5.

(a)

```
function A = FormMatrix(u, alpha)

    beta = 1e-6;
    n = size(u, 1);
    m = sqrt(n);
    v = reshape(u,[m,m])';
    h = 1/(1+m);

    aw = zeros(m);
    ae = zeros(m);
    as = zeros(m);
    an = zeros(m);
    ac = zeros(m);

    function s = get(i ,j)
        if (i>m || i<1 || j>m || j<1)
            s = 0;
        else
            s = v(i ,j);
        end
    end

    function s = calc(v1,v2,v3,v4)
        s = 1/(2*sqrt(((v1-v2)/h)^2+((v3-v4)/h)^2+beta));
    end

    for i = 1:m
        for j = 1:m
            aw(i,j) = (-alpha/h^2)*(calc(get(i,j), get(i-1,j), get(i,j), get(i,j-1))+...
                calc(get(i,j), get(i-1,j), get(i-1,j+1), get(i-1,j)));
            ae(i,j) = (-alpha/h^2)*(calc(get(i+1,j), get(i,j), get(i+1,j), get(i+1,j-1))+...
                calc(get(i+1,j), get(i,j), get(i,j+1), get(i,j)));
            as(i,j) = (-alpha/h^2)*(calc(get(i,j), get(i-1,j), get(i,j), get(i,j-1))+...
                calc(get(i+1,j-1), get(i,j-1), get(i,j), get(i,j-1)));
            an(i,j) = (-alpha/h^2)*(calc(get(i+1,j), get(i,j), get(i,j+1), get(i,j))+...
                calc(get(i,j+1), get(i-1,j+1), get(i,j+1), get(i,j)));
            ac(i,j) = -(aw(i,j)+ae(i,j)+as(i,j)+an(i,j))+1;
        end
    end

    wcol = reshape(aw(2:m,:)',[],1);
    ecol = reshape(ae(1:m-1,:)',[],1);
    scol = reshape(as',[],1);
    ncol = reshape(an',[],1);
    ccol = reshape(ac',[],1);
    A = sparse(diag(wcol,-m)+diag(ecol,m)+diag(scol(2:n),-1)+diag(ncol(1:n-1),1)+diag(ccol));
```

```
end
```

```
function u0 = FormRHS(X)
    u0 = reshape(X',[],1);
end
```

(b)

```
function [x,iter] = Jacobi(A,b,x_initial,maxiter,tol)
    D = diag(diag(A));
    x = x_initial;
    for iter = 0:maxiter
        r = b - A*x;
        if (norm(r)<=tol*norm(b))
            break
        end
        x = x + D^(-1)*r;
    end
end
```

```
function [x,iter] = GS(A,b,x_initial,maxiter,tol)
    D = diag(diag(A));
    L = tril(A);
    x = x_initial;
    for iter = 0:maxiter
        r = b - A*x;
        if (norm(r)<=tol*norm(b))
            break
        end
        x = x + (D+L) \ r;
    end
end
```

```
function [x,iter] = SOR(omega,A,b,x_initial,maxiter,tol)
    D = diag(diag(A)/omega);
    L = tril(A);
    x = x_initial;
    for iter = 0:maxiter
```

```
        r = b - A*x;
        if (norm(r)<=tol*norm(b))
            break
        end
        x = x + (D+L) \ r;
    end
end
```

```
function [x,iter] = CG(A,b,x_initial,maxiter,tol)
    x = x_initial;
    r = b - A*x;
    p = r;
    rnew = r'*r;
    for iter = 0:maxiter
        if (norm(r)<=tol*norm(b))
            break
        end
        alpha = rnew / (p'*A*p);
        x = x + alpha * p;
        r = r - alpha * A * p;
        rold = rnew;
        rnew = r' * r;
        beta = rnew / rold;
        p = r + beta * p;
    end
end
```

(c)

Find the best omega value for each image size:

```
function [] = findBest()
    tol = 1e-2;
    trial = 1:100;
    maxiter = 20000;
    k=8;

    size = [16, 32, 64, 128];
    alpha = [6.4e-2, 3.2e-2, 1.6e-2, 8e-3];

    for i=1:4
        m = size(i);
        a = alpha(i);
        bestOmega = 0;
        besttime = 1e8;
```

```
        bestiter=1e8;
        [~,z]=set_image(m);
        u0=FormRHS(z);
        for t=trial
            x=u0;
            totaliter=0;
            tic
            for j=1:k
                A=FormMatrix(u0,a);
                [x,iter] = SOR(t,A,u0,x,maxiter,tol);
                totaliter = totaliter+iter;
            end
            time = toc;
            if (totaliter<=bestiter && time<besttime)
                bestiter = totaliter;
                besttime = time;
                bestOmega = t;
            end
        end
        fprintf("grid size is: %d\r\n", m);
        fprintf("total time is: %d\r\n", besttime);
        fprintf("best omega is: %d\r\n\r\n", bestOmega);
    end
end
```

The best omega values are:

```
grid size is: 16

total time is: 2.746200e-03

best omega is: 14


grid size is: 32

total time is: 7.904280e-02

best omega is: 87


grid size is: 64

total time is: 1.099489e+00

best omega is: 79


grid size is: 128

total time is: 1.330076e+01

best omega is: 88
```

The denoise function is:

```
function [] = denoise()
    tol=1e-2;
    maxiter=20000;
    k=8;
    size = [16,32,64,128];
    alpha = [6.4e-2, 3.2e-2, 1.6e-2, 8e-3];
    omega = [14, 87, 79, 88];
    image_size = zeros(16,1);
    method = strings(16,1);
    times = zeros(16,1);
    iterations = zeros(16,1);

    for i=1:4
        m = size(i);
        a = alpha(i);
        o = omega(i);
        [~,z]=set_image(m);
```

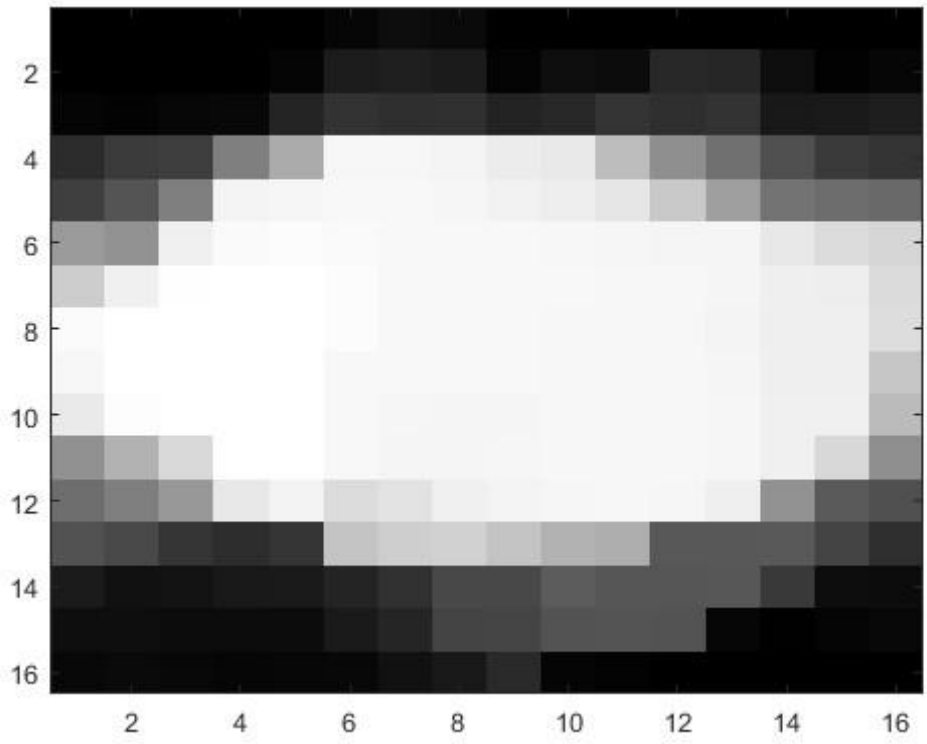
```
        u0 = FormRHS(z);
        iterjacobi = 0;
        itergs = 0;
        itersor = 0;
        itercg = 0;
        x_j = u0;
        x_g = u0;
        x_s = u0;
        x_c = u0;

        method((i-1)*4+1)="jacobi";
        image_size((i-1)*4+1)=m;
        tic
        for j=1:k
            A = FormMatrix(x_j,a);
            [x_j,iter] = Jacobi(A,u0,x_j,maxiter,tol);
            iterjacobi = iterjacobi + iter;
        end
        time = toc;
        times((i-1)*4+1)=time;
        iterations((i-1)*4+1)=iterjacobi;

        method((i-1)*4+2)="GS";
        image_size((i-1)*4+2)=m;
        tic
        for j=1:k
            A = FormMatrix(x_g,a);
            [x_g,iter] = GS(A,u0,x_g,maxiter,tol);
            itergs = itergs + iter;
        end
        time = toc;
        times((i-1)*4+2)=time;
        iterations((i-1)*4+2)=itergs;

        method((i-1)*4+3)="SOR";
        image_size((i-1)*4+3)=m;
        tic
        for j=1:k
            A = FormMatrix(x_s,a);
            [x_s,iter] = SOR(o,A,u0,x_s,maxiter,tol);
            itersor = itersor + iter;
        end
        time = toc;
        times((i-1)*4+3)=time;
        iterations((i-1)*4+3)=itersor;

        method((i-1)*4+4)="CG";
        image_size((i-1)*4+4)=m;
        tic
        for j=1:k
```

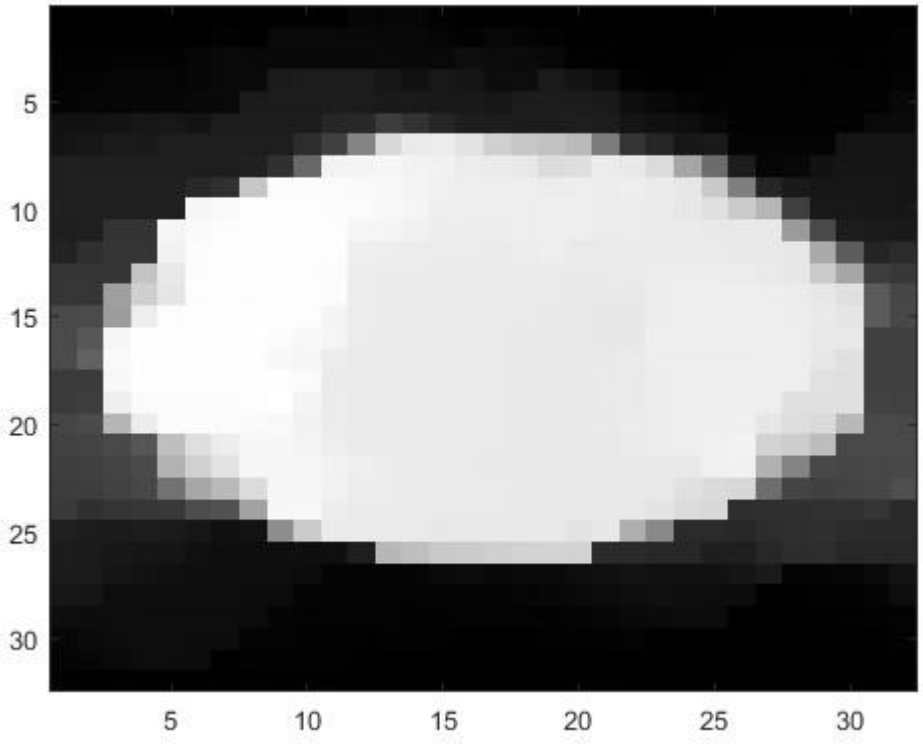
```
            A = FormMatrix(x_c,a);
            [x_c,iter] = CG(A,u0,x_c,maxiter,tol);
            itercg = itercg + iter;
        end
        time = toc;
        times((i-1)*4+4)=time;
        iterations((i-1)*4+4)=itercg;
    end
    t = table(method,image_size,times,iterations);
    disp(t);

end
```

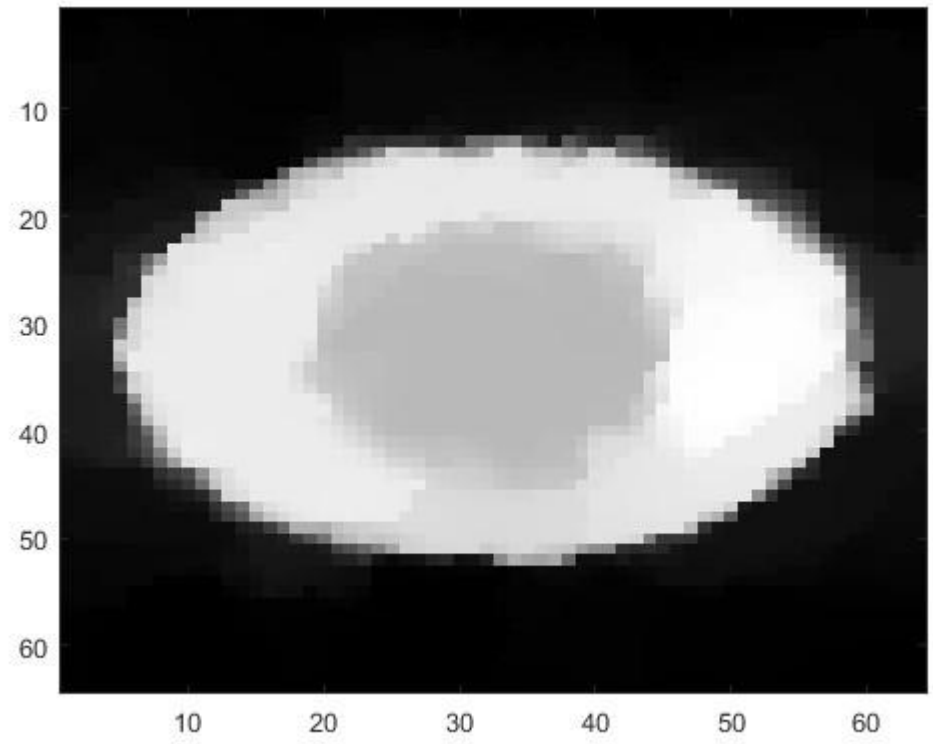
The table containg running time and iteration is shown below:

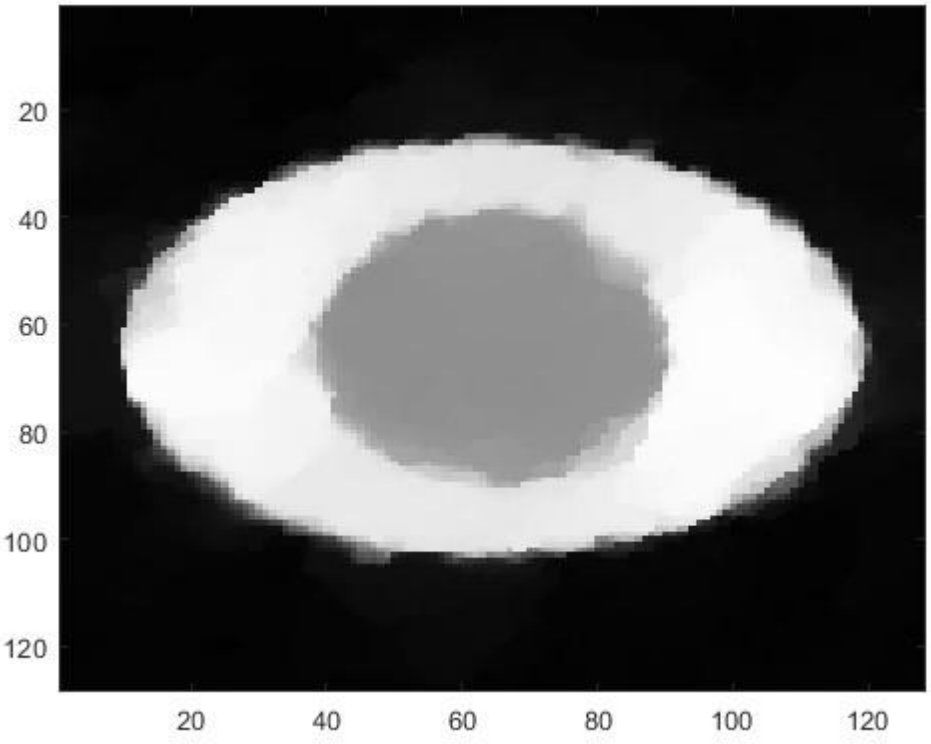
method	image_size	times	iterations
"jacobi"	16	0.055504	12877
"GS"	16	0.053268	9255
"SOR"	16	0.024585	3736
"CG"	16	0.014355	932
"jacobi"	32	0.40106	26689
"GS"	32	0.31576	13207
"SOR"	32	0.1752	4917
"CG"	32	0.12043	1838
"jacobi"	64	2.4562	31003
"GS"	64	2.6238	22288
"SOR"	64	1.6841	8268
"CG"	64	1.3745	3949
"jacobi"	128	19.959	32901
"GS"	128	23.32	33218
"SOR"	128	17.693	12131
"CG"	128	15.893	5471

And the result denoised images are like:


2
4
6
8
10
12
14
16
2
4
6
8
10
12
14
16


5
10
15
20
25
30
5
10
15
20
25
30


10
20
30
40
50
60
10
20
30
40
50
60


20
40
60
80
100
120
20
40
60
80
100
120

Comments:

From the table, we see that the running costs of four methods show a clear pattern with respect to four different image sizes. The time cost of Jacobi is the highest, while the CG method is the lowest. SOR method has a considerably lower cost and iterations than Jacobi and GS, and CG is much lower than all three. However, with the image size increases, we see that the GS method becomes more and more inefficient. When the image size is 128, the cost of running GS and iterations are actually higher than Jacobi.