Stationary and non-stationary sinusoidal model synthesis with phase vocoder and FFT^{-1}

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Abstract

The present document is to serve as both a technical report and a documentation for the code produced during the long project. As such the first part will explain the theoretical framework and the state-of-the-art of the field. The second part will give some insight about the code structure and the conventions that were adopted and last but not least, the third part will serve as an actual documentation and details every class, methods and attributes.

AudioGaming
NATURAL BORN INTERACTIVE

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Contents

Part I
Introduction

1 The company

For this project we work with Audiogaming. Audiogaming is a start-up company which creates audio plug-in for movies and video games. They are base in Toulouse and Paris.

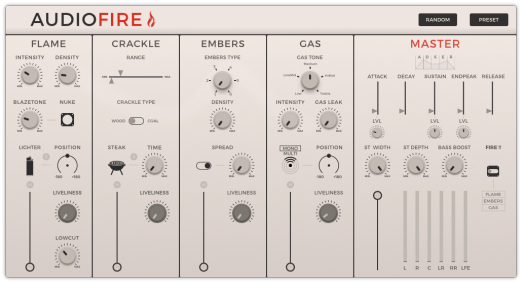


Figure 1: Audiofire: audio plug-in that recreates fire sound

2 Objective

The objective of the project is to use inverse Fourier transform to synthesize a sound. There are two parts in this process: the Analysis and the Synthesis.

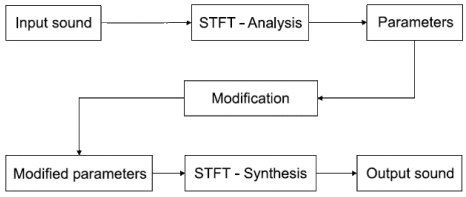


Figure 2: General approach for modifying a sound in the spectral domain

Our objectives were to focus on the synthesis. We supposed that the analysis part is done and that we know all the parameters. The work is divided into two parts : the stationary synthesis and the non-stationary synthesis.

3 Context of the project

We had 6 weeks to do this work. As a beginning, we had the work of the previous group which worked on this project in 2015. They had done an analysis estimator of sinus parameters and sinus generation with those parameters (only stationary) in Python and some research on the Non-stationary synthesis with the LUT of lobes in Matlab. We imposed to ourself an Object Oriented Programming tree structure in Python, in that way the code will be easier to use by the client and the next group which will work on this project. This structure will be described in details in this report.

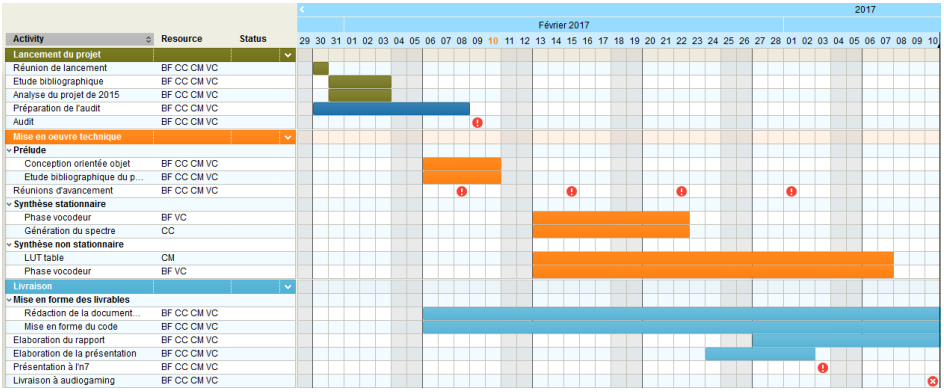
4 Work environment and project management

4.1 Work environment

We used PyCharm to program in Python, Slack to communicate with our supervisor (ask questions, send some quick results, plan a meeting...). An other important tool was GitHub, we used it to share our codes and to work all at the same time in the same project.

Figure 3: *PyCharm* as Python IDE , *Slack* to communicate, *GitHub* to stock the codes and have a versionning, *Freedcamp* to plan the project events

4.2 Project management (Gantt chart)



Part II
General analysis/synthesis approach

5 Additive Synthesis (Time Domain)

In signal processing, about as much work has been done in analysing sounds as in synthesizing them. The most common approach is to assimilate the sound to a finite sum of time varying sinusoids, which are called partials [1] added to a residual, that is, a stochastic process equivalent to a noise:

$$s(t) = \sum_{i=1}^{K} \alpha_i(t) \sin(2\pi f_i(t)t + \phi_i) + e(t) \tag{1}$$

Where the amplitude $\alpha_i(t)$ and the frequency $f_i(t)$ are smooth slowly varying functions.
Although in most cases, that is speech or musical instruments synthesis, the residual contribution is not negligible compared to the partials [1, 2] one can, in a first time, set the residual term to zero[1].
This is motivated by the fact that a great amount of research has been done towards analysing such sound models [2, 4]. The most common and obvious approach consists in adding a number of independent oscillator to reproduce the sound, each one of them being controlled independently.

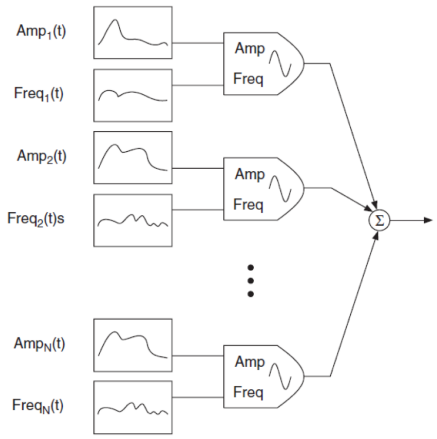


Figure 4: The additive synthesis

[1] In fact, the sinusoidal plus residual model is a direct consequence of the source/filter model [3], the synthesis of the residual term is thus more or less equivalent to filtering a white noise, which is, in its naive resolution, simpler than the sinusoids synthesis. This is why we mainly focus on the sinusoidal synthesis despite the non negligible nature of the residual term.

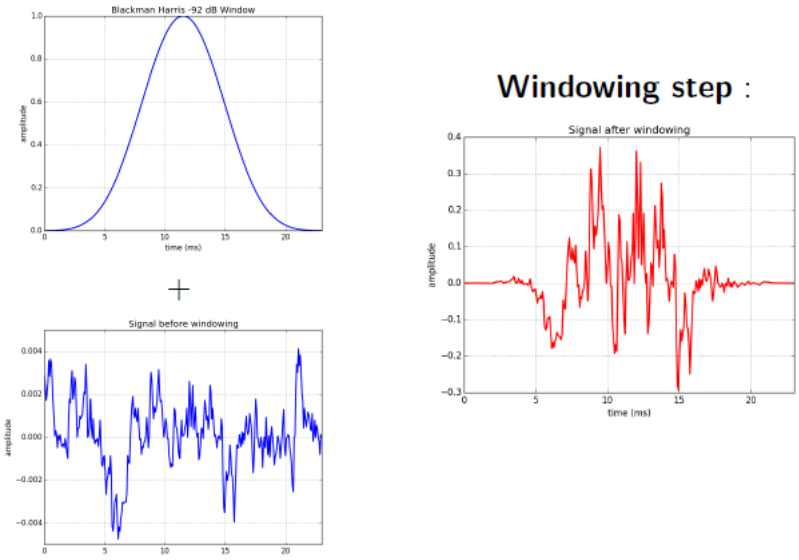


Figure 5: Windowing step

The problem of this method is that it is very costly to implement, so it is impossible to use in real-time applications. A solution would be to generate the sines in frequency domain, using the parameters given by the analysis.

6 General framework

The first step of the method is the analysis. The aim is to extract the parameters of the signal (magnitude, phase, frequency). First of all, we window the signal in order to maximize the energy in the main lobe, using a 92 dB Blackman-Harris window for instance:

Then, we apply a STFT (Short Time Fourier Transform) to extract the coefficients. We assume each peak represents a sinusoid, and we use a particular case of STFT called STPT, that detects peaks in the signal and that applies the STFT in the areas that are close to the peaks and sets the rest of the signal to zero. It is a simplest version of STFT, because we neglect the residual part of the signal by only considering the periodic part of the signal. This is less costly and works quite well, except in case of transitional signal because it is very difficult to model with a finite sum of sines.

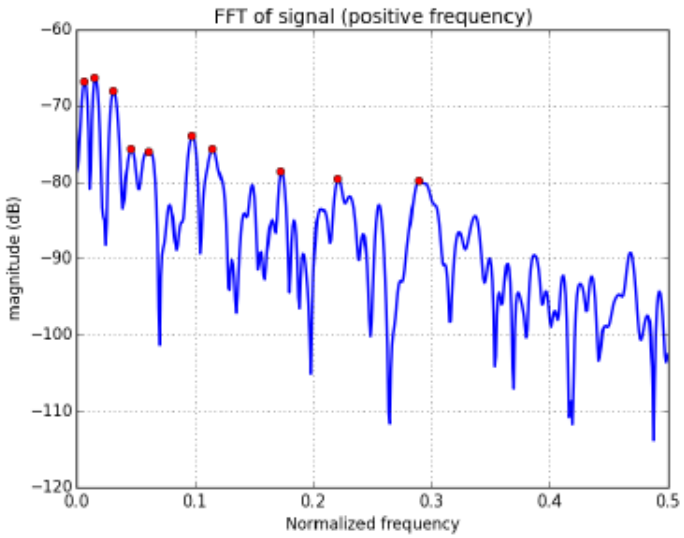


Figure 6: Peaks detection

Finally, we synthesize the new signal by FFT^{-1} and we compare it to the original signal we applied the analysis to.

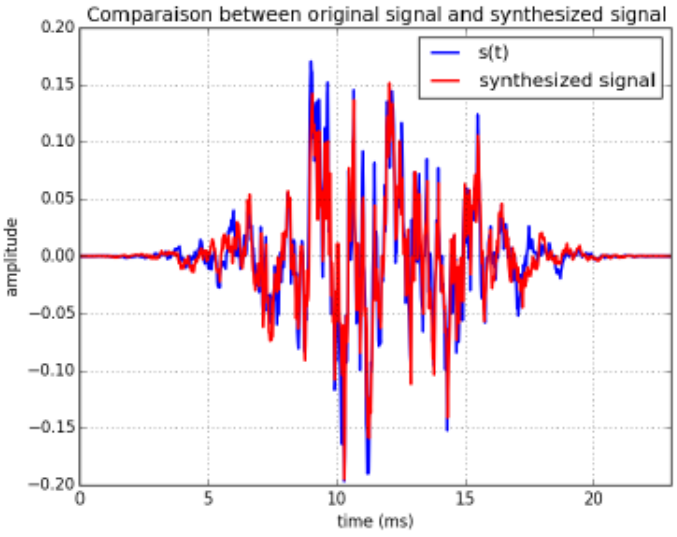


Figure 7: Synthesized frame vs Original frame

7 Phase Coherence

The phase becomes a problem when we are synthesizing a signal, even a simple sinusoid, because for each frame we create a "different" sinusoid with the same parameters. That is not what we want, we want to create a single sinusoid :

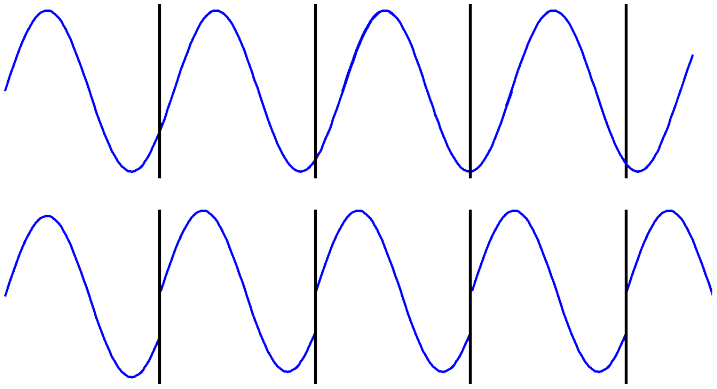


Figure 8: Coherent phase (above) vs. Incoherent phase (below) between frames

When we listen to a signal which phase is incoherent between frames (vertical jump) it will sound as "pops". This is something very unpleasant to listen to. We need to do something to make sure there is no discontinuity. Thus, correcting the phase is absolutely necessary.
In the stationary case, it is something easy to do. Knowing the exact frequency and for each frame, we advance the phase of $2\pi\tilde{f}R_a$ [2].

[2]To see details of the theoretical phase advance for the stationary case, please refer to the appendix A.2 page 27.

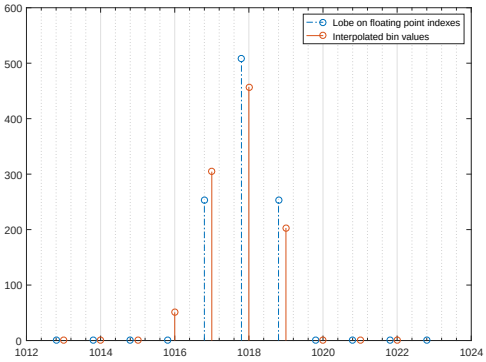


Figure 9: Interpolation of stored lobe values onto the FFT bins.

Part III
Sinusoidal model synthesis in the frequency domain

8 Stationary Case

Most stationary signals can be decomposed in a sum of partials with seldom loss in perception :

$$s(n) = \sum_{i=1}^{K} \alpha_i \sin(2\pi \tilde{f}_i n + \phi_i) \tag{2}$$

If all three parameters α_i, $\tilde{f}_i$ and ϕ_i are known for each partial, perfect reconstruction is thus possible.
It is first needed to compute the Fourier transform of a given frame. Let $w_{t_a}(n)$ be an analysis window starting at $t_a = kR_a$ where R_a is called the *hop size*. The resulting spectral frame is:

$$S_{t_a}(m) = \mathrm{DFT}[s](m) * W_{t_a}(m) \tag{3}$$

Where $W_{t_a}(m)$ is the DFT of the window w_{t_a}. Under the right assumptions on the window phase[3], the spectrum consists in main and secondary lobes (which originates directly from $W_{t_a}(m)$) convolved at the sinus amplitude, phase and frequency.
However since most of the energy is concentrated in the main lobe[4], one can omit the side lobes with little to no error, ever more so since most of the error is

[3] Which involves circular shift to cancel the phase shift [5].
[4] At the condition that w_{t_a} is not rectangular.

made on the edges of the temporal frame where the amplitude is small. In the stationary case, every main lobes are identical in shape, this means that one can synthesise a spectrum by copying a precomputed lobes onto the relevant bins of the FFT. This makes the process computationally light since only 9 points of the main lobes are sufficient [2], although our method used 11 for further precision.
Interpolation is necessary to fill the bins because the sinusoid frequency bin $\hat{f}_i$ is unlikely to fall on an integer bin number ($\hat{f}_i = \tilde{f}_i \times N$ with N the fft size) as illustrated in figure 9, this step allows for a good precision in frequency without the need for zero-padding the spectrum, thus increasing performance.
The phase is set at a common value for every bins of the lobe, on the rational that they all represents the contribution of the same sinusoid. Note that the phase used is the original phase ϕ_i corrected from the windowing effects and advanced by the time passed, to ensure phase coherence as explained in section 7, that is to say that:

$$\phi_{lobe} = \hat{\phi}_i + k \times 2\pi \tilde{f}_i R_a \tag{4}$$

Where $\hat{\phi}_i$ denote the window-corrected phase, $2\pi \tilde{f}_i R_a$ the phase advance for a single hop at frequency $\tilde{f}_i$ and k s.t $t_a = k \times R_a$.

9 Quasi-Stationary Case

9.1 Theory

9.2 phase vocoder

10 Non-Stationary Case

10.1 Theory

10.2 phase vocoder

Part IV
Results

11 Stationary Case

12 Non-Stationary Case

Part V
Code structure and conventions

13 Conventions

In this section we remind the reader of a few coding convention necessary to ensure a seamless work flow and a bug free program as much as possible. Files should contains an entire module (as described in 14.1) not just a single class to limit the number of files and ease the bug tracking. Imports in files should be kept to a minimum and left in namespaces (do not use the **from** module **import** $*$ syntax). It is preferable to import a whole module if more than three elements from the module are needed in the file, otherwise consider the **from** module **import** element1 , element2 syntax to avoid unnecessary memory flooding. If conflicts exists, notwithstanding the number of elements needed, the whole modules are to be loaded with a namespace. Namespaces may be abbreviated to the programmer's convenience however some abbreviation are to be universally respected :

(i) `numpy` should always be imported as `np`

(ii) `matplotlib.pyplot` should always be imported as `plt`

Finally math functions should always come from the numpy module and not python's math module to guarantee a universal behaviour across the program.

13.1 Naming conventions

Naming conventions are freely adapted from Python recommended conventions defined in PEP8 [6], as such :

(i) *Class* should be named in `CapitalizedWord`

(ii) *Methods* and *functions* should be named in `lower_case_with_underscores`

(iii) *Attributes* and *variables* should be names in `lower_case_with_underscores`

(iv) *Instantiation* following the fact that everything is an object in python should be named as *variables*.

Moreover during class declaration, the following principles should be adopted :

(i) Non-public methods and attributes should use one leading underscore.

(ii) Elements that conflicts with python reserved name should use one trailing underscore rather that simplification or a misspelling.

(iii) Accessors or mutators using one leading underscore should be interpreted as properties of their associated attribute. As such it should be guaranteed that they induce a low computational cost.

(iv) Non-public elements that should not be inherited or may cause conflicts during inheritance should use two leading underscore and make use of Python name-mangling.

To seamlessly manipulate both *stationary* and *non stationary* models, class that are inherited in two versions are preceded with either `Stationary` are `NonStationary` respectively.

13.2 Spectrum and sinusoids parameters classes

Because many spectra, main lobes and sinusoidal model parameters have to be traded between modules we created two classes, respectively `Spectrum` and `Parameters`. They mainly serve as containers, holding the data and returning them in a point wise fashion. This way we can prevent conflicts and errors that would come from a non-uniform data sharing protocol and as well ensure that every operation performed on either spectra or parameters are made following the same principles and algorithms.

13.2.1 Spectrum

Spectrum	
_amplitude : np.array	
_phase : np.array	
_nfft : int	
__init__(self, amplitude, phase)	
__add__(self, other)	
__iadd__(self, other)	
__mul__(self, other)	
__imul__(self, other)	
from_complex_spectrum(cls, complex_spectrum)	@classmethod
void_spectrum(cls)	@classmethod
set_spectrum(self, amplitude, phase, start_bin=None, stop_bin=None)	
set_complex_spectrum(self, complex_spectrum, start_bin=None, stop_bin=None)	
get_amplitude(self, k)	
get_phase(self, k)	
get_nfft(self)	

The Spectrum class stores a spectrum in amplitude and phase, however it may be created or changed from a complex `np.array` respectively with the class method `from_complex_spectrum` and the method `set_complex_spectrum`. Those two methods may take optional parameters start_bin and stop_bin if one need to update only a part of the spectrum, for example a single lobe. The class checks that the given data are consistent upon instantiation. The + operation as well as the + = operation have been defined between two `Spectrum` objects and between a `Spectrum` object and an array of complex numbers.

The $\times$ operation as well as the $\times =$ operation have been defined between a `Spectrum` object and an array of complex numbers. Addition and multiplication attempts between other data type will result in a `NotImplementedError` exception.

13.2.2 Parameters

Parameters
_amplitudes : np.array _frequencies : np.array _phases : np.array _number_sinuses : int
__init__(self, amplitudes, frequencies, phases) get_amplitude(self, k) get_frequency(self, k) get_phase(self, k) get_number_sinuses(self)

The Parameters class is more of a structure than a class and only contains the stationary sinusoidal model parameters and their respective accessors. It also stores the number of sinuses and checks that the given data are consistent upon instantiation.
In the stationary sinusoidal model the signal $s(t)$ is defined as follow[1] :

$$s(n) = \sum_{i=1}^{N_{sinus}} \alpha_i \sin(2\pi \tilde{f}_i n + \phi_i)$$

with $\tilde{f}_i = \frac{f_i}{f_s}$ the normalised frequency.
We then store the parameters as follows :

_amplitudes stores the α_i

_frequencies stores the $\tilde{f}_i$

_phases stores the ϕ_i

13.2.3 NonStationaryParameters

NonStationaryParameters(Parameters)
_acrs : np.array _fcrs : np.array
__init__(self, amplitudes, frequencies, phases, acrs, fcrs) get_acr(self, k) get_fcr(self, k)

The stationary sinusoidal model can be extended to the first order development

to better model fast amplitude and frequency change over time. The signal $s(t)$ can then be expressed as a sum of linearly varying chirps[1] :

$$s(n) = \sum_{i=1}^{N_s inus} (\alpha_i + \mu_i \cdot nT_s) \sin(2\pi \tilde{f}_i n + \frac{\psi_i}{2}(nT_s)^2 + \phi_i)$$

Where we define the **Amplitude Change Rate** μ and the **Frequency Change Rate** ψ.
Thus we inherit the `Parameters` class to add the two additional parameters as follow :

_acrs stores the μ_i

_fcrs stores the ψ_i

14 Code structure

14.1 General structure

14.2 Class structure

[1] Please look up section 10 page 12 for more details

Part VI
Documentation

15 Core module

15.1 Synthesizer

Synthesizer
_window_size : float _window_type : float _nfft : float _analysis_hop : float _synthesis_hop : float _current_parameters : Parameters _past_parameters : Parameters _fs : float _past_spectrum : Spectrum _current_spectrum : Spectrum
__init__(self, window_size, window_type, zero_padding_factor, analysis_hop, synthesis_hop, current_parameters, fs=None) __del__(cls) reset_synthetizer (self, window_size, window_type, zero_padding_factor, analysis_hop, synthesis_hop, current_parameters, fs) get_next_frame(self) set_next_frame(self, next_parameters) _set_window_size(self, window_size) _set_window_type(self, window_type) _set_analysis_hop(self, analysis_hop) _set_synthesis_hop(self, synthesis_hop) _set_current_parameters(self, current_parameters) _get_current_spectrum(self) _set_current_spectrum(self, new_spectrum)

15.2 StationarySynthesizer

StationarySynthesizer
_window_size : float _window_type : float _nfft : float _analysis_hop : float _synthesis_hop : float _current_parameters : Parameters _past_parameters : Parameters _fs : float _past_spectrum : Spectrum _current_spectrum : Spectrum _spectrum_generator : StationarySpectrumGenerator
__init__(self, window_size, window_type, zero_padding_factor, analysis_hop, synthesis_hop, current_parameters, fs=None) get_next_frame(self) inverse_fft(self, current_spectrum)

15.3 NonStationarySynthesizer

NonStationarySynthesizer
_window_size : float _window_type : float _nfft : float _analysis_hop : float _synthesis_hop : float _current_parameters : NonStationaryParameters _past_parameters : NonStationaryParameters _fs : float _past_spectrum : Spectrum _current_spectrum : Spectrum _spectrum_generator : StationarySpectrumGenerator
__init__(self, window_size, window_type, zero_padding_factor, analysis_hop, synthesis_hop, current_parameters, fs=None) get_next_frame(self) inverse_fft(self, current_spectrum)

16 Spectrum generation module

16.1 SpectrumGenerator

SpectrumGenerator
_parameters : Parameters _nfft : np.float _spectrum : Spectrum _window_size : np.float _analysis_hop : np.float
__init__(self, window_size, parameters, nfft, analysis_hop) _add_lobe(self, k, lobe) _set_window_size(self, window_size) _set_window_type(self, window_type) _get_parameters(self, new_parameters) _get_spectrum(self)

The aim of this class is to generate a synthetic spectrum from known parameters (amplitudes, phases, frequencies). This class is divided into two subclasses that generate stationary and non-stationary spectrums.

16.2 StationarySpectrumGenerator

StationarySpectrumGenerator (SpectrumGenerator)
_parameters : Parameters _nfft : np.float _spectrum : Spectrum _window_size : np.float _analysis_hop : np.float _lobe_generator : StationaryLobeGenerator
__init__(self, window_size, parameters, nfft, analysis_hop) _add_lobe(self, k, lobe)

16.3 NonStationarySpectrumGenerator

NonStationarySpectrumGenerator (SpectrumGenerator)
_parameters : Parameters _nfft : np.float _spectrum : Spectrum _window_size : np.float _analysis_hop : np.float _lobe_generator : NonSationaryLobeGenerator _regular_lut : np.array
__init__(self, window_size, parameters, nfft, analysis_hop) _add_lobe(self, k)

This class has not been made yet, but the goal is the same as for the StationarySpectrumGenerator.

16.4 LobeGenerator

LobeGenerator
_window_type : np.float _window_size : np.float _nfft : np.float _window : np.array _lobe : Spectrum
__init__(self, window_type, window_size, nfft) _set_window_size(self, window_size) _set_window_type(self, window_type) _gen_lobe(self) _get_lobe

This class generates a 11 points main lobe.

16.4.1 StationnaryLobeGenerator

StationaryLobeGenerator(LobeGenerator)
_window_type : np.float _window_size : np.float _nfft : np.float _window : np.array _lobe : Spectrum _gen_lobe : Spectrum
__init__(self, window_type, window_size, nfft) _gen_lobe(self)

16.4.2 NonStationaryLobeGenerator

NonStationaryLobeGenerator(LobeGenerator)
_abscisse : np.array _ordonnee : np.array _interpolated_lobe : np.array _regular_grid : np.array _domain : np.array _number_acr : np.float _number_fcr : np.float _number_points : np.float _LUT : np.array _gen_lobe : Spectrum
__init__(self, regular_grid, acr_domain, fcr_domain, number_acr, number_fcr, window_type, window_size, nfft, fs=None, method_a=None, method_p=None, method_f=None) _gen_uniform_lut(self) _gen_non_uniform_lut(self) _gen_lobes_legacy(self, i, j, acr, fcr, t, n) _gen_lobe(self) _get_lobe(self) _interpolate_lobe(self, acr, fcr, method_a=None, method_p=None, method_f = None))

The aim is to generate a LUT with a uniform or a non-uniform grid, and to generate a lobe for a given ACR/FCR couple by interpolating with existing lobes of the LUT. The user must chose the kind of interpolation he or she wants to use for magnitude, phase and frequency. Besides, he can choose the size of the LUT to build by giving the number of ACRs and FCRs.

17 Phase Vocoder module

17.1 PhaseVocoder

PhaseVocoder
_analysis_hop : int _synthesis_hop : int _omega : np.array _past_analysis_spectrum : Spectrum _past_synthesis_spectrum : Spectrum _current_analysis_spectrum : Spectrum current_synthesis_spectrum : Spectrum
__init__(self, analysis_hop, synthesis_hop, current_synthesis_spectrum) get_region(self, k) get_pv_spectrum(self, k)

This module is not used !
The PhaseVocoder file gathers both the Stationary Phase Vocoder and the Non-Stationary Phase.

17.2 StationaryPhaseVocoder

`StationaryPhaseVocoder`
`get_pv_spectrum(self)`

The Phase Vocoder algorithm is located here. It will correct the phase at each frequency bin k (remember that this module is not used at the end). To do so it proceed in 5 steps :
-Get the phase difference,
-Remove the expected phase difference,
-Map the phase shift to $[-\pi, \pi]$ range,
-Get the true frequency,
-And finally get the final phase.
For the details, please refer to [7–9] and annex .

17.3 NonStationaryPhaseVocoder

`NonStationaryPhaseVocoder`
`get_pv_spectrum(self)`

This module is not complete ! The detailed algorithm called Scaled-Phase Locking is also described in the paper [7]. The idea is to correct the phase on the peaks bin and then apply this correction on each bin of the region containing the peak. To do so, a peak trajectory detection is needed to know between two frames if the peak is the same one on the other frame, because it could have changed his frequency bins due to the non-stationary case. We also need to get the region (bins near the peak bin). And then we apply on this peak the phase vocoder and the corrected phase shift is applied on all bins of the region corresponding to this peak. A Matlab implementation can be found in [10].

Conclusion

A Phase advance and propagation along the signal

A.1 On the first attempt at Phase Vocoder use

In the stationary case, the Phase Vocoder is expressed in 3 equations as follow :

$$\Delta\Phi_k^u = \angle X(t_a^u, \Omega_k) - \angle X(t_a^{u-1}, \Omega_k) - R_a\Omega_k \tag{5}$$

$$\hat{\omega_k}(t_a^u) = \Omega_k + \frac{\Delta_p\Phi_k^u}{R_a} \tag{6}$$

$$\angle Y(t_s^u, \Omega_k) = \angle Y(t_s^{u-1}, \Omega_k) + R_s\hat{\omega_k}(t_a^u) \tag{7}$$

What (5) means is that we look for the true phase shift (the analysis phase shift) during the frame u and $u-1$ which is $\angle X(t_a^u, \Omega_k) - \angle X(t_a^{u-1}, \Omega_k)$ and we compute the error in phase shift, that is to say the difference between the *true* phase shift and the *expected* phase shift.
In (6) we use the error in phase shift to compute the deviation in frequency $\frac{\Delta_p\Phi_k^u}{R_a}$ from the expected frequency Ω_k and thus compute the *true* frequency at which the bin was excited between t_a^{u-1} and t_a^u.
Finally (7) assume the correct synthesis phase shift will be $R_s\hat{\omega_k}(t_a^u)$ that is the true frequency times the synthesis hop.

The issue with such an approach in our case is that we dropped the analysis. We want to use the Phase Vocoder to *create* phase shifts in our spectra, but the Phase Vocoder is in fact nothing but a fancy way to copy existing phase shifts while taking into account a different hop during analysis and synthesis. I will try and break down the issues I have into two cases.

A.1.1 "Pure" synthesis

We wish to synthesize a stationary or a sum of stationary sinusoid from scratch. For simplicity's sake and without loss of generality we will take the one sinusoid case. That is to say that we want, without prior knowledge to generate $s(n)$ such as :

$$s(n) = \alpha\cos(2\pi\tilde{f}n + \phi) \tag{8}$$

knowing only α, $\tilde{f}$ and ϕ.

We will also, to ease the process, assume that we know the application[5] $f_{w,\tilde{f}}(\phi) : \phi \mapsto \tilde{\phi}$ which takes into account the effect of windowing on the phase of the frame spectrum's phase[6].

[5] And this this is a very strong hypothesis in the sense that it will never be true, but this is not the core issue at stake here.

[6] For more details on the theory, please read part II page 7

The first step is then to generate a synthetic spectrum with the desired parameters. To do this we only generate a main lobe derived from the Fourier transform of the normalized window w supposedly[7] used during analysis, and place it at the right position on the spectrum. This involves to interpolate the relevant bins value if by any chance the wanted frequency $\tilde{f}$ is not exactly on a bin, that is to say if $\tilde{f} \notin \{\frac{2k\pi}{N}\}_{k=0...N-1}$. We then multiply the generated lobe by $\frac{A}{2}$ and set the lobe phase to $\tilde{\phi} + 2\pi\tilde{f}R_a$ [8]. We then wished to use the phase vocoder to advance the phase (compute the needed phase shift). To get the temporal frame, we theoretically only have to compute the inverse Fourier transform of the generated spectrum.
In order to use the Phase Vocoder we assumed the generated spectrum to be equivalent to the analysis spectrum $X(t_a^u)$ and the antecedently phase corrected spectrum to be equivalent to the past synthesis spectrum $Y(t_a^{u-1})$.
At the first iteration :

- $X(t_a^{u-1})$ is void because by hypothesis, nothing happened before.
- $X(t_a^u)$ is the freshly generated spectrum
- $Y(t_a^{u-1})$ is also void for the same reasons

Equation 5 gives, for $k \in 1...N-1$ s.t $X(t_a^u, \Omega_k)$ is a bin of the lobe a phase shift error of $\tilde{\phi}$.
Then after 6 and 7 we obtain :

$$\begin{aligned}\angle Y(t_s^u, \Omega_k) &= \angle Y(t_s^{u-1}, \Omega_k) + R_s\hat{\omega_k}(t_a^u) \\ &= \angle Y(t_s^{u-1}, \Omega_k) + \frac{R_s}{R_a}\tilde{\phi} + R_s\Omega_k\end{aligned}$$

If $R_s = R_a$ we have a perfect reconstruction of the time synthesized overlap-add test signal. However, in that case, the Phase Vocoder is perfectly irrelevant to the synthesis, indeed, since we have no *actual* analysis phase, we only need to modify R_a to change the length of the final signal.
At the following iteration, we have to update the phase of the generated spectrum to $\tilde{\phi} + 2 \times 2\pi\tilde{f}R_a$ instead of $\tilde{\phi} + 2\pi\tilde{f}R_a$ (as computed in (11)), recursively, we can define the phase of the lobe the ith generated spectrum as :

$$\begin{cases}\tilde{\phi}_i = \tilde{\phi}_{i-1} + 2\pi\tilde{f}R_a \\ \tilde{\phi}_0 = \tilde{\phi}\end{cases} \tag{9}$$

A.1.2 "Parametered" synthesis

In this case, we will not synthesize a truly stationary signal but we assume that the signal is quasi-stationary, which is to say that given a small enough analysis

[7] Because no actual analysis happened

[8] This is because we wish to generate frame spaced by R_a so we have to compensate the *expected* phase shift by hand. In fact, in the purely stationary case, the expected phase shift is the theoretical phase shift.

window, it can be considered stationary within that frame.

$$s(n) \cdot \mathbb{1}_{[t_a^{u-1}, t_a^u]} \simeq \alpha \cos(2\pi \tilde{f} n + \phi) \tag{10}$$

Note that the initial phase ϕ is constant from one frame to the other *by definition*, indeed, a change of phase is equivalent to a sweep in frequency. Also this is more of a constrain of stability on frequency than it is on amplitude because of the nature of the overlap-add process.
We can assume that under the right conditions the method developed in subsection A.1.1 above still holds, if we update the parameters $\tilde{f}$ and α in the spectrum generation.

A.2 Theoretical phases advance

A.2.1 Stationary case

For stationary signals expressed as in (8) the phase advance from one frame to the other is elementary to compute:

$$\begin{aligned} \Delta\Phi_f^{t_a} &= \Phi_f^{t_a + H_a} - \Phi_f^{t_a} \\ &= 2\pi f(t_a + H_a) + \phi - 2\pi f t_a - \phi \\ &= 2\pi f H_a = 2\pi \tilde{f} R_a \end{aligned} \tag{11}$$

Where H_a is the hop-size in seconds that is $H_a = \frac{R_a}{f_s}$.

A.2.2 Non-stationary case

We can not say much about the phase shift of non-stationary signals without making a few assumptions about the frequency modulation.
We assume that given a small enough window the non-stationary signal can be expressed as the following Taylor expansion:

$$s(t)\mathbb{1}_{[t_a^{u-1}, t_a^u]} \simeq (\alpha + \mu t)\sin(2\pi f t + \frac{\psi}{2} t^2 + \phi_a) \tag{12}$$

This is equivalent to assume that the signal is a linear chirp in the window, with f the instantaneous frequency and α the instantaneous amplitude at the start of the analysis window, and ϕ_a is the phase at the start of the window.
The phase advance is thus computed in the same way as in the stationary case:

$$\begin{aligned} \Delta\Phi_f^{t_a} &= \Phi_f^{H_a} - \Phi_f^0 \\ &= 2\pi f H_a + \frac{\psi}{2}(H_a)^2 + \phi_a - \phi_a \\ &= 2\pi f H_a + \frac{\psi}{2} H_a^2 \end{aligned} \tag{13}$$

References

[1] T. F. Quatieri and R. J. McAulay, "Audio signal processing based on sinusoidal analysis/synthesis," in *Applications of digital signal processing to audio and acoustics*. Springer, 2002, pp. 343–416.

[2] X. Rodet and P. Depalle, "Spectral envelopes and inverse fft synthesis," in *Audio Engineering Society Convention 93*. Audio Engineering Society, 1992.

[3] R. McAulay and T. Quatieri, "Speech analysis/synthesis based on a sinusoidal representation," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 34, no. 4, pp. 744–754, 1986.

[4] P. Depalle, G. Garcia, and X. Rodet, "Tracking of partials for additive sound synthesis using hidden markov models," in *Acoustics, Speech, and Signal Processing, 1993. ICASSP-93., 1993 IEEE International Conference on*, vol. 1. IEEE, 1993, pp. 225–228.

[5] D. Arfib, F. Keiler, and U. Zölzer, "Time-frequency processing," *DAFX: digital audio effects*, pp. 237–297, 2002.

[6] G. Van Rossum, B. Warsaw, and N. Coghlan, "Style guide for python code," Aug. 2013.

[7] T. Karrer, E. Lee, and J. O. Borchers, "Phavorit: A phase vocoder for real-time interactive time-stretching." in *ICMC*, 2006.

[8] D. Barry, D. Dorran, and E. Coyle, "Time and pitch scale modification: A real-time framework and tutorial," in *Conference papers*, 2008, p. 16.

[9] F. Hammer, "Time-scale modification using the phase vocoder," *Institute for Electr. Music and Ac, Graz Univ. of Dramatic Arts, Graz*, 2001.

[10] J. Grünwald, "Theory, implementation and evaluation of the digital phase vocoder in the context of audio effects," *Institute for Electr. Music and Ac, Graz Univ. of Dramatic Arts, Graz*, 2010.