# THÈSE

Pour obtenir le grade de

## DOCTEUR DE L'UNIVERSITE GRENOBLE ALPES

Spécialité : Mathématiques et Informatique

Arrêté ministériel : 25 mai 2016


Présentée par

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préparée au sein du **Laboratoire des Sciences pour la Conception, l'Optimisation et la Production de Grenoble**
dans **l'École Doctorale Mathématiques, Sciences et technologies de l'Information, Informatique**


# Classes héréditaires de graphes : De la structure vers la coloration

# Hereditary classes of graphs: From structure to coloring

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# Chapter 1

# Abstract

## Abstract

This thesis deals with the structure of some hereditary classes of graphs. A class of graphs is *hereditary* if it is closed under vertex deletion. A better understanding of the structure of graphs contained in certain hereditary classes sometime yields results on optimisation problems such as the coloring problem. We focus on three hereditary classes, the first being a subclass of even-hole-free graphs and the other two being minimal open cases for the complexity of the coloring problem when restricted to classes defined by excluding subgraphs of order 4.

First we provide a structural result for the class of graphs $\mathscr{C}_k$ that is the class of graphs where every hole has length $k$. Using earlier results on other related classes of graphs, we obtain a structural theorem for the graphs in $\mathscr{C}_k$ when $k$ is odd and at least 7. The theorem states that a graph in $\mathscr{C}_k$ with no clique cutset and no universal vertex is either a ring or belongs to a new class of graphs named *blowup of template* that we fully described.

Secondly, we study the structure of graphs in Free$\{C_4, 4K_1\}$. We first focus on *fixers*. A graph $H$ in Free$\{C_4, 4K_1\}$ is a fixer if any graph in Free$\{C_4, 4K_1\}$ containing $H$ as an induced subgraph is an extended proper blowup of $H$. We prove that the icosahedron is a fixer. In addition the icosahedron minus one vertex has a similar property. It follows that graphs in Free$\{C_4, 4K_1\}$ that contain an icosahedron minus one vertex have bounded clique-width and can be colored in polynomial time. We provide a program that computes all fixers of small fixed order. Next we observe that for any graph $G$ in Free$\{C_4, 4K_1\}$, every subgraph of $G$ induced by two disjoint cliques is a half graph. We give some thoughts on the study of the structure of graphs in Free$\{C_4, 4K_1\}$ whose vertex sets can be partitioned into 3 cliques.

The last class that we are interested in, is the class of antiprismatic graphs. We prove that the coloring problem is polynomial-time solvable when restricted to non-orientable antiprismatic graphs. The proof is largely based on the structural result provided by Chudnovsky and Seymour on the complement class: the prismatic graphs. Using this result, we prove that every non-orientable prismatic graph has at most 10 pairwise disjoint triangles. This yields a $\mathcal{O}(n^{7.5})$ algorithm that solves the clique cover problem in non-orientable prismatic graph. We also give an $\mathcal{O}(n^5)$ algorithm for solving the problem of finding a maximum number of vertex-disjoint triangles in both orientable and non-orientable prismatic graphs.

## Résumé

Cette thèse porte sur la structure des classes de graphes héréditaires. Une classe de graphes est *héréditaire* si elle est fermée par suppression de sommet. Une meilleure compréhension de la structure des graphes dans une telle classe peut conduire à des résultats pour certains problèmes d'optimisation comme le problème de la coloration. Dans cette thèse, nous nous concentrons sur trois classes héréditaires de graphes. La première est une sous-classe des graphes sans trou pair. Les deux autres sont des cas ouverts minimaux pour la complexité du problème de coloration restreint aux classes de graphes définies en excluant des sous-graphes d'ordre 4.

Tout d'abord, nous donnons un résultat structurel pour la classe de graphes $\mathscr{C}_k$. C'est la classe des graphes dont tous les trous ont longueur $k$. En utilisant des résultats antérieurs sur des classes de graphes reliées, nous donnons un théorème de structure pour les graphes dans $\mathscr{C}_k$ pour $k$ impair et au moins 7. Le théorème stipule qu'un graphe dans $\mathscr{C}_k$ qui ne contient ni un clique cutset ni un sommet universel est un ring ou appartient à une nouvelle classe nommée *blowup de templates* que nous décrivons complètement.

Ensuite, nous étudions la structure des graphes dans Free$\{C_4, 4K_1\}$. Nous

nous concentrons, tout d'abord, sur les *fixers*. Un graphe $H$ est un fixer si tout graphe dans Free$\{C_4, 4K_1\}$ contenant $H$ comme sous-graphe induit est un blowup propre étendu de $H$. Nous prouvons que l'icosaèdre est un fixer. De plus l'icosaèdre moins un sommet possède des propriétés similaires. Il en résulte qu'un graphe dans Free$\{C_4, 4K_1\}$ contenant un icosaèdre moins un sommet comme sous-graphe induit, a une clique-width bornée et donc peut être colorié en temps polynomial. Nous donnons un programme qui génère tous les fixers d'un petit ordre donné en entrée. Par la suite, nous observons que dans tout graphe $G$ dans Free$\{C_4, 4K_1\}$, tout sous-graphe induit par des cliques disjointes est un half graph. Nous donnons quelques idées pour l'étude de la structure des graphes dans Free$\{C_4, 4K_1\}$ dont l'ensemble des sommets peut se partitionner en 3 cliques.

La dernière classe qui nous intéresse est celle des graphes anti-prismatiques. Nous prouvons que le problème de coloration restreint aux graphes anti-prismatiques non-orientables peut se résoudre en temps polynomial. La preuve s'appuie largement sur un résultat structurel de Chudnovsky et Seymour sur la classe complémentaire : les graphes prismatiques. Grace à ce résultat nous prouvons que tout graphe prismatique non-orientable a au plus 10 triangles deux à deux disjoints. Cela conduit à un algorithme en $\mathcal{O}(n^{7.5})$ qui résout le problème de la couverture par cliques dans les graphes prismatiques non-orientables. Nous donnons aussi un algorithme en $\mathcal{O}(n^5)$ pour résoudre le problème du nombre maximum de triangles disjoints dans tout graphe prismatique, tant orientable que non-orientable.

# Chapter 2

# Introduction

## 2.1 The schedules, what a problem!

Consider the following scheduling problem: a certain number of courses ($n$) are all characterised by a starting time and an end time. One wants to know the minimum number of rooms needed to assign a room to each course in such a way that two courses that overlap do not share the same room.

If the number of courses is small, it is possible to answer this question "by hand", but providing an answer becomes more difficult as the number of classes increases. That is where come mathematical modelling and computer science.

Mathematical models are aimed to represent a certain kind of real situations. The study of a mathematical model leads to a general study of all kinds of situations that can be represented by this model. Hence, all results found on the model can naturally be used for all situations that can be represented by this model. The models that we are interested in are called graphs.

A *graph* is a set of nodes named *vertices* that are pairwise linked or not by *edges*. Graphs are usually used to represent binary interactions. The vertices usually represent objects that are pairwise connected or not, depending on whether the vertices are linked by an edge or not. Hence solving problems in graphs leads to solutions to some related problems in real situations independently of what is represented by vertices.

In this thesis, graphs are *undirected*. This means that an edge does not encode any order between its two endpoints. Except when it is specified they are also *simple*, i.e. edges are undirected and every pair of vertices is connected by at most one edge.

Consider again the scheduling problem mentioned above. Represent every course by a vertex. Add an edge between two vertices if the corresponding courses overlap. The graph $G$ that we obtain is a good representation of the situation. Now look at the following question : what is the minimum number of colors needed in order to color every vertex of the graph such that any two vertices linked by an edge are of different colors? Such a coloring is called a *proper coloring*. The *vertex coloring problem* is the problem of finding the minimum number of colors needed to color a given graph. An answer to the coloring problem in $G$, yields an answer to the scheduling problem by assigning the same room to every courses whose representative vertices are colored with the same color.

## 2.2 Computational complexity

For a given problem with a given algorithm that solves it, two theoretical aspects are of importance: verify that the proposed algorithm does indeed perform correctly and analyse how efficient it is. The *computational complexity* of an algorithm is the number of basic operations required for its execution in the worst case. For simplicity we just call it complexity of the algorithm. The number of basic operations is obviously linked to the size of the input. In graph theory, when the input is a graph, the size of the input depends on how it is represented in the computer, but all classical representations are equivalent up to a factor of at most $\mathcal{O}(n^2)$, where $n$ denotes the number of vertices of the input graph. We say that an algorithm $A$ is a *polynomial-time algorithm* if the complexity of $A$ is bounded (from above) by a polynomial in the size of its input. Polynomial-time algorithms are obviously more efficient than exponential-time algorithms for sufficiently large inputs. This is why polynomiality became an important criterion to classify algorithms and problems. The complexity of a problem is the complexity of the best possible algorithm that solves it.

All problems that can be solved by a polynomial-time algorithm are said to be *polynomial*. The class of all polynomial problem is the class $\mathcal{P}$.

A *decision problem* is a problem whose answer is either "yes" or "no". The class of problem $\mathcal{NP}$ (nondeterministic polynomial time) is the class of all decision problems such that given any instance with "yes" answer there is a certificate validating this fact which can be checked in polynomial time.

From the definitions it is obvious that $\mathcal{P} \subseteq \mathcal{NP}$. In 1971, Cook Edmonds and Levin asked the following question : is $\mathcal{P} = \mathcal{NP}$ ? This problem is one of the seven *Millennium Prize Problems* and it is still open. In this thesis we work under the hypothesis $\mathcal{P} \neq \mathcal{NP}$.

The class of *NP-complete* problems is the class of decision problems in $\mathcal{NP}$ which are "at least as hard to solve" as any problem in $\mathcal{NP}$. Under the hypothesis $\mathcal{P} \neq \mathcal{NP}$ it follows that there does not exist any polynomial algorithm that solves NP-complete problems.

The decision form of the coloring problem is NP-complete [26]. Hence, under the hypothesis $\mathcal{P} \neq \mathcal{NP}$, there is no polynomial algorithm that solves the coloring problem. But when we add some restrictions on the input graph, the coloring problem can become polynomial. A trivial example is when we bound the number of vertices of the graph by a constant. In the case of our scheduling problem, any graph that corresponds to such situation belongs to the famous class of interval graphs. It is classical that the coloring problem is polynomial time solvable when restricted to interval graphs. An interesting question is to know, what are, in general, the restrictions on the input graphs such that the coloring problem remains NP-complete and the restrictions on the input graphs such that the coloring problem become polynomial.

In this thesis, we look at the general structure of graphs in certain classes of graphs that will be presented in Chapter 3. A better understanding of the structure of graphs under certain hypotheses can yield to some complexity results on optimisation problems in graphs. Considering the coloring problem, if the input graph is restricted to a certain class of graphs $\mathcal{H}$ then, a better understanding of the structure of graphs in $\mathcal{H}$ can yield to define a polynomial-time algorithm or a proof of NP-completeness.

## 2.3 More formal definitions

In this document when $G$ is a graph, $V(G)$ denotes the set of vertices of $G$ and $E(G)$ the set of edges of $G$.

In a graph $G$, a *stable set* is a set of vertices that are pairwise non-adjacent.

A *clique* is a set of vertices that are pairwise adjacent. In a graph, we view the empty set as a clique. The size of a maximum clique in $G$ is denoted by $\omega(G)$. The size of a maximum independent set in $G$ is denoted by $\alpha(G)$.

A *proper coloring* of a graph $G$ is a function $f\colon V(G) \to \mathbb{N}$ such that for all $uv \in E(G)$, $f(u) \neq f(v)$. The size of a proper coloring given by $f$ is $|f(V(G)|$. For any positive integer $k$, a *$k$-coloring* of a graph $G$ is a proper coloring of $G$ of size $k$.

The vertex-coloring problem in its optimisation form is the problem of, given a graph, finding the minimum number of colors of a proper coloring. In this thesis, we omit the term vertex in vertex coloring problem (just *coloring problem*). The decision form of the coloring problem is the following :

COLORING PROBLEM

| | |
|---|---|
| INPUT: | A graph $G$ and a positive integer $k$ |
| OUTPUT: | *Yes* if there exists a proper coloring of $G$ that uses at most $k$ colors and *No* otherwise. |

The coloring problem is NP-complete [26].

Here, the number of colors needed is part of the input. For an integer $k$, the $k$-coloring problem is the same problem but, the number of colors is fixed.

$k$-COLORING PROBLEM

| | |
|---|---|
| INPUT: | A graph $G$ |
| OUTPUT: | *Yes* if there exists a proper coloring of $G$ that uses at most $k$ colors and *No* otherwise. |

When $k$ is at least 3, the $k$-coloring problem is NP-complete [26].

For a graph $G$, the *chromatic number* (minimum number of colors needed to have a proper coloring of $G$) is denoted by $\chi(G)$. It is easy to see that $\chi(G)$ is at least equal to the size of the maximum clique of $G$ ($\chi(G) \geq \omega(G)$ ). This is because all vertices of the cliques have to use different colors in any proper coloring of the graph.

The *order* of a graph is its number of vertices. For any subset $V'$ of vertices of $V(G)$, the subgraph of $G$ induced by $V'$ denote by $G[V']$ is the graph with

$V'$ as vertex set and all edges in $E(G)$ that are incident to two vertices in $V'$, and only those edges. A graph $H$ is an *induced subgraph* of a graph $G$ if there exists $V' \subseteq V(G)$ such that $G[V']$ is isomorphic to $H$. In this thesis, when we say that a graph $G$ *contains* a graph $H$, we always mean as an induced subgraph.

The *complement* of the graph $G$, denoted by $\overline{G}$, is the graph with the same set of vertices as $G$ and such that two vertices in $\overline{G}$ are adjacent if and only if they are non-adjacent in $G$. When $\mathscr{H}$ is a class of graph, we set $\overline{\mathscr{H}} = \{\overline{G} : G \in \mathscr{H}\}$.

A graph is *connected* if every pair of distinct vertices is joined by a path. It is *anticonnected* if its complement is connected. A *connected component* of a graph $G$ is a subset $X$ of $V(G)$ such that $G[X]$ is connected and $X$ is maximal with respect to this property. An *anticonnected component* of a graph $G$ is a subset $X$ of $V(G)$ such that $G[X]$ is anticonnected and $X$ is maximal with respect to this property.

When $x$ is a vertex of a graph $G$ and $A$ is a subset of vertices of $G$ or a subgraph of $G$, we denote by $N_A(x)$ the *neighbourhood* of $x$ in $A$, i.e.: the set of neighbours of $x$ that are in $A$. Note that $x \notin N_A(x)$. We set $N_A[x] = \{x\} \cup N_A(x)$, the *closed neighbourhood* of $x$ in $A$. If $X \subseteq V(G)$, we set $N_A(X) = (\cup_{x \in X} N_A(x)) \setminus X$ and $N_A[X] = N_A(X) \cup X$. We sometimes write $N$ instead of $N_{V(G)}$ (when there is no risk of confusion). If $e = uv$ is an edge of $G$, we say that the vertices $u$ and $v$ are the *endpoints* of $e$ and that $e$ is *incident* to $u$ and $v$.

The *degree* of a vertex $x$ in $G$ is equal to $|N_G(x)|$. A vertex $x$ is *isolated* in $G$ if $x$ has no neighbour in $G$, i.e.: if $x$ has degree 0 in $G$. A vertex that is adjacent to all other vertices of $G$ is *universal* in $G$. We also use the adjective universal for a set of vertices that only contains universal vertices, for example, a *universal clique* $W$ in a graph $G$ is a clique that is complete to $V(G) \setminus W$. Two distinct vertices $x$ and $y$ in a graph $G$ are *twins* in $G$ if $N_G[x] = N_G[y]$ (in particular, $x$ and $y$ are adjacent). A graph is *twinless* if it contains no twins.

A set $X \subseteq V(G)$ is *complete* to a set $Y \subseteq V(G)$ if they are disjoint and every vertex of $X$ is adjacent to every vertex of $Y$. A set $X \subseteq V(G)$ is *anticomplete* to a set $Y \subseteq V(G)$ if they are disjoint and no vertex of $X$ is adjacent to a vertex of $Y$. We sometimes say that $x$ is complete (resp. anticomplete) to $Y$ to mean that $\{x\}$ is complete (resp. anticomplete) to $Y$.

A graph $G$ is *bipartite* if $V(G)$ can be partitioned into two stable sets $V_1$ and $V_2$. In addition, if $V_1$ and $V_2$ are complete to each other, $G$ is a *complete bipartite graph*. Observe that bipartite graphs can be colored with 2 colors. For $k \geq 1$, we denote by $K_{k,l}$ the complete bipartite graph with one side of the partition of size $k$ and the other of size $l$.

When $G$ and $H$ are graphs, we denote by $G+H$ their disjoint union. When $G$ is a graph and $k$ an integer, we denote by $kG$ the disjoint union of $k$ copies of $G$.

For $k \geq 1$, we denote by $P_k$ the *path* on $k$ vertices, that is, the graph with vertex-set $\{p_1, \dots, p_k\}$ and edge-set $\{p_1p_2, \dots, p_{k-1}p_k\}$. It is sometimes denoted by $p_1p_2 \dots p_k$. If $1 \leq i \leq j \leq k$, we then denote by $p_iPp_j$ the path $p_ip_{i+1} \dots p_j$. Given a graph $G$ and two vertices $u$ and $v$ in $V(G)$, a *uv-path* is a path with endpoints $u$ and $v$ that is induced in $G$.

For $k \geq 3$, we denote by $C_k$ the cycle on $k$ vertices; that is, the graph with vertex-set $\{p_1, \dots, p_k\}$ and edge-set $\{p_1p_2, \dots, p_{k-1}p_k, p_kp_1\}$. We denote it by $p_1p_2 \dots p_kp_1$. When $C_k$ is a subgraph of a graph $G$ (possibly not induced), an edge with both ends in $\{p_1, ..., p_k\}$ that is not an edge of $C_k$ is called a *chord* of $C_k$. A *hole* in a graph $G$ is a cycle of length at least 4 without any chord in $G$. Note that, unlike cycles, holes have to be induced. An *antihole* is the complement graph of a hole. The *length* of a path or a hole is the number of its edges. Holes, paths and cycles are *even* or *odd* depending on the parity of their lengths.

For $k \geq 1$, we denote by $K_k$ the *complete graph* on $k$ vertices that is the graph with all possible edges.

In the figure 2.1, notations for some special graphs of order 4 are presented. Note that the graph $\overline{2P_1 + P_2}$ is also named the *diamond*, the graph $K_{1,3}$ is also named the *claw* and the graph $\overline{P_3 + K_1}$ is also named the *paw*.

## 2.4 Problems related to coloring

In this thesis, we also mention other problems than the coloring problem. They are described in this section.

The *clique covering* problem is the problem of, given a graph $G$, partition $V(G)$ into the minimum number of cliques. The decision form of the clique covering problem is the following :

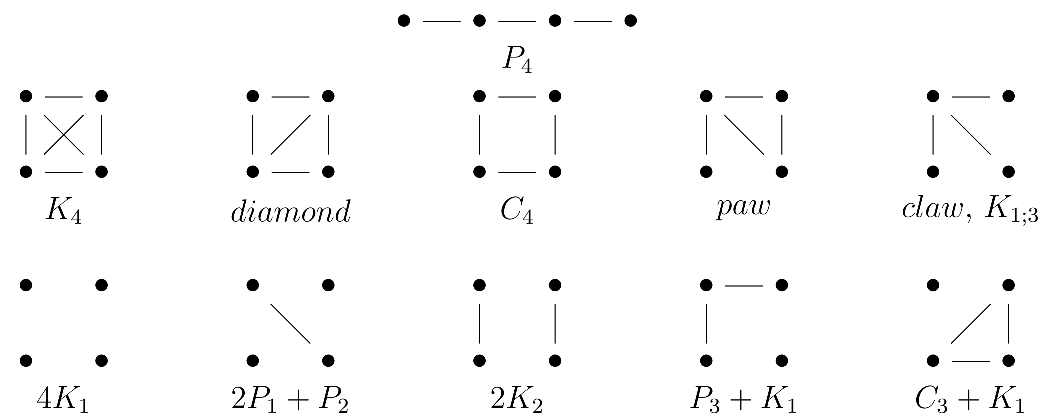



Figure 2.1: Graphs of order 4

CLIQUE COVERING PROBLEM

INPUT: A graph $G$, a positive integer $k$

OUTPUT: *Yes* if $V(G)$ can be partitioned into at most $k$ disjoint cliques and *No* otherwise.

In the general case, the clique covering problem is NP-complete [26]. It is easy to see that solving the clique covering problem in a graph $G$ is equivalent to solving the coloring problem in $\overline{G}$ and vice versa. Like the coloring problem the integer $k$ can be fixed in the definition of the problem. This gives the *$k$-clique covering problem*

$k$-CLIQUE COVERING PROBLEM

INPUT: A graph $G$

OUTPUT: *Yes* if $V(G)$ can be partitioned into at most $k$ disjoint cliques and *No* otherwise.

When $k$ is at least 3, the $k$-clique covering problem is NP-complete [26]. The size of the minimum clique cover is denoted by $\theta(G)$. Note that $\theta(G) = \chi(\overline{G})$.

The *edge coloring problem* is the problem of coloring every edge of a given graph $G$ such that two incident edges do not share the same color, andn the number of colors used is minimum. The decision form of the edge coloring problem is the following :

EDGE COLORING PROBLEM

INPUT: A graph $G$, a positive integer $k$

OUTPUT: *Yes* if $G$ admits an edge coloring using at most $k$ colors, and *No* otherwise.

In the general case, the edge coloring problem is NP-complete [33].

The *vertex-disjoint triangles problem* is the problem of finding, in a given graph $G$, the maximum number of vertex-disjoint triangles. The decision form of the vertex-disjoint triangles problem is the following :

VERTEX-DISJOINT TRIANGLES PROBLEM

INPUT: A graph $G$, a positive integer $k$

OUTPUT: *Yes* if $G$ contains at least $k$ disjoint triangles and *No* otherwise.

In the general case, the vertex-disjoint triangles problem is NP-complete [31].

The *maximum matching problem* is the problem of, given a graph $G$, find the maximum number of pairwise non-incident edges. The decision form of the maximum matching problem is the following :

MAXIMUM MATCHING PROBLEM

INPUT: A graph $G$, a positive integer $k$

OUTPUT: *Yes* if there exist at least $k$ pairwise disjoint edges in $G$ and *No* otherwise.

A set of edges that are pairwise non-incident are called a *matching*. The maximum matching is polynomial time solvable [20].

## 2.5 Outline

Chapter 3 is about hereditary classes of graphs. We give some intuitions about what leads us to work on them and survey known results. We motivate the choice of the three classes that are studied in the rest of this thesis.

Chapter 4 is devoted to our results on the structure of graphs where every hole has the same length.

Chapter 5 is devoted to our study on the structure of graphs in Free$\{C_4, 4K_1\}$. We also give some partial results on the complexity of the coloring problem when restricted to graphs in Free$\{C_4, 4K_1\}$.

Chapter 6 is devoted to the complexity of the coloring problem when restricted to antiprismatiques graphs.

At the end of the document, a table of notations and an index are provided.

# Chapter 3

# Hereditary classes of graphs

## 3.1 Hereditary classes of graphs

In modelling, vertices of a graph usually represent objects and edges represent constraints. It is more understandable to consider the removal of an object than to consider the removal of a single constraint. Hence, it is natural to consider classes of graphs closed under vertex deletion. Such classes are called *hereditary* classes of graphs: a class of graphs $\mathscr{H}$ is *hereditary* if for every graph $G$ in $\mathscr{H}$, every induced subgraph of $G$ is also in $\mathscr{H}$.

We begin with an example. A graph $G$ is an *interval graph* if every vertex of $G$ corresponds to an interval on the real line. Two vertices of $G$ are adjacent if and only if the corresponding intervals intersect (see figure 3.1). Considering, once again the example of the scheduling problem presented in chapter 2, it is easy to note that the corresponding graph is an interval graph. The removal of a vertex in this graph corresponds to the removal of one course, which make sense. But if we remove a unique edge, that means that two courses that originally intersect do not intersect anymore without changing other things. It does not really make sense in real life. More formally, the removal of a vertex of an interval graph $G$ corresponds to the removal of one interval in the set of intervals that correspond to $G$. Therefore, it is obvious that every graph obtained from $G$ by removing some vertices is also an interval graph. Hence, the class of interval graphs is a hereditary class of graphs. Removing an edge from $G$ does not make a lot of sense. It would mean that two intervals that

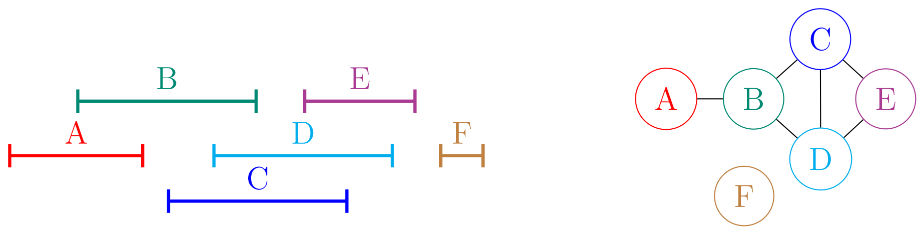



Figure 3.1: Interval graph


Figure 3.2: Graph that is not an interval graph

intersect at the beginning do not intersect anymore. It is possible that this new graph does not even correspond to any set of intervals on the real line. For example, in Figure 3.2, there is no way to move the interval corresponding to vertex $D$ such that this interval intersects those that correspond to $B$ and $E$ but not the interval that intersects $C$. If we remove the edge $CD$, the graph is no longer an interval graph.

A nice property for any hereditary classes of graphs is that they are characterised by a set of forbidden induced subgraphs. Given a set of graphs $\mathscr{F}$, denote by $\mathrm{Free}\,\mathscr{F}$ the class of graphs containing no graph from $\mathscr{F}$.

**Property 3.1.1**
A class $\mathscr{H}$ of graphs is hereditary if and only if there exists a set $\mathscr{F}$ of graphs such that $\mathscr{H} = \mathrm{Free}\,\mathscr{F}$.

*Proof.* By definition, if $\mathscr{H} = \mathrm{Free}\,\mathscr{F}$ then $\mathscr{H}$ is hereditary. The converse is also obvious by defining $\mathscr{F} = \{G : G \notin \mathscr{H}\}$. $\square$

A graph $G$ is a *minimal forbidden induced subgraph* for a hereditary class

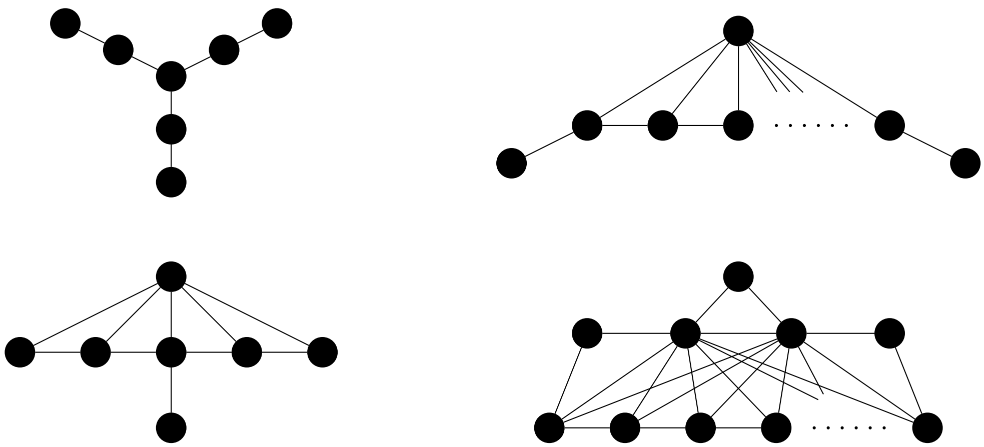


Figure 3.3: Forbidden induced subgraph for interval graphs

of graphs $\mathscr{H}$ if the only induced subgraph of $G$ not in $\mathscr{H}$ is $G$ itself.

**Property 3.1.2**
If $\mathscr{M}$ is the set of all minimal forbidden induced subgraphs of a hereditary class of graphs $\mathscr{H}$, then $\mathscr{H} = \text{Free}\,\mathscr{M}$. Moreover, $\mathscr{M}$ is inclusion-wise minimal with respect to this property.

We omit the proof of property 3.1.2 that can be found, as the previous results and definitions in [37].

In order to illustrate the two previous properties, consider again the class of interval graphs. Let $\mathscr{F}$ be the set of graphs that are not interval graphs. It is obvious, but not really interesting, that the class of interval graphs is the class $\text{Free}\,\mathscr{F}$. Lekkerkerker and Boland [39] proved that when $\mathscr{G}$ is the set of all $C_n$ for $n \geq 4$ and all graphs in figure 3.3, then $\mathscr{G}$ is the minimal set of graphs such that the class of interval graphs is the class $\text{Free}\,\mathscr{G}$. Now consider a subclass of interval graphs : the *unit interval graphs*. A graph is a unit interval graph, if it is an interval graph with all the corresponding intervals having the same length. Bogart and West proved the following :

**Property 3.1.3 ([5])**
An interval graph $G$ is a unit interval graph if and only if $G$ does not contain $K_{1,3}$.

Using property 3.1.3 and the result of Lekkerkerker and Boland we can conclude that the class of unit interval graphs is the class Free$\{K_{1,3}, C_n\ (n \geq 4)\}$. This is because every graph in Figure 3.3 contains a claw. It is easy to see that every disjoint union of paths is a unit interval graph. Since all induced subgraphs of $K_{1,3}$ and of cycles are disjoint union of paths, the set of graphs $\{K_{1,3}, C_n\ (n \geq 4)\}$ is the set of all minimal forbidden induced subgraphs for the class of unit interval graphs.

We add some properties for a better understanding of hereditary classes of graphs. The proofs are easy and not given.

**Property 3.1.4**

1. If $\mathscr{H}$ is a hereditary class of graphs then $\overline{\mathscr{H}}$ is also a hereditary class of graphs.

2. If $\mathscr{H} = \text{Free}\,\mathscr{F}$ for a set of graphs $\mathscr{F}$, then $\overline{\mathscr{H}} = \text{Free}\,\overline{\mathscr{F}}$.

3. If every graph in $\mathscr{F}_2$ contains a graph of $\mathscr{F}_1$ then $\text{Free}\,\mathscr{F}_1 \subseteq \text{Free}\,\mathscr{F}_2$.

4. If $\mathscr{F}_1 \subseteq \mathscr{F}_2$ then $\text{Free}\,\mathscr{F}_2 \subseteq \text{Free}\,\mathscr{F}_1$.

Several problems are known to be "difficult" when considering any general graph as an input. It is natural to ask the following question : is there a more restricted setting in which the question is "easier" and still interesting? A good candidate is the restriction of the graphs considered to some hereditary family. While this very concrete condition makes these graphs relatively easy to handle, this still leads to numerous interesting problems.

Another related question, when restricting a problem to a certain class of graphs is to decide whether a graph is the class or not. This is the *recognition problem*. About hereditary classes of graphs, when the number of minimal forbidden induced subgraphs is finite, deciding if a graph is in the class or not can be done in polynomial time.

## 3.2 Coloring problem in hereditary classes

As explained in the previous section, a hereditary class of graphs is defined by a set $\mathscr{M}$ of forbidden induced subgraphs. In the following, we always consider that $\mathscr{M}$ is minimal and so it does not contain two graphs such that one is an induced subgraph of the other. An interesting issue is to determine, for every such set of graphs $\mathscr{M}$, if the coloring problem is polynomial time solvable or NP-complete when restricted to Free $\mathscr{M}$.

For a better understanding of the mechanisms that are involved in this problem, we need the following observations. If the coloring problem is polynomial time solvable in a class of graphs $\mathscr{H}$ then it is polynomial time solvable for all subclasses of $\mathscr{H}$. If the coloring problem is NP-complete in a class of graphs $\mathscr{H}$ then it is NP-complete when extended to any classes of graphs that contains $\mathscr{H}$ as a subclass. The next property follows directly from 3.1.4.

**Property 3.2.1**
Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two sets of graphs such that every graph in $\mathscr{F}_2$ contains a graph from $\mathscr{F}_1$.

- If the coloring problem is polynomial time solvable in Free $\mathscr{F}_2$ then it is also polynomial time solvable in Free $\mathscr{F}_1$.

- If the coloring problem is NP-complete in Free $\mathscr{F}_1$ then it is also NP-complete in Free $\mathscr{F}_2$.

In 2001, Král, Kratochvil, Tuza and Woeginger [38], closed the dichotomy of the coloring problem when restricted to Free$\{H\}$, with $H$ being a unique given graph. The result is the following:

**Theorem 3.2.2 ([38])** *The vertex coloring problem is polynomial time solvable in* Free$\{H\}$ *when $H$ is an induced subgraph of $P_4$ or $P_3 + K_1$ and it is NP-complete otherwise.*

The big steps of the proof are interesting to understand the mechanisms used in the study of the complexity of the coloring problem in classes defined by forbidden induced subgraphs. Let us now describe them.

The polynomial part is proved in the following way. Seinsche [51] proved that if $G \in$ Free$\{P_4\}$ then $\chi(G) = \omega(G)$. Note that graphs in Free$\{P_4\}$ are also called *cograph*. Seinsche proved that a graph $G$ is a cograph if and only if every induced subgraph of $G$ on at least two vertices is either not connected or not anticonnected. Hence the coloring problem is polynomial time solvable in Free$\{P_4\}$. To prove that the coloring problem is polynomial time solvable in Free$\{P_3 + K_1\}$, Král et al. use a structural result from Olariu [45] on the complement. It states that every connected component of a graph in Free$\{\overline{P_3 + K_1}\}$ either does not contain triangles, or is a complete multipartite graph. In both cases, the clique covering problem is polynomial time solvable. Hence, the coloring problem is polynomial time solvable in Free$\{P_3 + K_1\}$. Those two results prove the first assertion of Theorem 3.2.2.

To prove the NP-complete part, Král et al. considered 3 cases :

1. $H$ contains an induced cycle,

2. $H$ is a forest with a vertex of degree at least three,

3. $H$ is a disjoint union of paths.

To deal with case 1, they proved that the coloring problem is NP-complete in Free$\{H\}$ if $H$ contains a cycle by reducing to 3-coloring problem in the general case. To deal with case 2, they use the fact that the coloring problem is NP-complete in claw-free graphs. We will explain this case in the section 3.4. To deal with case 3, they proved that the clique covering problem is NP-complete in Free$\{C_4, diamond, K_4, C_5\}$ by using some more restrictive SAT problem (known to be NP-complete). It proves that coloring problem is NP-complete in Free$\{2P_2, 2K_1 + P_2, 4K_1, C_5\}$. We use a similar method to prove Lemma 5.3.3 in chapter 5. Hence, if $H$ contains one of $2P_2$, $2K_1 + P_2$, $4K_1$ or $C_5$, the coloring problem is NP-complete for Free$\{H\}$.

Since we are in case 3, $H$ is a disjoint union of paths. If $H$ have more than three connected components then $4K_1 \subseteq_i H$. If $H$ have exactly three connected components then either $2K_1 + P_2 \subseteq_i H$, or $H = 3K_1$ (and the problem is polynomial). If $H$ has two connected components, then either $2P_2 \subseteq_i H$ or $H \subseteq_i P_3 + K_1$ (and the problem is polynomial). Lastly, if $H$ is a path then either $H \subseteq_i P_4$ (and the problem is polynomial) or $P_5 \subseteq_i H$ and so $2P_2 \subseteq_i H$. That concludes the proof.

An interesting point about Theorem 3.2.2 is that the graphs $H$ such that coloring is polynomial in Free$\{H\}$ have at most four vertices. Lozin and Malyshev [40] studied the complexity of the coloring problem when restricted to the classes defined by forbidden induced subgraphs of order 4. Observe that when $P_4$ or $P_3 + K_1$ are forbidden, the coloring problem is polynomial by Theorem 3.2.2.

Lozin and Malyshev started by the following observation : in [7], the classes of graphs defined by forbidden induced subgraphs of order 4 are completely characterised in terms of bounded or unbounded *clique-width*. The clique-width is a parameter that describes the structural complexity of a graph. It will be explained in chapter 5. In [49], the following result is proved :

**Theorem 3.2.3 ([49])** *The vertex coloring problem is polynomial time solvable in classes of graphs with bounded clique-width.*

Lozin and Malyshev focused their work on the seven minimal classes of graphs with unbounded clique-width that are defined by forbidden induced subgraphs of order 4. They found the complexity of the coloring problem in some of them by using structural results and bounds on $\chi$. They end-up with three minimal classes where the complexity of the coloring problem is still unknown: $\text{Free}\{K_{1,3}, 2P_1 + P_2\}$, $\text{Free}\{K_{1,3}, 4K_1\}$ and $\text{Free}\{C_4, 4K_1\}$. Furthermore, they proved the following (the proof is not difficult) :

**Lemma 3.2.4 ([40])** *If a graph $G \in \text{Free}\{K_{1,3}, 2P_1 + P_2\}$ contains a $4K_1$, then $G$ is edgeless.*

By this result, the vertex coloring problem in $\text{Free}\{K_{1,3}, 2P_1+P_2\}$ is polynomially equivalent to the same problem in $\text{Free}\{K_{1,3}, 2P_1+P_2, 4K_1\}$. Therefore, it is considered that there are three minimal open cases for the complexity of the coloring problem when restricted to classes of graphs defined by forbidden induced subgraphs of order 4:

- $\mathscr{H}_1 = \text{Free}\{K_{1,3}, 4K_1\}$,
- $\mathscr{H}_2 = \text{Free}\{K_{1,3}, 2P_1 + P_2, 4K_1\}$,
- $\mathscr{H}_3 = \text{Free}\{C_4, 4K_1\}$.

Other hereditary classes of graphs are interesting considering the complexity of the vertex coloring problem. In this section we focused on the classes where the set of forbidden induced subgraphs is finite and small. But a non-finite number of graphs can be excluded and still the complexity of the coloring problem being unknown. For example, see some subclasses of even-hole-free graphs that are presented in section 3.3. For more details about the complexity of the coloring problem and its relative problems when restricted to hereditary classes of graphs, a reader can look at the survey of Golovach, Johnson, Paulusma and Song [28].

A way to study the coloring problem in a class of graphs is to study the specific structure of graphs in this class. For example, Král et al. used the structural result of Olariu to prove Theorem 3.2.2. A usual way is to decompose a graph into subgraphs in the expectation that the problem can be solved in all such subgraphs. The decomposition is interesting if the answer is preserved when the original graph is rebuilt. For example, an obvious way to decompose a graph is to consider all its connected components. It is easy to see that once we have an optimal coloring of all connected components of a graph, we directly obtain an optimal coloring of the entire graph.

Another way to decompose a graph is to look at the cutsets. A *cutset* in a graph $G$ is a set $S$ of vertices such that $G \setminus S$ is disconnected. In the graphs in figure 3.4, two cutsets are evidenced. A *clique cutset* in a graph $G$ is a cutset $S$ such that $G[S]$ is a clique. Clique cutsets are convenient for coloring because of the following tool :

Let $G$ be a graph with a clique cutset $S$. Denote by $V_1$ and $V_2$ the two anticomplete sets of vertices such that $V_1$, $V_2$ and $S$ is a partition of $V(G)$. Start with an optimal proper coloring of $G[V_1 \cup S]$, and an optimal proper coloring of $G[V_2 \cup S]$. Since $S$ is a clique, in any optimal coloring of any subgraphs of $G$ containing $S$, all vertices of $S$ have their own color. Hence, it is possible to relabel the colors of the optimal coloring of $G[V_1 \cup S]$ in order that the colors of vertices of $S$ coincide in the optimal colorings of $G[V_1 \cup S]$ and $G[V_2 \cup S]$. In that way, we obtain a proper optimal coloring of $G$. See figure 3.5.

This tool is particular to clique cutsets and does not work for any other cutset. Sometimes, there is no way to relabel the vertices of the cutset in order that they correspond. As an example, a *star cutset* is a cutset that contains a vertex adjacent to all other vertices of the cutset. The blue cutset of figure 3.4 is a star cutset that will not preserve the coloring. See figure 3.6.

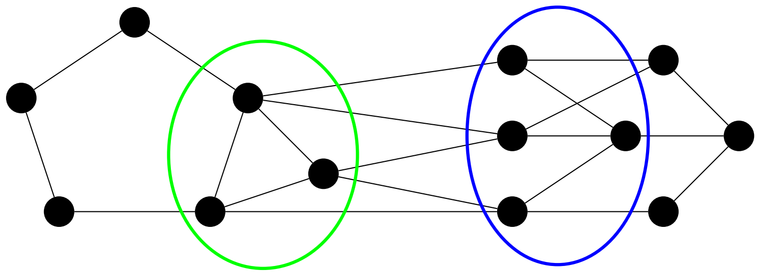
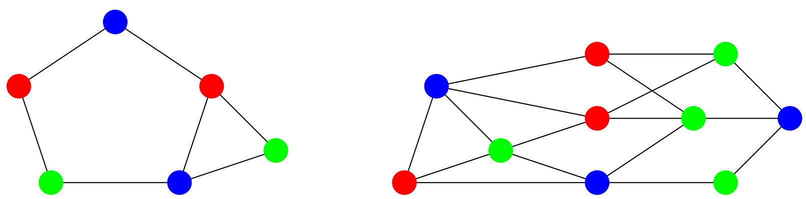
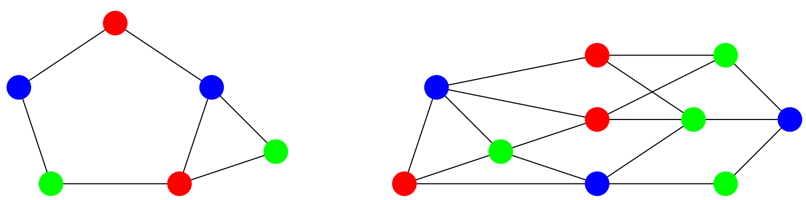


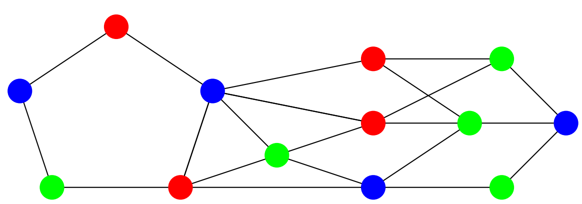
Figure 3.4: Cutsets



Optimal coloring of the subgraphs



Relabeling of the colours of a subgraphs



Optimal coloring of the graphs

Figure 3.5: Clique cutset

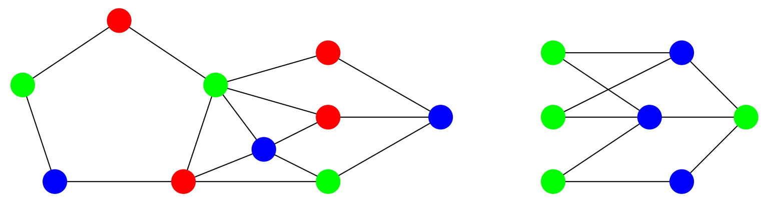


Figure 3.6: Star cutset

## 3.3 Even-hole-free graphs

As presented in section 2.3, for any graph $G$, $\chi(G) \geq \omega(G)$. An obvious related question is to characterise graphs such that $\chi(G) = \omega(G)$. The hereditary class of graphs associated with this property is the one such that $\chi(G') = \omega(G')$ for every induced subgraph $G'$ of $G$. This is the famous class of *perfect graphs*.

By Theorem 3.1.1, there exists a minimal set of graphs $\mathscr{F}$ such that the class of perfect graphs is the class Free $\mathscr{F}$. Berge conjectured that $\mathscr{F}$ is made of every odd hole and every odd antihole. Later, it became customary to call *Berge graphs* the graphs that do not contain an odd hole or an odd antihole. Chudnovsky, Robertson, Seymour and Thomas proved the famous Strong Perfect Graph Theorem [9]. It states that a graph is perfect if and only if it is a Berge graph. The proof of this result first conjectured by Claude Berge, relies on the understanding of the structure of such graphs through a decomposition theorem. This theorem states that every Berge graph is either basic, or has a 2-join, a complement 2-join, a homogeneous pair or a balanced skew partition.

An algorithm to color perfect graphs in polynomial time using the ellipsoid method [30] was presented by Grötschel, Lovász, and Schrijver in 1980s. The decomposition theorem mentioned above, stated by Chudnovsky, Robertson, Seymour and Thomas does not help, so far, to devise a *combinatorial* algorithm to color perfect graph. This is mostly because of the balanced skew partition because all other conclusions are well behaved regarding coloring. Indeed, Chudnovky, Trotignon, Trunck and Vušković [13] found an $\mathscr{O}(n^7)$ algorithm to color perfect graphs with no balanced skew-partition. This algorithm relies on a more precise decomposition theorem. An even more precise decomposition theorem for Berge graphs is given by Trotignon in [52] but the balanced skew partition is still there. This leads to consider that a better understanding of

Berge graphs in general is needed.

The class of *even-hole-free graphs* is the class of graphs that does not contain even-hole. Noting that forbidding $C_4$ as an induced subgraph implies forbidding any antihole of order at least 6, the relationship between Berge graphs and even-hole-free graphs is obvious. Hence, even-hole-free graphs are the graphs without even-hole or even-antihole except $2K_2$. Hence, one can think that a better understanding of even-hole-free graphs would yield to a better understanding of Berge graphs.

A decomposition theorem for even-hole-free graphs is proved by Da Silva and Vušković [17]. It states that a connected even-hole-free graph is either some basic graph or it has a 2-join or a star cutest. It yields to a recognition algorithm in $\mathscr{O}(n^{19})$ by bypassing the problem of the star-cutset.

The star cutset prevents to use the decomposition theorem to find a polynomial time algorithm to color even-hole-free graphs as evidenced by Kristina Vušković in her survey published in 2010 [53]. The complexity of the coloring problem for even-hole-free graphs is still unknown and we think that a better understanding of their structure can help to find an answer to this question.

Another interesting point about the class of even-hole-free graphs is that it contains the class of $\beta$-perfect graphs. A graph $G$ is *$\beta$-perfect* if any induced subgraph $G'$ of $G$, $\beta(G') = \chi(G')$ with $\beta(G) = \max\{\delta(G')+1 : G' \subseteq_i G\}$. Since for an even integer $k$, $\beta(C_k) = 3$ and $\chi(C_k) = 2$, it follows that a $\beta$-perfect graph is an even-hole-free graph.

In chapter 4, we present a decomposition theorem for a subclass of even-hole-free graphs that is the class of graphs with every hole having the same length that is odd. We believe that this result could yield to a better understanding of even-hole-free graphs.

## 3.4 Claw-free graphs

The class of *claw-free graphs* is the class $\text{Free}\{K_{1,3}\}$. It is an interesting generalisation of the class of *line graphs* where the line graph of a graph $G = (V(G), E(G))$ is the graph with edges of $G$ as vertices ($V(L(G)) = E(G)$) and such that two vertices are adjacent if and only if, the two corresponding edges in $G$ are incident to a same vertex. The line graph of $G$ is denoted by $L(G)$. Beineke [2] characterised the class of line graphs by 9 minimal forbidden induced sugraphs. The smallest being the claw, the class of line graphs is a subclass of the class of claw-free graphs.

Line graphs are interesting because solving problems related to edges in a graph $G$ is equivalent to solve the vertex version of the problem in the line graph $L(G)$. For example, given a graph $G$, solving the maximum stable set problem in $L(G)$ is equivalent to solve the maximum matching problem in $G$. The maximum matching of a graph can be computed in polynomial time thanks to Edmond's algorithm [20]. Hence the maximum stable set problem can be solved in polynomial time for line graphs. Sbihi [50] and Minty [44] generalised this result for claw-free graphs.

Solving the coloring problem for a line graph $L(G)$ is equivalent to solving the edge coloring problem in $G$. Holyer [33] proved that the edge coloring problem is NP-complete. Hence the coloring problem is NP-complete for line graphs. Since the class of line graphs is a subclass of claw-free graphs, by property 3.2.1, the coloring problem is NP-complete for claw-free graphs. When we restrict the coloring problem to some subclasses of claw-free graphs it can become polynomial. For example, Hsu presented in [35], an algorithm that finds a minimum proper coloring for claw-free perfect graphs in $\mathcal{O}(n^4)$.

Chudnovsky and Seymour published a series of seven papers about claw-free graphs. They started this study after their work about perfect graphs (see section 3.3). The idea was to construct a decomposition theorem for claw-free graphs using the same tools used to decompose perfect graphs. The first five papers show this decomposition theorem for claw-free graphs. The details are particularly complicated and we will not discuss them here. A survey of those results can be found in [10].

One of the classes presented is the class of *antiprismatic graphs*. The class of antiprismatic graphs is exactly the class Free$\{K_{1,3}, P_2 + 2K_1, 4K_1\}$. As presented in section 3.2, it is one of the classes where graphs of order four are forbidden as an induced subgraphs and where the complexity of the coloring problem is still unknown. Among all basic subclasses of claw-free graphs, Chudnovsky and Seymour dedicated the first two papers of the series to describe the structure of antiprismatic graphs [11, 12].

Furthermore, all the minimal forbidden induced subgraphs for line graphs that are not the claw contain a *diamond*. Also, it is easy to see that if a graph does not contain a *paw*, then its line graph is in Free$\{K_{1,3}, \mathit{diamond}\}$. By

Lemma 3.2.4, graphs in Free$\{K_{1,3}, diamond\}$ that contains a $4K_1$ are edgeless. Therefore if a graph $G$ does not contain a *paw* then either $L(G)$ is antiprismatic or it is edgeless.

In Chapter 6 we use those results to study the complexity of the coloring problem for antiprismatic graphs.

# Chapter 4

# When all holes have the same length

## 4.1 Introduction

In chapter 3, we presented two important hereditary classes of graphs: the perfect graphs and the even-hole free graphs. They both exclude holes depending on the parity of their length. Until now, the class of perfect graphs and the class of even-hole free graphs do not have structural theorem precise enough to be used for providing a coloring algorithm (or an NP-completeness proof in the case of even-hole free graphs). It could be interesting to restrict once again these classes by excluding some additional holes. A radical approach consists in excluding all holes. This is the class of chordal graphs that will be presented in section 4.2. Chordal graphs have a structure well known that easily yields a polynomial-time algorithm for coloring.

The next approach consists in excluding all holes except the ones with a certain length. For an integer $k \geq 4$, we denote by $\mathcal{C}_k$, the class of graphs where every hole has length $k$. Note that when $k$ is odd, $\mathcal{C}_k$ is a subclass of even-hole-free graphs and when $k$ is even and at least 6, $\mathcal{C}_k$ is a subclass of perfect graphs. In this chapter, for every integer $\ell \geq 3$, we give the following structural description of the class of graphs whose holes all have length $2\ell + 1$ (definitions will be given later).

**Theorem 4.1.1** *Let $\ell \geq 3$ be an integer. If $G$ is a graph in $\mathcal{C}_{2\ell+1}$ then one of the following holds:*

1. *$G$ is a ring of length $2\ell + 1$;*
2. *$G$ is a proper blowup of a twinless odd $\ell$-template;*
3. *$G$ has a universal vertex or*
4. *$G$ has a clique cutset.*

Some subclasses of $\mathcal{C}_k$ have been already studied and results have been obtained. In [47], the class of Free$\{4K_1, C_4, C_6, C_7\}$ graphs is studied. It is a subclass of $\mathcal{C}_5$. In [6], the class of rings of length $k$ is defined for every integer $k \geq 4$ (see Section 4.2.2 for the definition), and it is used as a basic class for several decomposition theorems. Rings of length $k$ form a subclass of $\mathcal{C}_k$. In [42], a polynomial time algorithm that colors every ring is given. In [34], it is proved that for every fixed integer $k$, there exists rings of length $k$ of arbitrarily large rankwidth.

In this chapter, we will use the notion of *hypergraph*; that is, a structure similar to graphs except that the edges (called *hyperedges*) may contain an arbitrary positive number of vertices. While all the graphs that we use are simple, in hypergraphs, we allow hyperedges that contain a single vertex and multiple hyperedges (that is, there can be different hyperedges on the same set of vertices). Observe that we do not allow an empty hyperedge.

Lemma 4.1.3 gives a recognition algorithm with complexity $\mathcal{O}(n^{19})$ for the class $\mathcal{C}_k$. It uses the following result from Berger, Seymour and Spirkl [4].

**Theorem 4.1.2 ([4])** *There exists an algorithm that, given a graph $G$ and $u, v \in V(G)$, decides whether there exists a path from $u$ to $v$ that is not a shortest path in time $\mathcal{O}(|V(G)|^{16})$.*

**Lemma 4.1.3** *There is an algorithm to recognise $\mathcal{C}_k$ with a running time $\mathcal{O}(n^{19})$.*

*Proof.* The following algorithm is called algorithm $\mathcal{A}$.

- INPUT : a graph $G = (V, E)$

- For all $P_3$ $abc$ contained in $G$ :
    1. Set $G' = G[(V \setminus N_G[b]) \cup \{a, c\}]$
    2. If a shortest $ac$-path from in $G'$ has length different from $k-2$ then return "No" and Stop.
    3. If non-shortest $ac$-path in $G'$ exists then return "No" and Stop.
- Return "Yes".

Algorithm $\mathcal{A}$ runs in polynomial time. Enumerating all $P_3$'s can trivially be done in time $\mathcal{O}(n^3)$. Finding a shortest path can be done with breadth first search algorithm in time $\mathcal{O}(|V(G)|^2)$ (precisely in time $(|V(G)|+|E(G)|)$). Finding a non-shortest path can be done in time $\mathcal{O}(|V(G)|^{16})$ by Theorem 4.1.2. The complexity of algorithm $\mathcal{A}$ is $\mathcal{O}(|V(G)|^{19})$

Regarding the correctness, we prove that the algorithm returns "No" if and only if $G$ is not in $\mathcal{C}_k$. If the algorithm returns "No", then there exists an $ac$-path in $G'$ of length different from $k-2$. Therefore, $abcPa$ is a hole of length different from $k$. Suppose now that the input graph $G$ has a hole $H$ of length different from $k$. Let $abc$ be a $P_3$ in $H$. Denote by $P$ the path induced by $V(H) \setminus \{b\}$. The path $P$ has length different from $k-2$ and is contained in $G' = G[(V \setminus N_G(b)) \cup \{a, c\}]$. If $P$ is a shortest $ac$-path, algorithm $\mathcal{A}$ outputs "No" at step 2. If $P$ is not a shortest path then there is at least one non-shortest $ac$-path and the algorithm $\mathcal{A}$ outputs "No" at step 3. In both cases, the algorithm answer "No". □

We believe that Theorem 4.1.1 could give a recognition algorithm with running time faster that $\mathcal{O}(n^{19})$.

### 4.1.1 Outline

In section 4.2, we present several known classes of graphs and their properties that are used in this chapter.

In section 4.3, a new structure called template is defined and fully described.

Section 4.4 is about the operation of blowup on templates.

Sections 4.5 and 4.6 contain the proof of the main Theorem (Theorem 4.1.1).

This chapter is a joint work with Jake Horsfield, Myriam Preissmann, Ni Luh Dewi Sintiari, Nicolas Trotignon and Kristina Vušković. It will be submitted with Linda Cook and Paul Seymour who did simultaneously a similar work. The final joint paper will include a part on the even case.

## 4.2 A survey of some classes of graphs

### 4.2.1 Classes of perfect graphs

A graph is *chordal* if it is hole-free. A graph is a *split graph* if it is in Free$\{C_4, C_5, 2K_2\}$. A graph is a *quasi-threshold graph* if it is Free$\{P_4, C_4\}$ (quasi-threshold graphs are sometimes called *trivially perfect graphs*, see [29]). A graph is a *threshold graph* if it is in Free$\{P_4, C_4, 2K_2\}$ (threshold graphs are sometimes called *graphs with Dilworth number 1*). A graph is a *half graph* if it is in Free$\{3K_1, C_4, C_5\}$. Observe that this is not the class of half graphs defined by Erdős but they are closed in some point of view ([21]).

Observe that these six classes are all classes of perfect graphs. The classes of split graphs and threshold graphs are self-complementary while the classes of chordal graphs, quasi-threshold and half graphs are not. In Figure 4.1, a Venn diagram of seven graph classes is represented ($\overline{\text{chordal}}$ and $\overline{\text{quasi} - \text{threshold}}$ mean complements of chordal and quasi-threshold graphs respectively). In every set, a typical example of the class is represented. The diagram provides several alternative definitions of the classes we work on (for instance, a split graph is a chordal graph whose complement is chordal, and so on). All the informations given by Figure 4.1 are easily recovered from the definitions of the corresponding classes.

**Theorem 4.2.1 ([19])** *A graph $G$ is chordal if and only if every non-complete induced subgraph of $G$ has a clique cutset.*

**Theorem 4.2.2 ([23])** *A graph $G$ is a split graph if and only if $V(G)$ can be partitioned into a (possibly empty) clique and a (possibly empty) stable set.*

The *line graph* of a hypergraph $\mathcal{H}$ is the graph $G$ whose vertex-set is $E(\mathcal{H})$ and where two hyperedges of $\mathcal{H}$ are adjacent vertices of $G$ whenever their intersection is non-empty. Recall that in this paper, hypergraphs may have

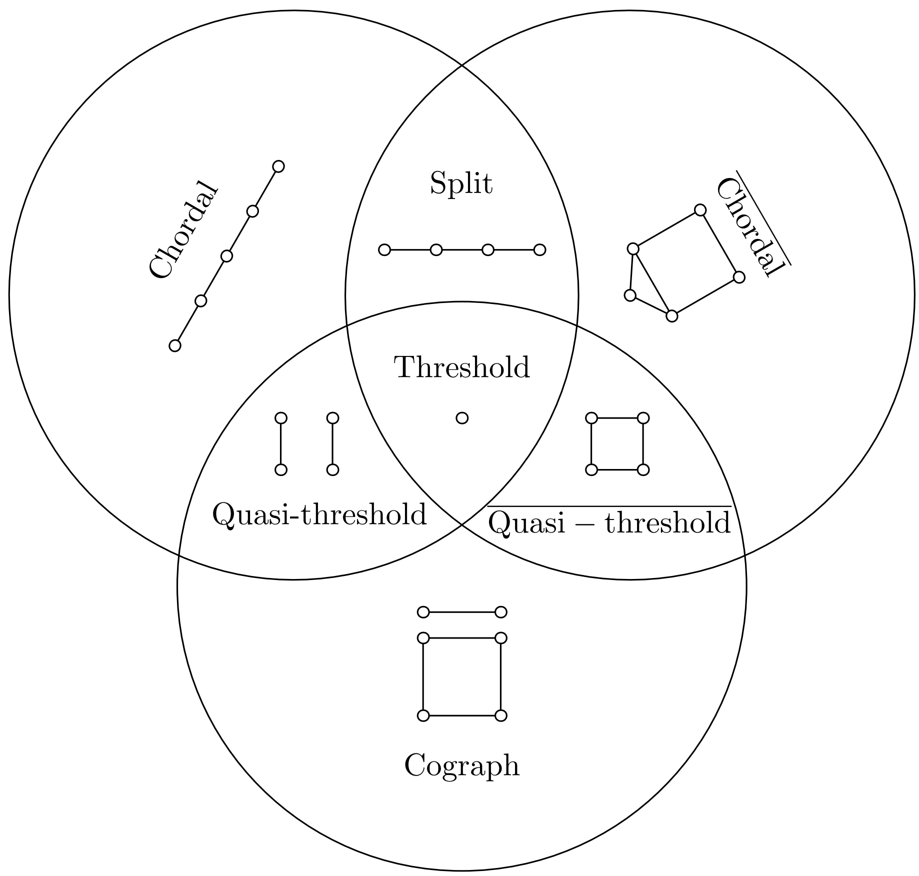



Figure 4.1: Venn diagram of seven classes of graphs

multiple hyperedges (that are distinct hyperedges with the same vertices in them). A hypergraph is *laminar* is for every pair $X, Y$ of hyperedges, either $X \subseteq Y$ or $Y \subseteq X$ or $X \cap Y = \emptyset$.

**Theorem 4.2.3 ([55])** *For all graphs $G$ the following statements are equivalent.*

1. *$G$ is a quasi-threshold graph.*
2. *Every induced subgraph of $G$ is disconnected or has a universal vertex.*
3. *$G$ is the line graph of a laminar hypergraph.*

**Theorem 4.2.4 ([14])** *For all graphs $G$ the following statements are equivalent.*

1. *$G$ is a threshold graph.*
2. *Every induced subgraph of $G$ has an isolated vertex or a universal vertex.*
3. *For all vertices $u$ and $v$ of $G$, $N_G(u) \subseteq N_G(v)$ or $N_G(v) \subseteq N_G(u)$*

It is convenient to sort the vertices of a threshold graph. Formally, an ordering $v_1, \dots, v_k$ such that $N_G(v_i) \subseteq N_G[v_j]$ for all integers $i$ and $j$ satisfying $1 \leq i \leq j \leq k$ is called a *domination ordering*. There is another convenient ordering of the vertices of a threshold graph. By characterization (2) in Theorem 4.2.4, every threshold graph can be obtained by the following inductive process: start with a vertex $u_1$, assume for some $k \geq 1$ that vertices $u_1, \dots, u_k$ are already constructed, and then add a vertex $u_{k+1}$ that is either complete or anticomplete to $\{u_1, \dots, u_k\}$. The order $u_1, \dots, u_n$ is then called an *elimination ordering* of the threshold graph (and it is not a domination ordering in general).

An example is represented in Figure 4.2. On the top, a threshold graph $J$ on $\{v_1, \dots, v_{10}\}$ is represented for which $(v_1, \dots, v_{10})$ is a domination ordering. Vertices are circles with a number in them that gives the place of the vertex in the elimination ordering. On the bottom, the complement $J'$ of $J$ is represented. It is also a threshold graph but the domination ordering is reversed (it is $(v_{10}, \dots, v_1)$), while the elimination ordering remains the same.

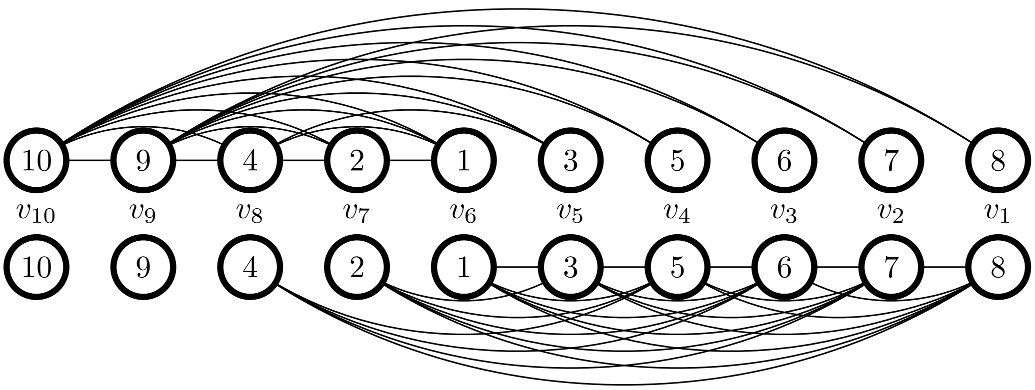



Figure 4.2: A threshold graph and its complement

### 4.2.2 Classes defined by excluding Truemper configurations

*Truemper configurations* are graphs that play a role in many decomposition theorems, see [54]. They are the long prisms, thetas, pyramids and wheels. Let us define them.

A *long prism* is a graph made of three vertex-disjoint paths $P_1 = a_1 \dots b_1$, $P_2 = a_2 \dots b_2$, $P_3 = a_3 \dots b_3$ of length at least 1, such that $a_1a_2a_3$ and $b_1b_2b_3$ are triangles and no edges exist between the paths except those of the two triangles. In chapter 6, we call *prism*, the graph that is a long prism with all paths of length 1.

A *pyramid* is a graph made of three paths $P_1 = a \dots b_1$, $P_2 = a \dots b_2$, $P_3 = a \dots b_3$ of length at least 1, two of which have length at least 2, vertex-disjoint except at $a$, and such that $b_1b_2b_3$ is a triangle and no edges exist between the paths except those of the triangle and the three edges incident to $a$. The vertex $a$ is called the *apex* of the pyramid.

A *theta* is a graph made of three internally vertex-disjoint paths $P_1 = a \dots b$, $P_2 = a \dots b$, $P_3 = a \dots b$ of length at least 2 and such that no edges exist between the paths except the three edges incident to $a$ and the three edges incident to $b$.

Observe that the lengths of the paths used in the three definitions above are designed so that the union of any two of the paths induce a hole. A long prism, pyramid or theta is *balanced* if the three paths in the definition are of the same length. It is *unbalanced* otherwise.

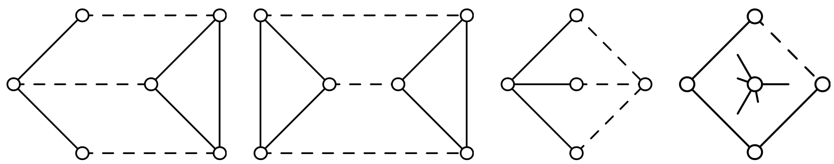


Figure 4.3: Pyramid, long prism, theta and wheel (dashed lines represent paths)

A *wheel* $W = (H, c)$ is a graph formed by a hole $H$ (called the *rim*) together with a vertex $c$ (called the *center*) that has at least three neighbors in the hole.

A wheel is a *universal wheel* if the center is adjacent to all vertices of the rim. A wheel is a *twin wheel* if the center is adjacent to exactly three vertices of the rim and they induce a $P_3$. A wheel is *proper* if it is neither a twin wheel nor a universal wheel.

Truemper configurations are of interest here because of the following easy observation.

**Lemma 4.2.5** *Every unbalanced long prism, every unbalanced pyramid, every unbalanced theta and every proper wheel contains holes of different lengths.*

*Every pyramid contains an odd hole. Every long prism and every theta contains an even hole.*

*Proof.* In a long prism, pyramid or theta, the union of any two paths used in the definition induces a hole. Paths of different lengths are then easily used to provide holes of different lengths. In a proper wheel, the rim and a shortest hole are holes of different lengths.

In a pyramid, paths of the same parity, that exist since there are three paths, induce an odd hole. In thetas and long prisms, they induce an even hole. □

The following variant is more useful for our study.

**Lemma 4.2.6** *If $\ell \geq 2$ is an integer and $G \in \mathcal{C}_{2\ell+1}$, then every Truemper configuration of $G$ is a twin wheel, a universal wheel or a pyramid whose three paths all have length $\ell$.*

*Proof.* Clear from Lemma 4.2.5. □

A graph $G$ is *universally signable* if $G$ is in Free{long prism, pyramid, theta, wheel}.

**Theorem 4.2.7 ([15])** *A graph $G$ is universally signable if and only if every induced subgraph of $G$ is a hole, a complete graph or has a clique cutset.*

A graph $G$ is a *ring* if its vertex-set can be partitioned into $k \geq 4$ sets $K_1, \dots, K_k$ such that (with subscripts understood to be taken modulo $k$):

1. $K_1, \dots, K_k$ are cliques;
2. for all $i \in \{1, \dots, k\}$, $K_i$ is anticomplete to $V(G) \setminus (K_{i-1} \cup K_i \cup K_{i+1})$;
3. for all $i \in \{1, \dots, k\}$, some vertex of $K_i$ is complete to $K_{i-1} \cup K_{i+1}$;
4. for all $i \in \{1, \dots, k\}$ and all $x, x' \in K_i$, either $N_G(x) \subseteq G(x')$ or $N_G(x') \subseteq G(x)$ .

The integer $k$ in the definition above is the *length of the ring*. Observe that when $k \geq 4$, the hole $C_k$ is a ring of length $k$. Observe also that, by Theorem 5.3.1, for any integer $1 \leq i \leq k$, the graph $G[K_i \cup K_{i+1}]$ is a half graph. We refer to the cliques $K_1, \dots, K_k$ as the *cliques of the ring $G$*.

The following is a corollary of Theorem 1.6 from [6].

**Theorem 4.2.8** *If $G$ is in* Free{*long prism, theta, pyramid, proper wheel*, $C_4, C_5$}, *then one of the following holds.*

1. *$G$ is a ring of length at least 6;*
2. *$G$ has a clique cutset;*
3. *$G$ has a universal vertex.*

## 4.3 Odd templates

Here we define and study the main basic class of Theorem 4.1.1.

### 4.3.1 Modules in threshold graphs

Let $G$ be a graph. A *module* of $G$ is a set $X \subseteq V(G)$ such that every vertex in $V(G) \setminus X$ is either complete or anticomplete to $X$. Observe that all subsets of $V(G)$ of cardinality 0, 1 or $|V(G)|$ are modules of $G$. We will use the notion of module only in the context of threshold graphs. The reader can check that sets of vertices that are intervals for both elimination and domination orderings are modules. We omit the proof since we do not need this formally. We now state three lemmas.

**Lemma 4.3.1** *Let $J$ be a threshold graph and $X \subseteq V(J)$ such that $|X| \geq 2$. Then $\bar{J}$ is a threshold graph, $X$ is a module of $J$ if and only if it is a module of $\bar{J}$, and exactly one of $J[X]$ and $\bar{J}[X]$ is anticonnected.*

*Proof.* Being a threshold graph and module are properties that are closed under taking the complement. By Theorem 4.2.4, exactly one of $J[X]$ or $\bar{J}[X]$ contains an isolated vertex, and the other one contains a universal vertex. Hence, since $|X| \geq 2$, exactly one of $J[X]$ or $\bar{J}[X]$ is connected and the other one is anticonnected. □

**Lemma 4.3.2** *Let $J$ be a threshold graph. If $X$ is an anticonnected module of $J$ that contains at least two vertices, then $N(X)$ is a clique that is complete to $X$. Moreover, $N_J(X) \subseteq N(N(X))$.*

*Proof.* Since $X$ is a module, $N(X)$ is complete to $X$. Suppose that $N(X)$ is not a clique and let $u$ and $v$ be two non-adjacent vertices in $N(X)$. Since $|X| \geq 2$ and $X$ is anticonnected, $X$ contains two non-adjacent vertices $u', v'$ that together with $u$ and $v$ form a $C_4$ in $J$. This contradicts $J$ being a threshold graph.

Suppose that $N_J(X) \subseteq N(N(X))$ does not hold. So, there exists $u \in N(X)$ and $v \in X$ with $N_J(u) \subseteq N_J(v)$. Since $u \in N(X)$, $u$ is complete $X$, so $v$ is complete to $X \setminus \{v\}$. This contradicts $X$ being anticonnected. □

**Lemma 4.3.3** *Let $J$ be a threshold graph and $X \subseteq V(J)$ a module of $J$. If $X$ contains some isolated vertices of $J$, then either $X$ contains only isolated vertices of $J$, or $X$ contains all non-isolated vertices of $J$.*

*Proof.* Let $S$ be the set of all isolated vertices of $J$ and $T = V(J) \setminus S$. By assumption, $X$ contains a vertex of $S$. If $X$ contains only vertices of $S$, then the conclusion holds, so suppose that $X$ contains at least one vertex of $T$. Suppose for a contradiction that $X$ does not contain all of $T$. Since $J[T]$ is connected (because it is a threshold graph with no isolated vertices and hence by Theorem 4.2.4 it contains a universal vertex), there exists an edge $uv$ of $J$ with $u \in T \cap X$ and $v \in T \setminus X$. Since $X$ contains isolated vertices, this contradicts $X$ being a module. □

### 4.3.2 Templates

For an integer $\ell \geq 2$, an *odd $\ell$-template* is any graph $G$ that can be built according to the following process.

1. Choose a threshold graph $J$ on vertex set $\{1, \ldots, k\}$, $k \geq 3$.
2. Choose a laminar hypergraph $\mathcal{H}$ on vertex set $\{1, \ldots, k\}$ such that :
   (a) every hyperedge $X$ of $\mathcal{H}$ is a module of $J$ of cardinality at least 2 and
   (b) at least one hyperedge $W$ of $\mathcal{H}$ contains all vertices of $\mathcal{H}$.
3. For each $i \in \{1, \ldots, k\}$, $G$ contains two vertices $v_i$ and $v'_i$ that are linked by a path of $G$ of length $\ell - 1$. The $k$ paths built at this step are vertex disjoint and are called the *principal paths* of the odd template.
4. The set of vertices of $G$ is $V(G) = A \cup A' \cup B \cup B' \cup I$ where:
   (a) $I$ is the set of all internal vertices of the principal paths,
   (b) $A = \{v_1, \ldots, v_k\}$,
   (c) $A' = \{v'_1, \ldots, v'_k\}$,
   (d) $B = \{v_X : X$ hyperedge of $\mathcal{H}$ such that $J[X]$ is anticonnected$\}$,

(e) $B' = \{v'_X : X$ hyperedge of $\mathcal{H}$ such that $\bar{J}[X]$ is anticonnected$\}$.

Note that by Lemma 4.3.1, for every hyperedge $X$ of $\mathcal{H}$, either $v_X \in B$ or $v'_X \in B'$ (and not both).

5. The set of edges of $G$ is defined as follows.

   (a) for every $v_i, v_j \in A$, $v_iv_j \in E(G)$ if and only if $ij \in E(J)$,

   (b) for every $v'_i, v'_j \in A'$, $v'_iv'_j \in E(G)$ if and only if $ij \notin E(J)$,

   (c) for every $v_X, v_Y \in B$, $v_Xv_Y \in E(G)$ if and only if $X \cap Y \neq \emptyset$,

   (d) for every $v'_X, v'_Y \in B'$, $v'_Xv'_Y \in E(G)$ if and only if $X \cap Y \neq \emptyset$,

   (e) for every $v_i \in A$, $v_X \in B$, $v_iv_X \in E(G)$ if and only if $i \in N_J[X]$,

   (f) for every $v'_i \in A'$, $v'_X \in B'$, $v'_iv'_X \in E(G)$ if and only if $i \in N_{\bar{J}}[X]$,

   (g) for every $v \in I$, $v$ is incident to exactly two edges (those in its principal path).

The following notation is convenient.

**Notation:** For every vertex $x \in B$ such that $x = v_X$ where $X$ is a hyperedge of $\mathcal{H}$, we set $H_x = \{v_i : i \in X\}$. Similarly, For every vertex $x \in B'$ such that $x = v'_X$ where $X$ is a hyperedge of $\mathcal{H}$, we set $H'_x = \{v'_i : i \in X\}$.

We now list some properties of templates that follow directly from the definition.

1. $G[A]$ is a threshold graph isomorphic to $J$ and $G[A']$ is a threshold graph isomorphic to $\bar{J}$ (and hence to the complement of $G[A]$).

2. For all $x \in B$, $H_x$ is a module of $G[A]$ and $G[H_x]$ is anticonnected. Also for all $x \in B'$, $H'_x$ is a module of $G[A']$ and $G[H'_x]$ is anticonnected.

3. $G[B]$ is isomorphic to the line graph of the hypergraph $\mathcal{H}_B$ on vertex set $A$ and hyperedge set $\{H_x : x \in B\}$. Also $G[B']$ is isomorphic to the line graph of the hypergraph $\mathcal{H}_{B'}$ on vertex set $A'$ and hyperedge set $\{H'_x : x \in B'\}$. Hence $G[B]$ and $G[B']$ are a quasi-threshold graphs by Theorem 4.2.3.

4. There is an edge between $v_i \in A$ and $x \in B$ if and only if $v_i \in N_A[H_x]$, and there is an edge between $v'_i \in A'$ and $x \in B'$ if and only if $v'_i \in N_{A'}[H'_x]$.

**Lemma 4.3.4** *There exist vertices $w$ and $w'$ that are universal vertices in respectively $G[A \cup B]$ and $G[A' \cup B']$, and such that either $w \in A$ and $w' \in B'$, or $w \in B$ and $w' \in A'$.*

*Proof.* By Theorem 4.2.4, $G[A]$ contains a vertex $u$ that is either universal or isolated. If $u$ is universal, then for every $x \in B$, $u \in N(H_x)$ ($u$ cannot be in $H_x$ since $G[H_x]$ is anticonnected by property (2) of templates). So, $u$ is adjacent to $x$ by property (4) of templates. Hence, $u$ is a universal vertex of $G[A \cup B]$.

Otherwise, $u$ is an isolated vertex of $G[A]$. So, $G[A]$ is anticonnected. Hence, the vertex $w$ corresponding to the hyperedge $W$ from condition (2b) of templates is in $B$. By property (4) of templates, $w$ is a universal vertex of $G[A \cup B]$.

The proof for $G[A' \cup B']$ is similar. So, $w$ and $w'$ exist, and by the way we construct them, we see that either $w \in A$ and $w' \in B'$, or $w \in B$ and $w' \in A'$.□

Let $w$ and $w'$ be as in Lemma 4.3.4. The 7-tuple $(A, B, A', B', I, w, w')$ is then called an *$\ell$-partition* of $G$.

Let us give a simple example. Consider an integer $\ell \geq 2$ and a threshold graph $J$ on three vertices $\{1, 2, 3\}$ with no edges. So, $G[A]$ has no edges, $G[A']$ is a triangle on three vertices $v'_1, v'_2, v'_3$, and for $i = 1, 2, 3$, there is a path of length $\ell - 1$ from $v_i$ to $v'_i$. Consider $\mathcal{H}$ the hypergraph on $\{1, 2, 3\}$ with a unique hyperedge that is $\{1, 2, 3\}$. We now see that $G$ is a balanced pyramid with apex $w$ and triangle $v'_1v'_2v'_3$. Under these circumstances, the sets $A = \{v_1, v_2, v_3\}$, $B = \{w\}$, $A' = \{v'_1, v'_2, v'_3\}$, $B' = \emptyset$, $I = V(G) \setminus (A \cup B \cup A' \cup B')$, $w$ and $v'_3$ form an $\ell$-partition of $G$.

It is worth noting that the $\ell$-partition above is not unique. Here is another one. Call $u$ the neighbor of $v'_3$ in the path from $v_3$ to $v'_3$ with interior in $I$ (possibly, $u = v_3$). Set $A_1 = \{w, v_1, v_2\}$, $B_1 = \emptyset$, $A'_1 = \{u, v'_1, v'_2\}$, $B'_1 = \{v'_3\}$ and $I_1 = V(G) \setminus (A_1 \cup B_1 \cup A'_1 \cup B'_1)$. It can be checked that $A_1$, $B_1$, $A'_1$, $B'_1$, $I_1$, $w$ and $v'_3$ form another $\ell$-partition of $G$. See Figure 4.4. Some edges are dashed in several ways, this will be explained later, so far, they are just edges of $G$.

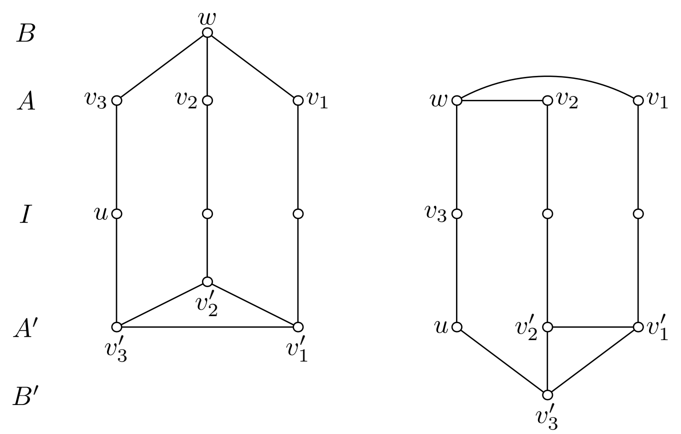



Figure 4.4: Two 3-partitions of a pyramid whose paths all have length 3

**Lemma 4.3.5** *For all integers $\ell \geq 2$, every pyramid $\Pi$ such that $\Pi \in \mathcal{C}_{2\ell+1}$ is an odd $\ell$-template.*

*Proof.* Since $\Pi \in \mathcal{C}_{2\ell+1}$, its three paths have length $\ell$. The explanations above show it is an odd $\ell$-template. □

We now give a more complicated example represented in Figure 4.5. The threshold graph $J$ has 10 vertices. Each vertex of $G[A]$ and $G[A']$ is represented with a number in a circle that represents the elimination ordering of the threshold graph it belongs to. The hypergraph $\mathcal{H}$ has the following hyperedges: $X_1 = \{1, 2\}$, $X_2 = \{1, 2, 3\}$, $X_3 = \{9, 10\}$, $X_4 = \{5, 6, 7\}$, $X_5 = \{5, 6, 7, 8\}$, $X_6 = \{4, 5, 6, 7, 8\}$ and $X_7 = \{1, \dots, 10\}$. The vertex of $B \cup B'$ corresponding to a hyperedge $X_i$ is denoted by $x_i$.

### 4.3.3 Structure of odd templates

Throughout this subsection, $\ell \geq 2$ is an integer and $(A, B, A', B', I, w, w')$ is an $\ell$-partition of an odd $\ell$-template $G$.

**Lemma 4.3.6** *If $x \in B$ (resp. $x \in B'$), then $H_x$ (resp. $H'_x$) is the unique anticomponent of $G[N_A(x)]$ (resp. $G[N_{A'}(x)]$) that contains at least two vertices.*

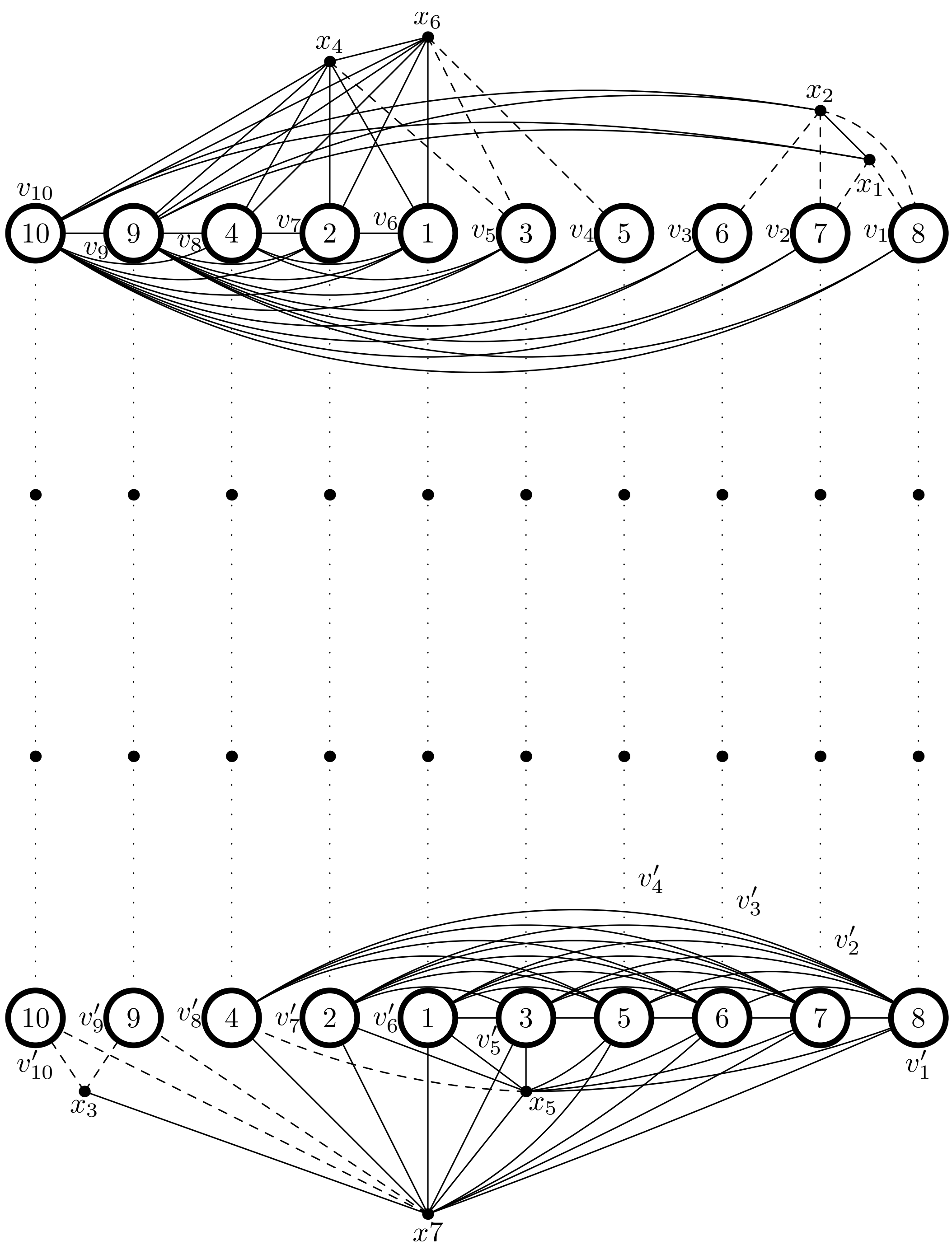



Figure 4.5: An odd 4-template

*Proof.* Since $G$ is in Free$\{C_4\}$ and $N_A(x)$ contains at least two non-adjacent vertices, $G[N_A(x)]$ contains a unique anticomponent $X$ of size at least 2. Since by property (2) of templates, $H_x$ is an anticonnected module of $G[A]$, it is also an anticonnected module of $G[N_A(x)]$. Since every vertex of $N_A(x)$ is either in $H_x$ or complete to $H_x$, $H_x$ must be an anticomponent of $G[N_A(x)]$, and since it contains at least two vertices, it is equal to $X$. □

**Lemma 4.3.7** *If $x, y \in B$ (resp. $\in B'$) are such that $xy \notin E(G)$, then $H_x \cup \{x\}$ (resp. $H'_x \cup \{x\}$) is anticomplete to $H_y \cup \{y\}$ (resp. $H'_y \cup \{y\}$).*

*Proof.* Suppose $x, y \in B$ and $xy \notin E(G)$. Then by condition (5c) of templates, $H_x$ and $H_y$ are disjoint.

Suppose there is at least one edge from $H_x$ to $H_y$. Since they are both modules of $G[A]$, it follows that $H_x$ is complete to $H_y$, so $H_y \subseteq N(H_x)$. Hence, by Lemma 4.3.2, $H_y$ is a clique. Since $H_y$ contains at least two vertices, this contradicts $H_y$ being anticonnected. So, $H_x$ is anticomplete to $H_y$. Hence, by property (4) of templates, $x$ is anticomplete to $H_y$ and $y$ is anticomplete to $H_x$. So, $H_x \cup \{x\}$ is anticomplete to $H_y \cup \{y\}$ because $xy \notin E(G)$ holds from our assumption.

The proof for $x, y \in B'$ is similar. □

**Lemma 4.3.8** *Every vertex of $G$ has degree at least 2 and every vertex of $B \cup B'$ has degree at least 3.*

*Proof.* Vertices in $I$ are all in the interior of some path, so they have degree at least 2.

Vertex $w$ has degree at least 2 since $|A| \geq 3$. A vertex $v \in A \setminus \{w\}$ therefore has degree at least 2 (one neighbor in $I$, and $w$). So every vertex of $A$ has degree at least 2. The proof for $A'$ is similar.

Let $x$ be a vertex of $B$. If $x = w$, then $x$ has degree at least 3 (because $|A| \geq 3$), so we may assume $x \neq w$. By, property (2) of templates, $|H_x| \geq 2$ and $w \notin H_x$ because $H_x$ is anticonnected. So $x$ has degree at least 3 as claimed (at least two neighbors in $H_x$, and $w$). The proof for $x \in B'$ is similar. □

The following shows that odd templates can be considered as a generalization of balanced pyramids (we do not need it and include it because we believe it helps understanding the structure of the class we work on).

**Lemma 4.3.9** *For every integer $\ell \geq 2$, every odd $\ell$-template $G$ contains a pyramid.*

*Proof.* Consider three vertices $v_i, v_j$ and $v_h$ in $A$ and the corresponding vertices $v'_i, v'_j$ and $v'_h$ in $A'$. Exactly one of $G[\{v_i, v_j, v_h\}]$ and $G'[\{v'_i, v'_j, v'_h\}]$ is connected (because they have three vertices and one is isomorphic to the complement of the other). So, up to symmetry, we may assume that $G[\{v_i, v_j, v_h\}]$ is disconnected (and therefore contains at most one edge).

Note that $w$ is distinct from $v_i, v_j$ and $v_h$ since $G[\{v_i, v_j, v_h\}]$ is disconnected. We see that $w$ and the three principal paths linking $\{v_i, v_j, v_h\}$ to $\{v'_i, v'_j, v'_h\}$ form a pyramid (if $G[\{v_i, v_j, v_h\}]$ contains one edge $e$, then the triangle is formed by $e$ and $w$, and otherwise it is $v'_i v'_j v'_h$). $\square$

From the definition of odd $\ell$-templates, every vertex $x \in B$ corresponds to a set $H_x \subseteq A$. These sets form a hypergraph $\mathcal{H}_B$ on the vertex-set $A$ (that is isomophic to a sub-hypergraph of $\mathcal{H}$). Let us build an extention $\mathcal{H}_A$ of $\mathcal{H}_B$ by adding more hyperedges: for every vertex $v \in A$, we add the hyperedge

$$H_v = N_A[v] \cap \{u \in A : N_A(u) \subseteq N_A(v)\}.$$

Note that $v \in H_v$.

**Lemma 4.3.10** *$\mathcal{H}_A$ is a laminar hypergraph and $G[A \cup B]$ is isomorphic to its line graph (in particular, $G[A \cup B]$ is a quasi-threshold graph and therefore a chordal graph). A similar statements holds for $G[A' \cup B']$.*

*Proof.* By construction, every vertex of $A \cup B$ corresponds to a hyperedge of $\mathcal{H}_{\mathcal{A}}$. We have to check that the ends of every edge of $G[A \cup B]$ correspond to hyperedges of $\mathcal{H}_A$ that are included one in the other, and that the ends of every non-edge correspond to a pair of disjoint hyperedges. This will prove that $G[A \cup B]$ is isomorphic to the line graph of $\mathcal{H}_A$ and that $\mathcal{H}_A$ is laminar. Let us check all the cases.

For $x, y \in B$, since $\mathcal{H}_B$ is laminar and $G[B]$ is isomorphic to its line graph, we have nothing to prove.

Let $u, v \in A$. By Theorem 4.2.4, we may assume up to symmetry that $N_A(v) \subseteq N_A(u)$. If $uv \in E(G)$, then clearly $H_v \subseteq H_u$. Suppose $uv \notin E(G)$, and let $t$ be a vertex of $A$ such that $N_A(t) \subseteq N_A(v)$. So, $N_A(t) \subseteq N_A(u)$. If $tv \in E(G)$, then $uv \in E(G)$, a contradiction. So, $tv \notin E(G)$ and $H_v = \{v\}$. Since $v \notin N[u]$, we have that $H_u \cap H_v = \emptyset$.

Consider finally vertices $u \in A$ and $x \in B$. Suppose first that $ux \in E(G)$. By property (4) of templates, we have that $u \in N_A[H_x]$. If $u \in H_x$, then $H_u \subseteq H_x$ by Lemma 4.3.2 (specifically, we use $N(H_x) \subseteq N_A(N_A(H_x))$ to conclude that $N_A(H_x) \cap H_u = \emptyset$, and then since $H_u \subseteq N_A[H_x]$ it follows that $H_u \subseteq H_x$). If $u \in N_A(H_x)$, then by Lemma 4.3.2 (again also using that $N(H_x) \subseteq N_A(N_A(H_x)))$, $H_x \subseteq H_u$. So, an edge indeed yields an inclusion of the corresponding hyperedges.

Suppose now that $ux \notin E(G)$. So, $u \notin N[H_x]$. Since $u$ is not in $H_x$ and has no neighbor in $H_x$, it follows that $H_u$ is disjoint from $H_x$.

So $\mathcal{H}_A$ is a laminar hypergraph and $G[A \cup B]$ is isomorphic to its line graph. It follows from Theorem 4.2.3 that $G[A \cup B]$ is a quasi-threshold graph and therefore a chordal graph. □

**Lemma 4.3.11** *Every hole of $G$ is formed by two principal paths of $G$ and a single vertex of $A \cup B \cup A' \cup B'$ that does not belong to these principal paths (it therefore has length $2\ell + 1$).*

*Proof.* By Lemma 4.3.10, a hole $C$ of $G$ cannot contain only vertices of $A \cup B$, and similarly, it cannot contain only vertices of $A' \cup B'$. So it must contain vertices of some principal path, and also of a second principal path. In fact, $C$ must go through exactly two principal paths, since $G[A]$ is isomorphic to the complement of $G[A']$, if three paths are involved, there would be a vertex of $C$ with three neighbors in $C$, a contradiction.

Since $G[A]$ is isomorphic to the complement of $G[A']$, up to a symmetry, for some nonadjacent vertices $u, v \in A$, the hole $C$ is made of a path $P = u \dots v$ with interior in $I \cup A'$ (whose length is $2\ell - 1$) and a path $Q = u \dots v$ of $G[A \cup B]$. By Lemma 4.3.10, $Q$ has length at most 2 (because a quasi-threshold graph is in Free$\{P_4\}$), and since $uv \notin E(G)$, it has length 2. So, $C$ has length $2\ell + 1$ as claimed. □

### 4.3.4 Connecting vertices of a template

**Lemma 4.3.12** *If $x \in B$ and $y \in B'$, then there exists in $G$ two paths $P$ and $Q$ of length $\ell + 1$ from $x$ to $y$ such that $P$ (resp. $Q$) contains a principal path $P_0$ (resp. $Q_0$), and $P_0 \neq Q_0$.*

*Proof.* We set $X = \{i \in \{1, \dots, k\} : v_i \in H_x\}$ and $Y = \{i \in \{1, \dots, k\} : v'_i \in H'_y\}$. So, $X$ and $Y$ are hyperedges of $\mathcal{H}$ and since $\mathcal{H}$ is laminar, either $X \subseteq Y$, $Y \subseteq X$ or $X \cap Y = \emptyset$.

If $X \subseteq Y$, then let $i, j$ be distinct members of $X$ (and therefore of $Y$). The paths $xv_iP_iv'_iy$ and $xv_jP_jv'_jy$ are the paths we are looking for. The proof is similar when $Y \subseteq X$.

If $X \cap Y = \emptyset$, then let $i, j, q, r$ be distinct integers such that $i, j \in X$ and $q, r \in Y$. Since $G[A]$ is isomorphic to the complement of $G[A']$, we may assume up to symmetry that $v_iv_q \in E(G)$. So, $v'_iv'_q \notin E(G)$. Since $H'_y$ is a module of $G[A']$, $v'_iv'_r \notin E(G)$. It follows that $v_iv_r \in E(G)$. So, $v_r, v_q \in N_A(H_x)$. Hence, by property (4) of templates, $xv_r, xv_q \in E(G)$. It follows that $xv_qP_qv'_qy$ and $xv_rP_rv'_ry$ are the two paths we are looking for. □

**Lemma 4.3.13** *If $x \in A \cup B$ and $y \in A' \cup B'$, then there exists in $G$ a path $P$ of length $\ell - 1$, $\ell$ or $\ell + 1$ from $x$ to $y$ that contains a principal path.*

*More specifically:*

- *If $x \in A$ and $y \in A'$, then $P$ has length $\ell - 1$ or $\ell$.*
- *If $x \in A$ and $y \in B'$, or if $x \in B$ and $y \in A'$, then $P$ has length $\ell$ or $\ell + 1$.*
- *If $x \in B$ and $y \in B'$, then $P$ has length $\ell + 1$.*

*Proof.* Suppose first that $x \in A$, say $x = v_i$. If $y \in A'$, then set $y = v'_j$. If $i = j$, then $P_i$ has length $\ell - 1$. If $i \neq j$, then one of $v_iv_jP_jv'_j$ or $v_iP_iv'_iv'_j$ is a path of length $\ell$. If $y \in B'$, then one of $v_iP_iv'_iy$ or $v_iP_iv'_iw'y$ is the path we are looking for. The proof is similar when $y \in A'$.

We may therefore assume that $x \in B$ and $y \in B'$. So one of the two paths obtained in Lemma 4.3.12 can be chosen. □

### 4.3.5 Odd pretemplates

Checking that a graph is an odd $\ell$-template is tedious. We now introduce a simpler notion that is in some sense equivalent. For every integer $\ell \geq 3$, an *odd $\ell$-pretemplate* is a graph $G$ whose vertex-set can be partitioned into five sets $A$, $B$, $A'$, $B'$ and $I$ with the following properties.

1. $N(B) \subseteq A$ and $N(A \cup B) \subseteq I$.
2. $N(B') \subseteq A'$ and $N(A' \cup B') \subseteq I$.
3. $|A| = |A'| = k \geq 3$, $A = \{v_1, \dots, v_k\}$ and $A' = \{v'_1, \dots, v'_k\}$.
4. For every $i \in \{1, \dots, k\}$, there exists a unique path $P_i$ from $v_i$ to $v'_i$ whose interior is in $I$.
5. Every vertex in $I$ has degree 2 and lies on a path from $v_i$ to $v'_i$ for some $i \in \{1, \dots, k\}$.
6. All paths $P_1, \dots, P_k$ have length $\ell - 1$.
7. $G[A \cup B]$ and $G[A' \cup B']$ are both connected graphs.
8. Every vertex of $B$ is in the interior of a path of $G[A \cup B]$ with both ends in $A$.
9. Every vertex of $B'$ is in the interior of a path of $G[A' \cup B']$ with both ends in $A'$.

We then say that $(A, B, A', B', I)$ is an *$\ell$-pretemplate partition* of $G$. Note that templates are defined for all integers $\ell \geq 2$, while pretemplates are defined only when $\ell \geq 3$. In fact we do not need odd 2-templates, we defined them for possible later use.

It is easy to check that when $\ell \geq 3$, the five first elements of every $\ell$-partition of $G$ is an $\ell$-pretemplate partition. The condition on the connectivity of $G[A \cup B]$ and $G[A' \cup B']$ follows Lemma 4.3.4. The condition (8) follows from the fact for every $x \in B$, $H_x$ contains two non-adjacent vertices, so a vertex $x \in B$ lies on a path of length 2 with ends in $A$ and condition (9) holds similarly. Conversely, we prove the following lemma (it is important to note that $\ell \geq 3$).

**Lemma 4.3.14** *Let $\ell \geq 3$ be an integer. If $G \in \mathcal{C}_{2\ell+1}$ is an odd $\ell$-pretemplate, then $G$ is an odd $\ell$-template. Moreover, for every odd $\ell$-pretemplate partition $(A, B, A', B', I)$ of $G$, there exist $w$ and $w'$ in $V(G)$ such that $(A, B, A', B', I, w, w')$ is an $\ell$-partition of $G$.*

*Proof.* Let $(A, B, A', B', I)$ be an $\ell$-pretemplate partition of $G$. We first study the structure of $G[A]$ and $G[A']$.

*(1) For all distinct $i, j \in \{1, \dots, k\}$, $v_iv_j \in E(G)$ if and only if $v'_iv'_j \notin E(G)$. In particular, $G[A]$ is isomorphic to the complement of $G[A']$.*

If $v_iv_j, v'_iv'_j \in E(G)$, then $P_i$ and $P_j$ form a hole of even length, a contradiction. If $v_iv_j, v'_iv'_j \notin E(G)$, then $P_i$, $P_j$, a path from $v_i$ to $v_j$ in $G[A \cup B]$ and a path from $v'_i$ to $v'_j$ in $G[A' \cup B']$ form a hole of length at least $2\ell+2$, a contradiction. This proves (1).

*(2) Every path of $G[A \cup B]$ with both ends in $A$ is of length at most 2.*

Let $P = v_i \dots v_j$ be a path of $G[A \cup B]$ with both ends in $A$. If $P$ has length at least 3, then by (1), paths $P$, $P_i$ and $P_j$ form a hole of length at least $2\ell + 2$, a contradiction. This proves (2).

*(3) $G[A]$ is a threshold graph.*

$G[A]$ is obviously Free$\{C_4\}$. Since the complement of $C_4$ is $2K_2$ and since $G[A']$ is also Free$\{C_4\}$, it follows by (1) that $G[A]$ is Free$\{2K_2\}$. By (2), $G[A]$ is Free$\{P_4\}$. So $G[A]$ is in Free$\{P_4, C_4, 2K_2\}$ and is therefore a threshold graph. This proves (3).

We now study the structure of $G[B]$ and its relation with $G[A]$.

*(4) For every vertex $x \in B$, $G[N_A(x)]$ has a unique anticonnected component of size at least 2.*

By the definition of odd pretemplates, $x$ is in the interior of a path $P = v_i \dots v_j$ of $G[A \cup B]$ with both ends in $A$. By (2), $P$ has length 2, so $x$ is adjacent to $v_i$ and $v_j$. Hence $G[N_A(x)]$ has an anticonnected component of size at least 2. It is unique, for otherwise $G[A]$ contains a $C_4$. This proves (4).

For all $x \in B$, we define $H_x$ to be the anticonnected component of $G[N_A(x)]$

of size at least 2 whose existence follows from (4).

*(5) For every $x$ in $B$, $H_x$ is a module of $G[A]$.*

Otherwise, since $H_x$ is anticonnected and is not a module, there exists $v_h \in A \setminus H_x$ and non-adjacent $v_i, v_j \in H_x$ such that $v_iv_h \in E(G)$ and $v_jv_h \notin E(G)$. Note that $xv_h \notin E(G)$ because otherwise, $v_h$ would be in $H_x$. Hence, $v_i$, $x$, $P_j$ and $P_h$ form a hole of length $2\ell + 2$, a contradiction. This proves (5).

*(6) If $xy$ is an edge of $G[B]$, then $H_x \subseteq H_y$ or $H_y \subseteq H_x$.*

Up to symmetry, we may assume that $N_A(x) \subseteq N_A(y)$, for otherwise vertices $v_i \in N_A(x) \setminus N_A(y)$ and $v_j \in N_A(y) \setminus N_A(x)$ either form a $C_4$ with $x$ and $y$ or a hole of length $2\ell + 2$ with $P_i$ and $P_j$.

By (4), $G[N_A(y)]$ has only one anticonnected component of size at least 2, namely $H_y$. Since $H_x$ is anticonnected, has size at least 2 and is included in $N_A(y)$, it must be included in $H_y$. This proves (6).

*(7) If $x$ and $y$ are non-adjacent vertices of $B$, then $H_x$ and $H_y$ are disjoint.*

On the contrary, suppose that $x$ and $y$ are nonadjacent vertices of $B$ but there exists a vertex $v \in H_x \cap H_y$. Since $H_x$ is anticonnected and of size at least 2, there exists $v_i \in H_x$ non-adjacent to $v$. Note that $v_iy \notin E(G)$, for otherwise $x$, $y$, $v_i$ and $v$ form a $C_4$. Similarly, there exists a vertex $v_j \in H_y$ that is non-adjacent to $v$ and to $x$. If $v_iv_j \in E(G)$, then $\{x, y, v, v_i, v_j\}$ induces a $C_5$, a contradiction. Otherwise, $P_i$, $P_j$, $x$, $y$ and $v$ form a hole of length $2\ell + 3$, a contradiction. This proves (7).

We are now ready to define the hypergraph $\mathcal{H}$. For every $x \in B$, we defined a set $H_x \subseteq A$. We may define similarly a set $H'_x \subseteq A'$ for every $x \in B'$. From (6) and (7), the sets $H_x$ for $x \in B$ form a laminar hypergraph $\mathcal{H}_B$ (with vertex set $A$). Symetrically, the sets $H'_x$ for $x \in B'$ form a laminar hypergraph $\mathcal{H}_{B'}$ (with vertex set $A'$). Let $\mathcal{H}$ be the hypergraph whose vertex set is $\{1, \dots, k\}$ and such that $H \subseteq \{1, \dots, k\}$ is a hyperedge of $\mathcal{H}$ if and only if $H = \{i : v_i \in H_x\}$ for some $x \in B$ or $H = \{i : v'_i \in H'_x\}$ for some $x \in B'$.

*(8) The hypergraph $\mathcal{H}$ is laminar.*

If $\mathcal{H}$ is not laminar, then there exist $X, Y \in E(\mathcal{H})$ such that $X \setminus Y$, $Y \setminus X$ and $X \cap Y$ are all non-empty. Since $\mathcal{H}_B$ and $\mathcal{H}_{B'}$ are both laminar, there exists

$x \in B$ such that $H_x = \{v_i : i \in X\}$ and $y \in B'$ such that $H'_y = \{v'_i : i \in Y\}$.

We set $H_y = \{v_i : i \in Y\}$. Note that $H_x \setminus H_y$, $H_y \setminus H_x$ and $H_x \cap H_y$ are all non-empty. Also, because of the properties of $H'_y$ and by (1), $G[H_y]$ is connected (because $G[H'_y]$ is anticonnected) and $H_y$ is a module of $G[A]$.

Since $G[H_x]$ is anticonnected, there exist non-adjacent vertices $u \in H_x \setminus H_y$ and $v \in H_x \cap H_y$. Since $G[H_y]$ is connected, there exists a path from $v$ to $t \in H_y \setminus H_x$ and we may assume that $vt$ is an edge. Since $H_y$ is a module of $G[A]$, $ut \notin E(G)$. So, $t$ is adjacent to $v$ and non-adjacent to $u$. This contradicts $H_x$ being a module of $G[A]$. This proves (8).

We may now finish the proof of Lemma 4.3.14. We show how $G$ can be built by the process described in the definition of odd templates. We start by setting $V(J) = \{1, \dots, k\}$, and by making $i$ adjacent to $j$ in $J$ if and only if $v_iv_j \in E(G)$. By (3), $J$ is a threshold graph as required. Clearly condition (4) of odd $\ell$-templates holds, the paths linking $A$ to $A'$ are as in condition (3) of odd $\ell$-templates and condition (5g) of odd $\ell$-templates holds. By (1), conditions (5a) and (5b) of odd $\ell$-templates hold. We then consider the hypergraph $\mathcal{H}$ defined above. It is laminar by (8). By (5), condition (2a) of templates is satisfied.

By definition of $H_x$, for every $x$ in $B$, $N_A(x) \subseteq N_A[H_x]$. Suppose that there exists $u \in N_A[H_x] \setminus N_A(x)$. Since by (5) $H_x$ is a module, it follows from Lemma 4.3.2 that $u$ is complete to $H_x$, so $x$ and $u$ together with two non-adjacent vertices from $H_x$ induce a $C_4$, a contradiction. Hence, $N_A(x) = N_A[H_x]$ and condition (5e) of odd templates is satisfied.

By (6) and (7), condition (5c) of templates is satisfied.

By symmetry and by (1), conditions (5d) and (5f) of templates are satisfied. Therefore condition (5) of templates is satisfied.

To conclude the proof, let us check condition (2b) of templates. By (1), (3) and Theorem 4.2.4, up to symmetry, we may assume that $G[A]$ contains an isolated vertex $v_i$. Since $G[A \cup B]$ is connected and $|A| \geq 3$ by the definition of odd pretemplates, there exists a path $P$ in $G[A \cup B]$ from $v_i$ to a vertex $u \in A \setminus \{v_i\}$. By (2) and since $v_i$ has no neighbor in $A$, we have that $P = uyv_i$ where $y \in B$. So, $H_y$ contains $v_i$. We may therefore consider the hyperedge $W$ of $\mathcal{H}$ that contains $i$ and that is inclusion wise maximal w.r.t. this property. If there exists $j \in \{1, \dots, k\} \setminus W$, since $v_jv_i \notin E(G)$, we deduce as above that $\mathcal{H}$ has a hyperedge $Z$ that contains $i$ and $j$. Because of $j$, $Z \subseteq W$ is impossible;

because of $i$, $W \cap Z = \emptyset$ is impossible; and because of the maximality of $W$, $W \subsetneq Z$ is impossible. Hence, $W$ and $Z$ contradict $\mathcal{H}$ being laminar. This proves that $W = \{1, \dots, k\}$, as claimed in condition (2b) of templates.

Hence, $G[A \cup B]$ has universal vertex $w$. Also, $G[A' \cup B']$ has a universal vertex $w'$ (we may apply Lemma 4.3.4 since we now know that $G$ is an odd $\ell$-template). So, $(A, B, A', B', I, w, w')$ is an $\ell$-partition of $G$. □

### 4.3.6 Twins and proper partitions

**Lemma 4.3.15** *Let $(A, B, A', B', I, w, w')$ be an $\ell$-partition of an odd $\ell$-template $G$. Two vertices $x$ and $y$ of $G$ are twins if and only if $x, y \in B$ and $H_x = H_y$, or $x, y \in B'$ and $H'_x = H'_y$.*

*Proof.* If $x, y \in B$ and $H_x = H_y$, or $x, y \in B'$ and $H'_x = H'_y$, then $x$ and $y$ are obviously twins.

We claim that for all $x \in A \cup I \cup A'$, there exist two vertices $a, b \in N_G(x)$ such that $N[a] \cap N[b] = \{x\}$. If $x \in I$ choose $a$ and $b$ to be the only two neighbors of $x$. If $x \in A$, then set $a = w$ when $x \neq w$, and choose for $a$ any vertex of $A \setminus \{x\}$ when $x = w$. Choose for $b$ the neighbor of $x$ in $I$. In both cases, by condition (e7) of templates, $N_G[a] \cap N_G[b] = \{x\}$. The proof is similar when $x \in A'$. So, $x$ has no twin in $G$. □

An $\ell$-partition $(A, B, A', B', I, w, w')$ of an odd $\ell$-template $G$ is *proper* if one of $G[A]$ or $G[A']$ contains at least two isolated vertices.

**Lemma 4.3.16** *For all integers $\ell \geq 3$, every twinless odd $\ell$-template $G$ admits a proper $\ell$-partition.*

*Proof.* Let $(A, B, A', B', I, w, w')$ be an $\ell$-partition of $G$ such that the number $M$ of isolated vertices of $G[A]$ is maximum. We suppose that $v_1, \dots, v_k$ is a domination ordering of $G[A]$.

By Theorem 4.2.4 and since we may swap $A, B, w$ and $A', B', w'$, by the maximality of $M$, $v_1$ is an isolated vertex of $G[A]$. It follows that $w \in B$. By definition of templates, $v'_1$ is a universal vertex of $G[A' \cup B']$. Suppose for a contradiction that $v_2$ is not isolated in $G[A]$. So, $M = 1$.

Let $H_x$ be any hyperedge of $\mathcal{H}_B$ containing $v_1$. By Lemma 4.3.3, since $H_x$ contains a non-isolated vertex of $G[A]$, it contains all of them. So, $H_x = A$. Hence, $N[x] = N[w]$, so $x = w$ since $G$ is twinless. This proves that $N(v_1) = \{w, v_1^+\}$ where $v_1^+$ is the neighbor of $v_1$ in $I$. Let $v_1'^+$ be the neighbor of $v_1'$ in $I$. We now describe a new partition of the vertices of $G$. We set:

- $A_1 = \{w, v_2, \dots, v_k\}$,
- $B_1 = B \setminus \{w\}$,
- $A_1' = \{v_1'^+, v_2', \dots, v_k'\}$,
- $B_1' = B' \cup \{v_1'\}$ and
- $I_1 = \{v_1\} \cup I \setminus \{v_1'^+\}$.

All conditions of the definition of a pretemplate are easily checked to be satisfied by $(A_1, B_1, A_1', B_1', I_1)$. By Lemma 4.3.11, every hole in $G$ has length $2\ell + 1$. We may therefore apply Lemma 4.3.14 to prove that $(A_1, B_1, A_1', B_1', I_1, w, v_1')$ is an $\ell$-partition of $G$. So, $(A_1', B_1', A_1, B_1, I_1, v_1', w)$ contradicts that maximality of $M$ since $G[A_1']$ has two isolated vertices, namely $v_1'^+$ and $v_k'$ (note that since $M = 1$, it follows by Theorem 4.2.4 that $G[A \setminus \{v_1\}]$ has a universal vertex ; in particular $v_k$ is a universal vertex of $G[A \setminus \{v_1\}]$ and hence $v_k'$ is an isolated vertex of $G[A_1']$). □

**Lemma 4.3.17** *Every proper $\ell$-partition $(A, B, A', B', I, w, w')$ of a twinless odd $\ell$-template satisfies one of the following:*

- *$w \in B$, $w$ is the unique universal vertex of $G[A \cup B]$, $G[A]$ contains at least two isolated vertices, $H_w = A$, $w' \in A'$ and $A' \setminus \{w'\}$ contains at least one universal vertex of $G[A']$.*
- *$w \in A$, $A \setminus \{w\}$ contains at least one universal vertex of $G[A]$, $w' \in B'$, $w'$ is the unique universal vertex of $G[A' \cup B']$, $G[A']$ contains at least two isolated vertices and $H'_{w'} = A'$.*

*Proof.* Since $(A, B, A', B', I, w, w')$ is a proper $\ell$-partition, up to symmetry, we may assume that $G[A]$ contains two isolated vertices. So, $w \in B$. By definition of $\ell$-partitions, it follows that $w' \in A'$. Since $G$ is twinless and $G[A]$ contains isolated vertices, $w$ is the unique universal vertex of $G[A \cup B]$. By Lemma 4.3.6, $H_w = A$. Also $A' \setminus \{w'\}$ contains at least one universal vertex of $G[A']$ since $G[A]$ contains two isolated vertices. □

We do not use the following lemma formally, but it illustrates a key property of proper partitions. In non-proper partitions, there may exist vertices in $A$ that have degree 2 and have one neighbor in $A$ and one in $I$. These are hard to think of, because they yield edges with both ends in $A$ that can be "blown up" into a general half graph as we will see in the next section. The next lemma states that this situation does not occur with proper partitions.

**Lemma 4.3.18** *Suppose $\ell \geq 3$ and $G$ is an odd $\ell$-template with a proper $\ell$-partition $(A, B, A', B', I, w, w')$. If a vertex $v$ in $A \cup B \cup A' \cup B'$ has degree 2 (in $G$), then $v \in A \cup A'$ and $v$ is adjacent to a vertex of $B \cup B'$ and has its other neighbor in $I$.*

*Proof.* Up to symmetry, suppose that $G[A]$ contains at least two isolated vertices. So, $w \in B$. Consider a vertex $v \in A \cup B \cup A' \cup B'$ that is of degree 2 in $G$. By Lemma 4.3.8, $v \notin B \cup B'$. Since $G[A']$ has two universal vertices, every vertex in $A'$ has degree at least 3, so $v \in A$, and $v$ is adjacent to $w \in B$ and to some vertex in $I$ as claimed. □

## 4.4 Blowup

Our goal in this section is to see how a bigger graph can be obtained from a template $G$ by turning every vertex into a non empty clique. This will be called *blowing up* $G$. In the blowup operation, non-adjacent vertices yield cliques that are anticomplete to each other. Adjacent vertices $u$ and $v$ yield cliques that are complete to each other in some situations (when $uv$ is a so-called *solid edge* of the template), but in some other situations, they may yield pairs of cliques that induce a more general half graph, like when a ring is obtained from "blowing up" a chordless cycle. This happens when $uv$ is a so-called *flat* or *optional edge* of the template. We now define all this formally.

Throughout all this section, $\ell \geq 3$ is an integer, $G$ is a an odd $\ell$-template with a fixed an $\ell$-partition $(A, B, A', B', I, w, w')$.

### 4.4.1 Flat, optional and solid edges

An edge of $G$ is *flat* if at least one of its end is in $I$. An edge of $G$ is *optional* if one end is a vertex $x \in B$ (resp. $x \in B'$) and the other end is a vertex $u \in H_x$ that is an isolated vertex of $G[H_x]$ (resp. a vertex $u \in H'_x$ that is an isolated vertex of $G[H'_x]$). An edge that is neither flat nor optional is *solid*. See Figure 4.5 where solid edges are represented by solid lines, flat edges by doted lines and optional edges by dashed lines.

Observe that the status of an edge depends on the $\ell$-partition of the odd $\ell$-template. See Figure 4.4, where the same template is represented with two different $\ell$-partitions. Recall that throughout this section, the $\ell$-partition is fixed, and so is the status of the edges.

**Lemma 4.4.1** *If $ux$ is an optional edge of $G$ with $u \in A$ and $x \in B$, then $N_A(H_x) = N_A(u)$. Moreover, if $y \in B \setminus \{x\}$ and $yu \in E(G)$, then $H_x \subseteq H_y$ or $H_y \subseteq H_x$ (in particular, $xy \in E(G)$).*

*Proof.* Since $H_x$ is a module of $G[A]$ and $u$ is isolated in $H_x$, we have $N_A(H_x) = N_A(u)$. If the second conclusion fails, then since $\mathcal{H}_B$ is laminar, $H_y \cap H_x = \emptyset$. So $xy \notin E(G)$. Since $yu \in E(G)$ and $u \notin H_y$, we have $u \in N_A(H_y)$, so $u$ is complete to $H_y$ since $H_y$ is a module. So $H_y \subseteq N_A(u) = N_A(H_x)$, which is a clique by Lemma 4.3.2. This contradicts $H_y$ being anticonnected. □

A clique of $G$ is *solid* if all its edges are solid.

**Lemma 4.4.2** *If $ux$ is an optional edge of $G$ such that $u \in A$ and $x \in B$, then $N_{A \cup B}(u)$ is a solid clique of $G$.*

*Proof.* By Lemma 4.4.1, $N_A(H_x) = N_A(u)$. By Lemma 4.3.2, $N_A(H_x) = N_A(u)$ is a clique. It is solid because edges with both ends in $A$ are solid. Hence $N_A(u)$ is a solid clique.

By Lemma 4.4.1 all vertices from $N_B(u)$ are adjacent since they correspond to hyperedges of $\mathcal{H}_B$ that are included in each other. Therefore, $N_B(u)$ is a clique and it is solid because edges with both ends in $B$ are solid. Hence $N_B(u)$ is a solid clique.

It remains to prove that $N_A(u)$ is complete to $N_B(u)$ and that all edges between these two sets are solid. So let $y \in B$ and $v \in A$ be two neighbors of $u$. Note that $v \notin H_x$ and possibly $y = x$. If $u \in H_y$, then $vy$ is an edge because $v \in N_A(u)$ (and so $v \in N_A[H_y]$), and it is a solid edge because $v$ is not an isolated vertex of $H_y$. If $u \notin H_y$, then by Lemma 4.4.1, $H_y \subseteq H_x$. So, $u \in N_A(H_y)$ since $uy \in E(G)$, and this contradicts $u$ being isolated in $H_x$. $\square$

**Lemma 4.4.3** *Let $C$ be a cycle of $G$ of length at least 4 with no solid chord. If $C$ is not a hole then there exist three consecutive vertices $x$, $y$, $u$ in $C$ such that:*

*- $u \in A$, $x, y \in B$, $\{u\} \subseteq H_y \subseteq H_x$ and $u$ is an isolated vertex of $H_x$, or*

*- $u \in A'$, $x, y \in B'$, $\{u\} \subseteq H'_y \subseteq H'_x$ and $u$ is an isolated vertex of $H'_x$. In particular $ux$ is an optional edge of $G$ and a chord of $C$.*

*Proof.*

We may assume that $C$ has a chord $e$ for otherwise it is a hole. This chord cannot be a flat edge of $G$ because a flat edge contains a vertex of $I$, so a vertex of degree 2, and it therefore cannot be a chord of any cycle. Hence, $e$ is an optional edge of $G$. So, up to symmetry, we may assume that $e = ux$ with $u \in A$ and $x \in B$. By definition of optional edges, $u$ is an isolated vertex of $G[H_x]$.

Let $u'$ and $y$ be the two neighbors of $u$ along $C$. If $u', y \in A \cup B$, then by Lemma 4.4.2, $u'y$ is a solid chord of $C$, a contradiction. So, up to symmetry, $y \in A \cup B$ and $u' \in I$.

Suppose first that $y \in A$. Since $uy \in E(G)$ and $u$ is isolated in $H_x$, we have that $y \in N_A(H_x)$. If follows that $xy \in E(G)$, and moreover, $xy$ is a solid edge since $y \notin H_x$. Since $x$ and $y$ are both in $C$ and $C$ has no solid chord, $C$ visits consecutively $u'$, $u$, $y$ and $x$. Let $x'$ be the neighbor of $x$ in $C \setminus y$. If $x' \in B$ then $H_x \cap H_{x'} \neq \emptyset$, and since $y$ is complete to $H_x$, $y$ has a neighbor in $H_{x'}$. It follows that $yx'$ is an edge of $G$, the edge $yx'$ is solid, and is therefore a solid chord of $C$, a contradiction. Hence $x' \in A$, and so since $xx'$ is an edge,

$x' \in N_A[H_x]$. If $x' \in H_x$, then $yx'$ is a solid chord of $C$, and if $x' \in N(H_x)$, then (since $H_x$ is a module of $G[A]$) $x'u$ is a solid chord of $C$, in each case a contradiction.

Suppose now that $y \in B$. If $H_y \subseteq H_x$, then $xy$ is an edge that is solid and hence is an edge of $C$, so the conclusion of the lemma holds. So we may assume by Lemma 4.4.1 that $H_x \subseteq H_y$. In particular, $xy$ is an edge, and since it is solid, $u$, $y$ and $x$ are consecutive along $C$. Let $v$ be the neighbor of $x$ in $C \setminus y$. If $v \in B$, then $H_v \cap H_x \neq \emptyset$, so $H_v \cap H_y \neq \emptyset$, showing that $yv$ is a solid chord of $G$, a contradiction. Hence, $v \in A$. We have $uv \notin E(G)$ for otherwise $uv$ would be a solid chord of $G$. Hence, $v \in H_x$ since $vx \in E(G)$ and $H_x$ is a module of $G[A]$. So, $v \in H_y$ (and hence $vy \in E(G)$) and $v$ is an isolated vertex of $H_y$, for otherwise $vy$ would be a solid chord of $G$.

Now, we have three consecutive vertices $y$, $x$, $v$ in $C$ such that: $v \in A$, $y, x \in B$, $\{v\} \subseteq H_x \subseteq H_y$ and $v$ is an isolated vertex of $H_y$. So, the conclusion of the lemma is satisfied again with these three vertices. □

### 4.4.2 Blowups and holes

Let $G$ be a twinless odd $\ell$-template with an $\ell$-partition $(A, B, A', B', I, w, w')$ A *blowup* of $G$ is any graph $G^*$ that satisfies the following:

1. For every vertex $u$ of $G$ there is a clique $K_u$ in $G^*$ on $k_u \geq 1$ vertices $u_1, \dots, u_{k_u}$ such that $u_{k_u} = u$ ; for distinct vertices $u, v$ of $G$, $K_u \cap K_v = \emptyset$ and $V(G^*) = \cup_{u \in V(G)} K_u$, so $V(G) \subseteq V(G^*)$.

2. For all vertices $u \in V(G)$ and all integers $1 \leq i \leq j \leq k_u$, in $G^*$ $N[u_i] \subseteq N[u_j]$ (in particular, for all $u, v \in V(G)$, $G^*[K_u \cup K_v]$ is a half graph).

3. If $u$ and $v$ are non-adjacent vertices of $G$, then $K_u$ is anticomplete to $K_v$ (in particular $uv \notin E(G^*)$).

4. If $uv$ is a solid edge of $G$, then $K_u$ is complete to $K_v$ (in particular $uv \in E(G^*)$).

5. If $uv$ is a flat edge of $G$, then $u$ is complete to $K_v$ and $v$ is complete to $K_u$ (in particular $uv \in E(G^*)$).

6. If $ux$ is an optional edge of $G$ with $u \in A$ and $x \in B$ (resp. $u \in A'$ and $x \in B'$), then $u$ is complete to $K_x$ (in particular $uv \in E(G^*)$).

7. If $ux$ and $uy$ are optional edges of $G$ with $u \in A$, $x, y \in B$ and $H_y \subsetneq H_x$ (resp. $u \in A'$, $x, y \in B'$ and $H'_y \subsetneq H'_x$), then every vertex of $K_u$ with a neighbor in $K_y$ is complete to $K_x$.

8. $w$ (resp. $w'$) is a universal vertex of $G^*[\cup_{u \in A \cup B} K_u]$ (resp. $G^*[\cup_{u \in A' \cup B'} K_u]$).

Observe that $G = G^*[V(G)]$ follows clearly from the definition, so $G$ is an induced subgraph of $G^*$. For every vertex $u$ of $G$, the clique $K_u$ is called a *blown up clique*, more specifically the *clique blown up from $u$*.

Note that to define the blowup of a graph, it is first needed to fix an $\ell$-partition of it. Also, it should be stressed that the blowup is defined only for twinless graphs. Hence, in condition (7) of the definition, since $G$ is twinless, when $x \neq y$, $H_y \subsetneq H_x$ is equivalent to $H_y \subseteq H_x$ because $H_x = H_y$ would imply that $x$ and $y$ are twins.

**Lemma 4.4.4** *A hole $C$ in a blowup of a twinless odd $\ell$-template contains at most one vertex in each blown up clique.*

*Proof.* Since a hole is triangle-free, $C$ intersects any clique in at most two vertices. So suppose for a contradiction that some blown up clique $K_v$ contains two vertices $x$ and $y$ of $C$. Let $x'$ be the neighbor of $x$ in $C \setminus y$ and $y'$ be the neighbor of $y$ in $C \setminus x$. Since by condition (2) of the definition of the blowup we have that in $G^*$, $N[x] \subseteq N[y]$ or $N[y] \subseteq N[x]$, one of $xyx'$ or $xyy'$ is a triangle of $C$, a contradiction. $\square$

**Lemma 4.4.5** *In a blowup $G^*$ of a twinless odd $\ell$-template $G$, every hole has length $2\ell + 1$.*

*Proof.* Let $C^*$ be a hole in $G^*$. By Lemma 4.4.4, it contains at most one vertex in each blown up clique. Let $C$ be the subgraph of $G$ that is induced by all vertices $v$ such that some vertex of $C^*$ is in $K_v$. By Lemma 4.4.4, $|V(C^*)| = |V(C)|$. By the definition of blowup (specifically conditions (3) and (4)), $C^*$ is isomorphic to some graph obtained from $C$ by removing optional or flat edges of $G$. Hence, $C$ is a cycle of $G$ with no solid chord. If $C$ is a hole of $G$, then since it has the same length as $C^*$, by Lemma 4.3.11, $C^*$ has length $2\ell + 1$. Hence, we may assume that $C$ has chords, so by Lemma 4.4.3, without loss of generality, $C$ contains three consecutive vertices $x$, $y$, $u$ such that: $u \in A$, $x, y \in B$, $\{u\} \subsetneq H_y \subsetneq H_x$ and $u$ is an isolated vertex of $H_x$. Note that it follows that both $ux$ and $uy$ are optional edges of $G$. Because of $C^*$, the vertex $u_i$ of $K_u \cap V(C^*)$ has a neighbor in $K_y$. So, by condition (7) of blowups, $u_i$ is complete to $K_x$. Hence, $C^*$ has a chord, a contradiction. □

### 4.4.3 Preblowup

Checking that a graph is the blowup of a template is tedious. Here we provide a simpler notion and prove it is in some sense equivalent.

A *preblowup* of an odd $\ell$-template $G$ with an $\ell$-partition $(A, B, A', B', I, w, w')$ is any graph $G^*$ obtained from $G$ as follows. Every vertex $u$ of $A \cup A' \cup I$ is replaced by a clique $K_u$ on $k_u \geq 1$ vertices such that $u \in K_u$. We denote by $A^*$ the set $\cup_{u \in A} K_u$ and use a similar notation $A'^*$ and $I^*$. The set $B$ (resp. $B'$) is replaced by a set $B^*$ (resp. $B'^*$) of vertices such that $B \subseteq B^*$ (resp. $B' \subseteq B'^*$). So, $V(G^*) = A^* \cup B^* \cup A'^* \cup B'^* \cup I^*$. The sets $A^*$, $B^*$, $A'^*$, $B'^*$, $I^*$ are disjoint. Vertices of $G$ are adjacent in $G^*$ if and only if they are adjacent in $G$, so $G$ is an induced subgraph of $G^*$. Finally, we require that the following conditions hold (throughout $N$ refers to the neighborhood in $G^*$):

1. For all $u \in A$, $N(K_u) \subseteq A^* \cup B^* \cup K_{u^+}$ where $u^+$ is the neighbor of $u$ in $I$ and:
   (a) For every $u^* \in K_u$, $N_A(u^*) = N_A[u]$.
   (b) Every vertex of $K_u$ has a neighbor in $K_{u^+}$.
2. $N(B^*) \subseteq A^*$ and:
   (a) If $w \in B$, then there exists $w^* \in B^*$ that is complete to $A^*$.

   (b) If $u^* \in B^*$, then there exist non-adjacent $a, b \in A$ such that $u^*$ has neighbors in both $K_a$ and $K_b$.

9. For all $u \in I$, $N(K_u) \subseteq K_a \cup K_b$ where $a$ and $b$ are the neighbors of $u$ in $G$, and:

   (a) Every vertex $u^* \in K_u$ has at least one neighbor in each of $K_a$ and $K_b$.

   Conditions analogous to (1) and (2) hold for $A'$ and $B'$.

Recall that to blowup (resp. preblowup) a template, one needs to first fix an $\ell$-partition. If this partition is proper, the blowup (resp. preblowup) is *proper*. Recall that by Lemma 4.3.16, a proper $\ell$-partition $(A, B, A', B', I, w, w')$ exists for every twinless odd $\ell$-template $G$ (but this remark will be used only in the next section, so far we just assume the $\ell$-partition we work with is proper).

When $G^*$ is a preblowup of a template $G$, the *domination score* of $G$ w.r.t. $G^*$ is (where $N$ refers to the neighborhood in $G^*$):

$$s(G, G^*) = \sum_{x \in A \cup A' \cup I} |\{x^* \in K_x : N[x^*] \subseteq N[x]\}|$$

Observe that the blowup is defined only for twinless templates while the preblowup is defined for any template. It is straightforward to check that a blowup is a particular preblowup. The following is a converse of this statement.

**Lemma 4.4.6** *Let $\ell \geq 3$ and let $G^*$ be a proper preblowup of an odd $\ell$-template with $k \geq 3$ principal paths. If $G^* \in \mathcal{C}_{2\ell+1}$, then $G^*$ is a proper blowup of a twinless odd $\ell$-template $G$ with $k$ principal paths (in particular, $G$ is an induced subgraph of $G^*$).*

*Proof.* Among all the induced subgraphs of $G^*$ that are odd $\ell$-templates and for which $G^*$ is a proper preblowup, we suppose that $G$ is one that maximizes $s(G, G^*)$. We denote by $(A, B, A', B', I, w, w')$ the proper $\ell$-partition of $G$ that is used for its preblowup and by $(A^*, B^*, A'^*, B'^*, I^*)$ the corresponding partition of the vertices of $G^*$.

*(1) There exist vertices $w^*$ and $w'^*$ that are complete to respectively $A^* \setminus \{w^*\}$ and $A'^* \setminus \{w'^*\}$, and such that either $w^* \in B^*$ and $w'^* \in A'^*$, or $w^* \in A^*$ and $w'^* \in B'^*$.*

If $w \in A$, then from the definition of $w$ (see Lemma 4.3.4), the definition of $A^*$ and condition (1a), it follows that $w^* = w$ is complete to $A^* \setminus \{w^*\}$. If $w \in B$, by condition (2a) there exists $w^* \in B^*$ that is complete to $A^*$.

The statement about $w'^*$ holds by symmetry. The last statement comes from the fact that by Lemma 4.3.4 exactly one of $w, w'$ is in $A \cup A'$, and the other one is in $B \cup B'$. This proves (1).

*(2) For every principal path $P_u = u \dots u'$ of $G$ and $u^* \in K_u$, there exists in $G^*$ a path $P_{u^*}$ of length $\ell - 1$ from $u^*$ to some $u'^* \in K_{u'}$ whose interior is in $\cup_{x \in I \cap V(P_u)} K_x$. Moreover, the interior of $P_{u^*}$ is anticomplete to $V(G^*) \setminus \cup_{v \in V(P_u)} K_v$.*

The existence of a path from $u^*$ to some $u'^* \in K_{u'}$ whose interior is in $\cup_{x \in I \cap V(P)} K_x$ follows from conditions (1), (9), (9a), and (1b) of preblowups. Its length is $\ell - 1$ by condition (3) of templates. The statement about its interior follows from conditions (1), (2) and (9) of preblowups. This proves (2).

*(3) For all $u, v \in A$ such that $uv \notin E(G)$, $K_u$ is anticomplete to $K_v$. A similar statement holds for $A'$.*

Suppose that there exists $u^* \in K_u$ and $v^* \in K_v$ such that $u^*v^* \in E(G^*)$. By condition (1a) of preblowups, $u \neq u^*$ and $v \neq v^*$. Let $P_u = u \dots u'$ and $P_v = v \dots v'$ be principals paths. Denote by $u^+$ the neighbor of $u$ in $P_u$ and by $v^+$ the neighbor of $v$ in $P_v$. By property (1) of a template, $u'v' \in E(G)$. Hence $uP_uu'v'P_vvv^*u^*u$ is a cycle $C$. By conditions (1) and (1a) of preblowup, the only possible chords in $C$ are $u^+u^*$ and $v^+v^*$. Without loss of generality, we may assume that $u^*u^+ \in E(G^*)$ for otherwise $C$ is a hole of length $2\ell + 2$, a contradiction.

Let $P_{v^*}$ be a path of length $\ell - 1$ from $v^*$ to $v'^*$ as defined in (2). Since $v'^* \in K_{v'}$ and by (1a) applied to $A'$, $v'^*u' \in E(G^*)$ and $v^*P_{v^*}v'^*u'P_uu^+u^*v^*$ is a hole of length $2\ell$, a contradiction.

The result for $A'$ holds symmetrically. This proves (3).

*(4) For all $u, v \in A$ such that $uv \in E(G)$, $K_u$ is complete to $K_v$. A similar statement holds for $A'$.*

Suppose that there exists $u^* \in K_u$ and $v^* \in K_v$ such that $u^*v^* \notin E(G^*)$. Let $P_{u^*} = u^* \dots u'^*$ and $P_{v^*} = v^* \dots v'^*$ be defined as in (2). Observe that $u'^* \in K_{u'}$

and $v'^* \in K_{v'}$. Furthermore $u'v' \notin E(G)$ by property (1) of templates. Hence, by (3), $u'^*v'^* \notin E(G^*)$.

We claim that there exists a vertex $a \in (A \cup B) \setminus \{u, v\}$ that is adjacent to both $u^*$ and $v^*$. If $w^* \neq u, v$, then by (1) and condition (1b), we may choose $a = w^*$. Otherwise, up to symmetry, $w^* = u$. Since the $\ell$-partition of $G$ is proper, by Lemma 4.3.17, $A$ contains a universal vertex $x$ distinct from $w^* = u$. If $x \neq v$, we set $a = x$. If $x = v$, then both $u$ and $v$ are universal vertices of $G[A]$ and we may choose for $a$ any vertex of $A \setminus \{u, v\}$. This proves our claim.

Now, $au^*P_{u^*}u'^*w'^*v'^*P_{v^*}v^*a$ is a hole of length $2\ell + 2$, a contradiction. The result for $A'$ holds symmetrically. This proves (4).

*(5) For all $u \in I$ and $u_1, u_2 \in K_u$, either $N[u_1] \subseteq N[u_2]$ or $N[u_2] \subseteq N[u_1]$.*

Otherwise, there exists $x_1^* \in N[u_1] \setminus N[u_2]$ and $x_2^* \in N[u_2] \setminus N[u_1]$. Note that $x_1^*x_2^* \notin E(G^*)$ for otherwise, $\{x_1^*, x_2^*, u_1, u_2\}$ induces a $C_4$. It follows that $x_1^*$ and $x_2^*$ belong respectively to distinct cliques $K_{x_1}$ and $K_{x_2}$, where $x_1$ and $x_2$ are the two neighbors of $u$ along some principal path $P = v \dots v'$ of $G$. Because of $x_1^*$, $x_2^*$ and condition (9a) of preblowups, there exists a path $P^*$ of length $\ell$ from some $v^* \in K_v$ to some $v'^* \in K_{v'}$ whose interior is in $\cup_{x \in I \cap V(P)} K_x$.

Let $q \neq v$ be a vertex of $A$ and $Q = q \dots q'$ be a principal path of $G$, and suppose up to symmetry that $qv \notin E(G)$. Now, by conditions (9) and (1a) of preblowups and (1), $P^*$, $Q$ and $w^*$ form a hole of length $2\ell + 2$. This proves (5).

*(6) For all $u \in I$ and $u^* \in K_u$, $N[u^*] \subseteq N[u]$.*

Otherwise, by (5), there exists a vertex $u^* \in K_u$ such that $N[u] \subsetneq N[u^*]$. Hence $(V(G \setminus \{u\}) \cup \{u^*\}$ induces a subgraph $G_0$ of $G^*$ and it is easy to verify that $G^*$ is a preblowup of $G_0$. This contradicts to the maximality of $s(G, G^*)$. This proves (6).

By (5), for every $u \in I$, the clique $K_u$ can be linearly ordered by the inclusion of the neighborhoods as $u_1, \dots, u_{k_u}$ with $u = u_{k_u}$ by (6) (so, for $1 \leq 1 \leq j \leq k_u$, $N[u_i] \subseteq N[u_j]$). From condition (9) of the preblowup it also follows that, in $G^*$, $u$ is complete to the cliques associated to its two neighbors in $G$.

*(7) For every $u \in A$ and $u_1, u_2 \in K_u$, either $N[u_1] \subseteq N[u_2]$ or $N[u_2] \subseteq N[u_1]$.*

*A similar statement holds for $A'$.*

Otherwise, there exist $x_1 \in N[u_1] \setminus N[u_2]$ and $x_2 \in N[u_2] \setminus N[u_1]$. Note that $x_1x_2 \notin E(G^*)$ for otherwise, $\{x_1, x_2, u_1, u_2\}$ induces a $C_4$.

Observe first that by (3) and (4), $N_{A^*}[u_1] = N_{A^*}[u_2]$. Hence by (1) of preblowup, $x_1, x_2 \in B^* \cup K_{u^+}$ where $u^+$ is the neighbor of $u$ in the principal path that contains $u$. Without loss of generality and since $K_{u^+}$ is a clique, $x_1 \in B^*$.

By condition (2b), there exist non-adjacent $a, b \in A$ such that $x_1$ has neighbors $a^* \in K_a$ and $b^* \in K_b$, and by (3) $a^*b^* \notin E(G^*)$. Note that $a^*, b^* \neq u_2$ because $u_2x_1 \notin E(G^*)$. If $u_2$ is complete to $\{a^*, b^*\}$, then $\{u_2, a^*, x_1, b^*\}$ induces a $C_4$, a contradiction. So, up to symmetry $u_2a^* \notin E(G)$. So, $a^* \notin K_u$ and by (4) and (3), $a^*u_1 \notin E(G^*)$. Observe that $x_2a^* \notin E(G^*)$ for otherwise $\{a^*, x_1, u_1, u_2, x_2\}$ induces a $C_5$.

Suppose that $x_2 \in B^*$. As above, we can show that $x_2$ has a neighbor $c^* \in A^*$ that is anticomplete to $\{u_1, u_2, x_1\}$. Note that $a^*c^* \notin E(G^*)$ for otherwise $\{x_1, a^*, c^*, x_2, u_2, u_1\}$ induces a $C_6$. Let $P_{a^*} = a^* \dots a'^*$ and $P_{c^*} = c^* \dots c'^*$ be defined as in (2).

By (3) and (4) and since $a^*c^* \notin E(G^*)$, $a'^*c'^* \in E(G^*)$. So, by conditions (1), (2) and (9), $u_1x_1a^*P_{a^*}a'^*c'^*P_{c^*}c^*x_2u_2u_1$ is a hole of length $2\ell + 4$, a contradiction.

So $x_2 \in K_{u^+}$. Hence by condition (9a) of preblowups, there exists a path $Q$ of length $\ell - 2$ from $x_2$ to some $u'^* \in K_{u'}$. Now $x_2Qu'^*a'^*P_{a^*}a^*x_1u_1u_2x_2$ is a hole of length $2\ell + 2$, a contradiction.

The result for $A'$ holds symmetrically. This proves (7).

*(8) For all $u \in A$ and $u^* \in K_u$, $N[u^*] \subseteq N[u]$. A similar statement holds for $A'$.*

Otherwise, there exists a vertex $u^* \in K_u$ such that $N[u] \subsetneq N[u^*]$. Hence, $(V(G) \setminus \{u\}) \cup \{u^*\}$ induces a subgraph $G_0$ of $G^*$ and it is easy to verify that $G^*$ is a preblowup of $G_0$ (that is a template by Lemma 4.3.14 and whose partition is proper by (3) and (4)). This contradicts the maximality of $s(G, G^*)$. The result for $A'$ holds symmetrically. This proves (8).

By (7), for every $u \in A \cup A'$, the clique $K_u$ can be linearly ordered by the inclusion of the neighborhoods as $u_1, \dots, u_{k_u}$, and by (8) $u_{k_u} = u$ (so, for

$1 \leq 1 \leq j \leq k_u$, $N[u_i] \subseteq N[u_j]$).

*(9) If $xy$ is an edge of $G[B^*]$, then either $N_{A^*}(x) \subseteq N_{A^*}(y)$ or $N_{A^*}(y) \subseteq N_{A^*}(x)$.*

Otherwise, there exists $u^* \in N_{A^*}(x) \setminus N_{A^*}(y)$ and $v^* \in N_{A^*}(y) \setminus N_{A^*}(x)$. Note that $u^*v^* \notin E(G)$ for otherwise $\{u^*, x, y, v^*\}$ induces a $C_4$. So, for some $u, v \in A$, we have $u^* \in K_u$ and $v^* \in K_v$. Hence, by (4), $uv \notin E(G)$. Let $P_{u^*} = u^* \dots u'^*$ and $P_{v^*} = v^* \dots v'^*$ be defined as in (2). So, $xu^*P_{u^*}u'^*v'^*P_{v^*}v^*yx$ form a hole of length $2\ell + 2$, a contradiction. This proves (9).

*(10) For every $x \in B^*$, there exist non-adjacent $u, v \in A$ such that $xu, xv \in E(G^*)$.*

This follows from condition (2b) of preblowups and from (8). This proves (10).

Two vertices $x, y$ in $B^*$ are *equivalent* if $N_A(x) = N_A(y)$.

*(11) If $x$ and $y$ are equivalent vertices of $B^*$, then $xy \in E(G^*)$.*

If $xy \notin E(G^*)$, then $x$, $y$ and two of their neighbors provided by (10) induce a $C_4$. This proves (11).

Vertices of $B^*$ are partitioned into equivalence classes. By (11), each equivalence class is a clique $X$, and by (9), vertices of $X$ can be linearly ordered according to the inclusion of neighborhoods in $A^*$. In each such a clique $X$ we choose a vertex $x$ maximal for the order and call $B_1$ the set of these maximal vertices. For every $x \in B_1$, we denote by $K_x$ the clique of $B^*$ of all vertices equivalent to $x$. Observe that if $w^* \in B$, then $w^*$ is a maximal vertex of its clique. Hence, we can set $w^* \in B_1$.

So, for every $u \in B_1$, the clique $K_u$ can be linearly ordered by the inclusion of the neighborhod in $A^*$ as $u_1, \dots, u_{k_u}$ with $u = u_{k_u}$ (so, for $1 \leq 1 \leq j \leq k_u$, $N_{A^*}(u_i) \subseteq N_{A^*}(u_j)$).

Statements similar to (9), (10), (11) hold for $B'^*$ and we define $B'_1$ as well.

We set $G_1 = G^*[A \cup B_1 \cup A' \cup B'_1 \cup I]$ and claim that $(A, B_1, A', B'_1, I)$ is an $\ell$-pretemplate partition of $G_1$. Since $G_1[A \cup I \cup A']$ is exactly $G[A \cup I \cup A']$, conditions (3), (4), (5) and (6) hold. Adding the fact that $N_{G_1}(B_1) \subseteq A^* \cap V(G_1) = A$ by condition (2) of preblowup, condition (1) for a pretemplate holds and symmetrically also condition (2). Now condition (7) holds because $w^*$ and $w'^*$ are complete to respectively $A \cup B_1$ and $A' \cup B'_1$. By (10), the last two conditions for a pretemplate are fulfilled by $(A, B_1, A', B'_1, I)$. Hence, by Lemma 4.3.14, $G_1$ is a an odd $\ell$-template. It is twinless by Lemma 4.3.15. We also notice that by construction $w^*$ and $w'^*$ belong to $G_1$. Furthermore, by (1), $w^*$ (respectively $w'^*$) is complete to $A \setminus \{w^*\}$ (respectively $A' \setminus \{w'^*\}$). From the definition of a template it is easy to conclude that $w^*$ (respectively $w'^*$) is universal in $G_1[A \cup B_1]$ (respectively $G_1[A' \cup B'_1]$). Hence $(A, B_1, A', B'_1, I, w^*, w'^*)$ is a proper $\ell$-partition of $G_1$.

We now prove that $G^*$ is a proper blowup of $G_1$.

By the definition of a preblowup and by (11), for all $u \in V(G_1)$, $K_u$ is a clique and $V(G^*) = \cup_{u \in V(G_1)} K_u$

*(12) If $u, v \in V(G_1)$ and $uv \notin E(G_1)$, then $K_u$ is anticomplete to $K_v$.*

Suppose $u, v \in V(G_1)$ and $uv \notin E(G_1)$. If $u \in I$ or $v \in I$, the conclusion follows directly from condition (9) of preblowups. So we may assume up to symmetry that $u \in A \cup B_1$. By conditions (1) and (2) of preblowups, we may assume $v \in A \cup B_1$. If $u, v \in A$, then the result follows from (3), so we may assume that $v \in B_1$.

Now suppose for a contradiction that there exist $u^* \in K_u$ and $v^* \in K_v$ such that $u^*v^* \in E(G_1)$. By the choice of vertices in $B_1$, for all $v^* \in K_v$, $N[v^*] \subseteq N[v]$. So $u^*v \in E(G_1)$. For the same reason or by (8), for all $u^* \in K_u$, $N[u^*] \subseteq N[u]$. Hence $uv \in E(G_1)$, a contradiction. This proves (12).

*(13) If $uv$ is a solid edge of $G_1$ then $K_u$ is complete to $K_v$.*

Otherwise, let $u^* \in K_u$ and $v^* \in K_v$ such that $u^*v^* \notin E(G)$. Since $uv$ is a solid edge, up to symmetry, $u, v \in A$ or $u, v \in B_1$ or $u \in A$, $v \in B_1$ and in this last case $u$ is not an isolated vertex of $G[H_v]$.

By (4) the case where $u$ and $v$ are in $A$ cannot happen. Assume then that $v \in B_1$. By Lemma 4.3.6, there exist $a, b \in H_v$ (and hence in $A$) that are not adjacent. Assume that $u$ is also in $B_1$. Since $u$ and $v$ are adjacent, by (9) we may assume without loss of generality that $H_v \subseteq H_u$ and so $a$ and $b$ belong to $H_u$ too. Then, by the definition of $K_u$ and $K_v$, we get a $C_4$ induced by $\{u^*, v^*, a, b\}$, a contradiction. So $u$ should be in $A$, and to avoid a $C_4$ induced by $\{u^*, v^*, a, b\}$, $u^*$ should be non-adjacent to at least one of $a$ and $b$, say $a$. In particular, $a \neq u$. Then, by (4), $ua \notin E(G_1)$. So $u$ does not belong to $N(H_v)$ and since $uv$ is an edge of $G_1$, we get that $u \in H_v$. Since $uv$ is solid, $u$ has at least one neighbor in $H_v$, and it is not adjacent to $a \in H_v$. Hence, as $H_v$ is anticonnected, there exist non-adjacent vertices $c, d \in H_v$ such that $uc \notin E(G_1)$ and $ud \in E(G_1)$. Now $u^*P_{u^*}u'^*c'P_c cv^*du^*$ is a hole of length $2\ell+2$, a contradiction again.

This proves (13).

*(14) For all $u \in V(G_1)$ and $1 \leq i \leq j \leq k_u$, $N[u_i] \subseteq N[u_j]$.*

The result follows from how vertices are ordered after the proof of (5) (vertices in $I$), (7) (vertices in $A$ or $A'$) and (11) (vertices in $B_1$ or $B_1'$). This proves (14).

*(15) If $uv$ is a flat edge of $G_1$, then $u$ is complete to $K_v$ and $v$ is complete to $K_u$.*

By definition of a flat edge, either $u$ and $v$ are in $I$ or one is in $I$ and the other is in $A$ or in $A'$. The result follows from (6) , (8), and conditions (1b) (applied to $A$ or $A'$) and (9a) of the preblowup. This proves (15).

*(16) If $ux$ is an optional edge of $G_1$ with $u \in A$ and $x \in B_1$ (resp. $u \in A'$ and $x \in B_1'$), then $u$ is complete to $K_x$.*

The result follows from the definition of $K_x$ when $x \in B_1$. This proves (16).

*(17) If $ux$ and $uy$ are optional edges with $u \in A$, $x, y \in B_1$ and $H_y \subsetneq H_x$ (resp. $u \in A'$, $x, y \in B_1'$ and $H_y' \subsetneq H_x'$), then every vertex of $K_u$ with a neighbor in $K_y$ is complete to $K_x$.*

Otherwise, let $u^*$ be a vertex in $K_u$ that has a neighbor $y^*$ in $K_y$ and a non-neighbor $x^*$ in $K_x$. Since $H_x$ and $H_y$ are not disjoint, $xy$ is a solid edge of $G_1$ and by (13), $x^*y^* \in E(G_1)$.

Since $x$ and $y$ are not equivalent, there exists a vertex $a$ such that $a \in N_A(y) \setminus N_A(x)$ or $a \in N_A(x) \setminus N_A(y)$. In the first case, by definition of a template, $a \in A \setminus N_A[H_x]$. Then since $H_y \subsetneq H_x$ and $H_x$ is a module of $A$ we get that $a$ is anticomplete to $H_x$ and hence to $H_y$. So $a \notin N_A(y)$, a contradiction; we may then conclude that $a \in N_A(x) \setminus N_A(y)$

By definition of the cliques in $B$, $x^*a \in E(G^*)$ and $y^*a \notin E(G^*)$. Therefore, to avoid a $C_4$ induced by $\{x^*, y^*, u^*, a\}$, it should be that $u^*a \notin E(G^*)$.

Now $aP_aa'u'^*P_{u^*}u^*y^*x^*a$ is a hole of length $2\ell + 2$ a contradiction. This proves (17).

*(18) $w^*$ (resp. $w'^*$) is a universal vertex of $G^*[\cup_{u\in A\cup B_1} K_u]$ (resp. $G^*[\cup_{u\in A'\cup B'_1} K_u]$).*

By (1), $w^*$ is complete to $A^* \setminus \{w^*\}$ and so to $\cup_{u\in A} K_u \setminus \{w^*\}$. Furthermore, from the definition of $G_1$ we know that $w^*$ is complete to $B_1 \setminus \{w^*\}$. If $w^* \in B_1$, since all edges between vertices in $B_1$ are solid, by (13), $w^*$ is complete to $B^* \setminus \{w^*\}$. Similarly, if $w^* \in A$, by (13) and (16), we get that $w^*$ is complete to $B^*$. In both cases $w^*$ is a universal vertex of $G^*[\cup_{u\in A\cup B} K_u]$. The proof for $w'^*$ is symmetric. This proves (18).

From all the claims above, $G^*$ satisfies all conditions to be a proper blowup of $G_1$. □

## 4.5 Graphs in $\mathcal{C}_{2\ell+1}$ that contain a pyramid

The goal of this section is to prove the the following.

**Lemma 4.5.1** *Let $\ell \geq 3$ be an integer. If $G$ is a graph in $\mathcal{C}_{2\ell+1}$ and $G$ contains a pyramid, then one of the following holds:*

1. *$G$ is a proper blowup of a twinless odd $\ell$-template;*
2. *$G$ has a universal vertex;*
3. *$G$ has a clique cutset.*

The rest of this section is devoted to the proof of Lemma 4.5.1. So from here on $\ell \geq 3$ is an integer and $G$ is graph in $\mathcal{C}_{2\ell+1}$ that contains a pyramid $\Pi$. By Lemma 4.2.6, the three paths of $\Pi$ have length $\ell$. By Lemma 4.3.5, $\Pi$ is an odd $\ell$-template. Hence, we may define an integer $k$ and a sequence $F_0, F_1, F_2$ of induced subgraphs of $G$ as follows.

- $k$ is the maximum integer such that $G$ contains an odd $\ell$-template with $k$ principal paths. Observe that by Lemma 4.3.15, $G$ in fact contains a twinless template with $k$ principal paths, because twins can be eliminated from templates by deleting hyperedges with equal vertex-set while there are some.

- In $G$, pick a proper blowup $F_1$ of a twinless odd $\ell$-template $F_0$ with $k$ principal paths. Note that $F_0$ exists and the proper $\ell$-partition needed for the proper blowup exists by Lemma 4.3.16.

- Suppose that $F_0$ and $F_1$ are chosen subject to the maximality of the vertex-set of $F_1$ (in the sense of inclusion). Note that possibly $F_0$ is not a maximal template in the sense of inclusion, it can be that a smaller template leads to a bigger blowup (but $F_0$ has $k$ principal paths).

- $F_2$ is obtained from $F_1$ by adding all vertices of $G \setminus F_1$ that are complete to $F_1$.

**Lemma 4.5.2** *$V(F_2) \setminus V(F_1)$ is a (possibly empty) clique that is complete to $F_1$.*

*Proof.* Otherwise, $G$ contains a $C_4$. □

We now introduce some notation. We denote by $(A, B, A', B', I, w, w')$ the proper $\ell$-partition that is used to blowup $F_0$. When $u$ is a vertex of $F_0$, we denote by $K_u$ the clique of $F_1$ that is blown up from $u$. We set $A^* = \cup_{u \in A} K_u$. We use a similar notation $B^*$, $A'^*$, $B'^*$ and $I^*$.

### 4.5.1 Technical lemmas

We now prove lemmas that sum up several structural properties of $G$.

**Lemma 4.5.3** *If $u \in A \cup A' \cup I \cup \{w, w'\}$ and $v \in N_{V(F_0)}(u)$, then $u$ is complete to $K_v$.*

*Proof.* We prove this lemma using the conditions from the definition of blowups. If $u \in \{w, w'\}$, then the result follows form condition (8). If $u \in A \cup A'$, then the conclusion follows from conditions (4), (5) and (6). If $u \in I$, then the conclusion follows from condition (5). □

Very often, Lemma 4.5.3 will be used in the following way. Suppose there exists a principal path $P = u \dots u'$ of $F_0$. Suppose there exists a vertex $x$ of $P$ and $x^* \in K_x$. Then by Lemma 4.5.3 and condition (3) of blowups, $\{x^*\} \cup (V(P) \setminus \{x\})$ induces a path of $F_1$. If $y \neq x$ is a vertex of $P$ and $y^* \in K_y$, then $\{x^*, y^*\} \cup (V(P) \setminus \{x, y\})$ might fail to induce a path of $F_1$, because it is possible that $xy \in E(G)$ while $x^*y^* \notin E(G)$. But under the assumption that $x^*y^* \in E(G)$ or $xy \notin E(G)$, we do have that $\{x^*, y^*\} \cup (V(P) \setminus \{x, y\})$ induces a path of $F_1$. Several variant of this situation will appear soon and we will simply justify them by refering to Lemma 4.5.3.

When $u$ is a vertex in $A$, we denote by $P_u$ the unique principal path of $F_0$ that contains $u$. Its end in $A'$ is then denoted by $u'$. We denote by $u^+$ the neighbor of $u$ in $P_u$. We denote by $u^{++}$ the neighbor of $u^+$ in $P_u \setminus u$. Note that $u^+ \in I$ and $u^{++} \in I \cup A'$ ($u^{++} \in A'$ if and only if $\ell = 3$).

For any distinct $u, v \in A$, from the definition of templates, exactly one of $V(P_u) \cup V(P_v) \cup \{w\}$ or $V(P_u) \cup V(P_v) \cup \{w'\}$ induces a hole that is denoted by $C_{u,v}$. Such a hole is called a *principal hole*.

Note that there are two kinds of principal holes: those that contain $w$, and those that contain $w'$. Recall that by Lemma 4.3.11, every hole of a template contains two principal paths plus an extra vertex, but it may fail to be a principal hole (because it may fail to contain $w$ or $w'$). Though we do not use this information formally, it is worth noting that by Lemma 4.5.3, when $C$ is a principal hole, $\cup_{v \in V(C)} K_v$ induces a ring. But when $C$ is a non-principal hole, it may happen that $\cup_{v \in V(C)} K_v$ does not induce a ring (because there might be in $C$ an optional edge $uv$ with $u \in A$ and $v \in B$, and after the blowup process, there might be no vertex in $K_v$ that is complete to $K_u$).

**Lemma 4.5.4** *If $u \in V(F_0)$ and $u^* \in K_u$, then $u^*$ has two neighbors in $V(F_0) \setminus K_u$ that are not adjacent.*

*Proof.* If $u \in I$, then let $P$ be the principal path that contains $u$. By Lemma 4.5.3, $u^*$ is adjacent to the two neighbors of $u$ in $P$.

If $u \in A \cup A'$, say $u \in A$ up to symmetry, then we claim that $u$ has a neighbor $z$ in $A \cup B$. This is clear if $u$ is not isolated in $A$ and otherwise we set $z = w$. By Lemma 4.5.3, $z$ and $u^+$ are non-adjacent neighbors of $u^*$.

If $u \in B$, then by the definition of a template, $H_u$ contains two non adjacent vertices $a$ and $b$ that are neighbors of $u$. By Lemma 4.5.3, $a$ and $b$ are both adjacent to $u^*$. □

**Lemma 4.5.5** *If $uv$ is an edge of $F_0[A \cup A' \cup I \cup \{w, w'\}]$, then some principal hole of $F_0$ goes through $uv$.*

*Proof.* If at least one of $u$, $v$ is in $I$ then $uv$ is an edge of a principal path and we know that this principal path belongs to a principal hole. Else, since $A \cup \{w\}$ is anticomplete to $A' \cup \{w'\}$, up to symmetry both $u$ and $v$ are in $A$ or $u = w \in B$ and $v \in A$.

If $u, v \in A$ then $C_{u,v}$ is a principal hole containing $uv$.

If $u = w \in B$ and $v \in A$ : since $w$ is in $B$, $G[A]$ has no universal vertex and there exists $a \in A$ which is not adjacent to $v$. Now $w, P_v, P_a$ form a principal hole containing the edge $uv$. □

**Lemma 4.5.6** *If $K$ is a clique of $F_0$, $K^* = \cup_{v \in K} K_v$ and $D$ is a connected induced subgraph of $G \setminus F_2$ such that $N_{V(F_1)}(D) \subseteq K^*$, then $N_{V(F_1)}(D)$ is a clique.*

*Proof.* For suppose not. This means that there exists $u^*, v^* \in K^*$ and $x_u, x_v \in D$ such that $u^*v^* \notin E(G)$ and $x_u u^*, x_v v^* \in E(G)$ (possibly $x_u = x_v$). Since $D$ is connected, there exists a path $P$ in $D$ from $x_u$ to $x_v$. Suppose that $u^*$, $x_u$, $v^*$, $x_v$ and $P$ are chosen subject to the minimality of $P$. It follows that $u^* x_u P x_v v^*$ is a path, and recall that by assumption its interior is anticomplete to $F_1 \setminus K^*$.

Since $u^*v^* \notin E(G)$, $u^*$ and $v^*$ are in different blown-up cliques. Denote by $K_u$ and $K_v$ the blown-up cliques such that $u^* \in K_u$ and $v^* \in K_v$. By hypothesis, $uv \in K$ and so $uv \in E(G)$. Since $u^*v^* \notin E(G)$, by condition (4) of blowups, $uv$ is not a solid edge of $G$.

If $uv$ is a flat edge of $F_0$, then by Lemma 4.5.5 a principal hole $C$ goes through $uv$. Note that apart from $u$ and $v$, no vertex of $C$ is in $K$ since $K$ is a clique. By Lemma 4.5.3, in $G$, $(\{u^*, v^*\}) \cup V(C)) \setminus \{u, v\}$ induces a path $Q$ of length $2\ell$. So $P$ and $Q$ form a hole of length at least $2\ell + 2$, a contradiction.

If $uv$ is an optional edge of $F_0$, say with $u \in A$ and $v \in B$, then $u \in H_v$, and there exists $a$ in $H_v$ such that $au \notin E(F_0)$. Therefore, $P_u$, $P_a$ and $v$ form a hole $C^*$. By condition (6) of blowups (if $va$ is optional), or by condition (4) (if $va$ is solid), $a$ is complete to $K_v$. By Lemma 4.5.3 it follows that $(\{u^*, v^*\}) \cup V(C^*)) \setminus \{u, v\}$ induces a path $Q$ of length $2\ell$. So $P$ and $Q$ form a hole of length at least $2\ell + 2$, a contradiction again. □

When $C$ is a hole of $G$, a vertex $v$ of $V(G) \setminus V(C)$ is *minor* w.r.t. $C$ if $N_{V(C)}(v)$ is included in a 3-vertex path of $C$. A vertex of $V(G) \setminus V(C)$ that is not minor w.r.t. $C$ is *major* w.r.t. $C$.

**Lemma 4.5.7** *If $x \in V(G) \setminus V(F_2)$ and $C$ is a principal hole of $F_0$, then $x$ is minor w.r.t. $C$.*

*Proof.* Suppose up to symmetry that $w \in V(C)$ and suppose $C = C_{u,v}$. If $x$ is major w.r.t. $C$, then $C$ and $x$ form a theta or a wheel that is not a twin-wheel. So by Lemma 4.2.6, $x$ and $C$ form a universal wheel. Let $P_t = t \dots t'$ be a principal path where $t \neq u, v$. If $t$ is complete to $\{u, v\}$, then $xt \in E(G)$ for otherwise $\{t, u, v, x\}$ induces a $C_4$. Hence $x$ has at least 4 neighbors in $C_{u,t}$, so by Lemma 4.2.6, $x$ is complete to $P_t$. If $t$ is not complete to $\{u, v\}$, say $tu \notin E(G)$, then $x$ again has at least 4 neighbors in $C_{u,t}$ because $w \in V(C_{u,t})$, so again $x$ is complete to $P_t$.

We proved that $x$ is complete to all principal paths, so to $I \cup A \cup A'$. Let $y \in B \cup B'$. By definition of a template $y$ has two neighbors $a$ and $b$, both in $A$ or both in $A'$, that are non-adjacent. Therefore $a$, $b$, $v$ and $x$ form a $C_4$, unless $x$ is adjacent to $y$. This proves that $x$ is complete to $B \cup B'$, and so to $V(F_0)$.

Let $z$ be a vertex of $F_0$ and $z^* \in K_z$. By Lemma 4.5.4, there exists $a, b \in V(F_0)$ such that $z^*a, z^*b \in E(G)$ and $ab \notin E(G)$, so since there is no $C_4$ in $G$ it should be that $xz^* \in E(G)$. This proves that $x$ is complete to $F_1$. Hence, $x \in V(F_2)$, a contradiction. $\square$

**Lemma 4.5.8** *Let $a$ and $b$ be two non-adjacent vertices of some principal hole $C$ of $F_0$. If some vertex $x$ of $V(G) \setminus V(F_2)$ has neighbors in both $K_a$ and $K_b$, then $a$ and $b$ have a common neighbor $c$ in $C$, $x$ is adjacent to $c$, and $x$ is anticomplete to every $K_d$ such that $d \in V(C) \setminus \{a, b, c\}$.*

*Proof.* Let $a^* \in K_a$ and $b^* \in K_b$ be two neighbors of $x$. Since $ab \notin E(G)$, by Lemma 4.5.3, $\{a^*, b^*\} \cup V(C) \setminus \{a, b\}$ induces a hole $C^*$. Since $x$ is adjacent to $a^*$ and $b^*$, by Lemma 4.2.6, $x$ has another neighbor $c$ in $C^*$ (and in fact in $C$ since $c \neq a^*, b^*$). If $c$ is not adjacent to $a^*$ and $b^*$, then $x$ is major w.r.t. $C^*$, so by Lemma 4.2.6, $C^*$ and $x$ form a universal wheel. It follows that $x$ is major w.r.t. $C$, a contradiction to Lemma 4.5.7.

We proved that $a$ and $b$ have a common neighbor $c$ in $C$ and that $x$ is adjacent to $c$. Suppose for a contradiction that $x$ has a neighbor $d^* \in K_d$ where $d \in V(C) \setminus \{a, b, c\}$. By the same argument as above, since $x$ has neighbors in $K_d$ and $K_c$, $c$ and $d$ must have a common neighbor in $C$, and this common neighbor must be $a$ or $b$, say $a$ up to symmetry. So, $x$ has neighbors in $K_d$ and $K_b$ while $b$ and $d$ have no common neighbors in $C$, so we may reach a contradiction as above. □

### 4.5.2 Connecting vertices of $F_1$

We here explain how lemmas of Subsection 4.3.4 are extended from $F_0$ to $F_1$.

**Lemma 4.5.9** *If $u^* \in A^* \cup B^*$ and $v^* \in A'^* \cup B'^*$, then there exists in $F_1$ a path $P^*$ of length $\ell - 1$, $\ell$ or $\ell + 1$ from $u^*$ to $v^*$ that contains the interior of a principal path.*

*More specifically:*

- *If $u^* \in A^*$ and $v^* \in A'^*$, then $P^*$ has length $\ell - 1$ or $\ell$.*
- *If $u^* \in A^*$ and $v^* \in B'^*$, or if $u^* \in B^*$ and $v^* \in A'^*$, then $P^*$ has length $\ell$ or $\ell + 1$.*
- *If $u^* \in B^*$ and $v^* \in B'^*$, then $P^*$ has length $\ell + 1$.*

*Proof.* Let $u$ and $v$ be such that $u^* \in K_u$ and $v^* \in K_v$. Let $P$ be a path in $F_0$ like in Lemma 4.3.13 from $u$ to $v$ (so $P$ contains the interior of some principal path $Q$). By Lemma 4.5.3, $\{u^*, v^*\} \cup V(P) \setminus \{u, v\}$ induces a path of the same length as $P$ that contains the interior of $Q$. □

**Lemma 4.5.10** *If $u^* \in B^*$ and $v^* \in B'^*$, then there exist in $G$ two paths $P^*$ and $Q^*$ from $u^*$ to $v^*$ both of length at most $\ell + 1$ such that $P^*$ (resp. $Q^*$) contains the interior of a principal path $P$ (resp. $Q$), and $P \neq Q$.*

*Proof.* Let $u$ and $v$ be such that $u^* \in K_u$ and $v^* \in K_v$. Let $P = u \dots v$ and $Q = u \dots v$ be as in the conclusion of Lemma 4.3.12. By Lemma 4.5.3, $\{u^*, v^*\} \cup V(P) \setminus \{u, v\}$ and $\{u^*, v^*\} \cup V(Q) \setminus \{u, v\}$ are the desired paths. □

**Lemma 4.5.11** *If some vertex $x$ of $G$ is adjacent to the ends of a path $P$ of length at most $\ell + 1$ of $G \setminus x$, then $x$ is complete to $V(P)$.*

*Proof.* Otherwise, a shortest cycle in $G[V(P) \cup \{x\}]$ has length at least 4 and at most $\ell + 3$. Since $\ell \geq 3$ implies $\ell + 3 < 2\ell + 1$, this is a contradiction. $\square$

### 4.5.3 Attaching a vertex to $F_1$

In this subsection, we show that for all vertices $x$ of $G \setminus F_2$, $N_{V(F_1)}(x)$ is a clique (see Lemma 4.5.15). In Figure 4.6, several situations where $N_{V(F_1)}(x)$ is not a clique are represented and we explain informally how they lead to a contradiction. The first figure is an odd 3-template $F_0$ with its vertices $w$ and $w'$, and here $F_1 = F_0$. Then, vertex $x_1$ can be included in $K_{y_1}$, a contradiction to the maximality of $F_1$ (see Lemma 4.5.12). The vertex $x_2$ cannot be included in an existing blown up clique, but it can be added to $F_0$ to yield a bigger template (see Lemma 4.5.12). The vertex $x_3$ can be added to $K_{u_6}$ (see Lemma 4.5.13). The vertex $x_4$ can be added to $K_{u_1}$, but at the expense of modifying the template (see Lemma 4.5.13). The vertex $x_5$ can be added to $K_{i_3}$ (see Lemma 4.5.14).

The vertex $x_6$ is kind of pathological because it cannot be added to any blown-up clique, and does not increase the template. The idea for this one is to observe that $\{x_6\} \cup V(F_0) \setminus \{y_1, u_6^+\}$ induces a template and that $y_1$ can be incorporated in the set $K_{u_6}$ and $K_{x_6} = K_{u_6^+} \cup \{x_6\}$ (see Lemma 4.5.15). Note that in this case, we increase the size of the blowup while decreasing the size of the template.

In each case, we prove that adding $x$ yields a preblowup of $F_0$, so that the maximality of $F_1$ is contradicted.

**Lemma 4.5.12** *If $x \in G \setminus F_2$ has no neighbor in $I^*$, then $N_{V(F_1)}(x)$ is a clique.*

*Proof.* Suppose for a contradiction that $N_{V(F_1)}(x)$ is not a clique.

*(1) We may assume that $N_{V(F_1)}(x) \subseteq A^* \cup B^*$.*

If $x$ has neighbors in both $A^* \cup B^*$ and $A'^* \cup B'^*$, then consider a path $P$ as in Lemma 4.5.9 from a neighbor of $x$ in $A^* \cup B^*$ to a neighbor of $x$ in $A'^* \cup B'^*$. By Lemma 4.5.11, $x$ is complete to $V(P)$. This is a contradiction since $x$ has no neighbor in $I^*$. Hence $x$ does not have neighbors in both $A^* \cup B^*$ and $A'^* \cup B'^*$, and our claim follows up to symmetry. This proves (1).

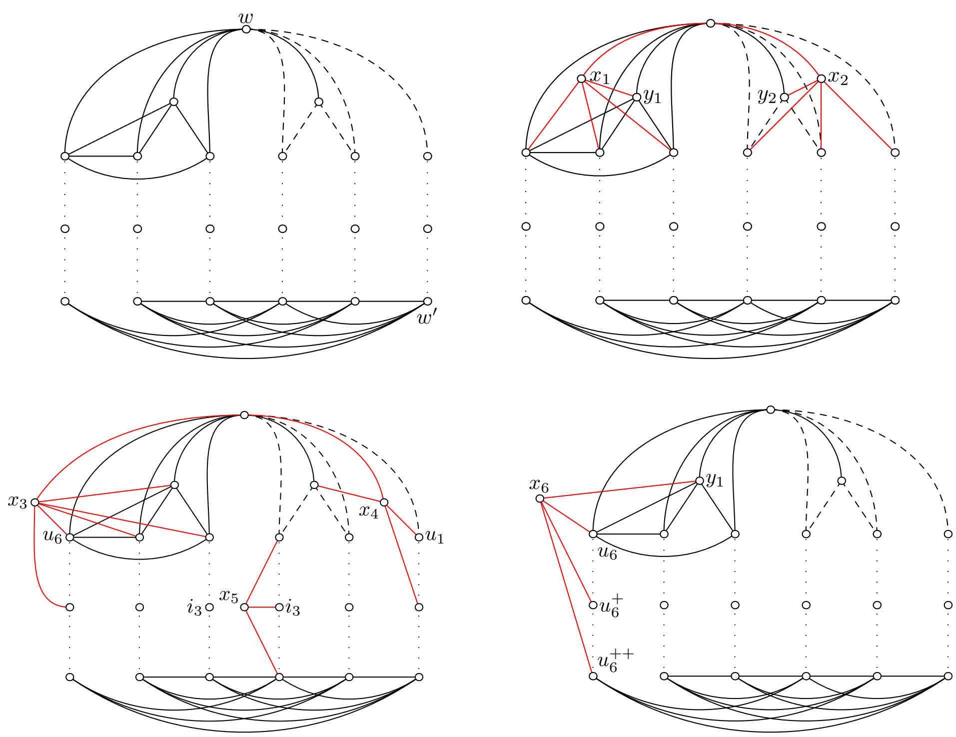



Figure 4.6: Vertices attaching to an odd 3-template

*(2) There exist non-adjacent $a, b \in A$ such that $x$ has neighbors in both $K_a$ and $K_b$.*

By Lemma 4.5.6, since $N_{V(F_1)}(x)$ is not a clique, there should exist two non-adjacent vertices $a, b \in V(F_0)$ such that $x$ has a neighbor $a^* \in K_a$ and a neighbor $b^* \in K_b$. By (1), $a, b \in A \cup B$.

If $a, b \in A$, then our conclusion holds, so we may assume that $b \in B$.

If $a \in A$, then since $ab \notin E(G)$, $H_b$ is anticomplete to $a$. Let $P_a^*$ be the path induced by $\{a^*\} \cup (V(P_a) \setminus \{a\})$. Let $v \in H_b$. We may assume that $xv \notin E(G)$ for otherwise our claim holds (with $a$ and $v$). Note that since $ab, av \notin E(G)$, by (3) of blowup, $a^*b^*, a^*v \notin E(G)$. Now, the paths $P_a^*$, $P_v$, $a^*xb^*v$ form a hole of length $2\ell+2$, a contradiction. Hence, we may assume $a \in B$.

Since $ab \notin E(G)$, by Lemma 4.3.7, $\{a\} \cup H_a$ is anticomplete to $\{b\} \cup H_b$. We may assume that $x$ is anticomplete to $H_a \cup H_b$ for otherwise we may apply the proofs above. Hence, for $u \in H_a$ and $v \in H_b$, the two paths $P_u$ and $P_v$ together with the path $ua^*xb^*v$ form a hole of length $2\ell+3$. This proves (2).

Now the sets $K_u$ for all $u \in A \cup A' \cup I$, $B^* \cup \{x\}$ and $B'^*$ form a preblowup of $F_0$. All conditions are easily checked. In particular $x$ satisfies condition (2) by (1) and (2b) by (2)). So, by Lemma 4.4.6, $G[V(F_1) \cup \{x\}]$ is a proper blowup of some $\ell$-template with $k$ principal paths. This contradicts the maximality of $F_1$. □

**Lemma 4.5.13** *If there exist $x \in V(G) \setminus V(F_2)$ and $u \in A$ such that $x$ has neighbors in both $K_u$ and $K_{u^+}$ and is anticomplete to $K_{u^{++}}$, then $N_{V(F_1)}(x)$ is a clique.*

*Proof.* Suppose for a contradiction that $N_{V(F_1)}(x)$ is a not clique.

*(1) $x$ is anticomplete to $A'^* \cup B'^* \cup (I^* \setminus K_{u^+})$.*

If $x$ has a neighbor $t^*$ in some $K_t$ such that $t \in (A' \cup I) \setminus \{u^+\}$, then note that $t \neq u^{++}$ by assumption. Let $C$ be a principal hole that contains $t$ and $u$. There is a contradiction to Lemma 4.5.8 because by (3) of blowup $u$, $u^+$ and $t$ cannot be consecutive along $C$.

It remains to prove that $x$ is anticomplete to $B'^*$. Otherwise, $x$ has a neighbor $t \in B'^*$. Consider a path $P$ from $t$ to the neighbor of $x$ in $K_u$ as in Lemma 4.5.9 and let $Q$ be the principal path whose interior is contained in $P$.

By Lemma 4.5.11, $x$ is complete to $V(P)$. This is a contradiction because if $Q = P_u$ then $x$ is anticomplete to $K_{u^{++}}$, and if $Q \neq P_u$ then we already proved that $x$ is anticomplete to $(A'^* \cup I^*) \setminus K_{u^+}$. This proves (1).

From here on, $u^*$ and $u^{+*}$ are neighbors of $x$ in respectively $K_u$ and $K_{u^+}$. Note that $x$ has a neighbor $y^* \in K_y$ for some $y \in A \cup B \setminus \{u\}$, for otherwise, by (1), $N_{V(F_1)}(x) \subseteq K_u \cup K_{u^+}$ and by Lemma 4.5.6, $N_{V(F_1)}(x)$ is a clique, a contradiction.

*(2) If $w \in B$, then $x$ has a neighbor $w^* \in B^*$ that is complete to $A^*$.*

We may assume that $x$ is non-adjacent to $w$, for otherwise by condition (8) of blowups, we may choose $w^* = w$. In particular $y^* \neq w$.

We claim that we may assume that $y^*$ has a non-neighbor $v^*$ such that $v^* \in K_v$, $v \in A$ and $v \neq u$.

If $y^* \in B^*$, this is because we may assume that $y^*$ has a non-neighbor $v^* \in A^*$ (so $v \in K_v$ for some $v \in A$) for otherwise we choose $w^* = y^*$ from the start. It remains to check that $u \neq v$. This is because if $u = v$, then there exists a path $Q$ of length 1, 2 or 3 from $x$ to $v^*$ with interior in $K_{u^+}$ (through $xv^*$, $u^+$, $u^{+*}$ or $u^+u^{+*}$). Hence, $xQv^*wy^*x$ is a hole of length 4, 5 or 6, a contradiction.

If $y^* \in A^*$, then $u^*y^* \in E(G)$ for otherwise, $\{x, y^*, w, u^*\}$ induces a $C_4$. By condition (3) of blowups, $uy \in E(G)$. It follows that none of $u$ and $y$ is isolated in $G[A]$, so the existence of $v^*$ follows from Lemma 4.3.17 that guarantees the existence of isolated vertices in $G[A]$ since $w \in B$ by assumption.

So, our claim is proved. Note that $xv^* \notin E(G)$ for otherwise $\{x, y^*, w, v^*\}$ induces a $C_4$. Now either $xy^*wv^*v^+P_vv'w'u'P_uu^{++}u^{+*}x$ is a hole of length $2\ell+3$ (in case $u'v' \notin E(G)$) or $xy^*wv^*v^+P_vv'u'P_uu^{++}u^{+*}x$ is a hole of length $2\ell + 2$ (in case $u'v' \in E(G)$). In both cases we get a contradiction. This proves (2).

*(3) $N_A(x) \setminus \{u\} = N_A(u)$.*

If there exists $v \in N_A(x) \setminus N_A[u]$, then $vP_vv'u'P_uu^{++}u^{+*}xv$ is a hole of length $2\ell$, a contradiction.

Conversely, suppose there exists $v \in N_A(u) \setminus N_A(x)$. We claim that there exists a path $Q$ of length 2 from $x$ to some $z \in N_A(u)$ with interior in $(A^* \cup B^*) \setminus (K_u \cup K_z)$.

If $w \in B$, then we may choose $z = v$ and $Q = xw^*z$ by (2).

Otherwise, $w \in A$. So, by Lemma 4.3.17, $G[A]$ contains at least two universal vertices. So, let $t \in A \setminus \{u, v\}$ be adjacent to $u$ and $v$ (if $u$ and $v$ are the universal vertices of $G[A]$, $t$ can be any vertex of $A \setminus \{u, v\}$ and otherwise choose $t$ to be a universal vertex).

If $x$ has a neighbor $t^*$ in $K_t$, then we choose $Q = xt^*v$. So, suppose $x$ is anticomplete to $K_t$ (in particular, $y \neq t$). If $x$ has a neighbor $v^*$ in $K_v$, then we choose $Q = xv^*t$. So, suppose $x$ is anticomplete to $K_v$ (in particular, $y \neq v$). Now, by the way we chose $v$ and $t$, one of $v$ or $t$ is a universal vertex of $G[A]$ and therefore a universal vertex of $G[A^* \cup B^*]$. So, we may choose $Q = xy^*v$ or $Q = xy^*t$.

So, our claim is proved. Hence $z'P_z zQxu^{+*}u^{++}P_u u'w'z'$ is a hole of length $2\ell + 2$, a contradiction. This proves (3).

*(4) $x$ is complete to $K_u$.*

Suppose there exists $r \in K_u$ such that $rx \notin E(G)$. We claim that $x$ and $r$ have a common neighbor $z$ in $(A^* \cup B^*) \setminus K_u$.

If $w \in B$, then $rw^* \in E(G)$ by (2) so we may choose $z = w^*$. If $w \in A$, then by Lemma 4.3.17, some vertex $z \in A \setminus \{u\}$ is a universal vertex of $G[A]$, and by (3), $z$ is adjacent to $x$. So, $z$ exists as claimed.

If $xu^+ \in E(G)$ then $\{r, z, u^+, x\}$ induces a $C_4$, a contradiction. Hence $xu^+ \notin E(G)$. Now by condition (5) of blowups, either $\{x, z, r, u^{+*}\}$ induces a $C_4$ or $\{x, z, r, u^+, u^{+*}\}$ induces a $C_5$. This proves (4).

Now, the sets $K_v$ for all $v \in (A \setminus u) \cup I \cup A'$, $K_u \cup \{x\}$, $B^*$ and $B'^*$ form a preblowup of $F_0$. All conditions are easy to check. In particular, $K_u \cup \{x\}$ is a clique by (4), conditions (1), (2) and (9) follows from (1), condition (1a) from (3), condition (2a) from (2) and condition (1b) from our assumptions.

Hence, by Lemma 4.4.6 $G[V(F_1) \cup \{x\}]$ is a proper blowup of some twinless odd $\ell$-template with $k$ principal paths that is an induced subgraph of $G$ a contradiction to the maximality of $F_1$. □

**Lemma 4.5.14** *If $x \in V(G) \setminus V(F_2)$ has no neighbor in $B^* \cup B'^*$, then $N_{V(F_1)}(x)$ is a clique.*

*Proof.* Suppose for a contradiction that $N_{V(F_1)}(x)$ is not a clique. By Lemma 4.5.12, $x$ has neighbors in $I^*$. So $x$ has a neighbor in a clique blown up from an internal vertex of some principal path $P_v = v \dots v'$. Let $a$ (resp. $b$) be the vertex of $P_v$ closest to $v$ (resp. to $v'$) along $P_v$ and such that $x$ has a neighbor in $K_a$ (resp. $K_b$).

Suppose first that $a = b$ (so $a \in I$). Then $x$ has a neighbor in some $K_y$ with $y \in V(F_0) \setminus \{a\}$, and since by assumption $x$ has no neighbor in $B^* \cup B'^*$, $y \in A \cup A' \cup I$. So, $y$ and $a$ are non-adjacent members of some principal hole. By Lemma 4.5.8, $x$ has a neighbor in some clique $K_d$ where $d$ is adjacent to $a = b$, a contradiction to $a = b$.

Suppose now that $ab \in E(G)$. If both $a$ and $b$ are internal vertices of $P_v$, then as in the previous paragraph, by Lemma 4.5.8, $N_{V(F_1)}(x) \subseteq K_a \cup K_b$. So, by Lemma 4.5.6, $N_{V(F_1)}(x)$ is clique, a contradiction. It follows that at least one of $a$ or $b$ is an end of $P_v$. Up to symmetry, we may assume that $a = v$ and $b = v^+$. Note that $x$ is then anticomplete to $K_{v^{++}}$. Hence, by Lemma 4.5.13, $N_{V(F_1)}(x)$ is a clique, a contradiction.

Hence, $a \neq b$ and $ab \notin E(G)$. So, by Lemma 4.5.8, $a$ and $b$ have a common neighbor $u$ in $P_v$. So, $a$, $u$ and $b$ are consecutive along $P_v$ (in particular, $u \in I$).

*(1) $x$ is complete to $K_u$.*

Otherwise, let $u^* \in K_u$ be a non-adjacent to $x$. There exists a path $Q_a$ of length 2 or 3 from $u^*$ to $x$ with interior in $K_a$ (either $xa^*u^*$, or $xa^*au^*$ for some $a^*$ in $K_a$). There exists a similar path $Q_b$. So, $Q_a$ and $Q_b$ form a hole of length 4, 5 or 6, a contradiction. This proves (1).

*(2) $x$ is anticomplete to $V(F_1) \setminus (K_a \cup K_u \cup K_b)$.*

This follows from Lemma 4.5.8 and from the fact that $x$ is anticomplete to $B^* \cup B'^*$. This proves (2).

*(3) $x$ has neighbors in each of $K_a$, $K_b$.*

This follows from the definition of $a$ and $b$. This proves (3).

Now the sets $K_v$ for all $v \in (A \cup A' \cup I) \setminus \{u\}$, $K_u \cup \{x\}$, $B^*$ and $B'^*$ form a preblowup of $F_0$. All conditions are easily checked, in particular $K_u \cup \{x\}$ is a clique by (1), it satisfies condition (9) by (2) and condition (9a) by (3).

Hence, by Lemma (4.4.6) $G[V(F_1) \cup \{x\}]$ is a proper blowup of some twinless odd $\ell$-template with $k$ principal paths that is an induced subgraph of $G$. This contradicts the maximality of $F_1$. □

**Lemma 4.5.15** *For all vertices $x$ of $G \setminus F_2$, $N_{V(F_1)}(x)$ is a clique.*

*Proof.* Suppose for a contradiction that $N_{V(F_1)}(x)$ is not a clique.

*(1) There exists a principal path $P_u = u \dots u'$ of $F_0$ such that $x$ is anticomplete to $I^* \setminus \cup_{v \in V(P_u)} K_v$.*

Otherwise, there exists two principal paths $P$ and $Q$ of $F_0$, $a$ in the interior of $P$ and $b$ in the interior of $Q$ such that $x$ has neighbors in both $K_a$ and $K_b$. Note that $P$ and $Q$ are in some principal hole $C$ of $F_0$. By Lemma 4.5.8, $a$ and $b$ have a common neighbor $c$ in $C$. This contradicts $a$ and $b$ being in the interior of distinct principal paths. This proves (1).

*(2) We may assume that $x$ has no neighbor in $B'^*$ and has a neighbor $y^* \in K_y$ where $y \in B$.*

Suppose that $x$ has a neighbor $u^* \in B^*$ and a neighbor $v^* \in B'^*$. Let $P$ and $Q$ be like in Lemma 4.5.10. By Lemma 4.5.11, $x$ is complete to both $V(P)$ and $V(Q)$. In particular, $x$ has neighbors in the interior of two distinct principal paths, a contradiction to (1). So, up to symmetry, we may assume that $x$ has no neighbor in $B'^*$. Hence, by Lemma 4.5.14, $x$ has neighbors in $B^*$. This proves (2).

*(3) $x$ is adjacent to $u$, $u^+$ and has a neighbor in $K_{u^{++}}$. Moreover, $x$ is anticomplete to $(A^* \cup I^* \cup A'^* \cup B'^*) \setminus (K_u \cup K_{u^+} \cup K_{u^{++}})$.*

By Lemma 4.5.12, $x$ has at least one neighbor in $I^*$ and by (1), such a neighbor is in a clique blown up from an internal vertex of $P_u$. So, let $v$ be the vertex of $P_u$ closest to $u'$ along $P_u$ such that $x$ has a neighbor $v^* \in K_v$. So $v \neq u$ and $v \in A' \cup I$. We set $Q = y^* u P_u v$ if $y^* u \in E(G)$ and $Q = y^* w u P_u v$ otherwise. Let $Q^*$ be the path induced by $\{v^*\} \cup (V(Q) \setminus \{v\})$ and observe that $Q^*$ has length at most $\ell + 1$. By Lemma 4.5.11, $x$ is complete to $Q^*$. If $v \notin \{u^+, u^{++}\}$, then $x$ has neighbors in at least 4 cliques blown up from vertices of $P_u$ and this contradicts Lemma 4.5.8. If $v = u^+$, $x$ is adjacent to $u$ (since $x$ is complete to $Q^*$) and anticomplete to $K_{u^{++}}$, so by Lemma 4.5.13, $N_{V(F_1)}(x)$ is a clique, a contradiction. So, $v = u^{++}$, meaning that $x$ is adjacent to $u$ and $u^+$, and is anticomplete to $I^* \setminus (K_{u^+} \cup K_{u^{++}})$ by (1).

If $x$ has neighbors in some $K_a$ for $a \in A \setminus \{u\}$ then $x$ and $C_{u,a}$ contradict Lemma 4.5.8. Hence $x$ is anticomplete to $A^* \setminus \{K_u\}$.

By (2), $x$ is anticomplete to $B'^*$. It remains to check that $x$ is anticomplete to $A'^* \setminus K_{u^{++}}$. So, suppose $x$ has a neighbor $z^*$ in some $K_z$ where $z \in A' \setminus \{u^{++}\}$. Then a principal hole that contains $z$ and $u$ contradicts Lemma 4.5.8. This proves (3).

Let $u^{++*}$ be a neighbor of $x$ in $K_{u^{++}}$ and $P_u^*$ be the path induced by $(V(P_u) \setminus \{u^{++}\}) \cup \{u^{++*}\}$.

*(4) For every $z \in B$ such that $x$ is adjacent to some $z^*$ in $K_z$ we have $N_A(z) = N_A[u]$ (in particular $N_A(y) = N_A[u]$).*

Suppose there exists $v \in N_A(z) \setminus N_A[u]$. By condition (4) or (6) of blowups, $vz^* \in E(G)$. So, by (3), $xz^*vP_vv'u'P_u^*u^{++*}x$ is a hole of length $2\ell$, a contradiction. This proves that $N_A(z) \subseteq N_A[u]$. In particular, $u$ has at least one neighbor in $H_z$, so by condition (5e) of templates, $uz \in E(G)$.

Suppose there exists $v \in N_A(u) \setminus N_A(z)$ (so $z$ and $v$ are not universal vertices of $G[A \cup B]$). By condition (3) of blowups, $vz^* \notin E(G)$. By (3), $xv \notin E(G)$. Hence $xd \in E(G)$ for every universal vertex $d$ of $G[A \cup B]$, for otherwise $xz^*dvP_vv'w'u'P_u^*u^{++*}x$ is a hole of length $2\ell + 2$.

Now, by (3) and Lemma 4.3.17, $w \in B$. So, there exists an isolated vertex $c \in A$. Again by (3), $xc \notin E(G)$ and $xwcP_cc'u'P_u^*u^{++*}x$ is a hole of length $2\ell$, a contradiction. This proves (4).

*(5) $N_{F_1}(x) \subseteq K_{u^{++}} \cup K_{u^+} \cup K_u \cup K_y$*

By (3) $N_{F_1}(x) \subseteq K_{u^{++}} \cup K_{u^+} \cup K_u \cup B^*$. Suppose there exists $z^* \in K_z$ such that $xz^* \in E(G)$ and $z \in B \setminus \{y\}$. By (4), $N_A(z) = N_A[u]$ and $N_A(y) = N_A[u]$. So, by Lemma 4.3.15, $y$ and $z$ are twins of $F_0$, a contradiction. This proves (5).

*(6) $y \neq w$.*

If $y = w$, then $w \in B$. So by Lemma 4.3.17, there exist isolated vertices in $G[A]$. But by (4), $N_A(w) = N_A[u]$ so $u$ is a universal vertex of $G[A]$, so $G[A]$ has a universal vertex and an isolated vertex, a contradiction. This proves (6).

*(7) $N_{K_y}(x)$ is complete to $N_A[u]$.*

By (4), $N_A(y) = N_A[u]$. The result follows from conditions (4) and (6) of blowups. This proves (7).

*(8) $x$ is complete to $K_{u^+}$.*

By (3), $ux \in E(G)$. Suppose for a contradiction that there exists $u^{+*} \in K_{u^+}$ non-adjacent to $x$. By condition (5) of blowups, $u^{+*}u, u^{+*}u^{++} \in E(G)$. Hence $xu^{++} \notin E(G)$ for otherwise $\{x, u^{++}, u^{+*}, u\}$ induces a $C_4$. But now, either $\{x, u^{++*}, u^{+*}, u\}$ induces a $C_4$ (if $u^{+*}u^{++*} \in E(G)$) or $\{x, u^{++*}, u^{++}, u^{+*}, u\}$ induces a $C_5$ (if $u^{+*}u^{++*} \notin E(G)$), a contradiction. This proves (8).

*(9) $K_u \cup K_y$ is a clique.*

Since by (4) $N_A(y) = N_A[u]$, $u$ cannot be an isolated vertex of $H_y$. Hence, $uy$ is a solid edge. So, by condition (4) of blowups, $K_u$ is complete $K_y$. This proves (9).

We define $B_0 = B^* \setminus N_{K_y}(x)$.

Now the sets $K_v$ for all $v \in (A \cup I \cup A') \setminus \{u, u^+\}$, $K_u \cup N_{K_y}(x)$, $K_{u^+} \cup \{x\}$, $B_0$ and $B'^*$ form a preblowup of $F_0$. All conditions are easy to check. In particular, $K_u \cup N_{K_y}(x)$ is a clique by (9), $K_{u^+} \cup \{x\}$ is a clique by (8), conditions (1), (2) and (9) follows from (5), condition (1a) from (7), condition (1b) holds because $x$ is complete to $N_{K_y}(x)$, condition (9a) follows from (3) and condition (2a) holds because (6) implies that if $w \in B$ then $w \in B_0$.

Hence, by Lemma 4.4.6, $G[V(F_1) \cup \{x\}]$ is a proper blowup of some twinless odd $\ell$-template with $k$ principal paths that is an induced subgraph of $G$, a contradiction to the maximality of $F_1$. $\square$

### 4.5.4 Attaching a component

**Lemma 4.5.16** *If $D$ is a connected component of $G \setminus F_2$, then $N(D)$ is a clique.*

*Proof.* Suppose that $N(D)$ is not a clique. By Lemma 4.5.2, $N_{V(F_1)}(D)$ is not a clique. So, there exist $a$ and $b$ in $D$ such that $N_{V(F_1)}(a) \cup N_{V(F_1)}(b)$ is not a clique, and a path $P$ from $a$ to $b$ in $D$. We choose $a$ and $b$ subject to the minimality of the length of $P$. By Lemma 4.5.15, $a \neq b$ (so $P$ has length at least 1).

We set $S^*_a = N_{V(F_1)}(a)$ and $S^*_b = N_{V(F_1)}(b)$. By Lemma 4.5.15, $S^*_a$ and $S^*_b$ are both cliques. Note that possibly $S^*_a \cap S^*_b \neq \emptyset$. We denote by $\mathrm{int}(P)$ the set of the internal vertices of $P$. We set $S^*_\circ = N_{V(F_1)}(\mathrm{int}(P))$.

We set $S_a = \{t \in V(F_0) : S_a^* \cap K_t \neq \emptyset\}$. We define $S_b$ and $S_\circ$ similarly. Note that $S_a$ is possibly not included in $S_a^*$, and the same remark holds for $S_b$ and $S_\circ$.

*(1) There exist non-adjacent $x_a^* \in S_a^*$ and $x_b^* \in S_b^*$. Moreover, for all such $x_a^*$ and $x_b^*$, $x_a^*aPbx_b^*$ is a path.*

The existence of $x_a^*$ and $x_b^*$ follows from the definition of $a$ and $b$, and $x_a^*aPbx_b^*$ is a path because of the minimality of $P$. This proves (1).

*(2) $S_a^* \cup S_\circ^*$ and $S_b^* \cup S_\circ^*$ are cliques (in particular, $S_\circ^*$ is a (possibly empty) clique of $F_1$ that is complete to both $S_a^* \setminus S_\circ^*$ and $S_b^* \setminus S_\circ^*$).*

If $S_a^* \cup S_\circ^*$ is not a clique, then let $x^*y^*$ be a non-edge in $S_a^* \cup S_\circ^*$. Since $S_a^*$ is a clique by Lemma 4.5.15, we may assume $y^* \in S_\circ^*$. By definition of $S_\circ^*$, $y^*$ has a neighbor in $\text{int}(P)$, and then $x^*, y^*$ and some subpath of $P$ contradict the minimality of $P$.

The proof is similar for $S_b^* \cup S_\circ^*$. This proves (2).

Note that while $S_a^* \cup S_b^*$ is not a clique by assumption, it might be that $S_a \cup S_b$ is a clique (for instance when $S_a = \{u\}$, $S_b = \{v\}$ and $uv$ is an optional edge of $F_0$).

*(3) $S_a \cup S_\circ$ and $S_b \cup S_\circ$ are cliques of $F_0$ (in particular, $S_a$ and $S_b$ are (non-empty) cliques of $F_0$ and $S_\circ$ is a (possibly empty) clique of $F_0$ that is complete to both $S_a \setminus S_\circ$ and $S_b \setminus S_\circ$).*

If $S_a \cup S_\circ$ is not a clique, then let $xy$ be a non-edge of $S_a \cup S_\circ$. Since $x \in S_a \cup S_\circ$, there exists $x^* \in K_x \cap (S_a^* \cup S_\circ^*)$ and $y^* \in K_y \cap (S_a^* \cup S_\circ^*)$. By condition (3) of blowups, since $xy \notin E(G)$, $K_x$ is anticomplete to $K_y$. So, $x^*y^* \notin E(G)$, a contradiction to (2).

The proof is similar for $S_b \cup S_\circ$. This proves (3).

*(4) If a hole $C$ of $F_1$ contains two non adjacent vertices $x \in S_a^*$ and $y \in S_b^*$, then $P$ and $C$ form a pyramid $\Pi_{C,x,y}$. More specifically, $C$ contains a vertex $z$ such that either:*

- $S_a^* \cap V(C) = \{x, z\}$, $S_b^* \cap V(C) = \{y\}$ *; the apex of $\Pi_{C,x,y}$ is $y$, its triangle is $axz$, and its three paths, all of length $\ell$, are the path from $x$ to $y$ in*

*$C \setminus z$, the path from $y$ to $z$ in $C \setminus x$, and the path from $a$ to $y$ obtained by adding the edge $by$ to $P$ ; or*

- *$S_b^* \cap V(C) = \{y, z\}$, $S_a^* \cap V(C) = \{x\}$ ; the apex of $\Pi_{C,x,y}$ is $x$, its triangle is $byz$, and its three paths, all of length $\ell$, are the path between $y$ and $x$ in $C \setminus z$, the path from $z$ to $x$ in $C \setminus y$, and the path from $b$ to $x$ obtained by adding the edge $ax$ to $P$.*

Note that since $S_a^*$ is a clique, $S_a^* \cap V(C)$ contains $x$ and at most one other vertex which should be adjacent to $x$. The same holds for $S_b^*$ and $y$.

Let us assume that $S_\circ^* \cap V(C) \neq \emptyset$. Then by (2), there exists a unique vertex $t \in S_\circ^* \cap V(C)$, $S_a^* \cap V(C) \subseteq \{x, t\}$ and $S_b^* \cap V(C) \subseteq \{y, t\}$. Hence $C$ and $P$ form a proper wheel centered at $t$, a contradiction to Lemma 4.2.6. So, $S_\circ^* \cap V(C) = \emptyset$.

If $a$ and $b$ have a common neighbor $t$ in $C$, then $x$ and $y$ are the two neighbors of $t$ in $C$ and so, $C$ and $P$ form a proper wheel centered at $t$, again a contradiction to Lemma 4.2.6. So the neighborhoods of $a$ and $b$ in $C$ are disjoint.

From this, we obtain that $C$ and $P$ form a theta, a long prism or a pyramid. So, by Lemma 4.2.6, $C$ and $P$ form a pyramid whose three paths have length $\ell$. This can happen only if we are in one of the two cases described in (4). This proves (4).

*(5) $S_a \cap I = S_b \cap I = \emptyset$.*

Otherwise, up to symmetry, $S_a \cap I \neq \emptyset$. So, there exists a principal path $P_u = u \dots u'$ of $F_0$ whose interior intersects $S_a$. By (3), $S_a$ is a clique, so $1 \leq |S_a| \leq 2$ and $S_a \subseteq V(P_u)$. We now break into three cases.

**Case 1:** $S_b \subseteq V(P_u)$.

By (1) there exist vertices $x_a$ and $x_b$ of $P_u$ such that there exist non adjacent vertices $x_a^* \in S_a^* \cap K_{x_a}$ and $x_b^* \in S_b^* \cap K_{x_b}$.

We first show that there exist such $x_a$ and $x_b$ that are not adjacent. Otherwise, and since $S_a, S_b \subseteq V(P_u)$, we have that $S_a \cup S_b = \{x_a, x_b\}$. By replacing $x_a$ and $x_b$ by $x_a^*$ and $x_b^*$ in any principal hole $C$ containing $P_u$ we obtain a path $P_C$ of length $2\ell$ and $V(P_C) \cup V(P)$ induces a path of length at least $2\ell + 3$, a contradiction. So we may assume that $x_a$ and $x_b$ are not adjacent.

Let $C$ be any principal hole of $F_0$ that contains $P_u$. By Lemma 4.5.3, $\{x_a^*, x_b^*\} \cup (V(C) \setminus \{x_a, x_b\})$ induces a hole $C^*$. Let us apply (4) to $C^*$, $x_a^*$ and $x_b^*$. We obtain that the shortest path in $C^*$ between $x_a^*$ and $x_b^*$ has length $\ell$. However $x_a^*$ and $x_b^*$ both belong to the path of length $\ell - 1$, contained in $C^*$, which is obtained from $P_u$ by replacing $x_a$ by $x_a^*$ and $x_b$ by $x_b^*$, a contradiction.

**Case 2:** $S_b$ contains a vertex of some principal path $P_v$ distinct from $P_u$. Up to symmetry, since $S_b$ is a clique (by (3)), we assume that $b$ is anticomplete to $K_{v'}$.

Let $y$ be the vertex of $P_u$ closest to $u'$ such that $a$ has a neighbor $y^* \in K_y$. Let $z$ be the vertex of $P_v$ closest to $v$ such that $b$ has a neighbor $z^* \in K_z$. Possibly $y = u'$ and $z = v$, but $z \neq v'$ and $y \neq u$ since $a$ has a neighbor in $I^*$ by assumption. In particular, $yz \notin E(G)$. By condition (3) of blowups, $y^*z^* \notin E(G)$.

Let $C$ be the principal hole of $F_0$ that contains $P_u$ and $P_v$. By Lemma 4.5.3, $\{y^*, z^*\} \cup (V(C) \setminus \{y, z\})$ induces a hole $C^*$. Applying (4) to $C^*$, $y^*$ and $z^*$, we obtain that $P$ has length $\ell - 1$. We denote by $P_u^*$ the path obtained from $P_u$ by replacing $y$ by $y^*$ and by $P_v^*$ the path obtained from $P_v$ by replacing $z$ by $z^*$. Let $P^*$ be the path $vP_v^*z^*bPay^*P_u^*u'$ (in case $z = v$ one should replace $vP_v^*z^*$ by $z^*$, and in case $y = u'$ one should replace $y^*P_u^*u'$ by $y^*$). The length of $P^*$ is at least $\ell + 1$.

Consider now any principal path $P_r$ for $r \in A \setminus \{u, v\}$. Depending on the adjacencies of $r$ with $u$ and $v$, one of $rvP^*u'r'P_rr$ or $rwvP^*u'r'P_rr$ or $rvP^*u'w'r'P_rr$ or $rwvP^*u'w'r'P_rr$ (with possibly $u'$ replaced by $y^*$ when $u' = y$) is a cycle of length at least $2\ell + 2$ with at most one chord that must be $br$ (observe that $ar'$ cannot be an edge since $S_a \subseteq V(P_u)$). The only possibility which avoids a hole of forbidden length is if $z = v$, $y = u'$ and $br, u'r', vr$ are edges of $G$. This proves that $v$ is complete to $A \setminus \{u, v\}$ and $u'$ is complete to $A' \setminus \{u', v'\}$.

Hence, $G[A]$ has at most one isolated vertex (namely $u$), and $G[A']$ has at most one isolated vertex (namely $v'$). This contradicts $(A, B, A', B', I, w, w')$ being a proper $\ell$-partition of $F_0$.

**Case 3:** we are neither in Case 1 nor in Case 2.

Since we are not in Case 1, $S_b$ contains a vertex of $F_0 \setminus P_u$, and since we are not in Case 2, this vertex must be in $B \cup B'$. Up to symmetry, we assume that $S_b \cap B \neq \emptyset$. Since $S_b$ is a clique (by (3)), $S_b \cap (B' \cup A' \cup I) = \emptyset$. Since we are not in Case 2, $S_b \cap (A \setminus \{u\}) = \emptyset$. Hence, $S_b \subseteq B \cup \{u\}$.

Suppose that some vertex $x \in B \cap S_b$ is such that $u \in H_x$ and let $x^* \in K_x \cap S_b^*$. Let $v \in H_x$ be non-adjacent to $u$ (this is possible since $G[H_x]$ is anticonnected). So, $P_u$, $P_v$ and $x$ form a hole $C$ (possibly not principal). Let $z$ be the vertex in $I \cap S_a$ which is the closest to $u$ in $P_u$ and let $z^* \in K_z \cap S_a^*$. By Lemma 4.5.3, $\{x^*, z^*\} \cup V(C) \setminus \{x, z\}$ induces a hole $C^*$ of $F_1$ ($ux^*, vx^* \in E(G)$ by condition (6) of blowups). The distance between $x^*$ and $z^*$ in $C^*$ is at most $\ell - 1$ since $P_u$ has length $\ell - 1$, a contradiction to (4) applied to $C^*$, $x^*$ and $z^*$ (because one path of the pyramid obtained by (4) is a subpath of $x^* u P_u z^*$, even when $bu \in E(G)$). Hence, from here on, we may assume that no vertex $x \in B \cap S_b$ is such that $u \in H_x$.

Let $u_a$ be the vertex of $S_a$ which is the closest to $u$ in $P_u$ and let $u'_a$ be the vertex of $S_a$ which is the closest to $u'$ in $P_u$. Notice that, since $S_a$ is a clique (by (3)), either $u_a = u'_a$ or $u_a u'_a$ is an edge. So it may be that $u_a = u$ or $u'_a = u'$ but not both. Let now $u_a^* \in K_{u_a} \cap S_a^*$ and $u'^*_a \in K_{u'_a} \cap S_a^*$. We denote by $P_u^*$ the path obtained from $P_u$ by replacing $u_a$ and $u'_a$ by respectively $u_a^*$ and $u'^*_a$ (note that $u_a^* u'^*_a \in E(G)$ since $S_a^*$ is a clique).

Suppose that some vertex $x \in B \cap S_b$ is such that $u \in N(H_x)$ and let $x^* \in K_x \cap S_b^*$. Since $S_b \subseteq B \cup \{u\}$, we have $S_b \cap H_x = \emptyset$. Depending on whether $b$ is adjacent to $u$ or not, one of $u_a^* a P b u P_u^* u_a^*$ or $u_a^* a P b x^* u P_u^* u_a^*$ (in case $u_a = u$ one should replace $u P_u^* u_a^*$ by $u_a^*$) is a hole, implying that $P$ has length at least $\ell$. Let $v \in H_x$, then $uv$ is an edge (since $u \in N(H_x)$) and $u'v'$ is not an edge by definition of the template. So, $x^* b P a u'^*_a P^* u' w' v' P_v v x^*$ (in case $u'^*_a = u'$ one should replace $u'^*_a P^* u'$ by $u'^*_a$) is a hole of length at least $2\ell + 4$, a contradiction.

Since we are in Case 3, there exists some vertex $x \in B \cap S_b$ and by the two paragraphs above, $u \notin N[H_x]$ and $S_b \cap N[H_x] = \emptyset$. Hence $x^* \neq w$ where $x^* \in K_x \cap S_b^*$. Depending on whether $b$ is adjacent to $u$ or not, one of $u_a^* a P b x^* w u P_u^* u_a^*$ or $u_a^* a P b u P_u^* u_a^*$ is a hole (in case $u_a = u$ one should replace $u P_u^* u_a^*$ by $u_a^*$), implying that $P$ has length at least $\ell - 1$. So, for some $v \in H_x$, the hole $x^* b P a u'^*_a P_u^* u' v' P_v v x^*$ (in case $u'_a = u'$ one should replace $u'^*_a P^* u'$ by $u'^*_a$) has length at least $2\ell + 2$, a contradiction. This proves (5).

*(6) We may assume that $S_a \subseteq A \cup B$ and $S_b \subseteq A' \cup B'$.*

Otherwise, by (5) and since $S_a$ and $S_b$ are cliques (by (3)), we may assume that $S_a, S_b \subseteq A \cup B$.

We claim that there exist non-adjacent vertices $x^* \in S_a^*$ and $y^* \in S_b^*$, and a path $Q^*$ from $x^*$ to $y^*$ of length at least $2\ell - 1$ that forms a hole together with $P$. This is a contradiction because it implies that $P$ has length at most 0. So, to conclude the proof, it remains to prove the existence of $Q^*$.

By (1), there exist non-adjacent $x_a^* \in S_a^*$ and $x_b^* \in S_b^*$. Let $x_a$ and $x_b$ be the vertices of $F_0$ such that $x_a^* \in K_{x_a}$ and $x_b^* \in K_{x_b}$. Note that possibly $x_a x_b$ is an edge, but this happens only if $x_a x_b$ is an optional edge of $F_0$ (since $x_a^* x_b^*$ is not an edge). We break into three cases.

**Case 1:** $x_a, x_b \in A$.

Then $x_a x_b \notin E(G)$ (otherwise it would be a solid edge of $F_0$), so from the definition of templates, there exists a path $Q$ of length $2\ell - 1$ from $x_a$ to $x_b$ whose interior is in $I \cup A'$. By Lemma 4.5.3, $\{x_a^*, x_b^*\} \cup (V(Q) \setminus \{x_a, x_b\})$ induces the path $Q^*$ that we are looking for. Note that $Q^*$ and $P$ form a hole by (2) and our assumption that $S_a, S_b \subseteq A \cup B$.

**Case 2:** $x_a \in A$ and $x_b \in B$.

Whether $x_a x_b$ is an optional edge or a non-edge, an immediate consequence of the definition of a template is that there exists a vertex $z \in H_{x_b}$ that is non-adjacent to $x_a$. We may furthermore assume that $z \notin S_b$ since else we are in the same situation as in Case 1. By definition of a template, there exists a path $Q_0$ of length $2\ell - 1$ between $x$ and $z$ whose interior is in $I \cup A'$. Then $x_b z Q_0 x_a$ is a path of length $2\ell$ and by replacing in this path $x_a$ and $x_b$ by respectively $x_a^*$ and $x_b^*$, we obtain by Lemma 4.5.3 a path $Q^*$ of the same length. Note that $Q^*$ and $P$ form a hole by (2) and our assumption that $S_a, S_b \subseteq A \cup B$, $z \notin S_b$ and $z \notin S_a$ since $S_a$ is a clique.

**Case 3:** $x_a, x_b \in B$.

Then $x_a x_b \notin E(G)$ (otherwise it would be a solid edge of $F_0$). Hence, by Lemma 4.3.7, $H_{x_a} \cup \{x_a\}$ is anticomplete to $H_{x_b} \cup \{x_b\}$. So, let $u_a \in H_{x_a}$ and $u_b \in H_{x_b}$, there exists then a path $Q_0 = u_a \dots u_b$ of length $2\ell - 1$ with interior in $I \cup A'$. By Lemma 4.5.3, $Q^* = x_a^* u_a Q_0 u_b x_b^*$ is also a path, it is of length $2\ell + 1$. We may assume that $u_a \notin S_a$ and $u_b \notin S_b$ since else we are in the same situation as in Case 2. Now, by (2), $Q^*$ and $P$ form a hole of length at least $2\ell + 4$.

This proves (6).

*(7)* $S_\circ = \emptyset$.

By (6) and (3), if $S_\circ \neq \emptyset$, then $\ell = 3$, and there exists a principal path $P_u = u \dots u'$ of $F_0$ such that $S_a = \{u\}$, $S_b = \{u'\}$ and $S_\circ = \{c\}$ where $c$ is the unique internal vertex of $P_u$. Let $u^* \in K_u \cap S_a^*$, $c^* \in S_\circ^*$ and $u'^* \in K_{u'} \cap S_b^*$. Observe that, from the rules of the blowup, each of $u^*c^*, u'^*c^*$ may be an edge or a non-edge of $G$. By definition, $c^*$ has a neighbor in $\text{int}(P)$.

We claim that $c^*u^*, c^*u'^* \in E(G)$. If $c^* = c$ this follows from condition (5) of blowups, so suppose $c^* \neq c$. Then $P$, $c$, $c^*$, $u^*$ and $u'^*$ form a theta or a non-twin wheel $W$ centered at $c^*$. So, by Lemma 4.2.6, $W$ is a universal wheel and again $c^*u^*, c^*u'^* \in E(G)$.

Let $P_v = v \dots v'$ be a principal path distinct from $P_u$ and suppose up to symmetry that $uv \in E(G)$. Now, $P_v$, $P$, $u^*$, $u'^*$ $w'$ and $c^*$ form a proper wheel centered at $c^*$, a contradiction to Lemma 4.2.6. This proves (7).

*(8) $P$ has length $\ell - 1$, or $P$ has length $\ell - 2$ and we may assume that $S_a \cap A = \emptyset$.*

By (1) and (6), consider $x \in S_a \cap (A \cup B)$ and $y \in S_b \cap (A' \cup B')$. Let $x^* \in K_x \cap S_a^*$ and $y^* \in K_y \cap S_b^*$.

If $x \in A$ and $y \in A'$, then let $C$ be a principal hole that contains $x$ and $y$. By Lemma 4.5.3, $\{x^*, y^*\} \cup (V(P) \setminus \{x, y\})$ induces a hole $C^*$. We may apply (4) to $C^*$, $x^*$ and $y^*$. It follows that $P$ has length $\ell - 1$. By symmetry we may therefore assume from here on that $S_a \cap A = \emptyset$.

Let $y'^*$ be a vertex in $S_b^*$ which is the closest to $x^*$ in $F_1$. By Lemma 4.5.9, there exists a path $Q$ in $F_1$ from $x^*$ to $y'^*$ of length $\ell$ or $\ell + 1$. From our assumption on $y'^*$ we get that $Q$ and $P$ form a hole (since $S_\circ = \emptyset$ by (7)). Therefore, if $Q$ has length $\ell$, then $P$ has length $\ell - 1$ and if $Q$ has length $\ell + 1$, then $P$ has length $\ell - 2$. This proves (8).

We may now conclude the proof.

If $P$ has length $\ell - 1$, then we set $A_0 = A \cup \{a\}$, $A'_0 = A' \cup \{b\}$ and $I_0 = I \cup \text{int}(P)$. We claim that $(A_0, B, A'_0, B', I_0)$ is an $\ell$-pretemplate partition of $G[A_0 \cup B \cup A'_0 \cup B' \cup I_0]$. All conditions are easily checked to hold (in particular conditions (1), (2) and (7) are satisfied because by (6), $a$ (resp. $b$) has a neighbor in $G[A \cup B]$ (resp. $G[A' \cup B']$), condition (5) holds by (7) and conditions (8) and (9) hold because they hold in $F_0$). Then, by Lemma 4.3.14, $G$ contains an odd $\ell$-template with $k + 1$ principal paths, a contradiction to the maximality of $k$.

By (8), $P$ has length $\ell - 2$ and we may assume that $S_a \cap A = \emptyset$ and $S_a \cap B \neq \emptyset$ (recall that by (6), $S_a \subseteq A \cup B$ and $S_b \subseteq A' \cup B'$). Let us choose $x \in S_a \cap B$ such that $H_x$ is maximal (note that $x$ is unique because $S_a \cap B$ is a clique and $F_0$ is twinless). Let $x^* \in K_x \cap S_a^*$. We set $A_0 = A \cup \{x^*\}$, $B_0 = B \setminus S_a$, $A'_0 = A' \cup \{b\}$ and $I_0 = I \cup \text{int}(P) \cup \{a\}$. Note that the path $x^*aPb$ has length $\ell - 1$ and has interior in $I_0$. We break into two cases.

**Case 1:** $b$ has a neighbor in $A' \cup B'^*$.

We claim that in that case $(A_0, B_0, A'_0, B'^*, I_0)$ is an $\ell$-pretemplate partition of $G[A_0 \cup B_0 \cup A'_0 \cup B'^* \cup I_0]$. All conditions are easily checked to hold (in particular condition (7) is satisfied for $A_0 \cup B_0$ because if $x = w$, then $x^*$ is complete to $(A_0 \cup B_0) \setminus \{x^*\}$, and otherwise, by the maximality of $H_x$, $w \in A_0 \cup B_0$, condition (7) is satisfied for $A'_0 \cup B'^*$ because $b$ has a neighbor in $A' \cup B'^*$ and by the rules of the blowup, conditions (1), (2), (8) and (9) hold because they hold in $F_0$ and by the rules of the blowup). Then, by Lemma 4.3.14, $G$ contains an odd $\ell$-template with $k + 1$ principal paths, a contradiction to the maximality of $k$.

**Case 2:** $b$ has no neighbor in $A' \cup B'^*$.

Then, by (6) there exists $x'^* \in K_{x'} \cap S_b^*$ for some $x' \in A'$.

Let $A'_1 = (A'_0 \cup \{x'^*\}) \setminus \{x'\}$. If $w' \in B'$ we set $B'_1 = \{w'\}$ and else we set $B'_1 = \emptyset$. We claim that $(A_0, B_0, A'_1, B'_1, I_0)$ is an $\ell$-pretemplate partition of $G[A_0 \cup B_0 \cup A'_1 \cup B'_1 \cup I_0]$.

Most conditions are easily checked to hold as in the previous case. Notice that conditions (7) and (9) hold because $x'^*$ is by definition adjacent to $b$ and by the rules of the blowup, $G[A'_1 \setminus \{b\}]$ is isomorphic to $G[A'_0 \setminus \{b\}]$ and $x'^*$ is adjacent to $w'$. Then, by Lemma 4.3.14, $G$ contains an odd $\ell$-template with $k + 1$ principal paths, a contradiction to the maximality of $k$. □

### 4.5.5 End of the proof

We may now conclude the proof of Lemma 4.5.1. If $G \setminus F_1$ is empty, then conclusion (1) holds. If $G \setminus F_1$ is non-empty and $G \setminus F_2$ is empty, then conclusion (2) holds. Otherwise, we consider a connected component $D$ of $G \setminus F_2$ and apply Lemma 4.5.16. We then see that $G$ has a clique cutset, so conclusion (3) holds.

## 4.6 Proof of Theorem 4.1.1

**Theorem 4.6.1** *Let $\ell \geq 3$ be an integer. If $G$ is a graph in $\mathcal{C}_{2\ell+1}$ then one of the following holds:*

1. *$G$ is a ring of length $2\ell + 1$;*
2. *$G$ is a proper blowup of a twinless odd $\ell$-template;*
3. *$G$ has a universal vertex or*
4. *$G$ has a clique cutset.*

*Proof.* By Lemma 4.2.6, $G$ contains no long prism no theta and no proper wheel. Also, clearly $G$ contains no $C_4$ and no $C_5$. Hence, by Theorem 4.2.8, we may assume that $G$ contains a pyramid for otherwise one of the conclusions (1), (3) or (4) holds. The result then follows from Lemma 4.5.1. □

## 4.7 Further work

In this chapter, we presented the structure of graphs in $\mathcal{C}_k$ when $k$ is odd. One of the basic classes is new and fully described. Theorem 4.1.1 could be used as a decomposition theorem.

As said in the section 4.1, the final work with Linda Cook, Jake Horsfield, Myriam Preissmann, Paul Seymour, Ni Luh Dewi Sintiari, Nicolas Trotignon and Kristina Vušković will include a similar result for $\mathcal{C}_k$ when $k$ is even and at least 8. We are also working on an algorithm that would recognize graphs in $\mathcal{C}_k$ with a running time smaller than the one presented in section 4.1.

We wonder if the result on the structure of graphs in $\mathcal{C}_k$ when $k$ is odd could be generalized to have a structural theorem for even-hole free graphs in Free$\{C_5, \text{proper wheel}\}$ by relaxing the constraints on the length of the principal paths.

# Chapter 5

# The class Free$\{C_4, 4K_1\}$

## 5.1 Introduction

In this chapter, we focus on the class Free$\{C_4, 4K_1\}$. It is one of the three minimal open cases identified by Lozin and Malyshev [40] (see Chapter 3) where the complexity of the coloring problem is still unknown.

An interesting fact about Free$\{C_4, 4K_1\}$ is that, for every graph $G$ in the class, the only possible holes in $G$ are $C_5$, $C_6$ and $C_7$. This is because, every hole of length at least 8 contains a $4K_1$. Furthermore, $G$ does not contain antiholes of length at least 6 because every such antihole contains a $C_4$. Hence the class of graphs Free$\{C_4, 4K_1, C_6\}$ is a subclass of even-hole-free graphs and the class of graphs Free$\{C_4, 4K_1, C_5, C_7\}$ is a subclass of perfect graphs.

In [25], Fraser et al. proved that graphs in Free$\{C_4, 4K_1, C_5\}$ that contain a $C_7$ have bounded clique-width. Hence, by Theorem 3.2.3, the coloring problem is polynomial time solvable when restricted to graphs in Free$\{C_4, 4K_1, C_5\}$ that contain a $C_7$. Since graphs in Free$\{C_4, 4K_1, C_5, C_7\}$ are perfect and since perfect graphs can be colored in polynomial time (see section 3.3), it follows that the coloring problem is polynomial time solvable for graphs in Free$\{C_4, 4K_1, C_5\}$. Later the same authors ([24]) proved that the coloring problem is polynomial time solvable for graphs in Free$\{C_4, 4K_1, C_6\}$ that contain a $C_7$. They also proved that the coloring problem is polynomial time solvable for graphs in Free$\{C_4, 4K_1, C_6, C_5 - twin\}$. A $C_5 - twin$ is displayed on figure 5.1.

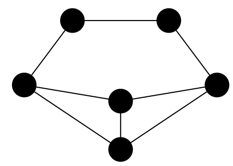


Figure 5.1: $C_5 - twin$

In this chapter, we present partial results and some thoughts about two subclasses of Free$\{C_4, 4K_1\}$. The first is the class of graphs in Free$\{C_4, 4K_1\}$ that contain an icosahedron (see figure 5.2). The second is the class of graphs in Free$\{C_4, 4K_1\}$ whose vertex set can be partitioned into 3 cliques. Both of these subclasses contain $C_5$ and $C_6$.

## 5.2 When the graph contains an icosahedron

Given a graph $H$, a *perfect blowup* of $H$ is any graph obtained from $H$ by replacing each vertex $v \in V(H)$ by a clique $K_v$ with the following property: $v \in K_v$ and for all $u, v \in V(H)$, $K_u$ and $K_v$ are complete to each other if $uv \in E(H)$, and anticomplete otherwise. In other words, a graph $G$ is a perfect blowup of a graph $H$, if it contains $H$ and if there exists a partition of $V(G)$ into $|V(H)|$ cliques such that, if we take one vertex in each clique, the subgraph of $G$ induced by those vertices is isomorphic to $H$. In this chapter, the notion of perfect blowup is similar but not the same as the notion of blowup in chapter 4. Indeed, in this case, two blown-up cliques are either complete or anticomplete to each other (depending on the adjacency between the two initial vertices).

A graph $G$ is an *extended perfect blowup* of a graph $H$ if there exists $V' \subseteq V(G)$ such that $G[V']$ is a perfect blowup of $H$ and $V(G) \setminus V'$ is a clique complete to $V'$ (called a universal clique). It is easy to see that if $H$ is in Free$\{C_4, 4K_1\}$ then any extended perfect blowup of $H$ is also in Free$\{C_4, 4K_1\}$.

For a set of graphs $\mathcal{H}$, a graph $H$ is a *$\mathcal{H}$-fixer* if $H$ is twinless and every graph in Free $\mathcal{H}$ containing $H$ is an extended perfect blowup of $H$. In this section we are interested in $\{C_4, 4K_1\}$-fixers.

The *clique-width* of a graph $G$ , denoted by $cw(G)$ is the minimum number of labels necessary to build $G$ using the four following operations:

1. Create a vertex $u$ labelled with integer $\ell$.
2. Make the disjoint union of two already built graphs.
3. Add edges between all vertices with label $i$ and all vertices with label $j$.
4. Relabel all vertices of label $i$ with label $j$.

As explained in Chapter 3, the following result is proved in [49].

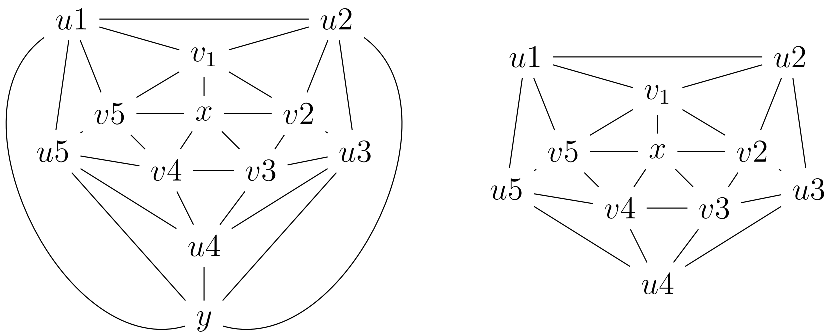
**Theorem 5.2.1 ([49])** *The coloring problem is polynomial time solvable in classes of graphs with bounded clique-width.*

Among other results, it is proved in [7] that the class Free$\{C_4, 4K_1\}$ has unbounded clique-width. But some subclasses might have bounded clique-width. A fact evidenced in [18] is that for any graph $G$, $cw(G) \geq cw(H)$ for every graph $H$ contained in $G$. The following lemma is a corollary from Proposition 1 in [8] and was proved in [16].

**Lemma 5.2.2** *For every graph $H$ with at least one edge and every graph $G$ that is an extended perfect blowup of $H$, $cw(G) = cw(H)$.*

*Proof.* By definition of extended perfect blowup, there exists $V' \subseteq V(G)$ such that $G[V']$ is a perfect blowup of $H$ and $V(G) \setminus V'$ is a universal clique. Set $G' = G[V']$. Note that $cw(G')$ is at least 2 because $H$ has at least one edge. Observe that $cw(G) = cw(G')$. This is because it is always possible to add a universal vertex to a graph of clique width at least 2 without using an additional label.

Proposition 1 in [8] states that the clique-width of a graph is the maximum of the clique-width of its *prime* subgraphs (graphs that only have trivial modules, i.e. modules of cardinality at most 1 or equal to the entire graph). It is easy to see that any prime graph either is a twinless graph or has at most 3 vertices. By the definition of a perfect blowup, the only subgraphs of $G'$ with more than 3 vertices that are prime are induced by taking at most one vertex from each blown up clique. Since all graphs of order 3 have clique width at most 2, it follows that all prime subgraphs of $G$ of clique-width at least 3 are




$ICO^-$

Icosahedron

Figure 5.2: Icosahedron and $ICO^-$

induced subgraphs of $H$. Hence the maximum of the clique-width of all prime subgraphs of $G'$ is lower or equal to $cw(H)$ and so $cw(G') \leq cw(H)$. Since $H$ is contained in $G$, $cw(H) \leq cw(G)$ and so $cw(G) = cw(G') = cw(H)$. $\square$

By Lemma 5.2.2 any graph $G$ in Free$\{C_4, 4K_1\}$ containing a $\{C_4, 4K_1\}$-fixer $H$, has $cw(G) = cw(H)$. By Lemma 5.2.1, it follows that the coloring problem is polynomial-time solvable for graphs in Free$\{C_4, 4K_1\}$ containing a given $\{C_4, 4K_1\}$-fixer.

The *icosahedron* (see figure 5.2) is the skeleton graph of the platonic solid of the same name. It is vertex-transitive and edge-transitive. The neighbourhood of each of its vertices is exactly a $C_5$. The icosahedron is a graph in Free$\{C_4, 4K_1\}$. We denote by $ICO^-$ the graph obtained from the icosahedron by deleting one vertex (recall that all vertices of an icosahedron are equivalent). We now prove that the icosahedron is a $\{C_4, 4K_1\}$-fixer and that $ICO^-$ is a $\{C_4, 4K_1, \text{icosahedron}\}$-fixer.

**Lemma 5.2.3** *The icosahedron is a $\{C_4, 4K_1\}$-fixer.*

*Proof.* Let $G$ be a graph in $Free\{C_4, 4K_1\}$ that contains an icosahedron $H$. We prove that $G$ is an extended proper blowup of the icosahedron.

Vertices of $H$ are denoted by $\{x, v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5, y\}$ as displayed in Figure 5.2. Denote by $H^*$ a blowup of $H$ contained in $G$, with a maximum number of vertices. The blown-up cliques in $H^*$ are denoted by $\{X, V_1, V_2, V_3, V_4, U_5, U_1, U_2, U_3, U_4, U_5, Y\}$ such that, for all $i \in \{1, 2, 3, 4, 5\}$ we have that $x \in X$, $u_i \in U_i$, $v_i \in V_i$ and $y \in Y$. Denote by $W$ the set of vertices in $V(G) \setminus V(H^*)$ that are complete to $H$.

Observe first that $G[W]$ is a clique for otherwise two nonadjacent vertices of $G[W]$ with any two nonadjacent vertices of $H$ induce a $C_4$, a contradiction. Furthermore, $G[W]$ is complete to $H^*$ because every vertex $a$ in $H^*$ has two nonadjacent neighbours $b$ and $c$ in $H$. Hence, if there is a vertex $w \in W$ nonadjacent to $a$, then $\{a, b, w, c\}$ induces a $C_4$, a contradiction.

If $V(G) = V(H^*) \cup W$ then $G$ is an extended perfect blowup of the icosahedron and we are done, so let $w$ be a vertex in $V(G) \setminus (V(H^*) \cup W)$.

Since $w \notin W$, $w$ has at least one non-neighbour in $H$. Furthermore, $w$ has at least one neighbour in $H$ for otherwise $\{w, u_1, u_3, x\}$ induces a $4K_1$, a contradiction. Since the icosahedron is connected, there is an edge $e$ in $H$ such that $w$ is adjacent to one end of $e$ and not adjacent to the other end of $e$. Since $H$ is vertex-transitive and edge-transitive, we may assume without loss of generality that $e = v_1x$ and that $wv_1 \in E(G)$ and $wx \notin E(G)$.

It follows that $w$ has at least one neighbour among $u_3$ and $u_5$, for otherwise $\{w, x, u_3, u_5\}$ induces a $4K_1$. By symmetry, choose $wu_5 \in E(G)$.

The conclusion follows from the next sequence of facts.

- $w$ is anticomplete to $V_3$ and to $V_4$ for otherwise it induces a $C_4$ with $x$, $v_1$ and a vertex in $V_3 \cup V_4$.
- $w$ is complete to $U_1$ and to $V_5$ for otherwise it induces a $C_4$ with $v_1$, $u_5$ and a vertex in $U_1 \cup V_5$.
- $w$ is anticomplete to $V_2$ for otherwise it induces a $C_4$ with $x$, any vertex of $V_5$ and a vertex in $V_2$.
- $w$ is anticomplete to $U_3$ for otherwise it induces a $C_4$ with $v_1$, any vertex of $V_2$ and a vertex in $U_3$.
- $w$ is anticomplete to $U_4$ for otherwise it induces a $C_4$ with any vertex of $V_4$, any vertex of $V_5$ and a vertex in $U_4$.
- $w$ is complete to $Y$ for otherwise it induces a $4K_1$ with any vertex of $V_2$, any vertex of $V_4$ and a vertex in $Y$.

- $w$ is complete to $U_2$ for otherwise it induces a $C_4$ with $v_1$, any vertex of $Y$ and a vertex in $U_2$.
- $w$ is complete to $U_5$ for otherwise it induces a $C_4$ with any vertex of $V_5$, any vertex of $Y$ and a vertex in $U_5$.
- $w$ is complete to $V_1$ for otherwise it induces a $C_4$ with any vertex of $U_2$, any vertex of $V_5$ and a vertex in $V_1$.
- $w$ is anticomplete to $X$ for otherwise it induces a $C_4$ with any vertex of $U_2$, any vertex of $V_2$ and a vertex in $X$.

By all the previous observations $w$ is a twin of $u_1$ and so, $H^* \cup \{w\}$ is a blowup of the icosahedron, a contradiction to the maximality of $V(H^*)$. □

**Lemma 5.2.4** *$ICO^-$ is a $\{C_4, 4K_1, icosahedron\}$-fixer.*

*Proof.* Let $G$ be a graph in $Free\{C_4, 4K_1, \text{icosahedron}\}$ that contains an $ICO^-$ denoted by $H$. We prove that $G$ is an extended perfect blowup of $ICO^-$ and we are done.

Since $G$ contains an $ICO^-$ then $G$ contains a blowup of $ICO^-$ denoted $H^*$. Choose $H^*$ subject to the maximality of its vertex set. Denote by $\{X, U_1, U_2, U_3, U_4, U_5, V_1, V_2, V_3, V_4, V_5\}$ the blown up cliques of $H^*$. Let $W$ be the set of vertices of $G$ that are complete to $H^*$.

Let $w$ be a vertex in $V(G) \setminus (V(H^*) \cup W)$. It exists for otherwise $V(G) = V(H^*) \cup W$ and we are done.

Let $H$ be an $ICO^-$ induced in $H^*$. Choose $H$ such that $N_H(w)$ is maximal. Therefore, if $w$ is nonadjacent to a certain vertex in $H$, $w$ is anticomplete to the corresponding blown up clique. Denote by $\{x, v_1, v_2, v_3, v_4, v_5, u_1, u_2, u_3, u_4, u_5\}$ the vertices of $H$ as displayed in Figure 5.2. Denote by $U$ the $C_5$ induced by $\{u_1, u_2, u_3, u_4, u_5\}$.

In all this prof, subscripts have to be considered modulo 5.

(*1) $w$ has at least two adjacent neighbours in $U$.*

We first show that $w$ has at least two neighbours in $U$. For otherwise, without loss of generality, $w$ is anticomplete to $\{u_1, u_2, u_3, u_4\}$ and $w$ is anticomplete to $\{U_1, U_2, U_3, U_4\}$ by choice of $H$. Therefore $w$ is complete to $X$ for otherwise,

$\{u_1, u_3, w\}$ with a non-neighbour of $w$ in $X$ induces a $4K_1$. Furthermore, if $w$ as a non-neighbour $a$ in $V_i$ for any $i \in \{2, 4, 5\}$ then $\{u_{i-1}, u_{i+2}, a, w\}$ induces a $4K_1$. Hence $w$ is complete to $\{V_2, V_4, V_5\}$. If $w$ as a non-neighbour $a$ in $V_i$ for any $i \in \{1, 3\}$ then $\{u_{i-1}, u_{i+1}, a, w\}$ induces a $C_4$. Hence $w$ is complete to $\{V_1, V_3\}$. Finally, $w$ is anticomplete to $U_5$ for otherwise $\{u_1, v_1, w\}$ with a non-neighbour of $w$ in $U_5$ induces a $C_4$. Now $w$ is a twin of $x$ and $H^*$ with $w$ induces a blowup of $ICO^-$, a contradiction with the maximality of $H^*$. Therefore $w$ has at least two neighbours in $U$.

Since $U$ is a $C_5$, if $w$ has at least three neighbours in $U$ then two of them are adjacent. If $w$ has exactly two neighbours that are nonadjacent, then there exists $i \in \{1, 2, 3, 4, 5\}$ such that $u_{i-1}$ and $u_{i+1}$ are the neighbours of $w$ in $U$. Now $\{u_{i-1}, u_i, u_{i+1}, w\}$ induces a $C_4$. Hence $w$ has at least two adjacent neighbours in $U$. This proves (1).

*(2) $w$ has at least two adjacent non-neighbours in $U$.*

Suppose that $w$ is not complete to $U$ and set $i \in \{1, 2, 3, 4, 5\}$ such that $wu_i \notin E(G)$. It follows that $w$ has a non-neighbour among $\{u_{i-1}, u_{i+1}\}$ for otherwise $\{u_{i-1}, u_i, u_{i+1}, w\}$ induces a $C_4$. Hence, if $w$ is not complete to $U$, the claim is proved.

Suppose that $w$ is complete to $U$. Since $G$ contains no icosahedron, $w$ has at least one neighbour in $\{v_1, v_2, v_3, v_4, v_5, x\}$. If $wx \notin E(G)$ then, by symmetry, $v_1w \in E(G)$. It follows that $wv_2 \in E(G)$ for otherwise $\{w, v_1, v_2, u_3\}$ induces a $C_4$. Therefore $v_3w \in E(G)$ for otherwise $\{v_2, v_3, u_4, w\}$ induces a $C_4$. But now, $\{w, v_1, x, v_3\}$ induces a $C_4$, a contradiction. So $wx \in E(G)$. For all $i \in \{1, 2, 3, 4, 5\}$ $w$ is adjacent to all $a \in V_i$, for otherwise, $\{w, u_i, a, x\}$ induces a $C_4$.

Hence $w$ is complete to $X$ since otherwise for any $x^* \in X$ nonadjacent to $w$, $\{w, v_1, x^*, v_3\}$ induces a $C_4$. Now for all $i \in \{1, 2, 3, 4, 5\}$, $w$ is adjacent to all $b \in U_i$, for otherwise, $\{w, u_{i+1}, v_{i-1}, a\}$ induces a $C_4$. Hence, $w$ is complete to $H^*$, a contradiction to the choice of $w$ not in $W$. This proves (2).

By (2) and without loss of generality, set $wu_1, wu_2 \notin E(G)$ and by the choice of $H$, $w$ is anticomplete to $U_1$ and $U_2$. By (1) and by symmetry, $u_3w, u_4w \in E(G)$. Now $w$ is anticomplete to $V_1$ for otherwise $\{w, v_1, u_2, u_3\}$ induces a $C_4$.

Suppose that $wx \in E(G)$. The contradiction follows from the next sequence of implications :

- $w$ is complete to $V_2$ and to $V_3$ for otherwise it induces a $C_4$ with $x$, $u_3$ and a vertex in $V_2 \cup V_3$.
- $w$ is complete to $V_4$ for otherwise it induces a $C_4$ with $u_4$, $x$ and a vertex in $V_4$.
- $w$ is anticomplete to $V_5$ for otherwise it induces a $C_4$ with $v_1$, $v_2$ and a vertex in $V_5$.
- $w$ is anticomplete to $U_5$ for otherwise it induces a $C_4$ with $v_5$, $x$ and a vertex in $U_5$.
- $w$ is complete to $X$ for otherwise it induces a $C_4$ with $v_2$, $v_4$ and a vertex in $X$.
- $w$ is complete to $U_3$ for otherwise it induces a $C_4$ with $v_2$, $u_4$ and a vertex in $U_3$.
- $w$ is complete to $U_4$ for otherwise it induces a $C_4$ with $v_4$, $u_3$ and a vertex in $U_4$.

Now $w$ is a twin of $v_3$, a contradiction to the maximality of $H^*$. Therefore $wx \notin E(G)$ and by the choice of $H$, $w$ is anticomplete to $X$.

- $w$ is complete to $U_4$ and $U_5$ for otherwise it induces a $4K_1$ with $x$, $u_2$ and a vertex in $U_4 \cup U_5$.
- $w$ is complete to $U_3$ for otherwise it induces a $4K_1$ with $x$, $u_1$ and a vertex in $U_3$.

If $w$ is adjacent to both $v_2$ and $v_5$ then $\{x, v_2, v_5, w\}$ induces a $C_4$. Without loss of generality, $wv_2 \notin E(G)$ and by the choice of $H$, $w$ is anticomplete to $V_2$.

- $w$ is complete to $V_4$ for otherwise it induces a $4K_1$ with $u_1$, $v_2$ and a vertex in $V_4$.
- $w$ is complete to $V_3$ for otherwise it induces a $C_4$ with $u_3$, $v_4$ and a vertex in $V_3$.

- $w$ is anticomplete to $V_5$ for otherwise it induces a $C_4$ with $x$, $v_3$ and a vertex in $V_5$.

By all the previous observations $w$ is a twin of $u_4$ and so, $H^* \cup \{w\}$ is an induced subgraph of a blowup of the icosahedron, a contradiction to the maximality of $V(H^*)$. □

The following is a corollary of Lemma 5.2.4.

**Corollary 5.2.5** *Graphs in* Free$\{C_4, 4K_1\}$ *containing* $ICO^-$ *have clique-width bounded by 12.*

*In addition the coloring problem is polynomial time solvable when restricted to graphs in* Free$\{C_4, 4K_1\}$ *containing* $ICO^-$.

*Proof.* If the graph contains an icosahedron, the result follows from Lemma 5.2.3 and Lemma 5.2.2.

If the graph does not contain an icosahedron, the result follows from Lemma 5.2.4 and Lemma 5.2.2.

The second part of the lemma follows from Theorem 5.2.1. □

Fixers of small order are interesting because any graph in the class containing a fixer has a rigid structure. The program in Appendix A searches for $\{C_4, 4K_1\}$-fixers. It generates all graphs of $G$ of order $n$ in Free$\{C_4, 4K_1\}$ and tries to attach an additional vertex $v$ with the following constraints : the resulting graph is in Free$\{C_4, 4K_1\}$, $v$ is not complete to $G$ and $v$ has no twin in $G$. After running the program in appendix A for all graphs of order at most 12, we were able to assert the following facts : There exists 16 $\{C_4, 4K_1\}$-fixers of order at most 12 for Free$\{C_4; 4K_1\}$ and they all have 12 vertices.

## 5.3 Partitioning into cliques

In this section, we say that a graph is $k$-CP ($k$ clique partitionable) for a fixed positive integer $k$, if its vertex set can be partitioned into $k$ cliques i.e. if the graph admits a $k$-clique cover. In [27], Gaspers and Huang proved that graphs in Free$\{2P_2, K_4\}$ are 4-colorable. It implies that graphs in Free$\{C_4, 4K_1\}$ are 4-CP. In this section, we focus on 3-CP graphs in Free$\{C_4\}$. Those graphs are trivially in Free$\{C_4, 4K_1\}$.

Recall that a graph is a *half graph* if it is in Free$\{3K_1, C_4, C_5\}$. For any two disjoints cliques $K_1$ and $K_2$ contained in a graph $G \in$ Free$\{C_4, 4K_1\}$, it is easy to see that $G[K_1 \cup K_2]$ belongs to Free$\{3K_1, C_5\}$ (because $C_5$ is not 2-CP). If, in addition, $G$ is also a graph in Free$\{C_4\}$ then $G[K_1 \cup K_2]$ is a half graph. It follows directly that any 2-CP graph in Free$\{C_4\}$ is a half graph.

Recall also that in any graph $G$ an ordering on the vertices $v_1, \ldots, v_k$ such that $N_G(v_i) \subseteq N_G[v_j]$ for all integers $i$ and $j$ satisfying $1 \leq i \leq j \leq k$ is called a *domination ordering* (see Chapter 4 section 4.2).

**Theorem 5.3.1 (Folklore)** *A graph $G$ is a half graph if and only if $V(G)$ can be partitioned into two (possibly empty) cliques $K_1$ and $K_2$ such that, for $i \in \{1, 2\}$ vertices in $K_i$ admits a domination ordering (i.e. for any couple of vertices $x$ and $y$ in $K_1$ (resp. in $K_2$), either $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$).*

*Proof.* If $G$ is a half graph, then the complement of $G$ contains (as a subgraph, not necessarily induced) no cycle of odd length because a shortest such cycle cannot have length 3 (it would yield a $3K_1$ in $G$), cannot have length 5 (it would yield a $C_5$ in $G$) and cannot have length at least 7 (it would yield a $C_4$ in $G$). It follows that the complement of $G$ is a bipartite graph (because a graph is bipartite if and only if it does not contain an odd cycle [1]), so $V(G)$ can be partitioned into two cliques as claimed. The condition on the neighbourhoods then follows from the fact that $G$ contains no $C_4$.

The converse statement is clear. □

There are several ways to see that the coloring problem is in $\mathcal{P}$ when restricted to half graphs. One way is to see that they are perfect (because complement of bipartite graphs). Another way is to see that half graphs are graphs with Dilworth number 2 and so, by [3] they are interval graphs. Since any 2-CP graph in Free$\{C_4\}$ is a half graph, it follows that these graphs can be colored in polynomial time. But it is interesting to note that, using clique-width operations, it is possible to build any half graph with 3 labels. In addition, the two cliques always remain with two distinct labels.

**Lemma 5.3.2** *Half graphs have clique-width at most 3.*

*Proof.* We prove by induction on the number of vertices the following property: Any half graph $G$ with cliques partition $K_1$, $K_2$ can be built with clique-width operations using 3 labels such that all vertices of $K_1$ end with label 1 and all vertices with label $K_2$ end with label 2.

It is easy to see that a simple edge is a half graph that satisfies the property.

Let $G$ be a half graph with clique partition $K_1$, $K_2$. By definition of half graphs, there exists a vertex $v$ that is either in $K_1$ and complete to $K_2$ or in $K_2$ and anticomplete to $K_1$.

The graph $G[V \setminus \{v\}]$ is a half graph with clique partition $K_1 \setminus \{v\}$ and $K_2 \setminus \{v\}$. By the induction hypothesis, it is possible to build $G[V \setminus \{v\}]$ with clique-width operations using 3 labels (1,2 and 3) such that all vertices in $K_1 \setminus \{v\}$ end with label 1 and all vertices in $K_2 \setminus \{v\}$ end with label 2. Build $v$ with label 3. If $v \in K_1$ then $v$ is complete to $K_2$. Add an edge between all vertices with label 3 and all vertices with labels 1 and 2. Relabel all vertices of label 3 with label 1. If $v \in K_2$ then $v$ is anticomplete to $K_1$. Add an edge between all vertices with label 3 and all vertices with label 2. Relabel all vertices of label 3 with label 2. In both cases $G$ is build with clique-width operations using 3 labels such that all vertices of $K_1$ end with label 1 and all vertices with label $K_2$ end with label 2. $\square$

Recall that a graph in Free$\{C_4, 4K_1\}$ is 4-CP ([27]). Hence, graphs in Free$\{C_4, 4K_1\}$ are graphs whose vertex set is made of at most 4 cliques, each pair of cliques inducing a half graph. It could be nice to generalise Lemma 5.3.2 in order to bound the clique-width in some particular subclasses of Free$\{C_4, 4K_1\}$. But take three cliques $A$, $B$ and $C$ in any graph $G$ in Free$\{C_4, 4K_1\}$. We have that $G[A \cup B]$ is a half graph so as $G[A \cup C]$. But it is possible that the dominating order (induced by Theorem 5.3.1) on the vertices in $A$ is not the same depending on which half graph is considered. Hence it is possible that there is no vertex that is either complete or anti-complete to the other cliques as used in the proof of Lemma 5.3.2.

On another hand, let $G$ be a graph in Free$\{C_4, 4K_1\}$ that can be decomposed into $k$ cliques $K_1, \dots, K_k$ ($k \in \mathbb{N}$) such that, for each $K_i$ ($i \leq k$) there is at most one $j \leq k$ such that $K_i$ is neither complete nor anticomplete to $K_j$ (in that case, by some previous observation, $G[K_i \cup K_j]$ form a half graph). By Lemma 5.3.2, all such graphs have clique-width bounded by $k$ ( $k$ being the number of cliques). This is because one can build, using 3 labels, all couples of cliques $K_i$ and $K_j$ such that $G[K_i \cup K_j]$ is a half graph and with all vertices of $K_i$ ending with label $i$ and all vertices of $K_j$ ending with label $j$ (see proof of Lemma 5.3.2). At this point we just have to make complete the cliques that are complete and the graph is built.

A way to divide the class of Free$\{C_4, 4K_1\}$ into two interesting subclasses is to consider those that are 3-CP and those that are not 3-CP. In [34], Hoàng and Trotignon presented a construction of graphs (ring on 3 sets, see section 4.2.2) in Free$\{C_4, 4K_1\}$ that are 3-CP and have unbounded clique-width.

As a corollary of next Lemma 5.3.3, the coloring problem restricted to graphs that can be partitioned into 3 cliques is NP-complete. Recall that the class of 3-CP graphs in Free$\{C_4\}$ is the complement class of 3-colourable graphs without $2K_2$. To prove Lemma 5.3.3, we use the arguments used by Král et al. to prove that the clique covering problem is NP-complete to Free$\{C_4, diamond, K_4, C_5\}$ in [38].

**Lemma 5.3.3** *The clique covering problem is NP-Complete when restricted to 3-colorable graphs.*

*Proof.* We use a variant of the *satisfiability* problem: each clause contains at most three variables and each variable is in exactly three clauses, once positive and twice negated. The NP-completeness of this problem is proved in [22].

Let $I_s$ be an instance of the variant of the satisfiability problem described before, with a set of boolean variables denoted by $X$ and a set of clauses denoted by $C$.

From $I_s$ we built an instance $I_c$ of the clique covering in 3-colorable graphs and show that $I_s$ has a "Yes" answer if and only if $I_c$ has a "Yes" answer. An instance of $I_c$ is composed by an integer $k$ and a 3-colorable graph $G$ (see section 2.4). Set $k = |X| + 3|C|$ and built $G$ as follow.

For each variable $x \in X$, build a graph $G_x$ with vertex set $\{v_x, x^+, x_1^-, x_2^-\}$ and edge set $\{v_x x^+, v_x x_1^-, v_x x_2^-, x_1^- x_2^-\}$. For all variables $x \in X$, vertices $v_x$ are called *flags*.

For each clause $c$ build $G_c$ that is a copy of $C_7$ with every variable $x$ in $c$ corresponding to a vertex in $G_c$ labelled $c(x)$. We call such vertices *ports* in $G_c$. Two distinct ports are nonadjacent. If the clause $c$ contains two literals, $G_c$ has only two ports (at distance at least 2). Vertices that are not ports are called *normal*.

We now obtain $G$ in the following way: for every variable $x$ appearing

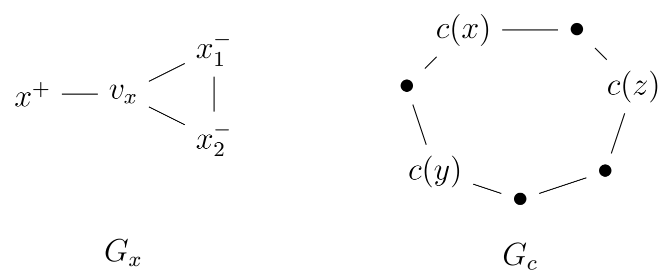



Figure 5.3: $G_x$ and $G_c$

positively in a clause $c$ and negated in the two clauses $d_1$ and $d_2$, identify $c(x)$ and $x^+$ into one vertex, $d_1(x)$ and $x_1^-$ into one vertex and $d_2(x)$ and $x_2^-$ into one vertex.

Observe that there is a proper coloring of $G$ that uses 3 colors. For every variable $x$ color $v_x$ with 1, $x_2^-$ with color 2 and both $x_1^-$ and $x^+$ with color 3. Observe that all normal vertices are of degree 2. Therefore, it is always possible to complete this coloring using colors 1, 2 and 3. Hence $G$ is 3-colorable.

Recall that a clique cover of a graph is a partition of its vertex set into cliques. Given a set of disjoint cliques $\mathcal{K}$, we say that a port $c(x)$ is checked by $\mathcal{K}$ if there is a clique in $\mathcal{K}$ containing both $v_x$ and $c(x)$.

Suppose that $I_s$ has a "Yes" answer and let $\mathcal{A}$ be a True/False assignment on the variables in $X$ that satisfied the formula. For every $x \in X$ let $K_x$ be the clique $\{v_x, x^+\}$ if $\mathcal{A}(x)$ =True and the clique $\{v_x, x_1^-, x_2^-\}$ otherwise. Set $\mathcal{K} = \{K_x : x \in X\}$ and observe that $|\mathcal{K}| = |X|$. For every clause $c \in C$, let $M_c$ be a set of cliques that covers the vertices of $G_c$ that are not included in any clique of $\mathcal{K}$. Observe that cliques in $M_c$ contains at most 2 vertices. Denote by $\mathcal{M}$ the disjoint union of all $M_c$ for $c \in C$. Since $\mathcal{A}$ satisfied the formula, every clause has at least one port checked by $\mathcal{K}$. Hence, for every clause $c \in C$, $|M_c| \leq 3$ and so $|\mathcal{M}| \leq 3|C|$. Now $\mathcal{K} \cup \mathcal{M}$ is a clique cover of $G$ of size at most $|X| + 3|C|$. Hence $I_c$ has a "Yes" answer.

Suppose now that $I_c$ has a "Yes" answer and let $\Phi$ be a set of $|X| + 3|C|$ cliques that covers $G$. Define $\mathcal{A}$, a True/False assignment on the variables in $X$ such that $\mathcal{A}(x)$ =True if and only if $\{v_x, x^+\} \in \Phi$. Observe that a clause

$c \in C$ is satisfied by $\mathcal{A}$ if at least one of the ports of $G_c$ is checked by $\Phi$. Hence, the formula of $I_s$ is satisfied if and only if, for all clauses $c \in C$, $G_c$ have at least one port checked by $\Phi$.

Denote by $K_1$ the set of cliques in $\Phi$ that contains a flag and denote by $K_2^c$ the cliques in $\Phi$ containing normal vertices of $G_c$ for every clause $c \in C$. Set $K_2 = \bigcup_{c \in C} K_2^c$. By construction of $G$ and since $\Phi$ is a partition of $V(G)$, $K_1$ and $K_2$ are disjoint and $|K_1| = |X|$. By construction, for $c_1$ and $c_2$, two distinct clauses in $C$, $K_2^{c_1}$ and $K_2^{c_2}$ are disjoint. Because for each clauses$c \in C$, $G_c$ has a least 3 nonadjacent normal vertices, it follows that $|K_2^c| \geq 3$ and so $|K_2| \geq 3|C|$. Hence $\Phi = K_1 \cup K_2$ and $|K_2^c| = 3$ for all $c \in C$. Since all cliques of size at least 3 in $G$ contains a flag, $K_2$ contains only cliques of size 2. Since, for all $c \in C$, $G_c$ is a $C_7$ that is covered by at most 3 cliques of size 2 from $K_2$, it follows that at least one vertex in $G_c$ is covered by a clique in $K_1$. Such a vertex has to be a port and is checked in $K_1$. Therefore $\mathcal{A}$ satisfies the formula and $I_s$ has a "Yes" answer.

We proved that $I_s$ have a "Yes" answer if and only $I_c$ have a "Yes" answer. Since passing from $I_s$ to $I_c$ can be done in polynomial time, it follows that the clique covering problem is NP-Complete when restricted to 3-colorable graphs. $\square$

As a corollary of Lemma 5.3.3, the coloring problem is NP-complete when restricted to the complement of 3-colourable graphs. Furthermore coloring Free$\{C_4\}$ graphs is also NP-complete by Theorem 3.2.2. We wonder if the intersection of these two problems, that is coloring 3-CP graphs in Free$\{C_4\}$, could yield to a polynomial result.

A nice way to understand 3-CP graphs in Free$\{C_4\}$, is to find the minimal set of graphs $\mathscr{H}^*$ such that 3-CP graphs in Free$\{C_4\}$ are exactly the graphs in Free$\mathscr{H}^*$. Observe that the icosahedron is not 3-CP. The following is a consequence of a result of Maffray and Morel [41] (notations correspond to the graphs displayed on Figures 5.4 ):

**Lemma 5.3.4** *The class of 3-CP graphs in* Free$\{C_4\}$ *is exactly the class* Free$\{C_4, 4K_1, Ico^{-2}, C_6^+, C_5 + K_1, C_7, \Pi_5, F_{13}\}$.

We observed that all $\{C_4, 4K_1\}$-fixers returned by the program in Ap-

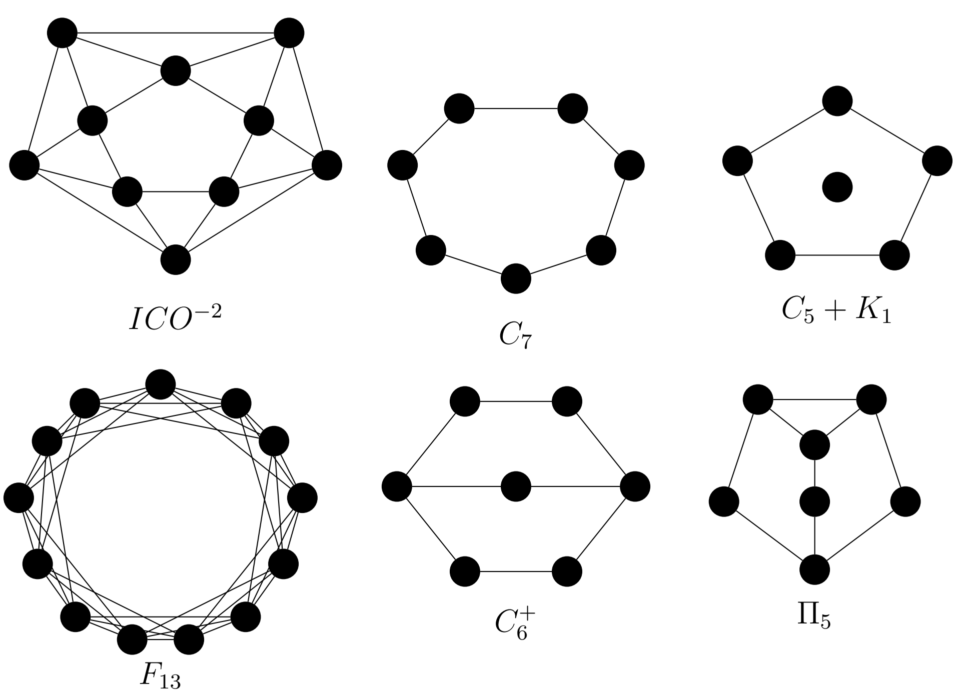



Figure 5.4: Graphs of lemma 5.3.4

pendix A (see end of section 5.2) contain at least one of the obstructions from Lemma 5.3.4. Hence, all $\{C_4, 4K_1\}$-fixers of order 12 are not 3-CP.

## 5.4 Further works

In this chapter we present some approaches in the study of graphs in Free$\{C_4, 4K_1\}$.

The next step could be the study of the structure of 3-CP graphs in Free$\{C_4\}$ that contain a $C_5$ using Lemma 5.3.4. Note that $Ico^{-2}$, $C_6^+$, $C_5 + K_1$, $\Pi_5$ and $F_{13}$ all contain a $C_5$. The goal is to answer the question : does this additional constraint lead to a polynomial result for coloring this subclass of Free$\{C_4, 4K_1\}$?

Another approach is to use the program in Appendix A in two directions. The first direction consists of running the program with graphs of order 13 excluding the ones that contains $\{C_4, 4K_1\}$-fixers of order 12. The second direction consists of going backward and looking for graphs of order 11 in Free$\{C_4, 4K_1\}$ such that, it is not possible to attach a vertex to them without creating a twin, a universal vertex or a $\{C_4, 4K_1\}$-fixer of order 12. It should output at least the icosahedron minus one vertex. These two approaches might give insight to possibly answer the questions : Are there finitely many $\{C_4, 4K_1\}$-fixers and how general are they?

# Chapter 6

# Coloring antiprismatic graphs

## 6.1 Introduction

A *triangle* in a graph is a set of three pairwise adjacent vertices. A graph $G$ is *prismatic* if for every triangle $T$ of $G$, every vertex of $G$ not in $T$ has a unique neighbour in $T$. In other words, the class of prismatic graphs is the class Free$\{C_3+K_1, \mathit{diamond}, K_4\}$. A graph is *antiprismatic* if its complement is prismatic. It is straightforward to check that antiprismatic graphs are precisely Free$\{K_{1,3}, 2P_1 + P_2, 4K_1\}$ graphs. Observe that if $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$ are two vertex-disjoint triangles in a prismatic graph $G$, then there is a perfect matching in $G$ between $\{s_1, s_2, s_3\}$ and $\{t_1, t_2, t_3\}$, so that $\{s_1, s_2, s_3, t_1, t_2, t_3\}$ induces the complement of a $C_6$, sometime called *prism*, see Figure 6.1. This is where the name prismatic comes from. Recall that the prism is a long prism with all paths of length 1 (cf section 4.2.2).

As presented in chapter 3, the complexity of coloring antiprismatic graphs is still unknown. It is exactly one of the three minimal open cases for the complexity of the coloring problem presented by Lozin et al. [40]. Remind that the *clique cover problem* is the problem of finding, in an input graph $G$, a minimum number of cliques that partition $V(G)$. It is equivalent to the coloring problem for the complement. It is therefore NP-complete in the general case. Our work is about the coloring problem for antiprismatic graphs. However, it is more convenient to view it as a study of the clique cover problem for prismatic graphs. Hence, from here on, we focus on the prismatic graphs and

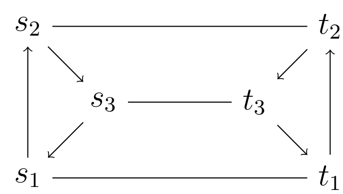



Figure 6.1: The prism

the clique cover problem.

Chudnovsky and Seymour gave a full structural description of prismatic graphs (and therefore of their complement). They showed that the class can be divided into two subclasses, to be defined later: the orientable prismatic graphs, and the non-orientable prismatic graphs. They described the structure of the two subclasses: orientable in [11] and non-orientable in [12]. Javadi and Hajebi [36] discovered a flaw in [12], that they could fix at the expense of adding a basic class in the structural description.

Our main result is that there exists an $\mathcal{O}(n^{7.5})$-time algorithm to solve the clique cover problem in non-orientable prismatic graphs. Our proof is based on the existence, in any non-orientable prismatic graph $G$ with more than 27 vertices, of a set of at most 5 vertices that intersects all triangles of $G$ (see Theorem 6.4.1). This follows directly from the structural description, but needs careful verification. For the orientable case, we could not settle the complexity of the clique cover problem, but we prove that a related problem can be solved in polynomial time: the *vertex-disjoint triangles problem.* (It consists in finding a maximum number of disjoint triangles in an input graph). This problem is known to be NP-hard in the general case [31].

Our algorithm for the clique cover problem in the non-orientable case relies on the existence of a hitting set of the triangles of bounded size. Sepehr Hajebi [32] observed that the existence of such a set can be proved with a short argument that relies on several lemmas of [12]. This argument gives a hitting set of size at most 15. The way we use hitting sets then provides an algorithm for the clique cover problem of complexity $\mathcal{O}(n^{17.5})$.

## Outline

In Section 6.2 we give the definitions specific to this chapter such as several results about prismatic graphs.

In Section 6.3, we give the structural description of non-orientable prismatic graphs from [12] and show that it implies the existence of a set of bounded number of vertices that intersects all triangles. Since our proof mostly relies on the structural description of Chudnovsky and Seymour, we have to give many long definitions extracted from their work.

In Section 6.4, we show that this yields a polynomial time algorithm for the clique cover problem in non-orientable prismatic graphs.

In Section 6.5, we prove that the vertex-disjoint triangles problem can be solved in polynomial time for all prismatic graphs. Our proof does not rely on the structural description from [11].

In Section 6.6, we describe Hajebi's approach.

Section 6.7 is devoted to concluding remarks.

# 6.2 Prismatic graphs

Recall that a *clique cover* of $G$ is a set of disjoint cliques of $G$ that partitions $V(G)$. A triangle in a graph $G$ is *covered* by a set $S$ of vertices if at least one vertex of the triangle is in $S$. A set $S \subseteq V(G)$ is a *hitting set of the triangles of* $G$ if every triangle in $G$ is covered by $S$. We often write *hitting set* instead of hitting set of the triangles.

### Orientable and non-orientable prismatic graphs

Let $T = \{a, b, c\}$ be a triangle in a graph $G$. There are two cyclic permutations of $T$, and we use the notation $a \to b \to c \to a$ to denote the cyclic permutation mapping $a$ to $b$, $b$ to $c$ and $c$ to $a$. Thus $a \to b \to c \to a$ and $b \to c \to a \to b$ mean the same permutation.

A prismatic graph $G$ is *orientable* if there is a choice of a cyclic permutation $O(T)$ for every triangle $T$ of $G$, such that if $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ are disjoint triangles and $s_i t_i$ is an edge for $1 \leq i \leq 3$, then $O(S)$ is $s_1 \to s_2 \to s_3 \to s_1$ if and only if $O(T)$ is $t_1 \to t_2 \to t_3 \to t_1$. In that case, the set of permutations containing $O(T)$ for every triangle $T$ of $G$ is called a *correct orientation* of $G$.

A graph $G$ is *non-orientable* if there exists no correct orientation of $G$.

Orientable and non-orientable prismatic graphs have very different structures. By Theorem 6.4.1, a non-orientable prismatic graph contains at most 9 disjoint triangles. It might seem surprising that having a tenth triangle in a prismatic graph implies the existence of an orientation. This is because having a large number of disjoint triangles in a prismatic graph entails so many constraints that the only way to satisfy them all is in an orientable prismatic graph.

There is a nice characterisation of orientable prismatic graphs. The *rotator* and the *twister* are the graphs represented on Figure 6.2. In the rotator there exists one triangle that intersects all triangles, we call it the *center of the rotator*.

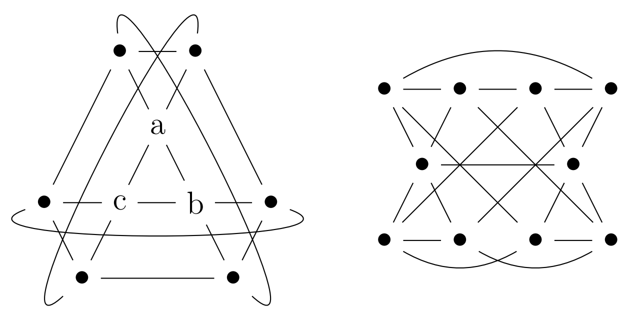



Figure 6.2: The rotator with center $\{a, b, c\}$ and the twister

**Theorem 6.2.1 (**6.1 **in [12])** *A prismatic graph is orientable if and only if it is* Free$\{$*rotator, twister*$\}$*.*

The structure of non-orientable prismatic graphs presented in [12] can be seen as a description of how the rotator and the twister can be completed in order to obtain all non-orientable prismatic graphs.

## 6.3 Non-orientable prismatic graphs

Our goal in this section is to prove the following.

**Lemma 6.3.1** *Every prismatic non-orientable graph contains a hitting set of the triangles of cardinality smaller or equal to 10.*

The *core* of a graph $G$ is the union of all triangles of $G$. Clearly, in a prismatic graph, deleting an edge between two vertices that are not in the core yields a prismatic graph. It follows that vertices not in the core are less structured than vertices in the core. Therefore, to prove Lemma 6.3.1, we may restrict our attention to the cores of the graphs in the class.

A prismatic graph $G$ with core $W$ is *rigid* if

- there do not exist distinct $u, v \in V(G) \setminus W$ adjacent to precisely the same vertices in $W$, and
- every two non-adjacent vertices of $G$ have a common neighbour in $W$.

*Replicating* a vertex $v$ in a graph $G$ means replacing $v$ by a stable set $S$ that is complete to $N(v)$ and anticomplete to $V(G) \setminus (N(v) \cup \{v\})$. We need the following.

**Theorem 6.3.2 (2.2 from [12])** *Every non-orientable prismatic graph can be obtained from a rigid non-orientable prismatic graph by replicating vertices not in the core, and then deleting edges between vertices not in the core.*

It follows, from this result, that to prove Lemma 6.3.1, it is enough to prove it for rigid non-orientable prismatic graphs.

Theorem 4.1 in [12] states that the class of rigid non-orientable prismatic graphs is included in the union of 13 classes. Javadi and Hajebi [36] discovered that one class is missing in Theorem 4.1, the so-called class $\mathcal{F}_0$. We describe these 14 classes whose union is called the *menagerie*.

In the definition of the menagerie, two operations are sometimes needed, the so-called *multiplication* and *exponentiation*.

The rest of the section is therefore organized as follows. The first two subsections describe the multiplication and exponentiation together with a proof that applying them under some specific hypotheses preserves the existence of a hitting set. The next 14 subsections each presents one class of the menagerie, together with a proof of the existence of a small hitting set. These subsections with Theorem 6.3.2 therefore form the proof of Lemma 6.3.1.

Before we start, we state the following lemma which is a direct consequence of the definition of prismatic graphs.

**Lemma 6.3.3** *If $v$ be a vertex of a prismatic graph $G$ then $N_G(v)$ is a hitting set of $G$.*

### 6.3.1 Multiplication

Let $H$ be a prismatic graph and $X$ be a subset of vertices of $H$. For each vertex $x \in X$, let $A_x$ be a set of vertices not in $V(H)$ such that for all distinct $x, x' \in X$, $A_x \cap A_{x'} = \emptyset$. Let $A = \cup_{x \in X} A_x$ and let $\varphi$ be a map from $A$ to the set of integers such that for all $x \in X$, $\varphi$ is injective on $A_x$.

Let now $G$ be the graph defined as follows:

- $V(G) = (V(H) \setminus X) \cup A$.

Let $v$ and $v'$ be two distinct vertices of $G$.

- If there is an $x \in X$ such that both $v$ and $v'$ are in $A_x$ then $v$ and $v'$ are not adjacent. $A_x$ is a stable set of $G$.
- If $v$ and $v'$ are in $V(H) \setminus X$ then $vv' \in E(G)$ if and only if $vv' \in E(H)$.
- If $v \in V(H) \setminus X$ and $v' \in A_x$ for some $x \in X$ then $vv' \in E(G)$ if and only if $vx \in E(H)$.
- If $v \in A_x$ and $v' \in A_{x'}$ where $x, x' \in X$ are distinct and adjacent in $H$, then $vv' \in E(G)$ if and only if $\varphi(v) = \varphi(v')$.
- If $v \in A_x$ and $v' \in A_{x'}$ where $x, x' \in X$ are distinct and nonadjacent in $H$, then $vv' \notin E(G)$ if and only if $\varphi(v) = \varphi(v')$.

The graph $G$ *is obtained from $H$ by multiplying $X$*. For $x \in X$, the set $A_x$ is the *set of new vertices corresponding to $x$* and $\varphi$ is the *corresponding integer map*. As noted in [12], the multiplication does not preserve being prismatic in general, but it is used only in situations where it does.

**Lemma 6.3.4** *If $H[X]$ is an induced subgraph of $C_4$ and non-adjacent vertices of $X$ have no common neighbours in $V(H)\backslash X$ then any hitting set of $H$ disjoint from $X$ is also a hitting set of $G$.*

*Proof.* Let $S_H$ be a hitting set of $S_H$. We prove that every triangle $\{u, v, w\}$ in $G$ is covered by $S_H$. If $\{u, v, w\} \subseteq V(H)$, then it is covered by $S_H$, so we may assume that $|\{u, v, w\} \cap A| > 0$.

**Case I:** $|\{u, v, w\} \cap A| = 1$

Suppose up to symmetry that there exists $x \in X$ such that $u \in A_x$ and $v, w \in V(H) \setminus X$. Then $\{x, v, w\}$ is a triangle in $H$ and it has to be covered by $S_H$. Since $X \cap S_H = \emptyset$, $v$ or $w$ belongs to $S_H$. Hence, $S_H$ covers $\{u, v, w\}$.

**Case II:** $|\{u, v, w\} \cap A| = 2$

Since for every $x \in X$, $A_x$ is a stable set, we may assume up to symmetry that there exist distinct $x, x' \in X$ such that $u \in A_x$, $v \in A_{x'}$ and $w \in V(H)\backslash X$. Since $w$ is a common neighbour of $u$ and $v$ in $G$, $w$ is a common neighbour of $x$ and $x'$ in $H$. From our assumptions, it follows that $x$ and $x'$ are adjacent in $H$. Hence $\{x, x', w\}$ is a triangle in $H$ and it has to be covered by $S_H$. Since $X \cap S_H = \emptyset$, $w \in S_H$, we have $S_H$ covers $\{u, v, w\}$.

**Case III:** $|\{u, v, w\} \cap A| = 3$

Since for every $x \in X$, $A_x$ is a stable set, there exist then distinct $x, y, z \in X$ such that $u \in A_x$, $v \in A_y$ and $w \in A_z$. Because of the hypothesis on $X$, $H[\{x, y, z\}]$ induces a $P_3$. Without loss of generality, suppose $xz \notin E[H]$.

Since $x$ and $y$ are adjacent in $H$, in order to have $u$ and $v$ also adjacent in $G$, we have $\varphi(u) = \varphi(v)$. Similarly, $\varphi(v) = \varphi(w)$. Hence $\varphi(u) = \varphi(w)$.

But since $x$ and $z$ are not adjacent, in order to have $u$ and $v$ adjacent in $G$, we need $\varphi(u) \neq \varphi(w)$, a contradiction. □

### 6.3.2 Exponentiation

A triangle $T = \{a, b, c\}$ of a graph $G$ is a *leaf triangle at $c$* if every triangle of $G$ distinct from $T$ contains neither a nor b.

Let $T = \{a, b, c\}$ be a leaf triangle at $c$ of a prismatic graph $H$. We define a partition of the neighbours of $c$, distinct from $a$ and $b$, into three disjoint sets:

$D_1$, $D_2$, and $D_3$ as follows (see Figure 6.3). Let $v \neq a, b$ be a vertex adjacent to $c$ in $H$, then :

- $v \in D_1$ if $v$ belongs to a triangle that does not contain $c$.
- $v \in D_2$ if $v \notin D_1$ and $v$ belongs to a triangle (then this triangle is unique and contains $c$).
- $v \in D_3$ if $v$ does not belong to any triangle.

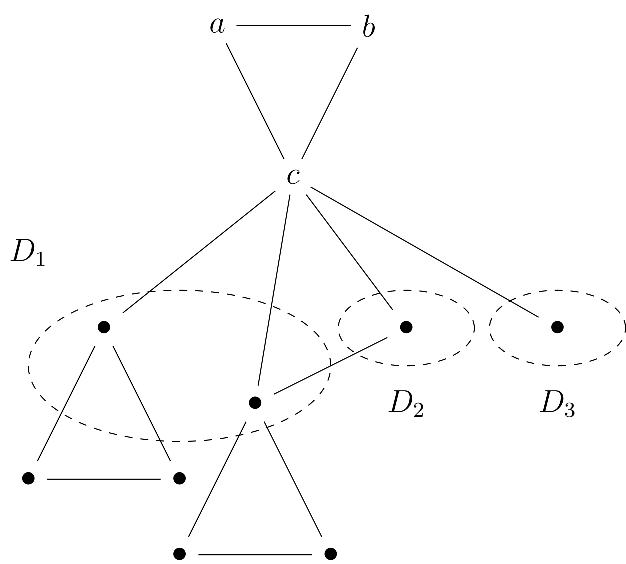



Figure 6.3: Neighbourhood of $c$

Let $A$, $B$ and $C$ be three pairwise disjoint sets of vertices.
The graph $G$ is defined as follows:

- $V(G) = (V(H) \setminus \{a, b\}) \cup A \cup B \cup C$

with the following adjacencies:

- Vertices in $V(H)\setminus\{a, b\}$ are adjacent in $G$ if and only if there are adjacent in $H$.

- $A$, $B$ and $C$ are stable sets.
- Every vertex in $A$ has at most one neighbour in $B$ and vice versa.
- Every vertex in $V(H) \setminus \{a, b\}$ adjacent (resp. non-adjacent) to $a$ in $H$ is complete (resp. anticomplete) to $A$ in $G$.
- Every vertex in $V(H) \setminus \{a, b\}$ adjacent (resp. non-adjacent) to $b$ in $H$ is complete (resp. anticomplete) to $B$ in $G$.
- $C$ is complete to $D_1 \cup D_3$ and anticomplete to $V(H) \setminus (\{a, b\} \cup D_1 \cup D_3)$.
- Every vertex in $C$ is adjacent to exactly one end of every edge between $A$ and $B$ and adjacent to every vertex in $A \cup B$ with no neighbour in $A \cup B$.

The graph $G$ is obtained from $H$ by *exponentiating the leaf triangle* $\{a, b, c\}$.

Before proving that the exponentiation preserves hitting sets, note that every prismatic graph $H$ with a leaf triangle $T = \{a, b, c\}$ at $c$ has a hitting set $S_H$ that contains $c$ but neither $a$ nor $b$.

**Lemma 6.3.5** *If $G$ is prismatic and $S_H$ is a hitting set of $H$ containing $c$ but neither $a$ nor $b$, then $S_H$ is also a hitting set of $G$.*

*Proof.*

Let $\{u, v, w\}$ be a triangle in $G$. We show that one of $u$, $v$ or $w$ belongs to $S_H$. Since $u, v, w \in V(G)$, none of them is $a$ or $b$.

**Case I:** $|\{u, v, w\} \cap (V(H) \setminus \{a, b\})| = 3$

This case is trivial because then $\{u, v, w\}$ is a triangle in $H$ and has to be covered by $S_H$.

**Case II:** $|\{u, v, w\} \cap (V(H) \setminus \{a, b\})| = 2$

Without loss of generality suppose $u \notin V(H)$ and $v, w \in V(H)$.

If $u$ belongs to $A$, that means that $\{a, v, w\}$ is a triangle in $H$. By our hypothesis this triangle should be T and $c \in \{u, v, w\}$. So $S_H$ covers $\{u, v, w\}$. The case where $u$ belongs to $B$ is similar.

If $u$ belongs to $C$, then $v$ and $w$ have to belong to $D_1 \cup D_3$ by definition of the neighbourhood of $C$. Then $\{c, v, w\}$ is a triangle in $G$. So $\{u, v, w, c\}$ is a diamond in $G$, a contradiction to $G$ being prismatic.

**Case III:** $|(\{u, v, w\} \cap V(H) \setminus \{a, b\})| = 1$

Without loss of generality suppose $w \in V(H) \setminus \{a \cup b\}$. Note that $A$, $B$ and $C$ are stable sets so $\{u, v, w\}$ contains at most one vertex of each.

Suppose that one of $u, v$ is in $C$ (so the other one is in $A \cup B$). Up to symmetry, we may assume that $u$ belongs to $A$ and $v$ belongs to $C$. Then $aw \in E(H)$ and $cw \in E(H)$. This means that $\{a, w, c\}$ is a triangle in $H$. This contradicts $\{a, b, c\}$ being a leaf triangle at $c$.

Suppose that none of $u$, $v$ belong to $C$. Up to symmetry, suppose $u \in A$ and $v \in B$. So, $\{a, b, w\}$ is a triangle of $H$ and this triangle can only be $\{a, b, c\}$. So $w = c \in S_H$.

**Case IV:** $|(\{u, v, w\} \cap H \setminus \{a, b\})| = 0$

This case cannot happen because $A$, $B$ and $C$ are stable sets and every vertex of $C$ has a unique neighbour in any edge of $G[A \cup B]$. $\square$

### 6.3.3 Schläfli-prismatic graphs

We have to define the Schläfli graph, and it is more convenient to work in the complement. The *complement of the Schläfli graph* has 27 vertices $r^i_j, s^i_j, t^i_j$, $1 \leq i, j \leq 3$ with adjacencies as follows. For $1 \leq i, i', j, j' \leq 3$:

- If $i \neq i'$ and $j \neq j'$, then $r^i_j$ is adjacent to $r^{i'}_{j'}$, $s^i_j$ is adjacent to $s^{i'}_{j'}$ and $t^i_j$ is adjacent to $t^{i'}_{j'}$.

- If $j = i'$, then $r^i_j$ is adjacent to $s^{i'}_{j'}$, $s^i_j$ is adjacent to $t^{i'}_{j'}$ and $t^i_j$ is adjacent to $r^{i'}_{j'}$.

There are no other edges.

This graph will be denoted by $\Sigma$ throughout the rest of the paper. We will often rely on the fact that $\Sigma$ is vertex-transitive.

We introduce more notation. We set $R = \{r^i_j : 1 \leq i, j \leq 3\}$, $S = \{s^i_j : 1 \leq i, j \leq 3\}$ and $T = \{t^i_j : 1 \leq i, j \leq 3\}$ and call *tile* each of the sets $R$, $S$, $T$. We call *line* $i$ of $R$ the set $\{r^i_j : 1 \leq j \leq 3\}$ and *column* $j$ of $R$ the set $\{r^i_j : 1 \leq i \leq 3\}$. We use a similar notation for $S$ and $T$.

By definition an edge between $u$ and $v$ in a same tile exists if and only if $u$ and $v$ are in different lines and columns. Edges between tiles are conveniently described as follows: for every $i = 1, 2, 3$, column $i$ of $R$ is complete to line $i$ of $S$, column $i$ of $S$ is complete to line $i$ of $T$ and column $i$ of $T$ is complete to line $i$ of $R$. There are no other edges.

A triangle in $\Sigma$ is *internal* if it is included in a tile, and *external* otherwise. The observations above show that an internal triangle is made of three vertices that are in three different lines, and also in three different columns of the tile. An external triangle $\{u, v, w\}$ satisfies $\{u, v, w\} = \{r^i_j, s^j_k, t^k_i\}$ for some $1 \leq i, j, k \leq 3$.

This shows that there exist 6 internal triangles in each tile and 27 external triangles, that gives 45 triangles in total. Each vertex lies in two internal triangles and three external triangles. Every edge is contained in exactly one triangle. See Figure 6.4.

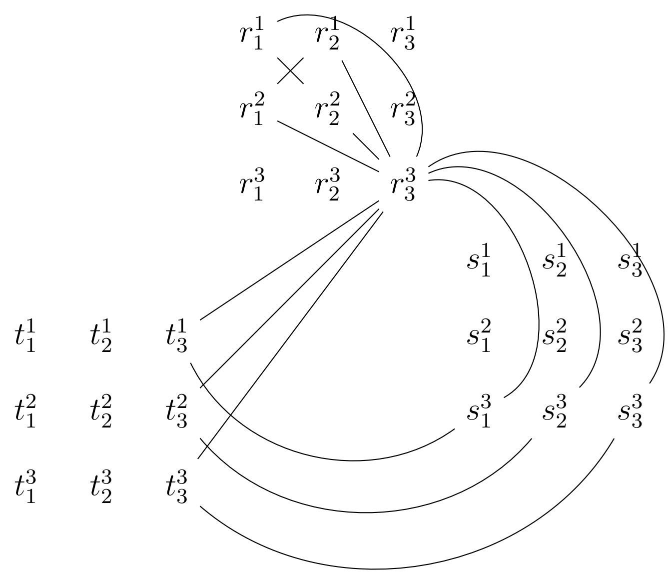



Figure 6.4: $\Sigma$, the complement of the Schläfli graph. (Only the 10 edges and the 5 triangles that contain $r^3_3$ are represented.)

We call *Schläfli-prismatic* graph every induced subgraph of $\Sigma$. It is easy to see that they are prismatic.

**Lemma 6.3.6** *A smallest hitting set of $\Sigma$ has cardinality 10.*

*Proof.* Every vertex $v$ in $\Sigma$ has degree 10, so by Lemma 6.3.3, $N_\Sigma(v)$ is a hitting set of $\Sigma$ of size 10.

Suppose for a contradiction that $W$ is a hitting set of $\Sigma$ and $|W| = 9$. Since $\Sigma$ contains 45 triangles and every vertex of $\Sigma$ is contained in exactly 5 triangles (2 internal and 3 external), no two vertices of $W$ hit the same triangle. Since every edge of $\Sigma$ is contained in a triangle, it follows that $W$ is a stable set.

For each tile $X$, a maximum stable set in $\Sigma$ has cardinality 3 and is a line or a column. It follows that for $X \in \{R, S, T\}$, $W \cap X$ is a line or a column of $X$.

By the pigeon hole principle, $W$ contains either two lines or two columns (of different tiles). This contradicts the fact that $W$ is a stable set, because between two lines (or two columns) of different tiles, there exists at least one edge.

□

### 6.3.4 Fuzzily Schläfli-prismatic graphs

Let $\{a, b, c\}$ be a leaf triangle at $c$ in a Schläfli-prismatic graph $H$. If a prismatic graph $G$ can be obtained from $H$ by multiplying $\{a, b\}$, and $A$, $B$ are the two sets of new vertices corresponding to $a$, $b$ respectively, the graph $G$ is a *fuzzily Schläfli-prismatic* graphs. Note that this operation is not iterated.

**Lemma 6.3.7** *Every fuzzily Schläfli-prismatic graph has a hitting set of cardinality smaller or equal to 5.*

*Proof.* Let $G$, $H$ and $\{a, b, c\}$ as in the definition.

Since $a$ belongs to exactly one triangle of $H$ and to exactly 5 triangles in $\Sigma$, we have $|N_H(a)| \leq |N_\Sigma(a)| - 4 = 6$.

By Lemma 6.3.3, $N_H(a)$ is a hitting set of $H$.

Since $b \in N_H(a)$ and since the unique triangle containing $b$ is $\{a, b, c\}$ which is already covered by $c$, we have that $N_H(a) \setminus \{b\}$ is a hitting set of $H$ of cardinality at most 5.

By Lemma 6.3.4 it is also a hitting set of $G$. $\square$

### 6.3.5 Graphs of parallel-square type

Let $X$ be the edge-set of some $C_4$ of the complete bipartite graph $K_{3,3}$, and let $z$ be the edge of $K_{3,3}$ disjoint from all edges in $X$. Thus $X$ induces a $C_4$ of the line graph $H$ of $K_{3,3}$. Any graph $G$ obtained from $H$ by multiplying $X$, and possibly deleting $z$, is prismatic and is called a graph of *parallel-square type*.

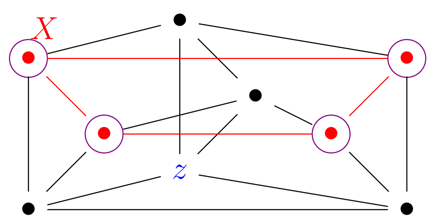



Figure 6.5: Line graph of $K_{3,3}$, $X$, $z$ as defined in section 6.3.5

**Lemma 6.3.8** *Every prismatic graph of parallel square type admits a hitting set of cardinality smaller or equal to 4.*

*Proof.* Let $H$, $X$, $z$ and $G$ as in the definition.

Let $S_H = V(H) \setminus (X \cup \{z\})$. Obviously $S_H$ is a hitting set of $H$.

The set $X$ induces a $C_4$ in $H$ and no two non-adjacent vertices of $X$ have common neighbours in $V(H) \setminus X$. By Lemma 6.3.4, $S_H$ is a hitting set of $G$.

Note that the deletion of $z$ does not change the result because $z$ is not in $S_H$. Since $|S_H| = 4$, the proof is completed. $\square$

### 6.3.6 Graphs of skew-square type

Let $K$ be a graph with five vertices $a$, $b$, $c$, $s$, $t$, where $\{s, a, c\}$ and $\{t, b, c\}$ are triangles and there is no more edge. Let $H$ be obtained from $K$ by multiplying $\{a, b, c\}$, let $A$, $B$, $C$ be the sets of new vertices corresponding to $a$, $b$, $c$ respectively, and let $\varphi$ be the corresponding integer map. Add three more vertices $d_1$, $d_2$, $d_3$ to $H$, with adjacency as follows:

- $d_1$, $d_2$, $d_3$, $s$, $t$ are pairwise non-adjacent,
- for $1 \leq i \leq 3$ and $v \in A \cup B$, $d_i$ is adjacent to $v$ if and only if $1 \leq \varphi(v) \leq 3$ and $\varphi(v) \neq i$,
- for $1 \leq i \leq 3$ and $v \in C$, $d_i$ is non-adjacent to $v$ if and only if $1 \leq \varphi(v) \leq 3$ and $\varphi(v) \neq i$.

Any graph obtained by this way is prismatic, and is called a graph of *skew-square type*.

**Lemma 6.3.9** *Every prismatic graph of skew-square type admits a hitting set of cardinality smaller or equal to 5.*

*Proof.* Let $G$ be a graph of skew-square type.

Let $S_K = \{s, t\}$. We first show that $S_K$ is a hitting set of $H$ and then prove that $S_K \cup \{d_1, d_2, d_3\}$ is a hitting set of $G$.

First it is obvious that $S_K$ is a hitting set of $K$. Furthermore, in $K$, $\{a, b, c\}$ induces a $P_3$ which is an induced subgraph of $C_4$ and vertices $a$ and $b$ do not have a common neighbour in $V(K) \setminus \{a, b, c\}$. We may therefore apply Lemma 6.3.4, showing that $S_K$ is a hitting set of $H$.

Since we just add three vertices $d_1$, $d_2$ and $d_3$ to construct $G$ from $H$, every triangle in $G$ either contains one vertex of $\{d_1, d_2, d_3\}$ or is a triangle in $H$ that is covered by $S_K$.

This shows that $S_K \cup \{d_1, d_2, d_3\}$ is a hitting set of $G$ of size 5. □

### 6.3.7 The class $\mathcal{F}_0$

Note that this class is defined in [36].

Let $H$ be a subgraph of $\Sigma$ induced by:

$$\{r^i_j : (i, j) \in I_1\} \cup \{s^i_j : (i, j) \in I_2\} \cup \{t^i_j : (i, j) \in I_3\}$$

where $I = \{(i, j) : 1 \leq i, j \leq 3\}$ and $I_1$, $I_2$, $I_3$ are subset of $I$ such that:

- $(1, 1), (1, 3), (2, 2), (2, 3), (3, 1)(3, 2) \in I_1$ and $(3, 3) \notin I_1$,
- $(1, 1), (2, 1), (3, 2) \in I_2$ and $(1, 2), (1, 3), (2, 2), (2, 3) \notin I_2$,

- $(1,3),(2,1),(2,2) \in I_3$ and $(1,1),(1,2),(3,1),(3,2) \notin I_3$ .

Let $G$ be the graph obtained from $H$ by adding the edges $s_1^3t_3^2$, $s_1^3t_3^3$ and $s_3^3t_3^2$ if the corresponding vertices are in $H$. We define $\mathcal{F}_0$ to be the class of all such graphs $G$.

**Lemma 6.3.10** *Every graph of the class $\mathcal{F}_0$ admits a hitting set of cardinality smaller or equal to 3.*

*Proof.* By Lemma 6.3.3, $N_\Sigma(s_1^3) \cap V(H) = S_H = \{r_3^1, r_3^2, t_3^1\}$ is a hitting set of $H$. Note that $s_1^3$ and $t_3^2$ do not have common neighbours in $H$ so as $s_1^3$ and $t_3^3$, and $s_3^3$ and $t_3^2$. Since $G[\{s_1^3, s_3^3, t_3^2, t_3^3\}]$ induces a $C_4$, we have that the addition of the new edges does not add any triangle in $G$. Therefore $S_H$ is also a hitting set of $G$. □

### 6.3.8 The class $\mathcal{F}_1$

Let $G$ be a graph with vertex set the disjoint union of sets $\{s,t\}$, $R$, $A$, $B$, where $|R| \leq 1$, and with edges as follows:

- $s$, $t$ are adjacent, both are complete to $R$, and $s$ is complete to $A$; $t$ is complete to $B$;
- every vertex in $A$ has at most one neighbour in $A$, and every vertex in $B$ has at most one neighbour in $B$;
- if $a, a' \in A$ are adjacent and $b, b' \in B$ are adjacent, then the subgraph induced by $\{a, a', b, b'\}$ is a cycle;
- if $a, a' \in A$ are adjacent and $b \in B$ has no neighbour in $B$, then $b$ is adjacent to exactly one of $a, a'$;
- if $b, b' \in B$ are adjacent and $a \in A$ has no neighbour in $A$, then $a$ is adjacent to exactly one of $b, b'$;
- if $a \in A$ has no neighbour in $A$, and $b \in B$ has no neighbour in $B$, then $a, b$ are adjacent

We define $\mathcal{F}_1$ to be the class of all such graphs $G$.

**Lemma 6.3.11** *Every prismatic graph of the class $\mathcal{F}_1$ admits a hitting set of cardinality smaller or equal to 2.*

*Proof.* We claim that $\{s, t\}$ is a hitting set of $G$. This is equivalent to the fact that in $G \setminus \{s, t\}$, the neighborhood of any vertex $v$ is a stable set. And this follows directly from the definition in all cases ($v = r$, $v$ in $A$ with no neighbor in $A$, $v$ in $A$ with one neighbor in $A$, symmetric cases with $v \in B$). □

Note that graphs in $\mathcal{F}_1$ can have arbitrarily large minimum degree.

### 6.3.9 The class $\mathcal{F}_2$

Let $K$ be the line graph of $K_{3,3}$ with vertices numbered $s^i_j$ ($1 \leq i, j \leq 3$), where $s^i_j$ and $s^{i'}_{j'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. Note that this is how usually the complement of the line graph of $K_{3,3}$ is defined, but since it is a self-complementary graph, it makes no difference.

Let H be a graph obtained from this by multiplying $\{s^1_2, s^1_3, s^2_1, s^3_1\}$, thus, $H$ is of parallel-square type. Let $A^1_2$, $A^1_3$, $A^2_1$, $A^3_1$ be the sets of new vertices corresponding to $\{s^1_2$, $s^1_3$, $s^2_1$, $s^3_1\}$ respectively, and let $\varphi$ be the corresponding integer map. Suppose that:

- there do not exist $u \in A^3_1$ and $v \in A^1_3$ with $\varphi(u) = \varphi(v)$;
- there exist $a^1_2 \in A^1_2$ and $a^2_1 \in A^2_1$ such that $\varphi(a^1_2) = \varphi(a^2_1) = 1$;
- $\varphi(v) \neq 1$ for all $v \in A^3_1 \cup A^1_3$.

Let $G$ be obtained from $H$ by exponentiating $\{a^1_2, a^2_1, s^3_3\}$, leaf triangle at $s^3_3$. We define $\mathcal{F}_2$ to be the class of all such graphs $G$.

**Lemma 6.3.12** *Every prismatic graph of the class $\mathcal{F}_2$ admits a hitting set of cardinality smaller or equal to 4.*

*Proof.* Let $S_K = \{s^3_2, s^2_3, s^2_2, s^3_3\}$. We can easily see that $S_K$ is a hitting set of $K$. We prove that $S_K$ is a hitting set of $G$.

By definition $H$ is obtained from $K$ by multiplying $\kappa = \{s_3^1, s_2^1, s_1^2, s_1^3\}$. Note that, in $K$, $\kappa$ induces a $C_4$, $\kappa \cap S_K = \emptyset$, $s_3^1, s_2^1$ do not have a common neighbours outside of $\kappa$ and $s_1^3, s_1^2$ do not have a common neighbours outside of $\kappa$. We may now apply Lemma 6.3.4, and conclude that $S_K$ is a hitting set of $H$.

We may apply Lemma 6.3.5, and we obtain that $S_K$ is a hitting set of $G$ (note that the fact that $H$ is a prismatic graph is not used in the proof of Lemma 6.3.5) . □

### 6.3.10 The class $\mathcal{F}_3$

Let $K$ be the line graph of $K_{3,3}$, with vertices numbered $s_j^i$ $(1 \leq i, j \leq 3)$, where $s_j^i$ and $s_{j'}^{i'}$ are adjacent if and only if $i' \neq i$ and $j' \neq j$. Let $H$ be obtained from $K$ by deleting the vertex $s_2^2$ and possibly $s_1^1$, and then multiplying $\{s_2^1, s_3^1, s_1^2, s_1^3\}$. Let $A_2^1$, $A_3^1$, $A_1^2$, $A_1^3$ be the sets of new vertices corresponding to $s_2^1$, $s_3^1$, $s_1^2$, $s_1^3$ respectively, and let $\varphi$ be the corresponding integer map. Suppose that

- there exist $a_2^1 \in A_2^1$ and $a_1^3 \in A_1^3$ such that $\varphi(a_2^1) = \varphi(a_1^3) = 1$;
- $\varphi(v) \neq 1$ for all $v \in A_3^1 \cup A_1^2$.
- there exist $a_3^1 \in A_3^1$ and $a_1^2 \in A_1^2$ such that $\varphi(a_3^1) = \varphi(a_1^2) = 2$;
- $\varphi(v) \neq 2$ for all $v \in A_2^1 \cup A_1^3$.

Let $G$ be obtained from $H$ by exponentiating $\{a_2^1, a_1^3, s_3^2\}$ and $\{a_3^1, a_1^2, s_2^3\}$, leaf triangles respectively at $s_3^2$ and $s_2^3$. We define $\mathcal{F}_3$ to be the class of all such graphs $G$.

**Lemma 6.3.13** *Every prismatic graph of the class $\mathcal{F}_3$ admits a hitting set of cardinality smaller or equal to 3.*

*Proof.* Let $S_H = \{s_3^3, s_2^3, s_3^2\}$. Note that in $K$ minus vertex $s_2^2$, $S_H$ is a hitting set, $X = \{s_3^1, s_1^3, s_2^1, s_1^2\}$ induces a $C_4$, $s_3^1, s_2^1$ do not share a common neighbour in $V(H) \setminus X$ so as $s_1^2, s_1^3$. We may apply Lemma 6.3.4. It follows that $S_H$ is a hitting set of $H$.

We may apply Lemma 6.3.5, and we conclude that $S_H$ is a hitting set of $G$. □

### 6.3.11 The class $\mathcal{F}_4$

Take $\Sigma$, with vertices numbered $r^i_j$, $s^i_j$, $t^i_j$ as usual. Let $H$ be the subgraph induced on

$$Y \cup \{s^i_j : (i,j) \in I\} \cup \{t^1_1, t^2_2, t^3_3\}$$

where $\emptyset \neq Y \subseteq \{r^3_1, r^3_2, r^3_3\}$ and $I \subseteq \{(i,j) : 1 \leq i, j \leq 3\}$ with $|I| \geq 8$ and including $\{(i,j) : 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 2\}$.

We consider $T = \{t^1_1, t^2_2, t^3_3\}$ which is a leaf triangle at $t^3_3$. Let $G$ be obtained from $H$ by exponentiating $T$. We define $\mathcal{F}_4$ to be the all such graphs $G$.

**Lemma 6.3.14** *Every prismatic graph of the class $\mathcal{F}_4$ admits a hitting set of cardinality smaller or equal to 4.*

*Proof.* Let $K$ be the subgraph of $\Sigma$ induced by the vertices: $\{r^3_1, r^3_2, r^3_3\} \cup S \cup \{t^1_1, t^2_2, t^3_3\}$.

By Lemma 6.3.3, $N_\Sigma(r^3_3) \cap K = S_K = \{s^3_1, s^3_2, s^3_3, t^3_3\}$ is a hitting set of $K$.

Since $H$ is a subgraph of $K$, $S_H = S_K \cap V(H)$ is a hitting set of $H$ and $|S_H| \leq 4$.

Since $t^3_3 \in S_H$, we can apply Lemma 6.3.5, and conclude that $S_H$ is a hitting set of $G$. $\square$

### 6.3.12 The class $\mathcal{F}_5$

Take $\Sigma$, with vertices numbered $r^i_j$, $s^i_j$, $t^i_j$ as usual. Let $H$ be the subgraph induced on

$$\{r^i_j : (i,j) \in I_1\} \cup \{s^i_j : (i,j) \in I_2\} \cup \{t^i_j : (i,j) \in I_3\}$$

where $I_1$, $I_2$, $I_3 \subseteq \{(i,j) : 1 \leq i, j \leq 3\}$ are chosen such that:

- $(1,1)$, $(3,1)$, $(3,2)$, $(3,3) \in I_1$ and $(2,2)$, $(2,3) \notin I_1$
- $(1,1) \notin I_2$
- $(1,2)$, $(1,3)$, $(2,3)$, $(3,3) \in I_3$ and $(2,1)$, $(3,1) \notin I_3$

Let $G$ be obtained from $H$ by adding the edge $r^1_1 t^1_2$. We define $\mathcal{F}_5$ to be the class of all such graphs $G$.

**Lemma 6.3.15** *Every prismatic graph in the class $\mathcal{F}_5$ admits a hitting set of cardinality smaller or equals to 5.*

*Proof.*
By Lemma 6.3.3, $N_\Sigma(r_1^1) \cap H = S_H = \{r_2^3, r_3^3, s_2^1, s_3^1, t_1^1\}$ is a hitting set of $H$.

In $\Sigma$, vertices $r_1^1$ and $t_2^1$ have as common neighbours the following vertices: $s_1^1$, $r_2^2$, $r_3^2$, $t_1^2$ and $t_1^3$. Since none of them are in $H$, the addition of the edge $r_1^1 t_2^1$ does not create another triangle.

This proves that $S_H$ is a hitting set of $G$. □

### 6.3.13 The class $\mathcal{F}_6$

Take $\Sigma$ with vertices numbered $r_j^i$, $s_j^i$, $t_j^i$ as usual. Let $H$ be the subgraph induced by

$$\{r_j^i : (i,j) \in I_1\} \cup \{s_j^i : (i,j) \in I_2\} \cup \{t_j^i : (i,j) \in I_3\}$$

where :

- $I_1 = \{(1,1),(1,2),(3,1),(3,2),(3,3)\}$,
- $I_2 = \{(1,2),(2,1),(2,2),(3,3)\}$,
- $I_3 = \{(1,2),(2,2),(1,3),(2,3),(3,3)\}$

Let $G$ be obtained from $H$ by adding the edge $r_1^1 t_2^1$ and then multiplying $\{r_3^3, t_3^3\}$. We define $\mathcal{F}_6$ to be the class of all such graphs $G$.

**Lemma 6.3.16** *Every prismatic graph in the class $\mathcal{F}_6$ admits a hitting set of cardinality smaller or equals to 3.*

*Proof.* By Lemma 6.3.3, $N_G(r_3^1) = \{r_1^3, r_2^3, s_3^3\}$ is a hitting set of $G$. □

### 6.3.14 The class $\mathcal{F}_7$

The *six-vertex prism* is the graph with six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and edges

$$a_1a_2, a_1a_3, a_2a_3, b_1b_2, b_1b_3, b_2b_3, a_1b_1, a_2b_2, a_3b_3$$

Let $K$ be a graph with six vertices, with the six-vertex prism as a subgraph. Construct a new graph $G$ as follows. The vertices of $G$ consist of $E(K)$ and some of the vertices of $K$, so $E(K) \subseteq V(G) \subseteq E(K) \cup V(K)$; two edges of $K$ are adjacent in $G$ if they have no common end in $K$; an edge and a vertex of $K$ are adjacent in $G$ if they are incident in $K$; and two vertices of $K$ are adjacent in $G$ if they are non-adjacent in $K$. The class of all such graphs $G$ is called $\mathcal{F}_7$ (they are all prismatic).

**Lemma 6.3.17** *Every prismatic graph in the class $\mathcal{F}_7$ admits a hitting set of cardinality at most* 5*.*

*Proof.*
Let $K$ and $G$ be as in the definition. Let us show that

$$S_G = E(K) \cap \{a_1a_2, a_1a_3, a_1b_1, a_1b_2, a_1b_3\}$$

is a hitting set of $G$. Let $T$ be a triangle in $G$. We now break into 4 cases.

- $|T \cap V(K)| = 1$: Such a triangle does not exist. Because if two edges in $K$ are adjacent in $G$ then they do not share a common vertex in $K$ and then they do not have in $G$ a common neighbour in $V(G) \cap V(K)$.

- $|T \cap V(K)| = 2$: Such a triangle does not exist. Indeed, if in $G$, there are two vertices $u, v \in V(K)$, both adjacent to $e \in E(K)$, then $e = uv$. A contradiction to $u$, $v$ being adjacent in $G$.

- $|T \cap V(K)| = 3$: Such a triangle does not exist. Otherwise it would induce in $K$ a stable set of cardinality 3 and there is no such stable set in the prism and hence in $K$.

- $|T \cap V(K)| = 0$: Every vertex of $G$ not in $V(K)$ is in $E(K)$. If such a triangle $T = \{e_1, e_2, e_3\}$, $(e_1, e_2, e_3 \in E(K))$ exists, then $e_1$, $e_2$ and $e_3$ do not have common ends in $K$. It follows that at least one of $e_1$, $e_2$ or $e_3$ has $a_1$ as an end point and therefore is included in $S_G$.

Hence $S_G$ is a hitting set of $G$ and $|S_G| \leq 5$.

□

### 6.3.15 The class $\mathcal{F}_8$

Let $H$ be the graph with nine vertices $v_1, \ldots, v_9$ and with edges as follows: $\{v_1, v_2, v_3\}$ is a triangle, $\{v_4, v_5, v_6\}$ is complete to $\{v_7, v_8, v_9\}$, and for $i = 1, 2, 3$, $v_i$ is adjacent to $v_{i+3}$, $v_{i+6}$. Note that $H$ is a rotator. Let $G$ be obtained from $H$ by multiplying $\{v_4, v_7\}$, $\{v_5, v_8\}$ and $\{v_6, v_9\}$. We define $\mathcal{F}_8$ to be the class of all such graphs $G$.

**Lemma 6.3.18** *Every prismatic graph in the class $\mathcal{F}_8$ admits a hitting set of cardinality smaller or equals to 3.*

*Proof.* We can easily see that $S_H = \{v_1, v_2, v_3\}$ is a hitting set of $H$. Since $\{v_4, v_7\}$, $\{v_5, v_8\}$ and $\{v_6, v_9\}$ are each not in $S_H$ and are each edges, we can successively apply Lemma 6.3.4, for each one separately and $S_H$ stays a hitting set of $G$ (note that the fact that $H$ is a prismatic graph is not used in the proof of Lemma 6.3.4). □

### 6.3.16 The class $\mathcal{F}_9$

Take $\Sigma$ with vertices numbered $r^i_j$, $s^i_j$, $t^i_j$ as usual. Let $H$ be the subgraph induced by

$$\{r^i_j : (i,j) \in I_1\} \cup \{s^i_j : (i,j) \in I_2\} \cup \{t^i_j : (i,j) \in I_3\}$$

where $I_1$, $I_2$, $I_3 \subseteq \{(i,j) : 1 \leq i, j \leq 3\}$ satisfy

- $(2,1)$, $(3,1)$, $(3,2)$, $(3,3) \in I_1$ and $I_1$ contains at least one of $(1,2)$, $(1,3)$ and $(1,1)$, $(2,2)$, $(2,3) \notin I_1$,
- $(1,1)$, $(2,2)$, $(3,3) \in I_2$ and $(1,2)$, $(1,3) \notin I_2$,
- $(1,3)$, $(2,3)$, $(3,3) \in I_3$, and $I_3$ contains at least one of $(1,2)$, $(2,2)$, $(3,2)$, and $(1,1)$, $(2,1)$, $(3,1) \notin I_3$,

- either $(1,2)$, $(1,3) \in I_1$ or $I_3$ contains $(1,2)$ and at least one of $(2,2)$, $(3,2)$.

Let $G$ be obtained from $H$ by adding a new vertex $z$ adjacent to $r_2^3$, $r_3^3$, $s_1^1$, and to $t_2^2$ if $(2,2) \in I_3$, and to $t_2^3$ if $(3,2) \in I_3$. We define $\mathcal{F}_9$ to be the class of all such graphs $G$.

**Lemma 6.3.19** *Every prismatic graph in the class $\mathcal{F}_9$ admits a hitting set of cardinality smaller or equals to 3.*

*Proof.* By Lemma 6.3.3, $N_G(r_1^1) = \{r_2^3, r_3^3, s_1^1\}$ is a hitting set of $G$. □

## 6.4 Clique cover of non-orientable prismatic graphs

In this section we show how the presence of a hitting set of cardinality bounded by a constant can be used for solving the clique cover problem. We have seen in the previous section that every non-orientable prismatic graph admits a hitting set of size at most 10. The following is more useful for algorithmic purposes.

**Theorem 6.4.1** *If $G$ is a non-orientable prismatic graph then $G$ admits a hitting set of cardinality at most 5 or $G$ is a Schläfli-prismatic graph.*

*Proof.* By Theorem 6.3.2, $G$ is obtained from a prismatic graph $H$ from the menagerie by replicating vertices not in the core of $H$ and then deleting edges between vertices not in the core.

It is easy to verify that $G$ and $H$ have exactly the same triangles. Therefore, it is obvious that if $S_H$ is a hitting set of $H$ then $S_H$ is a hitting set of $G$.

From Lemmas 6.3.7, 6.3.8, 6.3.9, 6.3.10, 6.3.11, 6.3.12, 6.3.13, 6.3.14, 6.3.15, 6.3.16, 6.3.17, 6.3.18, and 6.3.19, if $H$ is either Fuzzily Schläfli-prismatic, of parallel-square type, of skew-square type, or in $\mathcal{F}_i$, $i \in \{0, \ldots, 9\}$, then $G$ admits a hitting set of size at most 5.

It remains to consider the case where $H$ is Schläfli-prismatic. Hence, $H$ is an induced subgraph of $\Sigma$.

If every vertex of $H$ is contained in a triangle of $H$, then no vertex of $H$ can be replicated, so $G = H$ and $G$ is Schläfli-prismatic. Hence, we may assume that some vertex $v$ of $H$ is contained in no triangle of $H$.

Since $N_\Sigma(v)$ induces a matching of 5 edges in $\Sigma$ and $v$ is contained in no triangle of $H$, we see that $N_H(v)$ contains at most 5 vertices. By Lemma 6.3.3, $N_H(v)$ is a hitting set of $H$. Hence, $H$ (and therefore $G$) contains a hitting set of size at most 5. □

Observe that in a triangle-free graph, solving the clique cover problem is easily reducible to computing a matching of maximum cardinality by Edmonds' algorithm. Furthermore, the clique cover problem is solvable in constant time when the number of vertices of the input graph is bounded. Note that Schläfli-prismatic graphs have at most 27 vertices.

We need the following notation : $T(G)$ is a variant of the adjacency matrix of $G$. For $v, w \in V(G)$, the entry $(v, w)$ of $T(G)$ is 0 if $v$ and $w$ are not adjacent, 1 if they are adjacent but without common neighbour, and $x$ if they are adjacent and have $x$ as a common neighbour. Note that in the last case, if $G$ is diamond-free and $K_4$-free, then $x$ is unique.

This matrix can be computed in time $\mathcal{O}(n^3)$ at the beginning of an algorithm and used afterwards to find the triangles in $G$ or in any induced subgraph of $G$.

**Lemma 6.4.2** *Let $G \in$ Free$\{$diamond$, K_4\}$. There is an algorithm that finds a hitting set of $G$ of cardinality at most 5 if such a set exists and answers "no" otherwise. This algorithm has complexity $\mathcal{O}(n^7)$.*

*Proof.* First compute $T(G)$ in time $\mathcal{O}(n^3)$ as above. Enumerate each set $X$ of vertices of $G$ of size at most 5 in time $\mathcal{O}(n^5)$. For each $X$, test in time $\mathcal{O}(n^2)$ if $G \setminus X$ is triangle-free by checking if every entry of $T(G)$ reduced to $G \setminus X$ is either 0, 1 or an element of $X$. If no such $X$ exists answer "no" and otherwise output $X$. □

**Theorem 6.4.3** *The Clique Cover Problem for non-orientable prismatic graphs is solvable in time $\mathcal{O}(n^{7.5})$.*

*Proof.* Let $G$ be a prismatic non-orientable graph. The following method provides a minimum clique cover.

1. Compute the matrix $T(G)$ as previously defined. This can be done in time $\mathcal{O}(n^3)$.

2. Use the method in time $\mathcal{O}(n^7)$ described in Lemma 6.4.2. If the algorithm outputs a hitting set of $G$ of size at most 5 denoted by $S = \{s_1, \ldots, s_{i^*}\}$ ($i^* \leq 5$), then go to Step 4. Else by Theorem 6.4.1, $G$ is a Schläfli-prismatic graph and go to Step 3.

3. Since there is a bounded number of vertices in $G$ compute all possible clique covers of $G$ in constant time. Go to Step 5.

4. Enumerate all sets $X$ of at most 5 disjoint triangles of $G$, (this can trivially be done in time $\mathcal{O}(n^{15})$). We can do it in time $\mathcal{O}(n^5)$ as follows.

   Compute the set $\mathcal{T}_i$ of triangles containing $s_i$ for each $1 \leq i \leq i^*$. This can be done in $\mathcal{O}(n)$ by reading the line of $T(G)$ corresponding to $s_i$. Notice that there are at most $n/2$ triangles in each $\mathcal{T}_i$. Then compute all subsets $\mathcal{T}$ of triangles of $G$ obtained by choosing at most one triangle in each $\mathcal{T}_i$. For each such $\mathcal{T}$ which contains only pairwise vertex-disjoint triangles, compute by some classical algorithm a maximum matching $M_{\mathcal{T}}$ of $G \setminus (\cup_{T \in \mathcal{T}} T)$ and let $\mathcal{R}_{\mathcal{T}}$ be the vertices of $G$ that are neither in $\mathcal{T}$ nor in $\mathcal{M}_{\mathcal{T}}$. Notice that $\mathcal{T} \cup \mathcal{M}_{\mathcal{T}} \cup \mathcal{R}_{\mathcal{T}}$ is a clique cover of $G$ . Go to Step 5.

5. Among all the clique covers generated by the previous steps, let $\mathcal{C}^*$ be one of the smallest size. Return $\mathcal{C}^*$.

**Correctness:**

If $G$ has no hitting set of size at most 5 then the algorithm will consider all possible clique covers of $G$. Therefore, the algorithm will give a clique cover of minimum size.

Otherwise, let $\mathcal{C}$ be a minimum clique cover of $G$. Since $G$ is $K_4$-free, $\mathcal{C}$ is the union of a set $\mathcal{T}$ of vertex-disjoint triangles, a set $\mathcal{E}$ of vertex-disjoint edges and a set $\mathcal{R}$ of vertices. Since $\mathcal{C}$ is of minimum size, $\mathcal{E}$ should be a maximum matching in the subgraph of $G$ induced by the vertices not in any triangle of $\mathcal{T}$. Each triangle in $\mathcal{T}$ contains a vertex of the hitting set $S$ of $G$ obtained by Step 2 in the algorithm. Furthermore, each vertex of $S$ is contained in at most one triangle of $\mathcal{T}$. So the algorithm will consider $\mathcal{T}$ at some point in Step 4 and will compute a maximum matching in the remaining graph. Therefore it will return a clique cover $\mathcal{C}^*$ of same size as $\mathcal{C}$.

**Complexity**:

The procedure to enumerate all sets $\mathcal{T}$ takes time $\mathcal{O}(n^5)$. For each such set, the maximum matching can be found by Micali and Vazirani's algorithm [43] in time $\mathcal{O}(n^{2.5})$. Overall, a best clique cover is found in time $\mathcal{O}(n^{7.5})$. □

## 6.5 Orientable prismatic graphs

In the non-orientable case, we have shown that the clique cover problem is polynomial time solvable. We do not know the complexity of this problem in the orientable case. In this section we show that the vertex-disjoint triangles problem (the problem of finding a maximum number of vertex-disjoint triangles) is polynomial time solvable in prismatic graphs. As noted in the introduction, this problem is NP-hard in the general case [31].

Remark that solving the vertex-disjoint triangle problem is not sufficient to solve the clique cover problem in orientable prismatic graph. See Figure 6.6.

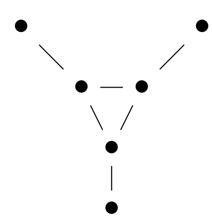
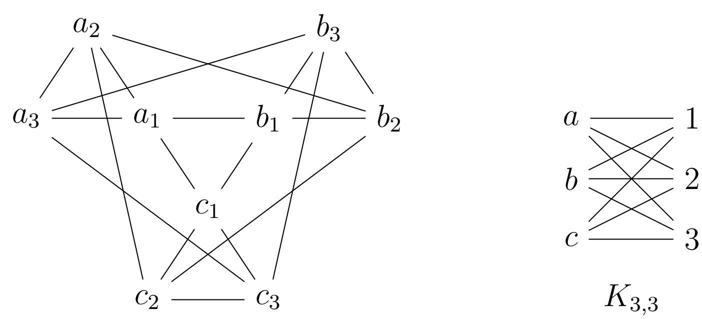


Figure 6.6: An orientable prismatic graph where the best clique cover is not obtained by first selecting the maximum number of disjoint triangles

The *derived graph* $D(G)$ of a graph $G$ is the intersection graph of the triangles of $G$. More formally, if $H = D(G)$, then $V(H) = \{T : T \text{ is a triangle in } G\}$. Two vertices of $H$ are adjacent if they are distinct triangles of $G$ sharing at least one vertex. Note that the class of derived graphs is not hereditary.




$L(K_{3,3})$; (See also Fig. 6.5)

Figure 6.7: $D(L(K_{3,3})) = K_{3,3}$

**Theorem 6.5.1** *Let $G$ an orientable prismatic graph. Every connected component of $D(G)$ is claw-free or is isomorphic to $K_{3,3}$.*

*Proof.* Let $D$ be a connected component of $D(G)$ containing a claw. Hence, $G$ has to contain 4 triangles as represented on Figure 6.8 (not all edges of $G$ are represented). We will use the notation given there and we denote by $K$ the set of vertices $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\}$.

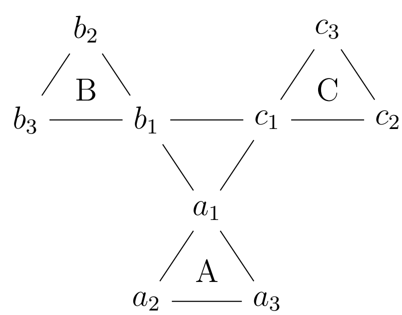



Figure 6.8: A graph whose derived graph is $K_{1,3}$

Since $G$ is prismatic there should be a matching between the extremities of the edge $a_2a_3$ and those of $b_2b_3$, $c_2c_3$.

Without loss of generality we may assume that these matching edges are: $a_2b_2$, $a_3b_3$ and $a_2c_2$, $a_3c_3$.

Now there are two possibilities for the matching between $b_2b_3$, $c_2c_3$. If $b_2c_3, b_3c_2 \in E(G)$, then $G[K]$ is a rotator of center $\{a_1, b_1, c_1\}$, a contradiction to $G$ being orientable (Theorem 6.2.1). We can now assume that $b_2c_2, b_3c_3 \in E(G)$. So $G[K]$ contains $L(K_{3,3})$.

It remains to show that there is no other vertex in $D$. Assume that it is not the case, then there is a triangle $T$ in $G$, that intersects $K$. Since in $G[K]$, every vertex and edge is in a triangle, $|T \cap K| = 1$.

Since $L(K_{3,3})$ is vertex transitive, we may assume up to symmetry that $T = \{u, v, b_2\}$. Since $T$ and $\{a_1, b_1, c_1\}$ form a prism, we may assume without loss of generality that $ua_1$ and $vc_1$ are edges of $G$. Now, because of triangles $T$ and $A$, $va_3 \in E(G)$ and because of triangles $T$ and $C$, $uc_3 \in E(G)$. Then $\{u, v, b_2, a_2, a_3, a_1, c_2, c_3, c_1\}$ induces a rotator with center $\{a_2, b_2, c_2\}$, a contradiction to $G$ being orientable. □

**Theorem 6.5.2** *The vertex-disjoint triangle problem is $\mathcal{O}(n^5)$-time solvable in prismatic graphs.*

*Proof.* Let G be a prismatic graph. If $G$ has at most 27 vertices, we solve the problem in constant time. Otherwise, we look by Lemma 6.4.2, for a hitting set of size at most 5 in $G$. If one exists, then we know that at most 5 disjoint triangles exist in $G$, and we find an optimal set of vertex-disjoint triangles of $G$ in time $\mathcal{O}(n^5)$ as in the proof of Theorem 6.4.3. Hence, we may assume that no hitting set of size at most 5 exists. By Theorem 6.4.1, $G$ is orientable.

A set $R$ is a stable set of $D(G)$ if and only if it is a set of vertex-disjoint triangles of $G$. Hence, it is enough to compute a maximum stable set in $D(G)$. Such a set can obviously be found by computing a maximum stable set in each connected component of $D(G)$. By Lemma 6.5.1, each such component is either isomorphic to $K_{3,3}$ or claw-free. The components that are isomorphic to $K_{3,3}$ are handled trivially. In the components that are claw-free, to find a maximum stable set, we may rely on the classical algorithm of Sbihi [50] ($\mathcal{O}(n^3)$). □

## 6.6 A shorter proof for a weaker result

Sepehr Hajebi [32] noted that the existence of a hitting set of bounded size (namely 15) in any non-orientable prismatic graph $G$ can be deduced from several parts of [12] with a small amount of additional work as follows.

Consider a non-orientable prismatic graph $G$. By Theorem 6.1 in [12], $G$ contains either a twister or a rotator.

If $G$ contains a twister $Z$ (so $|V(Z)| = 10$) but does not contain any rotator, then (5) in the proof of 7.2 from [12] shows that $V(Z)$ is a hitting set of $G$.

If $G$ contains a rotator, then Hajebi's strategy is to rely on results from Section 10 of [12]. This section is about graphs that contain a rotator and no so-called "square-forcer" (we do not need the definition), but the argument relies only on 10.3, where the assumption that there is no square-forcer is not used.

From here on, we use notation from [12].

In 10.3, the set of all triangles of $G$ is described, and it is proved that they all fall in one of the following categories: subsets of $S$ (where $S$ is a set of cardinality at most 9), $R$-triangles, $T$-triangles, diagonal triangles and marginal triangles. The definition of these categories implies that the set of vertices $K = \{r_1^3, r_2^3, r_3^3, t_3^1, t_3^2, t_3^3\} \cup S$ is a hitting set of $G$. It has size at most 15 (because $|S| \leq 9$).

## 6.7 Concluding remarks

Despite our efforts to use the results of [11], the complexity of the clique cover problem remains unknown for orientable prismatic graphs. As explained in the section 6.2, having an orientation implies that the triangles in the graph have to be "ordered" in some kind of way. It is quite logical to see that it would then be possible to have an arbitrary large number of disjoint triangles. It prevents the use of the same tools as in Theorem 6.4.3 to solve the clique covering problem in orientable prismatic graphs.

Chudnovsky and Seymour described a subclass of orientable prismatic graphs that intrigued us : the *path of triangle graphs*(the description is not given here and can be found in [11]). These graphs seem quite simple at first sight and we first believed that solving the clique cover in the path of triangle graphs would be easy. But even in those graphs with a simple structure, difficulties were encountered. The rigid structure around triangles have thwarted our NP-completeness hopes, while the freedom in the structure of vertices not in triangles was an obstacle for some polynomial results.

However, here is a simple remark. Suppose that $\{K_1, \dots, K_l\}$ is an optimal clique cover of a prismatic graph $G$. If for some $1 \leq i, j \leq l$, $K_i$ is a triangle and $K_j$ an isolated vertex $v$, then by prismaticity, $v$ has a neighbour $u$ in $K_i$. Hence, we may replace $K_i$ and $K_j$ by $K_i \setminus \{u\}$ and $K_j \cup \{u\}$. We may iterate this until the optimal cover does not contain simultaneously a triangle and an isolated vertex. It follows that for every prismatic graph, there exists an optimal cover that is either made only of triangles and edges, or made only of edges and isolated vertices. Observe that an optimal clique cover of the last kind is easily computable in polynomial time by some classical algorithm for the maximum matching. Hence, to solve the clique cover problem, it is enough to find an optimal clique cover of the first kind.

# Chapter 7

# Conclusion

Here, we give a short summary of the results obtained in this thesis and some directions for further research.

In Chapter 4 we provided a structural result for graphs with all holes having the same length that is odd and at least 7 (class $\mathcal{C}_k$ for an odd integer $k \geq 7$). One of the basic classes is new and fully described. The main theorem could be used as a decomposition theorem. The next step could be to generalise this result to have a structural theorem for even-hole-free graphs in Free$\{C_5, \text{proper wheel}\}$. The idea is to relax the constraints on the length of the principal paths. Another question is: Is it possible to use this structure for solving the coloring problem when restricted to $\mathcal{C}_k$ in polynomial time? The blow-up operation seems to be the main obstacle to that goal.

In Chapter 5 we presented a few partial results on the structure of graphs in Free$\{C_4, 4K_1\}$. It is one of the three minimal open cases considering the complexity of the coloring problem restricted to classes of graphs defined by excluding graphs of order 4. Two possibilities appear to us for a future work in closing the dichotomy of the complexity of the coloring problem when restricted to Free$\{C_4, 4K_1\}$. The first possibility is to use Lemma 5.3.4 for studying the structure of 3-CP graphs in Free$\{C_4\}$ that contain a $C_5$. Note that $Ico^{-2}$, $C_6^+$, $C_5 + K_1$, $\Pi_5$ and $F_{13}$ all contain a $C_5$. We wonder if for any graph $G$ in Free$\{C_4, 4K_1\}$ that is 3-CP (graphs whose vertex set can be partitioned into 3 cliques), there exists a partition of $V(G)$ into a bounded number of cliques that are either complete or anticomplete except of some disjoint couples. Using Lemma 5.3.2 it would answer the question whether the coloring problem is polynomial time solvable when restricted to 3-CP graphs in Free$\{C_4, 4K_1\}$. The other possibility is to use the program in appendix A with two directions. The first direction consists of running the program with graphs of order 13 excluding the ones that contain $\{C_4, 4K_1\}$-fixers of order 12. The second direction consists of taking a step back and looking for graphs of order 11 in Free$\{C_4, 4K_1\}$ such that the addition of any vertex to them results in the creation of a twin, a universal vertex, a graph not in the class or a $\{C_4, 4K_1\}$-fixer of order 12. It should output at least the icosahedron minus one vertex. The purpose of these two directions is to answer the questions : Are there finitely many $\{C_4, 4K_1\}$-fixers and how general are they?

In Chapter 6 we provided an $\mathcal{O}(n^{7.5})$-time algorithm to solve the clique cover problem in non-orientable prismatic graphs and a polynomial-time algorithm that solves the vertex-disjoint triangles problem in prismatic graphs. The class of prismatic graphs is the complement class of one of the three minimal open cases considering the complexity of the coloring problem restricted to classes of graphs defined by excluding graphs of order 4. Despite our efforts to use the results of [11], the complexity of the clique cover problem remains open for orientable prismatic graphs. A step that seems achievable is to answer the question whether the coloring problem is polynomial or NP-complete for path of triangles graphs.

# Table of Notations

| Notations about a graph $G$ | |
|---|---|
| $N_G(v)$ | set of neighbor of $v$ in $G$ |
| $G[V']$ | graph induced from $G$ by the set of vertices $V'$ |
| $\overline{G}$ | complement of $G$ |
| $L(G)$ | line graph of $G$ |
| $\chi(G)$ | minimum number of colors needed to have a proper coloring of $G$ |
| $\omega(G)$ | size of the maximum clique |
| $\alpha(G)$ | size of the maximum stable set |
| $\sigma(G)$ | size of the minimum clique cover |
| $cw(G)$ | clique-width of $G$ |

| Notations for some particular graphs | |
|---|---|
| $P_k$ | path with $k$ vertices |
| $K_k$ | complete graph with $k$ vertices |
| $C_k$ | cycle with $k$ vertices |
| $K_{k,l}$ | complete bipartite graph with one side of the bipartition of size $k$ and the other side of size $l$ |

| When all holes have the same length | |
|---|---|
| $\mathcal{C}_k$ | class of graph with all holes having length $k$ |
| $K_u$ | clique blown up from $u$ |
| $H_x$ | hyperedge corresponding to the vertex $x \in B \cup B'$ in a template |
| $s(G, G^*)$ | Domination score of $G$ w.r.t. $G^*$ |

| Coloring antiprismatic graphs | |
|---|---|
| $\Sigma$ | complement of the Schläfli graph |
| $R$ $S$ and $T$ | tile of a Schläfli prismatic graph |
| $\mathcal{F}_i$ | one of the 10 classes for the menagerie of prismatic graphs |
| $D(G)$ | derived graph for prismatic graphs |

# Index

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# Appendix A

# Code for Chapter 5

The python program presented in Chapter 5 can be downloaded using following the link: `https://github.com/CleopheeR/Code_de_These.git`

The program first generates all graphs in Free$\{C_4, 4K_1\}$ of order $n$. Once it is done, it checks if it is possible to add another vertex without creating a twin, a universal vertex and still obtain a graph in Free$\{C_4, 4K_1\}$. We used the graph tool library [46].