Quantile functions

6 marks

Quantile functions. Suppose a continuous random variate X has a strictly increasing cumulative distribution function $F_X(x)$, a continuous density $f_X(x)$, and a quantile function $Q_X(p)$.

a. **(2 marks)** Suppose $U \sim U(0,1)$. Define a random variate $Y = Q_X(U)$. Prove that $Pr(Y \leq a) = F_X(a)$ for any value of a, and hence that Y has the same distribution as does X.

$$
\begin{aligned}
Pr(Y \leq a) &= Pr(Q_x(U) \leq a) \\
&= Pr(F_X^{-1}(U) \leq a) \text{ as quantile is the inverse of cdf} \\
&= Pr(F_X(F^{-1}(U)) \leq F_X(a)) \\
&= Pr(U \leq F_X(a)) \\
&= F_X(a) \text{ as } U \sim U(0,1)
\end{aligned}
$$

Therefore, $Pr(Y \leq a) = F_X(a)$ as required.

b. **(4 marks)** Let $Y = aX + b$ for some constants $a > 0$ and b. Prove that the quantile function $Q_Y(p)$ for Y is related to that of X as

$$Q_Y(p) = aQ_X(p) + b.$$

$$
\begin{aligned}
\text{Since } Pr(Y \leq Q_Y(p)) &= p \text{ by definition of quantile function} \\
\text{and } Pr(X \leq Q_X(p)) &= p \text{ by definition of quantile function}
\end{aligned}
$$

$$
\begin{aligned}
\text{So, } Pr(Y \leq Q_Y(p)) &= Pr(X \leq Q_X(p)) \\
&= Pr(aX + b \leq aQ_X(p) + b) \\
&= Pr(Y \leq aQ_X(p) + b) \text{ as } Y = aX + b
\end{aligned}
$$

To make $Pr(Y \leq Q_Y(p)) = Pr(Y \leq aQ_X(p) + b)$, we must have $Q_Y(p) = aQ_X(p) + b$

Therefore, $Q_Y(p) = aQ_X(p) + b$ as required