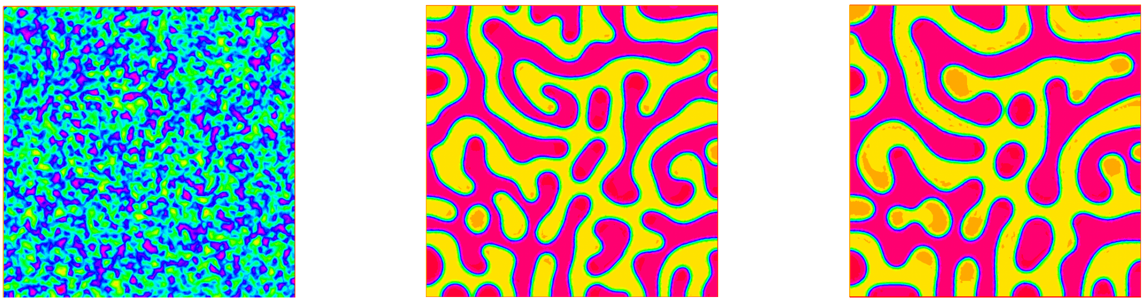
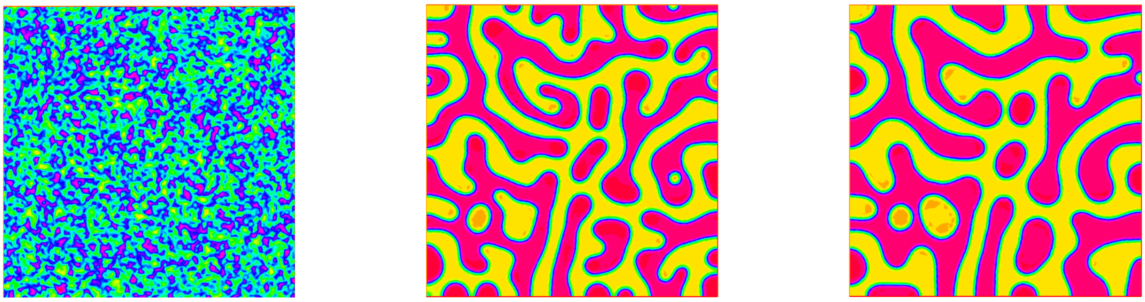
Exercice 1



(a) t = 0 (b) t = 0.05 (c) t = 0.1

FIGURE 1 – Schéma IMEX



(a) t = 0 (b) t = 0.05 (c) t = 0.1

FIGURE 2 – Schéma IMEX relaxé

Exercice 2.1

1. On pars de l'équation de base,

$$\partial_t u - \Delta(\Delta^2 u + 2\Delta u + \frac{1}{\epsilon^2} f(u)) = 0. \tag{1}$$

   Puis, on pose

$$w = \Delta^2 u + 2\Delta u + \frac{1}{\epsilon^2} f(u)$$

   Alors, on obtient le système d'équations

$$\begin{cases} \partial_t u - \Delta w = 0 \\ w = \Delta^2 u + 2\Delta u + \frac{1}{\epsilon^2} f(u) \end{cases}$$

   Avec, $u(x, 0) = u_0(x)$ et $w(x, 0) = w_0(x)$.
   De plus, on pose que

$$p = -\Delta u$$

   Alors, on a que

$$w = -\Delta p - 2\Delta p + \frac{1}{\epsilon^2} f(u)$$

   Donc, au final, on obtient le système d'équations

$$\begin{cases} \partial_t u - \Delta w = 0 \\ w = -\Delta p - 2\Delta p + \frac{1}{\epsilon^2} f(u) \\ p = -\Delta u \end{cases}$$

   Avec, $u(x, 0) = u_0(x)$, $w(x, 0) = w_0(x)$ et $p(x, 0) = p_0(x)$.

2. On pars de notre équation de la forme

$$\begin{cases} \partial_t u - \Delta w = 0 \\ w = -\Delta p - 2p + \frac{1}{\epsilon^2} f(u) \\ p = -\Delta u \end{cases}$$

Avec, $u(x,0) = u_0(x)$, $w(x,0) = w_0(x)$ et $p(x,0) = p_0(x)$.
auquel on associe la formulation variationnelle suivante

$$\begin{cases} (\partial_t u, \phi_1) + (\nabla w, \nabla \phi_1) = 0, \quad \forall \phi_1 \in V_1 \\ (w, \phi_2) = (\nabla p, \nabla \phi_2) - 2(p, \phi_2) + \frac{1}{\epsilon^2}(f(u), \phi_2), \quad \forall \phi_2 \in V_2 \\ (p, \phi_3) = (\nabla u, \nabla \phi_3), \quad \forall \phi_3 \in V_3 \end{cases}$$

De plus, on prendra $V_1 = V_2 = V_3 \in H^1(\Omega)$.

3. En utilisant la formulation faible précédente, on obtient la formulation faible approché

$$\begin{cases} (\partial_t u, \phi_1) + (\nabla w, \nabla \phi_1) = 0, \quad \forall \phi_1 \in V_{1h} \\ (w, \phi_2) = (\nabla p, \nabla \phi_2) - 2(p, \phi_2) + \frac{1}{\epsilon^2}(f(u), \phi_2), \quad \forall \phi_2 \in V_{2h} \\ (p, \phi_3) = (\nabla u, \nabla \phi_3), \quad \forall \phi_3 \in V_{3h} \end{cases}$$

avec, $V_{1h} = V_{2h} = V_{3h}$, des espaces d'éléments finies construits sur des éléments $\mathbb{P}_1$.
A l'aide de cette formulation faible approché, on peut construire le schéma IMEX suivant

$$\left(\frac{u_h^{k+1} - u_h^k}{\Delta t}, \phi_{1h}\right) - (\nabla w_h^{k+1}, \nabla \phi_{1h}) + (w_h^{k+1}, \phi_{2h}) - (\nabla p_h^{k+1}, \nabla \phi_{2h}) + 2(p_h^{k+1}, \phi_{2h}) - \frac{1}{\epsilon^2}(f(u_h^k), \phi_{2h})$$

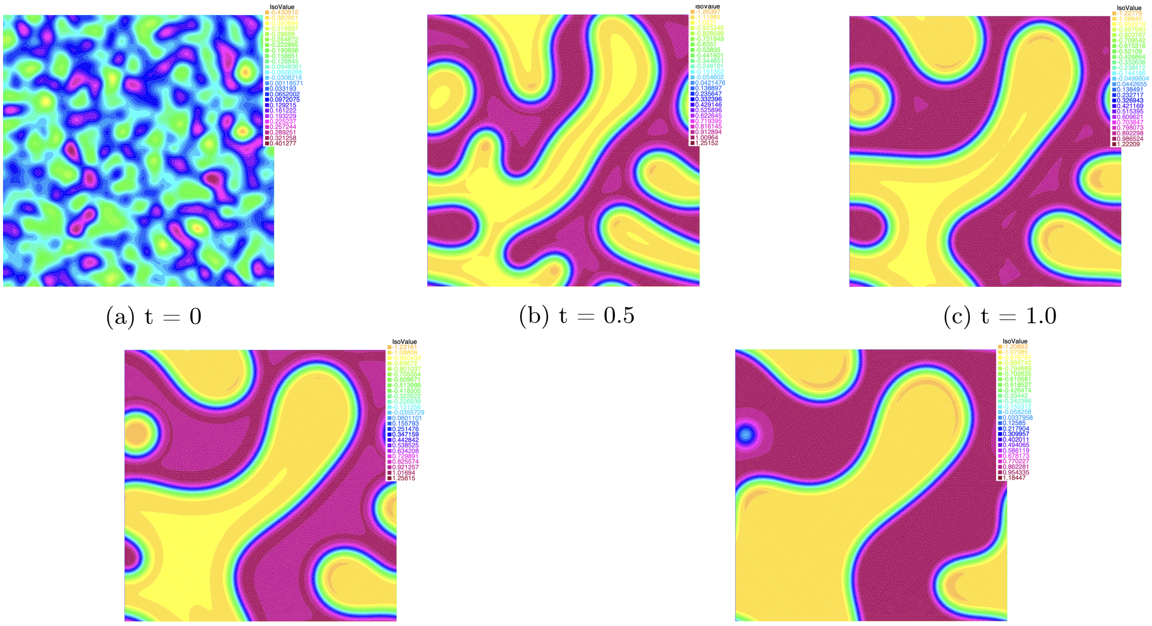
$$+(p_h^{k+1}, \phi_{3h}) - (\nabla u_h^{k+1}, \nabla \phi_{3h}) = 0$$

On applique ce schéma dans le programme FreeFem++ suivant :

```
// Parametres
int nx =100,ny = 100;
real L = 10;
real aa = -L, bb = L, cc = -L, dd = L;
real eps = 0.1;
real eps2 = 1/eps^2;
real T = 2;
real dt = 0.001;

// Creation des bordures du maillage
border AB(t = aa,bb){x = t; y=cc; label =1;};
border BC(t = cc,dd){x = bb; y=t; label =2;};
border CD(t = bb,aa){x = t; y=dd; label =3;};
border DA(t = dd,cc){x = aa; y=t; label =4;};

// Creation du maillage et de l'espace d'elements finis
mesh Th = buildmesh(AB(nx)+BC(ny)+CD(nx)+DA(ny));
fespace Vh(Th,P1,periodic = [[1,x],[3,x],[4,y],[2,y]]);

Vh u,w,p,uold,phi,psi,xi;

// Definition de la condition initiale
func u0 = 1-2*randreal1();

macro Grad(u)[dx(u),dy(u)] //

// Definition du probleme
problem CH([u,w,p],[phi,psi,xi]) =
```

```
  int2d(Th)(u*phi/dt)
  - int2d(Th)(uold*phi/dt)
  + int2d(Th)(Grad(w)'*Grad(phi))
  + int2d(Th)(w*psi)
  - int2d(Th)(Grad(p)'*Grad(psi))
  + int2d(Th)(2*p*psi)
  - int2d(Th)(eps2*(uold^3 - uold)*psi)
  + int2d(Th)(p*xi)
  - int2d(Th)(Grad(u)'*Grad(xi));

u = u0;
int k = 0;

// Iterations
for (real t=0;t<=T;t+=dt){

        uold = u;
        CH;

  if (k == 0)
  {
    plot(u,fill=true,value=true,nbiso= 25,wait = 0, ps = "exo2_00.eps");
  };
  if (k == 500)
  {
    plot(u,fill=true,value=true,nbiso= 25,wait = 0, ps = "exo2_05.eps");
  };
  if (k == 1000)
  {
    plot(u,fill=true,value=true,nbiso= 25,wait = 0, ps = "exo2_10.eps");
  };
  if (k == 1500)
  {
    plot(u,fill=true,value=true,nbiso= 25,wait = 0, ps = "exo2_15.eps");
  };
  if (k == 2000)
  {
    plot(u,fill=true,value=true,nbiso= 25,wait = 0, ps = "exo2_20.eps");
  };

  k = k+1;
}
```

4. A l'aide du programme précédent, on obtient



(a) t = 0 (b) t = 0.5 (c) t = 1.0 (d) t = 1.5 (e) t = 2.0

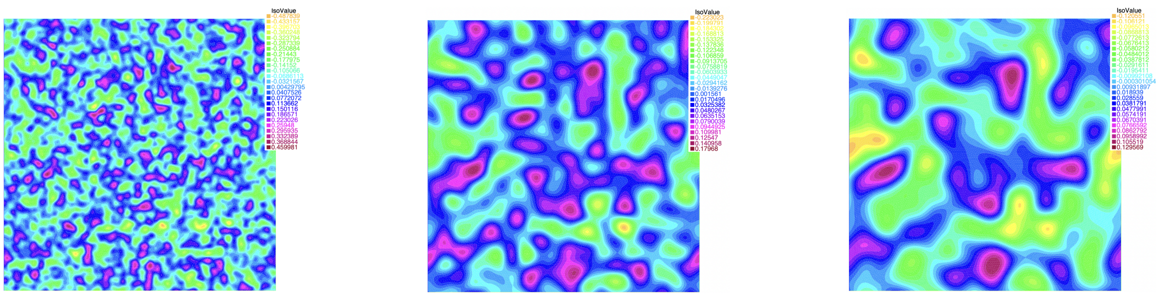
FIGURE 3 – Schéma IMEX de (1) pour $\Delta t = 10^{-3}$, $\epsilon = 0.1$ et $T = 2$

Exercice 2.2

```
// Parametres
int nx =149,ny = 149;
real L = ?;
real aa = ?, bb = ?, cc = ?, dd = ?;
real eps = ?;
real T = ?;
real dt = ?;

// Parametres probleme
real a20 = ?;
real a11 = ?;
real a02 = ?;
real a10 = ?;
real a01 = ?;

// Creation des bordures du maillage
border AB(t = aa,bb){x = t; y=cc; label =1;};
border BC(t = cc,dd){x = bb; y=t; label =2;};
border CD(t = bb,aa){x = t; y=dd; label =3;};
border DA(t = dd,cc){x = aa; y=t; label =4;};

// Creation du maillage et de l'espace d'elements finis
mesh Th = buildmesh(AB(nx)+BC(ny)+CD(nx)+DA(ny));
fespace Vh(Th,P1, periodic = [[1,x],[3,x],[4,y],[2,y]]);

Vh u,w,p,q,uold,phi1,phi2,phi3,phi4;

// Definition de la condition initiale
func u0 = ?;

// Definition de la fonction f
func real f(real c)
```

```
{
return c^3-c;
}

macro Grad(u)[dx(u),dy(u)] //

// Definition du probleme
problem CH([u,w,p,q],[phi1,phi2,phi3,phi4]) =
  int2d(Th)(u*phi1/dt)
  - int2d(Th)(uold*phi1/dt)
  - int2d(Th)(Grad(w)'*Grad(phi1))
  + int2d(Th)("""g(x,uold)"""*phi1/eps)
  + int2d(Th)(w*phi2)
  - int2d(Th)(a20*eps*dx(p)*dx(phi2))
  - int2d(Th)(a02*eps*dy(q)*dy(phi2))
  - int2d(Th)(a11*dy(p)*dy(phi2)/2)
  - int2d(Th)(a11*dx(q)*dx(phi2)/2)
  - int2d(Th)(a10*eps*p*phi2)
  - int2d(Th)(a01*eps*q*phi2)
  + int2d(Th)(f(uold)*phi2/eps)
  + int2d(Th)(p*phi3)
  + int2d(Th)(dx(u)*dx(phi3))
  + int2d(Th)(q*phi4)
  + int2d(Th)(dy(u)*dy(phi4));

u = u0;

//Iterations
for (real t=0;t<=T;t+=dt){
  // Calculs et redefinition
        uold = u;
        CH;

  //Affichage
  plot(u,fill=true,value=true,nbiso=20);
}
```

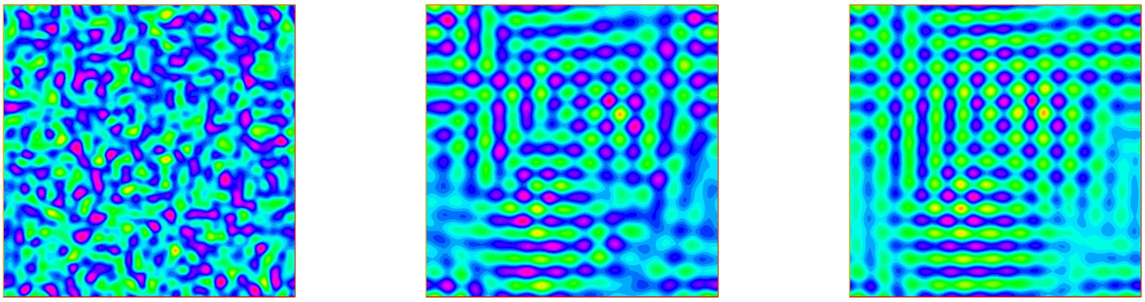
1. Première simulation :



(a) t = 0 (b) t = 10^{-6} (c) t = 5×10^{-6}

FIGURE 4 – **Cahn-Hilliard-Oono**. Condition initiale u_0 suivant une loi uniforme en $[-1, 1]$. $f = u^3 - u$, $g = 0.5u$, $\epsilon = 0.05$, $\Delta t = 5 \times 10^{-8}$.

2. Deuxieme simulation :

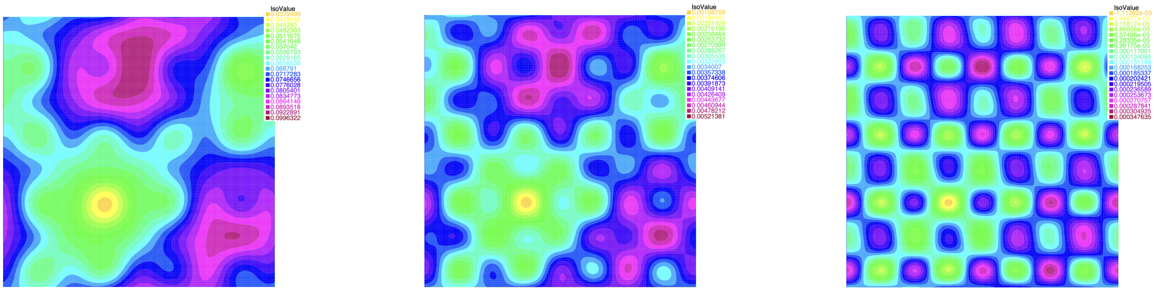


(a) t = 0 (b) t = 2.5×10^{-2} (c) t = 5×10^{-2}

(d) t = 7.5×10^{-2} (e) t = 10^{-1}

FIGURE 5 – **Phase-field crystal**. Condition initiale u_0 suivant une loi uniforme en $[-1, 1]$. $f = u^3 + (1 - 0.025)u$, $g = 2u$, $\epsilon = 1$, $\Delta t = 10^{-4}$.

3. Troisième simulation :



(a) t = 0 (b) t = 1.5 (c) t = 3.0

(d) t = 4.5 (e) t = 6

FIGURE 6 – **Phase-field crystal**.
$u_0 = 0.07 - 0.02cos(\frac{2\pi(x-12)}{32})sin(\frac{2\pi(y-1)}{32}) + 0.02cos^2(\frac{\pi(x+10)}{32})cos^2(\frac{\pi(y+3)}{32}) - 0.01sin^2(\frac{4\pi x}{32})sin^2(\frac{4\pi(y-6)}{32})$.
$f = u^3 + (1 - 0.025)u$, $g = 2u$, $\epsilon = 1$, $\Delta t = 10^{-3}$.