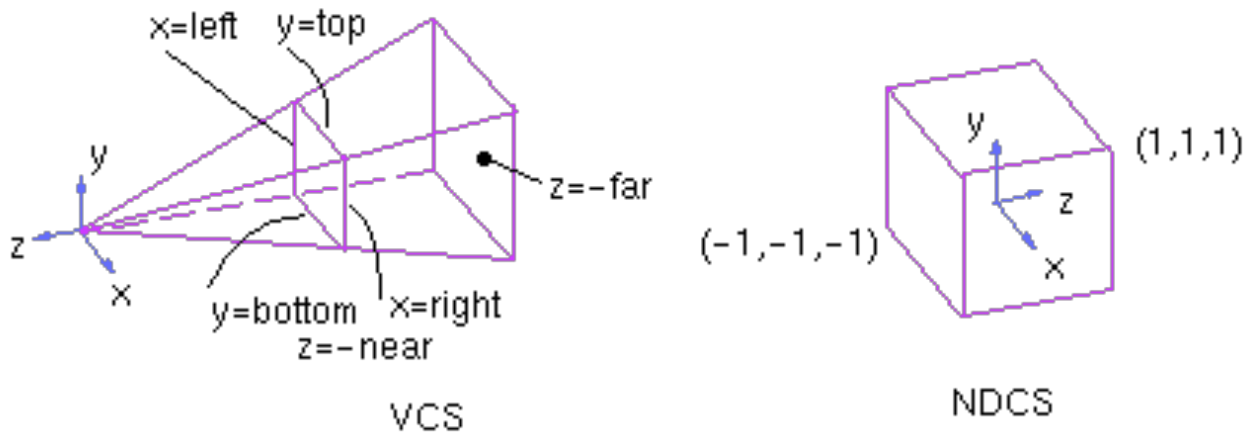
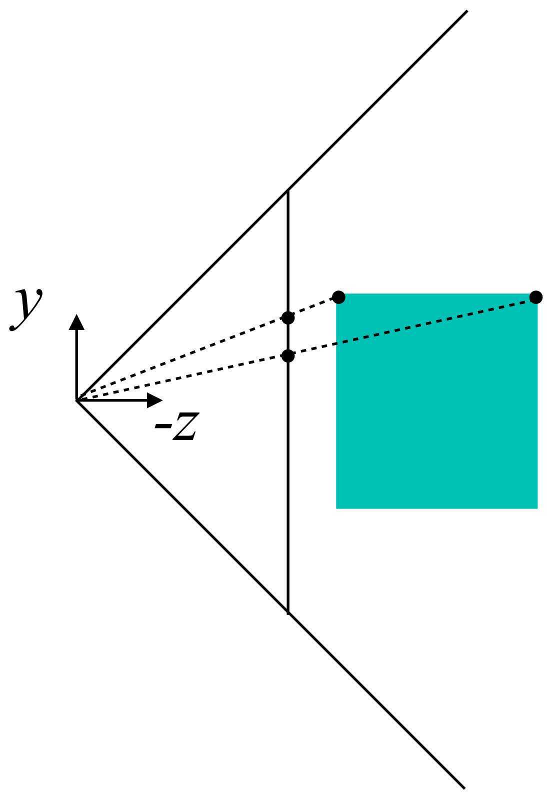
Derivation of the perspective transformation

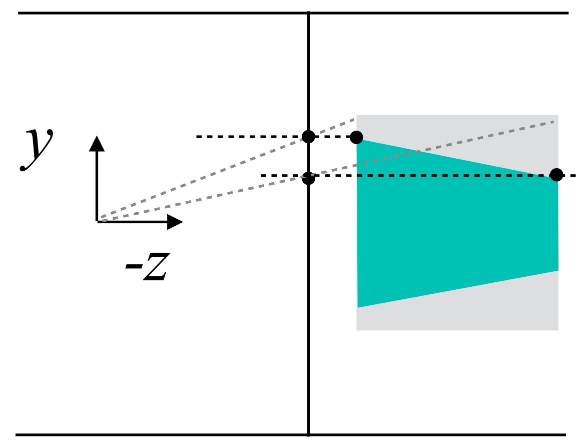


- It is basically a mapping of planes
- Normalized view volume is a left handed system
- However, there is an easier derivation

Interpretation of the Perspective transformation

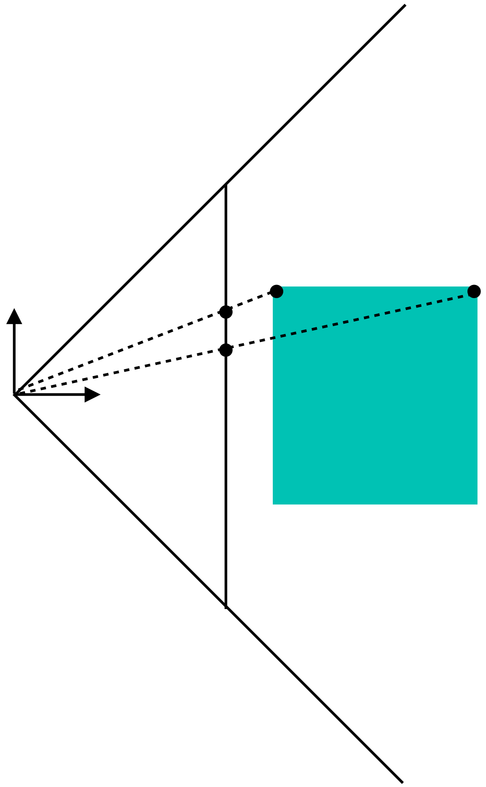


Divide the vertices by their z and then view with orthographic projection



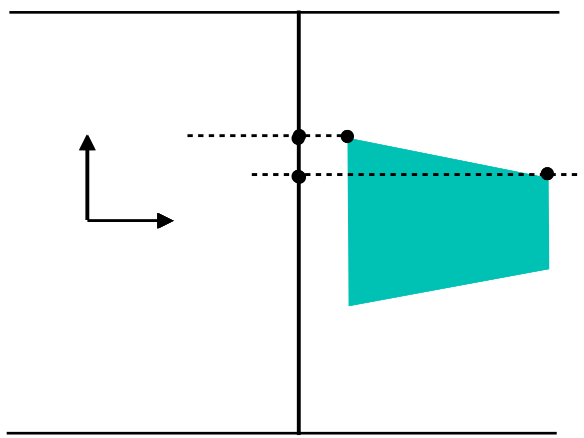
Think of the centre of projection moving at infinity

Interpretation of the Perspective transformation



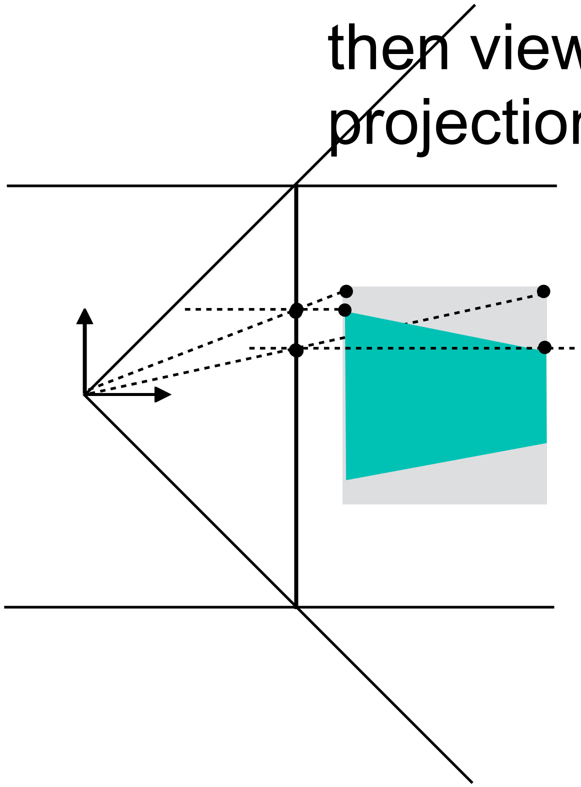
Interpretation of the Perspective transformation

Divide the vertices by their z and then view with orthographic projection

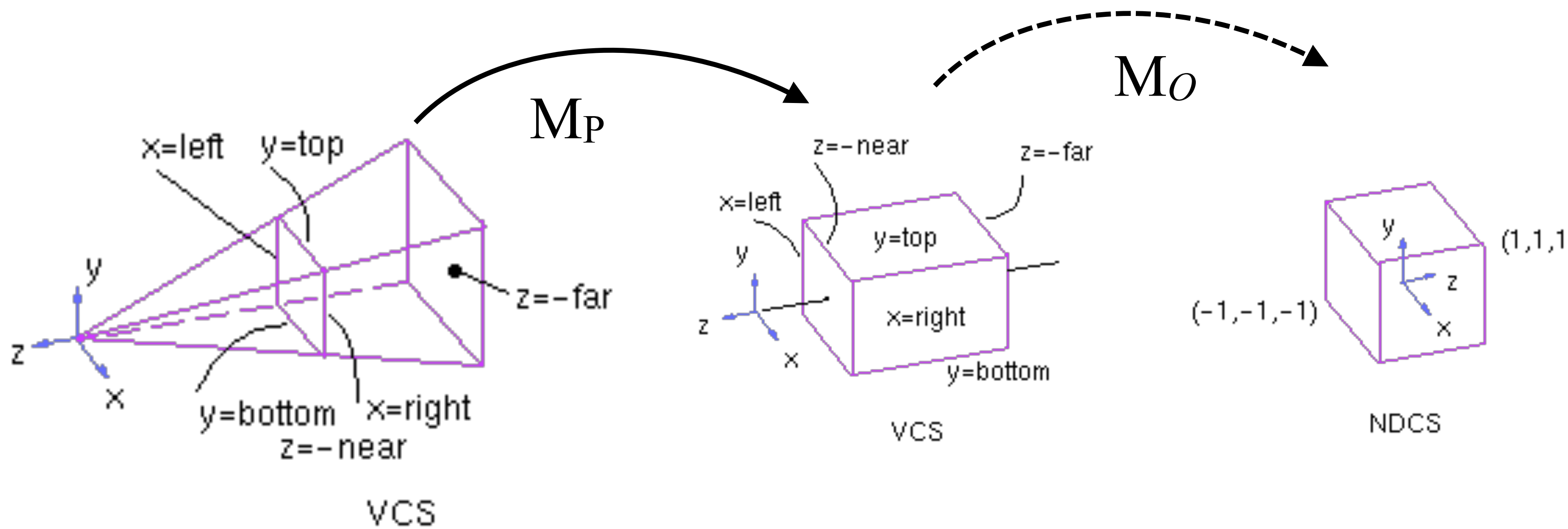


Interpretation of the Perspective transformation

Divide the vertices by their z and then view with orthographic projection

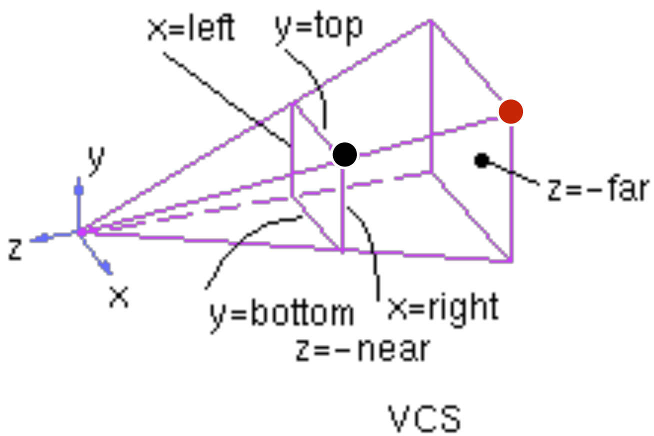


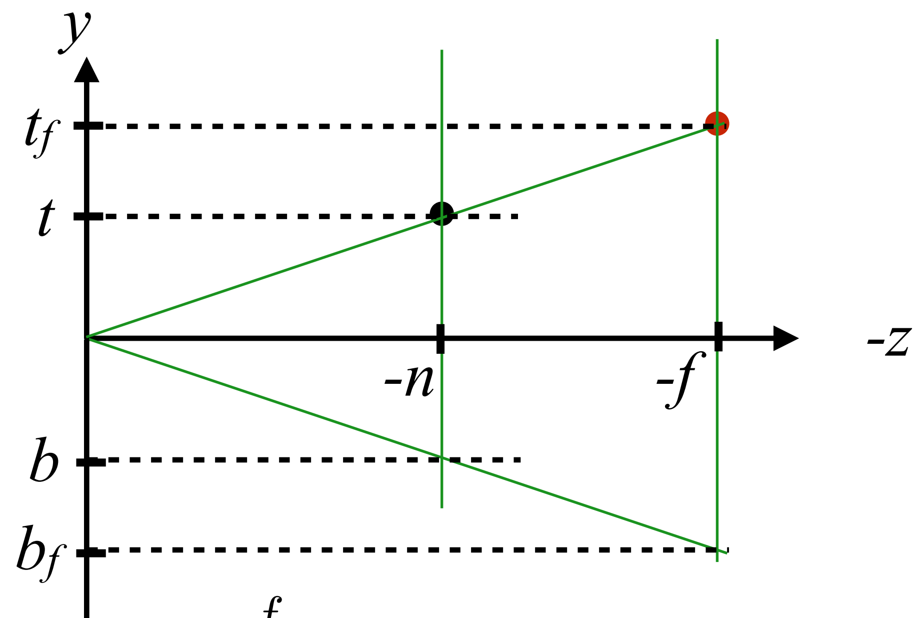
Two step derivation



- Start with the Canonical Perspective Projection matrix
- Adjust CPP to scale z so that after the application of CPP z = -near maps to z'=-near and z=-far maps to z'=-far)
- We now have the standard orthographic view volume
- Use the previously derived orthographic projection matrix M_O

First: What are the view volume points in viewing coordinates?



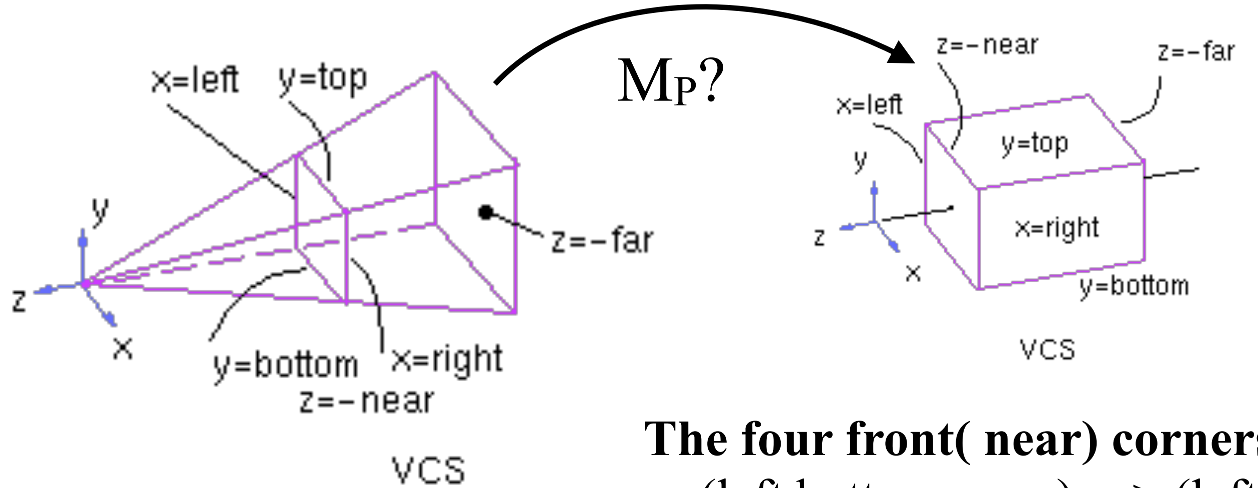


From similar triangles: $\frac{t}{-n} = \frac{t_f}{-f} \rightarrow t_f = t\frac{f}{n}$

Similarly $y = b$, and for $x = r$ and $x = l$

So for example: black point: $(r, t, -n) \rightarrow$ Red point: $(rf/n, tf/n, -f)$

First step: What does the CPP produce?



- $P'_x = n\,P_x/\,P_z$
- $P'_y = n\,P_y/P_z$
- $P'_z = \;-n$

Reminder:
CPP implements
Canonical equations

The four front(near) corners map as follows:

(left,bottom,-near) --> (left,bottom,-near)
(left,top,-near) --> (left,top,-near)
(right,bottom,-near) --> (right,bottom,-near)
(right,top,-near) --> (right,top,-near)

The four back (far) corners map to the same points

(l*f/n,b*f/n, -f) --> (l, b,-n) instead of (l,b,-f)
(l*f/n,t*f/n,-f) --> (l, t, -n) instead of (l,t,-f)
(r*f/n,b*f/n, -f) --> (r,b,-n) instead of (r,b,-f)
(r*f/n,t*f/n,-f) --> (r,t,-n) instead of (r,t,-f)

So we just have to adjust the matrix for z

First step:Adjust the matrix to keep and scale Z

- We want to keep z and
- We want $P'_z = -n$ for $P_z = -n$ and $P'_z = -f$ for $P_z = -f$ **after** the perspective division
- Where do we do the changes?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{n} & 0 \end{bmatrix} \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

First step: With some intuition...

Reminder: n,f are positive

$$\mathbf{M}_P \begin{bmatrix} P_x \\ P_y \\ P_z \\ 1 \end{bmatrix} = \begin{bmatrix} P_x \\ P_y \\ P_z \frac{n+f}{n} + f \\ -\frac{P_z}{n} \end{bmatrix} \xrightarrow{\texttt{Homogenize with } h = -P_z/n} \begin{bmatrix} \frac{-P_x n}{P_z} \\ \frac{-P_y n}{P_z} \\ -(n+f) - \frac{fn}{P_z} \\ 1 \end{bmatrix}$$

Therefore

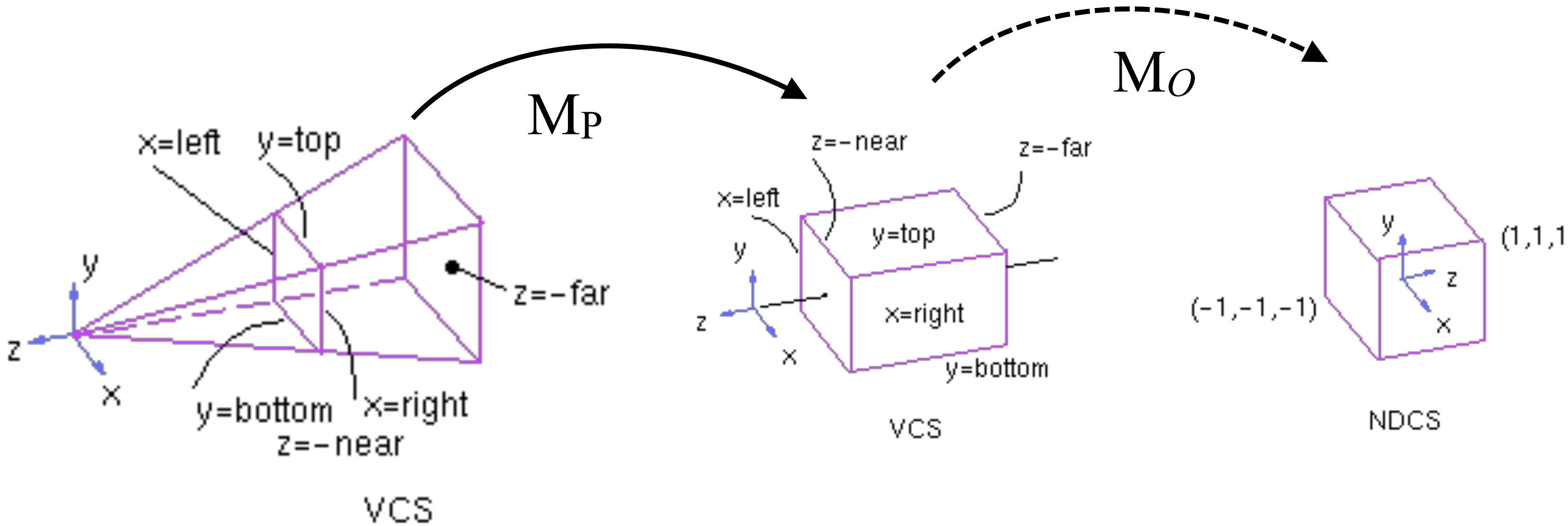
$$\mathbf{M}_P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{n+f}{n} & f \\ 0 & 0 & \frac{-1}{n} & 0 \end{bmatrix} \quad \text{or} \quad \mathbf{M}_P = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & fn \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Sanity check

- (l*f/n,b*f/n,-f) --> (l,b,-f)
- (l,b,-n) --> (l,b,-n)

First step: M_P

- Creates the previous orthographic view volume



- Which can then be transformed to NDCS with M_O

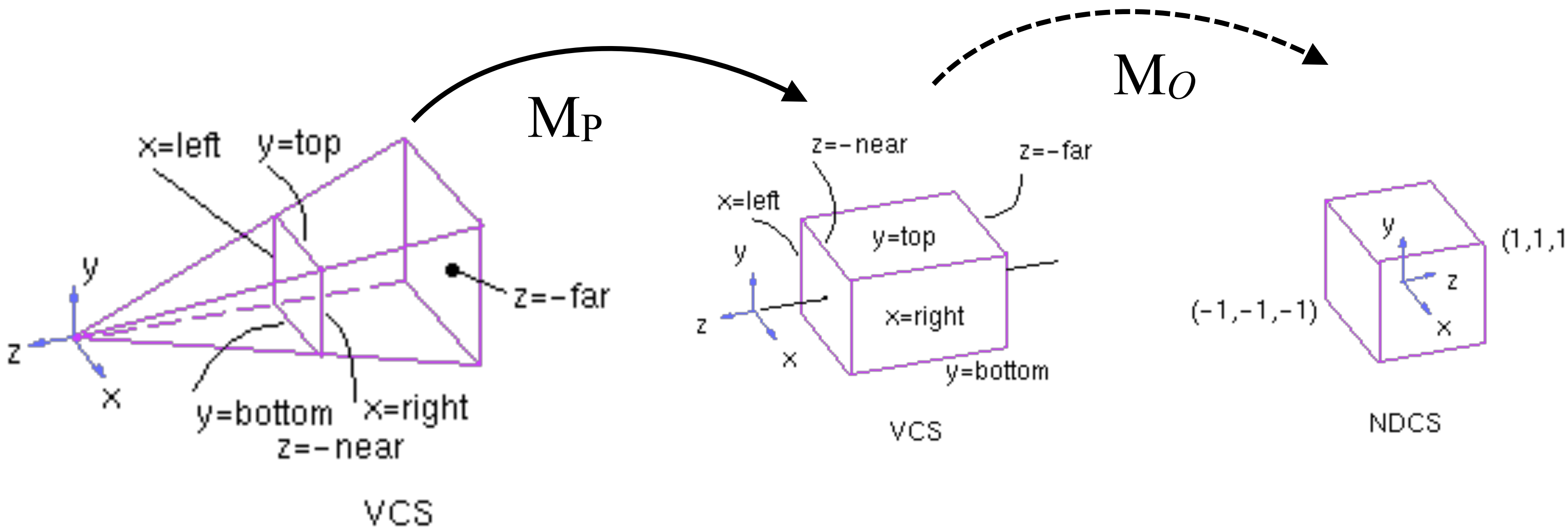
Second Step: Combine with Orthographic Projection Matrix

Now all we need to do is an orthographic transformation

- We can use matrix M_O that transforms an orthographic view volume to a normalized one (NDCS)

$$\mathrm{M}_O = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Second Step: Combine with Orthographic Projection Matrix



$$\mathrm{M}_{\mathtt{proj}} = \mathrm{M}_O \mathrm{M}_P$$

OpenGL Perspective Matrix

Old form

- Still widely used

$$\mathrm{M}_{\mathtt{OpenGL}} = \begin{bmatrix} \frac{2|n|}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2|n|}{t-b} & \frac{b+t}{t-b} & 0 \\ 0 & 0 & \frac{|n|+|f|}{|n|-|f|} & \frac{2|f||n|}{|n|-|f|} \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Projections in OpenGL (mimicking the old way)

New way

```
Not in our math  library but common in others
projMat = frustum(left, right, bottom, top, near, far);

In our math library:
projMat = ortho(left, right, bottom, top, near, far) ;
projMat = perspective(fov, aspect, near far) ;
```

near plane at z = -near

far plane at z = -far

Matrix Order

Normally projection has to apply to all objects (i.e. the entire scene) thus it must pre-multiply the modelview matrix of each object

- $M = M_{proj} M_{modelview}$ or
- $M = M_{proj} M_{view} M_{model}$

Important

Projection parameters are given in CAMERA Coordinate system (Viewing).

So if camera is at z = 50, is aligned with the world CS, and you give near = 10 where is the near plane with respect to the world?

Important

Projection parameters are given in CAMERA Coordinate system (Viewing).

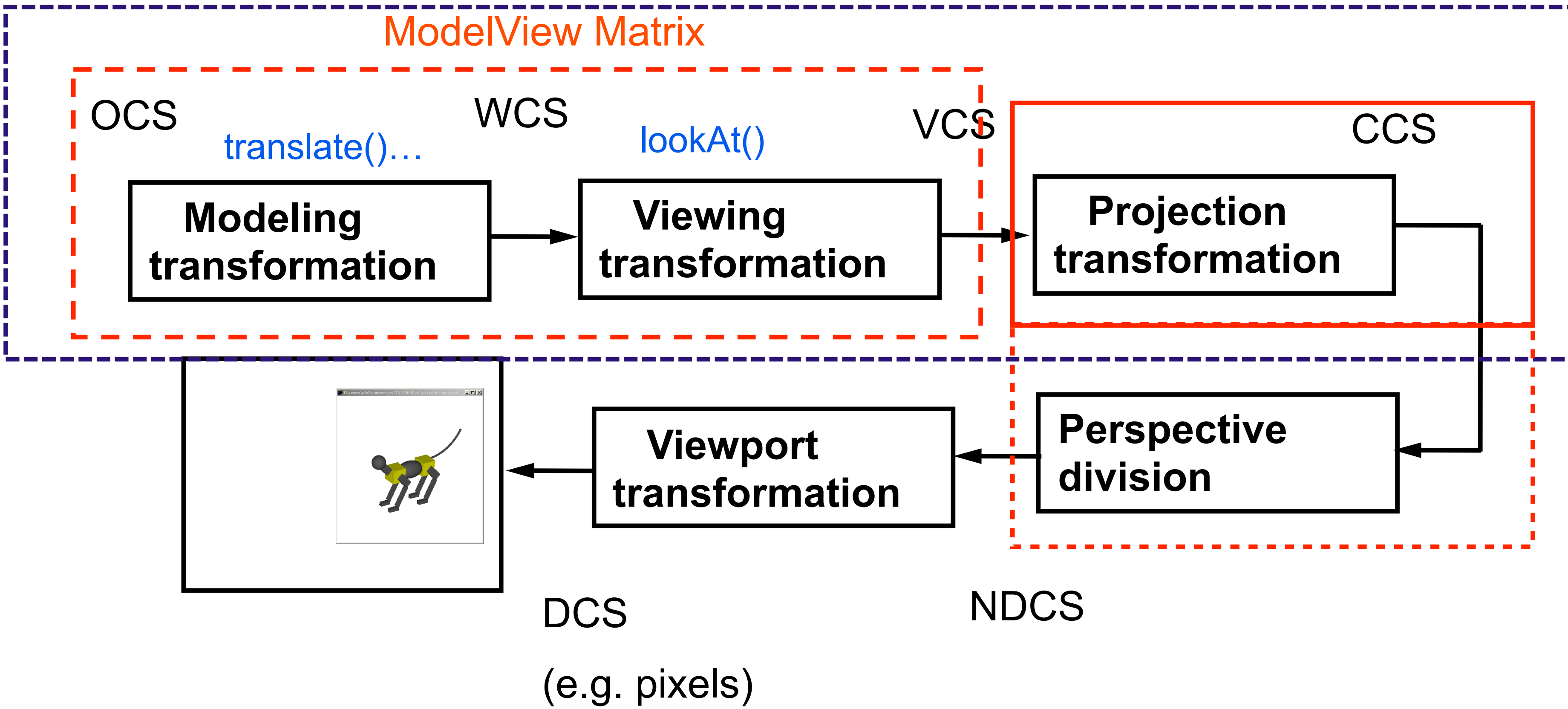
So if the camera is at z = 50, is aligned with the world CS, and you give |near| = 10 where is the near plane with respect to the world?

- Transformed by inverse(Mvcs)
- i.e. (0,0,40)

Perspective Division in Pipeline

The perspective division is done automatically

- Typically the vertex shaders write CCS in gl_Position.



Perspective Division in Pipeline

However, we can do it ourselves if we want.

The vertex shader has total freedom on how to deal with projections.