# CS 241, Lecture 7 - Non-Deterministic Finite Automata

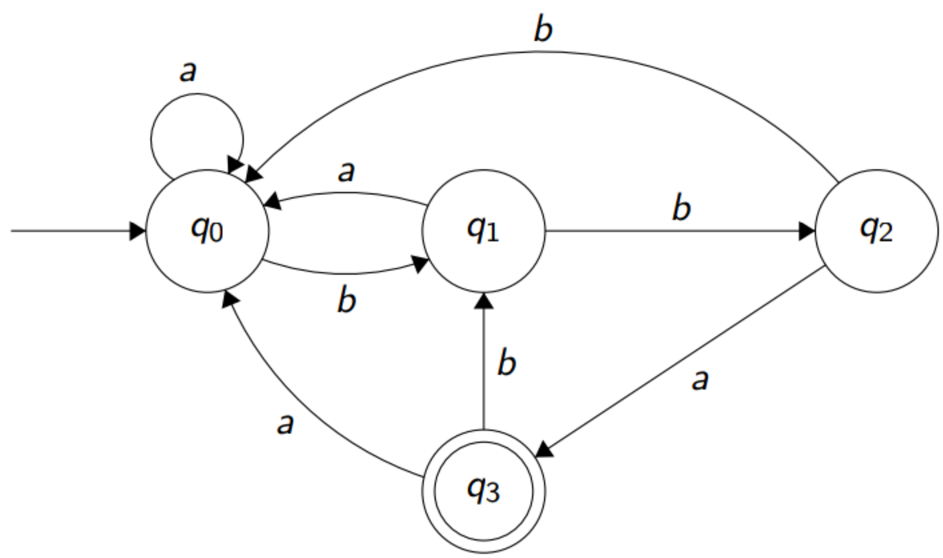
## 1 Quick review of regular languages

- For a formal language $L$, $L \cdot \varnothing$ is $\varnothing$! This is as we define a concatenation as $L_1 L_2 = \{xy : x \in L_1, y \in L_2\}$. If the set is empty, then nothing happens!

## 2 DFAs - cont.

- Warmup:

$L = \{w : w \text{ ends with } bba\}$




- We extend our definition of $\delta : (Q \times \sigma) \to Q$ to a fn defined over $(Q \times \sigma^*)$ via:

$$\begin{aligned} \delta : (Q \times \sigma^*) &\to Q \\ (q, \epsilon) &\mapsto q \\ (q, aw) &\mapsto \delta^*(\delta(q, a), w) \end{aligned}$$

where $a \in \sigma$ and $w \in \sigma^*$. $aw$ is the concatenation.

- A DFA given by $M = (\sigma, Q, q_0, A, \delta)$ **accepts a string** $w$ iff $\delta^*(q_0, w) \in A$.

- For example:

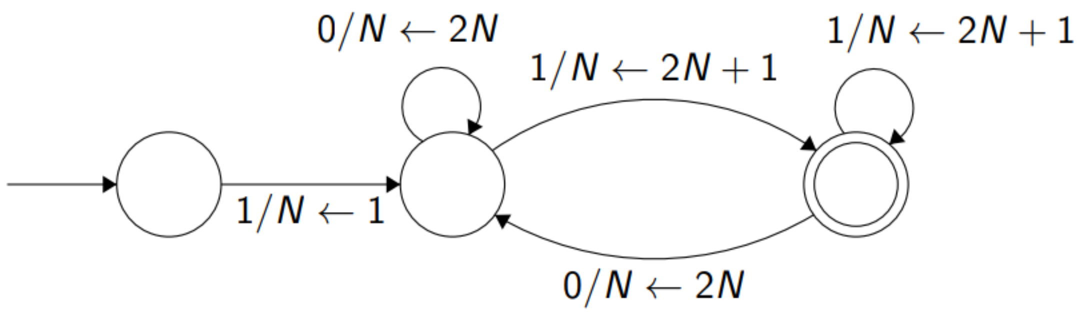
$$\begin{aligned}
\delta^*(q_0, abba) &= \delta^*(\delta(q_0, a), bba) \\
&= \delta^*(q_0, bba) \\
&= \delta^*(q_1, ba) \\
&= \delta^*(q_2, a) \\
&= \delta^*(q_3, \epsilon) \\
&= q_3
\end{aligned}$$

- Essentially, we define $\delta^*$ to be just $\delta$ but now, it supports more than one "character", allowing us to traverse the function.
- We define **the language of a DFA**, $M$, to be the set of all strings accepted by $M$, that is, $L(M) = \{w : M \text{ accepts } w\}$.
- **Kleene's Theorem:** $L$ is regular iff $L = L(M)$ for some DFA $M$. That is, the regular languages are precisely the languages that are accepted by DFAs.
- **Implementing a DFA:**

```
s = q_0
while not EOF do
    read character ch
    switch(s)
    case q_0:
        switch(ch)
        case ch = a_0:
            s = new_state_a_0
        case ch = a_1:
            s = new_state_a_1

        ...

        case ch = a_|σ|:
            s = new_state_a_sigma
        end switch
    case q_1:

        ...

    end switch
end while
```

- Alternatively, we could also use a LUT to store the appropriate states based on $a_x$ and $q_y$.

- We can extend our DFAs to attach actions to arcs.
- For example, consider $L = \{\text{binary numbers without leading zeros}\}$. We could create a DFA where we also compute the value of the number at the same time, then print the token.
- The regular language would be $1(0|1)^*1$.



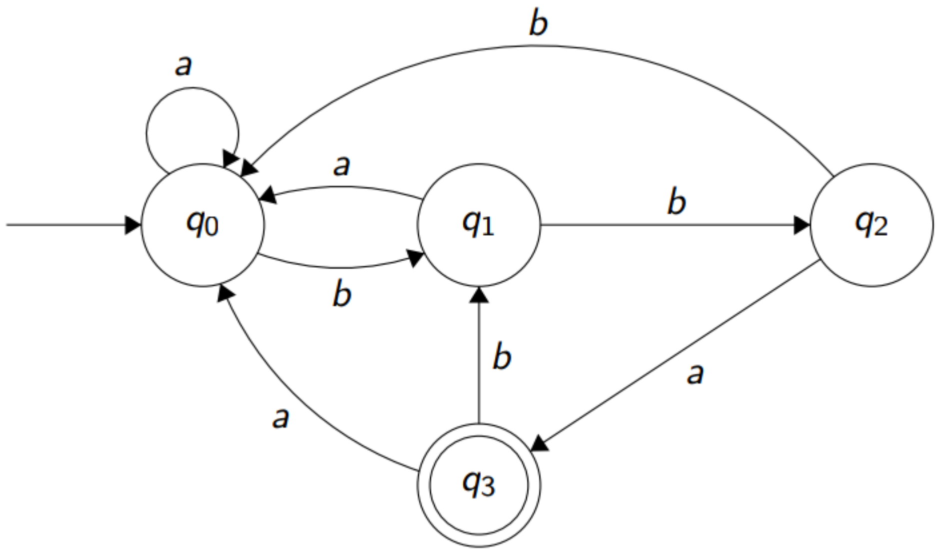

- For example, start with 101001:
  - 1 $\rightarrow$ N = 1
  - 10 $\rightarrow$ N = 2
  - 101 $\rightarrow$ N = 2 * 2 + 1 = 5
  - 1010 $\rightarrow$ N = 10
  - 10100 $\rightarrow$ N = 20
  - 101001 $\rightarrow$ N = 41

# 3 Non-deterministic Finite Automata (NFAs)

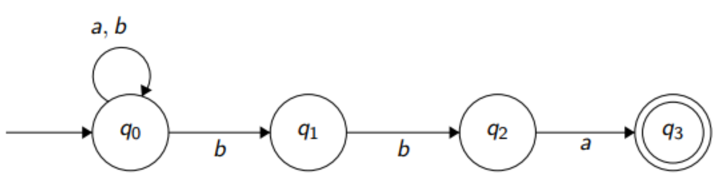
- What if we are allowed more than one transition from a state? That is, from, say, $q_0$, I could go to $q_1$ OR $q_2$ based solely on the value of, say, $a$?
- When we allow a state to thave multiple branches based on the same input, we say the machine *chooses* what path to go on - this is called **non-determinism**.
- We then say the machine accepts a word $w$ iff there exists **some** path that leads to an accepting state.

- We can simplify our warmup example by using an NFA:

$L = \{w : w \text{ ends with } bba\}$




$L = \{w : w \text{ ends with } bba\}$




- This makes it very easy to extend - if, for example, we wanted to extend this up to z, we can just say that the opening loop goes from a to z.
- An **NFA** is still a 5-tuple. Nothing changes, except $\delta$.
- $\delta$ is now defined as $\delta : (Q \times \sigma) \to 2^Q$, which is our total transition function.
- $2^Q$ denotes the **power set** of $Q$, that is, the set of all subsets of $Q$. This allows us to go to multiple states at once.
- For example, if our language is just $\{1, 2, 3\}$, then $2^Q = \{\{\}, \{1\}, \{2\}, \dots, \{2, 3\}, \{1, 2, 3\}\}$.
- Let $M$ be an NFA. We say $M$ **accepts** $w$ iff there exists some path through $M$ that leads to an accepting state.
- We denote the **language of an NFA** $M$ to be the set of all strings accepted by $M$, that is, $L(M) = \{w : M \text{ accepts } w\}$.

- We extend $\delta$ for an NFA:

$$\begin{aligned} \delta^* : (2^Q \times \sigma^*) &\to 2^Q \\ (S, \epsilon) &\mapsto S \\ (S, aw) &\mapsto \sigma^*(\bigcup_{q \in S} \sigma(q, a), w) \end{aligned}$$

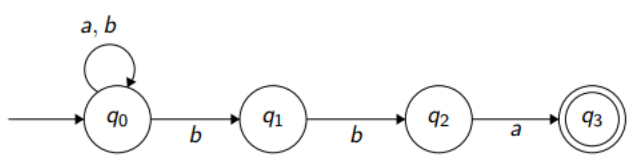
  where $a \in \sigma$.

- In other words, an NFA given by $M = (\sigma, Q, q_0, A, \delta)$ **accepts a string** $w$ iff $\delta^*(\{q_0\}, w) \cap A \neq \varnothing$.

- To simulate an NFA:

  S = $\{q_0\}$<br>
  while not EOF do:<br>
      c = read_char()<br>
      S = $\bigcup_{q \in S} \delta(q, c)$<br>
  end while<br>
  if $S \cap A \neq \varnothing$ then<br>
      Accept<br>
  else<br>
      Reject<br>
  end if

- Let us try simulating our warmup example with a NFA:




| Processed | Remaining | $S$ |
|---|---|---|
| $\epsilon$ | $abbba$ | $\{q_0\}$ |
| $a$ | $bbba$ | $\{q_0\}$ |
| $ab$ | $bba$ | $\{q_0, q_1\}$ |
| $abb$ | $ba$ | $\{q_0, q_1, q_2\}$ |
| $abbb$ | $a$ | $\{q_0, q_1, q_2\}$ |
| $abbba$ | $\epsilon$ | $\{q_0, q_3\}$ |

Since $\{q_0, q_3\} \cap \{q_3\} \neq \emptyset$, accept.