On Pseudovarieties of Finite Algebras

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Abstract

In 1976, Eilenberg & Schützenberger proved that a family of finite monoids is closed under the operations of passing to submonoids and homomorphic images and also under finite direct products if and only if each monoid in the family models all but finitely many of some sequence of identities [2]. In this paper we generalize this result to families of finite algebras in a finite signature. In 1988, Sapir proved that if $\mathbf{A}$ is a finite semigroup such that the generated pseudovariety $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based, then the generated variety $\mathcal{V}(\mathbf{A})$ must also be finitely based [3]. It follows from Sapir's proof that if $\mathbf{A}$ is a finite semigroup such that $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based and $\mathcal{V}(\mathbf{A})$ is not finitely based, then $\mathcal{V}(\mathbf{A})$ must be inherently non-finitely based, but not inherently non-finitely based in the finite sense. In this paper we generalize this result to finite algebras in a finite signature as well.

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Contents

1 Introduction **4**

2 Preliminaries **6**

2.1 Algebras and Signatures for Algebras 6
2.2 Terms and Identities 6
2.3 Homomorphic Images, Subalgebras, and Product Algebras 7
2.4 Kernels, Congruences, and Quotients 8
2.5 Homomorphism and Isomorphism Theorems 9
2.6 Finitely generated Algebras and Congruences 9
2.7 Models of Identities 10

3 An Example of a Pseudovariety **10**

4 Models of Which Identities? **14**

5 Free Algebra Construction **16**

6 Free Algebra Properties **18**

7 A Correspondence for Pseudovarieties of Finite Algebras **23**

8 Locally Finite Varieties and Pseudovarieties **27**

9 Finitely Based Varieties and Pseudovarieties **31**

10 Finitely Generated Varieties and Pseudovarieties **34**

1 Introduction

Given two groups $\mathbf{G}, \mathbf{H}$, one might ask if $\mathbf{H}$ is the homomorphic image of $\mathbf{G}$. We can look at it a different way. Suppose $\mathbf{G}$ is a group, and $\mathbf{H}$ is an algebra in the signature of groups, and suppose there exists a surjective function $h : G \to H$ that behaves like a homomorphism. Then $\mathbf{H}$ will turn out to be a group. This is because homomorphisms preserve group axioms, and the proof of this generalizes to homomorphic images of algebras with any set of identities:

Theorem 1. *Consider algebras $\mathbf{G}, \mathbf{H}$ in the same signature, and suppose there exists a surjective homomorphism $\mathbf{G} \to \mathbf{H}$. Then for any set of identities Σ,*

$$\mathbf{G} \models \Sigma \implies \mathbf{H} \models \Sigma.$$

Likewise, it turns out that just as passing to subgroups and passing to arbitrary direct products preserves group identities, passing to subalgebras and passing to arbitrary direct products preserves any set of identities:

Theorem 2. *Consider algebras $\mathbf{G}, \mathbf{H}$ in the same signature such that $\mathbf{H} \leq \mathbf{G}$. Then for any set of identities Σ,*

$$\mathbf{G} \models \Sigma \implies \mathbf{H} \models \Sigma$$

Theorem 3. *Let $\langle \mathbf{G}_i : i \in I \rangle$ be an indexed family of algebras in the signature, and let Σ be any set of identities. Then,*

$$(\forall i \in I,\ \mathbf{G}_i \models \Sigma) \implies \prod_{i \in I} \mathbf{G}_i \models \Sigma.$$

It follows that for any set of identities Σ, the family of its models $\mathbf{Mod}(\Sigma)$ is closed under the operations of passing to homomorphic images, subalgebras, and passing to arbitrary direct products. It turns out that this correspondence goes both ways.

Theorem 4. *[1] Consider a family of algebras K in the signature σ. There exists a set of identities Σ in the signature σ over a countably infinite set of variables for which $K = Mod(\Sigma)$ iff K is closed under the operations of passing to homomorphic images, subalgebras, and product algebras.*

In 1976, Eilenberg & Schützenberger prove a similar correspondence for families of finite monoids.

Theorem 5. *[2] Consider a family of finite monoids K. Then there exists a sequence of identities such that K is the class of finite models of all but finitely many of the identities iff K is closed under the operations of passing to homomorphic images, submonoids, and finite products.*

A class of finite monoids (or finite algebras) closed under homomorphic images, subalgebras and finite products is called a **pseudovariety**, and a class of finite monoids (or finite algebras) is said to be **ultimately defined by a sequence of identities** if it is the class of all finite monoids (or finite algebras) satisfying all but finitely many of the identities. Eilenberg & Schützenberger mention in their paper that the proof of Theorem 5 can be generalized to families of finite algebras, which this paper presents:

Theorem 6. *[Generalization of Eilenberg & Schützenberger] Consider a family of finite algebras K in a common signature. Then there exists a sequence of identities such that K is the class of finite models of all but finitely many of the identities iff K is closed under the operations of passing to homomorphic images, subalgebras, and finite product algebras.*

In the same paper, Eilenberg & Schützenberger consider a finite monoid $\mathbf{A}$, and observe that if the generated variety of monoids $\mathcal{V}(\mathbf{A})$ is finitely based, then the generated pseudovariety of monoids $\mathcal{V}_{fin}(\mathbf{A})$ must also be finitely based. They then pose the following open question:

> If $\mathbf{A}$ is a finite monoid such that the pseudovariety of monoids $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based, must the generated variety of monoids $\mathcal{V}(\mathbf{A})$ also be finitely based?

In 1988, Sapir proved that if $\mathbf{A}$ is a finite semigroup such that the generated pseudovariety $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based, then the generated variety $\mathcal{V}(\mathbf{A})$ must also be finitely based [3]. It follows from Sapir's proof that if $\mathbf{A}$ is a finite semigroup such that $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based and $\mathcal{V}(\mathbf{A})$ is not finitely based, then $\mathcal{V}(\mathbf{A})$ must be inherently non-finitely based, but not inherently non-finitely based in the finite sense. In this paper we generalize this result to finite algebras in a finite signature as well:

Theorem 7. *[Generalization of Sapir] If $\mathbf{A}$ is a finite algebra in a finite signature such that the generated pseudovariety $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based and the generated variety $\mathcal{V}(\mathbf{A})$ is not finitely based, then $\mathcal{V}(\mathbf{A})$ must be inherently non-finitely based, but not inherently non-finitely based in the finite sense.*

In the following section 2, we carefully define all the required notions and state results from universal algebra that we will use without proof. In section 3, we give an example of a pseudovariety that motivates and illustrates Theorem 6. In section 4, we give the particular identities needed to prove Theorems 4 and 6. In sections 5 and 6, we develop the theory of free algebras needed to prove Theorems 4 and 6. In section 7, we finish the proof of Theorem 6, along with some minor modifications which allow us to prove Theorem 4 along with a few more facts about generated varieties and finitely generated pseudovarieties. In section 8, we motivate and explore some properties of locally finite varieties and pseudovarieties. In section 9, we introduce the definitions needed to prove Theorem 7. Finally, in section 10, we prove Theorem 7, along with a few more facts that allow us to state Theorem 7 a different way.

2 Preliminaries

2.1 Algebras and Signatures for Algebras

Definition 1. *A* ***signature for algebras*** *is a pair $\sigma = (\mathcal{F}, \rho)$ where $\mathcal{F}$ is a set of function symbols, and where $\rho : \mathcal{F} \to \mathbb{N}_{\geq 0}$ is the arity function.*
In this paper, we assume $\mathcal{F}$ is finite.
We denote the set of constant symbols: $\mathcal{F}_0 = \{f \in \mathcal{F} \mid \rho(f) = 0\}$

Definition 2. *An* ***algebra*** *in the signature σ is a pair $\mathbf{A} = (A, F)$ where the* ***universe*** *A is a non-empty set, and the* ***basic operations*** *$F = \langle f^{\mathbf{A}} : A^{\rho(f)} \to A \mid f \in \mathcal{F}\rangle$ is an indexed family of functions.*
Each function symbol $f \in \mathcal{F}$ represents a function $f^{\mathbf{A}} \in F$ of arity $\rho(f)$.
The algebra is ***finite*** *if its universe is finite.*
We define $\mathcal{K}_\sigma$ to be the family of all algebras in the signature σ.
Unless otherwise stated, in this paper we assume $\sigma = (\mathcal{F}, \rho)$.

2.2 Terms and Identities

Definition 3. *Let $\sigma = (\mathcal{F}, \rho)$ be a signature, and let X be a set of* ***variables*** *disjoint from $\mathcal{F}$. We define $T_\sigma(X)$,* ***the set of terms over*** *X, to be the smallest set of finite length strings over $X \cup \mathcal{F}$ satisfying:*

1. *$X \cup \mathcal{F}_0 \subseteq T_\sigma(X)$;*
2. *If $f \in \mathcal{F} \backslash \mathcal{F}_0$ with $\rho(f) = k$, then*

$$s_1, \ldots, s_k \in T_\sigma(X) \implies f s_1 \ldots s_k \in T_\sigma(X).$$

Definition 4. *Let $t \in T_\sigma(X)$ be a term, and let $(x_1, \ldots, x_n)$ be a tuple of distinct variables from X such that the variables appearing in t are in the set $\{x_1, \ldots, x_n\} \subseteq X$, and $\mathbf{A} \in \mathcal{K}_\sigma$. We define an n-ary* ***term operation***

$$t^{\mathbf{A}} : A^n \to A$$

by induction on $|t|$ as follows.

1. *If $t \in X$, then $t = x_i$ for some (unique) $i = 1, \ldots, n$. We then define*

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = a_i.$$

2. *If $t = c \in \mathcal{F}_0$, then define*

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = c^{\mathbf{A}}.$$

3. *Suppose $t = fs_1 \ldots s_k$. Observe that the variables appearing in each s_i are elements of $\{x_1, \ldots, x_n\}$ and $s_i < |t|$. So, we may recursively define*

$$t^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{A}}(s_1^{\mathbf{A}}(a_1, \ldots, a_n), \ldots, s_k^{\mathbf{A}}(a_1, \ldots, a_n)).$$

Definition 5. *For any algebra $\mathbf{A}$ in the signature σ, if $s^{\mathbf{A}} = t^{\mathbf{A}}$ as term operations, we say the identity $s \approx t$ **is true** in the algebra $\mathbf{A}$ and we write*

$$\mathbf{A} \vDash s \approx t.$$

Definition 6. *Let $\sigma = (\mathcal{F}, \rho)$ be a signature and X be a set such that $X \cup \mathcal{F}_0 \neq \emptyset$. We define $\mathbf{T}_\sigma(X) \in \mathcal{K}_\sigma$, the **term algebra** in the signature σ over X, to be the algebra with universe $T_\sigma(X)$ and basic operations*

- $\forall c \in \mathcal{F}_0$
$$c^{\mathbf{T}_\sigma(X)} := c,$$
- *and $\forall f \in \mathcal{F} \backslash \mathcal{F}_0$ with $\rho(f) = k$,*
$$\forall s_1, \ldots, s_k \in T_\sigma(X) \qquad f^{\mathbf{T}_\sigma(X)}(s_1, \ldots, s_k) := fs_1 \ldots s_k.$$

Theorem 8. *Let X be a set of variables and let $\sigma = (\mathcal{F}, \rho)$ be a signature such that $X \cup \mathcal{F}_0 \neq \emptyset$. Let $t \in T_\sigma(X)$ be a term such that the variables appearing in t are in the set $\{x_1, \ldots, x_n\} \subseteq X$. Then,*

$$t = t^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n),$$

where the left hand side is the term $t \in T_\sigma(X)$ and $t^{\mathbf{T}_\sigma(X)}$ on the right hand side is the term operation, which takes as input the tuple $(x_1, \ldots, x_n)$.

Theorem 9 (Substitution Lemma)**.** *Let $t \in T_\sigma(Y)$ be a term where the variables that appear in t are in the set $\{y_1, \ldots, y_n\} \subseteq Y$. Let Z be a set of variables containing distinct variables $z_1, \ldots, z_n$ and let $t' \in T_\sigma(Z)$ be the term obtained by substituting the instances of y_i with z_i in t. Then for all $\mathbf{A} \in \mathcal{K}_\sigma$,*

$$t^{\mathbf{A}} = (t')^{\mathbf{A}}.$$

2.3 Homomorphic Images, Subalgebras, and Product Algebras

Definition 7. *Let $\mathbf{A}, \mathbf{B}$ be algebras in the same signature $\sigma = (\mathcal{F}, \rho)$. Then a function $h : A \to B$ is a **homomorphism** $\mathbf{A} \to \mathbf{B}$ if for any $f \in \mathcal{F}$ (say $\rho(f) = n$) and for any $a_1, \ldots, a_n \in A$,*

$$h \circ f^{\mathbf{A}}(a_1, \ldots, a_n) = f^{\mathbf{B}}(h(a_1), \ldots, h(a_n)).$$

Definition 8. *Let $\mathbf{A} = (A, F)$ be an algebra in the signature $\sigma = (\mathcal{F}, \rho)$ where $F = \langle f^{\mathbf{A}} : f \in \mathcal{F}\rangle$. Then an algebra $\mathbf{B} = (B, G)$ in the same signature is a **subalgebra** of $\mathbf{A}$ if $B \subseteq A$, and $\forall f \in \mathcal{F}$,*

$$f^{\mathbf{B}} = f^{\mathbf{A}} \restriction_{B^{\rho(f)}} .$$

If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, we write $\mathbf{B} \leq \mathbf{A}$.

Definition 9. *Let $\langle \mathbf{A}_i : i \in I\rangle$ be an indexed family of algebras in the signature $(\mathcal{F}, \rho)$. The **direct product** of this family is the algebra $\prod_{i\in I} \mathbf{A}_i$ in the same signature whose universe is $\prod_{i\in I} A_i$. This is the set of all functions α defined on I such that $\alpha(i) \in A_i$ for every $i \in I$ and whose operations are defined coordinatewise: if $f \in \mathcal{F}$ is n-ary and $\alpha_1, \ldots, \alpha_n \in \prod_{i\in I} A_i$, then*

$$f^{\prod_{i\in I} \mathbf{A}_i}(\alpha_1, \ldots, \alpha_n) : I \to \cup_{i\in I} A_i$$

is defined by

$$i \mapsto f^{\mathbf{A}_i}(\alpha_1(i), \ldots, \alpha_n(i)) \in A_i.$$

Definition 10. *Let $K \subseteq \mathcal{K}_\sigma$. We define*

$$\begin{aligned} H(K) &:= \textit{homomorphic images of algebras in } K; \\ S(K) &:= \textit{subalgebras of algebras in } K; \\ P(K) &:= \textit{arbitrary direct products of algebras in } K; \\ P_{fin}(K) &:= \textit{finite direct products of algebras in } K. \end{aligned}$$

Definition 11. *Let $K \subseteq \mathcal{K}_\sigma$. We say K is a **variety** if $H(K) \subseteq K$ and $S(K) \subseteq K$ and $P(K) \subseteq K$.*

Definition 12. *Let $K \subseteq \mathcal{K}_\sigma$ be a family of finite algebras. We say K is a **pseudovariety** if $H(K) \subseteq K$ and $S(K) \subseteq K$ and $P_{fin}(K) \subseteq K$.*

2.4 Kernels, Congruences, and Quotients

Definition 13. *If h is a function defined on A, the **kernel** of h is defined*

$$\ker(h) := \{(a, b) \in A^2 \mid h(a) = h(b)\}.$$

Definition 14. *If $\mathbf{A}$ is an algebra in the signature $\sigma = (\mathcal{F}, \rho)$ and θ an equivalence relation on A, θ is a **congruence** on $\mathbf{A}$ if for every $f \in \mathcal{F}$ with $n = \rho(f)$ and $\forall a_1, b_1, \ldots, a_n, b_n \in A$,*

$$(a_1, b_1), \ldots, (a_n, b_n) \in \theta \implies (f^{\mathbf{A}}(a_1, \ldots, a_n), f^{\mathbf{A}}(b_1, \ldots, b_n)) \in \theta.$$

Theorem 10. *If $h : \mathbf{A} \to \mathbf{B}$ is a homomorphism, then $\ker(h)$ is a congruence on $\mathbf{A}$.*

Definition 15. *If θ is a congruence on an algebra $\mathbf{A} \in \mathcal{K}_\sigma$, then the **quotient algebra** $\mathbf{A}/\theta \in \mathcal{K}_\sigma$ is the algebra with universe A/θ and operations given by*

$$f^{\mathbf{A}/\theta}(a_1/\theta, \ldots, a_n/\theta) := f^{\mathbf{A}}(a_1, \ldots, a_n)/\theta.$$

Theorem 11. *Let θ be a congruence on an algebra $\mathbf{A} \in \mathcal{K}_\sigma$. The natural map $\nu_\theta : A \to A/\theta$ defined by $a \mapsto a/\theta$ is a surjective homomorphism from $\mathbf{A}$ to $\mathbf{A}/\theta$.*

2.5 Homomorphism and Isomorphism Theorems

Theorem 12 (Homomorphism Theorem)**.** *Suppose $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ are homomorphisms where g is surjective and $\ker(g) \subseteq \ker(f)$. Then there exists a unique map $\bar{f} : C \to B$ such that $\bar{f} \circ g = f$.*
Also, $\bar{f} : \mathbf{C} \to \mathbf{B}$ is a homomorphism.
Also, if f is surjective and $\ker(g) = \ker(f)$, then $\bar{f}$ is an isomorphism.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, two heads, "g"] & B \\ C \arrow[ur, dotted, "\bar{f}"'] & \end{tikzcd}$$

Corollary 1 (First Isomorphism Theorem)**.** *If $h : \mathbf{A} \to \mathbf{B}$ is a surjective homomorphism, then $\mathbf{B} \cong \mathbf{A}/\ker(h)$.*

$$\begin{tikzcd} \mathbf{A} \arrow[r, two heads, "h"] \arrow[d, two heads, "\nu"] & \mathbf{B} \\ \mathbf{A}/\ker(h) \arrow[ur, dotted, two heads, "\cong"'] & \end{tikzcd}$$

2.6 Finitely generated Algebras and Congruences

Definition 16. *For any algebra $\mathbf{A}$ and any subset of the universe $X \subseteq A$, the subuniverse generated by X is defined as*

$$Sg(X) := \bigcap\{B : X \subseteq B,\ \mathbf{B} \leq \mathbf{A}\}.$$

*The algebra with universe $Sg(X)$ whose operations are defined by the operations on $\mathbf{A}$ restricted to $Sg(X)$ is **the subalgebra generated by** X denoted $\mathbf{Sg}(X)$. If there exists a finite subset $X \subseteq_{fin} A$ such that $A = Sg(X)$, we say $\mathbf{A}$ is **finitely generated**. If there exists an n-element set $X \subseteq_{fin} A$ such that $A = Sg(X)$, we say $\mathbf{A}$ is n**-generated**.*

Theorem 13. *If $\mathbf{A}$ is an algebra, and $X = \{a_1, \ldots, a_n\} \subseteq A$ and Y is a set of n variables,*

$$Sg(X) = \{t^{\mathbf{A}}(a_1, \ldots, a_n) \mid t \in T_\sigma(Y)\}.$$

Definition 17. *For any algebra* $\mathbf{A}$*, let* $\mathbf{Con}\ \mathbf{A}$ *denote the set of congruences on* $\mathbf{A}$*. For any binary relation* $W \subseteq A^2$*, we call the smallest congruence that contains* W *the **congruence generated by*** W*.*

$$\theta_W := \bigcap \{\theta \in \mathbf{Con}\ \mathbf{A} \mid W \subseteq \theta\}.$$

If $\theta = \theta_W$ for some finite binary relation $W \subseteq A^2$, we say θ is **finitely generated**. If θ has finitely many congruence classes (i.e. equivalence classes), we say θ has **finite index**.

2.7 Models of Identities

Definition 18. *For a set of identities* Σ *and an algebra* $\mathbf{A}$ *in the same signature* σ*, if all the identities in* Σ *are true in* $\mathbf{A}$*, we write*

$$\mathbf{A} \vDash \Sigma,$$

and for a family of algebras $K \subseteq \mathcal{K}_\sigma$*, if* $\mathbf{A} \vDash \Sigma$ *for all* $\mathbf{A} \in K$*, we write*

$$K \vDash \Sigma,$$

and we define the family of models of Σ *as*

$$\mathbf{Mod}(\Sigma) := \{\mathbf{A} \in \mathcal{K}_\sigma \mid \mathbf{A} \vDash \Sigma\}.$$

And the family of finite models of Σ *as*

$$\mathbf{Mod}_{fin}(\Sigma) := \{\mathbf{A} \in \mathcal{K}_\sigma \mid |A| < \omega \text{ and } \mathbf{A} \vDash \Sigma\}.$$

Definition 19. *For a sequence of identities* $\{s_i \approx t_i\}_{i \in \mathbb{N}}$*, we define the family **ultimately defined** by this sequence of identities to be the finite models of all but finitely many identities in the sequence:*

$$\bigcup_{n \in \mathbb{N}} \mathbf{Mod}_{fin}(\{s_i \approx t_i \mid i \geq n\})$$

3 An Example of a Pseudovariety

Perhaps the most obvious examples of pseudovarieties are the family of finite groups, or the family of finite rings.

Theorem 14. *Indeed, the finite models in any variety is a pseudovariety.*

Proof. Fix any variety V, and let V_{fin} denote the finite models in V. Fix any algebras $\mathbf{A}, \mathbf{B} \in V_{fin}$. As V is closed under arbitrary products, $\mathbf{A} \times \mathbf{B} \in V$. As $\mathbf{A} \times \mathbf{B}$ is finite, $\mathbf{A} \times \mathbf{B} \in V_{fin}$. This proves that V_{fin} is closed under the operation of passing to finite product algebras. The rest of the proof is similar. ∎

Definition 20. *A pseudovariety V is* ***equational*** *if there exists a variety K such that*

$$V = \{\mathbf{A} \in K \mid |A| < \omega\}.$$

However, we cannot generalize Theorem 4 in the following way:

Theorem 15 (An incorrect correspondence)**.** *Consider a family of finite algebras K in the signature σ. There exists a set of identities Σ in the signature σ over a countably infinite set of variables for which $K = Mod_{fin}(\Sigma)$ iff K is closed under the operations of H, S, P_{fin}.*

Here is an example of a non-equational pseudovariety to demonstrate this.

Definition 21. *Let $\mathcal{M}_{fin}$ denote the class of all finite groups in the signature of monoids. That is, for any finite monoid $\mathbf{M}$,*

$$\mathbf{M} \in \mathcal{M}_{fin} \iff \forall x \in M \exists y \in M \ s.t. \ x \cdot^{\mathbf{M}} y = y \cdot^{\mathbf{M}} x = e^{\mathbf{M}}.$$

While the family of finite groups forms an equational pseudovariety, the identities that characterize the existence of inverse elements relies on the inverse operation.

Theorem 16. *$\mathcal{M}_{fin}$ is closed under H, S, P_{fin} and is hence a pseudovariety, but is not equational.*

Proof. In this proof only we use $+$ for the binary monoid operation. We also use relaxed notation, and write $a+b$ in place of $+ab$ and $a+b+c$ in place of $((a+b)+c)$ and $(a+(b+c))$. We also use $\cdot$ to denote repeated use of the binary monoid operation. That is,

1. We write $n \cdot x$ in place of $x + \cdots + x$ where $+$ appears $n-1$ times in $x + \cdots + x$,

2. and $0 \cdot x = e$, the identity element of the monoid.

Closure under H, P_{fin} are straightforward. To prove closure under passing to subalgebras, suppose $\mathbf{A} \in \mathcal{M}_{fin}$ and fix any subalgebra $\mathbf{B} \leq \mathbf{A}$. As $\mathbf{A}$ is a monoid, $\mathbf{B}$ is a submonoid of $\mathbf{A}$. Fix any $x \in B$. Consider the submonoid of $\mathbf{B}$ generated by x:

$$\{k \cdot x \mid k \in \mathbb{N}_{\geq 0}\}.$$

This is a subalgebra of a finite algebra and is hence finite, so for some positive integers $l < m$, we have that $l \cdot x = (l+m) \cdot x$. As $\mathbf{A}$ is a group, we have the cancellation law hence $m \cdot x = e$. Then $x + (m-1) \cdot x = (m-1) \cdot x + x = e$, so $\mathbf{B}$ is also a group.
To prove that $\mathcal{M}_{fin}$ is non-equational, suppose for contradiction $\mathcal{M}_{fin} = \mathbf{Mod}_{fin}(\Sigma)$.
For each $n \in \mathbb{N}$ let $\mathbf{Z}_n$ denote the finite cyclic group of order n considered as a monoid, and consider

$$\mathbf{N} := Sg(1, 1, \dots) \leq \prod_{n \in \mathbb{N}_{\geq 2}} \mathbf{Z}_n.$$

The elements of the product algebra $\prod_{n\in\mathbb{N}_{\geq 2}} \mathbf{Z}_n$ are sequences whose i^{th} co-ordinate are elements of $\mathbf{Z}_{i+1}$, and we consider the subalgebra generated by the sequence $\mathbf{1} := (1, 1, \dots) \in \prod_{n\in\mathbb{N}_{\geq 2}} \mathbf{Z}_n$. As a finitely generated algebra, $\mathbf{N}$ has universe

$$\{t^{\mathbf{N}}(\mathbf{1}) \mid t \in T_\sigma(Y)\},$$

where $Y = \{y\}$ is some one variable set. We claim $t^{\mathbf{N}}(\mathbf{1}) = C(t) \cdot \mathbf{1}$ where $C(t)$ counts the number of occurrences of y in the term t.

Proof of this claim by induction on length of terms.
Base case: If $|t| = 1$, either $t = y$ so $t^{\mathbf{N}}(\mathbf{1}) = \mathbf{1} = 1 \cdot \mathbf{1}$
or $t = e$ so $t^{\mathbf{N}}(\mathbf{1}) = e^{\mathbf{N}} = 0 \cdot \mathbf{1}$.
Inductive case: If $|t| > 1$, $t = +s_1 s_2$ for terms $s_1, s_2 \in T_\sigma(Y)$. Then,

$$\begin{aligned} t^{\mathbf{N}}(\mathbf{1}) &= s_1^{\mathbf{N}}(\mathbf{1}) + s_2^{\mathbf{N}}(\mathbf{1}) \\ &= C(s_1) \cdot \mathbf{1} + C(s_2) \cdot \mathbf{1} && \textit{by induction, as } |s_1|, |s_2| < |t| \\ &= C(+s_1 s_2) \cdot \mathbf{1} \\ &= C(t) \cdot \mathbf{1}. \end{aligned}$$

□

So, $\mathbf{N}$ has universe

$$\{n \cdot \mathbf{1} \mid n \in \mathbb{N}_{\geq 0}\}.$$

Consider the equivalence relation on this set defined by $\theta = \{(\mathbf{0}, \mathbf{0})\} \cup \{n \cdot \mathbf{1} \mid n \in \mathbb{N}_{\geq 1}\}^2$, where $\mathbf{0} := (0, 0, \dots)$. We claim θ is a congruence relation.

Proof. We need to check θ is compatible with the monoid addition operation. Fix any pair of pairs $(i \cdot \mathbf{1}, j \cdot \mathbf{1}), (k \cdot \mathbf{1}, l \cdot \mathbf{1}) \in \theta$.
Case 1: $i = j = k = l = 0$

$$(+^{\mathbf{N}}(i \cdot \mathbf{1}, k \cdot \mathbf{1}), +^{\mathbf{N}}(j \cdot \mathbf{1}, l \cdot \mathbf{1})) = (\mathbf{0}, \mathbf{0}) \in \theta.$$

Case 2: $i = j = 0, k, l > 0$ (the case where $i, j \neq 0, k = l = 0$ is similar)

$$(+^{\mathbf{N}}(i \cdot \mathbf{1}, k \cdot \mathbf{1}), +^{\mathbf{N}}(j \cdot \mathbf{1}, l \cdot \mathbf{1})) = (k \cdot \mathbf{1}, l \cdot \mathbf{1}) \in \{n \cdot \mathbf{1} \mid n \in \mathbb{N}_{\geq 1}\}^2 \subseteq \theta.$$

Case 3: $i, j, k, l > 0$

$$(+^{\mathbf{N}}(i \cdot \mathbf{1}, k \cdot \mathbf{1}), +^{\mathbf{N}}(j \cdot \mathbf{1}, l \cdot \mathbf{1})) = ((i + k) \cdot \mathbf{1}, (j + l) \cdot \mathbf{1}) \in \{n \cdot \mathbf{1} \mid n \in \mathbb{N}_{\geq 1}\}^2 \subseteq \theta.$$

□

The quotient monoid $\mathbf{N}/\theta$ is the homomorphic image of $\mathbf{N}$ under the natural map $\nu_\theta : \mathbf{N} \to \mathbf{N}/\theta$, and as $\mathbf{Mod}(\Sigma)$ is closed under the operation of passing to homomorphic images, we have that $\mathbf{N}/\theta \in \mathbf{Mod}(\Sigma)$. As θ is a finite index congruence on $\mathbf{N}$, $\mathbf{N}/\theta \in \mathbf{Mod}_{fin}(\Sigma)$. θ is a congruence with 2 congruence classes, so we choose representatives $\mathbf{0}$ and $\mathbf{1}$. That is, the universe of $\mathbf{N}/\theta$ is

$$\{\mathbf{0}/\theta, \mathbf{1}/\theta\}.$$

However, $\mathbf{1}/\theta$ does not have an inverse element in $\mathbf{N}/\theta$:

Proof. The identity element is given by $e^{\mathbf{N}/\theta} = \nu_\theta(e^{\mathbf{N}}) = \nu_\theta(\mathbf{0}) = \mathbf{0}/\theta$.
However,

$$\mathbf{0}/\theta + \mathbf{1}/\theta = \mathbf{1}/\theta \neq \mathbf{0}/\theta$$

and

$$\mathbf{1}/\theta + \mathbf{1}/\theta = \mathbf{1}/\theta \neq \mathbf{0}/\theta$$

So, $\mathbf{1}/\theta$ has no inverse. □

This means that $\mathbf{N}/\theta \notin \mathcal{M}_{fin}$, contradicting our assumption that $\mathcal{M}_{fin} = \mathbf{Mod}_{fin}(\Sigma)$. □

A sequence of identities that do ultimately define this pseudovariety can be found from this characterization of the existence of inverse elements for finite groups:

Theorem 17. *A finite monoid $\mathbf{S} = (S, \cdot, e)$ is a group iff for all n sufficiently large,*

$$x^{n!} \approx e.$$

Proof. Suppose for all $n \geq N$,

$$x^{n!} \approx e.$$

Then in particular,

$$x^{N!} \approx e,$$

and hence

$$x \cdot x^{N!-1} \approx x^{N!-1} \cdot x \approx e,$$

and hence $\mathbf{S}$ is a group.
On the other hand, suppose $\mathbf{S}$ is a group.
Then for any $x \in S$, we may consider the generated subgroup $\langle x \rangle$. As $\mathbf{S}$ is finite, so is $\langle x \rangle$, and there exists some $n_x \in \mathbf{N}$ such that $x^{n_x} = e$.
So, let $N = \prod_{x \in S} n_x$. Then $x^N \approx e$, and hence $\forall n \geq N$,

$$x^{n!} \approx e.$$

□

So, $\mathcal{M}_{fin}$ as a family of finite monoids is ultimately defined by the sequence of identities $\{x^{k!} \approx e\}_{k\in\mathbb{N}}$. As a family of algebras in the signature of monoids, the defining sequence needs to have the monoid identities as well, repeated as constant subsequences.

Theorem 18.

$$\mathcal{M}_{fin} = \bigcup_{n=1}^{\infty} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}),$$

where $\forall i \in \mathbb{N}$

$$(s_i \approx t_i) = \begin{cases} (x \cdot (y \cdot z) \approx (x \cdot y) \cdot z) & \text{if } i = 4k \\ (e \cdot x \approx x) & \text{if } i = 4k+1 \\ (x \cdot e \approx x) & \text{if } i = 4k+2 \\ (x^{k!} \approx e) & \text{if } i = 4k+3 \end{cases}.$$

Proof. Suppose $\mathbf{S} \in \mathcal{M}_{fin}$. For some N, we have that $\forall n \geq N$,

$$x^{N!} \approx e.$$

We claim $\mathbf{S} \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq 4N\}) \subseteq \bigcup_{n=1}^{\infty} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\})$. Fix any $i \geq 4N$. If $i \not\equiv 3 \mod 4$, $s_i \approx t_i$ is a monoid identity hence $\mathbf{S} \vDash s_i \approx t_i$. If $i = 4k+3$ for some k, we have that $k \geq N$ and hence

$$x^{k!} \approx e$$

and hence $\mathbf{S} \vDash s_i \approx t_i$. On the other hand, suppose $\mathbf{S} \in \bigcup_{n=1}^{\infty} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\})$. As the union of an ascending chain, there exists N such that $\mathbf{S} \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq 4N\})$. As $\mathbf{S} \vDash s_{4N} \approx t_{4N}$ and $\mathbf{S} \vDash s_{4N+1} \approx t_{4N+1}$ and $\mathbf{S} \vDash s_{4N+2} \approx t_{4N+2}$ we have that $\mathbf{S}$ is a monoid. As $\forall n \geq N$, $\mathbf{S} \vDash s_{4n+3} \approx t_{4n+3}$, we have that that $\mathbf{S}$ is a group. $\square$

4 Models of Which Identities?

It follows from Theorems 1, 2, and 3 that for any set of identities Σ in the same signature σ, the family of its models $\mathbf{Mod}(\Sigma)$ is a variety. This same theorem allows us to prove that a family of finite algebras in the same signature ultimately defined by a sequence of identities is a pseudovariety.

Proof sketch that $K = \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}) \implies K$ *is a pseudovariety.*
We prove here that K *is closed under passing to finite products. The rest of the proof is similar.*

Suppose $\mathbf{A}, \mathbf{B} \in \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\})$. For some $n, m \in \mathbb{N}$, $\mathbf{A} \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\})$ and $\mathbf{B} \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq m\})$. Assume WLOG that $n \geq m$. So,

$$\begin{aligned}
& \mathbf{A}, \mathbf{B} \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}) \\
\implies & \mathbf{A}, \mathbf{B} \in \mathbf{Mod}(\{s_i \approx t_i | i \geq n\}) \\
\implies & \mathbf{A} \times \mathbf{B} \in \mathbf{Mod}(\{s_i \approx t_i | i \geq n\}) && \textit{by Theorem 3} \\
\implies & \mathbf{A} \times \mathbf{B} \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}) && \textit{since } \mathbf{A}, \mathbf{B} \textit{ are finite} \\
& \subseteq \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\})
\end{aligned}$$

$$\implies P_{fin}\left(\bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\})\right) \subseteq \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}).$$

□

On the other hand, given a variety V, we might ask, what set of identities Σ gives us $V = \mathbf{Mod}(\Sigma)$? Well, $V = \mathbf{Mod}(\Sigma)$ if Σ is the set of all identities true in V. Any identity needs only finitely many variables to be written, so a countably infinite set of variables will suffice. In this paper, let

$$X_\omega := \{x_1, x_2, \dots\}.$$

Then to formalize the notion of identities true in a family of algebras in the same signature, we introduce the following definition.

Definition 22. *For a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$, and a set of variables X such that $X \cup \mathcal{F}_0 \neq \emptyset$,*

$$Id_K(X) := \{s \approx t \mid s, t \in \mathbf{T}_\sigma(X) \text{ and } K \vDash s \approx t\}.$$

When K contains a single algebra $\mathbf{A}$, we write $\mathbf{A}$ in place of $\{\mathbf{A}\}$.

Then to prove Theorem 4, we show in section 7 that for any variety K,

$$K = \mathbf{Mod}(Id_K(X_\omega)).$$

Which sequence of identities ultimately define a given pseudovariety? To start, we note that a pseudovariety is countable up to isomorphism.

Theorem 19. *Consider a family of finite algebras in the same signature $K \subseteq \mathcal{K}_\sigma$. Then there exists a sequence of finite algebras $\mathbf{T}_1, \mathbf{T}_2, \dots \in \mathcal{K}_\sigma$ such that for any algebra $\mathbf{A} \in K$, there exists $n \in \mathbb{N}$ such that*

$$\mathbf{T}_n \cong \mathbf{A}.$$

Proof. For an n-element set A, each function symbol $f \in \mathcal{F}$ of arity $\rho(f)$ represents one of $n^{\left(n^{\rho(f)}\right)}$ possible functions $A^n \to A$, so there are $\prod_{f\in\mathcal{F}} n^{\left(n^{\rho(f)}\right)} \in \mathbb{N}$ many algebras possible in the signature σ with this set A as a universe, and hence the collection of finite algebras is countable up to isomorphism. □

In this paper, let

$$X_n := \{x_1, x_2, \dots, x_n\}.$$

Then, we consider the n-variable identities that are true in all the algebras $\mathbf{T}_1, \mathbf{T}_2, \dots \mathbf{T}_n$. That is, we consider

$$Id_{\{\mathbf{T}_1,\mathbf{T}_2,\dots\mathbf{T}_n\}}(X_n).$$

Definition 23. *A variety V is **finitely based** if*

$$V = \mathbf{Mod}(\Sigma)$$

for some finite set of identities Σ.

It turns out that $\mathbf{Mod}(Id_{\{\mathbf{T}_1,\mathbf{T}_2,\dots\mathbf{T}_n\}}(X_n))$ is finitely based, a fact to be proven in Theorem 26, say by some set of identities Σ_n. Then to prove Theorem 6, we will show in section 7 that any finite algebra in the pseudovariety is a model of all but finitely many identities from the countable set,

$$\bigcup_{n\in\mathbb{N}} \Sigma_n.$$

5 Free Algebra Construction

In this section we consider a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$, and a set of variables X such that $X \cup \mathcal{F}_0 \neq \emptyset$, and we construct the free algebra $\mathbf{F}_K(X)$, which has the following insightful property:

Theorem 20. *Consider a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$, and a set of variables X such that $X \cup \mathcal{F}_0 \neq \emptyset$, and any terms $p, q \in \mathbf{T}_\sigma(X)$.*

$$K \vDash p \approx q \iff \mathbf{F}_K(X) \vDash p \approx q.$$

To begin, recall the universal mapping property for term algebras:

Theorem 21 (Universal Mapping Property for Term Algebras)**.** *Suppose $X \cup \mathcal{F}_0 \neq \emptyset$. For any map $h_0 : X \to \mathbf{A} \in \mathcal{K}_\sigma$, there exists a unique homomorphism $h : \mathbf{T}_\sigma(X) \to \mathbf{A}$ such that $h \restriction_X = h_0$.*

Then we introduce the following congruence on the term algebra:

Definition 24. *For a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$, and a set of variables X such that $X \cup \mathcal{F}_0 \neq \emptyset$,*

$$\Phi_K(X) := \left\{ \varphi \subseteq T_\sigma(X)^2 \;\middle|\; \begin{array}{c} \varphi = \ker(h) \textit{ for some hom.} \\ h : \mathbf{T}_\sigma(X) \to \mathbf{B} \in K \end{array} \right\}.$$
$$\theta_K(X) := \bigcap \Phi_K(X).$$

When K contains a single algebra $\mathbf{A}$, we write $\mathbf{A}$ in place of $\{\mathbf{A}\}$.

By Theorem 10, $\Phi_K(X) \subseteq \mathbf{Con}\, \mathbf{T}_\sigma(X)$. $\mathbf{Con}\, \mathbf{T}_\sigma(X)$ is closed under arbitrary intersections, so $\theta_K(X)$ is a congruence on $\mathbf{T}_\sigma$. As a consequence of the universal mapping property for term algebras, we have the following theorem:

Theorem 22 (Consequence of UMP)**.**

$$\theta_K(X) = \{(s,t) \in T_\sigma(X)^2 \mid s \approx t \in Id_K(X)\} \textit{ and } Id_K(X) = \{s \approx t \mid (s,t) \in \theta_K(X)\}.$$

Proof. Fix any algebra $\mathbf{A} \in K$ and any $s, t \in T_\sigma(X)$. Suppose the variables appearing in s, t are in the set $\{x_1, \ldots, x_n\} \subseteq X$, so that by Theorem 8, $s = s^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n)$ and $t = t^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n)$. Let's prove the following **fact**:

$$s^{\mathbf{A}} = t^{\mathbf{A}} \iff \forall \textit{hom. } h : \mathbf{T}_\sigma(X) \to \mathbf{A},\ h(s) = h(t).$$

Proof of this fact. $\underline{s^{\mathbf{A}} = t^{\mathbf{A}} \implies \forall \textit{hom. } h : \mathbf{T}_\sigma(X) \to \mathbf{A},\ h(s) = h(t)}$
Fix any homomorphism $h : \mathbf{T}_\sigma(X) \to \mathbf{A}$.

$$\begin{aligned}
h(s) &= h(s^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n)) && \textit{by Theorem 8} \\
&= s^{\mathbf{A}}(h(x_1), \ldots, h(x_n)) && \textit{by induction on the definition of homomorphism} \\
&= t^{\mathbf{A}}(h(x_1), \ldots, h(x_n)) && \textit{by assumption} \\
&= h(t^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n)) && \textit{by induction on the definition of homomorphism} \\
&= h(t). && \textit{by Theorem 8}
\end{aligned}$$

$\underline{s^{\mathbf{A}} = t^{\mathbf{A}} \impliedby \forall \textit{hom. } h : \mathbf{T}_\sigma(X) \to \mathbf{A}\ h(s) = h(t)}$
Fix any $a_1, \ldots, a_n \in A$. We wish to show that $s^{\mathbf{A}}(a_1, \ldots, a_n) = t^{\mathbf{A}}(a_1, \ldots, a_n)$. Let $h_0 : X \to A$ be any map such that $\forall 1 \leq i \leq n\ h_0(x_i) = a_i$. By the UMP for term algebras, there exists a homomorphism $h : \mathbf{T}_\sigma(X) \to \mathbf{A}$ such that $\forall 1 \leq i \leq n\ h(x_i) = a_i$. Then,

$$\begin{aligned}
s^{\mathbf{A}}(a_1, \ldots, a_n) &= s^{\mathbf{A}}(h(x_1), \ldots, h(x_n)) \\
&= h(s^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n)) && \textit{by induction on the definition of homomorphism} \\
&= h(s) && \textit{by Theorem 8} \\
&= h(t) && \textit{by assumption} \\
&= h(t^{\mathbf{T}_\sigma(X)}(x_1, \ldots, x_n)) && \textit{by Theorem 8} \\
&= t^{\mathbf{A}}(h(x_1), \ldots, h(x_n)) && \textit{by induction on the definition of homomorphism} \\
&= t^{\mathbf{A}}(a_1, \ldots, a_n).
\end{aligned}$$

□

Returning to the proof of Theorem 22, the two statements are equivalent so we prove the first.

$$\begin{aligned}\theta_K(X) &= \bigcap \Phi_K(X) && \textit{by definition of } \theta_K(X)\\ &= \bigcap \left\{ \varphi \subseteq T_\sigma(X)^2 \,\middle|\, \begin{array}{c}\varphi = \ker(h) \textit{ for some hom.}\\ h : \mathbf{T}_\sigma(X) \to \mathbf{B} \in K\end{array} \right\}. && \textit{by definition of } \Phi_K(X)\end{aligned}$$

So, $\forall s, t \in T_\sigma(X)$,

$$\begin{aligned}&(s,t) \in \theta_K(X)\\ \iff& \forall \mathbf{A} \in K\ \forall \textit{hom. } h : \mathbf{T}_\sigma(X) \to \mathbf{A},\ h(s) = h(t)\\ \iff& \forall \mathbf{A} \in K\ s^{\mathbf{A}} = t^{\mathbf{A}} && \textit{by the fact above}\\ \iff& K \vDash s \approx t\\ \iff& (s,t) \in Id_K(X).\end{aligned}$$

□

So, the quotient algebra $\mathbf{T}_\sigma/\theta_K(X)$ can be thought of as the term algebra with pairs of terms identified with each other when the pair of terms as an identity is true in the family of algebras K. This is how we define the free algebra:

Definition 25. *For a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and a set of variables X such that $X \cup \mathcal{F}_0 \neq \emptyset$,*

$$\mathbf{F}_K(X) := \mathbf{T}_\sigma(X)/\theta_K(X).$$

When K contains a single algebra $\mathbf{A}$, we write $\mathbf{A}$ in place of $\{\mathbf{A}\}$.

6 Free Algebra Properties

Consider a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and a set of variables X such that $X \cup \mathcal{F}_0 \neq \emptyset$. By Theorem 20, we have that

$$\mathbf{F}_K(X) \vDash Id_K(X),$$

and by Theorem 1, the homomorphic images of $\mathbf{F}_K(X)$ model the identities over the set of variables X true in the family of algebras K as well. That is,

$$H(\mathbf{F}_K(X)) \vDash Id_K(X).$$

For a free algebra over a set of variables sufficiently large, the reverse is true too. The models of identities over the set of variables X true in the family of algebras K are homomorphic images of the free algebra:

Theorem 23. *For any family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and any algebra in the same signature $\mathbf{A} \in \mathcal{K}_\sigma$, if X is a set of variables such that $X \cup \mathcal{F}_0 \neq \emptyset$ and $|X| \geq |A|$, and if $\mathbf{A} \vDash Id_K(X)$, then there exists a surjective homomorphism $h : \mathbf{F}_K(X) \to \mathbf{A}$.*

Proof. Suppose $K \subseteq \mathcal{K}_\sigma$ a family of algebras in the same signature, and $\mathbf{A} \in \mathcal{K}_\sigma$ an algebra in the same signature, and X a set of variables such that $|X| \geq |A|$, and $\mathbf{A} \vDash Id_K(X)$. Let h_0 be any surjective map $X \to A$, which exists as $|X| \geq |A|$. By the universal mapping property for term algebras, there exists a homomorphism $h : \mathbf{T}_\sigma(X) \to \mathbf{A}$ which extends h_0, and is hence also a surjective map. Consider also the natural map $\nu_\theta : \mathbf{T}_\sigma(X) \to \mathbf{T}_\sigma(X)/\theta_K(X)$ which is also a surjective homomorphism. As $\mathbf{A} \vDash Id_K(X)$, it follows by Theorem 22 that

$$\theta_K(X) \subseteq \theta_{\mathbf{A}}(X) \subseteq \ker h.$$

So, by Theorem 12 there exists an induced surjective homomorphism $\mathbf{F}_K(X) = \mathbf{T}_\sigma(X)/\theta_K(X) \to \mathbf{A}$ as desired. $\square$

We will need in the proof of Theorem 4 that the free algebra over a variety is in the variety:

Theorem 24. *Let $K \subseteq \mathcal{K}_\sigma$ be a class of algebras in the same signature, and let X be a set with $\mathcal{F}_0 \cup X \neq \emptyset$. For some choice of algebras $\mathbf{A}_\varphi \in K$ indexed by $\varphi \in \Phi_K(X)$,*

$$\mathbf{F}_K(X) \in IS\left(\prod_{\varphi \in \Phi_K(X)} \mathbf{A}_\varphi\right).$$

That is, $\mathbf{F}_K(X) \in ISP(K)$.

Proof. Recall the definition of $\Phi_K(X)$:

$$\Phi_K(X) := \left\{ \varphi \subseteq T_\sigma(X)^2 \,\middle|\, \begin{array}{c} \varphi = \ker(h) \textit{ for some hom.} \\ h : \mathbf{T}_\sigma(X) \to \mathbf{B} \in K \end{array} \right\}.$$

It follows that for each $\varphi \in \Phi_K(X)$, we may choose some $\mathbf{A}_\varphi \in K$ and some homomorphism $h_\varphi : T_\sigma(X) \to \mathbf{A}_\varphi$ such that $\ker(h_\varphi) = \varphi$.
Consider the map $h : T_\sigma(X) \to \prod_{\varphi \in \Phi_K(X)} \mathbf{A}_\varphi$ defined by

$$h(t)(\varphi) = h_\varphi(t).$$

Proof that this map is a homomorphism. Suppose $\sigma = (\mathcal{F}, \rho)$ and fix any $f \in \mathcal{F}$ such that $\rho(f) = k$. Fix terms $s_1, \ldots, s_k \in T_\sigma(X)$. Then $\forall \varphi \in \Phi_K(X)$,

$$\begin{aligned} h(fs_1 \cdots s_k)(\varphi) &= h_\varphi(fs_1 \cdots s_k) && \textit{definition of } h \\ &= f^{\mathbf{A}_\varphi}(h_\varphi(s_1), \ldots, h_\varphi(s_k)) && \textit{since } h_\varphi \textit{ is a homomorphism} \\ &= f^{\mathbf{A}_\varphi}(h(s_1)(\varphi), \ldots, h(s_k)(\varphi)). && \textit{definition of } h \end{aligned}$$

$\square$

Notice that $\forall s, t \in T_\sigma(X)$,

$$\begin{aligned}
(s,t) \in \ker(h) &\iff h(s) = h(t) \\
&\iff h(s)(\varphi) = h(t)(\varphi) && \forall \varphi \in \Phi_K(X) \\
&\iff h_\varphi(s) = h_\varphi(t) && \forall \varphi \in \Phi_K(X) \\
&\iff (s,t) \in \ker(\varphi) && \forall \varphi \in \Phi_K(X) \\
&\iff (s,t) \in \theta_K(X).
\end{aligned}$$

So,

$$\begin{aligned}
\mathbf{F}_K(X) &= \mathbf{T}_\sigma(X)/\theta_K(X) \\
&= \mathbf{T}_\sigma(X)/\ker(h) \\
&\cong h\left(\mathbf{T}_\sigma(X)\right) && \textit{by the first isomorphism theorem} \\
&\leq \prod_{\varphi \in \Phi_K(X)} \mathbf{A}_\varphi.
\end{aligned}$$

□

Likewise, we will need in the proof of Theorem 6 that the free algebra over $\mathbf{Mod}(Id_{\{\mathbf{T}_1,\dots,\mathbf{T}_n\}}(X_n))$ is in the pseudovariety containing $\{\mathbf{T}_1, \dots, \mathbf{T}_n\}$. The following corollary establishes this.

Corollary 2. *If K is a finite set of finite algebras, and $X = \{x_1, \dots, x_n\}$ a finite set of variables, then $\Phi_K(X)$ is finite and hence $\mathbf{F}_K(X) \in ISP_{fin}(K)$*

Proof. Fix a finite set of finite algebras $K = \{\mathbf{T}_1, \dots, \mathbf{T}_k\} \subseteq \mathcal{K}_\sigma$ and any $1 \leq i \leq k$. The *uniqueness* part of the universal mapping property tells us that any homomorphism $h : \mathbf{T}_\sigma(X) \to \mathbf{T}_i$ is uniquely determined by $h(x_1), \dots, h(x_n)$. So, there are $|T_i|^n$ homomorphisms $\mathbf{T}_\sigma(X) \to \mathbf{T}_i$, hence finitely many homomorphisms from the term algebra to $\mathbf{T}_i$ for some i, hence $\Phi_K(X)$ is finite as desired. It follows by Theorem 24 that $\mathbf{F}_K(X) \in ISP_{fin}(K)$. □

We will need the next theorem to help us prove that $\mathbf{Mod}(Id_{\{\mathbf{T}_1,\dots\mathbf{T}_n\}}(X_n))$ is finitely based.

Theorem 25. *Suppose $\mathbf{A} \in \mathcal{K}_\sigma$ is a finitely generated algebra and $\theta \in$ **Con A** is a congruence of finite index. Then θ is finitely generated.*

Proof. Since $\mathbf{A}$ is finitely generated by some finite set $X = \{a_1, \dots, a_n\}$, by Theorem 13,

$$A = \{t^{\mathbf{A}}(a_1, \dots, a_n) \mid t \in T_\sigma(Y)\},$$

where Y is an n-variable set. So, we define $\ell : A \to \mathbb{N}$ by

$$\ell(a) := \min_{\substack{t \in T_\sigma(Y) \\ a = t^{\mathbf{A}}(a_1, \cdots, a_n)}} |t|.$$

Then as θ has finite index, we choose representatives for the congruence classes and write $A/\theta = \{w_1/\theta, \ldots, w_r/\theta\}$. Define

$$k := \max_{1 \leq i \leq r} \ell(w_i)$$

so that each congruence class contains an element w with length $\ell(w) \leq k$. As the set of function symbols in any signature is also finite, we define m to be the maximum arity

$$m := \max_{f \in \mathcal{F}} \rho(f) \in \mathbb{N}.$$

Define

$$W := \{(u, v) \in \theta \mid \ell(u) \leq mk + 1, \ell(v) \leq k\}.$$

Notice that any $a \in A$ has its length $\ell(a)$ determined by some term over the alphabet $Y \cup \mathcal{F}$ with length $\ell(a)$, which there are less than $(1 + n + |\mathcal{F}|)^{\ell(a)}$ possible values, and hence

$$|W| \leq (n + |\mathcal{F}| + 1)^{mk+k+1} \in \mathbb{N}.$$

Let θ_W be the congruence generated by W. We claim θ and θ_W are in fact the same congruence, hence θ is finitely generated. As $W \subseteq \theta$, we have that $\theta_W \subseteq \theta$. To prove that $\theta \subseteq \theta_W$, we first prove the following fact:

> For each $a \in A$, there exists $a' \in A$ such that $\ell(a') \leq k$ and $(a, a') \in \theta_W$.

Proof of this fact. We prove this by induction with respect to $\ell(a)$.

1. **Case 1:** If $\ell(a) \leq k$, $(a, a) \in \theta_W$ as congruences are reflexive.

2. **Case 2:** $\ell(a) > k$.
   We may assume for our induction hypothesis that for $b \in A$ such that $\ell(b) < \ell(a)$ there exists $b' \in A$ such that $\ell(b') \leq k$ and $(b, b') \in \theta_W$. For some term $t \in T_\sigma(Y)$ such that $|t| = \ell(a)$, we have that

   $$a = t^{\mathbf{A}}(a_1, \ldots, a_n).$$

   By the unique readability of terms, we may uniquely write

   $$t = f s_1 \cdots s_m$$

   for some function symbol $f \in \mathcal{F}$ of arity m' such that $s_1, \ldots, s_{m'} \in T_\sigma(X)$. For each $1 \leq i \leq m'$, define

   $$b_i = s_i^{\mathbf{A}}(a_1, \ldots, a_n).$$

   The term s_i witnesses that $\ell(b_i) < \ell(a)$, so by our induction hypothesis $(b_i, b_i') \in \theta_W$ for some b_i' with $\ell(b_i') \leq k$. By the definition of ℓ, there exist $s_1', \ldots, s_{m'}' \in T_\sigma(Y)$ such that each $b_i' = (s_1')^{\mathbf{A}}(a_1, \ldots, a_n)$ and $|s_i'| \leq k$. We define

   $$c := f^{\mathbf{A}}(b_1', \ldots, b_{m'}').$$

By definition of congruence $\forall 1 \leq i \leq m', (b_i, b_i') \in \theta_W \implies (a, c) \in \theta_W$ where $|fs_1' \cdots s_m'| \leq mk + 1$ and hence $\ell(c) \leq mk + 1$. k was chosen so that there exists some $a' \in c/\theta$ so that $\ell(a') \leq k$. Since

$$\begin{aligned} (a, c) &\in \theta_W \\ (c, a') &\in W && \textit{by the definition of } W \\ &\subseteq \theta_W \\ \therefore (a, a') &\in \theta_W \end{aligned}$$

as required.

□

So, this fact is true. Now assume $(u, v) \in \theta$. By this fact there exist u', v' such that $(u, u'), (v, v') \in \theta_W$ such that $\ell(u'), \ell(v') \leq k$. Since

$$\begin{aligned} (u', u) &\in \theta && \textit{since } \theta_W \subseteq \theta \\ (u, v) &\in \theta && \textit{by assumption} \\ (v, v') &\in \theta && \textit{since } \theta_W \subseteq \theta \\ \therefore (u', v') &\in \theta. \end{aligned}$$

and $\ell(u'), \ell(v') \leq k$, it follows that $(u', v') \in W$ and hence

$$\begin{aligned} (u, u') &\in \theta_W \\ (u', v') &\in \theta_W \\ (v', v) &\in \theta_W \\ \therefore (u, v) &\in \theta_W. \end{aligned}$$

So, θ and θ_W coincide and thus θ is finitely generated. □

We are now also ready to prove that $\mathbf{Mod}(Id_{\{\mathbf{T}_1, \mathbf{T}_2, \dots \mathbf{T}_n\}}(X_n))$ is finitely based.

Theorem 26. *For a finite set of finite algebras $\mathbf{T}_1, \mathbf{T}_2, \dots \mathbf{T}_n$ and for any $k \in \mathbb{N}_{\geq 0}$, $\mathbf{Mod}(Id_{\{\mathbf{T}_1, \mathbf{T}_2, \dots \mathbf{T}_n\}}(X_k))$ is finitely based.*

Proof. If $X \cup \mathcal{F}_0 = \emptyset$, $\mathbf{Mod}(Id_{\{\mathbf{T}_1, \dots, \mathbf{T}_n\}}(X_k))$ contains all of $\mathcal{K}_\sigma = \mathbf{Mod}(\emptyset)$, hence also has the empty set for a basis. Now suppose $X_k \cup \mathcal{F}_0 \neq \emptyset$. By Corollary 2, the free algebra $\mathbf{F}_{\{\mathbf{T}_1, \dots, \mathbf{T}_n\}}(X_k) = \mathbf{T}_\sigma(X_k)/\theta_{\{\mathbf{T}_1, \dots, \mathbf{T}_n\}}(X_k)$ is finite, which means the congruence $\theta_{\{\mathbf{T}_1, \dots, \mathbf{T}_n\}}(X_k)$ has finite index. So, by Theorem 25, as each $\mathbf{T}_\sigma(X_k)$ is finitely generated by X_k, $\theta_{\{\mathbf{T}_1, \dots, \mathbf{T}_n\}}(X_k)$ is generated by some finite set W. Let

$$\Sigma := \{s \approx t | (s, t) \in W\}.$$

For any algebra $\mathbf{A} \in \mathcal{K}_\sigma$,

$$\begin{aligned}
& \mathbf{A} \models \Sigma \\
\iff & \Sigma \subseteq Id_{\mathbf{A}}(X_k) \\
\iff & W \subseteq \theta_{\mathbf{A}}(X_k) && \textit{by Theorem } 22 \\
\iff & \theta_{\{\mathbf{T}_1,\dots,\mathbf{T}_n\}}(X_k) \subseteq \theta_{\mathbf{A}}(X_k) && \textit{since } W \textit{ generates } \theta_{\{\mathbf{T}_1,\dots,\mathbf{T}_n\}}(X_k) \\
\iff & \mathbf{A} \models Id_{\{\mathbf{T}_1,\dots,\mathbf{T}_n\}}(X_k). && \textit{by Theorem } 22
\end{aligned}$$

So, Σ is a finite basis for $\mathbf{Mod}(Id_{\{\mathbf{T}_1,\dots,\mathbf{T}_n\}}(X_k))$. □

7 A Correspondence for Pseudovarieties of Finite Algebras

We now have all the components necessary to complete the proof of Theorem 6, restated here:

Theorem 6. *[Generalization of Eilenberg & Schützenberger] Consider a family of finite algebras K in a common signature. Then there exists a sequence of identities such that K is the class of finite models of all but finitely many of the identities iff K is closed under the operations of passing to homomorphic images, subalgebras, and finite product algebras.*

Proof that K is a pseudovariety $\implies$ K is ultimately defined by some sequence of identities. Suppose K is a pseudovariety. By Theorem 19, there exists a sequence of finite algebras $\mathbf{T}_1, \mathbf{T}_2, \dots \in \mathcal{K}_\sigma$ such that for any algebra $\mathbf{A} \in K$, there exists $n \in \mathbb{N}$ such that

$$\mathbf{T}_n \cong \mathbf{A}.$$

As isomorphisms are surjective homomorphisms, this sequence of finite algebras $\mathbf{T}_1, \mathbf{T}_2, \dots$ are in fact in K. By Theorem 26, $\mathbf{Mod}(Id_{\{\mathbf{T}_1,\dots\mathbf{T}_n\}}(X_n))$ is finitely based, say by some set of equations Σ_n. That is, for any algebra $\mathbf{A} \in \mathcal{K}_\sigma$,

$$\mathbf{A} \models \Sigma_n \iff \mathbf{A} \models Id_{\{\mathbf{T}_1,\dots,\mathbf{T}_n\}}(X_n).$$

For any $n, \cup_{k<n}\Sigma_k$ is a finite set of identities, so it remains to be shown that

$$K = \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k).$$

$\underline{K \subseteq \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k)}$

Fix an algebra $\mathbf{A} \in K$. There exists some $n \in \mathbb{N}$ such that $\mathbf{A} \cong \mathbf{T}_n$.

$$\begin{aligned}
& \mathbf{A} \models Id_{\mathbf{T}_n}(X) && \textit{for any set of variables } X \\
\implies & \forall k \geq n\ (\mathbf{A} \models Id_{\{\mathbf{T}_1,\dots,\mathbf{T}_k\}}(X_k)) && \textit{since } Id_{\mathbf{T}_n}(X_k) \supseteq Id_{\{\mathbf{T}_1,\dots,\mathbf{T}_k\}}(X_k) \\
\implies & \forall k \geq n\ (\mathbf{A} \models \Sigma_k) && \textit{by Theorem } 26 \\
\implies & \mathbf{A} \models \cup_{k\geq n}\Sigma_k \\
\implies & \mathbf{A} \in \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k) \subseteq \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k).
\end{aligned}$$

$\underline{K \supseteq \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k)}$
Suppose $\mathbf{A} \in \bigcup_{n\in\mathbb{N}} \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k)$. For some $n \in \mathbb{N}, \mathbf{A} \in \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k)$. Let $N = \max\{n, |A|\}$. Then,

$$\begin{aligned}
& \mathbf{A} \in \mathbf{Mod}_{fin}(\cup_{k\geq n}\Sigma_k) \\
\implies & \mathbf{A} \models \cup_{k\geq n}\Sigma_k \\
\implies & \mathbf{A} \models \Sigma_N \\
\implies & \mathbf{A} \models Id_{\{\mathbf{T}_1,\ldots,\mathbf{T}_N\}}(X_N) && \textit{by Theorem } 26 \\
\implies & \mathbf{A} \in H(\mathbf{F}_{\{\mathbf{T}_1,\ldots,\mathbf{T}_N\}}(X_N)) && \textit{by Theorem } 23 \\
& \subseteq HISP_{fin}(\{\mathbf{T}_1,\ldots,\mathbf{T}_N\}) && \textit{by Corollary } 2 \\
& \subseteq K, && \textit{by definition of a pseudovariety}
\end{aligned}$$

□

We also have all the components necessary to complete the proof of Theorem 4 as well, restated here:

Theorem 4. *[1] Consider a family of algebras K in the signature σ. There exists a set of identities Σ in the signature σ over a countably infinite set of variables for which $K = Mod(\Sigma)$ iff K is closed under the operations of passing to homomorphic images, subalgebras, and product algebras.*

The only idea missing is that it takes only countably many variables to write any identity.

Proof that any variety $K = \mathbf{Mod}(Id_K(X_\omega))$.
$\underline{K \subseteq \mathbf{Mod}(Id_K(X_\omega))}$
Suppose $\mathbf{A} \in K$. Then as $\mathbf{A} \models Id_K(X_\omega)$, it follows that $\mathbf{A} \in \mathbf{Mod}(Id_K(X_\omega))$.
$\underline{K \supseteq \mathbf{Mod}(Id_K(X_\omega))}$
Suppose $\mathbf{A} \in \mathbf{Mod}(Id_K(X_\omega))$. Fix a set of variables Y such that $|Y| \geq |A|$.

Proof that $\mathbf{A} \models Id_K(Y)$. Fix any identity $s \approx t \in Id_K(Y)$. Suppose the variables that appear in s, t appear as some subset of $y_1, \ldots, y_n \in Y$, so that by Theorem 8, $s = s^{\mathbf{T}_\sigma(Y)}(y_1, \ldots, y_n)$ and $t = t^{\mathbf{T}_\sigma(Y)}(y_1, \ldots, y_n)$.
Let s', t' be the terms obtained by substituting the instances of y_i with $x_i \in X_\omega$.
Then, by the substitution lemma, for each $\mathbf{B} \in K$, $s^{\mathbf{B}} = (s')^{\mathbf{B}}$ *and* $t^{\mathbf{B}} = (t')^{\mathbf{B}}$, so

$$\begin{aligned}
& K \models s \approx t \\
\implies & K \models s' \approx t' && \textit{by substitution lemma} \\
\implies & (s' \approx t') \in Id_K(X_\omega) \\
\implies & \mathbf{A} \models s' \approx t' && \mathbf{A} \in \mathbf{Mod}(Id_K(X_\omega)) \\
\implies & \mathbf{A} \models s \approx t. && \textit{by substitution lemma}
\end{aligned}$$

□

So,

$$\begin{aligned}
\mathbf{A} &\in H(\mathbf{F}_K(Y)) && \textit{by Theorem 23} \\
&\subseteq HISP(K) && \textit{by Theorem 24} \\
&\subseteq K. && \textit{by definition of a variety}
\end{aligned}$$

□

We have in fact proven that given a variety K,

$$K \subseteq \mathbf{Mod}(Id_K(Y)) \subseteq HISP(K) \subseteq K.$$

We can generalize this.

Definition 26. *Given a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$, we define $\mathcal{V}(K)$, **the variety generated by** K, to be the smallest variety containing K. If V is a variety such that there exists a finite family of finite algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and $V = \mathcal{V}(K)$, we say V is a **finitely generated variety**.*

Theorem 27. *Given a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$,*

$$HSP(K) = \mathcal{V}(K) = \mathbf{Mod}(Id_K(X_\omega)).$$

Proof.
$\underline{HSP(K) \subseteq \mathcal{V}(K)}$

$$\begin{aligned}
K &\subseteq \mathcal{V}(K) && \textit{by definition of } \mathcal{V}(K) \\
\implies P(K) &\subseteq \mathcal{V}(K) && \textit{by definition of a variety} \\
\implies SP(K) &\subseteq \mathcal{V}(K) && \textit{by definition of a variety} \\
\implies HSP(K) &\subseteq \mathcal{V}(K). && \textit{by definition of a variety}
\end{aligned}$$

$\underline{\mathcal{V}(K) \subseteq \mathbf{Mod}(Id_K(X_\omega))}$
By Theorem 4, $\mathbf{Mod}(Id_K(X_\omega))$ is a variety, hence it suffices to show that $K \subseteq \mathbf{Mod}(Id_K(X_\omega))$
Suppose $\mathbf{A} \in K$. Then as $\mathbf{A} \vDash Id_K(X_\omega)$, it follows that $\mathbf{A} \in \mathbf{Mod}(Id_K(X_\omega))$.
$\underline{\mathbf{Mod}(Id_K(X_\omega)) \subseteq HSP(K)}$
Suppose $\mathbf{A} \in \mathbf{Mod}(Id_K(X_\omega))$. By the same argument as in Theorem 4, $\mathbf{A} \in HISP(K)$. But notice that as a surjective homomorphism composed with an isomorphism is just a surjective homomorphism, we have that $\mathbf{A} \in HSP(K)$. □

We also have a similar result for pseudovarieties:

Definition 27. *Given a family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$, we define $\mathcal{V}_{fin}(K)$,* ***the pseudovariety generated by*** *K, to be the smallest pseudovariety containing K. If V is a pseudovariety such that there exists a finite family of finite algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and $V = \mathcal{V}_{fin}(K)$, we say V is a* ***finitely generated pseudovariety****.*

Theorem 28. *Given a finite family of finite algebras in the same signature $K \subseteq \mathcal{K}_\sigma$,*

$$HSP_{fin}(K) = \mathcal{V}_{fin}(K) = \mathbf{Mod}_{fin}(Id_K(X_\omega)).$$

The second equality says that finitely generated pseudovarieties are equational.

Proof.
Fix a finite family of algebras $K = \{\mathbf{A}_1, \dots, \mathbf{A}_k\} \subseteq \mathcal{K}_\sigma$.
$\underline{\mathbf{Mod}_{fin}(Id_K(X_\omega)) \subseteq \mathcal{V}_{fin}(K)}$
By Theorem 6, there exists some sequence of identities $\{s_i \approx t_i | s_i, t_i \in \mathbf{T}_\sigma(X_\omega)\}_{i \in \mathbb{N}}$ for which

$$\mathcal{V}_{fin}(K) = \bigcup_{n=1}^{\infty} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}).$$

For each $1 \leq j \leq k$, there exists some $n_j \in \mathbb{N}$ such that $\mathbf{A}_j \in \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n_j\})$, and hence

$$\mathcal{V}_{fin}(K) \subseteq \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq \max\{n_j | 1 \leq j \leq k\}\}).$$

We also have that

$$\mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq \max\{n_j | 1 \leq j \leq k\}\}) \subseteq \bigcup_{n=1}^{\infty} \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq n\}) = \mathcal{V}_{fin}(K),$$

and so

$$\mathcal{V}_{fin}(K) = \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq \max\{n_j | 1 \leq j \leq k\}\}).$$

For each $i \geq \max\{n_j | 1 \leq j \leq k\}$ and each algebra $\mathbf{A} \in K$, the identity $s_i \approx t_i$ is true in $\mathbf{A}$, and hence $\{s_i \approx t_i | i \geq \max\{n_j | 1 \leq j \leq k\}\} \subseteq Id_K(X_\omega)$. So,

$$\mathbf{Mod}_{fin}(Id_K(X_\omega)) \subseteq \mathbf{Mod}_{fin}(\{s_i \approx t_i | i \geq \max\{n_j | 1 \leq j \leq k\}\}) = \mathcal{V}_{fin}(K),$$

as desired.
$\underline{\mathcal{V}_{fin}(K) \subseteq \mathbf{Mod}_{fin}(Id_K(X_\omega))}$
By Theorem 14, $\mathbf{Mod}_{fin}(Id_K(X_\omega))$ is a pseudovariety.
As $K \vDash Id_K(X_\omega), K \subseteq \mathbf{Mod}_{fin}(Id_K(X_\omega))$.
So, $\mathcal{V}_{fin}(K) \subseteq \mathbf{Mod}_{fin}(Id_K(X_\omega))$ by the definition of $\mathcal{V}_{fin}(K)$.
$\underline{HSP_{fin}(K) \subseteq \mathbf{Mod}_{fin}(Id_K(X_\omega))}$
Suppose $\mathbf{A} \in HSP_{fin}(K)$.

$$\begin{aligned} & \mathbf{A} \in HSP(K) && \textit{since finite products are products} \\ \Longrightarrow\ & \mathbf{A} \in \mathbf{Mod}(Id_K(X_\omega)) && \textit{by Theorem } 27 \\ \Longrightarrow\ & \mathbf{A} \in \mathbf{Mod}_{fin}(Id_K(X_\omega)). && \textit{since } \mathbf{A} \textit{ is finite} \end{aligned}$$

$\underline{\mathbf{Mod}_{fin}(Id_K(X_\omega)) \subseteq HSP_{fin}(K)}$
Suppose $\mathbf{A} \in \mathbf{Mod}_{fin}(Id_K(X_\omega))$. As A is finite, $|X_\omega| > |A|$, so

$$\begin{aligned}
&\mathbf{A} \in \mathbf{Mod}_{fin}(Id_K(X_\omega)) && \\
\implies &\mathbf{A} \models Id_K(X_\omega) && \\
\implies &\mathbf{A} \in H(\mathbf{F}_K(X_\omega)) && \text{by Theorem 23} \\
&\quad \subseteq HISP_{fin}(K). && \text{by Corollary 2}
\end{aligned}$$

But notice that as a surjective homomorphism composed with an isomorphism is just a surjective homomorphism, we have that $\mathbf{A} \in HSP_{fin}(K)$. □

8 Locally Finite Varieties and Pseudovarieties

Definition 28. *Let V be a variety, and let $n \in \mathbb{N}_{\geq 0}$ be a natural number. We define $V^{(n)}$ to be the models of the n-variable identities true in V:*

$$V^{(n)} := \mathbf{Mod}(Id_V(X_n)).$$

For any variety V, $V^{(n)}$ is the models of a set of identities. So, by Theorem 4, we have that $V^{(n)}$ is also a variety.

Theorem 29. *Furthermore, for any variety V, and for any $n \in \mathbb{N}_{\geq 0}$, we have that*

$$V = \bigcap_{n \in \mathbb{N}_{\geq 0}} V^{(n)}.$$

Proof. Fix any variety V and any $n \in \mathbb{N}_{\geq 0}$.

$$\begin{aligned}
X_\omega &= \bigcup_{n \in \mathbb{N}_{\geq 0}} X_n && \\
\implies Id_V(X_\omega) &= \bigcap_{n \in \mathbb{N}_{\geq 0}} Id_V(X_n) && \\
\implies V &= \mathbf{Mod}(Id_V(X_\omega)) && \text{by Theorem 4} \\
&= \bigcap_{n \in \mathbb{N}_{\geq 0}} \mathbf{Mod}(Id_V(X_n)) && \\
&= \bigcap_{n \in \mathbb{N}_{\geq 0}} V^{(n)} &&
\end{aligned}$$

□

Similarly, we can show that if $n, m \in \mathbf{N}_{\geq 0}$ such that $n \leq m$,

$$V^{(n)} \supseteq V^{(m)},$$

and so we have a chain of varieties

$$V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V.$$

We will return to discussing these varieties in section 9.

Theorem 30. *Fix an algebra* $\mathbf{A}$ *and a variety* V *in the same signature. The following are equivalent:*

1. $\mathbf{A} \in V$,
2. $\forall n \in \mathbb{N}_{\geq 0}$, $\forall a_1, \ldots, a_n \in A$,
$$\mathbf{Sg}(a_1, \ldots, a_n) \in V,$$
3. $\forall n \in \mathbb{N}_{\geq 0}$, $\forall a_1, \ldots, a_n \in A$,
$$\mathbf{Sg}(a_1, \ldots, a_n) \in V^{(n)},$$
4. $\forall n \in \mathbb{N}_{\geq 0}$,
$$\mathbf{A} \in V^{(n)}.$$

Proof.
$\underline{1 \implies 2}$
Suppose $\mathbf{A} \in V$, and fix any $n \in \mathbf{N}_{\geq}0$ and any $a_1, \ldots, a_n \in A$. Then,

$$\begin{aligned} \mathbf{Sg}(a_1, \ldots, a_n) &\in S(V) \\ &\subseteq V. && \textit{by Theorem } 2 \end{aligned}$$

$\underline{2 \implies 3}$
This follows from Theorem 29, which says that $V \subseteq V^{(n)}$.
$\underline{3 \implies 4}$
Suppose that $\forall n \in \mathbb{N}_{\geq 0}$, $\forall a_1, \ldots, a_n \in A$,

$$\mathbf{Sg}(a_1, \ldots, a_n) \in V^{(n)}.$$

Fix any $n \in \mathbb{N}_{\geq 0}$, any $a_1, \ldots, a_n \in A$, and any terms $s, t \in T_\sigma(X_n)$ such that $s \approx t \in Id_V(X_n)$.

$$\begin{aligned}
s^{\mathbf{A}}(a_1, \ldots, a_n) &= s^{\mathbf{Sg}(a_1,\ldots,a_n)}(a_1, \ldots, a_n) \\
&= t^{\mathbf{Sg}(a_1,\ldots,a_n)}(a_1, \ldots, a_n) && \textit{since } \mathbf{Sg}(a_1, \ldots, a_n) \in V^{(n)} \\
&= t^{\mathbf{A}}(a_1, \ldots, a_n) \\
\implies s^{\mathbf{A}} &= t^{\mathbf{A}} && \textit{since } a_1, \ldots, a_n \textit{ was chosen arbitrarily} \\
\implies \mathbf{A} &\models s \approx t \\
\implies \mathbf{A} &\models Id_V(X_n) \\
\implies \mathbf{A} &\in V^{(n)}
\end{aligned}$$

$\underline{4 \implies 1}$
This follows from Theorem 29, which says that if $\forall n \in \mathbf{N}_{\geq 0},\ \mathbf{A} \in V^{(n)}$, then

$$\mathbf{A} \in V.$$

$\square$

The above theorem says that to check for the membership for an algebra $\mathbf{A}$ in a variety V, it suffices to check the membership of each finitely generated subalgebra in the variety V.

Definition 29. *Let V be a variety. We define V_{fin} to be the finite models in V:*

$$V_{fin} := \{\mathbf{A} \in V \mid |A| < \omega\}.$$

By Theorem 14, we have that V_{fin} is an equational pseudovariety. Consider varieties V, W such that $V \subseteq W$. Then the finite models in V are finite models in W:

$$V_{fin} \subseteq W_{fin}.$$

Suppose instead that V, W are varieties such that $V_{fin} \subseteq W_{fin}$. What condition do we need so that $V \subseteq W$? Theorem 30 says that if $\mathbf{A} \in V$, then $\mathbf{A} \in W$ if the finitely generated subalgebras of $\mathbf{A}$ are in W. Since we have that the finite algebras in V are in W, let's assume the condition that the finitely generated algebras of V are finite:

Definition 30. *A variety V is* ***locally finite*** *if for any finitely generated algebra $\mathbf{A} \in V$,*

$$|A| < \omega.$$

Theorem 31. *Consider varieties V, W such that $V_{fin} \subseteq W_{fin}$. Then,*

$$V \textit{ is locally finite} \implies V \subseteq W.$$

Proof. Consider varieties V, W such that $V_{fin} \subseteq W_{fin}$. Suppose V is locally finite. Fix any algebra $\mathbf{A} \in V$, and any finitely generated subalgebra $\mathbf{B} \leq \mathbf{A}$.

$$\begin{aligned} \mathbf{B} &\in V && \textit{since } S(V) \subseteq V \\ \mathbf{B} &\in V_{fin} && \textit{since } V \textit{ is locally finite} \\ &\subseteq W_{fin} \\ &\subseteq W \\ \implies \mathbf{A} &\in W. && \textit{by Theorem } 30 \end{aligned}$$

□

It turns out that if a variety V is locally finite, then for any $d \in \mathbb{N}_{\geq 0}$, the d-generated algebras in V cannot be arbitrarily large:

Theorem 32. *Consider a locally finite variety V, and any $d \in \mathbb{N}_{\geq 0}$. Then there exists $N \in \mathbb{N}$ such that for any d-generated algebra $\mathbf{A} \in V$,*

$$|A| < N$$

Proof. Suppose V is a locally finite variety, and suppose for contradiction there exists $d \in \mathbb{N}_{\geq 0}$ and a sequence of d-generated algebras $\mathbf{A}_1, \mathbf{A}_2, \dots \in V$ such that for each $k \in \mathbb{N}$,

$$|A_k| > k.$$

Each algebra $\mathbf{A}_k$ is generated by some d element set $A_{k,gen} = \{a_{k,1}, \dots, a_{k,d}\}$. Let $\mathbf{S}$ be the subalgebra of $\prod_{k \in \mathbb{N}} \mathbf{A}_k$ generated by the d element set $S_{gen} = \{p_1, \dots, p_d\}$ where each

$$p_i = (a_{1,i}, a_{2,i}, \dots)$$

We claim that for any $k \in \mathbb{N}$, $\mathbf{A}_k$ is the homomorphic image of $\mathbf{S}$ under the projection map to the k^{th} co-ordinate.

Proof.
π_k is a homomorphism
Fix any n-ary operation $f \in \mathcal{F}$, and any n elements $s_1, \dots, s_n \in S \subseteq \prod_{j \in \mathbb{N}} A_j$, which we can write as sequences

$$s_i = (s_{1,i}, s_{2,i}, \dots),$$

where each $s_{j,i} \in \mathbf{A}_j$. Then,

$$\begin{aligned} \pi_k \circ f^{\mathbf{S}}(s_1, \dots, s_n) &= \pi_k \circ f^{\prod_{j \in \mathbb{N}} \mathbf{A}_j}(s_1, \dots, s_n) \\ &= \pi_k(f^{\mathbf{A}_1}(s_{1,1}, \dots, s_{1,n}), f^{\mathbf{A}_2}(s_{2,1}, \dots, s_{2,n}), \dots) \\ &= f^{\mathbf{A}_k}(s_{k,1}, \dots, s_{k,n}) \\ &= f^{\mathbf{A}_k}(\pi_k(s_1), \dots, \pi_k(s_n)). \end{aligned}$$

π_k is a surjective map
Fix any $a \in \mathbf{A}_k$. Then for some d-ary term $t \in \mathbf{T}_\sigma(X_d)$,

$$\begin{aligned} a &= t^{\mathbf{A}_k}(a_{k,1}, \dots, a_{k,d}) \\ &= t^{\mathbf{A}_k}(\pi_k(p_1), \dots, \pi_k(p_d)) && \textit{definition of projection} \\ &= \pi_k \circ t^{\mathbf{S}}(p_1, \dots, p_d) && \textit{by induction on the definition of homomorphism} \\ &\in \pi_k(S) \end{aligned}$$

□

So, S must have at least as many elements as $\mathbf{A}_k$ for any k, and is hence infinite. As $\mathbf{S}$ is also a d-generated algebra, this contradicts the local finiteness of V. □

Recall from the proof of Theorem 19 that for any $n \in \mathbb{N}$ and any signature σ, there are finitely many algebras in $\mathcal{K}_\sigma$ with an n-element universe up to isomorphism. So, a family of finite algebras has a finite upper bound on the cardinality of its members if and only if the family is finite.

Definition 31. *A pseudovariety W is **locally finite** if for any $d \in \mathbb{N}_{\geq 0}$, there exist finitely many d-generated algebras in W up to isomorphism.*

So, Theorem 32 says that if V is a locally finite variety, then V_{fin} is a locally finite pseudovariety.

9 Finitely Based Varieties and Pseudovarieties

Recall the definition of a finitely based variety:

Definition 23. *A variety V is **finitely based** if*

$$V = \mathbf{Mod}(\Sigma)$$

for some finite set of identities Σ.

Definition 32. *We say a locally finite variety V is **inherently non-finitely based** if for any locally finite variety W,*

$$V \subseteq W \implies W \textit{ is not finitely based.}$$

Inherently non-finitely based varieties are not finitely based, as if V were a locally finite, finitely based variety, then $W = V$ is a locally finite variety such that $V \subseteq W$ but W is not finitely based.

Definition 33. *We say an equational pseudovariety W is **finitely based** if there exists a finite set of identities Σ such that:*

$$W = \mathbf{Mod}_{fin}(\Sigma).$$

Theorem 33. *If V is a locally finite variety such that V_{fin} is finitely based, then V is either finitely based or inherently non-finitely based.*

Proof. Suppose V is a locally finite variety such that V_{fin} is finitely based. Suppose V is not inherently non-finitely based, so there exists a locally finite, finitely based variety W such that $V \subseteq W$.
As V_{fin} is finitely based,

$$V_{fin} = \mathbf{Mod}_{fin}(\Sigma_1)$$

for some finite set of identities Σ_1. As W is finitely based,

$$W = \mathbf{Mod}(\Sigma_2)$$

for some finite set of identities Σ_2. Let

$$K = \mathbf{Mod}(\Sigma_1 \cup \Sigma_2).$$

We claim that $K = V$, and hence V is finitely based.
$\underline{K \subseteq V}$
As $\Sigma_2 \subseteq \Sigma_1 \cup \Sigma_2$, we have that $K \subseteq W$. As W is locally finite, this means K is locally finite as well. As $\Sigma_1 \subseteq \Sigma_1 \cup \Sigma_2$, we have that $K_{fin} \subseteq V_{fin}$. So, by Theorem 31, we have that $K \subseteq V$.
$\underline{V \subseteq K}$
Fix any $\mathbf{A} \in V_{fin}$. As $\mathbf{A} \in V_{fin}$, we have that $\mathbf{A} \vDash \Sigma_1$. As $\mathbf{A} \in V_{fin} \subseteq V \subseteq W$, we have that $\mathbf{A} \vDash \Sigma_2$. So, $\mathbf{A} \vDash \Sigma_1 \cup \Sigma_2$, and hence $V_{fin} \subseteq K_{fin}$. We have by assumption that V is locally finite, so by Theorem 31, we have that $V \subseteq K$. ☐

Definition 34. *We say a locally finite variety V is **inherently non-finitely based in the finite sense** if for any locally finite equational pseudovariety W,*

$$V_{fin} \subseteq W \implies W \text{ is not finitely based.}$$

This is a stronger property than being inherently non-finitely based:

Theorem 34. *If a locally finite variety V is inherently non-finitely based in the finite sense, then it is also inherently non-finitely based.*

Proof. Suppose V is inherently non-finitely based in the finite sense. Fix any locally finite variety W such that $V \subseteq W$. Then, by Theorem 32, W_{fin} is a locally finite equational pseudovariety. As $V_{fin} \subseteq W_{fin}$, and as V is inherently non-finitely based in the finite sense,

W_{fin} must not be finitely based. This means W must not be finitely based as well: if we suppose for contradiction that $W = \mathbf{Mod}(\Sigma)$, where Σ is a finite set of identities, then $W_{fin} = \mathbf{Mod}_{fin}(\Sigma)$, making W_{fin} finitely based. So, as W must not be finitely based, so V is inherently non-finitely based. $\square$

In section 8, we introduced a sequence of varieties given a variety V:

$$V^{(n)} := \mathbf{Mod}(Id_V(X_n)).$$

If $n < m$, the n-variable identities true in V are m-variable identities true in V:

$$Id_V(X_0) \subseteq Id_V(X_1) \subseteq \cdots \subseteq Id_V(X_\omega),$$

and as a consequence, the sequence of varieties form a chain:

$$V^{(0)} \supseteq V^{(1)} \supseteq \cdots \supseteq V.$$

Theorem 29 gives us one sense in which this chain of varieties "approaches" the variety V. In a similar sense, the varieties of $V^{(n)}$ come as close as any finitely based variety containing V:

Theorem 35. *Consider varieties V, W such that $V \subseteq W$ and W is finitely based. There exists some $n \in \mathbb{N}_{\geq 0}$ such that*

$$V \subseteq V^{(n)} \subseteq W.$$

Proof. Suppose $W = \mathbf{Mod}(\Sigma)$ where Σ is a finite set of identities. Let n be the maximum number of variables that appear in any identity in Σ. So, by the substitution lemma, we may assume that each variable that appears in Σ comes from the set X_n. So, $\Sigma \subseteq Id_W(X_n) \subseteq Id_V(X_n)$, and so

$$V^{(n)} = \mathbf{Mod}(Id_V(X_n)) \subseteq \mathbf{Mod}(\Sigma) = W$$

as desired. $\square$

This theorem allows us to look at varieties that are inherently non-finitely based and varieties that are inherently non-finitely based in the finite sense in a different way:

Theorem 36. *Suppose V is a locally finite variety such that for each $n \in \mathbb{N}_{\geq 0}$, $V^{(n)}$ contains an infinite, finitely generated algebra. Then, V is inherently non-finitely based.*

Proof. Fix any finitely based variety W such that $V \subseteq W$. By Theorem 35, there exists n such that $V \subseteq V^{(n)} \subseteq W$. By assumption, $V^{(n)}$ contains an infinite, finitely generated algebra, which is in W as well, so W is not locally finite. So, as there does not exist a finitely based locally finite variety W such that $V \subseteq W$, V must be inherently non-finitely based. $\square$

Theorem 37. *Suppose V is a locally finite variety such that for each $n \in \mathbb{N}_{\geq 0}$, there exists $d > n$ such that $(V^{(n)})_{fin}$ contains infinitely many d-generated algebras, unique up to isomorphism. Then, V is inherently non-finitely based in the finite sense.*

Proof. Fix any finitely based equational pseudovariety W such that $V_{fin} \subseteq W$. That is, $W = \mathbf{Mod}_{fin}(\Sigma)$ for some finite set of identities Σ. By Theorem 35, there exists n such that $V \subseteq V^{(n)} \subseteq \mathbf{Mod}(\Sigma)$. By assumption, there exists $d > n$ such that $(V^{(n)})_{fin}$ contains infinitely many d-generated algebras, unique up to isomorphism, and hence $W = \mathbf{Mod}_{fin}(\Sigma) \supseteq (V^{(n)})_{fin}$ contains infinitely many d-generated algebras, unique up to isomorphism. So, W is not locally finite. So, as there does not exist a finitely based locally finite equational pseudovariety W such that $V_{fin} \subseteq W$, V must be inherently non-finitely based in the finite sense. □

10 Finitely Generated Varieties and Pseudovarieties

Recall that we say a variety V is finitely generated if there exists a finite family of finite algebras in the same signature $K = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ such that $V = \mathcal{V}(K)$. As varieties are closed under the operation of passing to products, $\mathcal{V}(K)$ contains the finite algebra $\mathbf{A}_1 \times \cdots \times \mathbf{A}_n$. Notice also that as $\pi_k(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n) = \mathbf{A}_k$, and as varieties are closed under the operation of passing to homomorphic images, $\mathcal{V}(\{\mathbf{A}_1, \ldots, \mathbf{A}_n\})$ contains finite family K. That is, $\mathcal{V}(K) = \mathcal{V}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$. Likewise, we say a pseudovariety V is finitely generated if there exists a finite family of finite algebras in the same signature $K = \{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ such that $V = \mathcal{V}_{fin}(K)$. Likewise, $\mathcal{V}_{fin}(K) = \mathcal{V}_{fin}(\mathbf{A}_1 \times \cdots \times \mathbf{A}_n)$. So, we have the following alternate definitions:

Definition 35. *A variety V is **finitely generated** if there exists a finite algebra $\mathbf{A}$ such that*

$$V = \mathcal{V}(\mathbf{A}).$$

Definition 36. *A pseudovariety V is **finitely generated** if there exists a finite algebra $\mathbf{A}$ such that*

$$V = \mathcal{V}_{fin}(\mathbf{A}).$$

In this section, we prove Theorem 7. We begin with a similar theorem to Theorem 23 which we restate here:

Theorem 23. *For any family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and any algebra in the same signature $\mathbf{A} \in \mathcal{K}_\sigma$, if X is a set of variables such that $X \cup \mathcal{F}_0 \neq \emptyset$ and $|X| \geq |A|$, and if $\mathbf{A} \vDash Id_K(X)$, then there exists a surjective homomorphism $h : \mathbf{F}_K(X) \to \mathbf{A}$.*

By taking $X = X_n$, Theorem 23 says that if $\mathbf{A}$ is an algebra with an n-variable universe and which models the n-variable identities true in some class of algebras K, then $\mathbf{A}$ is a homomorphic image of $\mathbf{F}_K(X_n)$. The following theorem says the same, but requires instead that $\mathbf{A}$ is an n-generated algebra.

Theorem 38. *For any family of algebras in the same signature $K \subseteq \mathcal{K}_\sigma$ and any n-generated algebra in the same signature $\mathbf{A} \in \mathcal{K}_\sigma$, if $\mathbf{A} \models Id_K(X_n)$, then there exists a surjective homomorphism $h : \mathbf{F}_K(X_n) \to \mathbf{A}$.*

Proof. Suppose $K \subseteq \mathcal{K}_\sigma$ a family of algebras in the same signature, and $\mathbf{A} \in \mathcal{K}_\sigma$ an n-generated algebra in the same signature, and $\mathbf{A} \models Id_K(X_n)$. There exists an n-element generating set $A_{gen} = \{a_1, \dots, a_n\} \subseteq_{fin} A$ such that $\mathbf{A} = \mathbf{Sg}(A_{gen})$. Let $h_0 : X_n \to A_{gen}$ be defined map

$$x_i \mapsto a_i.$$

By the universal mapping property for term algebras, there exists a homomorphism $h : \mathbf{T}_\sigma(X_n) \to \mathbf{A}$ which extends h_0. It turns out that h is also surjective:

Proof that h is surjective. By Theorem 13,

$$A = \{t^{\mathbf{A}}(a_1, \dots, a_n) \mid t \in T_\sigma(X_n)\}.$$

So, fix any $a \in A$. There exists some term $t \in T_\sigma(X_n)$ such that

$$\begin{aligned} a &= t^{\mathbf{A}}(a_1, \dots, a_n) && \textit{by Theorem } 13 \\ &= t^{\mathbf{A}}(h_0(x_1), \dots, h_0(x_n)) && \textit{by definition of } h_0 \\ &= t^{\mathbf{A}}(h(x_1), \dots, h(x_n)) && h \textit{ extends } h_0 \\ &= h \circ t^{\mathbf{T}_\sigma(X_n)}(x_1, \dots x_n) && h \textit{ is a homomorphism} \\ &\in h(\mathbf{T}_\sigma(X_n)). \end{aligned}$$

□

Consider also the natural map $\nu_\theta : \mathbf{T}_\sigma(X_n) \to \mathbf{T}_\sigma(X)/\theta_K(X_n)$ which is also a surjective homomorphism. As $\mathbf{A} \models Id_K(X_n)$, it follows by Theorem 22 that

$$\theta_K(X_n) \subseteq \theta_{\mathbf{A}}(X_n) \subseteq \ker h.$$

So, by Theorem 12 there exists an induced surjective homomorphism $\mathbf{F}_K(X_n) = \mathbf{T}_\sigma(X_n)/\theta_K(X_n) \to \mathbf{A}$ as desired. □

As a consequence of Theorem 38, we have that:

Theorem 39. *If $\mathbf{A}$ is a finite algebra, then $\mathcal{V}(\mathbf{A})$ is locally finite.*

Proof. Suppose $\mathbf{A}$ is a finite algebra, and suppose $\mathbf{B}$ is an n-generated algebra in $\mathcal{V}(\mathbf{A})$.

$$\begin{aligned} & \mathbf{B} \in \mathbf{Mod}(Id_{\mathbf{A}}(X_\omega)) && \textit{by Theorem } 27 \\ & \quad \subseteq \mathbf{Mod}(Id_{\mathbf{A}}(X_n)) && \textit{since } Id_{\mathbf{A}}(X_n) \subseteq Id_{\mathbf{A}}(X_\omega) \\ \Longrightarrow\ & \mathbf{B} \in H(\mathbf{F_A}(X_n)) && \textit{by Theorem } 38 \\ & \quad \subseteq HISP_{fin}(\mathbf{A}) && \textit{by Corollary } 2 \\ & \quad \subseteq HSP_{fin}(\mathbf{A}) && \textit{since } HI \subseteq H \\ & \quad = \mathcal{V}_{fin}(\mathbf{A}). && \textit{by Theorem } 28 \end{aligned}$$

So, as $\mathbf{B}$ turns out to be a finite algebra, $\mathcal{V}(\mathbf{A})$ is locally finite. □

So, all the theorems in section 8 that apply to locally finite varieties apply also to finitely generated varieties. We are now ready to prove Theorem 7, which we restate here:

Theorem 7. *[Generalization of Sapir] If $\mathbf{A}$ is a finite algebra in a finite signature such that the generated pseudovariety $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based and the generated variety $\mathcal{V}(\mathbf{A})$ is not finitely based, then $\mathcal{V}(\mathbf{A})$ must be inherently non-finitely based, but not inherently non-finitely based in the finite sense.*

Proof. Suppose $\mathbf{A}$ is a finite algebra. By Theorem 39, $\mathcal{V}(\mathbf{A})$ is locally finite. So, by Theorem 32, $\mathcal{V}_{fin}(\mathbf{A})$ is a locally finite pseudovariety. Also, by Theorem 28, $\mathcal{V}_{fin}(\mathbf{A})$ is an equational pseudovariety. Suppose also that $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based. Then as $\mathcal{V}_{fin}(\mathbf{A})$ is a locally finite, finitely based equational pseudovariety that contains itself, $\mathcal{V}(\mathbf{A})$ must not be inherently non-finitely based in the finite sense. Now suppose also that $\mathcal{V}(\mathbf{A})$ is not finitely based. Then by Theorem 33, $\mathcal{V}(\mathbf{A})$ must be inherently non-finitely based. □

Recall Theorem 26, which we restate here:

Theorem 26. *For a finite set of finite algebras $\mathbf{T}_1, \mathbf{T}_2, \ldots \mathbf{T}_n$ and for any $k \in \mathbb{N}_{\geq 0}$, $\mathbf{Mod}(Id_{\{\mathbf{T}_1,\mathbf{T}_2,\ldots\mathbf{T}_n\}}(X_k))$ is finitely based.*

This theorem is in fact about finitely generated varieties:

Theorem 40 (Same as Theorem 26)**.** *If V is a finitely generated variety, for any $n \in \mathbb{N}_{\geq 0}$, $V^{(n)}$ is finitely based.*

And as a consequence, we have the converse of the last two theorems of section 9 for finitely generated varieties:

Theorem 41. *Suppose V is a finitely generated variety. Then the following are equivalent:*

1. *V is inherently non-finitely based.*

2. *For each $n \in \mathbb{N}_{\geq 0}$, $V^{(n)}$ contains an infinite, finitely generated algebra. That is, $V^{(n)}$ is not locally finite.*

3. *For each $n \in \mathbb{N}_{\geq 0}$, there exists $d > n$ such that $V^{(n)}$ contains an infinite, d-generated algebra.*

Proof.
$\underline{1 \implies 2}$
Suppose for contradiction there exists n such that $V^{(n)}$ is locally finite. By Theorem 40, $V^{(n)}$ is a finitely based, locally finite variety that contains V, so V must not be inherently non-finitely based.
$\underline{2 \implies 3}$
As V is a finitely generated variety, $V = \mathcal{V}(\mathbf{A})$ for some finite algebra $\mathbf{A}$. Then:

$$\begin{aligned} \mathbf{A} &\in V \\ \implies Id_V(X_n) &\subseteq Id_{\mathbf{A}}(X_n) \\ \textit{also, } Id_V(X_n) &= Id_{HSP(\mathbf{A})}(X_n) && \textit{by Theorem } 27 \\ &\supseteq Id_{SP(\mathbf{A})}(X_n) && \textit{by Theorem } 1 \\ &\supseteq Id_{P(\mathbf{A})}(X_n) && \textit{by Theorem } 2 \\ &\supseteq Id_{\mathbf{A}}(X_n) && \textit{by Theorem } 3 \end{aligned}$$

We have by assumption that $V^{(n)}$ contains an infinite, finitely generated algebra, so fix such an algebra and call it $\mathbf{B}$. Suppose for contradiction $\mathbf{B}$ is d-generated for some $d \leq n$. Then:

$$\begin{aligned} \mathbf{B} &\in V^{(n)} \\ &= \mathbf{Mod}(Id_V(X_n)) && \textit{by definition of } V^{(n)} \\ &= \mathbf{Mod}(Id_{\mathbf{A}}(X_n)) \end{aligned}$$

We then follow a similar proof as for Theorem 39:

$$\begin{aligned} \mathbf{B} &\in H(\mathbf{F}_{\mathbf{A}}(X_n)) && \textit{by Theorem } 38 \\ &\subseteq HISP_{fin}(\mathbf{A}) && \textit{by Corollary } 2 \\ &\subseteq HSP_{fin}(\mathbf{A}) && \textit{since } HI \subseteq H \\ &= \mathcal{V}_{fin}(\mathbf{A}). && \textit{by Theorem } 28 \end{aligned}$$

But $\mathbf{B}$ being a finite algebra contradicts our assumption that $\mathbf{B}$ is infinite, so it must be the case that $\mathbf{B}$ is d-generated where $d > n$.
$\underline{3 \implies 1}$
d-generated algebras are finitely generated. This is Theorem 36. □

Theorem 42. *Suppose V is a finitely generated variety. Then the following are equivalent:*

1. *V is inherently non-finitely based in the finite sense.*
2. *For each $n \in \mathbb{N}_{\geq 0}$, there exists $d > n$ such that $(V^{(n)})_{fin}$ contains infinitely many d-generated algebras, unique up to isomorphism. That is, $(V^{(n)})_{fin}$ is not locally finite.*

Proof.
$\underline{1 \implies 2}$
Suppose for contradiction there exists n such that $(V^{(n)})_{fin}$ is locally finite. By Theorem 40, $V^{(n)}$ is a finitely based, so $(V^{(n)})_{fin}$ is a locally finite, finitely based pseudovariety that contains V_{fin}, so V must not be inherently non-finitely based in the finite sense.
$\underline{2 \implies 1}$
This is Theorem 37. $\square$

So, we can state Theorem 7 another way:

Theorem 43 (Same as Theorem 7)**.** *If $\mathbf{A}$ is a finite algebra in a finite signature such that the generated pseudovariety $\mathcal{V}_{fin}(\mathbf{A})$ is finitely based and the generated variety $\mathcal{V}(\mathbf{A})$ is not finitely based, then for any $n \in \mathbb{N}_{\geq 0}$, there exists $d > n$ such that the finitely based variety $\mathcal{V}(\mathbf{A})^{(n)}$ contains an infinite d-generated algebra, but there exists a particular $n \in \mathbb{N}_{\geq 0}$ for which $(\mathcal{V}(\mathbf{A})^{(n)})_{fin}$ contains, for any $d \in \mathbb{N}_{\geq 0}$, only finitely many d-generated finite algebras up to isomorphism.*

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