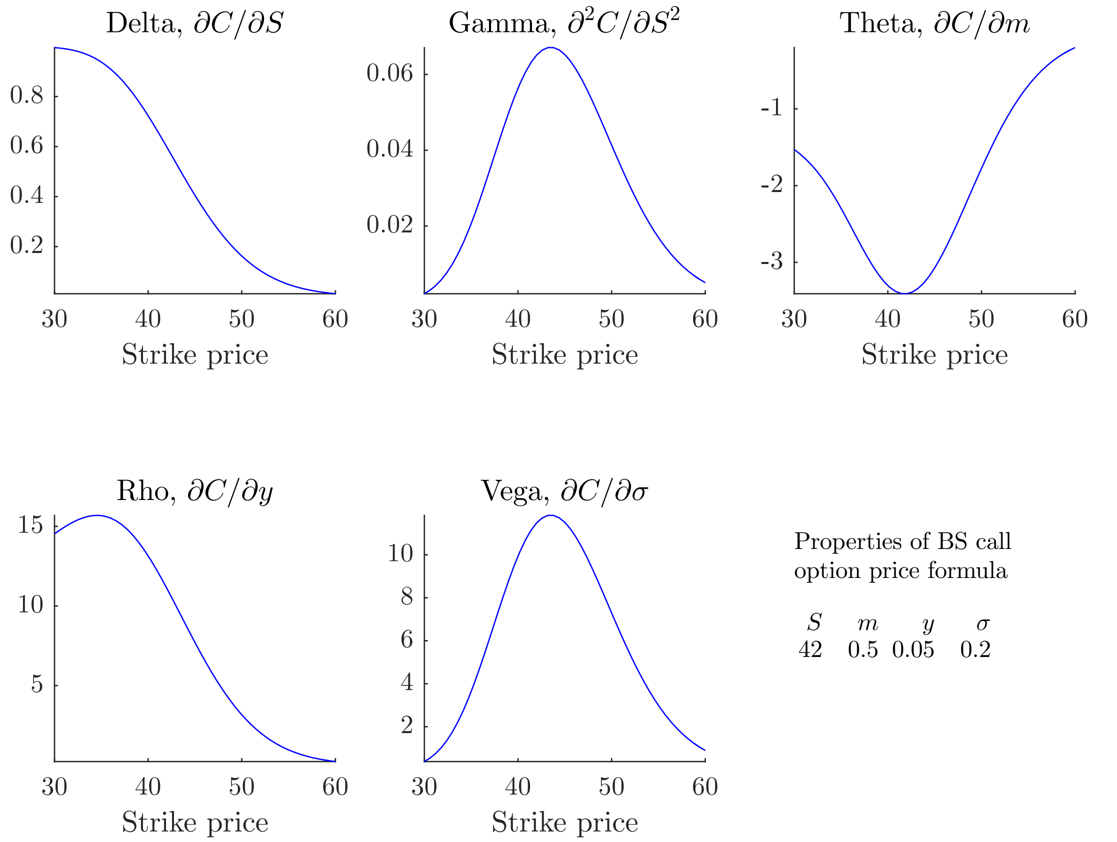


Figure 21.11: The Greeks in the Black-Scholes model as a function of strike price

This is a *delta hedge*. Clearly, the delta is likely to change from period to period, so the portfolio needs to be frequently rebalanced.

In practice, the hedging portfolio (also called the replicating portfolio) also includes a negative position in a short-term money market account ($M_t < 0$)—so the overall portfolio has a zero (initial) value

$$0 = \underbrace{\Delta_t S_t + M_t}_{\text{hedging portfolio}} - C_t, \text{ so } M_t = -\Delta_t S_t + C_t. \tag{21.19}$$

This means that we finance the purchase of the underlying asset with the proceeds from the selling the option and from borrowing. See Remark 21.9 for details on how the portfolio value changes over time.

Remark 21.9 *(Overall portfolio value over several subperiods*) Start by creating a hedge portfolio with a zero initial value as in (21.19). In $t+h$ (say, after one day so $h = 1/365$),* *this portfolio is worth (this is the marking-to-market)*

$$V_{t+h} = \Delta_t(D_{t+h} + S_{t+h}) + M_t e^{y_t h} - C_{t+h},$$

where the underlying pays a dividend ($D_{t+h} = 0$ if no dividends), the prices are measured after dividends and y_t is the interest rate. In $t + h$ we need Δ_{t+h} units of the underlying asset (value $\Delta_{t+h} S_{t+h}$). Since we already own Δ_t of the underlying asset, this means that we must withdraw an additional $(\Delta_{t+h} - \Delta_t)S_{t+h}$ from the money market account. On that account, we have since last period $M_t e^{y_t h} + \Delta_t D_{t+h}$ (old holdings with interest plus the dividends we received in cash), so our holdings in $t + h$ (after having rebalanced the holdings) is

$$M_{t+h} = M_t e^{y_t h} + \Delta_t D_{t+h} - (\Delta_{t+h} - \Delta_t)S_{t+h}.$$

The value of the overall portfolio in $t + 2h$ (marking-to-market) is computed as in the first equation, but with subscripts advanced one period. See Figure 21.13 for an illustration. In that figure, "m-to-m" stands for the marking-to-market stage (first equation in this remark) and "rebalancing" for the stage after rebalancing the portfolio (second equation in this remark).

In the Black-Scholes model for an asset with dividends, the delta is

$$\Delta = \frac{\partial C}{\partial S} = e^{-\delta m} \Phi(d_1), \tag{21.20}$$

where d_1 is given by (21.8), see Remark 21.8. Without dividends, just set $\delta = 0$.

Remark 21.10 *(Hedging a put option) Instead, if you want to hedge a put option, replace C by P in (21.17) and notice that the delta of a put option is*

$$\frac{\partial P}{\partial S} = e^{-\delta m}[\Phi(d_1) - 1] = -e^{-\delta m} \Phi(-d_1),$$

which is negative. (This result follows from the put-call parity and by using the the symmetry of the normal distribution. To see this, notice that the put-call parity says $P = C - Se^{-\delta m} + e^{-my} K$, so $\partial P / \partial S = \partial C / \partial S - e^{-\delta m}$.)

Example 21.11 *(Deltas) Using the same parameters as in Example 21.1 and $\delta = 0$, we have $\partial C / \partial S = 0.60$ and $\partial P / \partial S = -0.40$. The difference is clearly equal to one.*

See Figures 21.10–21.11 for an illustration of how Δ (and other derivatives) depend on the strike and underlying price. In particular, notice that $0 \leq \Delta \leq 1$ and that Δ

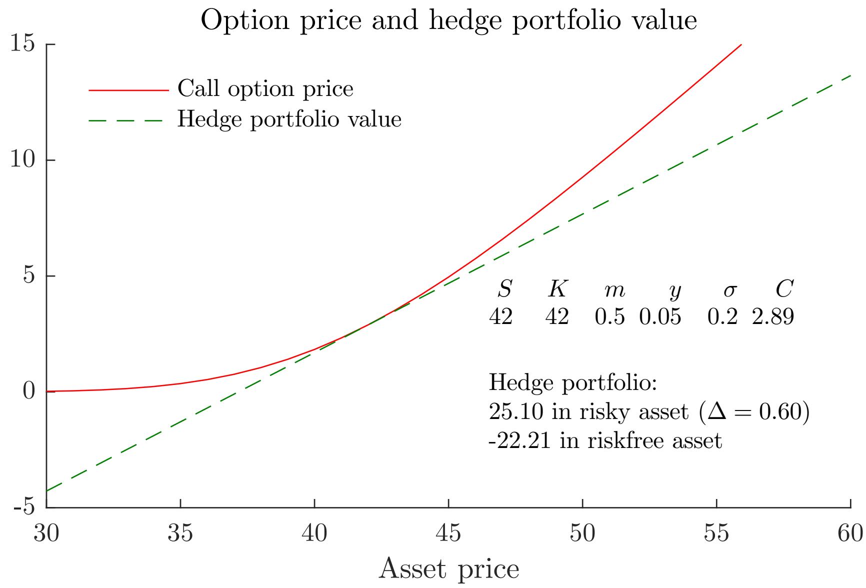


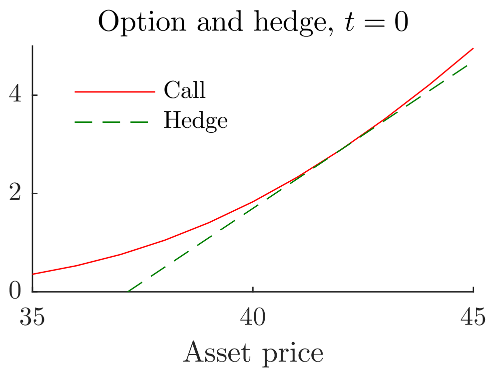
Figure 21.12: Delta hedging: approximating the option price change

increasing in the price of the underlying asset. Intuitively, an option that is deep out of the money will not be very sensitive to the asset price—since the chance of exercising is so low. Conversely, an option that is deep in the money moves almost in tandem with the asset price, since it will almost for sure be exercised.

See Figure 21.12 for how the hedging portfolio approximates the call option price (as well for numbers for the positions in the hedge). Also, see Figure 21.13 for how the positions of the hedge portfolio changes from day to day. In that figure, "m-to-m" stands for the marking-to-market stage and "rebalancing"for the stage after rebalancing the portfolio.

Example 21.12 *(Delta hedging) Using the same parameters as in Example 21.1 and $\delta = 0$, Figure 21.13 illustrates the initial positions (day 0), and two snap shots of the day after (day 1: after marking to market, day 1: after rebalancing). On day, the overall portfolio includes $\Delta = 0.6$ of the underlying asset (at a value of $0.6 \times 42 = 25.10$), -1 of the option (at the value -2.89) and the balance on a money market account ($-25.10 + 2.89 = -22.21$) so the total portfolio is worth zero. This clearly means that the investor has borrowed.*

Finally, see Figure 21.14 for an example of how a delta hedge works on real data.



S	K	m	y	σ	C
42	42	0.5	0.05	0.2	2.89

	S_0	Δ_0	$-C_0$	$\Delta_0 S_0$	M_0	V_0
Day 0	42.00	0.60	-2.89	25.10	-22.21	0.00
	S_1	Δ_0	$-C_1$	$\Delta_0 S_1$	$M_0 e^{yh}$	V_1
Day 1 (m-to-m)	43.00	0.60	-3.51	25.70	-22.22	-0.02
	S_1	Δ_1	$-C_1$	$\Delta_1 S_1$	M_1	V_1
Day 1 (rebalance)	43.00	0.66	-3.51	28.40	-24.91	-0.02

Figure 21.13: Delta hedging over time

21.3.2 Delta-Gamma Hedging*

Delta hedging can be imprecise if the price of the underlying asset changes much. A second-order Taylor approximation of the option price gives

$$C_{t+h} - C_t \approx \Delta_t \left(S_{t+h} - S_t\right) + \frac{1}{2}\Gamma_t \left(S_{t+h} - S_t\right)^2, \text{ where } \Delta_t = \frac{\partial C_t}{\partial S} \text{ and } \Gamma_t = \frac{\partial^2 C_t}{\partial S^2}. \tag{21.21}$$

The Δ and Γ of the Black-Scholes model are given in Remark 21.8, see Figures 21.10–21.11 for illustrations.

This movement can be hedged by holding v_t of the underlying asset and w_t of other option. Let Δ_t^* and Γ_t^* be the delta and gamma of this other option. A second-order Taylor approximation of the value of this portfolio (denoted U_t) is

$$U_{t+h} - U_t \approx v_t \left(S_{t+h} - S_t\right) + w_t \Delta_t^* \left(S_{t+h} - S_t\right) + \frac{1}{2} w_t \Gamma_t^* \left(S_{t+h} - S_t\right)^2. \tag{21.22}$$

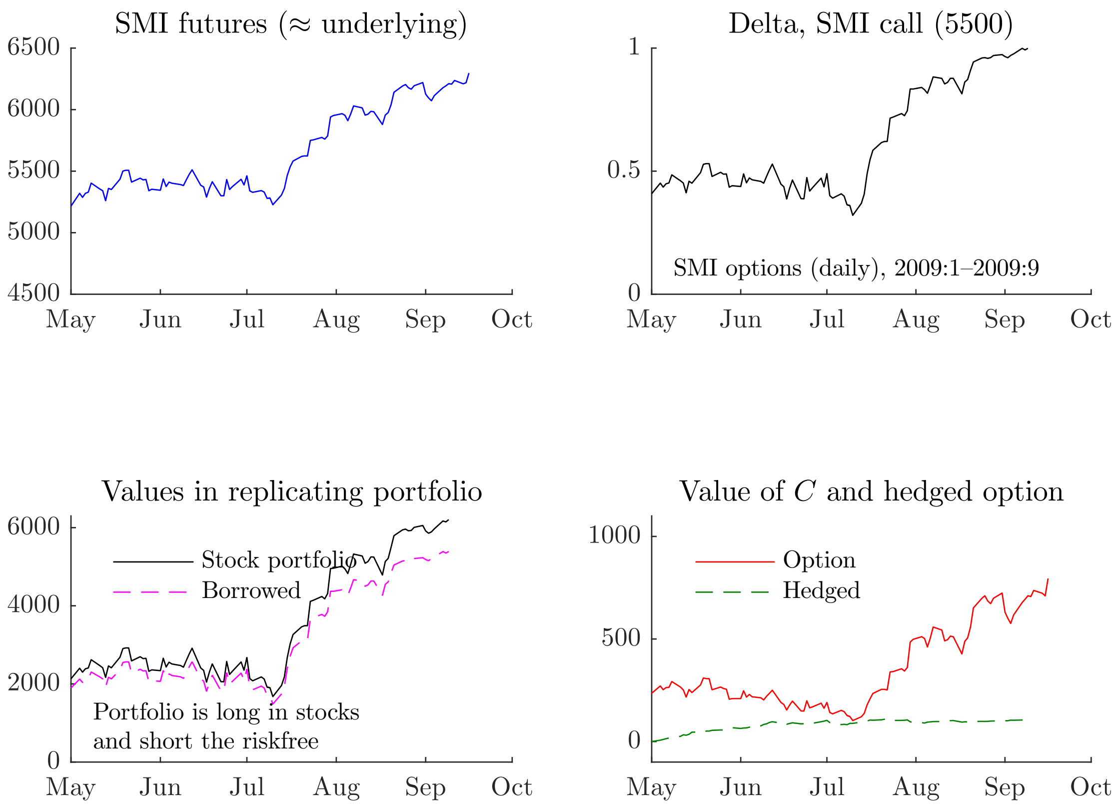


Figure 21.14: Delta hedging an SMI call option

Subtracting (21.21) from (21.22)

$$(U_{t+h} - U_t) - (C_{t+h} - C_t) \approx \underbrace{\left(v_t + w_t \varDelta_t^* - \varDelta_t\right)}_{A_t} (S_{t+h} - S_t) + \underbrace{\left(w_t \varGamma_t^* - \varGamma_t\right)}_{B_t} \frac{1}{2} (S_{t+h} - S_t)^2 . \tag{21.23}$$

By first choosing w_t to make the B_t term zero and then v_t to make the A_t term zero, we get a hedge. This clearly gives

$$w_t = \varGamma_t / \varGamma_t^*, \text{ and} \tag{21.24}$$

$$v_t = \varDelta_t - \left(\varGamma_t / \varGamma_t^*\right) \varDelta_t^*. \tag{21.25}$$

Example 21.13 *(Delta-gamma hedging) Suppose* $(\varDelta_t, \varGamma_t) = (0.5, 0.07)$ *and* $(\varDelta_t^*, \varGamma_t^*) = (0.3, 0.03)$*, then* $B_t = w_t 0.03 - 0.07 = 0$ *requires* $w_t = 2.33$ *and* $A_t = v_t + 2.33 \times 0.3 - 0.5 = 0$ *requires* $v_t = -0.2$*. Clearly, this is quite different from a delta hedge (which has* $v_t = 0.5$ *and* $w_t = 0$*). Here, the lower sensitivity (gamma) of the second option to the quadratic term, means that the hedge portfolio includes a lot of the second option. As a* *consequence, it becomes overexposed to the linear term, which is compensated for by a short position in the underlying asset.*

21.3.3 Delta-Vega Hedging*

The volatility of financial markets seems to change over time. To account for that, a first-order Taylor approximation of the call option price with respect to the underlying and the volatility is

$$C_{t+h} - C_t \approx \frac{\partial C_t}{\partial S}(S_{t+h} - S_t) + \frac{\partial C_t}{\partial \sigma}(\sigma_{t+h} - \sigma_t). \tag{21.26}$$

Consider a more sophisticated hedge portfolio with Δ_t of the underlying asset, v_t of a financial instrument that moves in tandem with the volatility (for instance a futures on the VIX) and short one call option. The value of the overall position is

$$V_t = \Delta_t S_t + v_t \sigma_t - C_t. \tag{21.27}$$

By comparing (21.26) and (21.27) it is clear that the value of the overall portfolio would be immune (to a first order approximation) against movements in both the underlying and the volatility if $\Delta_t = \partial C_t / \partial S$ and $v_t = \partial C_t / \partial \sigma$. Although the Black-Scholes model is inconsistent with the idea of a time-varying volatility, a starting point could still be to use the derivatives according to that model.

21.4 Estimating Riskneutral Distributions*

We have seen that the price of a derivative is a discounted risk-neutral expectation of the derivative payoff, see (21.10).

In the Black-Scholes model, this risk-neutral distribution is that $\ln S_m$ is normally distributed as in (21.2) except that the mean is different (this is the difference between the natural and the risk-neutral distribution). However, risk neutral distributions can be derived from other assumptions than those in the Black-Scholes model, and (21.10) would still be valid. For instance, it holds in the binomial model, whose distribution is not normal (unless we make the time steps very many and small). Alternatively, we could construct a binomial tree where the time steps have different volatilities (this is often done to fit the yield curve)—and even in the limit (with many and small time steps) the distribution would be non-normal. Once again, the Black-Scholes formula would not be exact, but (21.10) would still be true.

Example 21.14 *(Call prices, three states) Suppose that S_m only can take three values: 90, 100, and 110; and that the risk neutral probabilities for these events are: 0.5, 0.4, and 0.1, respectively. We consider three European call option contracts with the strike prices 89, 99, and 109. From (21.10) their prices are (if $y = 0$)*

$$
\begin{aligned}
C\,(K = 89) &= 0.5(90 - 89) + 0.4(100 - 89) + 0.1(110 - 89) = 7 \\
C\,(K = 99) &= 0.5 \times 0 + 0.4(100 - 99) + 0.1(110 - 99) = 1.5 \\
C\,(K = 109) &= 0.5 \times 0 + 0.4 \times 0 + 0.1(110 - 109) = 0.1.
\end{aligned}
$$

With prices on several options with different strike prices (but otherwise identical), it is possible to estimate the risk-neutral distribution.

Example 21.15 *(Extracting probabilities) Suppose we observe the option prices in Example 21.14, and want to use these to recover the probabilities. We know the possible states, but not their probabilities. Let $\Pr(x)$ denote the probability that $S_m = x$. From Example 21.14, we have that the option price for $K = 109$ equals*

$$
\begin{aligned}
C\,(K = 109) &= 0.1 \\
&= \Pr(90) \times 0 + \Pr(100) \times 0 + \Pr(110)(110 - 109),
\end{aligned}
$$

which we can solve as $\Pr(110) = 0.1$. We now use this in the expression for the option price for $K = 99$

$$
\begin{aligned}
C\,(K = 99) &= 1.5 \\
&= \Pr(90) \times 0 + \Pr(100)(100 - 99) + 0.1(110 - 99),
\end{aligned}
$$

which we can solve as $\Pr(100) = 0.4$. Since probabilities sum to one, it follows that $\Pr(90) = 0.5$.

A common approach is to make an assumption about the form of the distribution, for instance, a mixture of two normal distributions. The parameters of this distribution are then chosen (estimated) by minimizing the sum (across strike prices) of squared differences between observed and predicted prices. (This is like the minimization problem behind the least squares method in econometrics.) This allows the possibility to pick up skewed (downside risk different from upside risk?) and even bimodal distributions.

Remark 21.16 *Figure 21.15 shows some data and results (assuming a mixture of two normal distributions) for German bond options around the announcement of the very high money growth rate on 2 March 1994.*

Remark 21.17 *Figures 21.16–21.18 show results for the CHF/EUR exchange rate around the period of active (Swiss) central bank interventions on the currency market.*

Remark 21.18 *(Robust measures of the standard deviation and skewness) Let* P_α *be the* α*th quantile (for instance, quantile 0.1) of a distribution. A simple robust measure of the standard deviation is just the difference between two symmetric quantile,*

$$\text{Std} = P_{1-\alpha} - P_\alpha,$$

where it is assumed that $\alpha < 0.5$*. Sometimes this measure is scaled so it would give the right answer for a normal distribution. For instance, with* $\alpha = 0.1$*, the measure would be divided by 2.56 and for* $\alpha = 0.25$ *by 1.35.*
One of the classical robust skewness measures was suggested by Hinkley

$$Skew = \frac{(P_{1-\alpha} - P_{0.5}) - (P_{0.5} - P_\alpha)}{P_{1-\alpha} - P_\alpha}.$$

This skewness measure can only take on values between -1 *(when* $P_{1-\alpha} = P_{0.5}$*) and* 1 *(when* $P_\alpha = P_{0.5}$*). When the median is just between the two percentiles (*$P_{0.5} = (P_{1-\alpha} + P_\alpha)/2$*), then it is zero.*

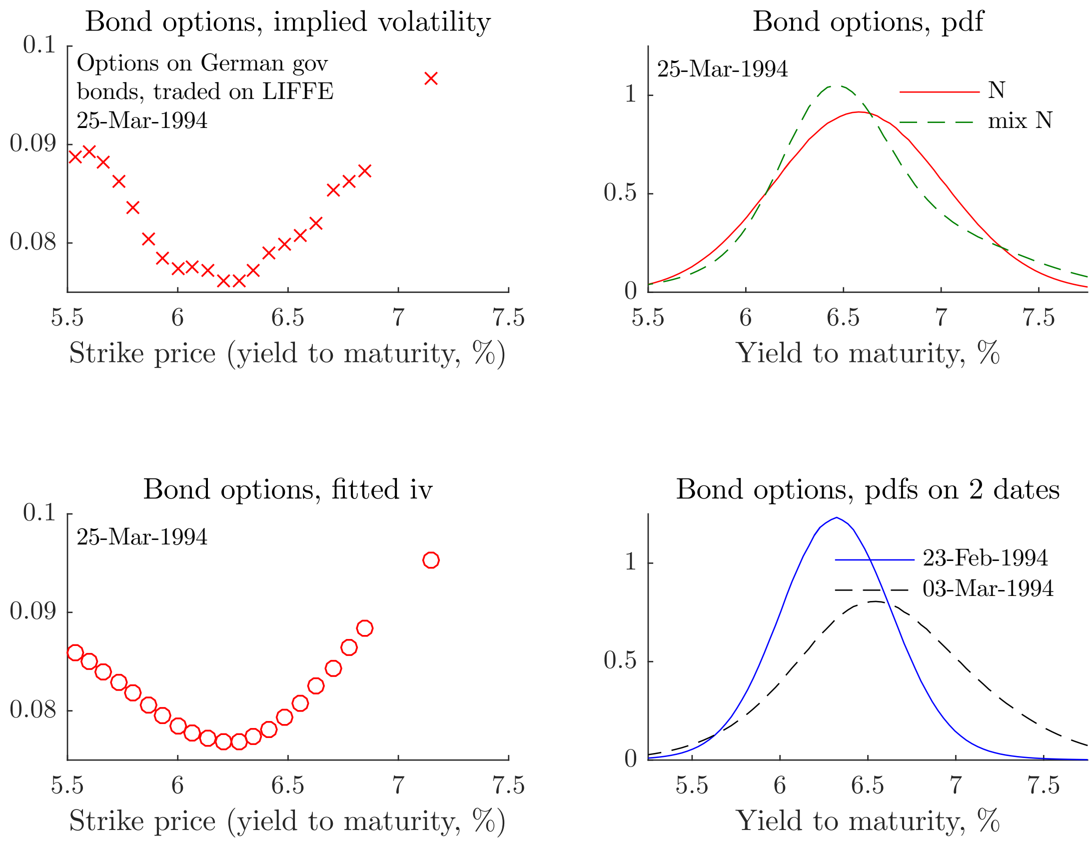


Figure 21.15: Bund options 23 February and 3 March 1994. Options expiring in June 1994.

21.5 Appendix: More Details on the Black-Scholes Model*

21.5.1 Limits of the Black-Scholes Formula when $\sigma = 0$ or $m = 0$

Remark 21.19 *(Black-Scholes formula when $\sigma = 0$*) From (21.4)* $\lim_{\sigma \to 0} d_1 = \lim_{\sigma \to 0} d_2 = \infty$ *if* $e^{ym}S \geq K$ *and* $-\infty$ *otherwise. Therefore,* $\lim_{\sigma \to 0} \Phi(d_1) = \lim_{\sigma \to 0} \Phi(d_2) = 1$ *if* $e^{ym}S \geq K$ *and* 0 *otherwise. The Black-Scholes call option price at* $\sigma = 0$ *is therefore* $\max(S - e^{-ym}K, 0)$.

Remark 21.20 *(Call option price when $\sigma = 0$, version 2*) When the underlying asset is riskfree ($\sigma = 0$), then its return (denoted μ in (21.2)) must equal the riskfree rate y, so the value of the underlying asset is $e^{ym}S$ at expiration. The present value of the known call payoff is* $e^{-ym}\max(e^{ym}S - K, 0)$*, which is the same as in the previous remark.*

Remark 21.21 *(Black-Scholes formula when $m = 0$*) From (21.4)* $\lim_{\sigma \to 0} d_1 = \lim_{\sigma \to 0} d_2 =$

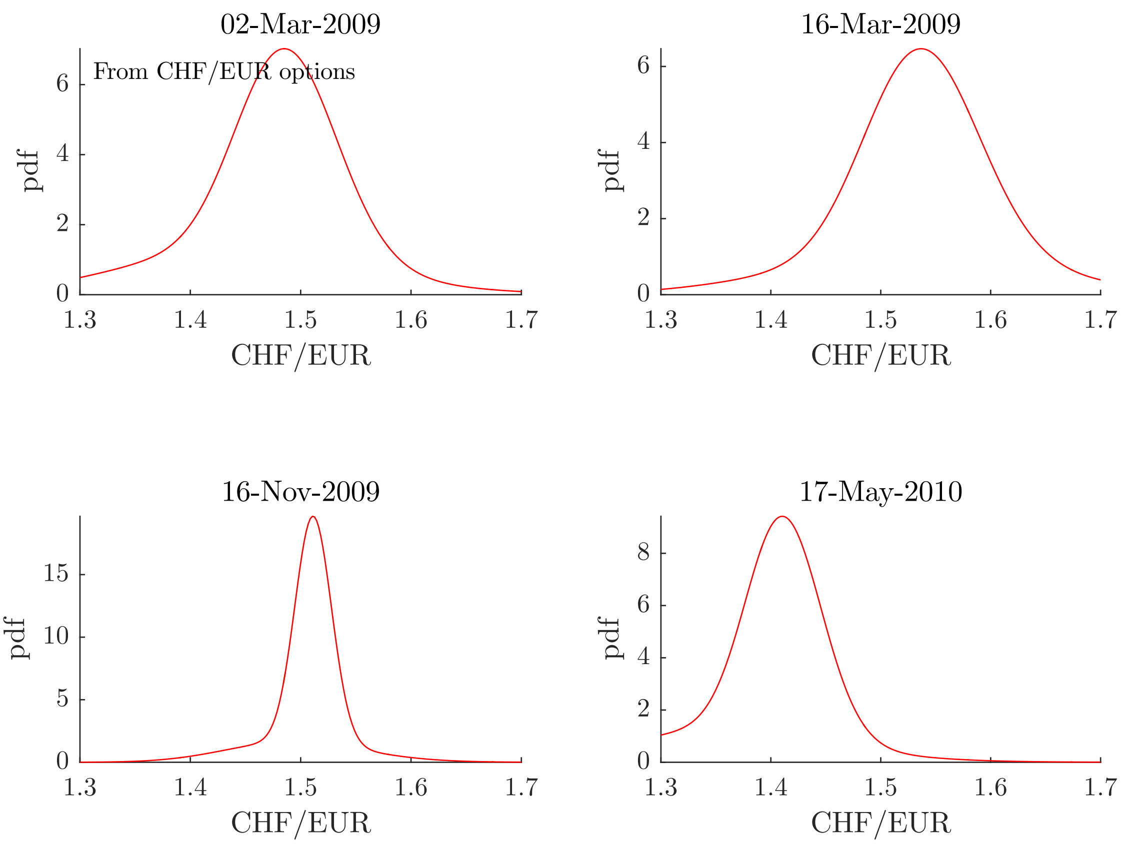


Figure 21.16: Riskneutral distribution of the CHF/EUR exchange rate

∞ *if* $S \geq K$ *and* $-\infty$ *otherwise. Therefore,* $\lim_{\sigma \to 0} \Phi(d_1) = \lim_{\sigma \to 0} \Phi(d_2) = 1$ *if* $S \geq K$ *and* 0 *otherwise. The Black-Scholes call option price at* $m = 0$ *is therefore* $\max(S - K, 0)$.

21.5.2 Calculating Black's model with Computer Code for the Black-Scholes Model

Remark 21.22 *(Approximation of option price*) A Taylor approximation gives that the call option price close to* $F = K$ *and* $\sigma = 0$ *is* $C \approx e^{-ym} F\sigma\sqrt{m/(2\pi)}$.

Remark 21.23 *(Practical hint: code for Black's model with a forward price*) Suppose you have a computer code for the BS model (21.3)—(21.4) which takes the inputs* (S, K, y, m, σ)*. To use that code for Black's model (21.5)–(21.6), substitute* $(F, 0)$ *for* (S, y) *and multiply the results by* e^{-ym}.

Remark 21.24 *(Practical hint: code for BS model with continuous dividends*) Suppose you have a computer code for the BS model (21.3)—(21.4) which takes the inputs*

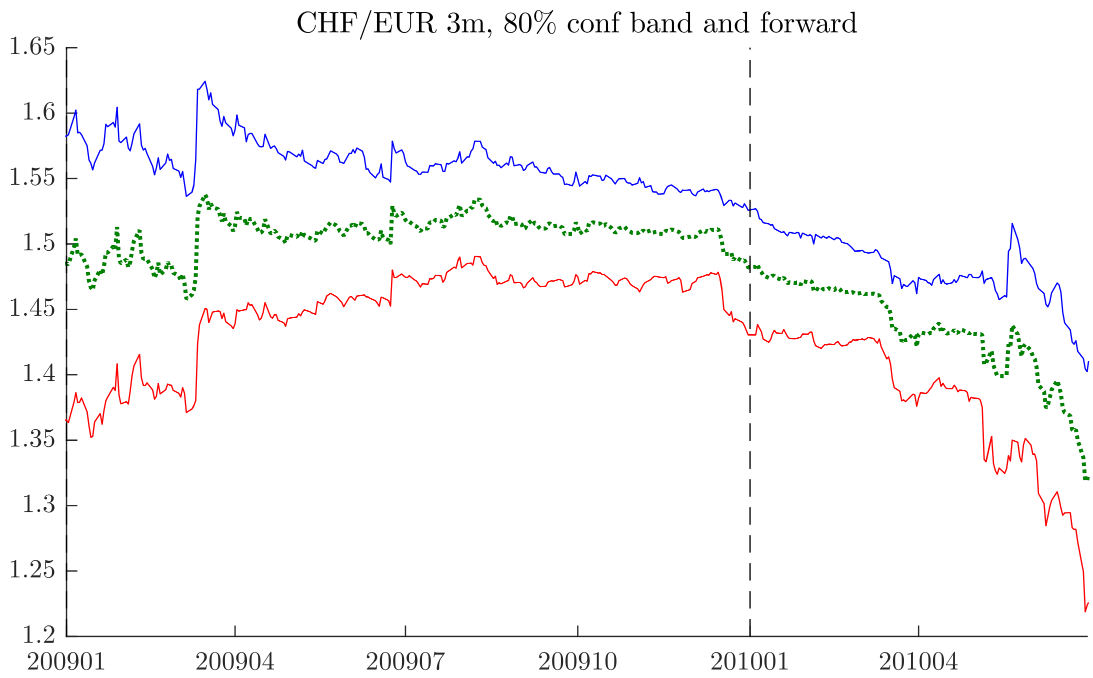


Figure 21.17: Riskneutral distribution of the CHF/EUR exchange rate

(S, K, y, m, σ). *To use that code for Black's model (21.5)–(21.6), substitute* $e^{-\delta m}S$ *for* S.

Remark 21.25 *(Practical hint: finding the dividend rate*) If you don't know what the dividend rate is, use the forward-spot parity,* $F = Se^{(y-\delta)m}$*, to calculate it as* $\delta = y - \ln(F/S)/m$.

21.5.3 Calculating the Implied Volatility

Remark 21.26 *(Practical hint: bisection method for finding* σ^**) The bisection method is a very simple (no derivatives are needed) and robust way to solve for the implied volatility. First, start with a lower (*σ_L*) and higher (*σ_H*) guesses which are known to bracket the true value, that is,* $C(\sigma_L) \leq C \leq C(\sigma_H)$ *where* C *is the observed call option price and* $C(\sigma)$ *denotes the Black-Scholes formula (as a function of* σ*). Recall that* $C(\sigma)$ *is increasing in* σ*. Second, calculate the option price at the average of the two guesses:* $C[(\sigma_L + \sigma_H)/2]$*. Third, replace either* σ_L *or* σ_H *according to: if* $C[(\sigma_L + \sigma_H)/2] \leq C$ *(so the midpoint is below the true volatility) then replace* σ_L *by* $(\sigma_L + \sigma_H)/2$ *(a higher value), but if* $C[(\sigma_L + \sigma_H)/2] > C$ *then replace* σ_H *by* $(\sigma_L + \sigma_H)/2$ *(a lower value). Fourth, iterate until* $\sigma_L \approx \sigma_H$*. See Figure 21.19.*

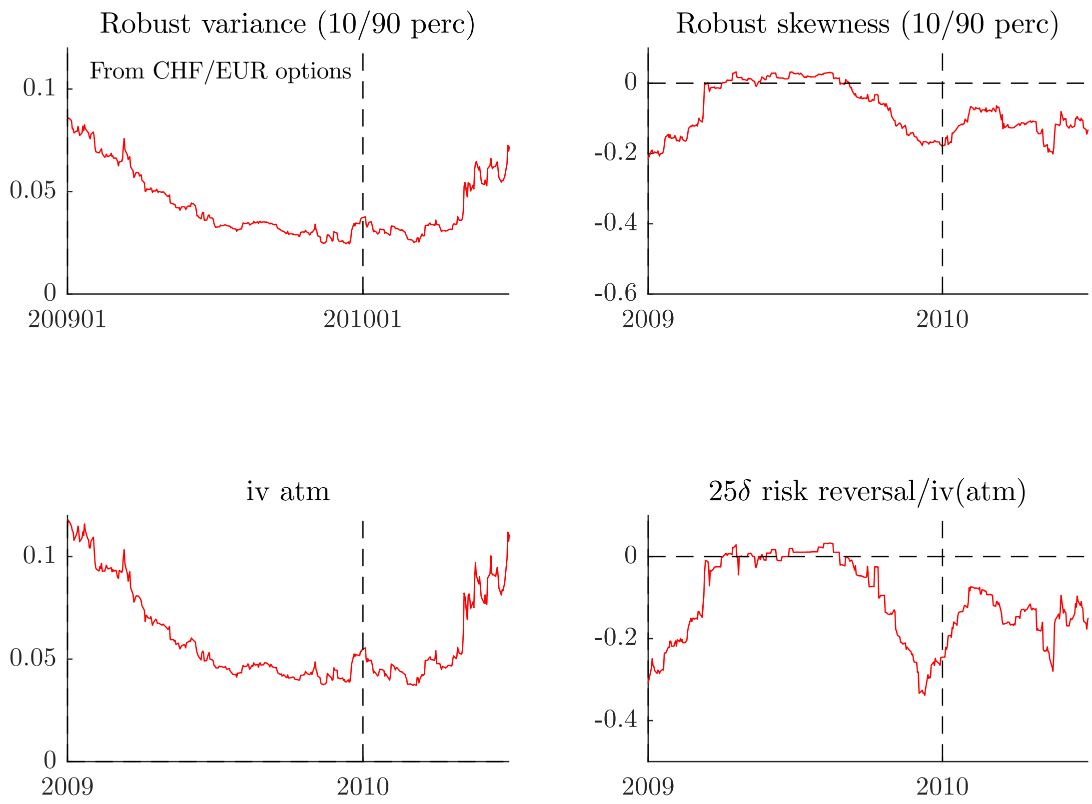


Figure 21.18: Riskneutral distribution of the CHF/EUR exchange rate

21.6 Appendix: The Probabilities in the BOPM and Black-Scholes Model*

The price of a European (call or put) option calculated by the binomial model converges to the Black-Scholes price as the number of subintervals increases (keeping the time to expiration constant, so the subintervals become shorter). This is illustrated in Figure 21.7.

Both the binomial option pricing model (BOPM) and the Black-Scholes model imply that the call option price can be written as the discounted risk neutral expected payoff (21.10). We can clearly rewrite (21.10) as

$$C = e^{-ym} \mathrm{E}^*(S_m - K | S_m > K) \Pr{}^* (S_m > K) \tag{21.28}$$

$$= e^{-ym} \mathrm{E}^*(S_m | S_m > K) \Pr{}^* (S_m > K) - e^{-ym} K \Pr{}^* (S_m > K) . \tag{21.29}$$

The first term is (the present value of) the risk neutral expected asset price conditional on exercise, times the risk neutral probability of exercise. The second term is (the present

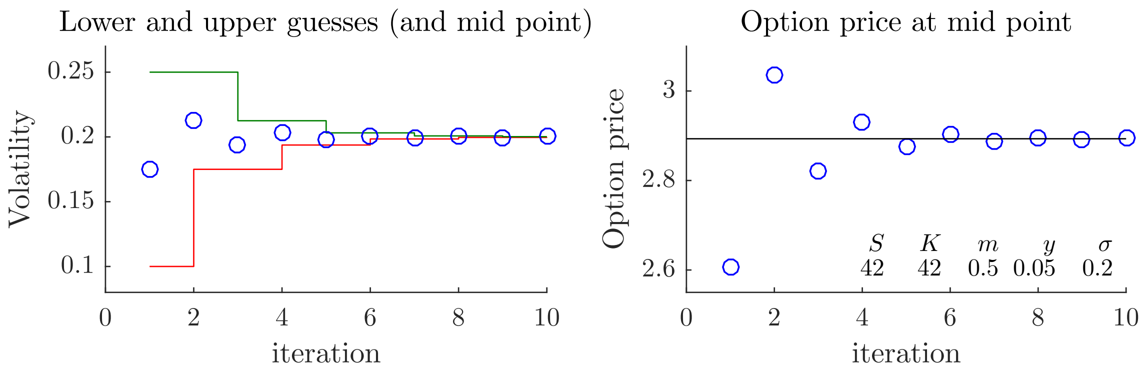


Figure 21.19: Bisection method for finding the implied volatility σ

value of) the strike price times the risk neutral probability of exercise.

The discussion below demonstrates that these probabilities are the same (in the limit) in the BOPM and the Black-Scholes models.

21.6.1 The Probabilities in the Binomial Tree

To understand the binomial model a bit better, consider a binomial tree with 2 subintervals ($n = 2$) of length h as illustrated in Figures 21.20–21.21.

The price of the call option is the discounted risk neutral expected value of the value in the next period

$$C = e^{-yh}[pC_u+(1-p)C_d], \begin{bmatrix} C_u = e^{-yh}[pC_{uu} + (1-p)C_{ud}] \\ C_d = e^{-yh}[pC_{du} + (1-p)C_{dd}] \end{bmatrix}, \text{ and } \begin{bmatrix} C_{uu} = \max(Suu - K, 0) \\ C_{ud} = \max(Sud - K, 0) \\ C_{dd} = \max(Sdd - K, 0) \end{bmatrix} \tag{21.30}$$

where $p = (e^{y/h} - d)/(u - d)$.

Remark 21.27 *(Probabilities for the final nodes) With two trials ($n = 2$), the probabilities for the final nodes are*

$$\Pr(uu) = p^2$$
$$\Pr(ud) = 2p(1-p)$$
$$\Pr(dd) = (1-p)^2.$$

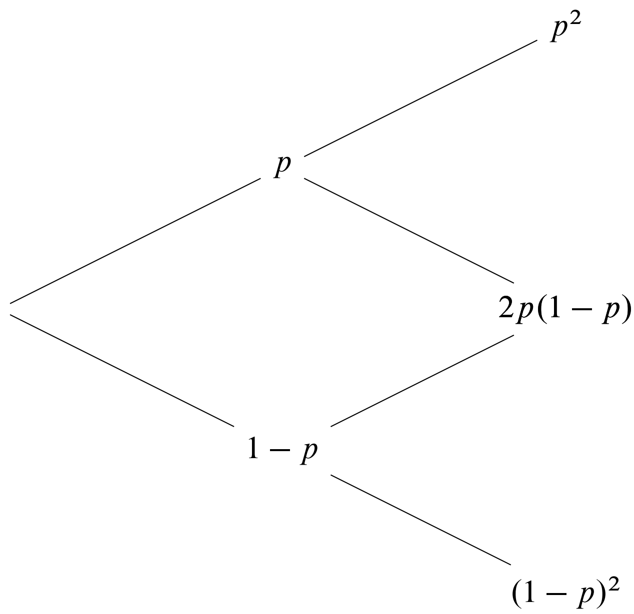


Figure 21.20: Probabilities of different nodes in a binomial tree

Combining (and using $2h = m$)

$$C = e^{-ym}\left[p^2 \max(Suu - K, 0) + 2p(1-p)\max(Sud - K, 0) + (1-p)^2 \max(Sdd - K, 0)\right], \tag{21.31}$$

which expresses the call option price as the discounted risk-neutral expectation of the option payoff.

Suppose only $Suu > K$, that is, it is only at the up and up branch, uu, that we exercise. Then

$$\begin{aligned} C &= e^{-ym} p^2 (Suu - K) \\ &= e^{-ym} \underbrace{Suu}_{\mathrm{E}^p(S_m|S_m>K)} \underbrace{p^2}_{\mathrm{Pr}^p(uu)} - e^{-ym} K \underbrace{p^2}_{\mathrm{Pr}^p(uu)}. \end{aligned} \tag{21.32}$$

The first term is the (discounted value of) the risk-neutral expected value of the asset price, conditional on being so high that we exercise the call option, times the risk neutral probability of that event. The second term is the (discounted value of) the strike price times the risk neutral probability of exercise. This clearly has the same form as (21.29). This extends to n steps, except that the expressions for the probabilities are more complicated.

Remark 21.28 *(Bernoulli and binomial distributions) The random variable X can only*

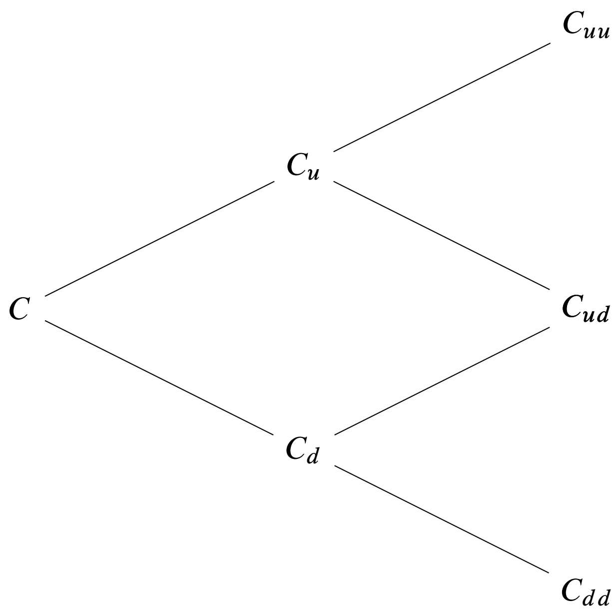


Figure 21.21: Binomial tree for derivative ($n = 2$)

take two values: 1 or 0, with probability p and $1 - p$ respectively. This gives $\mathrm{E}(X) = p$ *and* $\mathrm{Var}(X) = p(1-p)$*. After n independent trials, the number of successes (y) has the binomial pdf,* $n!/[y!(n-y)!]p^y(1-p)^{n-y}$ *for* $y = 0, 1, ..., n$*. This gives* $\mathrm{E}(Y) = np$ *and* $\mathrm{Var}(Y) = np(1-p)$*. To find the probability of at least z successes, sum the pdf over* $y = z, z+1, z+2, \ldots$

21.6.2 The Probabilities in the Black-Scholes Model

The following remark is useful for the proofs further on.

Remark 21.29 *(Properties of a lognormal distribution) Let* $x \sim N(\mu, s^2)$ *and define* $k_0 = (\ln K - \mu)/s$*. First,* $\Pr[\exp(x) > K] = \Phi(-k_0)$*. Second,* $\mathrm{E}[\exp(x)|\exp(x) > K] = \exp\left(\mu + s^2/2\right)\Phi(s - k_0)/\Phi(-k_0)$*. (To prove this, just integrate.)*

Proposition 21.30 *(Riskneutral probability of $S_m > K$) The $\Phi(d_2)$ term in the Black-Scholes formula (21.3)–(21.4) is the risk-neutral probability that $S_m > K$.*

Proposition 21.31 *($S\Phi(d_1)$ in Black-Scholes) The $S\Phi(d_1)$ term in the Black-Scholes formula (21.3)–(21.4) is (the present value of) the expected asset price conditional on exercise, times the probability of exercise, that is, the first term in (21.29).*

Proof. (of Proposition 21.30) The risk neutral probability of $\ln S_m$ is $N(\ln S + ym - \sigma^2 m/2, \sigma^2 m)$. To calculate the probability $\Pr[S_m > K] = \Phi(-k_0)$, notice that k_0 is

$$k_0 = \frac{\ln K - \overbrace{\left(\ln S + ym - \sigma^2 m/2\right)}^{\text{mean}}}{\underbrace{\sigma\sqrt{m}}_{\text{std}}}.$$

Clearly, $-k_0$ is then the same as the argument d_2 in (21.4)

$$d_2 = \frac{\ln(S/K) + \left(y - \sigma^2/2\right) m}{\sigma\sqrt{m}}.$$

■

Proof. (of Proposition 21.31) First, the first term in (21.29) can be written

$$FirstTerm = e^{-ym} \exp\left(\mu + s^2/2\right) \Phi(s - k_0),$$

since the two $\Phi(-k_0)$ terms cancel. Clearly,

$$\begin{aligned}
\mu + s^2/2 &= \ln S + ym, \\
s - k_0 &= \sigma\sqrt{m} - \frac{\ln K - \left(\ln S + ym - \sigma^2 m/2\right)}{\sigma\sqrt{m}} = d_1,
\end{aligned}$$

where the last line follows from comparing with (21.4). We can therefore write $FirstTerm$ as $S\Phi(d_1)$, since the $e^{-ym}e^{ym}$ term cancels. This is the same as in the Black-Scholes formula. ■

21.7 Appendix: Statistical Tables

	0.0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018
-2.9	0.0019	0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025
-2.8	0.0026	0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034
-2.7	0.0035	0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045
-2.6	0.0047	0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060
-2.5	0.0062	0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080
-2.4	0.0082	0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104
-2.3	0.0107	0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136
-2.2	0.0139	0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174
-2.1	0.0179	0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222
-2.0	0.0228	0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281
-1.9	0.0287	0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351
-1.8	0.0359	0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436
-1.7	0.0446	0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537
-1.6	0.0548	0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655
-1.5	0.0668	0.0681	0.0694	0.0708	0.0721	0.0735	0.0749	0.0764	0.0778	0.0793
-1.4	0.0808	0.0823	0.0838	0.0853	0.0869	0.0885	0.0901	0.0918	0.0934	0.0951
-1.3	0.0968	0.0985	0.1003	0.1020	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131
-1.2	0.1151	0.1170	0.1190	0.1210	0.1230	0.1251	0.1271	0.1292	0.1314	0.1335
-1.1	0.1357	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562
-1.0	0.1587	0.1611	0.1635	0.1660	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814
-0.9	0.1841	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.2090
-0.8	0.2119	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389
-0.7	0.2420	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709
-0.6	0.2743	0.2776	0.2810	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.3050
-0.5	0.3085	0.3121	0.3156	0.3192	0.3228	0.3264	0.3300	0.3336	0.3372	0.3409
-0.4	0.3446	0.3483	0.3520	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783
-0.3	0.3821	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.4090	0.4129	0.4168
-0.2	0.4207	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562
-0.1	0.4602	0.4641	0.4681	0.4721	0.4761	0.4801	0.4840	0.4880	0.4920	0.4960

Table 21.1: Values of the standard normal distribution function at x where x is the sum of the values in the first column and the first row.

	0.0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Table 21.2: Values of the standard normal distribution function at x where x is the sum of the values in the first column and the first row.

Chapter 22

FX and Interest Rate Options*

Main references: Hull (2009) 15 and Wystup (2006)

22.1 Forward Contract on a Currency

Proposition 22.1 *(Forward-spot parity, continuous dividends) When the dividend is paid continuously at the rate δ (of the price of the underlying asset), then*

$$e^{-ym}F = Se^{-\delta m}, \tag{22.1}$$

where m is the time to expiration of the forward contract, y is the continuously compounded interest rate (until expiration), F the forward price (to be paid at expiration) and S is the current asset price.

Investing in foreign currency effectively means investing in a foreign interest bearing instrument which earns the continuous interest rate ("dividend"), denoted δ in (22.1).

Consider a forward contract that expires in $t+m$, but where the contract was written at some earlier point in time ($\tau < t$). When written, it specified the forward price F_τ. The value of this contract in t is

$$W_t = e^{-ym}(F_t - F_\tau). \tag{22.2}$$

22.2 Summary of the Black-Scholes Model

For an asset with a continuous dividend rate of δ, the forward-spot parity says $F = Se^{(y-\delta)m}$. In this case the Black-Scholes formula can be written

$$C = e^{-\delta m} S \Phi(d_1) - e^{-ym} K \Phi(d_2), \text{ where} \tag{22.3}$$

$$d_1 = \frac{\ln(S/K) + (y - \delta + \sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{22.4}$$

When the asset is a currency (read: foreign money market account) and δ is the foreign interest rate, then this is the "Garman-Kolhagen" formula. The sensitivity to the underlying asset price is

$$\frac{\partial C}{\partial S} = e^{-\delta m} \Phi(d_1), \tag{22.5}$$

where d_1 is given by (22.4). Without dividends, just set $\delta = 0$. From the put-call parity, it is clear that the sensitivity of a put option is

$$\frac{\partial P}{\partial S} = e^{-\delta m}[\Phi(d_1) - 1] = -e^{-\delta m} \Phi(-d_1), \tag{22.6}$$

which is negative. The second equality follows from the symmetry of the normal distribution.)

Using $F = Se^{(y-\delta)m}$ in (22.3)–(22.4), we get *Black's model* for an option on a forward contract:

$$C = e^{-ym} F \Phi(d_1) - e^{-ym} K \Phi(d_2), \text{ where} \tag{22.7}$$

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)m}{\sigma\sqrt{m}} \text{ and } d_2 = d_1 - \sigma\sqrt{m}. \tag{22.8}$$

For an asset with a continuous dividend, this d_1 is the same as in (22.4). The sensitivity of the call option price to the forward price is

$$\frac{\partial C}{\partial F} = e^{-ym} \Phi(d_1), \tag{22.9}$$

where d_1 is given by (22.8). Similarly, the sensitivity of a put option is

$$\frac{\partial P}{\partial F} = e^{-ym}[\Phi(d_1) - 1] = -e^{-ym} \Phi(-d_1). \tag{22.10}$$

22.3 Hedging

22.3.1 Hedging with the Underlying Asset

Consider a portfolio with Δ_t of the underlying asset (the hedging portfolio) and short one call option. The value of the overall position is

$$V_t = \Delta_t S_t - C_t. \tag{22.11}$$

This portfolio has an almost stable value if

$$\Delta_t = \frac{\partial C_t}{\partial S}. \tag{22.12}$$

22.3.2 Hedging with a Forward Contract

Consider a portfolio with Δ_t of a forward contract and short one call option. The value of the overall position is

$$V_t = \Delta_t W_t - C_t, \tag{22.13}$$

where W_t is the value of an old forward contract as in (22.2). This portfolio is almost stable if

$$\Delta_t = e^{ym} \frac{\partial C_t}{\partial F}. \tag{22.14}$$

Proof. Assume that only the forward price can change (clearly not true, but at least a starting point for the analysis). A first-order Taylor approximation of the call option price is

$$C_{t+h} - C_t \approx \frac{\partial C_t}{\partial F} (F_{t+h} - F_t).$$

Use to approximate the change of the value of the overall portfolio as

$$\begin{aligned}
V_{t+h} - V_t &= \Delta_t (W_{t+h} - W_t) - C_{t+h} - C_t \\
&\approx \Delta_t e^{-ym} (F_{t+h} - F_t) - \frac{\partial C_t}{\partial F} (F_{t+h} - F_t) \\
&\approx 0 \text{ if } \Delta_t = e^{ym} \frac{\partial C_t}{\partial F}.
\end{aligned}$$

■

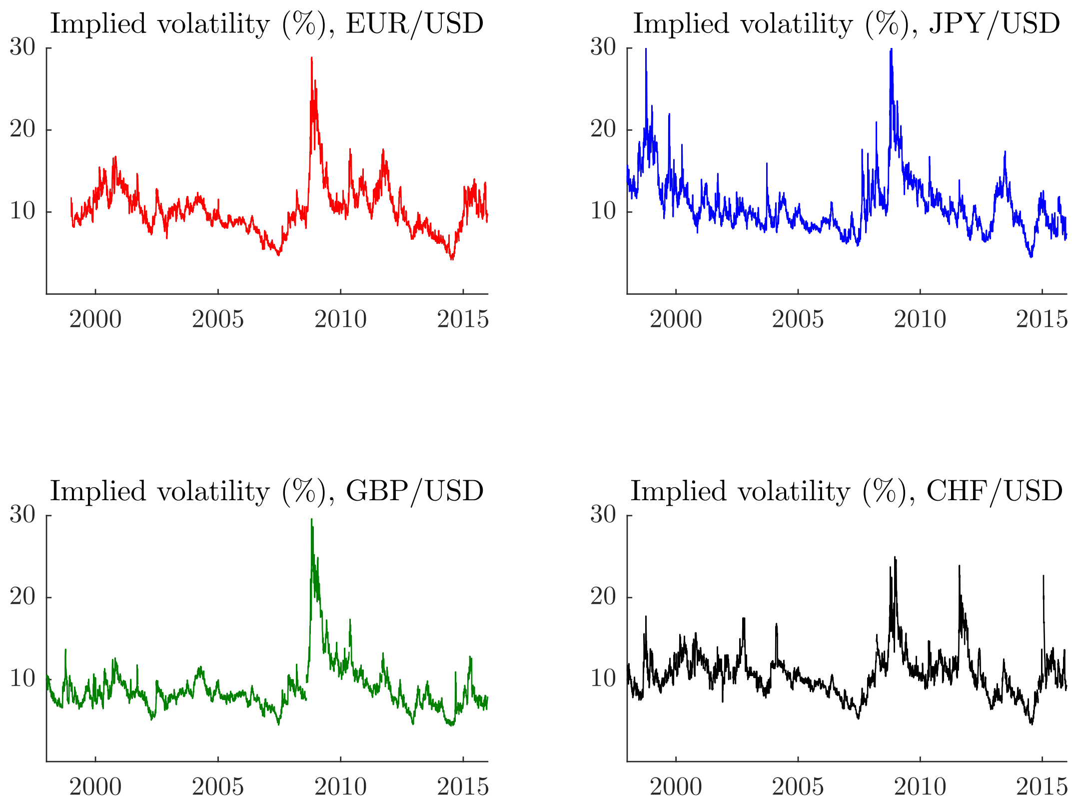


Figure 22.1: Implied volatility (atm) from different 1-month FX options

22.4 FX Options: Put or Call?

See Figure 22.1 for an empirical illustration of how the implied volatilities for different FX options have developed.

Buying one currency entails selling another. It should therefore come as no surprise that a call option on a currency is also a put option on the other currency. To be precise, the option prices are related according to

$$C_d(\text{strike} = K) = S_t K P_f(\text{strike} = 1/K). \tag{22.15}$$

On the left hand side, C_d is the domestic price of a call option on the foreign currency—with the strike price (K) is expressed in the domestic currency. On the right hand side, S_t is the current exchange rate (domestic price of one unit of the foreign currency), and P_f is the foreign price of a put option on the domestic currency—with the strike price $(1/K)$.

In particular, we can rewrite the expression as

$$P_f(\text{strike} = 1/K) = \frac{C_d(\text{strike} = K)}{S_t K}, \tag{22.16}$$

which is the price (measured in foreign currency) of the put on the domestic currency.

Example 22.2 *Let* $C_d = £0.01$ *for an option on US dollars and the strike price is £0.6 (to get one dollar). If the current exchange rate is £0.58 (per dollar), then the dollar price of a put option on GBP with a strike price of* $1/0.6$ *dollars per GBP is* $0.01/(0.58 \times 0.6) =$ \$0.0287.

Remark 22.3 *(Option price quoted in which currency?) In practice, it is important to consider which currency the option price is quoted in. For instance, most options involving the USD have option prices quoted in USD, while most options involving the EUR (unless also the USD is involved) have prices quoted in EUR.*

Proof. (of (22.15)) The payoff of a call option (denominated in the domestic currency) on foreign currency with strike price K is

$$\max(0, S_{t+m} - K),$$

where K is the strike price and S_{t+m} is the exchange rate at expiration—both expressed as the domestic price of one unit of foreign currency (for instance, GBP 0.6 per USD). The payoff is clearly expressed in the domestic currency. In contrast, the payoff of a put option (denominated in the foreign currency) on the domestic currency (with strike price $1/K$) has the payoff

$$\max(0, 1/K - 1/S_{t+m}),$$

which is clearly expressed in the foreign currency. Notice that both options are exercised when $S_{t+m} > K$. In fact, these options are identical, except for a scaling factor and the currency denomination. To see that, consider buying K of the foreign denominated options and then convert the payoff to the domestic currency (multiply by S_{t+m})

$$S_{t+m} K \max(0, 1/K - 1/S_{t+m}) = \max(0, S_{t+m} - K),$$

which is clearly the same as for the first option. For that reason, buying K of the foreign currency denominated put options should have the same price (when measured in domestic currency—multiply by S_t) as the domestically denominated call option. ■

22.5 FX Options: Risk Reversals and Strangles

Options on the FX (exchange rate) markets are often sold (on the OTC market) as special portfolios (consisting of straddles, risk-reversals and strangles) and quoted in terms of the implied volatilities. Apart from these conventions, options on exchange rates are no different from options on other assets (but, remember that currencies carry "dividends" since holding a currency in practice means holding a money market account in that currency).

A *delta-neutral straddle*, that is, a long position in a call and also in a put. To make it delta-neutral (with respect to the spot), we need

$$\frac{\partial C}{\partial S} + \frac{\partial P}{\partial S} = 0, \tag{22.17}$$

which from (22.5)–(22.6) gives (with d_1 defined by (22.8) or equivalently (22.4))

$$d_1 = 0, \text{ that is, } K_{atm} = Fe^{\sigma_{atm}^2 m/2} \text{ with } F = Se^{(y-\delta)m}. \tag{22.18}$$

This straddle is typically quoted in terms of the implied volatility (σ_{atm}) of an option at K_{atm}. A higher value of the straddle indicates more overall uncertainty. See Figure 22.2 for illustrations of the profits of different option portfolios.

A *25-delta risk reversal* is a portfolio of one call option with a strike price K_2 such that the delta is 0.25 and short one put option with a strike price K_1 such that the delta is -0.25. (Other values of the deltas are also used.) Both options are out of the money so the strike price for the put is lower than the forward price, which in turn is lower than the strike price of the call ($K_1 < F < K_2$). The risk reversal is typically quoted as the difference of the two implied volatilities

$$rr = \sigma_2 - \sigma_1, \tag{22.19}$$

where σ_2 and σ_1 are the implied volatilities of the options with strike prices K_2 and K_1 respectively (notice that, by the put-call parity, a put and a call with the same strike price have the same implied volatility). A higher value of the risk reversal indicates beliefs of an increase in the underlying—so it captures skewness of the exchange rate distribution.

A *25-delta strangle* has a long position in the 25-delta call and also in the 25-delta put. A *25-delta butterfly* is a portfolio that is long one 25-delta straddle and short one delta-neutral straddle. It is typically quoted as the average implied volatility of the K_2

and K_1 options (call and put, respectively) minus the at-the-money volatility

$$bf = \frac{\sigma_2 + \sigma_1}{2} - \sigma_{atm}. \tag{22.20}$$

An increase in bf signals a belief in fatter tails, so it captures kurtosis. Notice that a proportional increase of all volatilities does not change bf (it is "vega" neutral).

With the quotes on the risk reversal (22.19) and the butterfly (22.20), we can solve for the implied volatilities σ_1 and σ_2 as

$$\begin{aligned}\sigma_1 &= bf + \sigma_{atm} - rr/2 \\ \sigma_2 &= bf + \sigma_{atm} + rr/2.\end{aligned} \tag{22.21}$$

It is straightforward to invert the formulas for the deltas to derive what the strike prices are. If we use the convention that the deltas are with respect to the spot price, then by setting $\partial C/\partial S = \Delta$ (say, $\Delta = 0.25$) in (22.5) to derive the strike price K_2 and $\partial P/\partial S = -\Delta$ in (22.6) to derive the strike price K_1 we get the following strike prices (using $F = Se^{(y-\delta)m}$)

$$\begin{aligned}K_1 &= F \exp[\sigma_1 \sqrt{m} \Phi^{-1}(e^{\delta m} \Delta) + m\sigma_1^2/2] \\ K_2 &= F \exp[-\sigma_2 \sqrt{m} \Phi^{-1}(e^{\delta m} \Delta) + m\sigma_2^2/2],\end{aligned} \tag{22.22}$$

Clearly, by changing to $\Delta = 0.10$, we get the strikes for a 10-delta risk reversal.

See Figure 22.3 for how the strike prices are calculated and Figure 22.4 for an empirical illustration.

Example 22.4 *(σ_{atm}, rr and bf on 1 April 2005, 1-month EUR/GBP) For this particular date and contract σ_{atm} was 4.83%, the 25 delta risk reversal was 0.18% and the 25 delta strangle (really, a 25 delta butterfly) was 0.15%. (See Wystup (2006), tables 1.7–9.) This gives*

$$\begin{aligned}\sigma_1 &= 0.15 + 4.83 - 0.18/2 = 4.89 \\ \sigma_2 &= 0.15 + 4.83 + 0.18/2 = 5.07.\end{aligned}$$

The spot exchange rate was 0.6859 (the price of one EUR, in terms of GBP) and the 1-month interest rates were 4.87 in the UK and 2.10 in the euro zone, so the forward rate was $F = 0.6859 \times \exp[(0.0487 - 0.0210)/12] \approx 0.6875$. This gives $K_1 = 0.6811$, $K_{atm} = 0.6876$ and $K_2 = 0.6941$.

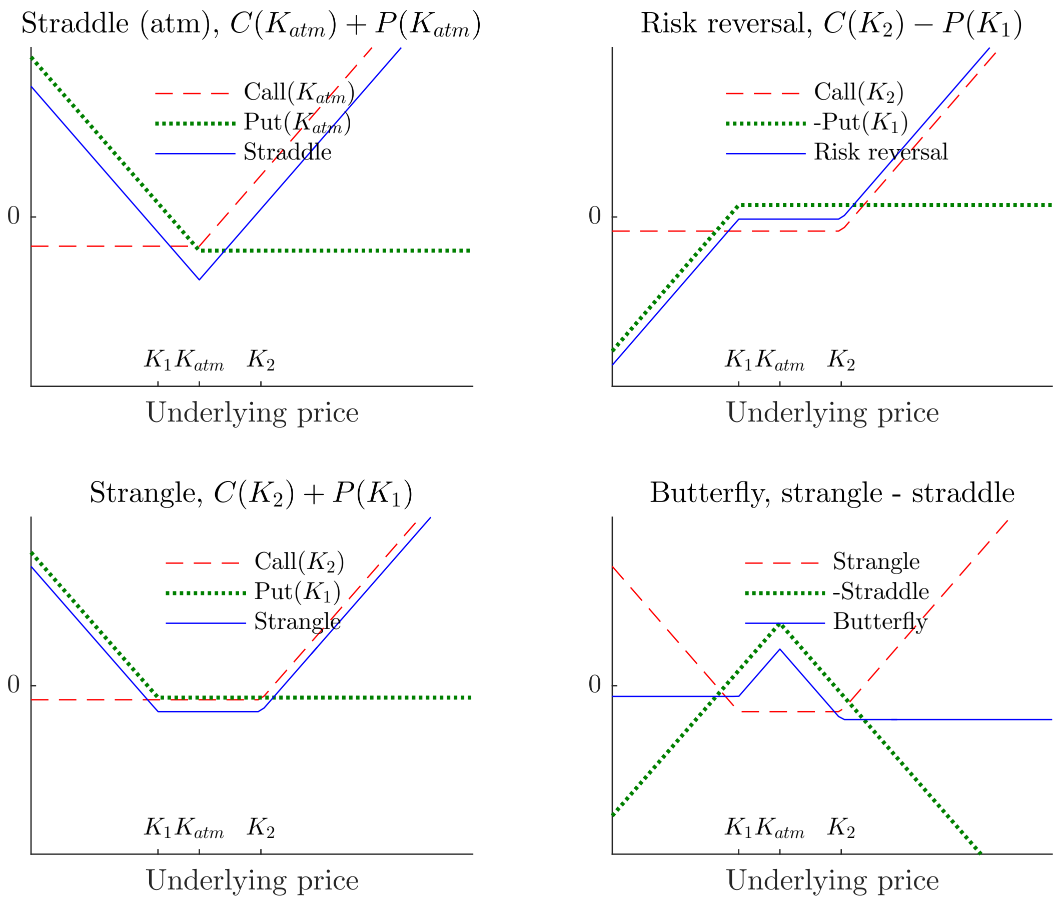


Figure 22.2: Profits diagrams for FX option portfolios

Remark 22.5 *(Deltas with respect to the forward price*) The market convention is that developed market currencies with time to expiration up to a year are quoted in deltas with respect to the spot price, while all other FX options are quoted in deltas with respect to the forward price. If we use the convention that the deltas are with respect to the forward price then K_{atm} is as in (22.18), but Δ is substituted for $e^{\delta m}\Delta$ in (22.22). Both conventions are used. (The forward deltas are more common for options with long time expiration and for emerging market currencies.)*

Proof. (of (22.18)) If we use spot deltas, then (22.5)–(22.6) give

$$\frac{\partial C}{\partial S} + \frac{\partial P}{\partial S} = e^{-\delta m}\Phi(d_1) - e^{-\delta m}\Phi(-d_1) = 0,$$

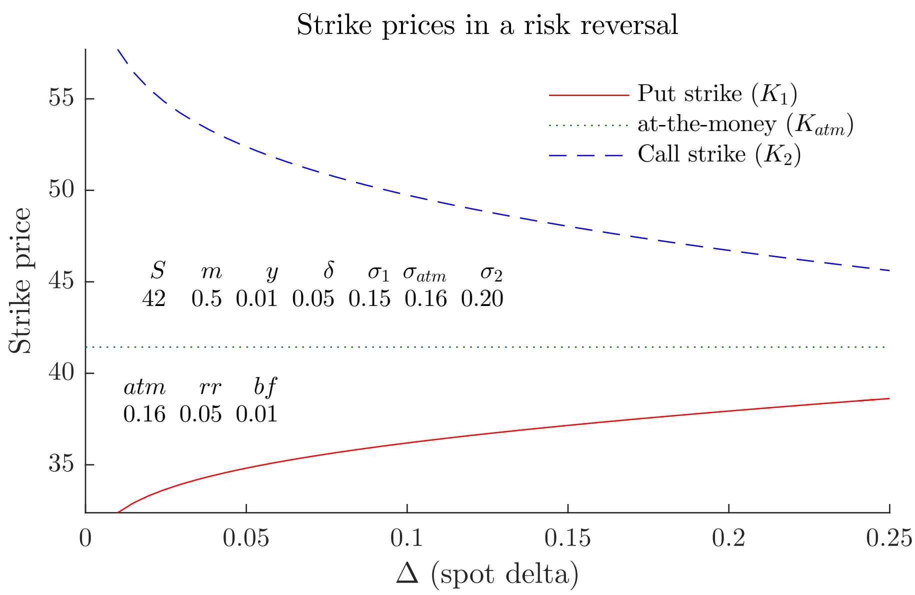


Figure 22.3: Strike prices in a risk reversal

which requires $d_1 = 0$. With d_1 defined by (22.4) we have

$$\ln K = \ln S + (y - \delta + \sigma^2/2)m = \ln F + (\sigma^2/2)m$$

If we instead use forward deltas, use (22.12) and (22.9)–(22.10) and set to zero

$$\frac{\partial C}{\partial F} + \frac{\partial P}{\partial F} = e^{ym}\left[e^{-ym}\Phi(d_1) - e^{-ym}\Phi(-d_1)\right] = 0,$$

which still requires $d_1 = 0$ (and d_1 is the same in (22.8) and (22.4)). ■

Proof. (of (22.22)) If we use the spot delta, then set (22.5) equal to 0.25

$$0.25 = e^{-\delta m}\Phi(d_1)\text{, so we need } d_1 = \Phi^{-1}\left(e^{\delta m}0.25\right)$$

With d_1 given by (22.4) we get

$$\ln K_2 = \ln F + (\sigma^2/2)m - \sigma\sqrt{m}\Phi^{-1}(e^{\delta m}0.25),$$

since $\ln F = \ln S + (y - \delta)m$. Instead, if we use the forward delta from using (22.9) in

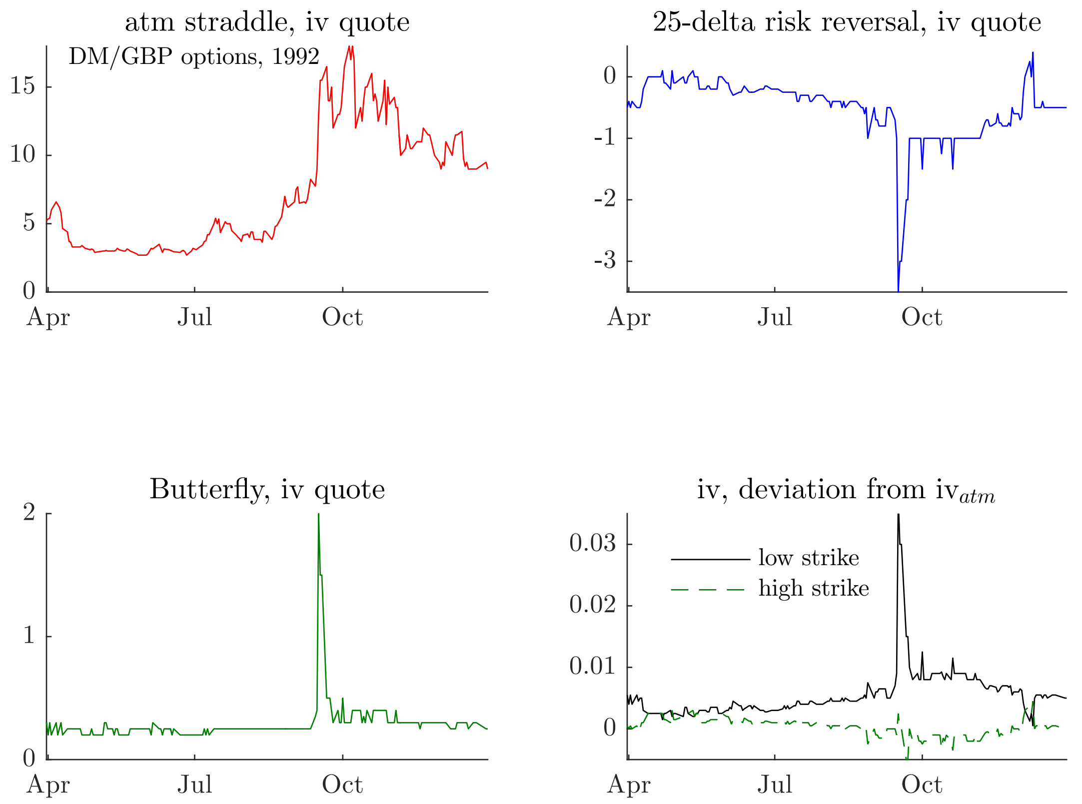


Figure 22.4: DM/GDP options, 1992

(22.14)

$$0.25 = e^{ym} \frac{\partial C_t}{\partial F} = \Phi\left(d_1\right), \text{ so}$$
$$d_1 = \Phi^{-1}(0.25).$$

With d_1 given by (22.8) we get

$$\ln K_2 = \ln F + (\sigma^2/2)m - \sigma\sqrt{m}\Phi^{-1}(0.25).$$

The calculations for the strike prices K_1 for the put are similar. ■

22.6 FX Options: Implied Volatility for Different Deltas

Another way to quote FX option prices is to list the implied volatility for different strike prices—but where the strike prices are expressed as deltas. For instance, $\Delta = (-0.25, 0, 0.25)$.

Often, these are labelled "$25\Delta P$", atm, and "$25\Delta C$", where $25\Delta P$ stands for the strike price where a put has a delta of -0.25, atm stands for the strike price at the money, and $25\Delta C$ is the strike price where a call has a delta of 0.25.

Typically, the atm strike price is as in (22.18), while the "$25\Delta P$" strike price is calculated as K_1 in (22.22) by setting $\Delta = 0.25$ and the "$25\Delta C$" strike price is calculated as K_2 in (22.22) by setting $\Delta = 0.25$. Other deltas are similar.

Remark 22.6 *(Premium-adjusted deltas) When the option price is quoted in the foreign currency, then the deltas reported do not correspond to (22.18) and (22.22). See Wystup (2006) for more details.*

22.7 Options on Interest Rates: Caps and Floors

Options on bonds are basically no different from options on equity, although bonds typically pay "dividends" (the coupons). For instance, a call option on a bond gives the right to buy the bond (at the expiration of the option) at the strike price.

Options on interest rates are also very similar, but often have a more complicated structure. A *caplet* is a call option that protects against higher interest rates (typically a floating 3-month market rate or similar). Let Z_{t+s} be the (annualized) market interest rate for a loan between $t+s$ and $t+s+m$ and let Z_K be the (annualized) cap rate. The payoff in $t+s+m$ (notice: paid at the end of the borrowing period) is

$$\max[0, m(Z_{t+s} - Z_K)]. \tag{22.23}$$

The second term is the interest rate cost for a loan (with a face value of unity) between $t+s$ and $t+s+m$ according to the market rate minus the same cost according to the cap rate. Clearly, buying such an option is a way to make sure that interest rate paid on a loan will not exceed the cap rate. If settled at $t+s$ the payoff is just the discounted value

$$\frac{\max[0, m(Z_{t+s} - Z_K)]}{1 + mZ_{t+s}}. \tag{22.24}$$

The payoff in (22.24) can be rewritten as

$$(1 + mZ_K)\max\left(0, \frac{1}{1 + mZ_K} - B_{t+s}(m)\right) \tag{22.25}$$

Notice that the max() term defines the payoff of a put option on an m-period bond in $t+s$ (whose value turns out to be $B_{t+s}(m) = 1/(1 + mZ_{t+s})$)—with a strike price of of

$1/(1+mZ_K)$. The caplet is therefore proportional to a put option on a bond.

Proof. (of (22.25)) Multiply and divide (22.24) by $(1+mZ_K)$ and rearrange

$$\begin{aligned}(1+mZ_K)\max&\left[0,\frac{mZ_{t+s}-mZ_K}{(1+mZ_{t+s})(1+mZ_K)}\right]\\ &=(1+mZ_K)\max\left(0,\frac{1}{1+mZ_K}-\frac{1}{1+mZ_{t+s}}\right).\end{aligned}$$

Notice that $B_{t+s}(m)=1/(1+mZ_{t+s})$. ■

We can apply the Black's formula (22.7)–(22.8) to price the caplet by assuming that a forward contract on either Z_{t+s} or (somewhat less often) B_{t+s} has a lognormal distribution. (These two assumptions are not compatible, since the latter is the same as assuming that $1+mZ_{t+s}$ has a lognormal distribution.)

Remark 22.7 *(Simple interest rates) If Z is a simple interest rates, then of a zero-coupon bond that gives unity at maturity is*

$$B(m)=\frac{1}{1+mZ(m)},\ \text{or}\ Z(m)=\frac{1/B(m)-1}{m}.$$

A simple forward rate for the period s to $s+m$ periods in the future is defined as

$$Z^f(s,s+m)=\frac{1}{m}\left[\frac{B(s)}{B(s+m)}-1\right].$$

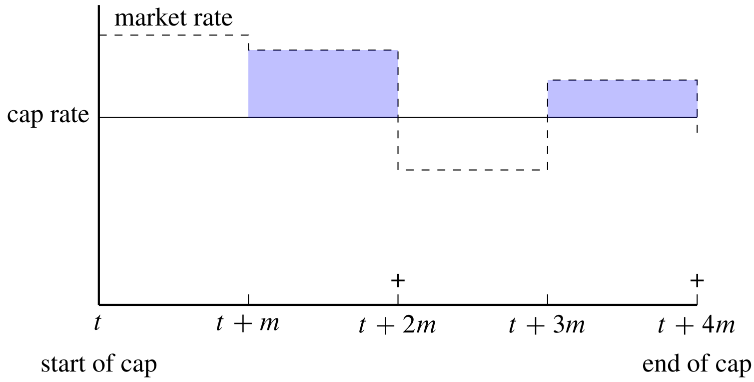
A forward rate (determined t) for the future investment period $t+s$ to $t+s+m$, denoted Z^f, clearly coincides with the market rate in $t+s$. We can therefore apply Black's formula to the underlying mZ^f by assuming that it is lognormally distributed—and using the strike "price" mZ_K. However, we need to discount by $\exp[-(s+m)y]$ instead of $\exp(-sy)$ since the payoff (22.23) is paid in $t+s+m$ (not in $t+s$). The value of this caplet is therefore

$$\text{Caplet}(s,m;\sigma,Z_K)=me^{-(s+m)y}[Z^f\Phi(d_1)-Z_K\Phi(d_2)],\ \text{where} \tag{22.26}$$

$$d_1=\frac{\ln(Z^f/Z_K)+(\sigma^2/2)s}{\sigma\sqrt{s}}\ \text{and}\ d_2=d_1-\sigma\sqrt{s}, \tag{22.27}$$

where σ is the (annualized) volatility of the log forward rate.

An *interest rate cap* is a portfolio of different caplets which protects the owner over several *tenors* (subperiods). Typically, the first caplet is deleted (as there is no uncertainty about what the short rate is today) and the last payment is done on the maturity date n. Therefore, the tenors are $[m,2m]$, $[2m,3m]$ and so forth until the last one which is



(The time of payments are marked by +)
(No payment before $t + 2m$)

Figure 22.5: Interest rate cap

$[n-m, n]$ so there are $n/m-1$ caplets. (The start/end of a tenor is called a reset/settlement date.) For instance, a 1-year cap on the 3-month Libor consists of 3 caplets. See Figure 22.5 for an illustration. (The cap could also be scheduled to start at a later date.)

If we apply the same volatility to all caplets ("flat volatilities"), then the price of a cap (according to the Black-Scholes model) starting now and ending in n, is

$$\text{Cap}(n, m; \sigma, Z_K) = \sum\nolimits_{i=1}^{n/m-1} \text{Caplet}(im, m; \sigma, Z_K). \tag{22.28}$$

Caps are often quoted in terms of the implied volatility (σ) that solves this equation—meaning that there is one implied volatility per cap contract, but it may differ across cap rates ("strike prices") and maturities. (If the cap is scheduled to start S periods ahead, instead of now, then im should be replaced by $S + im$.)

Example 22.8 *(1-year Cap starting now, 3-month tenors) Let $n = 1$ (1-year cap) and $m = 1/4$ (3-month tenors). The payoffs are based on the difference between the 3-month Libor and the cap rate at the beginning of the tenors $(1/4, 2/4, 3/4)$, but are paid one quarter later. Equation (22.28) is therefore*

$$Cap(1, 1/4; \sigma, Z_K) = Caplet(1/4, 1/4; \sigma, Z_K) + Caplet(2/4, 1/4; \sigma, Z_K) + Caplet(3/4, 1/4; \sigma, Z_K).$$

Floorlets and *floors* are similar to caplets and caps, except that they pay off when the interest goes below the cap rate.

Chapter 23

Trading Volatility

Reference: Gatheral (2006) and McDonald (2014) 29

More advanced material is denoted by a star (*). It is not required reading.

23.1 The Purpose of Trading Volatility

By using option portfolios (for instance, straddles) it is possible to create a position that is a bet on volatility—and is (in principle) not sensitive to the direction of change of the underlying. See Figure 23.1 for an illustration.

Volatility, as an asset class, has some interesting features. In particular, returns on the underlying asset and volatility are typically negatively correlated: very negative returns are typically accompanied by increases in future actual volatility as well as beliefs about higher future volatility (as priced into options). See Figure 23.2 for an illustration, where changes in the VIX are taken to proxy the one-day holding return on a straddle.

There are several ways of trading volatility: straddles (and other option portfolios), futures (and options) on the VIX, as well as volatility (and variance) swaps.

23.2 VIX and VIX Futures

The VIX is an index of volatility, calculated from 1-month options on S&P 500. It used to be calculated as an average of implied volatilities, but since 2003 the calculation is more complicated (the old series is now called VXO). It can be shown (although it is a bit tricky) that the VIX is a very good approximation to the square root of the variance swap rate (see below) for a 30-day contract. There are also futures contracts on VIX with payoff

$$\text{VIX futures payoff}_{t+m} = VIX_{t+m} - \text{futures price}_t. \tag{23.1}$$

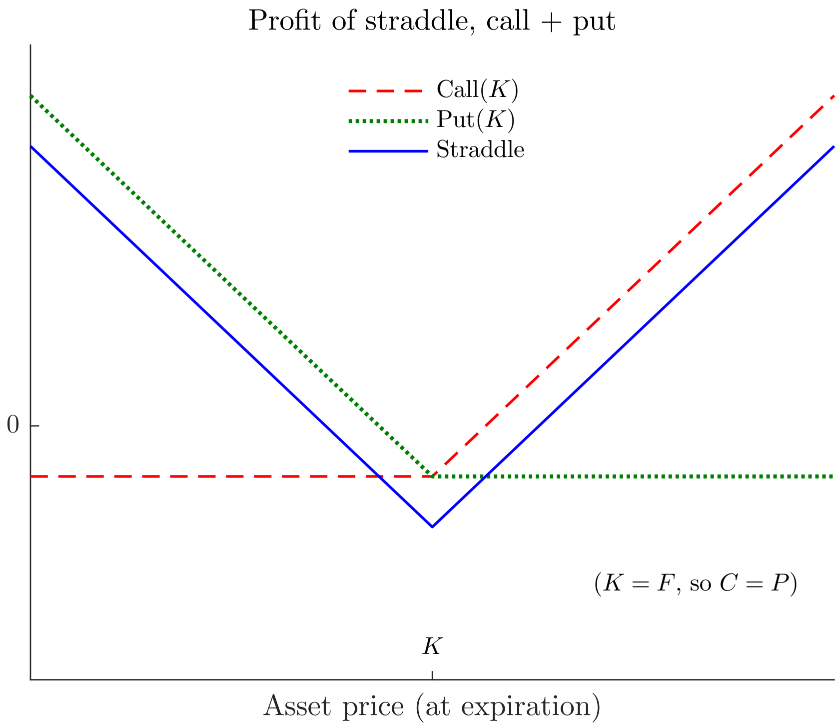


Figure 23.1: Profit of straddle

Notice that VIX_{t+m} is really a guess of what the volatility will be during the month after $t+m$, so the futures contract pays off when the expected volatility (in $t+m$) is higher than what was thought in t.

See Figures 23.3–23.4 for an empirical illustration. Notice that the futures prices indicate that volatility is mean reverting: high VIX levels are associated with negative spreads (the futures is lower than the current VIX). This indicates that market participants believe that volatility will settle down.

Remark 23.1 *(Calculation of VIX) Let F be the forward price, $\Delta K_i = (K_{i+1} - K_{i-1})/2$ and let K_0 denote the first strike price below F. Then, the VIX is calculated as*

$$VIX^2 = \frac{2}{m}\exp(ym)\sum_{K_i \le K_0} \frac{\Delta K_i}{K_i^2} P(K_i) + \frac{2}{m}\exp(ym)\sum_{K_i > K_0} \frac{\Delta K_i}{K_i^2} C(K_i) - \frac{1}{m}(F/K_0 - 1)^2,$$

where m is the time to expiration (around 1/12), y the interest rate, $P()$ the put price and $C()$ the call price.

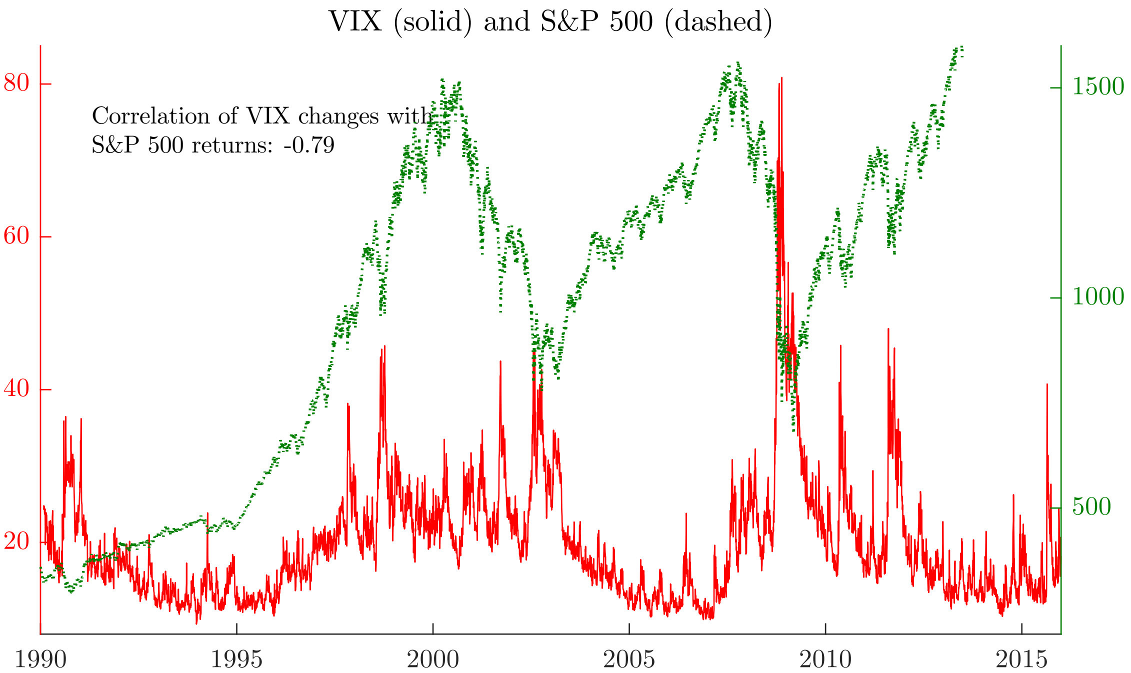


Figure 23.2: S&P 500 and VIX

23.3 Variance and Volatility Swaps

Instead of investing in straddles, it is also possible to invest in *variance swaps*. Such a contract has a zero price in inception (in t) and the payoff at expiration (in $t+m$) is

$$\text{Variance swap payoff}_{t+m} = \text{realized variance}_{t+m} - \text{variance swap rate}_t, \tag{23.2}$$

where the variance swap rate (also called the strike or forward price for) is agreed on at inception (t) and the realized volatility is just the sample variance for the swap period. Both rates are typically annualized, for instance, if data is daily and includes only trading days, then the variance is multiplied by 252 or so (as a proxy for the number of trading days per year).

A *volatility swap* is similar, except that the payoff it is expressed as the difference between the standard deviations instead of the variances

$$\text{Volatility swap payoff}_{t+m} = \sqrt{\text{realized variance}_{t+m}} - \text{volatility swap rate}_t, \tag{23.3}$$

If we use daily data to calculate the realized variance from t until the expiration(RV_{t+m}),

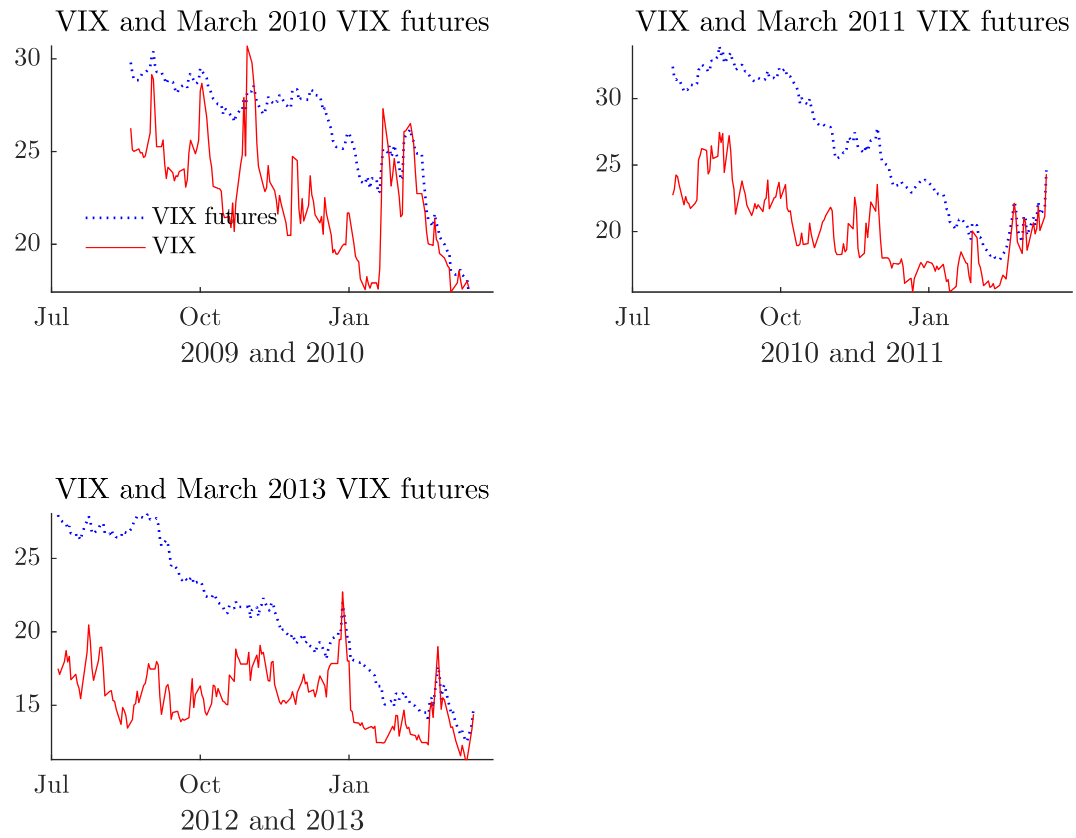


Figure 23.3: VIX and futures contract on VIX

then

$$RV_{t+m} = \frac{252}{m}\sum\nolimits_{s=1}^{m} R_{t+s}^2, \tag{23.4}$$

where R_{t+s} is the net return on day $t+s$. (This formula assumes that the mean return is zero—which is typically a good approximation for high frequency data. In some cases, the average is taken only over $m-1$ days.)

Notice that both variance and volatility swaps pay off if actual (realized) volatility between t and $t+m$ is higher than expected in t. In contrast, the futures on the VIX pays off when the expected volatility (in $t+m$) is higher than what was thought in t. In a way, we can think of the VIX futures as a futures on a volatility swap (between $t+m$ and a month later).

Since VIX^2 is a good approximation of variance swap rate for a 30-day contract, the return can be approximated as

$$\text{Return of a variance swap}_{t+m} = (RV_{t+m} - VIX_t^2)/VIX_t^2. \tag{23.5}$$

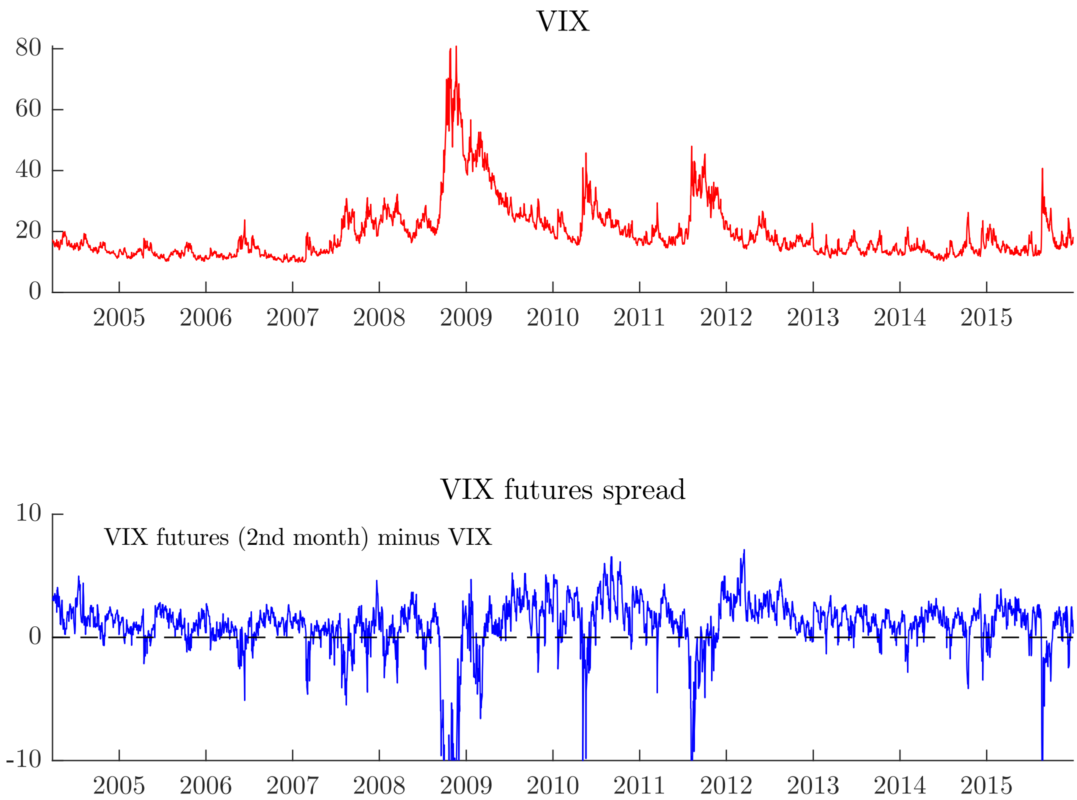


Figure 23.4: VIX futures spread

Figures 23.5–23.6 illustrate the properties for the VIX and realized volatility of the S&P 500. It is clear that the return of a variance swap (with expiration of 30 days) would have been negative on average. (Notice: variance swaps were not traded for the early part of the sample in the figure.) The excess return (over a riskfree rate) would, of course, have been even more negative. This suggests that selling variance swaps (which has been the specialty of some hedge funds) might be a good deal—except that it will incur some occasional really large losses (the return distribution has positive skewness). Presumably, buyers of the variance swaps think that this negative average return is a reasonable price to pay for the "hedging" properties of the contracts—although the data does not suggest a very strong negative correlation with S&P 500 returns.

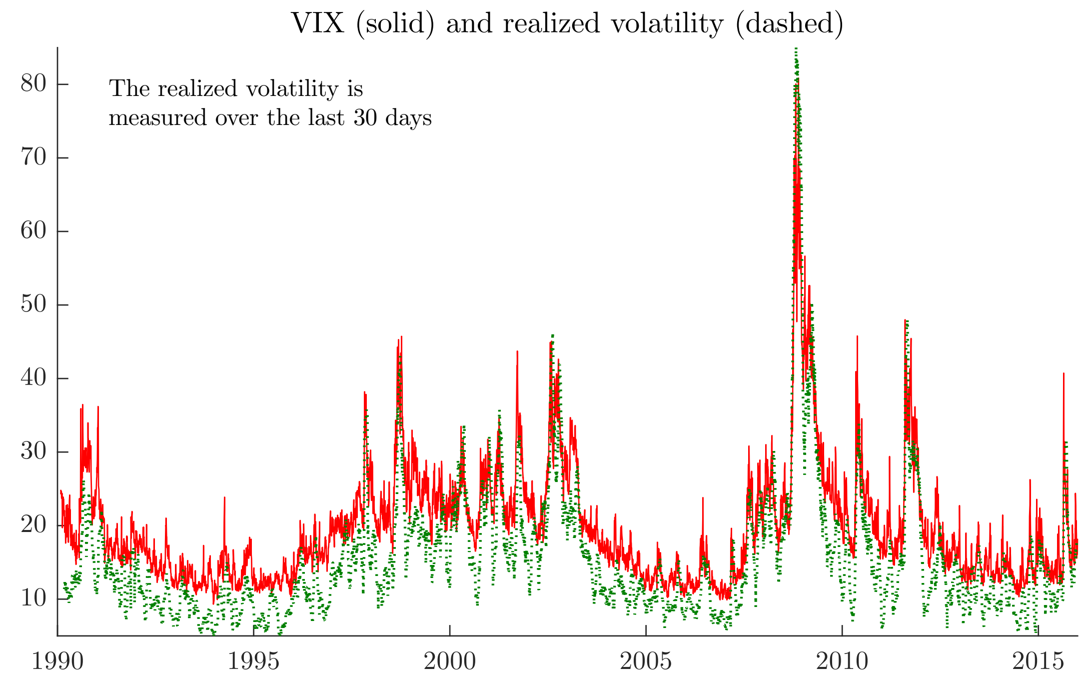


Figure 23.5: VIX and realized volatility (variance)

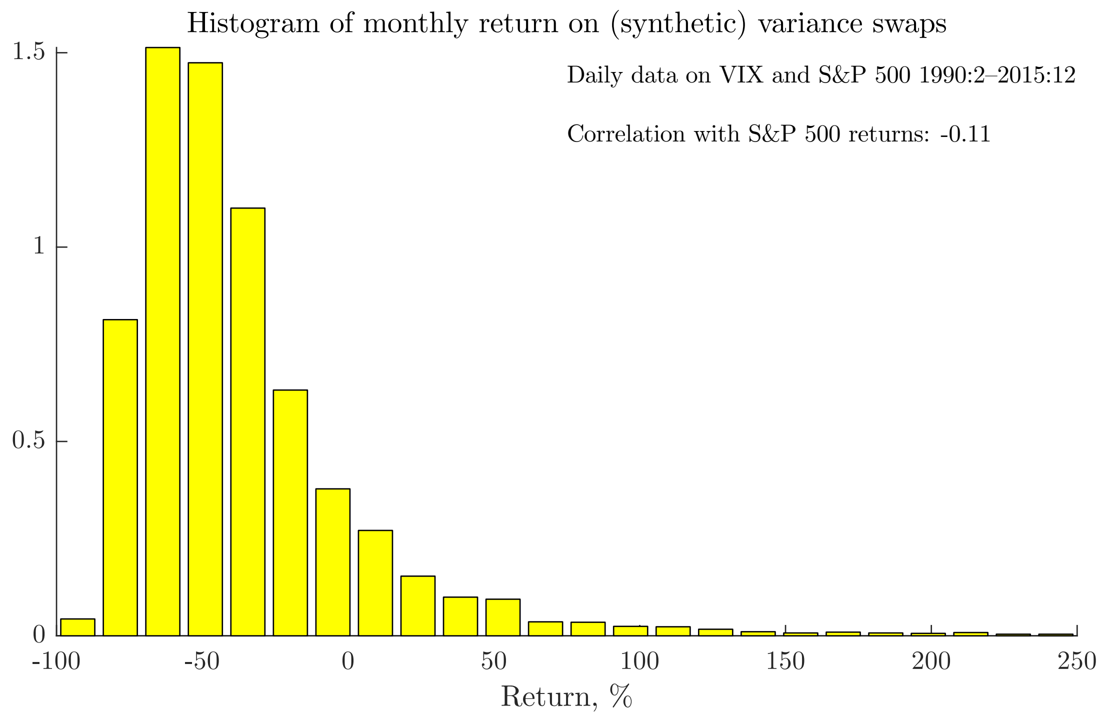


Figure 23.6: Distribution of return from investing in variance swaps

Bibliography

Blake, D., 1990, *Financial market analysis*, McGraw-Hill, London.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay, 1997, *The econometrics of financial markets*, Princeton University Press, Princeton, New Jersey.

Cochrane, J. H., 2001, *Asset pricing*, Princeton University Press, Princeton, New Jersey.

Cox, J. C., S. A. Ross, and M. Rubinstein, 1979, "Option pricing: a simplified approach," *Journal of Financial Economics*, 7, 229–263.

Deacon, M., and A. Derry, 1998, *Inflation-indexed securities*, Prentice Hall Europe, Hemel Hempstead.

Elton, E. J., M. J. Gruber, S. J. Brown, and W. N. Goetzmann, 2014, *Modern portfolio theory and investment analysis*, John Wiley and Sons, 9th edn.

Fabozzi, F. J., 2004, *Bond markets, analysis, and strategies*, Pearson Prentice Hall, 5th edn.

Gatheral, J., 2006, *The volatility surface: a practitioner's guide*, Wiley.

Hartzmark, M. L., 1991, "Luck versus forecast ability: determinants of trader performance in futures markets," *Journal of Business*, 64, 49–74.

Hull, J. C., 2009, *Options, futures, and other derivatives*, Prentice-Hall, Upper Saddle River, NJ, 7th edn.

Kolb, R. A., and H. O. Stekler, 1996, "How well do analysts forecast interest rates," *Journal of Forecasting*, 15, 385–394.

McCulloch, J., 1975, "The tax-adjusted yield curve," *Journal of Finance*, 30, 811–830.

McDonald, R. L., 2014, *Derivatives markets*, Pearson, 3rd edn.

Nelson, C., and A. Siegel, 1987, "Parsimonious modeling of yield curves," *Journal of Business*, 60, 473–489.

Svensson, L., 1995, "Estimating forward interest rates with the extended Nelson&Siegel method," *Quarterly Review, Sveriges Riksbank*, 1995:3, 13–26.

Wystup, U., 2006, *FX Options and Structured Products*, Wiley.