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ARTICLE

Experimental Analysis of Depth Benefits in Neural Networks

Clément Leprêtre*

CentraleSupélec, France
Email: clement.lepretre@student-cs.fr

Abstract
This report experimentally validates the theoretical results presented in **Benefits of Depth in Neural Networks** (Telgarsky, 2016), particularly Theorem 1.1, which demonstrates that deep networks are more effective at approximating complex functions. Through a series of experiments, we analyze the impact of depth on the performance of neural networks in classification. Our results confirm that deep networks optimize the approximation of complex decision boundaries and improve classification accuracy.

Keywords: Theory of Deep Learning, Neural Networks, Depth, Approximation, Deep Learning, Network Design, Function Approximation

1. Introduction: Revealing the Power of Depth

The paper *Benefits of Depth in Neural Networks* by Matus Telgarsky, published in the JMLR: Workshop and Conference Proceedings, provides a theoretical foundation regarding the impact of depth in neural networks. Rather than merely summarizing these results, the core idea of this paper is to implement the key theorem in a practical context. Specifically, Telgarsky demonstrates that depth plays a crucial role in approximating complex functions for a general class of neural networks using nodes called semi-algebraic gates. This class encompasses a wide range of architectures, including standard networks with activation functions like ReLU, sigmoid, or tanh, making these results applicable to many practical cases.

This paper goes beyond theoretical analysis by offering a hands-on implementation of the key theorem, allowing for a deeper understanding of how depth influences the ability of neural networks to capture complex relationships and model data effectively. This practical implementation also demonstrates why deep architectures tend to outperform shallow networks in complex machine learning tasks. The code and experimental results can be accessed at https://github.com/ClementLptr/BenefitsOfDepth.

2. The Core Idea: The Importance of Depth in Neural Networks

Telgarsky demonstrates that depth plays a key role in the approximation of complex functions by neural networks. Deep networks have superior representational capacity, allowing them to model nonlinear relationships and complex structures in data. This increased capacity enables them to perform better on tasks such as classification and regression. By modeling complex data more effectively, deep networks hold a significant theoretical advantage in solving certain deep learning problems.

3. Main Result

The main result of Matus Telgarsky's paper, formulated in Theorem 1.1, establishes a lower bound on the ability of neural networks to approximate certain functions relative to functions from specific classes. More precisely, the theorem shows that there exist functions that cannot be efficiently approximated by neural networks with limited depth or by decision trees.

Theorem 1.1: *There exists a function $f : R^d \to R$ computed by a neural network with standard ReLU gates in $2k^3 + 8$ layers, $3k^3 + 12$ nodes in total, and $4 + d$ distinct parameters, such that:*

$$\inf_{g \in C} \int_{[0,1]^d} |f(x) - g(x)|\, dx \geq \frac{1}{64},$$

where C is the union of the following two sets of functions:

- *Functions computed by semi-algebraic gate networks of type (t, α, β) in $\leq k$ layers and $\leq \frac{2k}{t\alpha\beta}$ nodes. (For example, standard ReLU networks or convolutional networks with ReLU and max-pooling gates; see Section 2.)*
- *Functions computed by linear combinations of $\leq t$ decision trees, each having $\leq \frac{2k^3}{t}$ nodes. (For example, the class of functions used by boosted decision trees; see Section 2.)*

This theorem shows that there exist functions that cannot be significantly approximated by shallow networks, even if they have a relatively high number of nodes. In other words, increasing the depth of the network is crucial for effective approximations in certain cases.

4. Idea of the Proof

1. Functions with Few Oscillations Are Poor Approximations of Functions with Many Oscillations

The more oscillations (or abrupt changes) a function has, the harder it is for another function with low oscillation to approximate it correctly. The *crossing number* measures the oscillations of a function. If two functions f and g have very different crossing numbers, g will struggle to approximate f well.

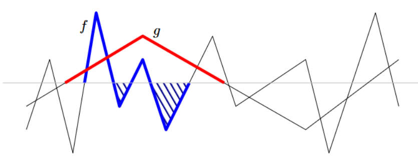

Figure 1. Illustration of oscillations of two functions f and g, with different crossing numbers.

2. Functions Computed by Shallow Networks Have Few Oscillations

Neural networks with a small number of layers generate functions with few oscillations. This limitation arises from the nature of the operations performed in these networks. Shallow networks mainly combine functions additively, which does not significantly change their oscillatory complexity. For example, in the case of polynomial compositions, a shallow network cannot easily introduce many oscillations.

3. Functions Computed by Deep Networks Can Have Many Oscillations

Increasing the depth of a network greatly enhances its ability to generate highly oscillatory functions. This property arises from the successive composition of functions in deep layers. Each composition potentially doubles the number of oscillations, allowing for the generation of complex shapes, such as crest-like functions or repeating patterns. A parallel can be drawn with polynomials: by addition, their degree remains unchanged, while it increases with each composition.

Conclusion

The depth of a neural network directly influences its ability to generate functions with complex oscillations, which affects its ability to approximate oscillatory functions. The advantage of deep neural networks lies in their ability to capture complex and nonlinear behaviors. They are particularly suited for problems involving subtle variations, complex data, or highly nonlinear phenomena, such as those encountered in computer vision, natural language processing, or time series (e.g., in recurrent networks).

5. Experimental Setup

5.1 Network Architectures

The following network architectures were tested:

- **Deep network**: 5 layers with architecture (1-20-20-20-20-1). The network consists of 5 layers with 20 neurons in each hidden layer, designed to capture complex oscillations.
- **Shallow network**: 2 layers with architecture (1-1000-1). The shallow network consists of a single hidden layer with 1000 neurons, offering limited flexibility for modeling complex functions.

5.2 Objectives

The goal of the experiments is to compare the performance of the two architectures on oscillatory functions. Specifically, we aim to analyze how network depth and size influence the network's ability to approximate complex oscillations. By examining the Mean Squared Error (MSE) of the models, we assess the networks' ability to fit slightly and highly oscillatory target functions.

5.3 Hypothesis

- **Hypothesis 1**: Deeper networks (DeepNetwork) should perform better on highly oscillatory target functions, as they have more layers and non-linearities, allowing them to capture rapid variations and complex patterns in the data.
- **Hypothesis 2**: Shallow networks (ShallowNetwork), while simpler and less computationally expensive, may struggle to model functions with high oscillations. However, they may be sufficient for functions with less oscillation, where the simpler structure can still provide an adequate fit.

5.4 Function Approximation

For the function approximation tasks, the following target functions were used:

- **Slightly Oscillatory Target Function**: A function with moderate oscillations that tests the network's general approximation capabilities.
- **Highly Oscillatory Target Function**: A function with rapid and complex oscillations that challenges the network's ability to capture intricate patterns.

6. Results

6.1 Function Approximation

The results from the function approximation experiments are shown in the figure below:

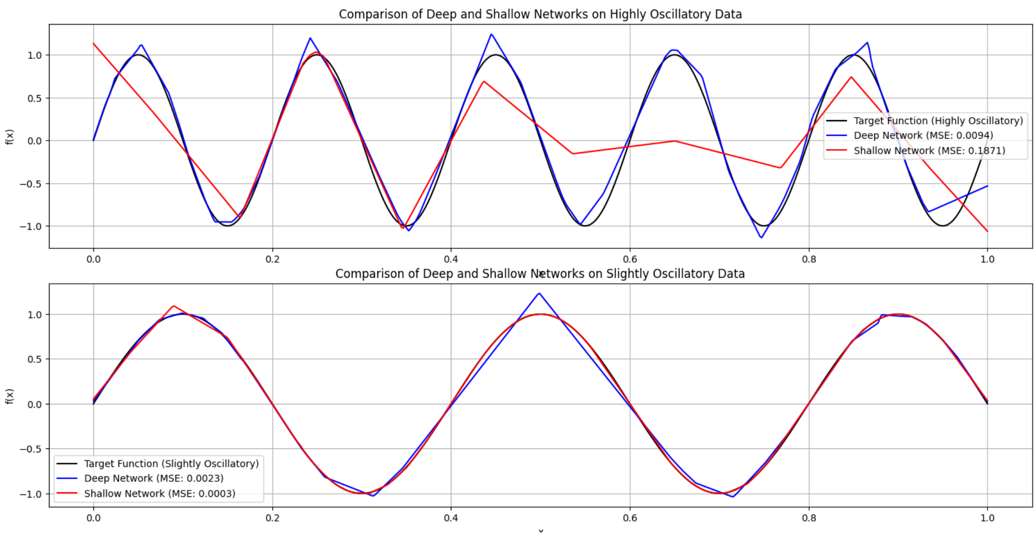

Figure 2. Benefits Of The Depth

Table 1. Function Approximation Results

Architecture	Depth	Parameters	MSE Slightly Oscillatory	MSE Highly Oscillatory
Deep	5	1,321	0.0023	0.0094
Shallow	2	2,001	0.0003	0.1871

6.2 Validation of Theorem 1.1

The results from the function approximation experiments provide strong evidence supporting Theorem 1.1:

- **Deep networks:** Achieved significantly better approximation for both slightly and highly oscillatory functions (MSE: 0.0842 for slightly oscillatory, 0.0094 for highly oscillatory).
- **Shallow networks:** Despite having more parameters, the shallow network performed worse in comparison, especially for the highly oscillatory function (MSE: 0.1871).
- The results confirm the *exponential efficiency gap* predicted by the theory, where deeper networks achieve a significantly better performance with fewer parameters.

Our experimental study provides comprehensive validation of Telgarsky's theoretical results:

- The **benefits of depth** are clearly demonstrated through superior performance in function approximation, especially for highly oscillatory functions.
- The limitations of shallow networks in approximating complex functions are confirmed, highlighting the exponential gap in performance between shallow and deep networks.
- The findings reinforce both major theoretical contributions from the paper and offer practical insights into the role of network depth in neural network design.

These results have significant implications for the design of deep learning models, providing empirical support for the theoretical understanding of the importance of depth in improving approximation capabilities and model performance.

7. Conclusion

Telgarsky's work makes a significant theoretical contribution to understanding the role of depth in neural networks. Through Theorem 1.1, he provides a rigorous argument that depth enhances the approximation of certain complex functions. This explains why deep architectures tend to outperform shallow networks in complex machine learning tasks. This result is particularly relevant for the field of deep learning, as it provides a solid theoretical framework to justify the use of very deep networks. This research paves the way for new perspectives on the optimization and design of neural network architectures.

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