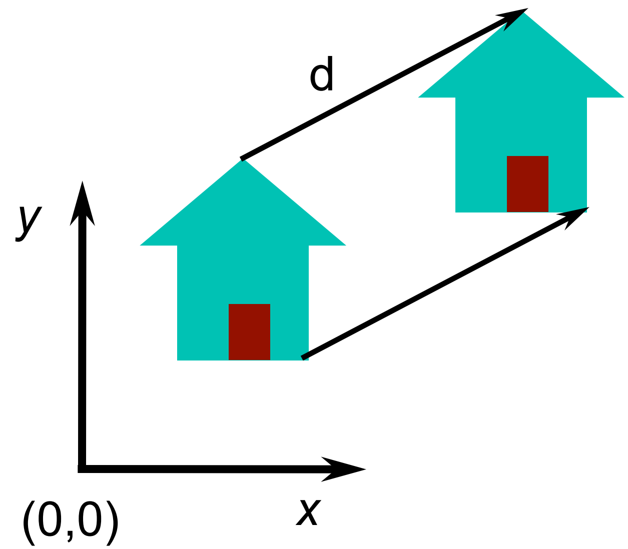
Elementary Affine Transformations

Any affine transformation is equivalent to a combination of four elementary affine transformations

- Translation
- Scaling
- Rotation
- Shear

Translation

$$Q = P + \mathbf{d}, \mathbf{d} = (T_x \;\; T_y)^T$$

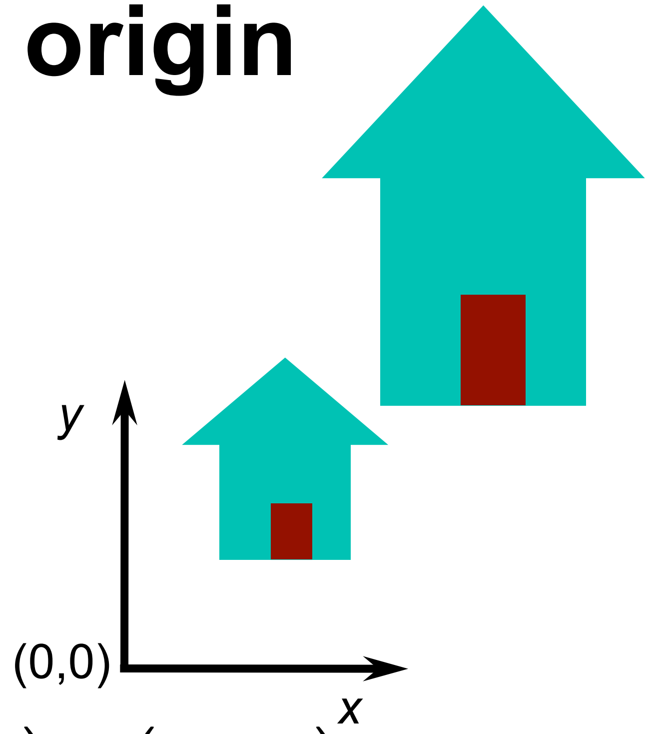


$$Q_x = P_x + T_x$$

$$Q_y = P_y + T_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Scaling around the origin



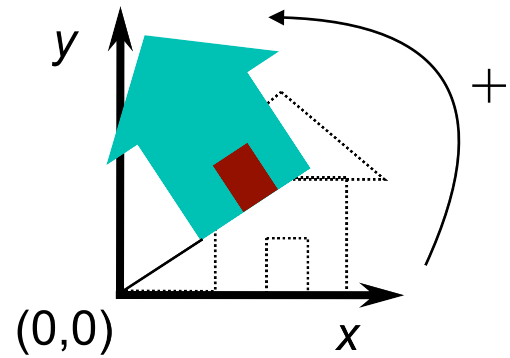
$$Q_x = s_x P_x$$

$$Q_y = s_y P_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

$$Uniform: s_x = s_y$$

Rotation around the origin



$$Q_x = cos\theta P_x - sin\theta P_y$$

$$Q_y = sin\theta P_x + cos\theta P_y$$

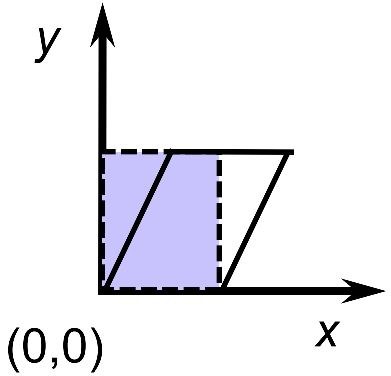
$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Exercise: Prove that the length of P is preserved, i.e. |Q| = |P|

Shear around the origin

In the x-direction

$$Q_x = P_x + aP_y$$
$$Q_y = P_y$$



$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Zero point

All elementary affine transformations have one or more ‘zero’ or ‘identity’ points **except translation**

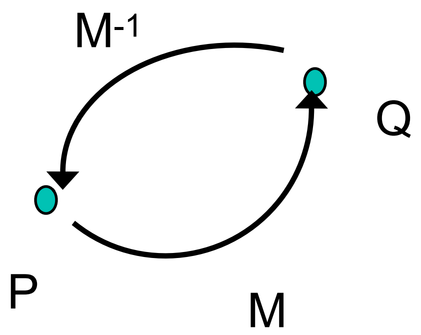
$$F(P) = P$$

What are these points for Shear(), Scale() and Rotate() ?

Inverse of a Transformation

Cramer's rule or we can be smarter

- Inverse transformation: $Q = MP$, $P = M^{-1}Q$



Inverse of Translation

$$Q = T(\mathbf{d})\mathbf{P} \rightarrow \mathbf{P} = \mathbf{T}(-\mathbf{d})\mathbf{Q}$$

$$\begin{pmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & -T_x \\ 0 & 1 & -T_y \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse of Scaling

$$Q = S(\mathbf{s})P \rightarrow P = S(1/s_x, 1/s_y)Q$$

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse of Rotation

$$Q = R(\theta)P \rightarrow P = R(-\theta)Q$$

$$\begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse of a Shear in x

$$Q = Sh_x(a)P \rightarrow P = Sh_x(-a)Q$$

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Composing 2D Affine Transformations

Composing two affine transformations produces an affine transformation

$$Q = T_2(T_1(P))$$

In matrix form:

$$Q = M_2(M_1 P) = (M_2 M_1)P = MP$$

Which transformation happens first?

Main Points

Any affine transformation can be performed as series of elementary transformations.

Affine transformations are the main modeling tool in graphics.

Make sure you understand the order.

Examples

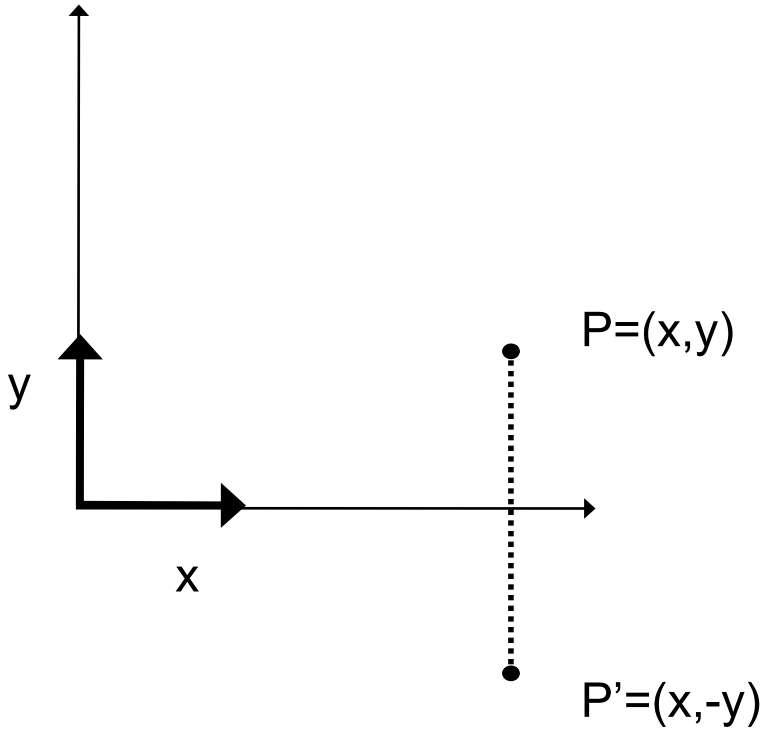
Reflection

Rotation about an arbitrary pivot point

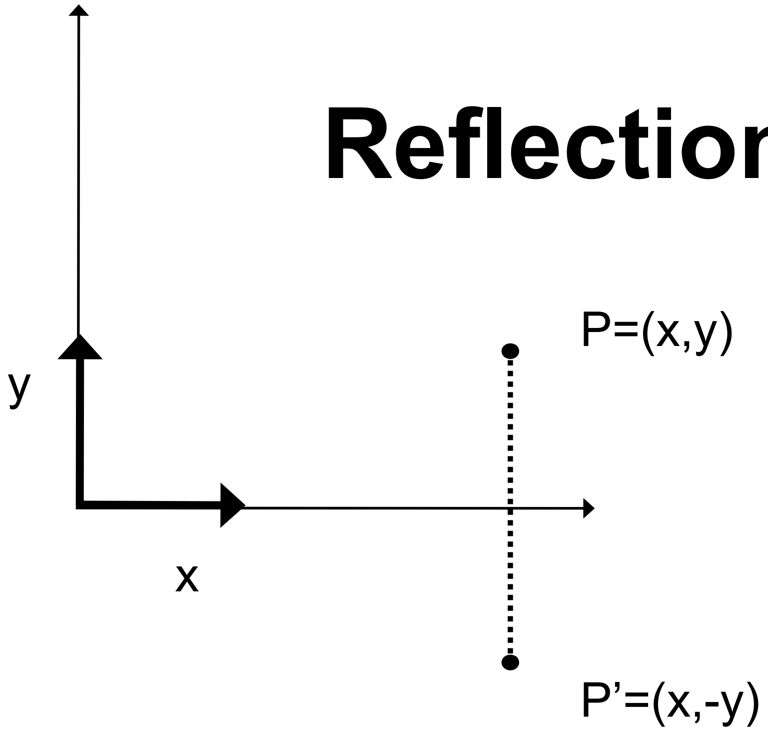
Scaling around an arbitrary point

Reflection about a tilted line

Reflection



Reflection

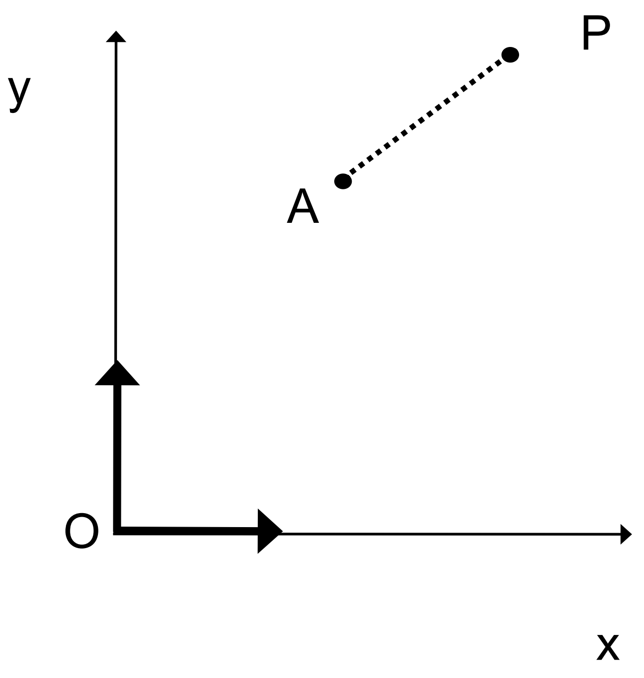


P'= Scale(1,-1)(P)=Reflect_y(P)

$$\begin{bmatrix} x \\ -y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

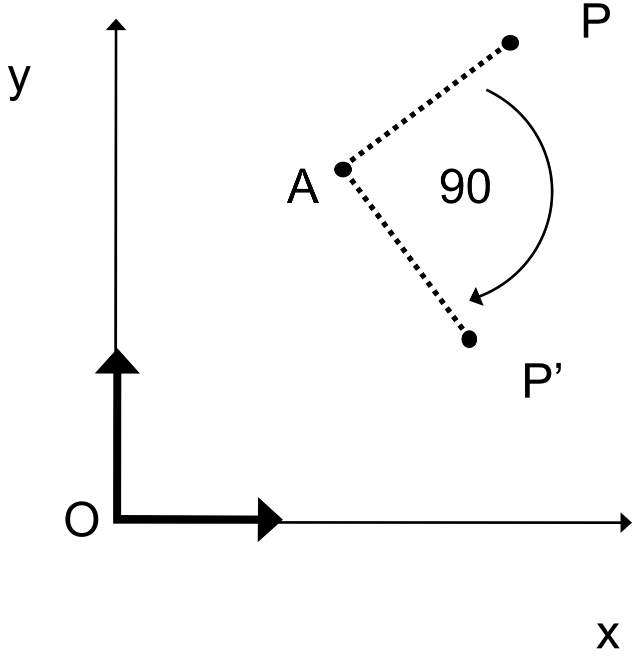
Rotate around an arbitrary point

Rotate around an arbitrary point O by -90 degrees:



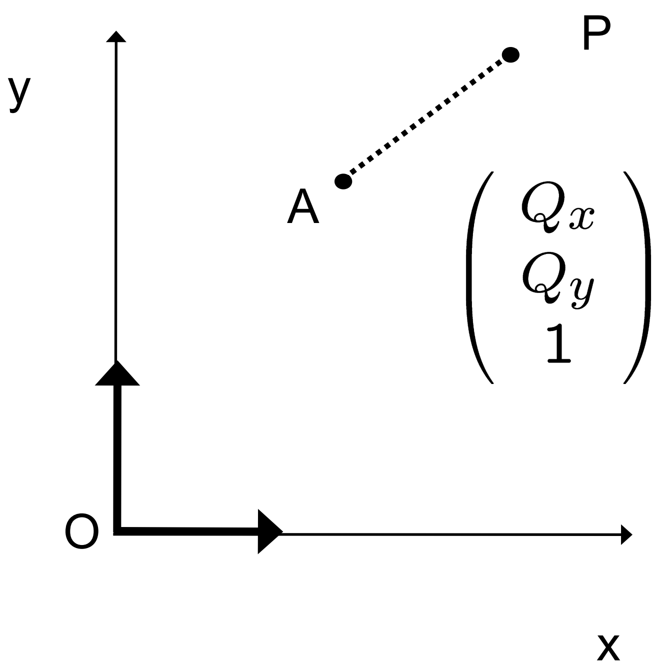
Rotate around an arbitrary point

Rotate around an arbitrary point A by -90 degrees:



Rotate around an arbitrary point

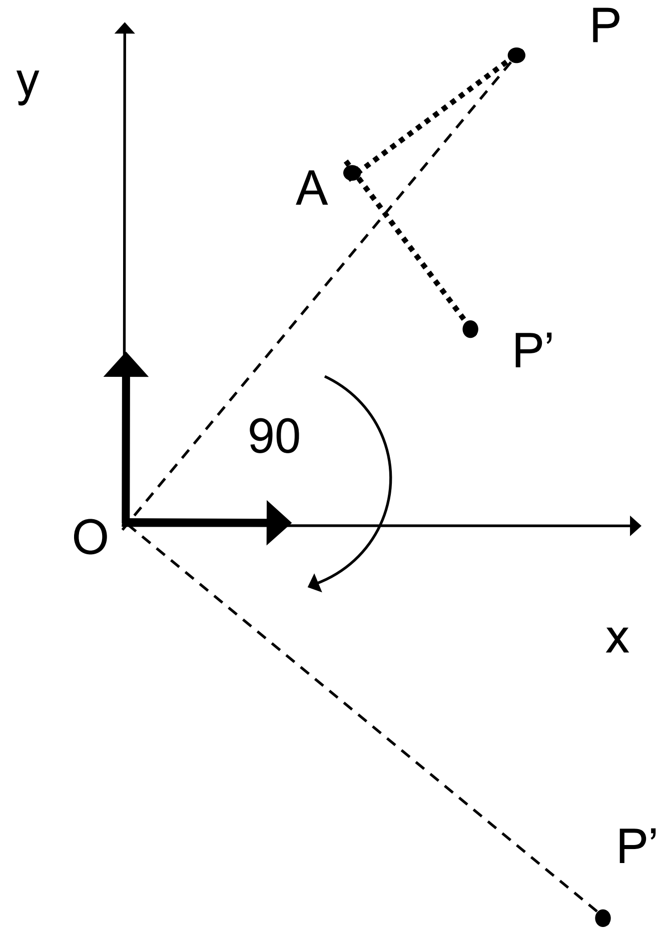
We know how to rotate around the origin



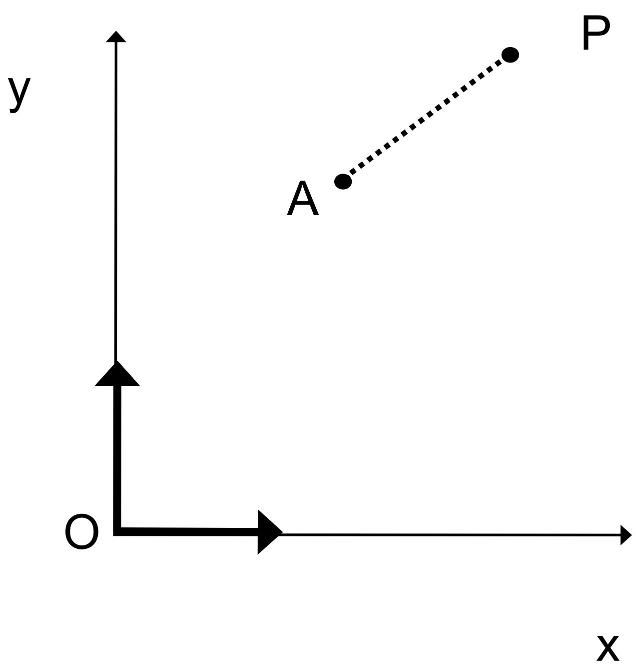
$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Rotate around an arbitrary point

…but that is not what we want to do!

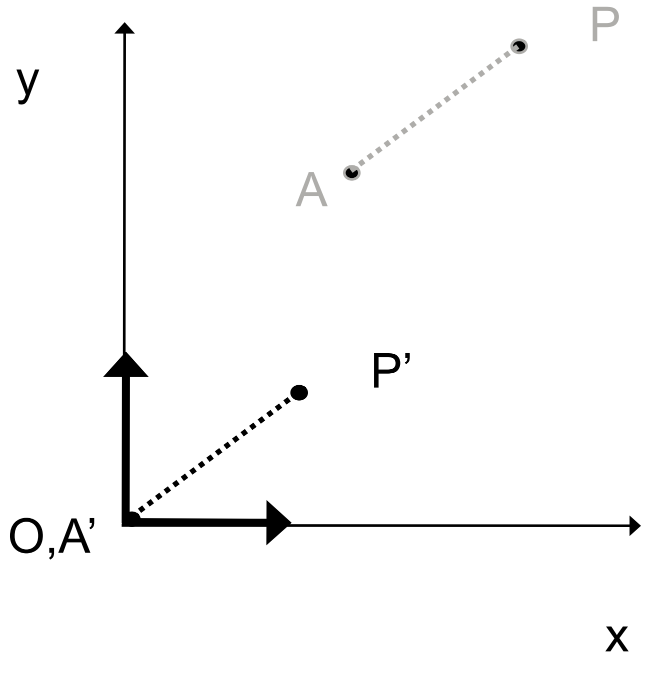


So what do we do?



Transform it to a known case

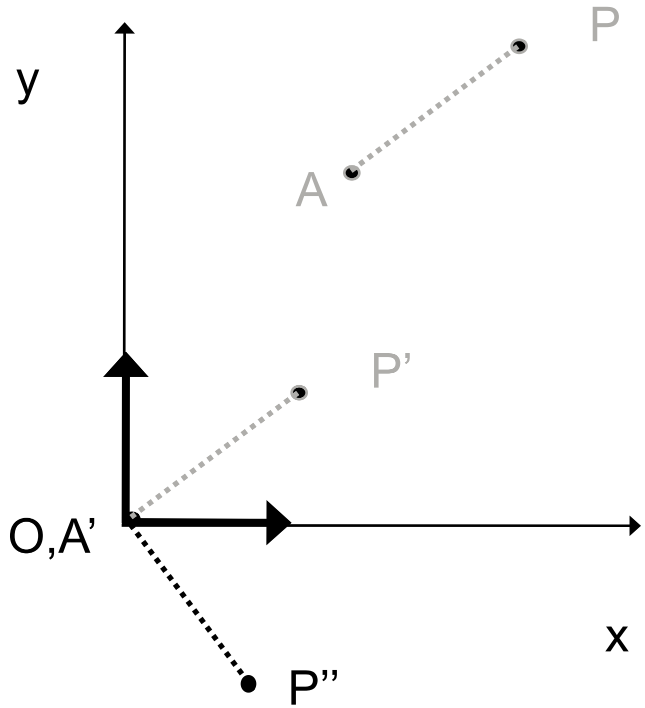
Translate(-Ax,-Ay)



Second step: Rotation

Translate(-Ax,-Ay)

Rotate(-90)

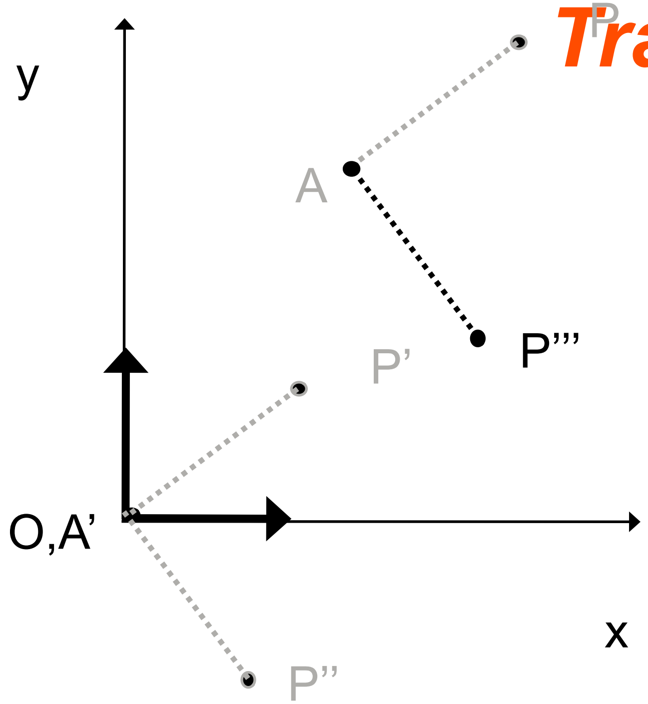


Final: Put everything back

Translate(-Ax,-Ay)

Rotate(-90)

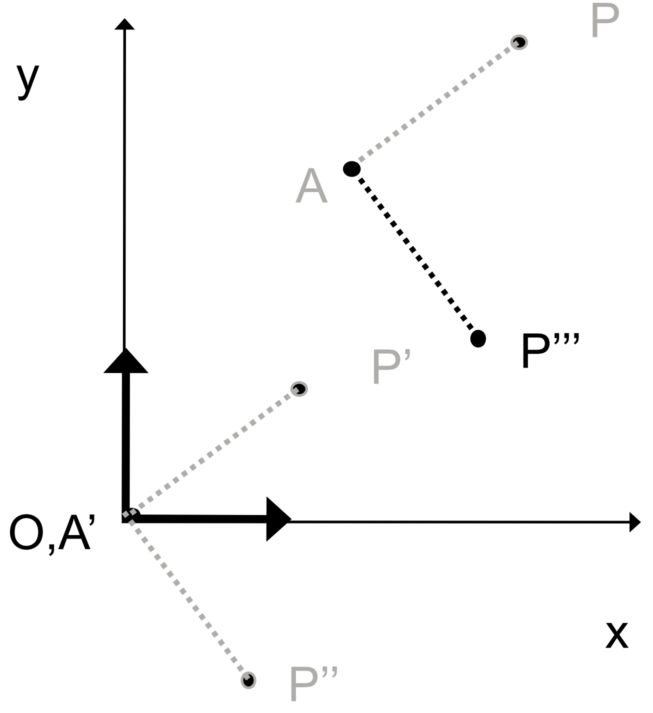
Translate(Ax,Ay)



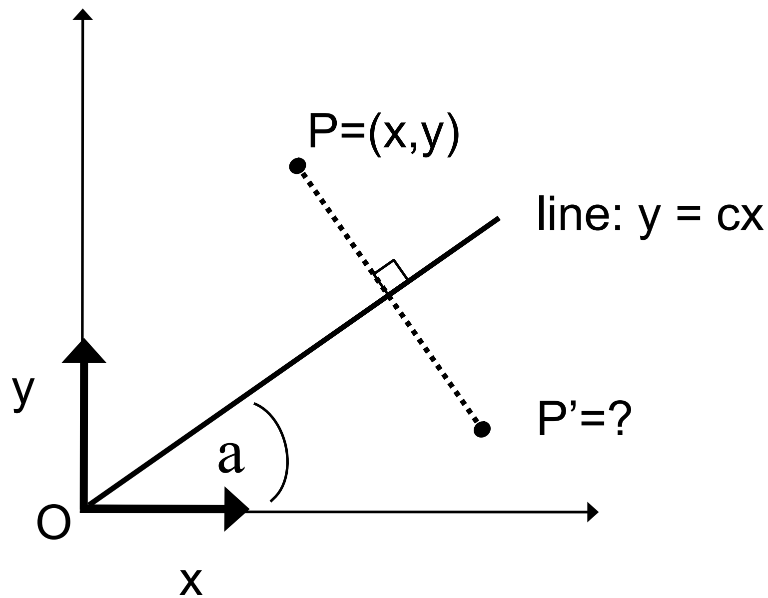
Rotation about arbitrary point

IMPORTANT!: Order

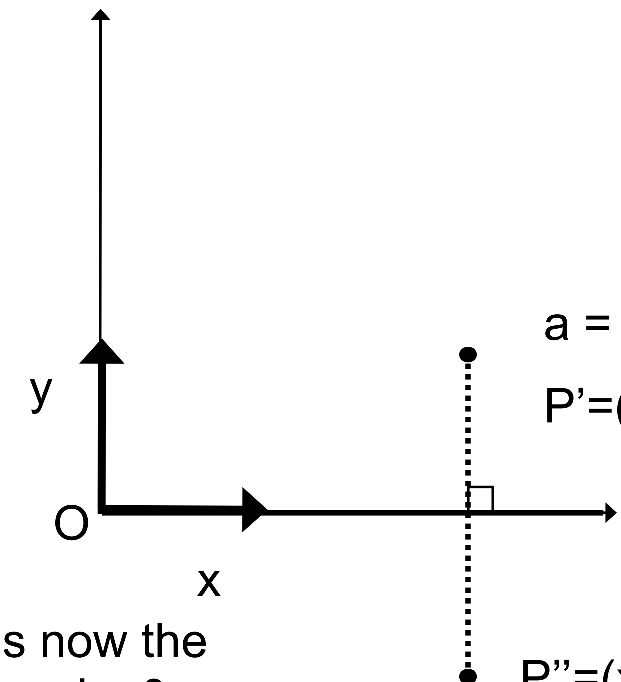
$$M = T(Ax,Ay)R(-90)T(-Ax,-Ay)$$



Reflection about a line through origin



Reflection about a line through origin



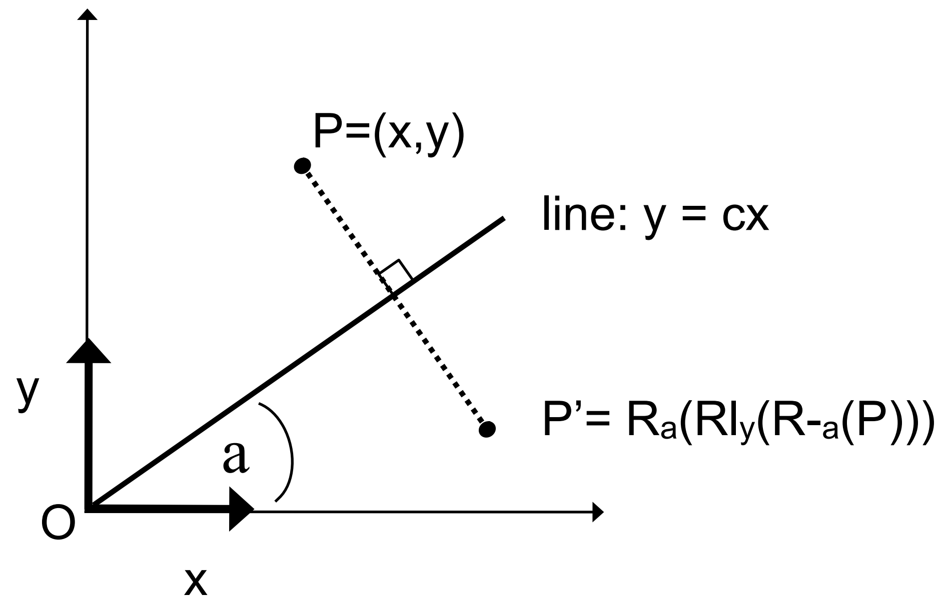
a = atan(c)

$P' = (x', y')$ = Rotate(-a)(P) = $R_{-a}(P)$

line is now the
x axis, y' = 0

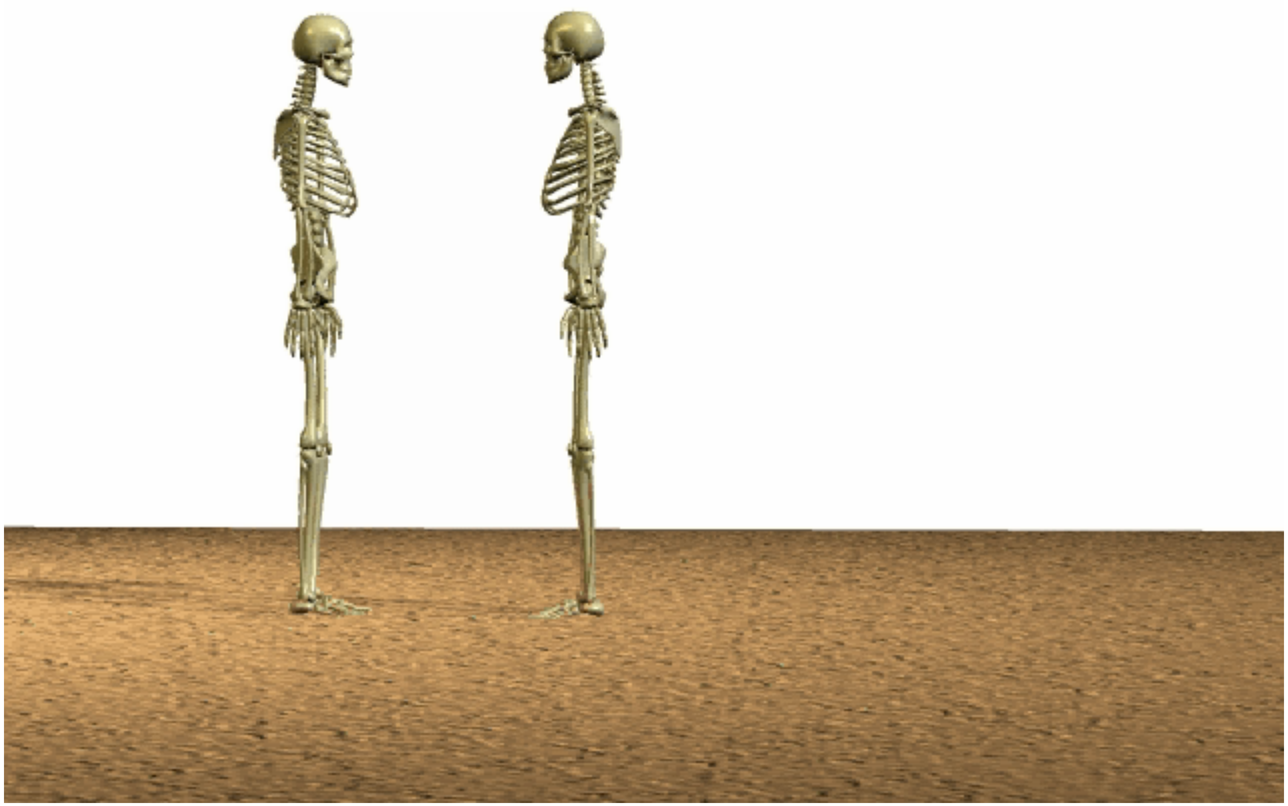
$P'' = (x', -y')$ = Reflect_y(P')) = $Rl_y(R_{-a}(P))$

Reflection about a line through origin



Put everything back and ignore intermediate points

Affine Transformations in 3D



Affine Transformations in 3D

General form

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Elementary 3D Affine Transformations

Translation

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

Scaling Around the Origin

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

- Uniform scaling: $s_x = s_y = s_z$

Shear around the origin

Along x-axis

$$\begin{pmatrix} Q_x \\ Q_y \\ Q_z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \\ 1 \end{pmatrix}$$

3 DRotation

Various representations

Decomposition into axis rotations (x-roll, y-roll, z-roll)

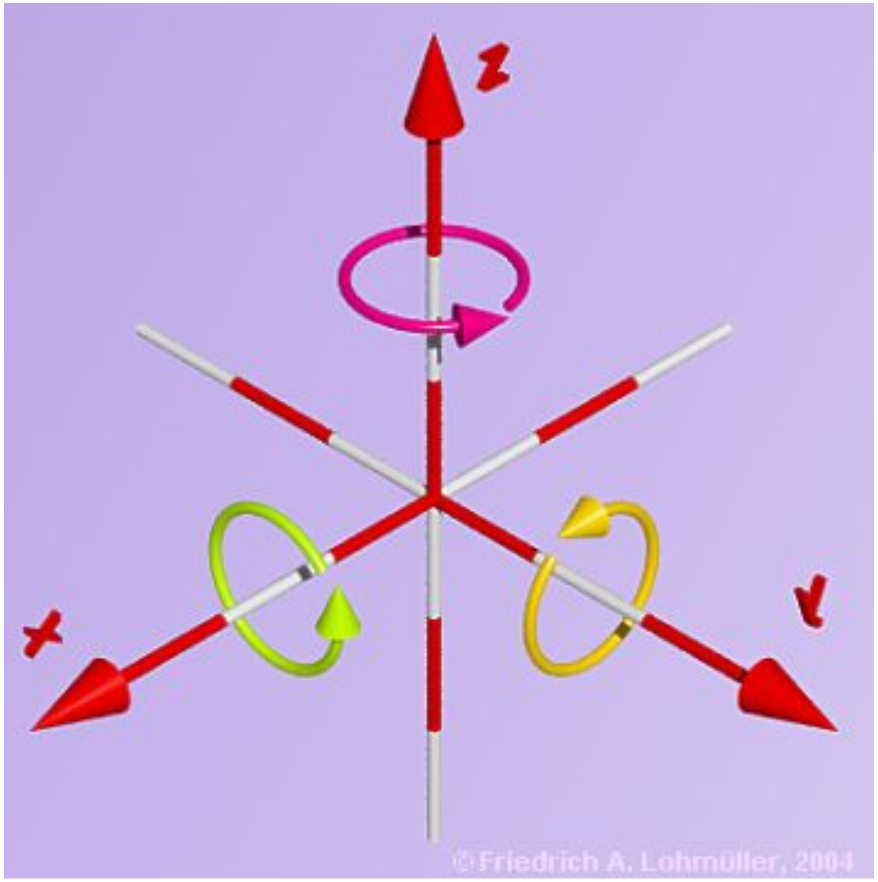
CCW positive assumption

Reminder 2D z-rotation

$$Q_x = cos\theta P_x - sin\theta P_y$$
$$Q_y = sin\theta P_x + cos\theta P_y$$

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Three axis to rotate around



Z-roll

$$Q_x = cos\theta P_x - sin\theta P_y$$
$$Q_y = sin\theta P_x + cos\theta P_y$$
$$Q_z = P_z$$

$$R_z(\theta) = \begin{pmatrix} cos(\theta) & -sin(\theta) & 0 & 0 \\ sin(\theta) & cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

X-roll

Cyclic indexing

$$x \rightarrow \boxed{y \rightarrow z \rightarrow x} \rightarrow y$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ \boxed{\begin{matrix} y \\ z \\ x \end{matrix}} \\ y \end{bmatrix}$$

$$Q_y = cos\theta P_y - sin\theta P_z$$
$$Q_z = sin\theta P_y + cos\theta P_z$$
$$Q_x = P_x$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Y-roll

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ \boxed{\begin{matrix} z \\ x \\ y \end{matrix}} \end{bmatrix}$$

$$Q_z = cos\theta P_z - sin\theta P_x$$

$$Q_x = sin\theta P_z + cos\theta P_x$$

$$Q_y = P_y$$

$$R_y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta) & 0 & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Inversion of transformations

Translation: $T^{-1}(a,b,c) = T(-a,-b,-c)$

Rotation: $R^{-1}_{axis}(b) = R_{axis}(-b)$

Scaling: $S^{-1}(sx,sy,sz) = S(1/sx,1/sy,1/sz)$

Shearing: $Sh^{-1}(a) = Sh(-a)$

Inverse of Rotations

Pure rotation only, no scaling or shear.

$$M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

$$M^{-1} = M^T$$

Composition of 3D Affine Transformations

The composition of affine transformations is an affine transformation.

Any 3D affine transformation can be performed as a series of elementary affine transformations.

Composite 3D Rotation around origin

$$R = R_z(\theta_3)R_y(\theta_2)R_x(\theta_1)$$

The order is important !!

It is often convenient to use other representations for 3D rotations….

Gimbal lock

$$R(\theta_1, \theta_2, \theta_3) = R_z(\theta_3) R_y(\theta_2) R_x(\theta_1)$$

$$\begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos(\theta_2) & 0 & sin(\theta_2) & 0 \\ 0 & 1 & 0 & 0 \\ -sin(\theta_2) & 0 & cos(\theta_2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta_1) & -sin(\theta_1) & 0 \\ 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R(\theta_1, 90^o, \theta_3) = R_z(\theta_3) R_y(90^o) R_x(\theta_1)$$

$$\begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta_1) & -sin(\theta_1) & 0 \\ 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & cos\theta_1) & -sin(\theta_1) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Loss of degree of freedom

$$R(\theta_1, 90^o, \theta_3) = \begin{pmatrix} cos(\theta_3) & -sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & sin(\theta_1) & cos(\theta_1) & 0 \\ 0 & cos\theta_1) & -sin(\theta_1) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & cos(\theta_3)sin(\theta_1) - sin(\theta_3)cos(\theta_1) & cos(\theta_3)cos(\theta_1) + sin(\theta_3)sin(\theta_1) & 0 \\ 0 & cos(\theta_3)cos(\theta_1) + sin(\theta_3)sin(\theta_1) & -cos(\theta_3)sin(\theta_1) + sin(\theta_3)cos(\theta_1) & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & sin(\theta_1 - \theta_3) & cos(\theta_1 - \theta_3) & 0 \\ 0 & cos(\theta_1 - \theta_3) & -sin(\theta_1 - \theta_3) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = R(\theta) \qquad (\theta_1, \theta_3) \to \theta = (\theta_1 - \theta_3)$$

Rotation around an arbitrary axis

Euler's theorem: Any rotation or sequence of rotations around a point is equivalent to a single rotation around an axis that passes through the point.

What does the matrix look like?