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Sample Average Approximation Method for Chance Constrained Programming: Theory and Applications

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Abstract We study sample approximations of chance constrained problems. In particular, we consider the *sample average approximation* (SAA) approach and discuss the convergence properties of the resulting problem. We discuss how one can use the SAA method to obtain good candidate solutions for chance constrained problems. Numerical experiments are performed to correctly tune the parameters involved in the SAA. In addition, we present a method for constructing statistical lower bounds for the optimal value of the considered problem and discuss how one should tune the underlying parameters. We apply the SAA to two chance constrained problems. The first is a linear portfolio selection problem with returns following a multivariate lognormal distribution. The second is a joint chance constrained version of a simple blending problem.

Keywords Chance constraints · Sample average approximation · Portfolio selection

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1 Introduction

We consider chance constrained problems of the form

$$\min_{x\in X} f(x), \quad \text{s.t. } \mathsf{prob}\{G(x,\xi)\le 0\}\ge 1-\alpha. \tag{1}$$

Here $X\subset\mathbb{R}^n$, ξ is a random vector[1] with probability distribution P supported on a set $\Xi\subset\mathbb{R}^d$, $\alpha\in(0,1)$, $f:\mathbb{R}^n\to\mathbb{R}$ is a real valued function and $G:\mathbb{R}^n\times\Xi\to\mathbb{R}^m$. Chance constrained problems were introduced in Charnes, Cooper and Symmonds [1] and have been extensively studied since. For a theoretical background we may refer to Prékopa [2] where an extensive list of references can be found. Applications of chance constrained programming include, e.g., water management [3] and optimization of chemical processes [4, 5].

Although chance constraints were introduced almost 50 years ago, little progress was made until recently. Even for simple functions $G(\cdot,\xi)$, e.g., linear, problem (1) may be extremely difficult to solve numerically. One of the reasons is that for a given $x\in X$ the quantity $\mathsf{prob}\{G(x,\xi)\le 0\}$ is hard to compute since it requires a multi-dimensional integration. Thus, it may happen that the only way to check feasibility, of a given point $x\in X$, is by Monte-Carlo simulation. In addition, the feasible set of problem (1) can be nonconvex even if the set X is convex and the function $G(x,\xi)$ is convex in x. Therefore the development went into two somewhat different directions. One is to discretize the probability distribution P and consequently to solve the obtained combinatorial problem (see, e.g., Dentcheva, Prékopa and Ruszczyński [6], Luedtke, Ahmed and Nemhauser [7]). Another approach is to employ convex approximations of chance constraints (cf., Nemirovski and Shapiro [8]).

In this paper we discuss the *sample average approximation* (SAA) approach to chance constrained problems. Such an approximation is obtained by replacing the actual distribution in chance constraint by an empirical distribution corresponding to a random sample. This approach is well known for stochastic programs with expected values objectives [9]. SAA methods for chance constrained problems have been investigated in [10] and [11].

The remainder of the paper is organized as follows. In Sect. 2 we provide theoretical background for the SAA approach, showing convergence of the optimal value of the approximation to the optimal value of the true problem. In addition, following [8] we describe how to construct bounds for the optimal value of chance constrained problems of the form (1). In Sect. 3, we present a chance constrained portfolio selection problem. We apply the SAA method to obtain upper bounds as well as candidate solutions to the problems. In addition we present several numerical experiments that indicate how one should tune the parameters of the SAA approach. In Sect. 4 we present a simple blending problem modeled as a joint chance constrained problem. Section 5 concludes the paper and suggests directions for future research.

We use the following notation throughout the paper. The integer part of number $a\in\mathbb{R}$ is denoted by $\lfloor a\rfloor$. By $\Phi(z)$ we denote the cumulative distribution function

[1] We use the same notation ξ to denote a random vector and its particular realization. Which of these two meanings will be used in a particular situation will be clear from the context.

(cdf) of standard normal random variable and by z_α the corresponding critical value, i.e., $\Phi(z_\alpha) = 1 - \alpha$, for $\alpha \in (0, 1)$,

$$B(k; p, N) := \sum_{i=0}^{k} \binom{N}{i} p^i (1-p)^{N-i}, \quad k = 0, \ldots, N, \tag{2}$$

denotes the cdf of binomial distribution. For sets $A, B \subset \mathbb{R}^n$, we denote by

$$\mathbb{D}(A, B) := \sup_{x \in A} \operatorname{dist}(x, B) \tag{3}$$

the *deviation* of set A from set B.

2 Theoretical Background

In order to simplify the presentation we assume in this section that the constraint function $G : \mathbb{R}^n \times \Xi \to \mathbb{R}$ is real valued. Of course, a number of constraints $G_i(x, \xi) \leq 0$, $i = 1, \ldots, m$, can be equivalently replaced by one constraint with

$$G(x, \xi) := \max_{1 \leq i \leq m} G_i(x, \xi) \leq 0.$$

Such operation of taking maximum preserves convexity of functions $G_i(\cdot, \xi)$. We assume that the set X is *closed*, the function $f(x)$ is *continuous* and the function $G(x, \xi)$ is a *Carathéodory function*, i.e., $G(x, \cdot)$ is measurable for every $x \in \mathbb{R}^n$ and $G(\cdot, \xi)$ continuous for a.e. $\xi \in \Xi$.

Problem (1) can be written in the following equivalent form:

$$\min_{x \in X} f(x), \quad \text{s.t. } p(x) \leq \alpha, \tag{4}$$

where

$$p(x) := P\{G(x, \xi) > 0\}.$$

Now let $\xi^1, \ldots, \xi^N$ be an *independent identically distributed* (iid) sample of N realizations of random vector ξ and $P_N := N^{-1} \sum_{j=1}^{N} \Delta(\xi^j)$ be the respective empirical measure. Here $\Delta(\xi)$ denotes measure of mass one at point ξ, and hence P_N is a discrete measure assigning probability $1/N$ to each point ξ^j, $j = 1, \ldots, N$. The sample average approximation $\hat{p}_N(x)$ of function $p(x)$ is obtained by replacing the 'true' distribution P by the empirical measure P_N. That is, $\hat{p}_N(x) := P_N\{G(x, \xi) > 0\}$. Let $\mathbb{1}_{(0,\infty)} : \mathbb{R} \to \mathbb{R}$ be the indicator function of $(0, \infty)$, i.e.,

$$\mathbb{1}_{(0,\infty)}(t) := \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Then, we can write that $p(x) = \mathbb{E}_P[\mathbb{1}_{(0,\infty)}(G(x, \xi))]$ and

$$\hat{p}_N(x) = \mathbb{E}_{P_N}[\mathbb{1}_{(0,\infty)}(G(x, \xi))] = \frac{1}{N} \sum_{j=1}^{N} \mathbb{1}_{(0,\infty)}(G(x, \xi^j)).$$

That is, $\hat{p}_N(x)$ is equal to the proportion of times that $G(x, \xi^j) > 0$. The problem associated with the generated sample $\xi^1, \ldots, \xi^N$, is

$$\min_{x \in X} f(x), \quad \text{s.t. } \hat{p}_N(x) \leq \gamma. \tag{5}$$

We refer to problems (4) and (5) as the true and SAA problems, respectively, at the respective significance levels α and γ. Note that, following [11], we allow the significance level $\gamma \geq 0$ of the SAA problem to be different from the significance level α of the true problem. Next we discuss the convergence of a solution of the SAA problem (5) to that of the true problem (4) with respect to the sample size N and the significance level γ. A convergence analysis of the SAA problem (5) has been given in [11]. Here we present complementary results under slightly different assumptions.

Recall that a sequence $f_k(x)$ of extended real valued functions is said to *epiconverge* to a function $f(x)$, written $f_k \xrightarrow{e} f$, if for any point x the following two conditions hold:

(i) for any sequence x_k converging to x, one has

$$\liminf_{k \to \infty} f_k(x_k) \geq f(x); \tag{6}$$

(ii) there exists a sequence x_k converging to x such that

$$\limsup_{k \to \infty} f_k(x_k) \leq f(x). \tag{7}$$

Note that, by the (strong) law of large numbers (LLN), we have that, for any x, $\hat{p}_N(x)$ converges w.p.1 to $p(x)$.

Proposition 2.1 *Let $G(x, \xi)$ be a Carathéodory function. Then the functions $p(x)$ and $\hat{p}_N(x)$ are lower semicontinuous, and $\hat{p}_N \xrightarrow{e} p$ w.p.1. Moreover, suppose that for every $\bar{x} \in X$ the set $\{\xi \in \Xi : G(\bar{x}, \xi) = 0\}$ has P-measure zero, i.e., $G(\bar{x}, \xi) \neq 0$ w.p.1. Then the function $p(x)$ is continuous at every $x \in X$ and $\hat{p}_N(x)$ converges to $p(x)$ w.p.1 uniformly on any compact set $C \subset X$, i.e.,*

$$\sup_{x \in C} |\hat{p}_N(x) - p(x)| \to 0, \quad \text{w.p.1}, \quad \text{as } N \to \infty. \tag{8}$$

Proof Consider function $\psi(x, \xi) := \mathbb{1}_{(0,\infty)}(G(x, \xi))$. Recall that $p(x) = \mathbb{E}_P[\psi(x, \xi)]$ and $\hat{p}_N(x) = \mathbb{E}_{P_N}[\psi(x, \xi)]$. Since the function $\mathbb{1}_{(0,\infty)}(\cdot)$ is lower semicontinuous and $G(\cdot, \xi)$ is a Carathéodory function, it follows that the function $\psi(x, \xi)$ is random lower semicontinuous[2] (see, e.g., [12, Proposition 14.45]). Then by Fatou's lemma we have for any $\bar{x} \in \mathbb{R}^n$,

$$\liminf_{x \to \bar{x}} p(x) = \liminf_{x \to \bar{x}} \int_{\Xi} \psi(x, \xi) dP(\xi)$$

[2] Random lower semicontinuous functions are called normal integrands in [12].

$$\geq \int_{\Xi} \liminf_{x \to \bar{x}} \psi(x, \xi) dP(\xi) \geq \int_{\Xi} \psi(\bar{x}, \xi) dP(\xi) = p(\bar{x}).$$

This shows lower semicontinuity of $p(x)$. Lower semicontinuity of $\hat{p}_N(x)$ can be shown in the same way.

The epiconvergence $\hat{p}_N \xrightarrow{e} p$ w.p.1 is a direct implication of Artstein and Wets [13, Theorem 2.3]. Note that, of course, $|\psi(x, \xi)|$ is dominated by an integrable function since $|\psi(x, \xi)| \leq 1$. Suppose, further, that for every $\bar{x} \in X$, $G(\bar{x}, \xi) \neq 0$ w.p.1, which implies that $\psi(\cdot, \xi)$ is continuous at $\bar{x}$ w.p.1. Then by the Lebesgue Dominated Convergence Theorem we have for any $\bar{x} \in X$,

$$\begin{aligned} \lim_{x \to \bar{x}} p(x) &= \lim_{x \to \bar{x}} \int_{\Xi} \psi(x, \xi) dP(\xi) \\ &= \int_{\Xi} \lim_{x \to \bar{x}} \psi(x, \xi) dP(\xi) = \int_{\Xi} \psi(\bar{x}, \xi) dP(\xi) = p(\bar{x}), \end{aligned}$$

which shows continuity of $p(x)$ at $x = \bar{x}$. Finally, the uniform convergence (8) follows by a version of the uniform law of large numbers (see, [9, Proposition 7, p. 363]). □

By lower semicontinuity of $p(x)$ and $\hat{p}_N(x)$ we have that the feasible sets of the 'true' problem (4) and its SAA counterpart (5) are closed sets. Therefore, if the set X is bounded (i.e., compact), then problems (4) and (5) have nonempty sets of optimal solutions denoted, respectively, as S and $\hat{S}_N$, provided that these problems have nonempty feasible sets. We also denote by ϑ^* and $\hat{\vartheta}_N$ the optimal values of the true and the SAA problems, respectively. The following result shows that for $\gamma = \alpha$, under mild regularity conditions, $\hat{\vartheta}_N$ and $\hat{S}_N$ converge w.p.1 to their counterparts of the true problem.

We make the following assumption.

(A) There is an optimal solution $\bar{x}$ of the true problem (4) such that for any $\varepsilon > 0$ there is $x \in X$ with $\|x - \bar{x}\| \leq \varepsilon$ and $p(x) < \alpha$.

In other words the above condition (A) assumes existence of a sequence $\{x_k\} \subset X$ converging to an optimal solution $\bar{x} \in S$ such that $p(x_k) < \alpha$ for all k, i.e., $\bar{x}$ is an accumulation point of the set $\{x \in X : p(x) < \alpha\}$.

Proposition 2.2 *Suppose that the significance levels of the true and SAA problems are the same, i.e., $\gamma = \alpha$, the set X is compact, the function $f(x)$ is continuous, $G(x, \xi)$ is a Carathéodory function, and condition (A) holds. Then $\hat{\vartheta}_N \to \vartheta^*$ and $\mathbb{D}(\hat{S}_N, S) \to 0$ w.p.1 as $N \to \infty$.*

Proof By the condition (A), the set S is nonempty and there is $x \in X$ such that $p(x) < \alpha$. We have that $\hat{p}_N(x)$ converges to $p(x)$ w.p.1. Consequently $\hat{p}_N(x) < \alpha$, and hence the SAA problem has a feasible solution, w.p.1 for N large enough. Since $\hat{p}_N(\cdot)$ is lower semicontinuous, the feasible set of SAA problem is compact, and hence $\hat{S}_N$ is nonempty w.p.1 for N large enough. Of course, if x is a feasible solution

of an SAA problem, then $f(x) \geq \hat{\vartheta}_N$. Since we can take such point x arbitrary close to $\bar{x}$ and $f(\cdot)$ is continuous, we obtain that

$$\limsup_{N\to\infty} \hat{\vartheta}_N \leq f(\bar{x}) = \vartheta^*, \quad \text{w.p.1.} \tag{9}$$

Now let $\hat{x}_N \in \hat{S}_N$, i.e., $\hat{x}_N \in X$, $\hat{p}_N(\hat{x}_N) \leq \alpha$ and $\hat{\vartheta}_N = f(\hat{x}_N)$. Since the set X is compact, we can assume by passing to a subsequence if necessary that $\hat{x}_N$ converges to a point $\bar{x} \in X$ w.p.1. Also we have that $\hat{p}_N \overset{\text{e}}{\to} p$ w.p.1, and hence

$$\liminf_{N\to\infty} \hat{p}_N(\hat{x}_N) \geq p(\bar{x}), \quad \text{w.p.1.}$$

It follows that $p(\bar{x}) \leq \alpha$ and hence $\bar{x}$ is a feasible point of the true problem, and thus $f(\bar{x}) \geq \vartheta^*$. Also $f(\hat{x}_N) \to f(\bar{x})$ w.p.1, and hence

$$\liminf_{N\to\infty} \hat{\vartheta}_N \geq \vartheta^*, \quad \text{w.p.1.} \tag{10}$$

It follows from (9) and (10) that $\hat{\vartheta}_N \to \vartheta^*$ w.p.1. It also follows that the point $\bar{x}$ is an optimal solution of the true problem and then we have $\mathbb{D}(\hat{S}_N, S) \to 0$ w.p.1. □

Condition (A) is essential for the consistency of $\hat{\vartheta}_N$ and $\hat{S}_N$. Think, for example, about a situation where the constraint $p(x) \leq \alpha$ defines just one feasible point $\bar{x}$ such that $p(\bar{x}) = \alpha$. Then arbitrary small changes in the constraint $\hat{p}_N(x) \leq \alpha$ may result in that the feasible set of the corresponding SAA problem becomes empty. Note also that condition (A) was not used in the proof of inequality (10). Verification of condition (A) can be done by ad hoc methods.

Suppose now that $\gamma > \alpha$. Then by Proposition 2.2 we may expect that with increase of the sample size N, an optimal solution of the SAA problem will approach an optimal solution of the true problem with the significance level γ rather than α. Of course, increasing the significance level leads to enlarging the feasible set of the true problem, which in turn may result in decreasing of the optimal value of the true problem. For a point $\bar{x} \in X$ we have that $\hat{p}_N(\bar{x}) \leq \gamma$, i.e., $\bar{x}$ is a feasible point of the SAA problem, iff no more than γN times the event "$G(\bar{x}, \xi^j) > 0$" happens in N trials. Since probability of the event "$G(\bar{x}, \xi^j) > 0$" is $p(\bar{x})$, it follows that

$$\mathsf{prob}\{\hat{p}_N(\bar{x}) \leq \gamma\} = B(\lfloor \gamma N \rfloor; p(\bar{x}), N). \tag{11}$$

Recall that, by the Chernoff inequality [14], for $k > Np$,

$$B(k; p, N) \geq 1 - \exp\{-N(k/N - p)^2/(2p)\}.$$

It follows that if $p(\bar{x}) \leq \alpha$ and $\gamma > \alpha$, then $1 - \mathsf{prob}\{\hat{p}_N(\bar{x}) \leq \gamma\}$ approaches zero at a rate of $\exp(-\kappa N)$, where $\kappa := (\gamma - \alpha)^2/(2\alpha)$. Of course, if $\bar{x}$ is an optimal solution of the true problem and $\bar{x}$ is feasible for the SAA problem, then $\hat{\vartheta}_N \leq \vartheta^*$. That is, if $\gamma > \alpha$, then the probability of the event "$\hat{\vartheta}_N \leq \vartheta^*$" approaches one exponentially fast. Similarly, we have that if $p(\bar{x}) = \alpha$ and $\gamma < \alpha$, then probability that $\bar{x}$ is a feasible point of the corresponding SAA problem approaches zero exponentially fast (cf., [11]).

In order to get a lower bound for the optimal value ϑ^* we proceed as follows. Let us choose two positive integers M and N, let

$$\theta_N := B\big(\lfloor \gamma N \rfloor; \alpha, N\big)$$

and let L be the *largest* integer such that

$$B(L-1; \theta_N, M) \le \beta. \tag{12}$$

Next, generate M independent samples $\xi^{1,m}, \ldots, \xi^{N,m}$, $m = 1, \ldots, M$, each of size N, of random vector ξ. For each sample, solve the associated optimization problem

$$\min_{x \in X} f(x), \quad \text{s.t.} \sum_{j=1}^{N} \mathbb{1}_{(0,\infty)}(G(x, \xi^{j,m})) \le \gamma N, \tag{13}$$

and hence calculate its optimal value $\hat{\vartheta}_N^m$, $m = 1, \ldots, M$. That is, solve M times the corresponding SAA problem at the significance level γ. It may happen that problem (13) is either infeasible or unbounded from below, in which case we assign its optimal value as $+\infty$ or $-\infty$, respectively. We can view $\hat{\vartheta}_N^m$, $m = 1, \ldots, M$, as an iid sample of the random variable $\hat{\vartheta}_N$, where $\hat{\vartheta}_N$ is the optimal value of the respective SAA problem at significance level γ. Next we rearrange the calculated optimal values in the nondecreasing order as follows $\hat{\vartheta}_N^{(1)} \le \cdots \le \hat{\vartheta}_N^{(M)}$, i.e., $\hat{\vartheta}_N^{(1)}$ is the smallest, $\hat{\vartheta}_N^{(2)}$ is the second smallest etc, among the values $\hat{\vartheta}_N^m$, $m = 1, \ldots, M$. We use the random quantity $\hat{\vartheta}_N^{(L)}$ as a lower bound of the true optimal value ϑ^*. It is possible to show that with probability at least $1 - \beta$, the random quantity $\hat{\vartheta}_N^{(L)}$ is below the true optimal value ϑ^*, i.e., $\hat{\vartheta}_N^{(L)}$ is indeed a lower bound of the true optimal value with confidence at least $1 - \beta$ (see[3] [8]). We will discuss later how to choose the constants M, N and γ.

3 Chance Constrained Portfolio Problem

Consider the following maximization problem subject to a single chance constraint

$$\max_{x \in X} \mathbb{E}[r^T x], \quad \text{s.t. } \mathsf{prob}\{r^T x \ge v\} \ge 1 - \alpha. \tag{14}$$

Here $x \in \mathbb{R}^n$ is vector of decision variables, $r \in \mathbb{R}^n$ is a random (data) vector (with known probability distribution), $v \in \mathbb{R}$ and $\alpha \in (0, 1)$ are constants, e is a vector whose components are all equal to 1 and

$$X := \{x \in \mathbb{R}^n : e^T x = 1,\ x \ge 0\}.$$

[3] In [8] this lower bound was derived for $\gamma = 0$. It is straightforward to extend the derivations to the case of $\gamma > 0$.

Note that, of course, $\mathbb{E}[r^T x] = \overline{r}^T x$, where $\overline{r} := \mathbb{E}[r]$ is the corresponding mean vector. That is, the objective function of problem.

The motivation to study (14) is the portfolio selection problem going back to Markowitz [15]. The vector x represents the percentage of a total wealth of one dollar invested in each of n available assets, r is the vector of random returns of these assets and the decision agent wants to maximize the mean return subject to having a return greater or equal to a desired level v, with probability at least $1-\alpha$. We note that problem (14) is not realistic because it does not incorporate crucial features of real markets such as cost of transactions, short sales, lower and upper bounds on the holdings, etc. However, it will serve to our purposes as an *example* of an application of the SAA method. For a more realistic model we can refer the reader, e.g., to [16].

3.1 Applying the SAA

First assume that r follows a multivariate normal distribution with mean vector $\overline{r}$ and covariance matrix Σ, written $r \sim \mathcal{N}(\overline{r}, \Sigma)$. In that case $r^T x \sim \mathcal{N}(\overline{r}^T x, x^T \Sigma x)$, and hence (as it is well known) the chance constraint in (14) can be written as a convex second order conic constraint (SOCC). Using the explicit form of the chance constraint, one can efficiently solve the convex problem (14) for different values of α. An *efficient frontier* of portfolios can be constructed in an objective function value versus confidence level plot, that is, for every confidence level α we associate the optimal value of problem (14). The efficient frontier dates back to Markowitz [15].

If r follows a multivariate lognormal distribution, then no closed form solution is available. The *sample average approximation* (SAA) of problem (14) can be written as

$$\max_{x \in X} \overline{r}^T x, \quad \text{s.t. } \hat{p}_N(x) \le \gamma, \tag{15}$$

where

$$\hat{p}_N(x) := N^{-1} \sum_{i=1}^{N} \mathbb{1}_{(0,\infty)}(v - r_i^T x)$$

and $\gamma \in [0, 1)$. The reason we use γ instead of α is to suggest that for a fixed α, a different choice of the parameter γ in (15) might be suitable. For instance, if $\gamma = 0$, then the SAA problem (15) becomes the linear program

$$\max_{x \in X} \overline{r}^T x, \quad \text{s.t. } r_i^T x \ge v, \quad i = 1, \ldots, N. \tag{16}$$

A recent paper by Campi and Garatti [17], building on the work of Calafiore and Campi [18], provides an expression for the probability of an optimal solution $\hat{x}_N$ of the SAA problem (5), with $\gamma = 0$, to be infeasible for the true problem (4). That is, under the assumptions that the set X and functions $f(\cdot)$ and $G(\cdot, \xi)$, $\xi \in \Xi$, are convex and that w.p.1 the SAA problem attains unique optimal solution, we have that, for $N \ge n$,

$$\mathsf{prob}\{p(\hat{x}_N) > \alpha\} \le B(n-1; \alpha, N), \tag{17}$$

and the above bound is tight. Thus, for a confidence parameter $\beta \in (0, 1)$ and a sample size N^* such that

$$B(n-1; \alpha, N^*) \leq \beta, \tag{18}$$

the optimal solution of problem (16) is feasible for the corresponding true problem (14) with probability at least $1 - \beta$.

For $\gamma > 0$, problem (15) can be written as the mixed-integer linear program

$$\max_{x,z} \quad \overline{r}^T x, \tag{19a}$$

$$\text{s.t.} \quad r_i^T x + v z_i \geq v, \tag{19b}$$

$$\sum_{i=1}^{N} z_i \leq N\gamma, \tag{19c}$$

$$x \in X,\ z \in \{0, 1\}^N, \tag{19d}$$

with one binary variable for each sample point. To see that problems (15) and (19) are equivalent, let $(x, z_1, \ldots, z_N)$ be feasible for problem (19). Then, from the first constraint of (19), we have $z_i \geq \mathbb{1}_{(0,\infty)}(v - r_i^T x)$, and so from the second constraint of (19) we have

$$\gamma \geq N^{-1} \sum_{i}^{N} z_i \geq N^{-1} \mathbb{1}_{(0,\infty)}(v - r_i^T x) = \hat{p}_N(x). \tag{20}$$

Thus, x is feasible to (15) and has the same objective value as in (19). Conversely, let x be feasible for problem (15). Defining $z_i = \mathbb{1}_{(0,\infty)}(v - r_i^T x),\ i = 1, \ldots, N$, we have that $(x, z_1, \ldots, z_N)$ is feasible for problem (19) with the same objective value.

Given a fixed α in (14), it is not clear what are the best choices of γ and N for (19). We believe it is problem dependent and numerical investigations will be performed with different values of both parameters. We know from Proposition 2.2 that, for $\gamma = \alpha$ the larger the N the closer we are to the original problem (14). However, the number of samples N must be chosen carefully because problem (19) is a binary problem. Even moderate values of N can generate instances that are very hard to solve.

3.2 Finding Candidate Solutions

First we perform numerical experiments for the SAA method with $\gamma = 0$ (problem (16)) assuming that $r \sim \mathcal{N}(\overline{r}, \Sigma)$. We considered 10 assets ($n = 10$) and the data for the estimation of the parameters was taken from historical monthly returns of 10 US major companies. We wrote the codes in GAMS and solved the problems using CPLEX 9.0. The computer was a PC with an Intel Core 2 processor and 2 GB of RAM.

Let us fix $\alpha = 0.10$ and $\beta = 0.01$. For these values, the sample size suggested by (18) is $N^* = 183$. We ran 20 independent replications of (16) for each of the sample

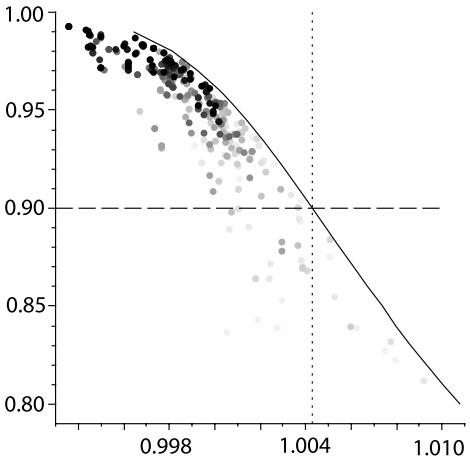

Fig. 1 Normal returns for $\gamma = 0$

sizes $N = 30, 40, \ldots, 200$ and for $N^* = 183$. We also build an *efficient frontier* plot of optimal portfolios with an objective value versus $\mathsf{prob}\{r^T x_\alpha \geq v\}$, where x_α is the optimal solution of problem (14) for a given α. We show in the same plot (Fig. 1) the corresponding objective function values and $\mathsf{prob}\{r^T \hat{x}_N \geq v\}$ for each optimal solution $\hat{x}_N$ found for the SAA (16). To identify each point with a sample size, we used a gray scale that attributes light tones of gray to smaller sample sizes and darker ones to larger samples. The efficient frontier curve is calculated for $\alpha = 0.8, 0.81, \ldots, 0.99$ and then connected by lines. The vertical and horizontal lines are for reference only: they represent the optimal value for problem (14) with $\alpha = 0.10$ and the 90% reliability level, respectively.

Figure 1 shows interesting features of the SAA (16). Although larger sample sizes always generate feasible points, the value of the objective function, in general, is quite small if compared with the optimal value 1.004311 of problem (14) with $\alpha = 0.10$. We also observe the absence of a convergence property: if we increase the sample size, the feasible region of problem (16) gets smaller and the approximation becomes more and more conservative and therefore suboptimal. The reason is that for increasingly large samples the condition $r_i^T x \geq v$ for all i approaches the condition $\mathsf{prob}\{r^T x \geq v\} = 1$.

We performed similar experiments for the lognormal case. For each point obtained in the SAA, we estimated the probability by Monte-Carlo techniques. The reader is referred to [20] for detailed instructions of how to generate samples from a multivariate lognormal distribution. Since in the lognormal case one cannot compute the efficient frontier, we also included in Fig. 2 the upper bounds for $\alpha = 0.02, \ldots, 0.20$, calculated according to (12). The detailed computation of the upper bounds will be given in the next subsection.

In order to find better candidate solutions for problem (14), we need to solve the SAA with $\gamma > 0$, (problem (19)), which is a combinatorial problem. Since our portfolio problem is a linear one, we still can solve problem (15) efficiently for a moderate number (e.g., 200 constraints) of instances. We performed tests for problem (15) with both distributions, fixing $\gamma = 0.05$ and 0.10 and changing N.

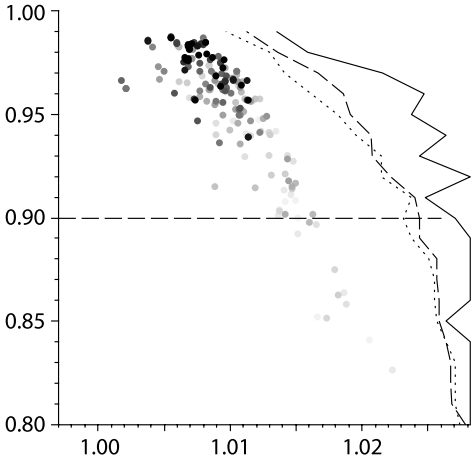

Fig. 2 Lognormal returns, $\gamma = 0$

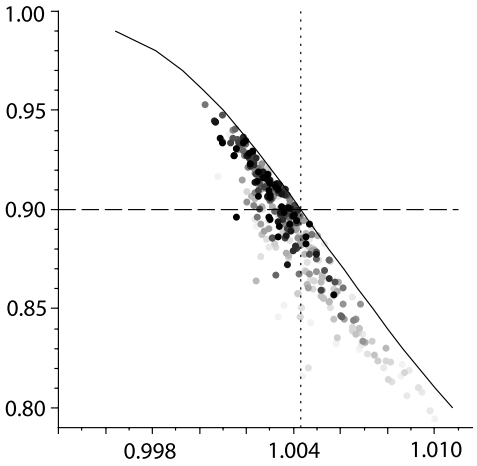

Fig. 3 Normal returns, $\gamma = 0.05$

The best candidate solutions to (14) were obtained with $\gamma = 0.05$. We considered different sample sizes from 30 to 200. Although several points are infeasible to the original problem, we observe in Figs. 3 and 4 that whenever a point is feasible it is close to optimal solution in the normal case or to the upper bound under lognormality. For $\gamma = 0.10$, almost all generated points were infeasible in both cases, as seen in Figs. 5 and 6.

To investigate the different possible choices of γ and N in problem (19), we created a three dimensional representation which we will call γN-plot. The domain is a discretization of values of γ and N, forming a grid with pairs (γ, N). For each pair we solve an instance of problem (19) with these parameters and stored the optimal value and the approximate probability of being feasible to the original problem (14). The z-axis represents the optimal value associated to each point in the domain in the grid. Finally, we created a surface of triangles based on this grid as follows. Let i be the index for the values of γ and j for the values of N. If candidate points associ-

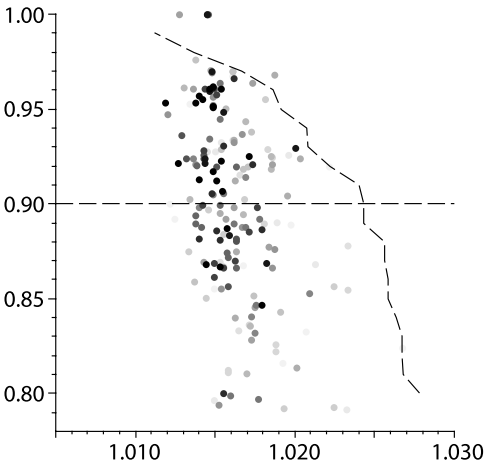

Fig. 4 Lognormal returns, $\gamma = 0.05$

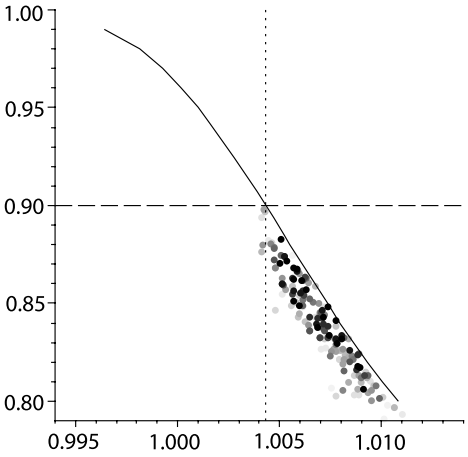

Fig. 5 Normal returns, $\gamma = 0.10$

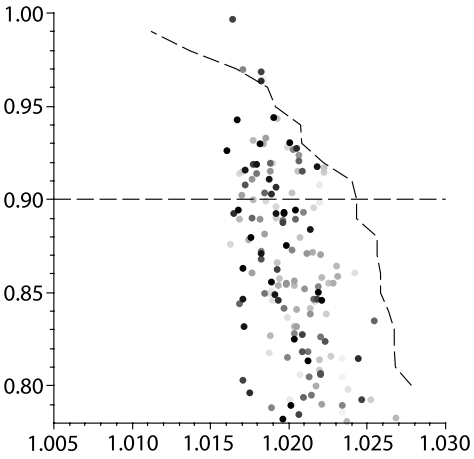

Fig. 6 Lognormal returns, $\gamma = 0.10$

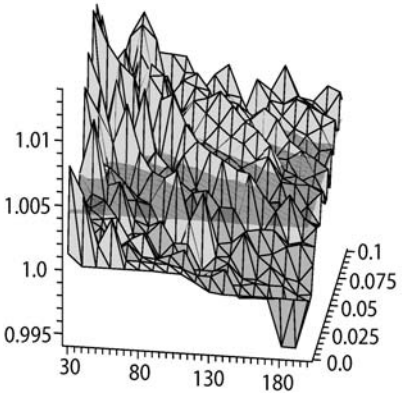

Fig. 7 γN-plot for the portfolio problem with normal returns

ated with grid members (i, j), $(i+1, j)$ and $(i, j+1)$ or $(i+1, j+1)$, $(i+1, j)$ and $(i, j+1)$ are feasible to problem (14) (with probability greater than or equal to $(1-\alpha)$), then we draw a dark gray triangle connecting the three points in the space. Otherwise, we draw a light gray triangle.

We created a γN-plot for problem (14) with normal returns. The result can be seen in Fig. 7, where we also included the plane corresponding to the optimal solution with $\alpha = 0.10$. The values for parameter γ were $0, 0.01, 0.02, \ldots, 0.10$ and for $N = 30, 40, 50, \ldots, 200$. There are several interesting features of Fig. 7 to be discussed. First, note that any fixed γ, small sample sizes tend to generate infeasible solutions and large samples feasible ones. As predicted by the results of Campi and Garatti, when $\gamma = 0$, large sample sizes generate feasible solutions, although they can be seen to be of poor quality judging by the low peaks observed in this region. The concentration of high peaks corresponds to γ values around $\alpha/2 = 0.05$ for almost all sample sizes, including small ones (varying from 50 until 120). We generated different instances of Fig. 7 and the output followed the pattern described here.

Even though there are peaks in other regions, Fig. 7 suggests a strategy to obtain good candidates for chance constrained problems: choose γ close to $\alpha/2$, solve instances of SAA with small sizes of N and keep the best solution. This is fortunate because SAA problems with $\gamma > 0$ are binary problems that can be hard to solve. Our experience with this problem and others is that this strategy works better than trying to solve large instances of SAA problems. Note that the choice $\gamma = \alpha/2$ is from our empirical experience. In general this choice depends on the underlying problem.

3.3 Upper Bounds

A method to compute lower bounds of chance constrained problems of the form (1) was suggested in [8]. We summarized that procedure at the end of Sect. 2 leaving the question of how to choose the constants L, M and N. Given β, M and N, it is straightforward to specify L: it is the largest integer that satisfies (12). For a given N, the larger M the better because we are approximating the L-th order statistic of the random variable $\hat{\vartheta}_N$. However, note that M represents the number of problems to be solved and this value is often constrained by computational limitations.

In [8] an indication of how N should be chosen is not given. It is possible to gain some insight on the magnitude of N by doing some algebra in inequality (12). With $\gamma = 0$, the first term ($i = 0$) of the sum (12) is

$$[1 - (1 - \alpha)^N]^M \approx [1 - e^{-N\alpha}]^M. \tag{21}$$

Approximation (21) suggests that for small values of α we should take N of order $O(\alpha^{-1})$. If N is much bigger than $1/\alpha$ then we would have to choose a very large M in order to honor inequality (12). For instance if $\alpha = 0.10$, $\beta = 0.01$ and $N = 100$ instead of $N = 1/\alpha = 10$ or $N = 2/\alpha = 20$, we need M to be greater than 100 000 in order to satisfy (12), which can be computationally intractable for some problems. If $N = 200$ then M has to be grater then 10^9, which is impractical for most applications.

In [11], the authors applied the same technique to generate bounds on probabilistic versions of the set cover problem and the transportation problem. To construct the bounds they varied N and used $M = 10$ and $L = 1$. For many instances they obtained lower bounds slightly smaller (less than 2%) or even equal to the best optimal values generated by the SAA. In the portfolio problem, the choice $L = 1$ generated poor upper bounds as we will see.

Since problem (14) is a maximization problem, we calculated upper bounds fixing $\beta = 0.01$ for all cases and by choosing three different values for the constants L, M and N. First we fixed $L = 1$ and $N = \lceil 1/\alpha \rceil$ (solid line in Fig. 2). The constant M was chosen to satisfy the inequality (12). The results were not satisfactory, mainly because M ended up being too small. Since the constant M defines the number of samples from $\hat{v}_N$ and since our problem is a linear one, we decided to fix $M = 1\,000$. Then we chose $N = \lceil 1/\alpha \rceil$ (dashed line) and $\lceil 2/\alpha \rceil$ (dotted line) in the next two experiments. The constant L was chosen to be the largest integer such that (12) is satisfied. Figure 2 shows the generated points for the SAA with $\gamma = 0$ along with the upper bounds.

It is harder to construct upper bounds with $\gamma > 0$. The difficulty lies in an appropriate choice of the parameters since we cannot have very large values of M or N when solving binary programs. Our experience shows that it is not significantly better than the bounds obtained with $\gamma = 0$.

4 Blending Problem

Let us consider a second example of chance constrained problems. Suppose a farmer has some crop and wants to use fertilizers to increase the production. He hires an agronomy engineer who recommends 7 g of nutrient A and 4 g of nutrient B. He has two kinds of fertilizers available: the first has ω_1 g of nutrient A and ω_2 g of nutrient B per kilogram. The second has 1 g of each nutrient per kilogram. The quantities ω_1 and ω_2 are uncertain: we will assume they are (independent) continuous uniform random variables with support in the intervals $[1, 4]$ and $[1/3, 1]$ respectively. Furthermore, each fertilizer has a unitary cost per kilogram.

There are several ways to model this blending problem. A detailed discussion can be found in [19], where the authors use this problem to motivate the field of stochastic

programming. We will consider a joint chance constrained formulation as follows:

$$\min_{x_1\geq 0, x_2\geq 0} x_1 + x_2, \quad \text{s.t. } \mathsf{prob}\{\omega_1 x_1 + x_2 \geq 7, \omega_2 x_1 + x_2 \geq 4\} \geq 1-\alpha, \tag{22}$$

where x_i represents the quantity of fertilizer i purchased, $i = 1, 2$, and $\alpha \in [0, 1]$ is the reliability level. The independence assumption allows us to convert the joint probability in (22) into a product of probabilities. After some calculations, one can explicitly solve (22) for all values of α. For $\alpha \in [1/2, 1]$, the optimal solution and optimal value are

$$x_1^* = \frac{18}{9+8(1-\alpha)}, \qquad x_2^* = \frac{2(9+28(1-\alpha))}{9+8(1-\alpha)}, \qquad v^* = \frac{4(9+14(1-\alpha))}{9+8(1-\alpha)}.$$

For $\alpha \in [0, 1/2]$, we have

$$x_1^* = \frac{9}{11-9(1-\alpha)}, \qquad x_2^* = \frac{41-36(1-\alpha)}{11-9(1-\alpha)}, \qquad v^* = \frac{2(25-18(1-\alpha))}{11-9(1-\alpha)}. \tag{23}$$

Our goal is to show that we can apply the SAA methodology to joint chance constrained problems. We can convert a joint chance constrained problem into a problem of the form (1) using the min (or max) operators. Problem (22) becomes

$$\min_{x_1\geq 0, x_2\geq 0} x_1 + x_2, \quad \text{s.t. } \mathsf{prob}\{\min\{\omega_1 x_1 + x_2 - 7, \omega_2 x_1 + x_2 - 4\} \geq 0\} \geq 1-\alpha. \tag{24}$$

It is possible to write the SAA of problem (24) as follows.

$$\min_{x_1\geq 0, x_2\geq 0} \quad x_1 + x_2, \tag{25a}$$

$$\text{s.t.} \quad u_i \leq \omega_1^i x_1 + x_2 - 7, \quad i = 1, \ldots, N, \tag{25b}$$

$$u_i \leq \omega_2^i x_1 + x_2 - 4, \quad i = 1, \ldots, N, \tag{25c}$$

$$u_i + K z^i \geq 0, \quad i = 1, \ldots, N, \tag{25d}$$

$$\sum_{i=1}^{N} z^i \leq N\gamma, \tag{25e}$$

$$z^i \in \{0, 1\}^N, \tag{25f}$$

where N is the number of samples, ω_1^i and ω_2^i are samples from ω_1 and ω_2, $\gamma \in (0, 1)$ and K is a positive constant positive constant greater or equal than 7.

4.1 Numerical Experiments

We performed experiments similar to the ones for the portfolio problem so we will present the results without details. In Fig. 8 we generated approximations for problem (22) with $\alpha = 0.05$ using the SAA. The sample points were obtained by solving a

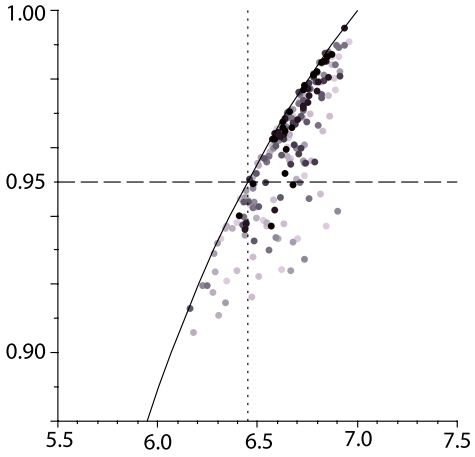

Fig. 8 Blending problem, $\gamma = 0.025$

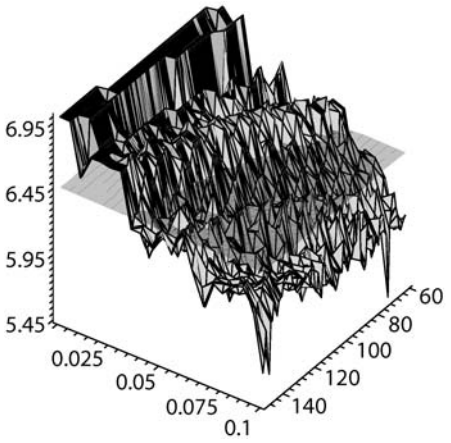

Fig. 9 γN-plot (blending problem)

SAA with $\gamma = 0.025$ and sample sizes $N = 60, 70, \ldots, 150$. The Campi-Garatti (18) suggested value is $N^* = 130$. We included the efficient frontier for problem (22). We will not show the corresponding figures for other values of γ, but the pattern observed in the portfolio problem repeated: with $\gamma = 0$ almost every point was feasible but far from the optimal, with $\gamma = \alpha = 0.05$ almost every point was infeasible. The parameter choice that generated the best candidate solutions was $\gamma = \alpha/2 = 0.025$.

We also show the γN-plot for the SAA of problem (22). We tested γ values in the range $0, 0.005, \ldots, 0.05$ and $N = 60, 70, \ldots, 150$. We included a plane representing the optimal value of problem (22) for $\alpha = 0.05$, which is readily obtained by applying formula (23).

In accordance with Fig. 7, in Fig. 9 the best candidate solutions are the ones with γ around 0.025. Even for very small sample sizes we have feasible solutions (dark gray triangles) close to the optimal plane. On the other hand, this experiment gives more evidence that the SAA with $\gamma = 0$ is excellent to generate feasible solutions (dark gray triangles) but the quality of the solutions is poor. As shown in Fig. 9, the

high peaks associated with $\gamma = 0$ persist for any sample size, generating points far form the optimal plane. In agreement with Fig. 7, the candidates obtained for $\gamma > \alpha$ are in their vast majority infeasible.

5 Conclusions

We have discussed chance constrained problems with a single constraint and proved convergence results about the SAA method. We applied the SAA approach to a portfolio chance constrained problem with random returns. In the normal case, we can compute the *efficient frontier* and use it as a benchmark solution. Experiments show that the sample size suggested by [17] was too conservative for our problem: a much smaller sample can yield feasible solutions. We observed that the quality of the solutions obtained was poor. Similar results were obtained for the lognormal case, where upper bounds were computed using a method developed in [8].

As another illustration of the use of the SAA method, we presented a two dimensional blending problem and modeled it as a joint chance constrained problem. We use the problem as a benchmark to test the SAA approach and also to show how one can use the SAA methodology to approximate joint chance constrained problems.

In both cases we observe that the choice $\gamma = \alpha/2$ gave very good candidate solutions. Even though it generated more infeasible points if compared to the choice $\gamma = 0$, the feasible ones were of better quality. Using the γN-plot we were able to confirm these empirical findings for our two test problems. Figures 7 and 9 tells us that relatively small sample sizes (e.g, if compared to Campi-Garatti estimates) can yield good candidate solutions. This is extremely important since for $\gamma > 0$ the SAA problem is an integer program and large values of N could quickly make the SAA approach intractable. Upper bounds were also constructed for the portfolio problem using the SAA with $\gamma = 0$ by solving several continuous linear programs. Since no closed solution is available for the portfolio problem with lognormal returns, having an upper bound is an important information about the variability of the solution.

We believe that the SAA methodology is suitable to approximate chance constrained problems. Such problems are usually impossible to be solved explicitly and the proposed method can yield good candidate solutions. One advantage of the method is the fairly general assumption on the distribution of the random variables of the problem. One must only be able to sample from the given distribution in order to build the corresponding SAA program. The main contribution of the paper is to present the method and its theoretical foundations and to suggest parameter choices for the actual implementation of the procedure.

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