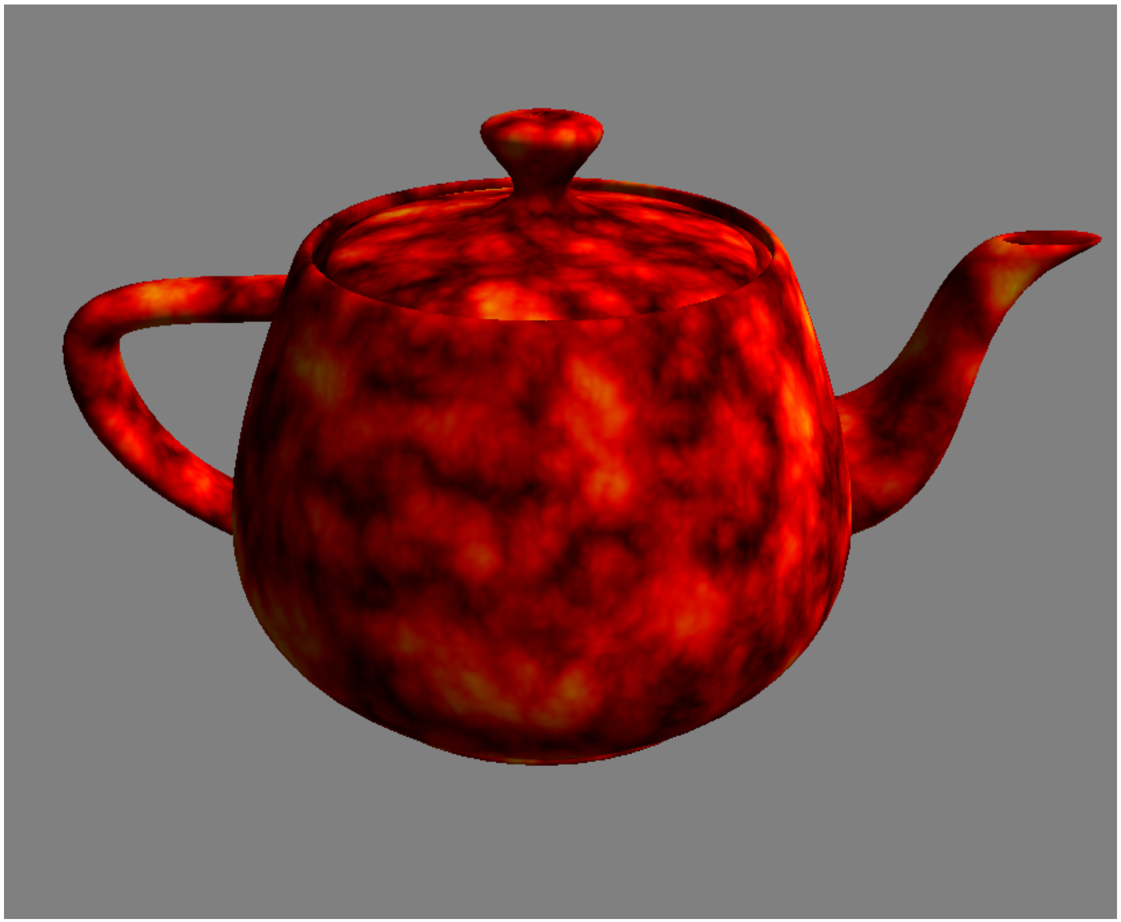
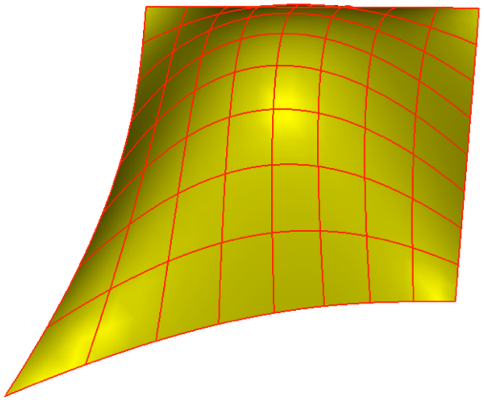
Surfaces





Formulations

Implicit: $f(x,y,z) = 0$

Normal $\mathbf{n} = \nabla(f)$

Explicit: $z = f(x,y)$

Parametric: $x = f_x(s,t)$, $y = f_y(s,t)$, $z=f_z(s,t)$

Quadric surfaces

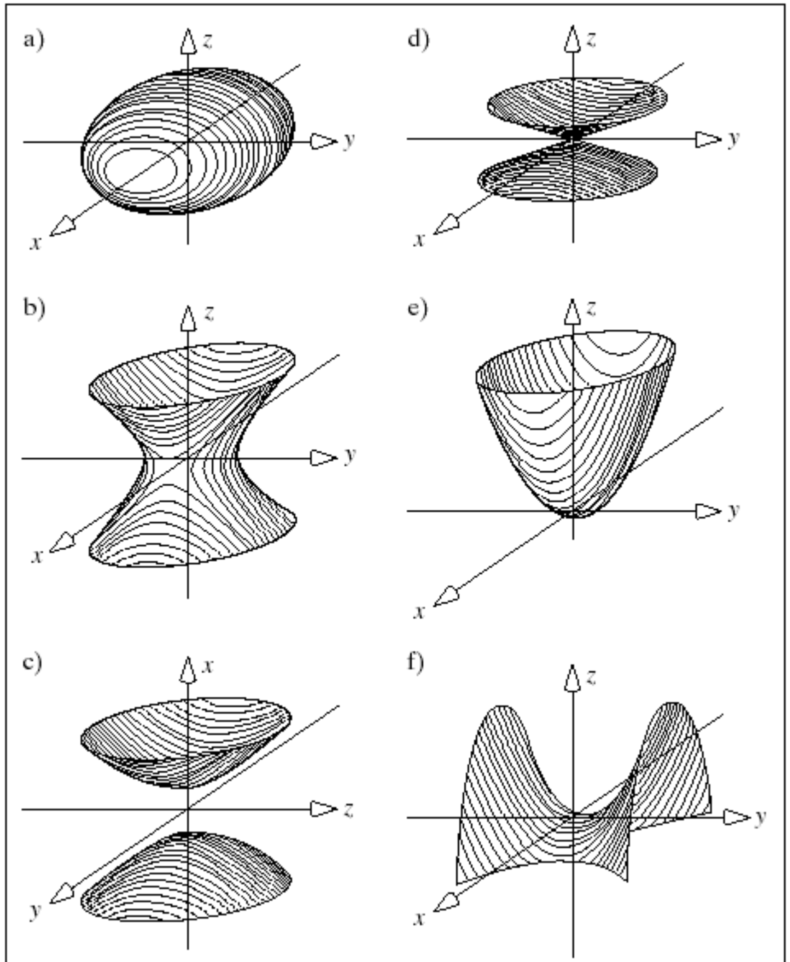


FIGURE 6.70 The six quadric surfaces: (a) Ellipsoid. (b) Hyperboloid of one sheet. (c) Hyperboloid of two sheets. (d) Elliptic cone. (e) Elliptic paraboloid. (f) Hyperbolic paraboloid.

Quadric surfaces

Sphere:

$$f(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

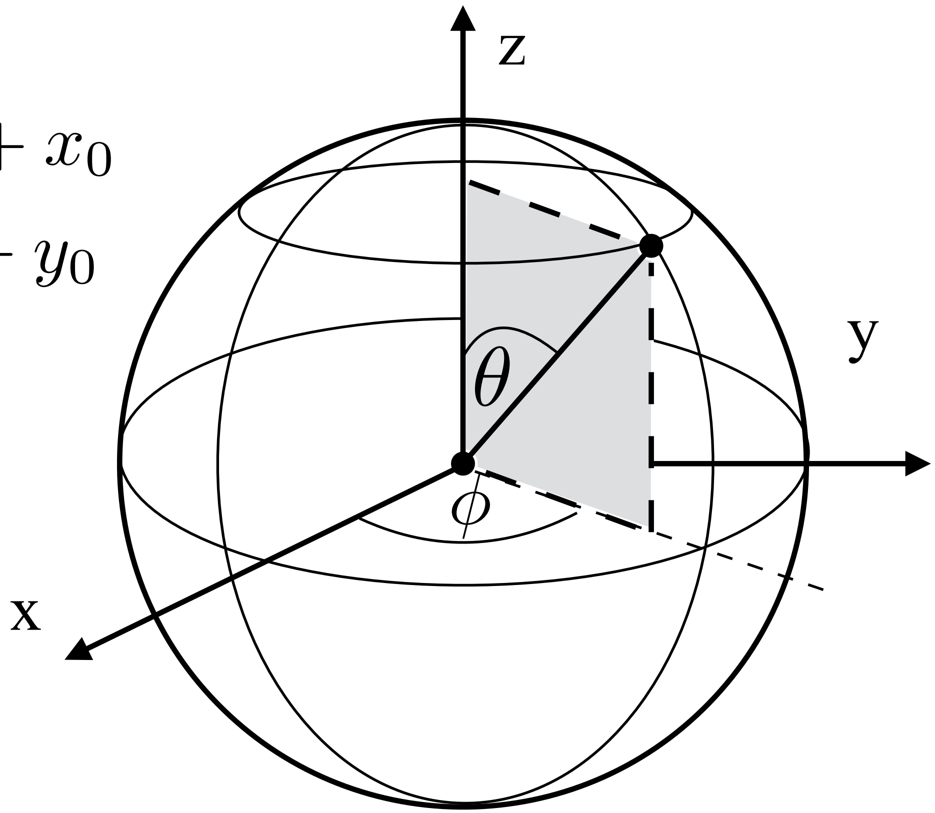
$$x(\phi, \theta) = R\sin(\theta)\cos(\phi) + x_0$$

$$y(\phi, \theta) = R\sin(\theta)\sin(\phi) + y_0$$

$$z(\phi, \theta) = R\cos(\theta) + z_0$$

$$0 \leq \phi \leq 2\pi$$

$$0 \leq \theta \leq \pi$$



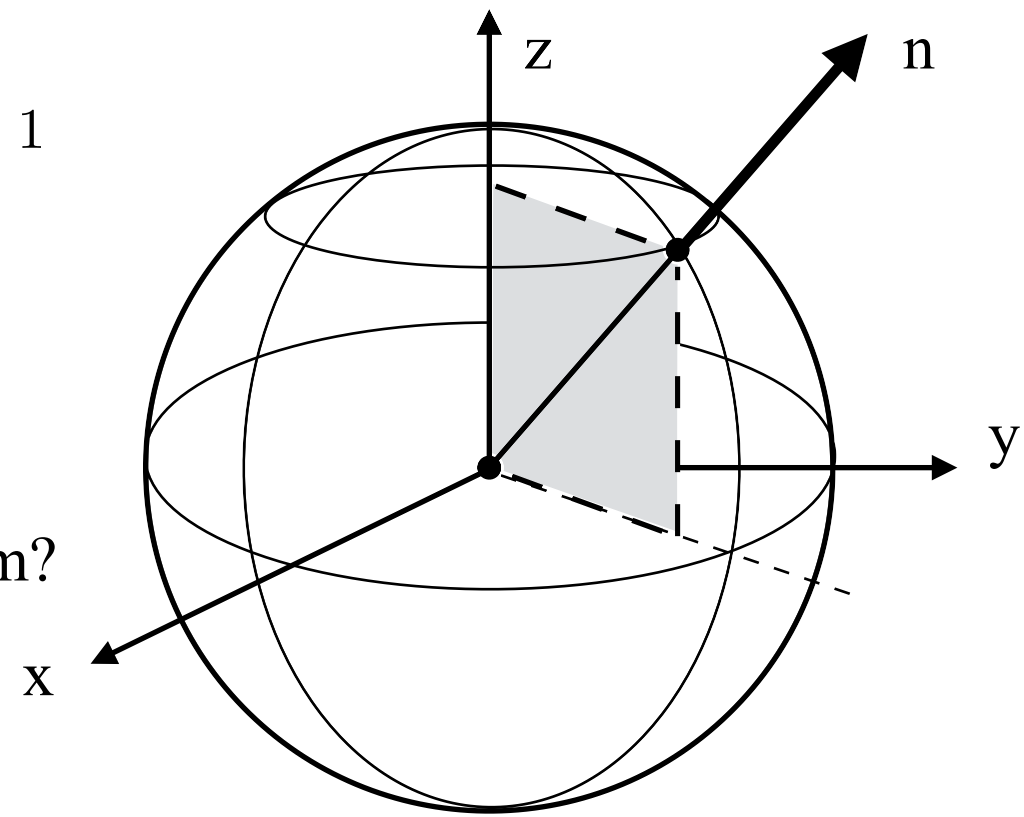
Normal on Sphere

$$\mathbf{n}(x,y,z) = \nabla f(x,y,z) = \left(\frac{\partial f(x,y,z)}{\partial x}, \frac{\partial f(x,y,z)}{\partial y}, \frac{\partial f(x,y,z)}{\partial z}\right)$$

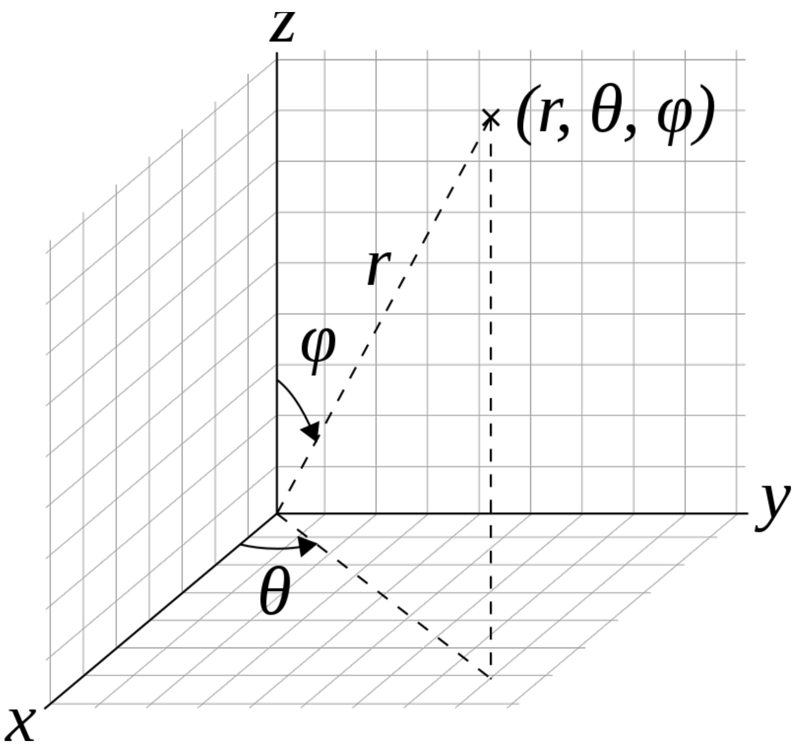
For $f(x,y,z) = x^2 + y^2 + z^2 - 1$

$\mathbf{n}(x,y,z) = (2x, 2y, 2z)$

Exercise:
a) Unit normal?
b) What about the general form?



Reminder: Spherical coordinates



r: radial distance
theta: azimuthal angle
phi: polar angle

Quadric surfaces

Ellipsoid

$$f(x,y,z) = (\frac{x-x_0}{R_x})^2 + (\frac{y-y_0}{R_y})^2 + (\frac{z-z_0}{R_z})^2 - 1$$

$$f(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = R^2$$

$$x(\phi,\theta) = R_x \sin(\theta)\cos(\phi) + x_0$$

$$y(\phi,\theta) = R_y \sin(\theta)\sin(\phi) + y_0$$

$$z(\phi,\theta) = R_z \cos(\theta) + z_0$$

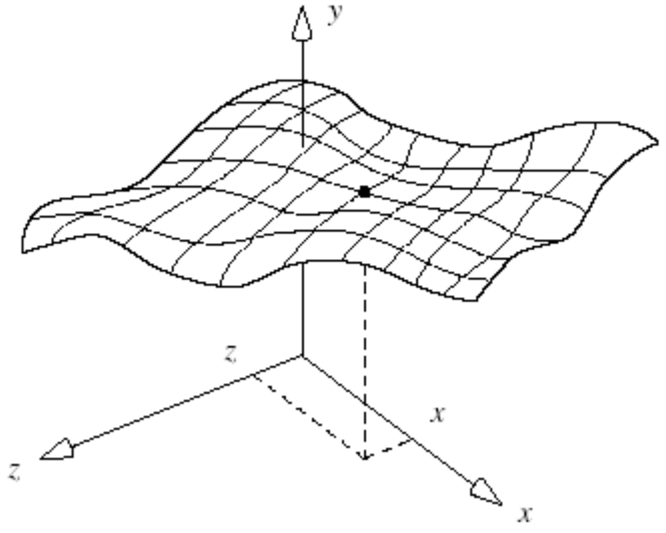
$$0 \leq \phi \leq 2\pi$$

$$0 \leq \theta \leq \pi$$

Normal vector?

Height fields

$y=f(x,z)$



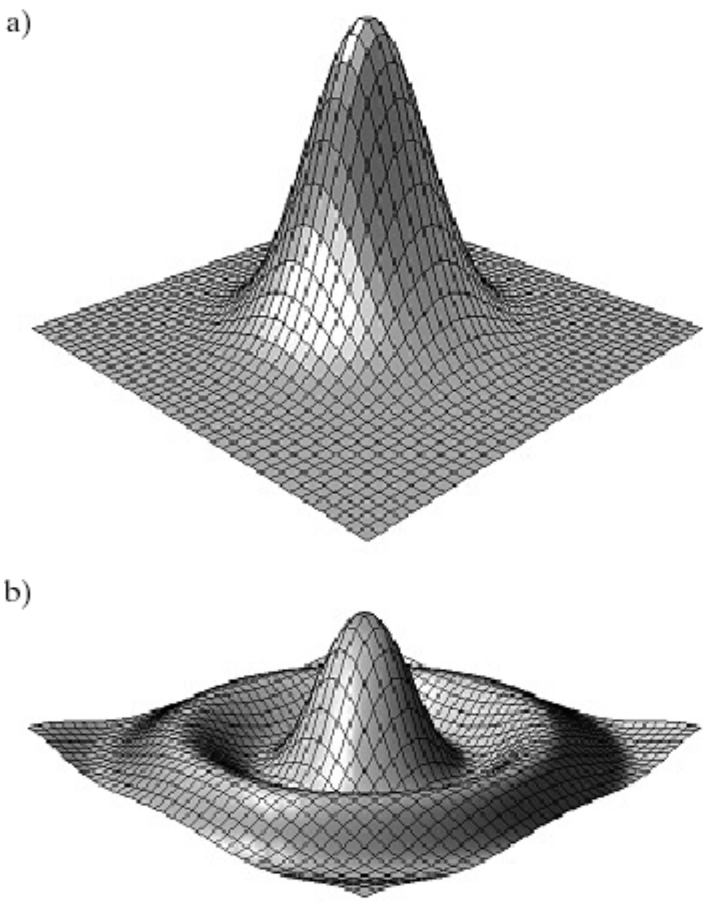
Typical height fields

Gaussian

$$y = f(x, z) = e^{-ax^2 - bz^2}$$

Sinc

$$y = f(x, z) = \frac{sin(\sqrt{x^2 + z^2})}{\sqrt{x^2 + z^2}}$$



Parametric formulations

Ruled surfaces:

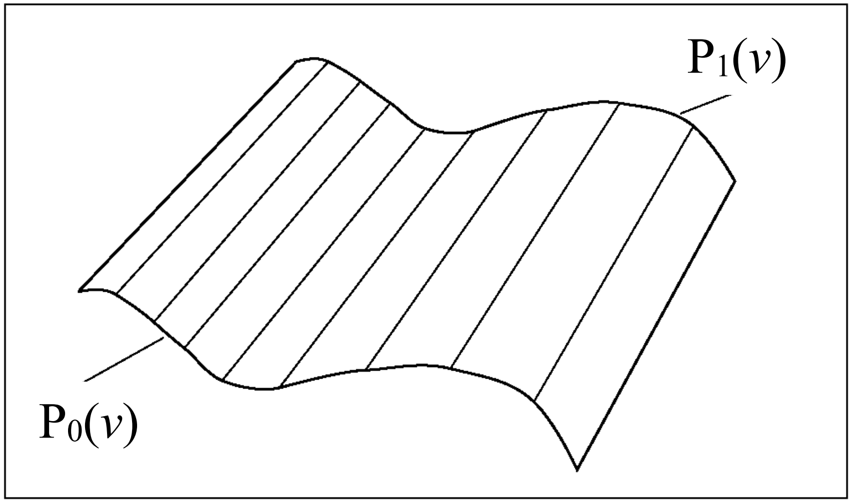
Linear combination of two curves

- Through every point on the surface there passes at least one line that lies on the surface

$$P(u) = (1-u)P_0 + uP_1$$

Making P_0 and P_1 curves:

$$P(u,v) = (1-u)P_0(v) + uP_1(v)$$

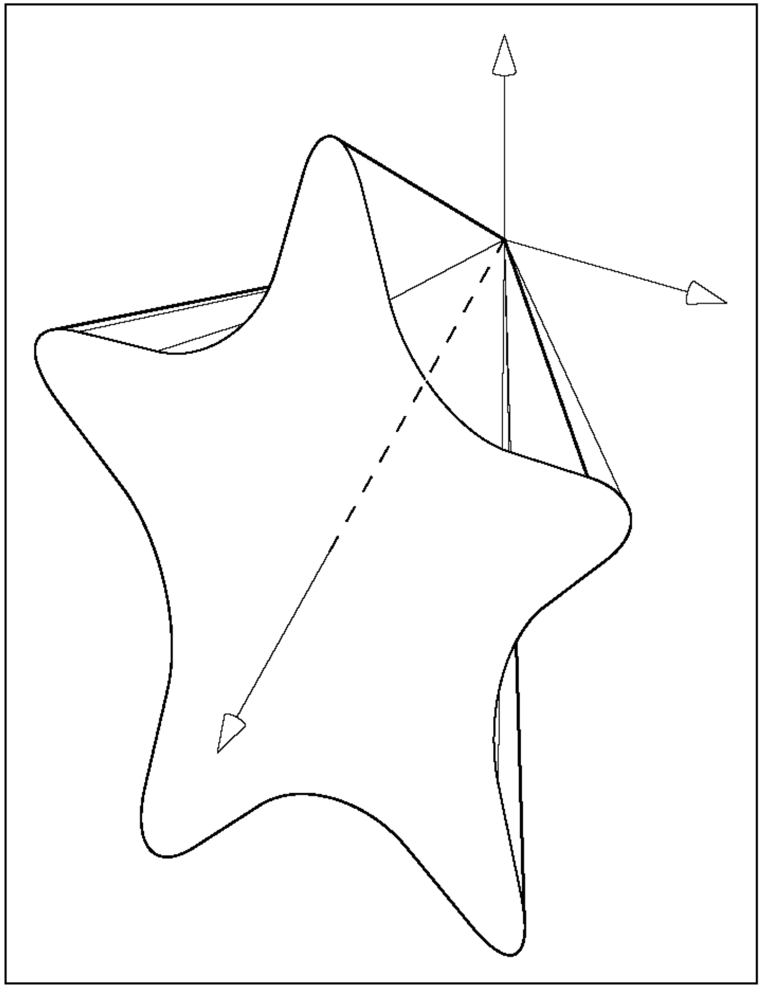


Special cases

General cone

$$P(u,v) = (1-u)P_0 + uP_1(v)$$

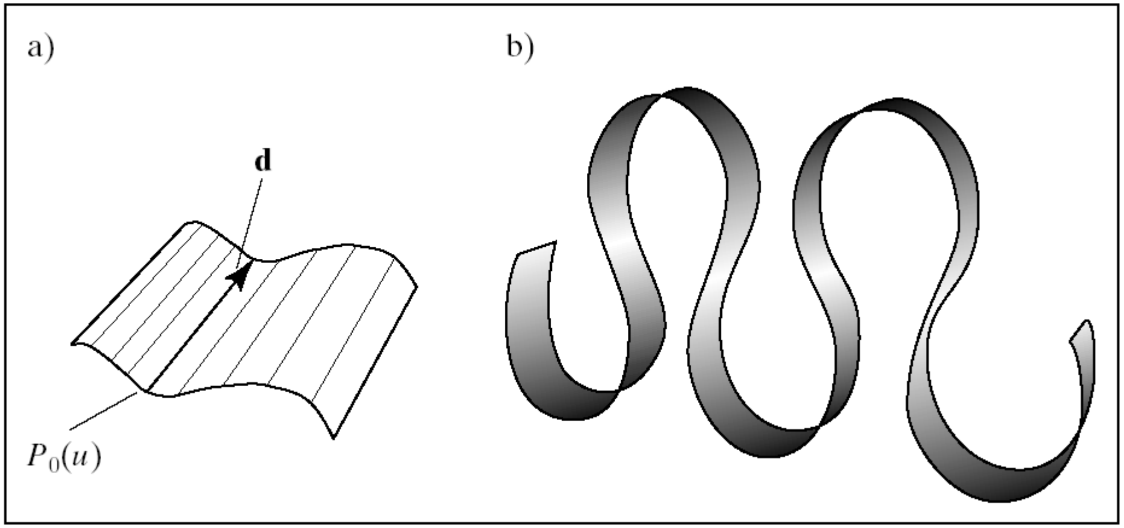
P_0 is the apex



General Cylinder

P_1 a translated version of P_0

$$P(u,v) = (1-u)P_0(v) + u(P_0(v) + \mathbf{d}) \Rightarrow P(u,v) = P_0(v) + u\mathbf{d}$$



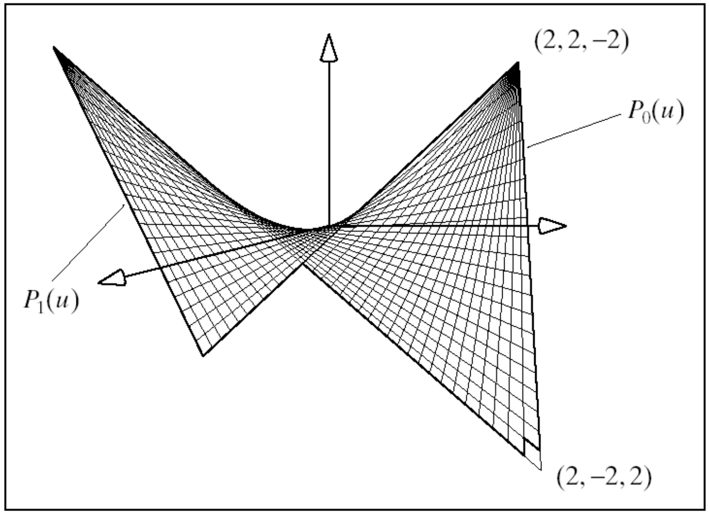
Bilinear patches

Both P_1 and P_0 are lines

$$P(u,v) = (1-u)P_0(v) + uP_1(v) \Rightarrow$$

$$P(u,v) = (1-u)[(1-v)P_{00} + vP_{01}] + u[(1-v)P_{10} + vP_{11}] \Rightarrow$$

$$P(u,v) = (1-u)(1-v)P_{00} + (1-u)vP_{01} + u(1-v)P_{10} + uvP_{11}$$

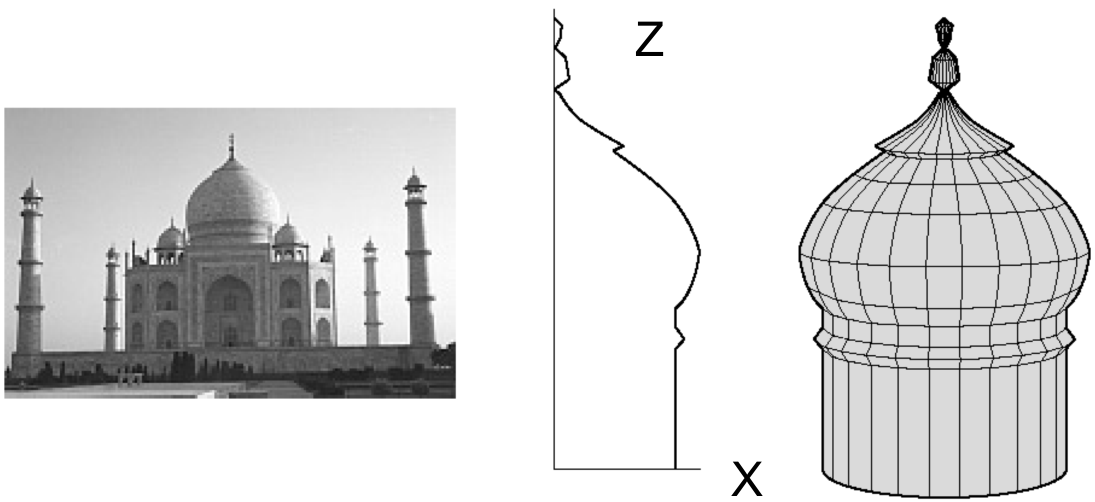


Surfaces of revolution

Sweep profile curve around an axis:

C(v) = (X(v),Z(v))

P(u, v)=(X(v)cos(u),X(v)sin(u), Z(v))

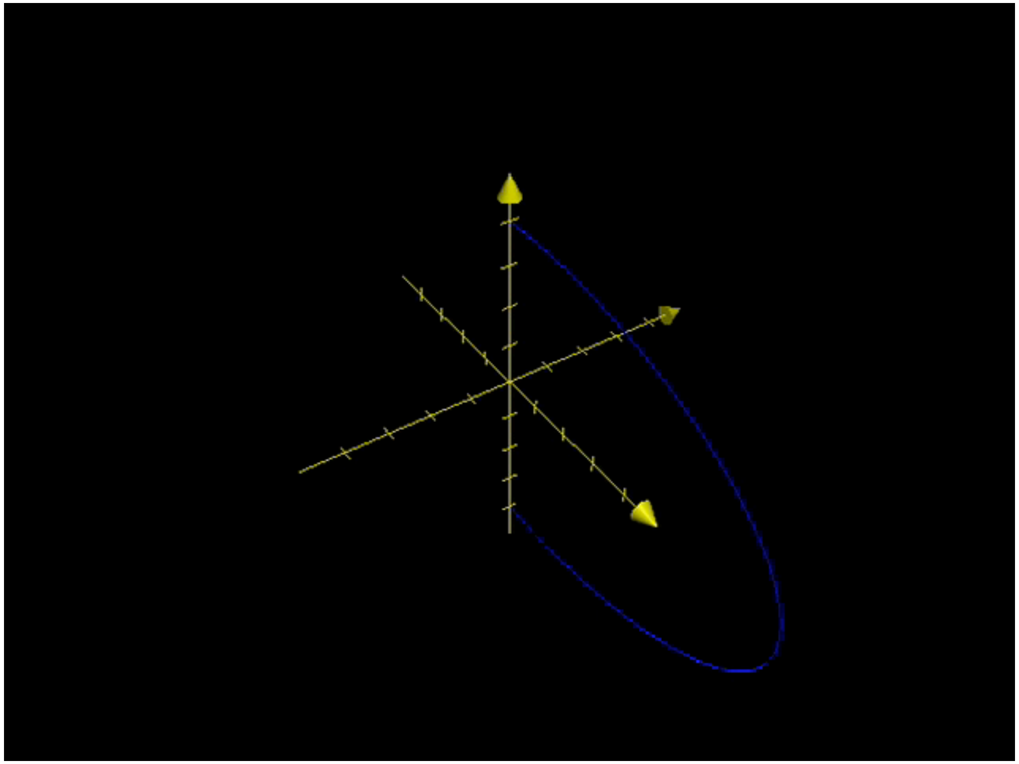


a) b) c)

Example

Curve

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4\cdot\cos(t) \\ 0 \\ 2\cdot\sin(t) \end{bmatrix}, t=-\frac{\pi}{2}\ldots\frac{\pi}{2}$$



Surface

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4\cdot\cos(t)\cdot\cos(u) \\ 4\cdot\cos(t)\cdot\sin(u) \\ 2\cdot\sin(t) \end{bmatrix}, t=-\frac{\pi}{2}\ldots\frac{\pi}{2}, u=0\ldots a$$

Parametric surfaces from control points (constraints)

Extension of the curve form to two dimensions

- Focus on cubic patches

General form of a cubic patch

$$P(s,t) = \sum_{i=0}^{15} B_i(s,t) G_i, \quad (s,t) \in [0,1] \times [0,1]$$

where

$B_i(s,t)$: Cubic polynomials in two variables

G_i : Point or tangent constraints

Parametric surfaces from control points (constraints)

Extension of the curve form to two dimensions

Curve: P*(s) = SMG = [s^3 s^2 s 1]MG with s in [0,1]*

Surface: P(s,t) = SMG(t) with s,t in [0,1]

Example: Bezier curve P (s) of four points P_1,P_2,P_3,P_4:

$$P(s) = S\mathbf{MG}, s \in [0,1] \text{ or}$$

$$\begin{bmatrix} x(s) & y(s) & z(s) \end{bmatrix} = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} G_x & G_y & G_z \end{bmatrix} \text{ or}$$

$$\begin{bmatrix} x(s) & y(s) & z(s) \end{bmatrix} = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} P_{1,x} & P_{1,y} & P_{1,z} \\ P_{2,x} & P_{2,y} & P_{2,z} \\ P_{3,x} & P_{3,y} & P_{3,z} \\ P_{4,x} & P_{4,y} & P_{4,z} \end{bmatrix}$$

Bezier Surfaces

Take a bezier curve P(s) and let its control points become bezier curves

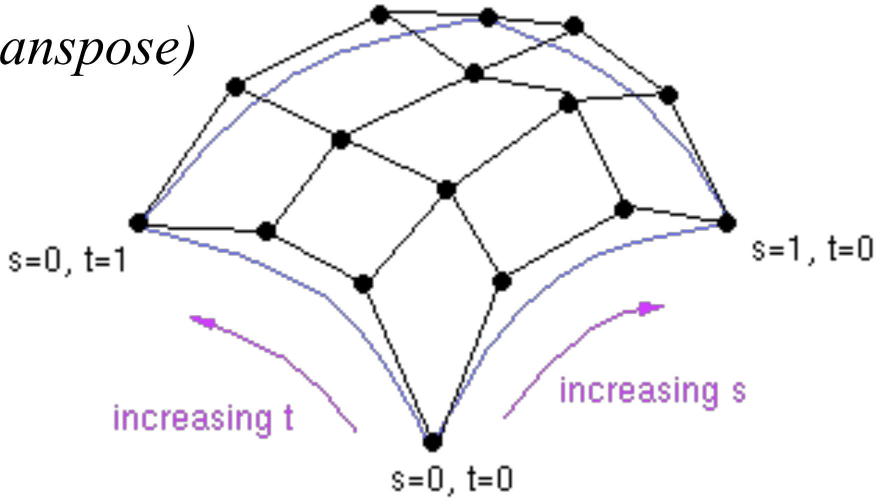
$P(s) = S\,\mathbf{M}\,\mathbf{G}(t)$

(transpose)

$\mathbf{G}(t) = [P_1(t)\;\; P_2(t)\;\; P_3(t)\;\; P_4(t)]^T$

Where:

$P_i(t) = T\,\mathbf{M}\,\mathbf{G}_i = T\mathbf{M}[\,P_{i1}\,P_{i2}\,P_{i3}\,P_{i4}]^T$



Total: 4x4 = 16 control points

$P_{ij},\ i=1,2,3,4,\ j=1,2,3,4$

Tensor product representation (easier per dimension)

$$P_x(s,t) = S\mathbf{M}G_x(t) = S\mathbf{M}\begin{bmatrix} P_{1,x}(t) \\ P_{2,x}(t) \\ P_{3,x}(t) \\ P_{4,x}(t) \end{bmatrix}, \text{ where}$$

$$P_{i,x}(t) = T\mathbf{M}G_{i,x} = G_{i,x}^T\mathbf{M}^T T^T = \left[\begin{array}{cccc} P_{i1,x} & P_{i2,x} & P_{i3,x} & P_{i4,x} \end{array}\right]\mathbf{M}^T T^T$$

Tensor product representation (easier per dimension)

$$P_x(s,t) = S\mathbf{M}G_x(t) = S\mathbf{M}\begin{bmatrix} P_{1,x}(t) \\ P_{2,x}(t) \\ P_{3,x}(t) \\ P_{4,x}(t) \end{bmatrix}, \text{ where}$$

$$P_{i,x}(t) = T\mathbf{M}G_{i,x} = G_{i,x}^T\mathbf{M}^T T^T = \begin{bmatrix} P_{i1,x} & P_{i2,x} & P_{i3,x} & P_{i4,x} \end{bmatrix}\mathbf{M}^T T^T$$

Together they give:

$$P_x(s,t) = S\mathbf{M}G_x(t) = S\mathbf{M}\begin{bmatrix} P_{1,x}(t) \\ P_{2,x}(t) \\ P_{3,x}(t) \\ P_{4,x}(t) \end{bmatrix} = S\mathbf{M}\begin{bmatrix} G_{1,x}^T\mathbf{M}^T T^t \\ G_{2,x}^T\mathbf{M}^T T^t \\ G_{3,x}^T\mathbf{M}^T T^t \\ G_{4,x}^T\mathbf{M}^T T^t \end{bmatrix} \rightarrow$$

$$P_x(s,t) = S\mathbf{M}\begin{bmatrix} P_{11,x} & P_{12,x} & P_{13,x} & P_{14,x} \\ P_{21,x} & P_{22,x} & P_{23,x} & P_{24,x} \\ P_{31,x} & P_{32,x} & P_{33,x} & P_{34,x} \\ P_{41,x} & P_{42,x} & P_{43,x} & P_{44,x} \end{bmatrix}\mathbf{M}^T T^T$$

$$P_x(s,t) = S\mathbf{M}\mathbf{G}_x\mathbf{M}^T T^T, (s,t) \in [0,1] \times [0,1]$$

Tensor product representation (easier per dimension)

$$P_x(s,t) = S\mathbf{M}G_x(t) = S\mathbf{M}\begin{bmatrix} P_{1,x}(t) \\ P_{2,x}(t) \\ P_{3,x}(t) \\ P_{4,x}(t) \end{bmatrix}, \text{ where}$$

$$P_{i,x}(t) = T\mathbf{M}G_{i,x} = G_{i,x}^T\mathbf{M}^T T^T = \begin{bmatrix} P_{i1,x} & P_{i2,x} & P_{i3,x} & P_{i4,x} \end{bmatrix}\mathbf{M}^T T^T$$

Together they give:

$$P_x(s,t) = S\mathbf{M}G_x(t) = S\mathbf{M}\begin{bmatrix} P_{1,x}(t) \\ P_{2,x}(t) \\ P_{3,x}(t) \\ P_{4,x}(t) \end{bmatrix} = S\mathbf{M}\begin{bmatrix} G_{1,x}^T\mathbf{M}^T T^t \\ G_{2,x}^T\mathbf{M}^T T^t \\ G_{3,x}^T\mathbf{M}^T T^t \\ G_{4,x}^T\mathbf{M}^T T^t \end{bmatrix} \rightarrow$$

$$P_x(s,t) = S\mathbf{M}\begin{bmatrix} P_{11,x} & P_{12,x} & P_{13,x} & P_{14,x} \\ P_{21,x} & P_{22,x} & P_{23,x} & P_{24,x} \\ P_{31,x} & P_{32,x} & P_{33,x} & P_{34,x} \\ P_{41,x} & P_{42,x} & P_{43,x} & P_{44,x} \end{bmatrix}\mathbf{M}^T T^T$$

$$P_x(s,t) = S\mathbf{M}\mathbf{G}_x\mathbf{M}^T T^T, (s,t) \in [0,1] \times [0,1]$$

Tensor product representation (easier per dimension)

$$P_x(s,t) = S\mathbf{M}\begin{bmatrix} P_{11,x} & P_{12,x} & P_{13,x} & P_{14,x} \\ P_{21,x} & P_{22,x} & P_{23,x} & P_{24,x} \\ P_{31,x} & P_{32,x} & P_{33,x} & P_{34,x} \\ P_{41,x} & P_{42,x} & P_{43,x} & P_{44,x} \end{bmatrix}\mathbf{M}^T T^T$$

$$P_x(s,t) = S\mathbf{M}\mathbf{G}_x\mathbf{M}^T T^T, (s,t) \in [0,1] \times [0,1]$$

Similarly:

$$P_y(s,t) = S\mathbf{M}\mathbf{G}_y\mathbf{M}^T T^T, (s,t) \in [0,1] \times [0,1]$$

$$P_z(s,t) = S\mathbf{M}\mathbf{G}_z\mathbf{M}^T T^T, (s,t) \in [0,1] \times [0,1]$$

Tensor product representation

More compactly:

$$P(s,t) = S\mathbf{M}\mathbf{G}\mathbf{M}^T T^T, (s,t) \in [0,1] \times [0,1] \text{ or}$$

$$P(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} \mathbf{M} \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix} \mathbf{M}^T \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

4x4x3 Tensor

Cubic Bezier patch forms

Pick the most convenient

a) $$P(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} MGM \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \quad (s,t) \in [0,1] \times [0,1]$$

b) $$P(s,t) = \sum_{i=0}^{3} B_i^3(s) \sum_{i=0}^{3} B_j^3(t) P_{ij}, \quad (s,t) \in [0,1] \times [0,1]$$

where the Bernstein polynomials are

$$B_0^3(v) = (1-v)^3, B_1^3(v) = 3(1-v)^2 v,$$

$$B_2^3(v) = 3(1-v)v^2, B_3^3(v) = v^3$$

Properties of Bezier surfaces

Affine Invariance

Convex Hull

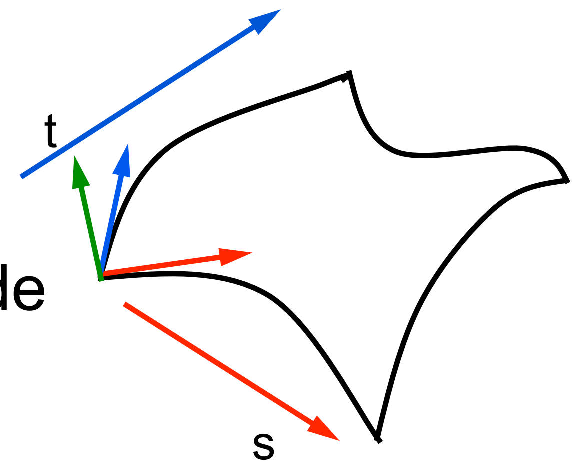
Plane precision

Variation diminishing

Hermite surfaces

Constraints at the four corners:

- Position, Tangent, Twist
- Same form, different basis matrix M_h
- Geometry vector in 4x4 form to fit in slide



points   tangents

$$G_x = \begin{bmatrix} x(0,0) & x(0,1) & \frac{\partial}{\partial t}x(0,0) & \frac{\partial}{\partial t}x(0,1) \\ x(1,0) & x(1,1) & \frac{\partial}{\partial t}x(1,0) & \frac{\partial}{\partial t}x(1,1) \\ \frac{\partial}{\partial s}x(0,0) & \frac{\partial}{\partial s}x(0,1) & \frac{\partial^2}{\partial s\,\partial t}x(0,0) & \frac{\partial^2}{\partial s\,\partial t}x(0,1) \\ \frac{\partial}{\partial s}x(1,0) & \frac{\partial}{\partial s}x(1,1) & \frac{\partial^2}{\partial s\,\partial t}x(1,0) & \frac{\partial^2}{\partial s\,\partial t}x(1,1) \end{bmatrix}$$

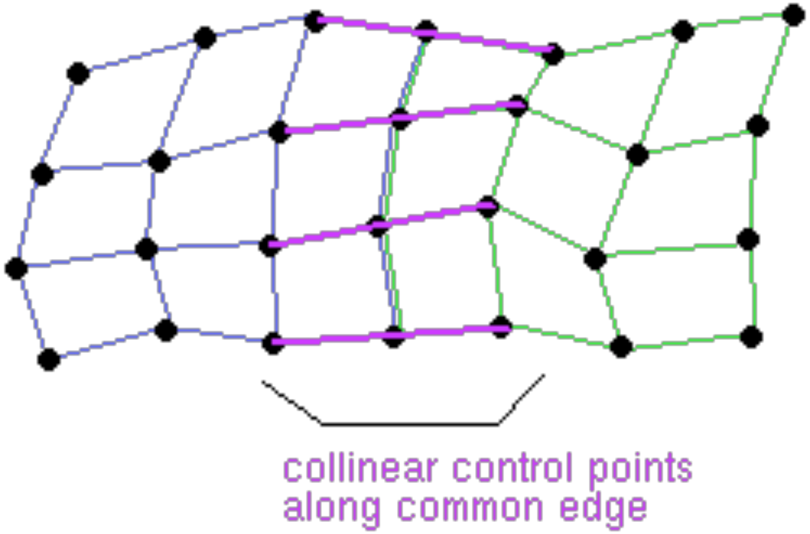
tangents   twist vectors

Piecewise cubic bezier surfaces

G1 continuity

Common edge

Make 2 sets of 4 control points collinear



Rendering parametric curves and surfaces

Transform into primitives we know how to handle

Curves

- Line segments

Surfaces

- Quadrilaterals
- Triangles

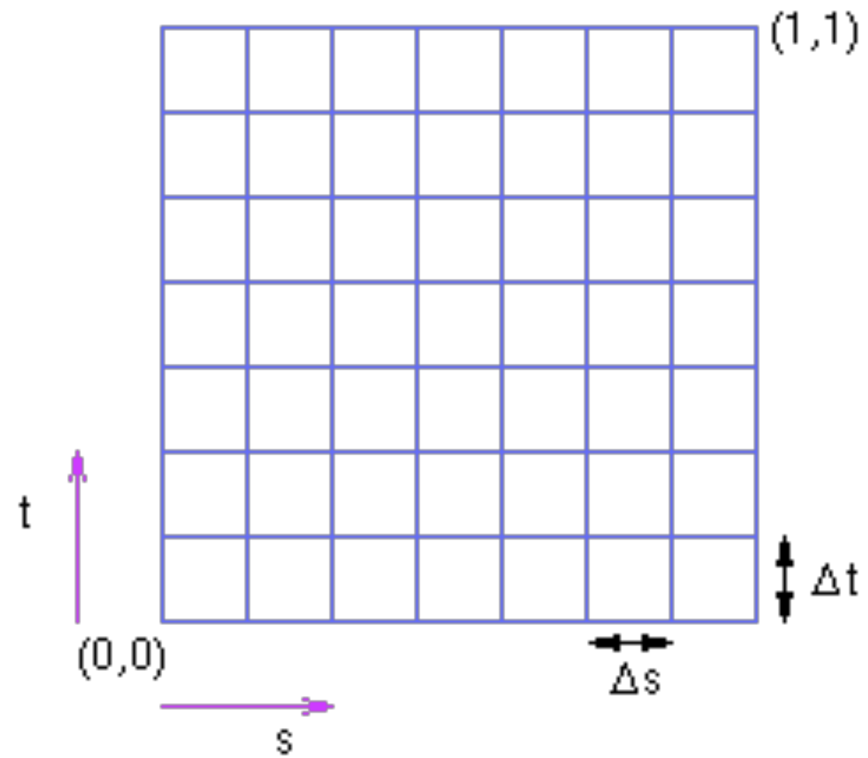
Converting to quadrilaterals

Straightforward

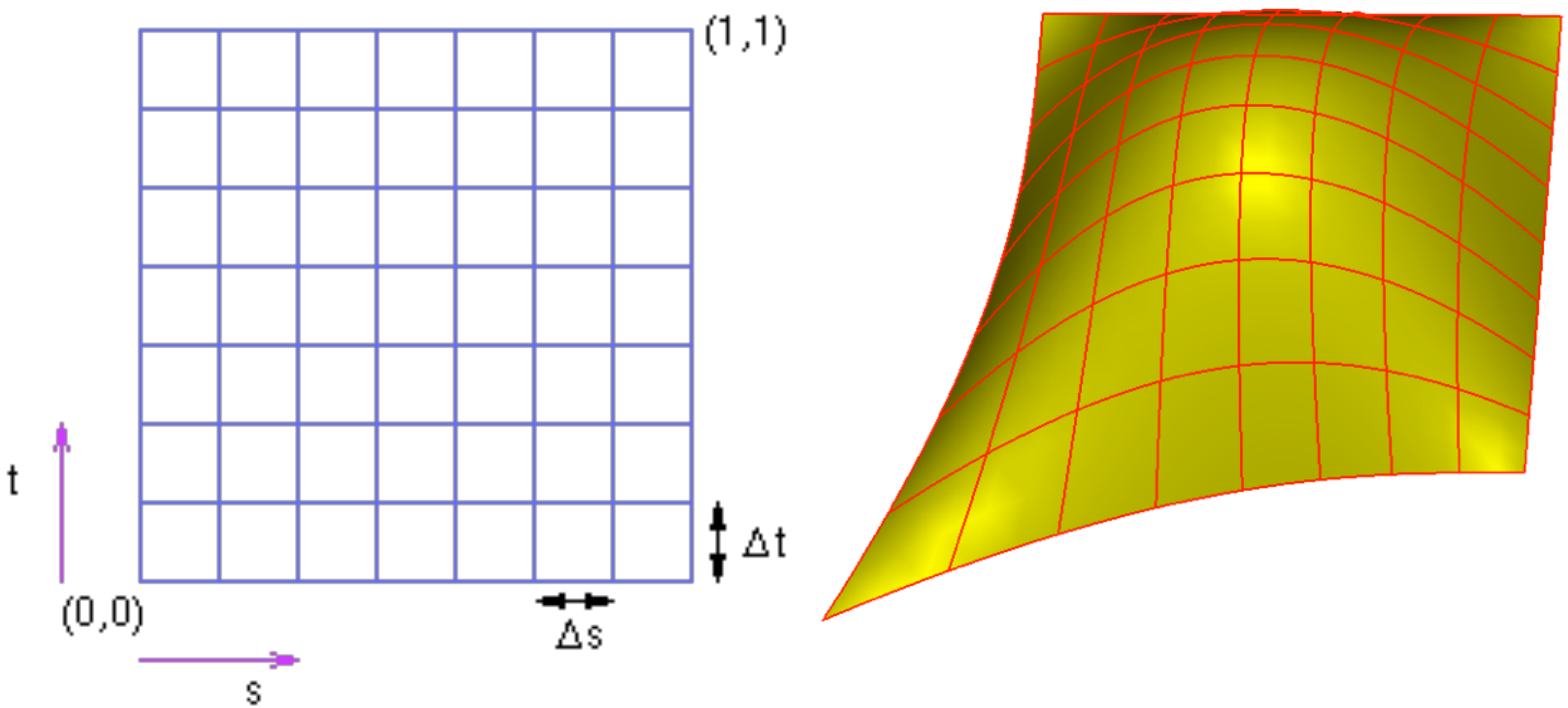
Uniform subdivision

Evaluation of P(s,t) at each grid point

Isoparametric lines (isolines) become isoparametric curves



Isolines



Optimizations

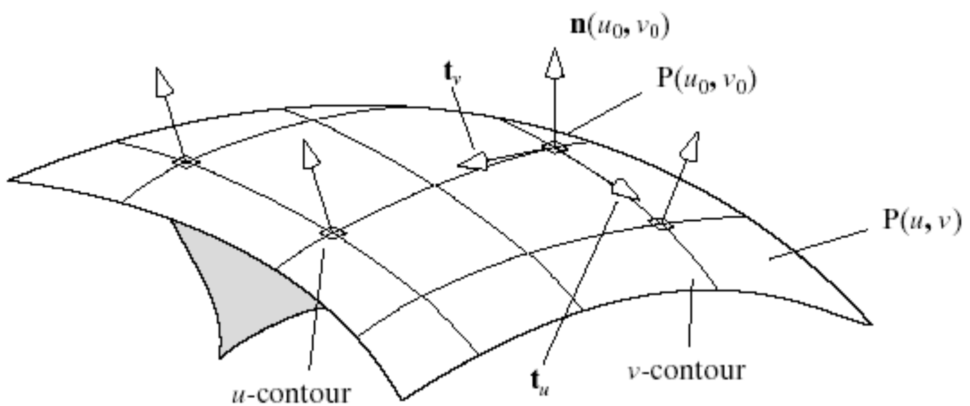
$x(s,t) = S\ M\ G_x\ M^T\ T^T$

- $M\ G\ M^T$ remains constant over patch: precompute
- $S\ M$ and $M^T\ T^T$ remain constant over all patches: precompute $S\ M$ and store in $Q[s]$
  $Q[t] = Q^T[s]$ assuming equal subdivisions in s and t

Computing surface normals

Parametric surface P(u,v)

$P(u,v) = U\ M\ G\ M^T\ V^T$



Normal Vector at u,v ?

$$\mathbf{N} = \frac{\partial P(u,v)}{\partial u} \times \frac{\partial P(u,v)}{\partial v}$$

Cubic Bezier patch forms (Again!)

Pick the most convenient

a) $$P(s,t) = \begin{bmatrix} s^3 & s^2 & s & 1 \end{bmatrix} MGM \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} \quad (s,t) \in [0,1] \times [0,1]$$

b) $$P(s,t) = \sum_{i=0}^{3} B_i^3(s) \sum_{i=0}^{3} B_j^3(t) P_{ij}, \quad (s,t) \in [0,1] \times [0,1]$$

where the Bernstein polynomials are

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General form of a cubic patch

$$P(s,t) = \sum_{i=0}^{15} B_i(s,t) G_i, \quad (s,t) \in [0,1] \times [0,1]$$

where

$B_i(s,t):$ Cubic polynomials in two variables

$G_i:$ Point or tangent constraints