Numerical Analysis Project 3

Direct and Iterative Methods to Solve Two-Point Boundary Value Problems

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Introduction

The objective of this project is to apply the conjugate gradient method to solve two point boundary value problems and compare the results of the algorithm when used as a direct method and as an iterative method for a linear system of equations of the form $Ay = b$, and $\hat{A}y = b$, where A and $\hat{A}$ are tridiagonal matrices and b is a given vector. The tridiagonal matrices A and $\hat{A}$, together with the vector b is given below

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

and

$$\hat{A} = \begin{bmatrix} 3 & -1 & & & \\ -1 & 3 & -1 & & \\ & -1 & 3 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 3 \end{bmatrix}$$

and

$$b^T = \begin{bmatrix} 1 + \frac{1^2}{(n+1)^4} & \frac{2^2}{(n+1)^4} & \frac{3^2}{(n+1)^4} & \cdots & \frac{(n-1)^2}{(n+1)^4} & 6 + \frac{n^2}{(n+1)^4} \end{bmatrix}$$

Both systems were adjusted for specific boundary conditions leading to modifications in the vector b. The boundary conditions for both systems are given below:
Boundary conditions for matrix A

$$\frac{d^2y}{dx^2} = -x^2, \quad y(0) = 1, \quad y(1) = 6,$$

using $h = \frac{1}{(n+1)}$

Boundary conditions for matrix $\hat{A}$

$$\frac{d^2y}{dx^2} = y - x^2, \quad y(0) = 1, \quad y(1) = 6,$$

using $h = \frac{1}{(n+1)}$

Both systems were applied to a specified n × n tridiagonal system, and their performance was analyzed in terms of accuracy, efficiency, and suitability for different system sizes.

Results for Matrix A

The Conjugate Gradient (CG) method was applied to solve the linear system $Ay = b$ derived from the finite difference approximation of the boundary value problem $\frac{d^2y}{dx^2} = -x^2$, with boundary conditions $y(0) = 1, \quad y(1) = 6$. The matrix A is symmetric and positive definite, which makes it well-suited for the CG method. The CG method usually converges in at most n iterations for such matrices.
The implementation's performance was evaluated across the residual, error norm, and the number of iterations required for convergence. The outcome for the CG method is given in the table below:

`Conjugate Gradient Method for Matrix A`

n	Iterations	Residual Norm	Error Norm
10	10	4.153204e-16	2.094765e-15
100	100	2.344330e-15	2.203658e-13
1000	1000	6.739607e-15	1.448283e-11
2000	2000	1.282298e-14	7.356420e-11

In the presented table, it can be noticed that the results for matrix A consistently requires exactly n iterations to converge for each matrix size, aligning with the theoretical maximum number of iterations required for convergence. The residual norms were remarkably low across all matrix sizes. This indicates that the conjugate gradient method effectively minimized the residuals to the required tolerance. While the error norms increased with the size of the matrix, they remained relatively small, suggesting that the computed solutions are accurate.

Results for Matrix $\hat{A}$

The Conjugate Gradient (CG) method was applied to solve the linear system $Ay = b$ derived from the finite difference approximation of the boundary value problem $\frac{d^2y}{dx^2} = y - x^2$, with boundary conditions

$y(0) = 1, \quad y(1) = 6$. The matrix $\hat{A}$ is symmetric and positive definite, which makes it well-suited for the CG method. The CG method usually converges in at most n iterations for such matrices. The implementation's performance was evaluated across the residual, error norm, and the number of iterations required for convergence. The outcome for the CG method is given in the table below:

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Conjugate Gradient Method for Matrix Ahat
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n	Iterations	Residual Norm	Error Norm
10	10	4.268896e-19	5.836719e-16
100	21	8.676350e-09	4.494450e-09
1000	21	8.672272e-09	4.492273e-09
2000	21	8.672224e-09	4.492247e-09

In the presented table, it can be noticed that the results for matrix $\hat{A}$ consistently requires fewer iterations across all tested sizes compared to matrix A. This can be attributed to the better conditioning of matrix $\hat{A}$ due to its increased diagonal dominance. The residual norms for $\hat{A}$ remain below 10^{-8} beyond $n = 10$, showing rapid convergence. Also, for larger matrix sizes, the error norm remains consistent, suggesting that the method maintains accuracy.

Question 3

(a). Number of Steps and Problem Size Dependence

For system A, the number of steps required to achieve a residual norm less than 10^{-8} is exactly equal to the dimension of the matrix, which is n steps. This indicates that for a well-conditioned matrix, the method performs as expected and convergence in at most n steps.
Also, For system $\hat{A}$, the method demonstrates a different behavior. The method reaches convergence in just 21 iterations, irrespective of the size of the system for $n \geq 100$. This indicates that the convergence rate is influenced by factors other than just the size of the matrix, which could be the structure of the matrix and the nature of the vector b. The residual norm for the conjugate gradient method decreases significantly, allowing for early termination of the method.

(b). Effectiveness of the Method as a Direct Method

For system A, the method is highly effective as a direct method. The residual norm $r^{(n)}$ after n steps is on the order of 10^{-16} to 10^{-14}. This implies that the solution found at the last iteration is very close to the true solution, which confirm the method's effectiveness as a direct solver for this type of matrix.
Also, For system $\hat{A}$, the method is better categorized as an iterative method due to its early convergence well before n steps. The residual norm $r^{(n)}$ after just 21 iterations is below 10^{-8}. This indicate the method's effectiveness in quickly finding an accurate solution, which further support the method's efficacy.
Finally, the residuals $r^{(n)}$ obtained for both systems align with theoretical expectations. For system A, where n iterations are performed, the residuals confirm the direct method's properties. Again, for system $\hat{A}$ the residuals after 21 iterations, while not adhering to the strict definition of a direct method that would require n iterations, are still indicative of a high-quality solution. Therefore, the conjugate gradient method can be regarded an effective method.

Conclusion

The objective of this project was to apply the conjugate gradient method to solve two point boundary value problems and compare the results of the algorithm when used as a direct method and as an iterative method for a linear system of equations of the form $Ay = b$, and $\hat{A}y = b$, where A and $\hat{A}$ are tridiagonal matrices and b is a given vector. The results show that matrix A consistently requires exactly n iterations to converge for each matrix size. The residual norms were remarkably low across all matrix sizes. This implies that the solution found is very close to the true solution, which confirm the method's effectiveness as a direct solver for this type of matrix.
Also, the results for matrix $\hat{A}$ consistently requires fewer iterations across all tested sizes compared to matrix A. The residual norms remain below 10^{-8} beyond $n = 10$, showing rapid convergence. The method is better categorized as an iterative method due to its early convergence well before n steps.
Hence, based on the results, the conjugate gradient method proves to be effective in solving two-point boundary value problems represented by the tridiagonal matrices A and $\hat{A}$. Despite minor differences in convergence behavior and solution accuracy between the matrices, the method consistently delivers accurate solutions.