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THE MAYOR'S DILEMMA

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Abstract

A simple sorting problem is considered that has been informally discussed in The Netherlands for some time (see Steutel (2008)). The context is as follows. The Mayor of Amsterdam wishes to meet the tallest inhabitant of his city, and to this end he asks the inhabitants to form a long line, where the people are numbered from one to a million, say. Some of the people will have to be paid for their trouble, so the Mayor will have to carry some money. He would hate to have too little, but he would rather not have much more than is necessary. The question then is: how much money should the Mayor carry? This leads to rather surprising answers, and to some not entirely trivial mathematics.

Keywords: Counting problem; sorting problem

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1. Introduction and summary

The Mayor proceeds as follows. Inhabitant number 1 is of course, the tallest so far. The Mayor takes him along the line until they find somebody taller. Then the Mayor takes leave of his companion, gives him a hundred-euro bill for his trouble, and resumes his walk with the taller inhabitant. This goes on until the Mayor reaches the end of the line with the tallest person in Amsterdam, who also receives a hundred-euro bill. The question is how many hundred-euro bills will the Mayor need? To solve the problem, we look at a city with n inhabitants rather then 1 000 000, and denote the number of bills needed by $K(n)$. We use simple induction to prove properties of $K(n)$. The answers are surprisingly attractive.

2. Expectation and variance of $K(n)$

Let $K(n)$ denote the (random) number of hundred-euro bills the Mayor will need, and let $\mathrm{E}(n)$ denote its expectation. If an $(n+1)$th inhabitant is added, the number of bills needed will not change with probability $n/(n+1)$, since the added inhabitant is the largest of all with probability $1/(n+1)$. So, we obtain

$$\mathrm{E}(n+1) = \frac{n}{n+1}\,\mathrm{E}(n) + \frac{1}{n+1}[\mathrm{E}(n)+1]\cdot \tag{1}$$

From (1), it follows that $\mathrm{E}(n+1) = \mathrm{E}(n) + 1/(n+1)$, and so, since $\mathrm{E}(1) = 1$,

$$\mathrm{E}(n) = \sum_{k=1}^{n} \frac{1}{k}.$$

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For large n, we thus have

$$E(n) \approx \log n + \gamma = \log n + 0.5772,$$

and so the average number of hundred-euro bills the Mayor will need when $n = 1\,000\,000$ equals

$$E(1\,000\,000) \approx 14.3727,$$

a surprisingly small number. On average, the Mayor would need only 15 hundred-euro bills. However, he would, of course, hate to have too little money. So he will need some information about the variance of $K(1\,000\,000)$. The variance of $K(n)$, denoted by $V(n)$, is obtained in the same way as the expectation. For the second moment of $K(n)$, we have

$$\mathrm{E}\,K(n+1)^2 = \frac{n}{n+1}\,\mathrm{E}\,K(n)^2 + \frac{1}{n+1}[\mathrm{E}\,K(n)+1]^2,$$

which leads to

$$V(n) = \sum_{k=1}^{n} \frac{1}{n} - \sum_{1}^{n} \frac{1}{n^2} \approx \log n + \gamma - \frac{\pi^2}{6} = \log n - 1.0677,$$

since $\sum_1^n 1/n^2 \approx \pi^2/6 = 1.6449$. For $n = 1\,000\,000$, we have

$$V(1\,000\,000) = 12.15 \quad \text{or} \quad \sigma K(1\,000\,000) = 3.717.$$

So, by the (in)famous six sigma rule, the Amsterdam Mayor should not need more than 37 hundred-euro bills, and probably much less.

3. The generating function and probabilities

Somewhat surprisingly, it is not hard to find the probability generating function of $K(n)$. Setting

$$\mathrm{P}(n; z) = \sum_{k=1}^{n} \mathrm{P}(K(n) = k) z^k, \qquad 0 < z \le 1,$$

we have the following recursion:

$$\mathrm{E}\,z^{K(n+1)} = \frac{n}{n+1}\,\mathrm{E}\,z^{K(n)} + \frac{1}{n+1}\,\mathrm{E}\,z^{K(n)+1},$$

and it follows that

$$\mathrm{P}(n+1, z) = \frac{n+z}{n+1}\mathrm{P}(n; z),$$

and, therefore,

$$\mathrm{P}(n; z) = \prod_{k=1}^{n} \frac{z+k-1}{k}, \qquad 0 < z \le 1.$$

It follows that $K(n)$ can be written (in distribution) as the sum of n independent random variables $X(k)$:

$$K(n) = X(1) + X(2) + \cdots + X(n), \tag{2}$$

with

$$\mathrm{P}(X(k) = 1) = 1 - \mathrm{P}(X(k) = 0) = \frac{1}{k}\cdot \tag{3}$$

From (2) and (3), the mean and variance of $K(n)$ easily follow, but it is hard to obtain probabilities. The Mayor of Amsterdam would like to know the probability that 20, 25, or 30 hundred-euro bills are sufficient. These probabilities can, however, be obtained with precision only by fairly advanced numerical methods. The results are

$$\begin{gathered}\mathrm{P}(K(10^6) \leq 20) = 0.950, \qquad \mathrm{P}(K(10^6) \leq 25) = 0.997, \\ \mathrm{P}(K(10^6) \leq 30) = 0.999\,96.\end{gathered} \tag{4}$$

Remark. With hindsight, (2) and (3), might have been used from the start. This can be seen as follows. Clearly, the probability that number k in the line will get 100 euros equals $1/k$. That these occurrences are independent follows from the symmetry of the problem; for a proof, see Tijms (2007, Problem 14.3).

Since the mean and variance of $k(n)$ are rather close, a Poisson distribution might be a fair approximation. This distribution could then be approximated again by the normal, so we try

$$\mathrm{P}(K(n) \leq k) \approx \Phi\left(\frac{k - \mathrm{E}\,K(n)}{\sigma K(n)}\right),$$

where Φ denotes the standard normal distribution function. For the case in which $n = 1\,000\,000$, this leads to the following approximations of the three probabilities above: 0.934, 0.998, and 0.999 987, respectively, which are satisfactorily close to the precise numbers in (4).

We may conclude that if the Mayor pockets 25 hundred-euro bills, he should not get into trouble.

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References

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