ESE 546, FALL 2023

HOMEWORK 5

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Solution 1. Your solution goes here.
(a)

$$
\begin{aligned}
KL(q||p) &= \sum_{w \in W} q(w) log \frac{q(w)}{p(w)} \\
&= \sum_{w \in W} q(w) log \frac{q(w)}{\frac{e^{-\beta\Phi(w)}}{Z(\beta)}} \\
&= \sum_{w \in W} q(w)(log q(w) - log \frac{e^{-\beta\Phi(w)}}{Z(\beta)}) \\
&= \sum_{w \in W} q(w)[log q(w) - (log e^{-\beta\Phi(w)} - log Z(\beta))] \\
&= \sum_{w \in W} q(w)[log q(w) - (-\beta\Phi(w) - log Z(\beta))] \\
&= \sum_{w \in W} q(w)[log q(w) + \beta\Phi(w) + log Z(\beta)] \\
&= \sum_{w \in W} q(w)\beta\Phi(w) + \sum_{w \in W} q(w) log q(w) + \sum_{w \in W} q(w) log Z(\beta) \\
&= \beta \sum_{w \in W} q(w)\Phi(w) + \sum_{w \in W} q(w) log q(w) + log Z(\beta) \sum_{w \in W} q(w) \\
&= \beta \sum_{w \in W} q(w)\Phi(w) + \sum_{w \in W} q(w) log q(w) + log Z(\beta)
\end{aligned}
$$

(b)

Since $logZ(\beta) = log \sum_{w \in W} \frac{e^{-\beta\Phi(w)}}{Z(\beta)}$, we have:

$$
\begin{aligned}
\frac{d}{d\beta} logZ(\beta) &= \frac{d}{d\beta} log \sum_{w \in W} e^{-\beta\Phi(w)} \\
&= \frac{1}{\sum_{w \in W} e^{-\beta\Phi(w)}} \frac{d}{d\beta} \sum_{w \in W} e^{-\beta\Phi(w)} \\
&= \frac{1}{\sum_{w \in W} e^{-\beta\Phi(w)}} \sum_{w \in W} -\Phi(w) e^{-\beta\Phi(w)} \\
&= -\frac{1}{\sum_{w \in W} e^{-\beta\Phi(w)}} \sum_{w \in W} \Phi(w) e^{-\beta\Phi(w)} \\
&= -\sum_{w \in W} \frac{\Phi(w) e^{-\beta\Phi(w)}}{\sum_{w \in W} e^{-\beta\Phi(w)}} \\
&= -\sum_{w \in W} \Phi(w) \frac{e^{-\beta\Phi(w)}}{\sum_{w \in W} e^{-\beta\Phi(w)}} \\
&= -\sum_{w \in W} \Phi(w) p(w) \text{ as } p(w) = \frac{e^{-\beta\Phi(w)}}{Z} \\
&= -E_{w \sim p(w)}[\Phi(w)]
\end{aligned}
$$

Therefore, $E_{w \sim p(w)}[\Phi(w)] = \frac{\partial logZ(\beta)}{\beta} \approx E_{w \sim q(w)}[\Phi(w)]$ $(p(w) \approx q(w))$ as required

(c)

$q^* = \min_{q \in Q_2} KL(q||p) = \sum_{w \in W} q(w) log \frac{q(w)}{p(w)}$. $p(w) = 0$ implies $q(w) = 0$ as well, or $KL(q||p) = \infty$. Since $w_i \in \{0, 1\}\, \forall i = 1....N$ and the probability of $w_i = 1$ is 0, $q(w = 1) = \prod_{i=1}^{N} q(w_i = 1) = 0$. Then, $q^*(w_i) = 1 - w_i$, and $q^*(w = 0) = \prod_{i=1}^{N} q(w_i = 0) = 1$, otherwise 0. Therefore, we have the distribution:

$$\begin{aligned} KL(q^*||p) &= \sum_{w \in W} q^*(w) log \frac{q^*(w)}{p(w)} \\ &= q^*(w = 0) log \frac{q^*(w = 0)}{p(w = 0)} \\ &= 1 \times log \frac{1}{\frac{1}{2^N - 1}} \\ &= log(2^N - 1) \end{aligned}$$

$q^*(w)$ is not even close to $p(w)$. $q^*(x)$ is not 0 only when all $w_i = 0$ and $p(x) = \frac{1}{2^N - 1}$ when all w_i is not 0.

(d)

$p(x) = (1 - \prod_i^N w_i)\frac{1}{2^N-1}$, then:

$$\begin{aligned}
KL(q||p) &= \sum_{w \in W} q(w) log \frac{q(w)}{p(w)} \\
&= \sum_{w} (1 - w_1) \prod_{i=2}^{N} log \frac{(1 - w_i) \prod_{i=2}^{N} q_i(w_i)}{(1 - \prod_{i=1}^{N} w_i)\frac{1}{2^N-1}} \\
&= \sum_{w_2..w_N} \prod_{i=2}^{N} q_i(w_i) log \prod_{i=2}^{N} q_i(w_i) \frac{1}{2^N - 1} \\
&= \sum_{w_2..w_N} \prod_{i=2}^{N} q_i(w_i) (\sum_{j=2}^{N} log q_j(w_j)) + log(2^N - 1)
\end{aligned}$$

The first term is the negative. the uniform distribution minimizes the KL divergence, where $q^*(w_i) = \frac{1}{2}$, then $q^*(w) = (1 - w_1) \prod_{i=2}^{N} q^*(w_i) = \frac{1-w_1}{2^{N-1}}$. Therefore:

$$\begin{aligned}
KL(q^*||p) &= -\sum_{w_2..w_N} log(2^{N-1}) \frac{1}{2^N - 1} + log(2^N - 1) \\
&= -log(2^{N-1}) + log(2^N - 1) \\
&= log \frac{2^N - 1}{2^{N-1}}
\end{aligned}$$

Now $q^*(w)$ looks similar to $p(w)$

Solution 2. Your solution goes here.
(i)

check code

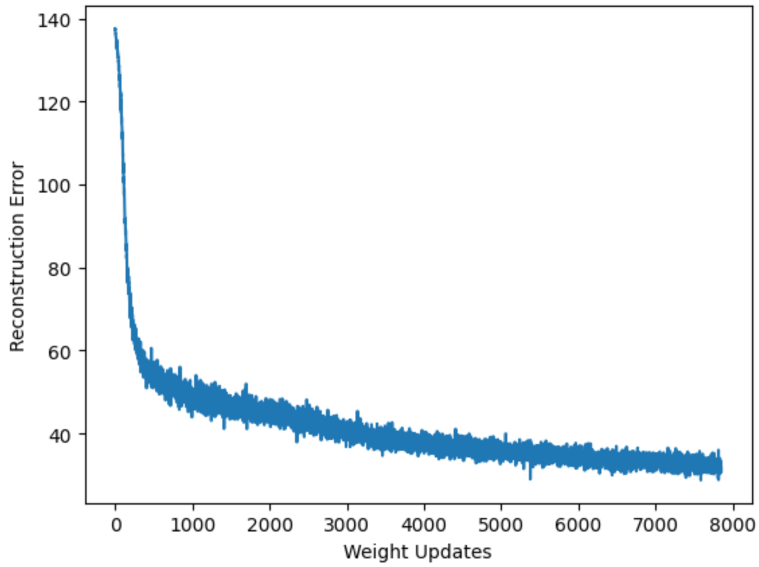
(ii)

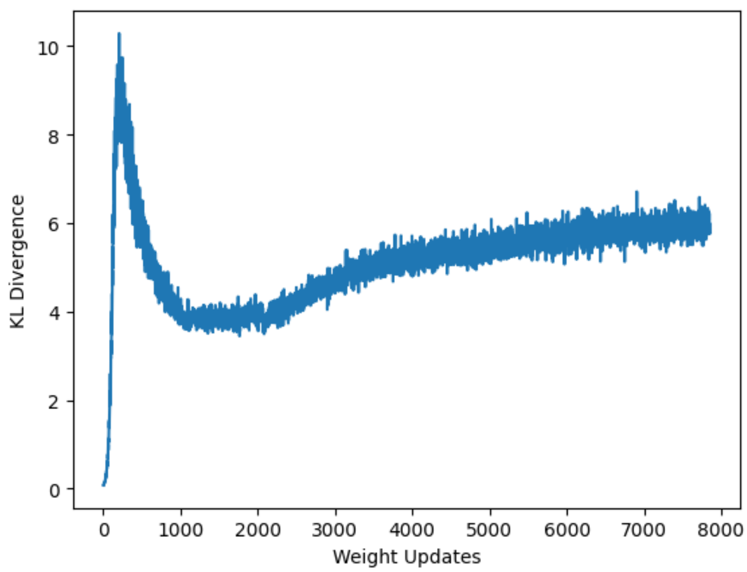
check code

```
Epoch [1/50], Loss: 116.9643
Epoch [2/50], Loss: 75.2627
Epoch [3/50], Loss: 63.2424
Epoch [4/50], Loss: 58.6469
Epoch [5/50], Loss: 56.1024
Epoch [6/50], Loss: 54.4040
Epoch [7/50], Loss: 53.0902
Epoch [8/50], Loss: 51.9848
Epoch [9/50], Loss: 51.2476
Epoch [10/50], Loss: 50.4601
Epoch [11/50], Loss: 49.9479
Epoch [12/50], Loss: 49.5056
Epoch [13/50], Loss: 49.1121
Epoch [14/50], Loss: 48.4276
Epoch [15/50], Loss: 47.8799
Epoch [16/50], Loss: 47.3366
Epoch [17/50], Loss: 46.7608
Epoch [18/50], Loss: 46.1849
Epoch [19/50], Loss: 45.4961
Epoch [20/50], Loss: 45.0622
Epoch [21/50], Loss: 44.5507
Epoch [22/50], Loss: 44.0811
Epoch [23/50], Loss: 43.7420
Epoch [24/50], Loss: 43.4028
Epoch [25/50], Loss: 43.0608
Epoch [26/50], Loss: 42.8091
Epoch [27/50], Loss: 42.4771
Epoch [28/50], Loss: 42.2195
Epoch [29/50], Loss: 42.0273
Epoch [30/50], Loss: 41.7019
Epoch [31/50], Loss: 41.4625
Epoch [32/50], Loss: 41.3099
Epoch [33/50], Loss: 41.0869
Epoch [34/50], Loss: 40.8619
Epoch [35/50], Loss: 40.6930
Epoch [36/50], Loss: 40.4849
Epoch [37/50], Loss: 40.2223
Epoch [38/50], Loss: 40.1150
Epoch [39/50], Loss: 39.9316
Epoch [40/50], Loss: 39.8060
Epoch [41/50], Loss: 39.5413
Epoch [42/50], Loss: 39.4373
Epoch [43/50], Loss: 39.2686
Epoch [44/50], Loss: 39.2336
Epoch [45/50], Loss: 38.9682
Epoch [46/50], Loss: 38.8976
Epoch [47/50], Loss: 38.7460
Epoch [48/50], Loss: 38.6323
Epoch [49/50], Loss: 38.4808
Epoch [50/50], Loss: 38.3998
```

(iii)

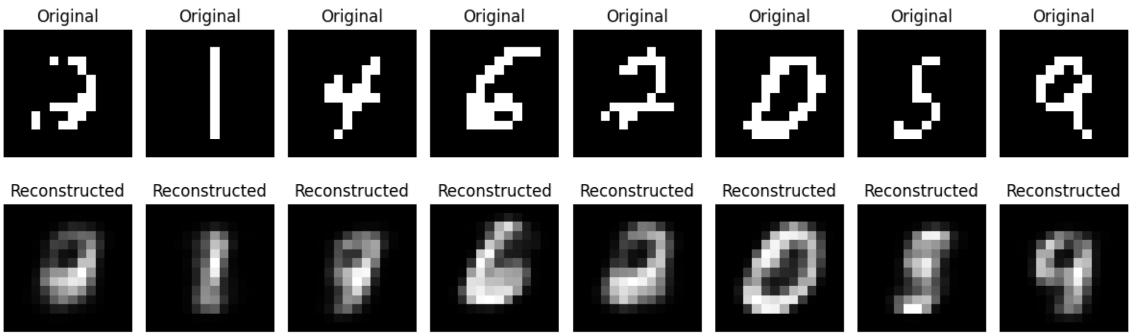
check code





(iv)

check code



(v)

check code