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BIFURCATIONS, PERIOD DOUBLINGS AND CHAOS IN CLARINET-LIKE SYSTEMS

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C. Maganza and R. Caussé,
IRCAM, 31 rue St-Merri, F-75004 Paris
and
F. Laloë
Laboratoire de Spectroscopie Hertzienne de l'E.N.S.
24 rue Lhomond, F-75005 Paris

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Wind instruments provide interesting hydrodynamical systems where non-linearities are important but well localized. A simple analysis shows that these systems should undergo Feigenbaum-type route to chaos, with a cascade of period doublings. Experiments have been performed with an acoustical resonator and an "artificial" excitation (non-linearities controlled by either analogic or digital devices); they have confirmed these predictions.

Many musical instruments are in essence non-linear physical systems since their role is to convert a (quasi) constant force, pressure, etc... applied by the player into an oscillatory sound wave. For a physicist, the study of wind instruments is actually nothing but a subfield of hydrodynamics, since it amounts to studying the motion of the air inside and outside the instrument; the vibrations of the walls of the acoustical cavity are known to play a negligible role in the direct radiation of sound in the air, at least for woodwind instruments [1]. Important progress has been made recently on non-linear hydrodynamics, instabilities, turbulence, etc... (for a review, see for example refs [2] or [3]) and one could wonder how these ideas could apply to the study of musical instruments. One specific property of wind instruments is that the non-linearities are indeed important, but well localized -mostly in the excitation sytem [4][5]; this contrasts with ordinary hydrodynamics where they are present at every point in the fluid [5]. The aim of the present work is to study how these localized non-linearities can drive instabilities and chaos. A related work on acoustical chaos, including sound propagation but not acoustical resonators, is described in ref. [6].

The physical study of musical instruments is not new and much basic work was already done in the last century [7][8] or 50 years ago [9]. More recently, a number of advances in our understanding of wind instruments have occurred; for general reviews, see for example refs [4], [11] or [1]. Examples of important new results are the studies of the role of the cut off frequency of hole lattices [4][10], and of good impedance peak cooperation [4][12]. Here, we take a model of the functioning of a wood-wind instrument which does not necessarily include all these physical effects, but gives the simplest description of non-linear generation of sound from a constant pressure. The model we use draws heavily from the work published in refs [13] and [14] (sections II-B and II-D); see in particular the short discussion of period doubling in Appendix A of ref. [14]. Among the instruments, we choose to concentrate on the clarinet because it is simpler in many respects: its acoustical cavity is approximately cylindrical and the excitation mechanism uses a simple reed; double reeds for the oboe or bassoon, or flute-like excitation mechanisms are intrinsically more complicated. A clarinet reed is merely a thin, almost flat, flexible wedge of cane which moves under the effect of the pressures difference applied on both its sides (it tends to "close" or "open"); this changes the coupling between the applied pressure P_o inside the mouth of the player and the pressure p inside the acoustical resonator. A simple way to characterize the reed is to use a non linear function, f, which gives the flow of air entering the clarinet as a function of the pressure difference $\delta p = P_o - p$; see fig. 1-a. For a detailed discussion of the shape of the curve which gives f as a function of δp, the reader is referred for instance to [4] and [11]. We simply note here that this curve includes negative (differential) impedance parts; as is well known in electronics for example, such negative impedances are required to sustain permanent oscillations.

In the model, we include only two variables, the pressure p and the flow f, which are both functions of time t; more precisely, we define f as the flow divided by the cross section of the cylindrical resonator (f then gives the air velocity). Since we ignore the dynamics of the reed, we simply write as a first equation:

$$f(t) = F\left[\, p(t);\ P_o \right] \qquad (1)$$

The second equation is obtained by writing a boundary condition which expresses the fact that the acoustical wave is a purely outgoing wave from the instrument (energy flows out from the bell and the open holes). If $f(\omega)$ and $p(\omega)$ are the Fourier transform of f(t) and p(t), this gives:

$$\bar{p}(\omega) = Z(\omega)\ \bar{f}(\omega)$$

where $Z(\omega)$ is the acoustical impedance of the resonator measured at a point close to the reed. If G(t) is the inverse Fourier transform of $Z(\omega)$, one then obtains:

$$p(t) = \int_0^\infty d\tau\ G(\tau)\ f(t-\tau) \qquad (2)$$

Equations (1) and (2) provide a closed system: (1) is non-linear, but local in time; on the other hand, (2) is linear but includes multiple time delays, as we now discuss.

An idealized resonator [15] can be obtained by considering a tube with constant section ending on a small acoustical impedance ϵZ_o, assumed to be real, positive, and frequency independent ($Z_o = \rho c$ is the acoustical impedance in the open air). If any other source of dissipation (viscosity, heat conduction) is ignored, one easily obtains:

$$G(t) = Z_o \left\{ \delta(t) + 2 \sum_{n=1}^{\infty} \left[- \frac{1-\epsilon}{1+\epsilon} \right]^n \delta\,(t-nT) \right\} \quad (3)$$

where δ is the Dirac peak function, and $T = 2L/c$ is the time taken by a sound wave to travel twice the length L of the tube. Clearly, this form of G(t) implies multiple delays, and complicates its solution. As discussed in [13] and [14], an elegant way to avoid this difficulty is to introduce two new variables X^+ and X^-, the amplitudes of the outgoing and ingoing waves:

$$X^{\pm}(t) = 2^{-1/2} \left[p(t) \pm Z_o\, f(t) \right] \quad (4)$$

Equation (2) then becomes :

$$X^-(t) = \int_0^{\infty} d\tau\, R(\tau)\, X^+(t-\tau) \quad (5)$$

where $R(\tau)$ is the so-called "reflexion function"; its Fourier transform $\overline{R}(\omega)$ is obtained from $Z(\omega)$ by the homographic transformation:

$$\overline{R}(\omega) = \left[Z(\omega) - Z_o \right] \left[Z(\omega) + Z_o \right]^{-1} \quad (6)$$

For the idealized resonator considered above, equation (3) leads to :

$$R(\tau) = -\delta(t-T) \times \left[Z_o - \epsilon \right] \left[Z_o + \epsilon \right]^{-1} \quad (7)$$

with only one time delay.

The convenient variables are therefore X^+ and X^- rather than p and f. Equation (1) can be rewritten:

$$X^+ = \hat{F} \left[X^-;\, P_o \right] \quad (8)$$

where the graph of $\hat{F}$ is obtained from that of F (with variables p and $Z_o f$) by a 45° rotation around the origin. Finally, if we set:

$$\mathcal{F}(x, P_o) = (Z_o - \epsilon)(Z_o + \epsilon)^{-1} \hat{F}(-x; P_o) \qquad (9)$$

and make use of (7), we ~~simply~~ obtain the final equation:

$$\left[-X^-(t) \right] = \mathcal{F} \left[-X^-(t-T) \right] \qquad (10)$$

The the approximations that we have made lead to a particularly simple result: the time dependence of the variable $Y = -X^-(t)$ is given by an iteration with the $\mathcal{F}$ function. We shall discuss here only particular solutions (it is easy to generalize), step-like solutions which remain constant and equal to Y_n in any time interval $nT < t < (n+1)T$, and then jump to the next value:

$$Y_{n+1} = \mathcal{F} \left[Y_n; P_o \right] \qquad (11)$$

The corresponding geometrical construction is shown in fig. 1-b. When the time dependence of X^- is obtained, equation (8) gives X^+ and (4) allows one to come back to p and f. The time variations obtained in this way are similar to those shown in fig. 5 of ref. [14].

Iterations and Feigenbaum type scenarios [16] appear often in the study of strongly dissipative systems [3]. This is not the case here, since ϵ is usually small (typically less than 0.1) and can even be put to zero. Another difference with the usual case is that the "control parameter" P_o does not correspond to a "gain" on the non-linear curve, but to a translation along a line at 45° to the axes. For small values of P_o, the fixed point (intersection of the 45° line with the curve) is stable since it has a slope between −1 and +1: no oscillation occurs. When it reaches the maximum of the p-f curve of fig. 1-a (that is when the differential impedance changes sign), the fixed point F becomes unstable: the clarinet starts to sound. As shown in fig. 2, the two values between which Y oscillates are obtained by intersecting the p-f curve with its symmetric with respect to the f axes. If the curve has a steeper slope in the small pressure difference region, as in fig. 2-a, small oscillations between almost equal values of Y are obtained just above threshold. On the other hand, if the steeper slope occurs in the region where the reed closes, the bifurcation takes a different character since the system jumps suddenly to a large level of oscillation (fig. 2-b). Fortunately, as noted by the authors of ref. [14], the p-f curve of real clarinets belongs to the first class of asymmetry [4], which ensures the possibility for small oscillations and pianissimo playing.

It is well known [16] that the stability of a 2-cycle of oscillations can conveniently be discussed in terms of the second iterate of $\mathcal{F}^1$, for which the 2-cycle corresponds to two stable fixed points Q and R. When the slope at these points exceeds 1 (in absolute value), the cycle becomes unstable and gives rise to a 4-cycle: a period doubling occurs in the system. Fig. 3 shows the results of a computer calculation where this phenomenon is visible: part a shows the time evolution of $Y=-X^-$; part b the second iterate of $\mathcal{F}^1$. Period doubling should therefore in principle be observable in clarinet-like systems. Nevertheless, in practice, with f-p characteristics similar to that of fig. 1-a, our numerical calculations have shown that period doubling can only be obtained in a rather narrow range of the parameters, which might explain why the phenomenon does not seem to be known among real clarinet players.

To do an experiment, it was therefore natural to try to obtain more flexibility on the non-linearity introduced by the acoustical excitation system. The principle involved is shown in fig. 4; an ordinary acoustical resonator was used, in several cases a real clarinet, but the non-linear reed excitation was replaced by an electroacoustic equivalent: a microphone measuring the acoustical pressure p inside the tube, a non-linear system to include controlled non-linearities in the feedback loop, and finally a loudspeaker to create a p depending air velocity f. The non-linear system was either an analog electronic device, as in ref. [6], or included digital processing of the data with a computer. The latter case gives even more flexibility on the choice of the non-linear function, but requires using analog-digital converters, which introduces time delays (of the order of 100μs in our experiment with two converters and the 4C computer of the IRCAM).

Fig. 5 shows the results obtained in an experiment where the non-linearity was obtained with an analog electronic circuit generating the function x^3-x; the resonator was simply a plastic tube (lentgh: 14cm; inner diameter: 2cm). The results shown in fig. 5 are the Fourier spectra provided by the spectrum analyzer; they correspond to increasing values of the gain in the non-linear reaction loop. A series of three successive period doublings can be seen, the last one resulting in a very low pitch sound (70Hz), which is very unusual with such a short resonator tube. All the period doublings could be clearly heard as sounds jumping one octave below, superimposed on some random acoustical noise, especially for the third bifurcation which is rather unstable. Beyond this bifurcation, acoustical chaos could be seen on the spectrum analyzer and heard in the laboratory. The number of observed period doublings might have been limited by the narrow domain of the experimental parameters in which they occur, but also by the poor efficiency of the electroacoustic components at low frequencies. The actual experiment is of course significantly different from an ideal experiment; for example phase rotations and band pass limitation are introduced by the loudspeaker. In practice, the time variations of the acoustical pressure have ~~of course~~ much smoother variations than in fig. 3-a. Also, in the period doubling regime before the chaos limit, some random noise was already present, making the phenomenon slightly less clearly audible (and probably less interesting musically!).

Similar results were also obtained with digitally generated non-linearities, including experiments with a real clarinet (with its mouthpiece removed). In the latter case, we never obtained more than two period doublings. This is not extremely surprising: with a real clarinet, the bore is not cylindrical (lateral holes, bell, etc...), and the reflexion function cannot be well approximated by one delta function. We were never able to observe any period tripling in any of the experimental conditions. A more detailed report on the experimental results obtained in various situations (e.g. with different sorts of non-linearities) will be published elsewhere.

In conclusion, woodwind instruments belong to a class of non-linear systems which exhibit interesting behaviour, especially if one allows for more flexible non-linearities. It has been known for a long time [7] that woodwind instruments can generate harmonic frequencies which do not correspond to any resonance of the resonator (e.g. the even harmonics in a clarinet tone: the effect is simply a frequency doubling occuring in the non-linear excitation). The period doubling effects observed here show that even the fundamental tone of the generated sound can also, in some cases, fall at a frequency which is completely much below any resonance of the tube.

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FIGURE CAPTIONS

Fig. 1 :

(a) Non-linear characteristics of the excitation system, giving the air flow f as a function of the pressure difference $\delta p=P_0-p$ across the reed. When $\delta p = P_c$, the reed closes and f vanishes; (b) Geometrical construction giving the successive values of δp and f, obtained by a non-linear iteration using the function of fig. 1-a after translation and a 45° rotation. 0, 1, 2, 3 are successive iterates. F is an (unstable) fixed point. The case shown corresponds to no dissipation ($\epsilon=0$).

Fig. 2 :

Geometrical construction of the permanent oscillation values just above threshold, obtained by intersecting the characteristic curve with its symmetric with respect to the f axis. In fig. a, small oscillations are obtained just above threshold, but not in fig. b.

Fig. 3 :

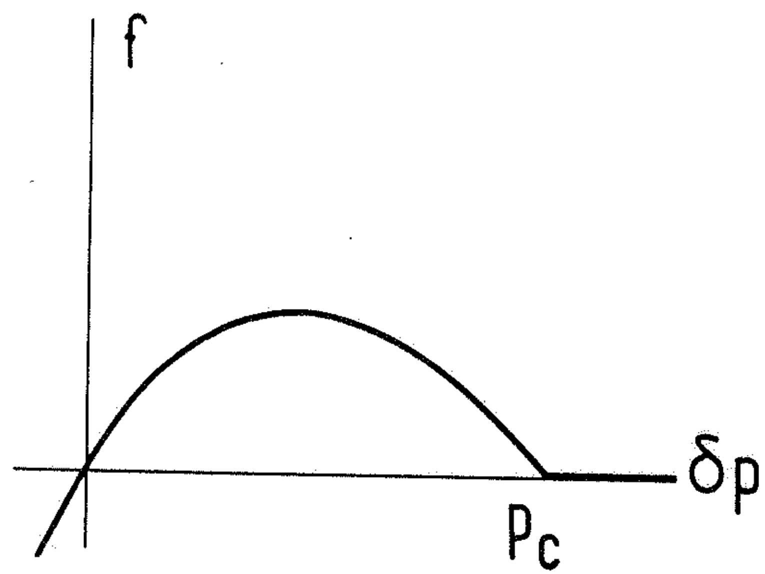
(a) Results of a computer iteration exhibiting permanent period doubling after a short transient regime. (b) The second iterate of $\mathcal{F}$; Q and R give the permanent regime of normal oscillation, until they become unstable (slope less than −1): then, period doubling occurs.

Fig. 4 :

Sketch of the experimental set up. The reaction loop includes a microphone M, a variable gain amplifier, a non-linear system N. L. (either an analogic circuit or a digital device including AD converters and a computer), a power amplifier A and finally a loudspeaker LS. The signals are analysed through a spectrum analyzer S. A.

Fig. 5 :

Results of our experiments showing the spectra obtained on the S. A. for increasing values of the gain in the non-linear feedback loop. Three period doublings can be seen (followed by acoustical chaos with a continuous spectrum, not shown here).



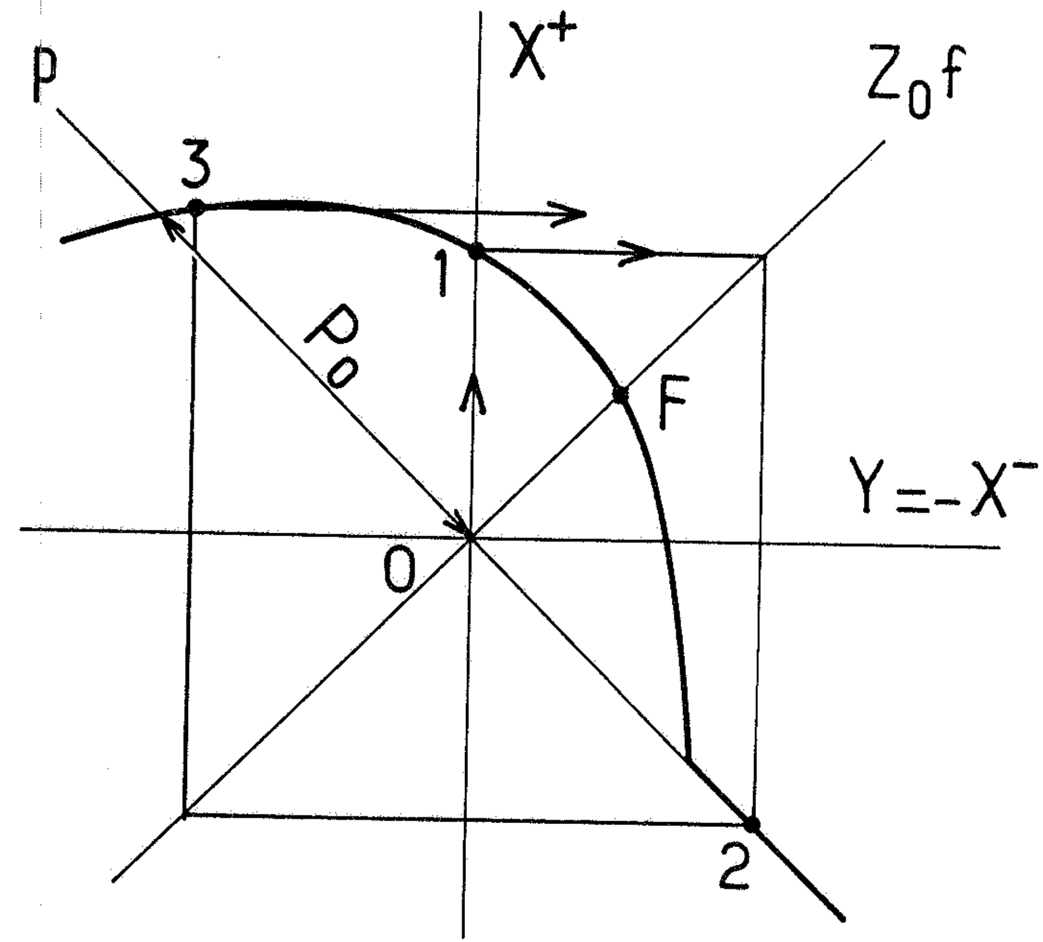


Fig 1

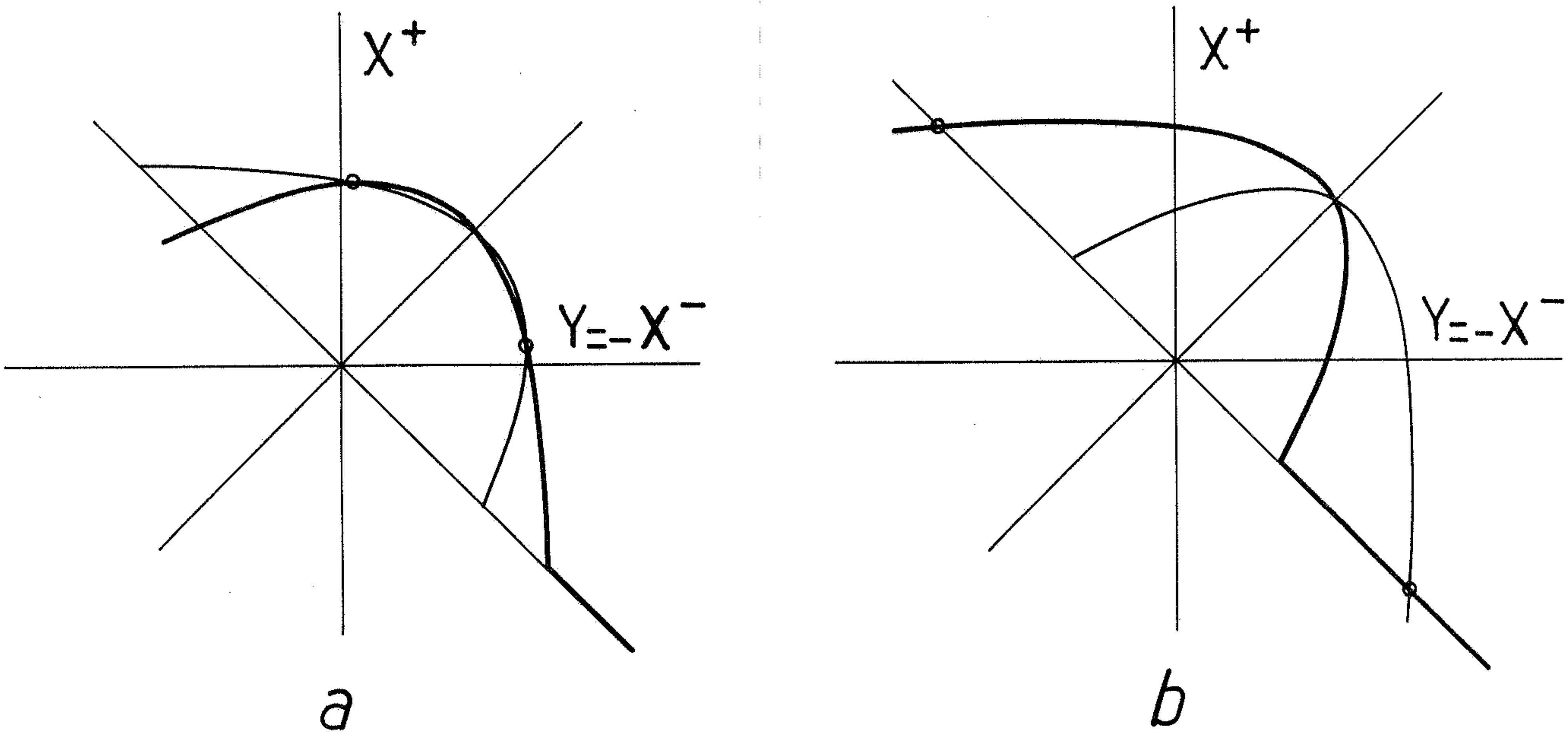


Fig 2

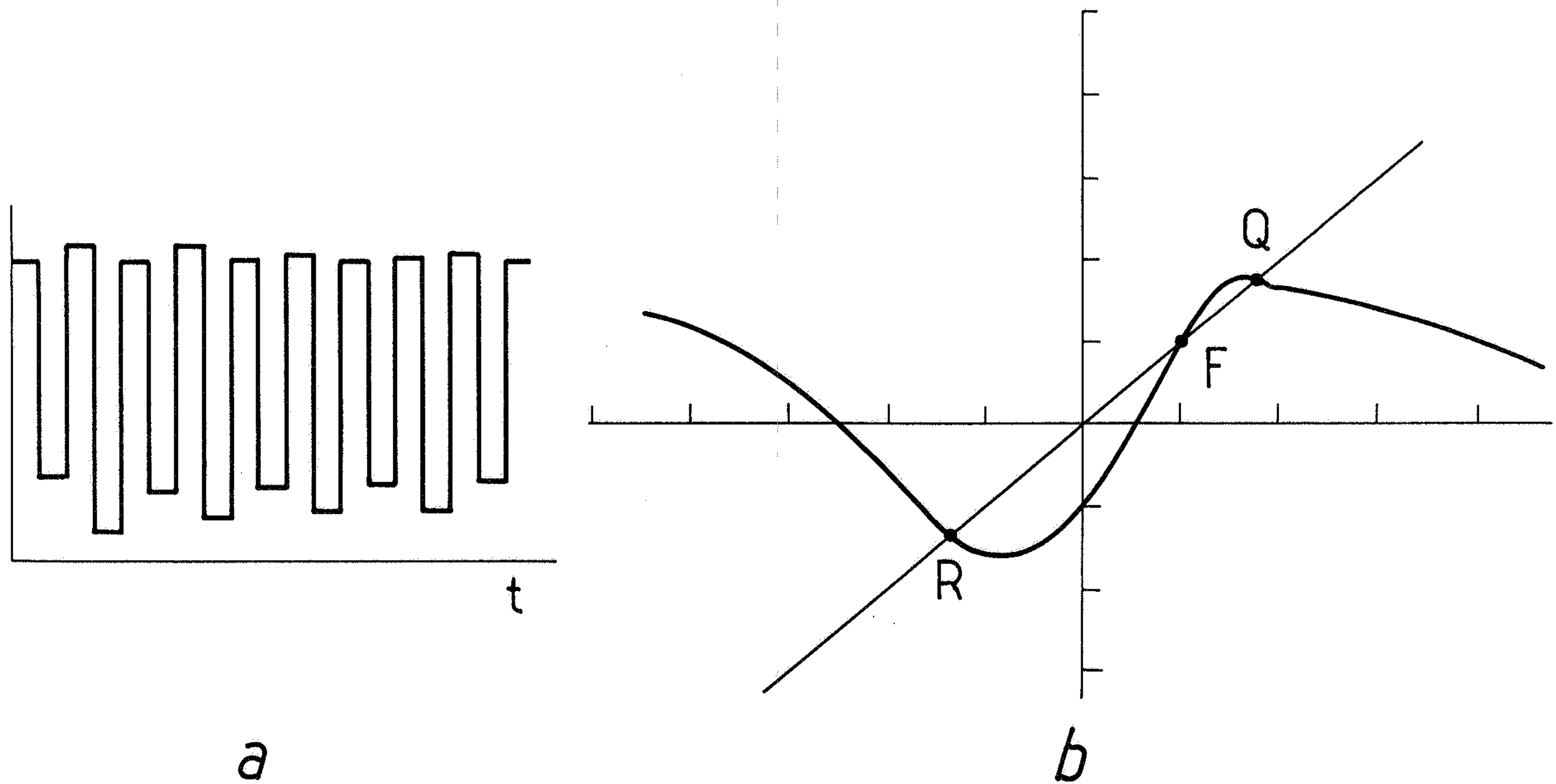


Fig 3

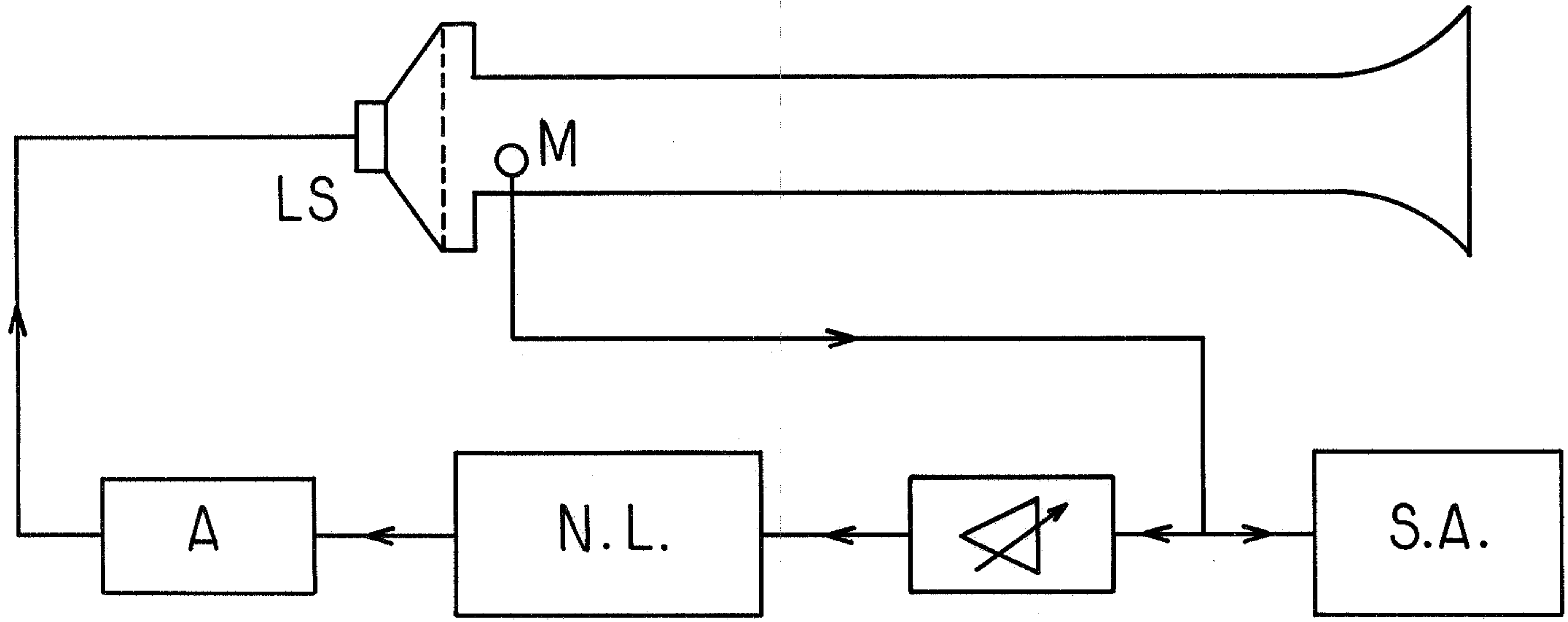
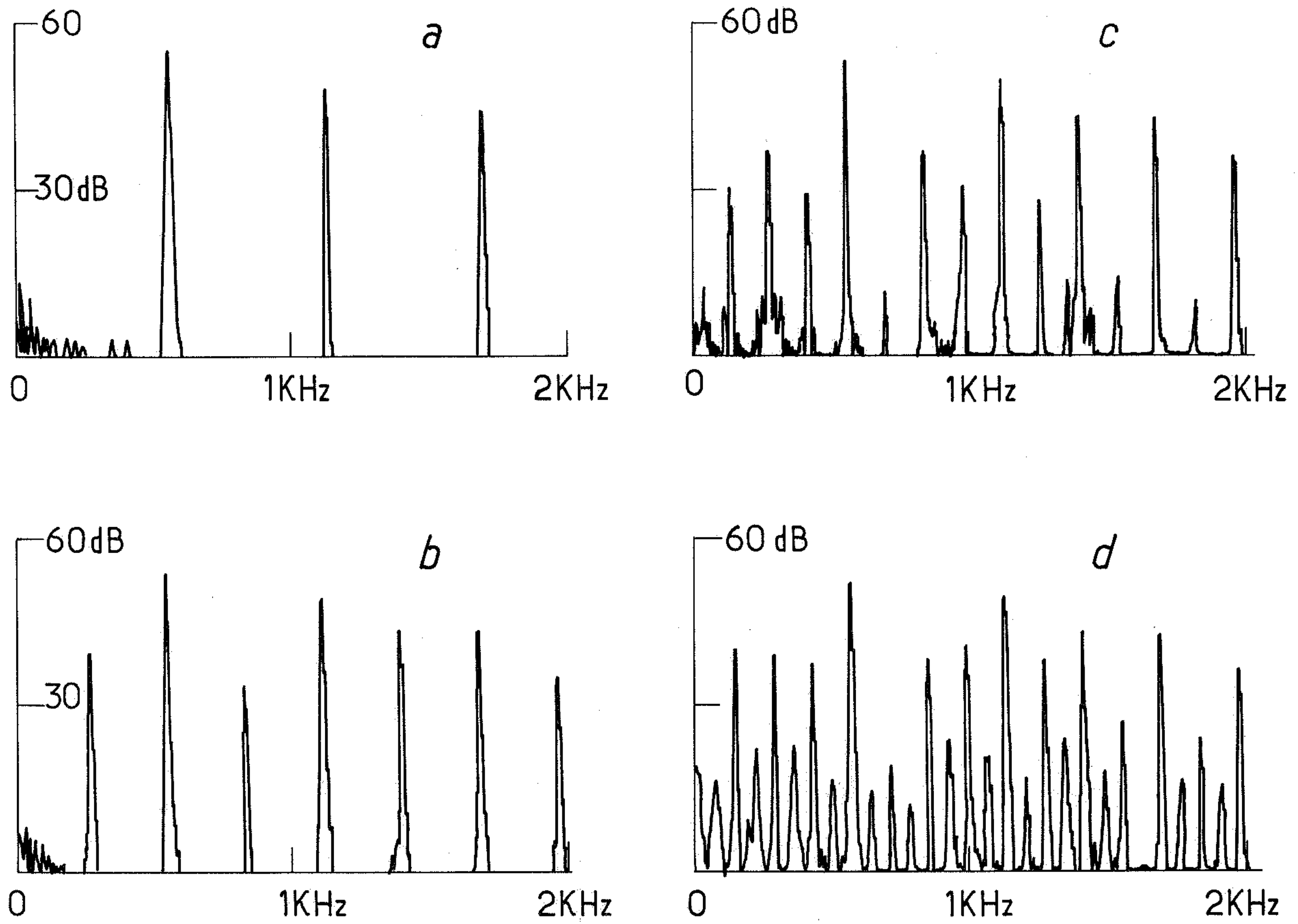


Fig 4


a
60
30dB
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1KHz
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b
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Fig 5