Vectors

***n*-tuple:**

$$\mathbf{v} \in \Re^n$$

$$\mathbf{v} = (x_1, x_2, \ldots, x_n), \quad x_i \in \Re$$

Terminology

Linear vector spaces

- Vector addition
- Multiplication of vector and a scalar

Euclidean vector spaces

- Vector spaces with definition of distance (norm)

Affine vector spaces

- Euclidean vector space with the notion of “point”

Vectors

***n*-tuple:**

$$\mathbf{v} = (x_1, x_2, \ldots, x_n), \quad x_i \in \Re$$

Magnitude:

$$|\mathbf{v}| = \sqrt{x_1^2 + \ldots + x_n^2}$$

Unit vectors

$$\mathbf{v} : |\mathbf{v}| = 1$$

Normalizing a vector

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Operations with vectors

Addition

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$$

Multiplication with scalar (scaling)

$$a\mathbf{x} = (ax_1, \ldots, ax_n), \quad a \in \Re$$

Properties

$$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$$

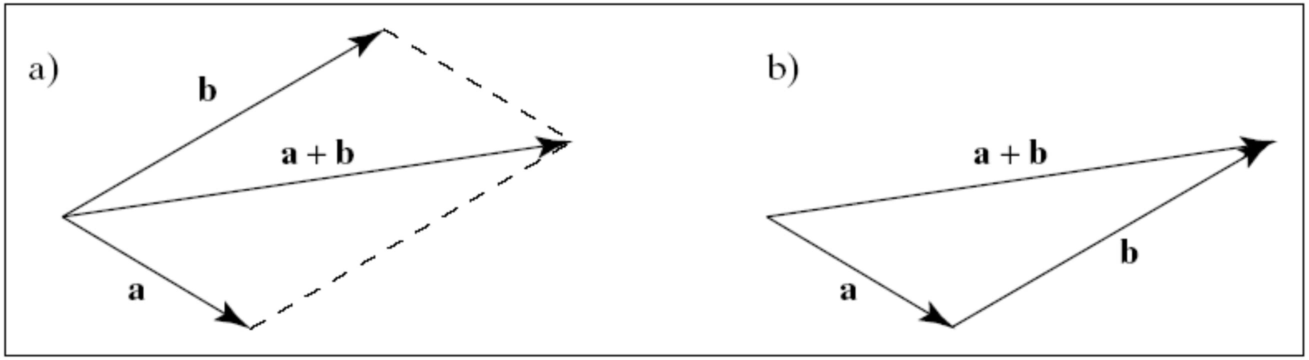
$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$$

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}, \quad a \in \Re$$

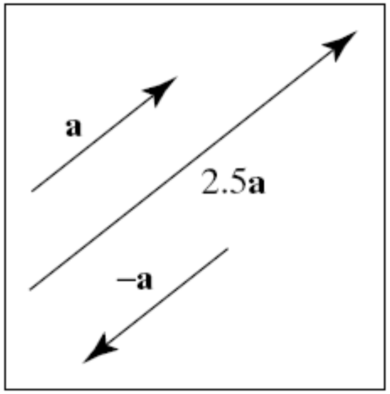
$$\mathbf{u} - \mathbf{u} = \mathbf{0}$$

Visualization for 2D and 3D vectors

Addition

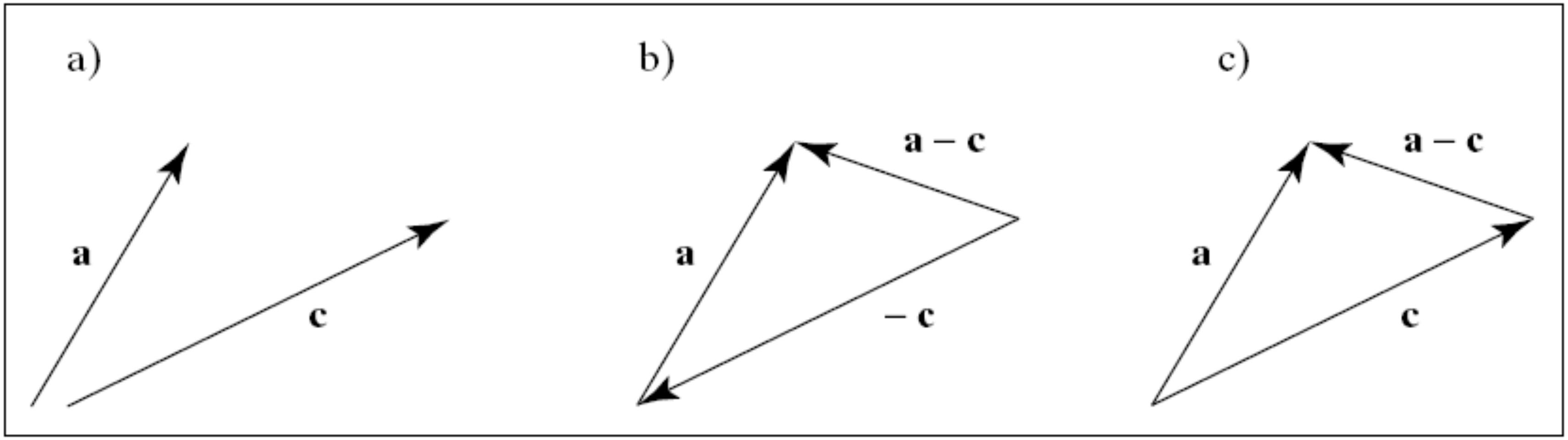


Scaling



Subtraction

Adding the negatively scaled vector



Linear combination of vectors

Definition

A linear combination of the m vectors $\mathbf{v}_1,\ldots,\mathbf{v}_m$ is a vector of the form:

$\mathbf{w} = a_1\mathbf{v}_1+\ldots a_m\mathbf{v}_m$, $a_1,\ldots,a_m$ in R

Special cases

Linear combination

$\mathbf{w} = a_1\mathbf{v}_1 + \ldots a_m\mathbf{v}_m$, $\quad a_1,\ldots,a_m$ in R

Affine combination:

A linear combination for which $a_1 + \ldots + a_m = 1$

Convex combination

An affine combination for which $a_i \geq 0$ for $i = 1,\ldots,m$

Linear Independence

For vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$

If $a_1\mathbf{v}_1 + \ldots a_m\mathbf{v}_m = \mathbf{0}$ iff $a_1 = a_2 = \ldots = a_m = 0$

then the vectors are linearly independent

- In other words no one vector can be written as a linear combination of the other vectors

Generators and Base vectors

How many vectors are needed to generate a vector space?

- Any set of vectors that generate a vector space is called a generator set.
- Given a vector space $\mathbf{R}^n$ we can prove that we need minimum of n, linearly independent, vectors to generate all vectors $\mathbf{v}$ in $\mathbf{R}^n$.
- A generator set with minimum size is called a base for the given vector space.

Representation of vectors through basis vectors

Given a vector space R^n, a set of basis vectors B {b_i in R^n, $i=1,\ldots n$} and a vector v in R^n we can always find scalar coefficients such that:

$$\mathbf{v} = a_1\mathbf{b}_1+\ldots+a_n\mathbf{b}_n$$

So, **v** with respect to *B* is:

$$\mathbf{v}_B = (a_1,\ldots,a_n)$$

The elements of a vector v in R^n are the scalar coefficients of the linear combination of the base vectors

Standard unit vectors

For any vector space R^n:

$$\mathbf{i}_1 = (1, 0, 0, \ldots, 0, 0)$$

$$\mathbf{i}_2 = (0, 1, 0, \ldots, 0, 0)$$

$$\ldots$$

$$\mathbf{i}_n = (0, 0, 0, \ldots, 0, 1)$$

The elements of a vector v in R^n are the scalar coefficients of the linear combination of the base vectors.

Standard unit vectors

The elements of a vector v in R^n are the scalar coefficients of the linear combination of the base vectors.

$$\mathbf{v} = (x_1, \ldots, x_n), \quad x_i \in \Re$$

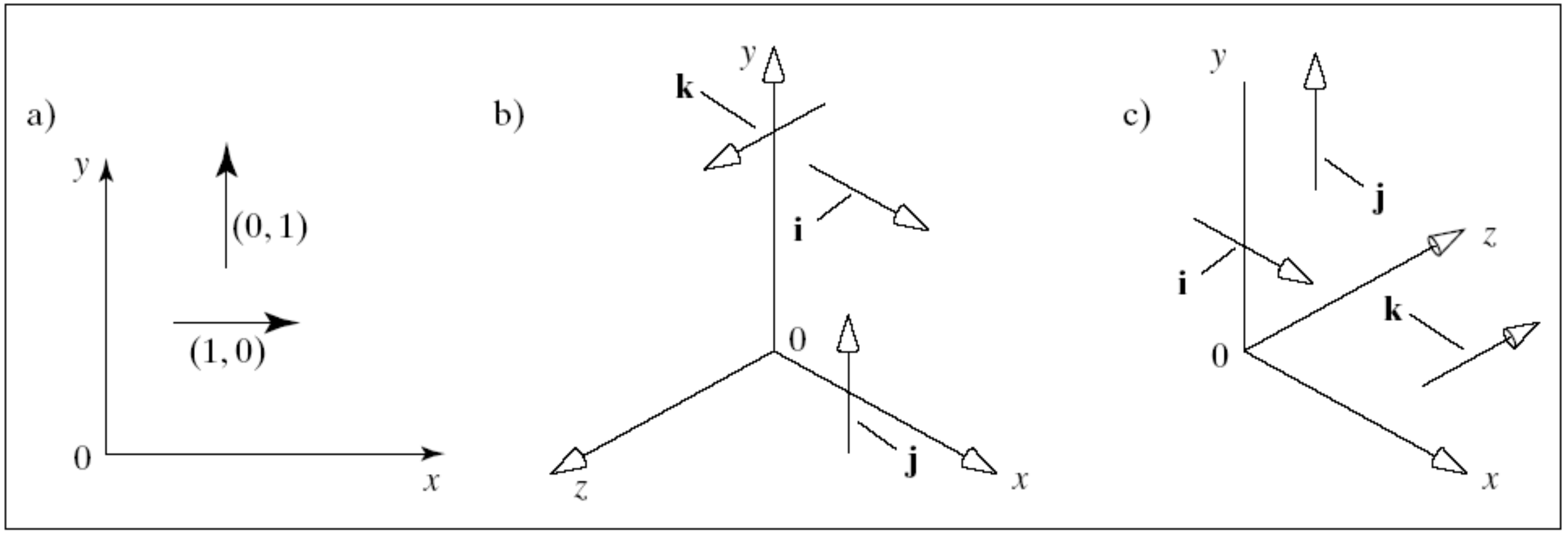
$$\begin{aligned}(x_1, x_2, ..., x_n) = \; & x_1(1, 0, 0, \ldots, 0, 0) \\ & +x_2(0, 1, 0, \ldots, 0, 0) \\ & \ldots \\ & +x_n(0, 0, 0, \ldots, 0, 1)\end{aligned}$$

Standard unit vectors in 3D

i = (1,0,0)

j = (0,1,0)

k = (0,0,1)



Right handed

Left handed

Dot Product

Definition:

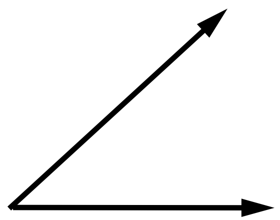
$$\mathbf{w}, \mathbf{v} \in \Re^n$$
$$\mathbf{w} \cdot \mathbf{v} = \sum_{i=1}^{n} w_i v_i$$

Properties

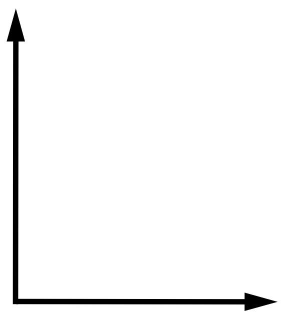
1. Summetry: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
2. Linearity: $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$
3. Homogeneity: $(s\mathbf{a}) \cdot \mathbf{b} = s(\mathbf{a} \cdot \mathbf{b})$
4. $|\mathbf{b}|^2 = \mathbf{b} \cdot \mathbf{b}$
5. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| cos(\theta)$

Dot product and perpendicularity

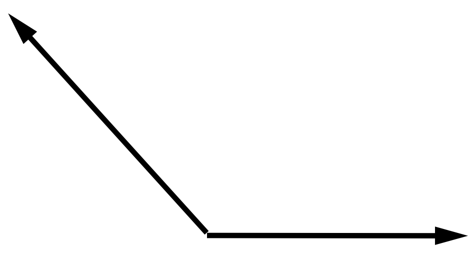
From Property 5:



$\mathbf{b} \cdot \mathbf{c} > 0$



$\mathbf{b} \cdot \mathbf{c} = 0$



$\mathbf{b} \cdot \mathbf{c} < 0$

Perpendicular vectors

Definition

Vectors $\mathbf{b}$ and $\mathbf{c}$ are perpendicular iff $\mathbf{b} \cdot \mathbf{c} = 0$

Also called normal or orthogonal

It is easy to see that the standard unit vectors form an orthogonal basis:

$\mathbf{i} \cdot \mathbf{j} = 0, \quad \mathbf{j} \cdot \mathbf{k} = 0, \quad \mathbf{i} \cdot \mathbf{k} = 0$

Cross product

Defined only for 3D Vectors and with respect to the standard unit vectors

Definition

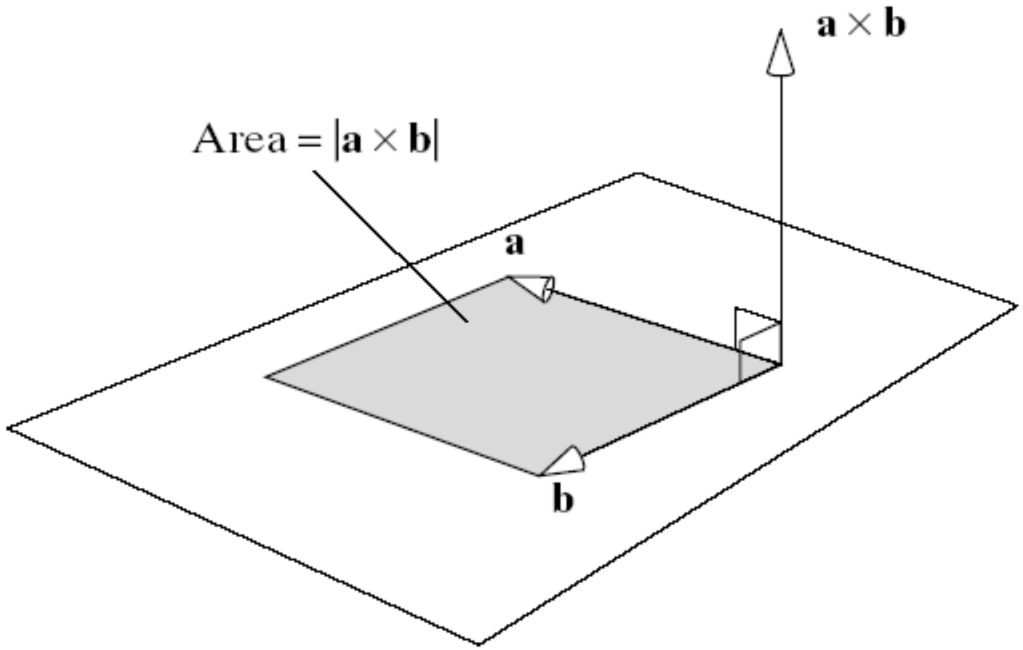
$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties of the cross product

1. $\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{i} \times \mathbf{j} = \mathbf{k}$.
2. Antisymmetry: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
3. Linearity: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
4. Homogeneity: $(s\mathbf{a}) \times \mathbf{b} = s(\mathbf{a} \times \mathbf{b})$.
5. The cross product is normal to both vectors:
$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.
6.$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|sin(\theta)$.

Geometric interpretation of the cross product



Clarification for the figure:
a and **b** need not be perpendicular

Recap

Vector spaces

Operations with vectors

Representing vectors through a basis

$\mathbf{v} = a_1\mathbf{b}_1 + \ldots a_n\mathbf{b}_n$, $\mathbf{v}_B = (a_1,\ldots,a_n)$

Standard unit vectors

Dot product

Perpendicularity

Cross product

Normal to both vectors

Points vs Vectors

What is the difference?

Points vs Vectors

What is the difference?

Points have location but no size or direction.

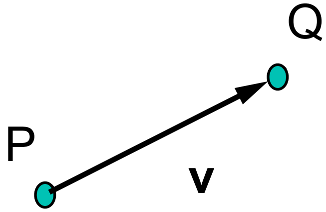
Vectors have size and direction but no location.

Problem: we represent both as triplets!

Relationship between points and vectors

A difference between two points is a vector:

$Q - P = \mathbf{v}$

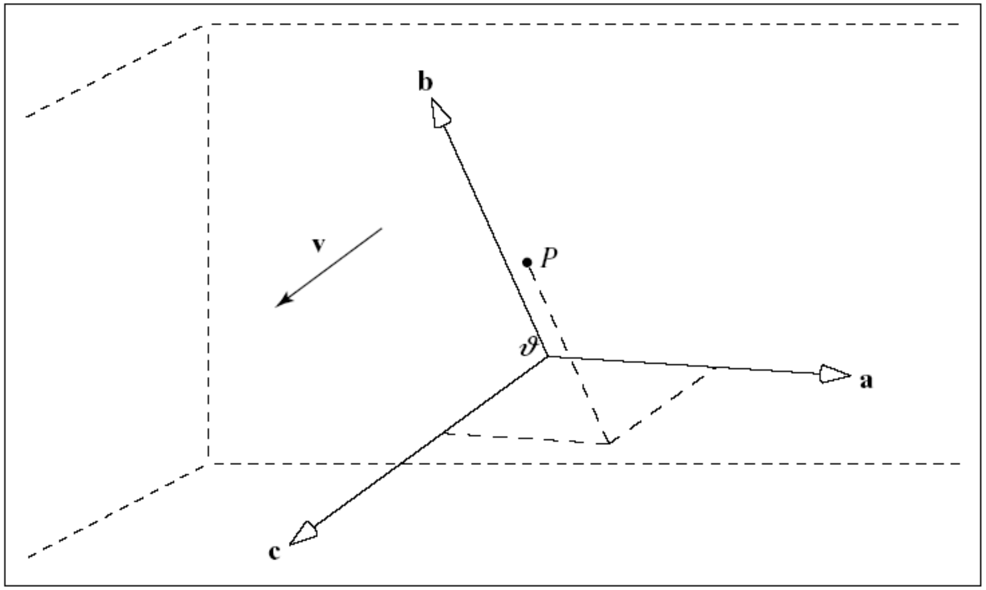


We can consider a point as a point plus an offset

$Q = P + \mathbf{v}$

Coordinate systems

Defined by: (**a**,**b**,**c**,θ)

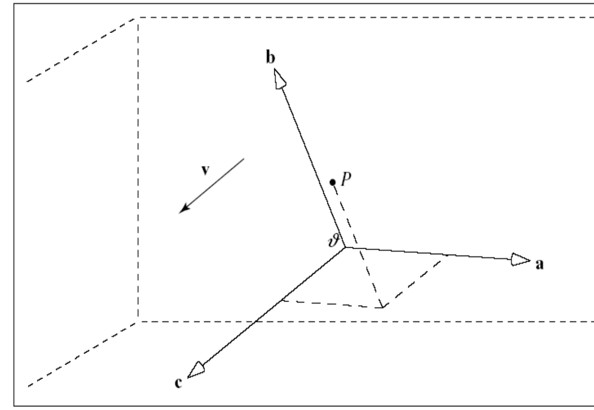


$$\mathbf{v} = v_1\mathbf{a} + v_2\mathbf{b} + v_3\mathbf{c}$$

$$P - \theta = p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c}$$

$$P = \theta + p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c}$$

The homogeneous representation of points and vectors



In the coordinate system $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \theta)$

$\mathbf{v} = v_1\mathbf{a} + v_2\mathbf{b} + v_3\mathbf{c}$, and

$P = \theta + p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c}$

Using the same base elements: $\mathbf{a}, \mathbf{b}, \mathbf{c}, \theta$

$\mathbf{v} = v_1\mathbf{a} + v_2\mathbf{b} + v_3\mathbf{c} + 0\theta$

$P = p_1\mathbf{a} + p_2\mathbf{b} + p_3\mathbf{c} + 1\theta$

or equivalently

$P = (p_1, p_2, p_3, 1), \quad \mathbf{v} = (v_1, v_2, v_3, 0)$

Switching coordinates

Normal to homogeneous:

- Vector: append as fourth coordinate 0

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix}$$

- Point: append as fourth coordinate 1

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \rightarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix}$$

Switching coordinates

Homogeneous to normal:

- Vector: remove fourth coordinate (0)

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- Point: remove fourth coordinate (1)

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

Does the homogeneous representation support operations?

Operations :

- $\mathbf{v} + \mathbf{w} = (v_1,v_2,v_3,0) + (w_1,w_2,w_3,0) =$
  $(v_1+w_1,\ v_2+w_2,\ v_3+w_3,\ 0)$ Vector!
- $a\mathbf{v} = a(v_1,v_2,v_3,0) = (av_1,\ av_2,\ av_3,\ 0),$ Vector!
- $a\mathbf{v} + b\mathbf{w} = a(v_1,v_2,v_3,0) + b(w_1,w_2,w_3,0) =$
  $(av_1+bw_1,\ av_2+bw_2,\ av_3+bw_3,\ 0)$ Vector!
- $P+\mathbf{v} = (p_1,p_2,p_3,1) + (v_1,v_2,v_3,0) =$
  $= (p_1+v_1,\ p_2+v_2,\ p_3+v_3,\ 1)$ Point!

Linear combination of points

Points *P*, *R* scalars *f,g*:

$$fP+gR = f(p_1,p_2,p_3,1) + g(r_1,r_2,r_3,1)$$
$$= (fp_1+gr_1,\ fp_2+gr_2,\ fp_3+gr_3,\ f+g)$$

What is it?

Linear combination of points

Points *P*, *R* scalars *f,g*:

$$fP+gR = f(p_1,p_2,p_3,1) + g(r_1,r_2,r_3,1)$$
$$= (fp_1+gr_1,\ fp_2+gr_2,\ fp_3+gr_3,\ f+g)$$

What is it?

- If $(f+g) = 0$ then vector!
- If $(f+g) = 1$ then point!

Affine combinations of points

Definition:

Points P_i: i = 1,…,n

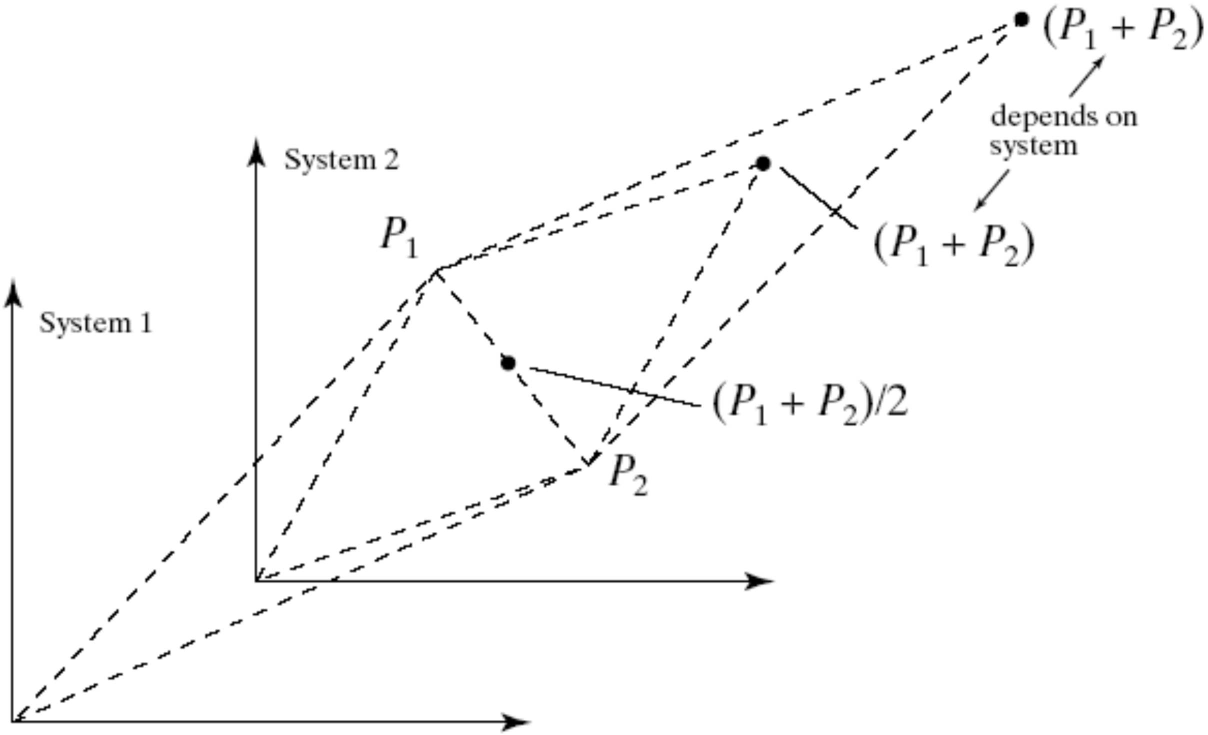
Scalars f_i: i = 1,…,n

$$f_1 P_1 + \ldots + f_n P_n \quad \text{iff} \quad f_1 + \ldots + f_n = 1$$

Example: $0.5P_1 + 0.5P_2$

$L(t) = (1-t)P_1 + t\, P_2$, t in [0,1]

Geometric explanation



Recap

Vector spaces

Dot product

Cross product

Coordinate systems (mostly orthonormal)

Homogeneous representations of points and vectors

Matrices

Rectangular arrangement of elements:

$$A_{3\times 3} = \begin{pmatrix} -1 & 2.0 & 0.5 \\ 0.2 & -4.0 & 2.1 \\ 3 & 0.4 & 8.2 \end{pmatrix}$$

$$A = (A_{ij})$$

Special square matrices

Symmetric: $(A_{ij})_{n \times n} = (A_{ji})_{n \times n}$

Zero: $A_{ij} = 0$, for all i,j

$$\text{Identity: } I_n = \begin{cases} I_{ii} = 1, \text{ for all } i \\ I_{ij} = 0 \text{ for } i \neq j \end{cases}$$

Operations with matrices

Addition:

$$A_{m\times n} + B_{m\times n} = (a_{ij} + b_{ij})$$

Properties:

1. $A + B = B + A.$
2. $A + (B + C) = (A + B) + C.$
3. $f(A + B) = fA + fB.$
4. Transpose: $A^T = (a_{ij})^T = (a_{ji}).$

Matrix Multiplication

Definition:

$$C_{m\times r} = A_{m\times n}B_{n\times r}$$

$$(C_{ij}) = (\sum_{k}^{n} a_{ik}b_{kj})$$

A few properties:

1. Not commutative: $AB \neq BA$.
2. Associative: $A(BC) = (AB)C$.
3. Compatible with Scalar multiplication:
   $f(AB) = (fA)B$ and $(AB)f = A(Bf)$.
4. Distributive:
   $A(B+C) = AB + AC$, and $(B+C)A = BA + CA$.
5. $(AB)^T = B^TA^T$.

Inverse of a square matrix

Definition

$MM^{-1} = M^{-1}M = I$

Important property (square matrices only)

$(AB)^{-1} = B^{-1} A^{-1}$

Convention

Vectors and points are represented as column matrices

$$P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 1 \end{pmatrix} \qquad v = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ 0 \end{pmatrix}$$

However, always keep track of the base, i.e. the corresponding coordinate system

Dot product as a matrix multiplication

A vector is a column matrix

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}^T\mathbf{b}\\ &= (a_1, a_2, a_3)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\\ &= a_1b_1 + a_2b_2 + a_3b_3.\end{aligned}$$

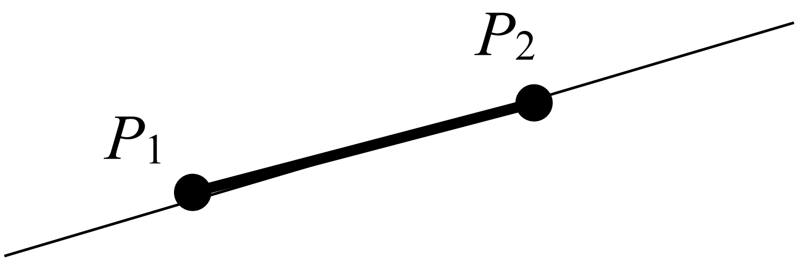
Lines and Planes

Usually defined by an appropriate number of points (vertices)

- Line from two points, or a point and a vector
- Plane from three points or a point and two vectors

- The three points of a triangle define a plane

2D Line



Points: $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$

Explicit: $y = ax + b$

Plug the points and you get:

$$\left\{ \begin{array}{l} y_1 = ax_1 + b \\ y_2 = ax_2 + b \end{array} \right\} \text{ Solve for } a, b \rightarrow y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

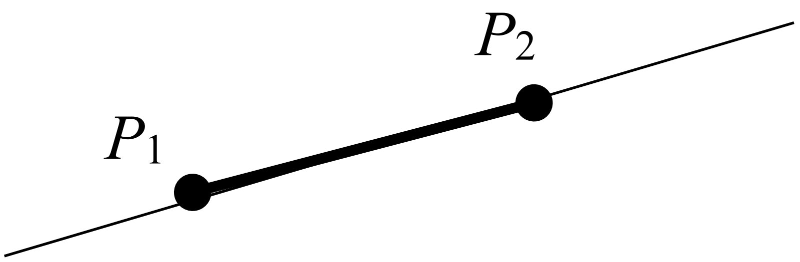
Implicit : $F(x, y) = 0$

$$F(x, y) = (y - y_1)(x_2 - x_1) - (y_2 - y_1)(x - x_1) = 0$$

if $F(x, y) < 0$ then point below or above the line depending on the sign

Parametric: $L(t) = P_1 + t(P_2 - P_1) = (1 - t)P_1 + tP_2, \;\; t \in \Re$

2D Line



Points: $P_1 = (x_1, y_1), P_2 = (x_2, y_2)$

Parametric:

$L(t) = P_1 + t(P_2 - P_1), \quad t \in \Re$

or

$L(t) = (1-t)P_1 + tP_2, \quad t \in \Re$

And per dimension

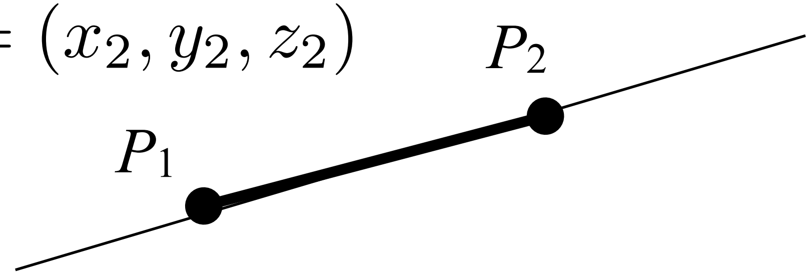
$$\begin{bmatrix} x(t) = (1-t)x_1 + tx_2 \\ y(t) = (1-t)y_1 + ty_2 \end{bmatrix}, \quad t \in \Re$$

Affine combination of points, extends to 3D and any dimension

Also referred to as linear interpolation

3D Line

Points: $P_1 = (x_1, y_1, z_1), P_2 = (x_2, y_2, z_2)$



Parametric:

$$L(t) = P_1 + t(P_2 - P_1), \quad t \in \Re$$

or

$$L(t) = (1-t)P_1 + tP_2, \quad t \in \Re$$

And per dimension

$$\begin{bmatrix} x(t) = (1-t)x_1 + tx_2 \\ y(t) = (1-t)y_1 + ty_2 \\ z(t) = (1-t)z_1 + tz_2 \end{bmatrix}, \quad t \in \Re$$

Planes

Plane equations

Implicit (next slide)

$F(x,y,z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$

Points on Plane $F(x,y,z) = 0$

Parametric

$Plane(s,t) = P_0 + s(P_1 - P_0) + t(P_2 - P_0)$

P_0, P_1, P_2 not colinear

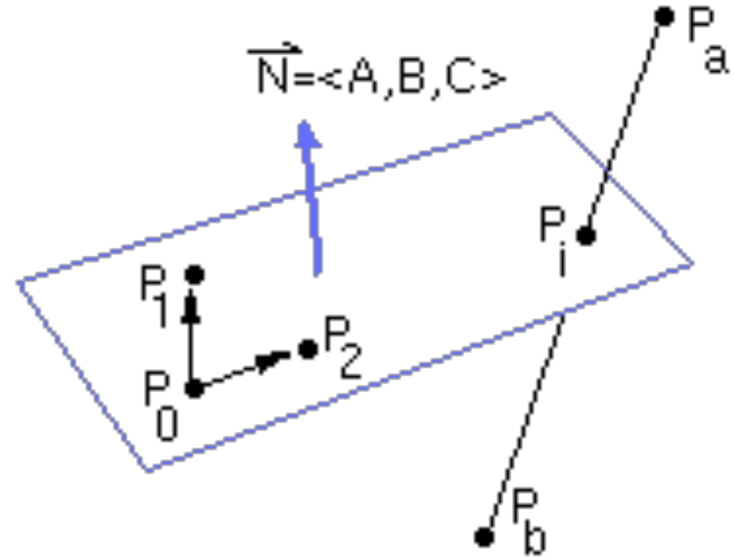
or

$Plane(s,t) = (1 - s - t)P_0 + sP_1 + tP_2$

$Plane(s,t) = P_0 + sV_1 + tV_2$ where V_1, V_2 basis vectors

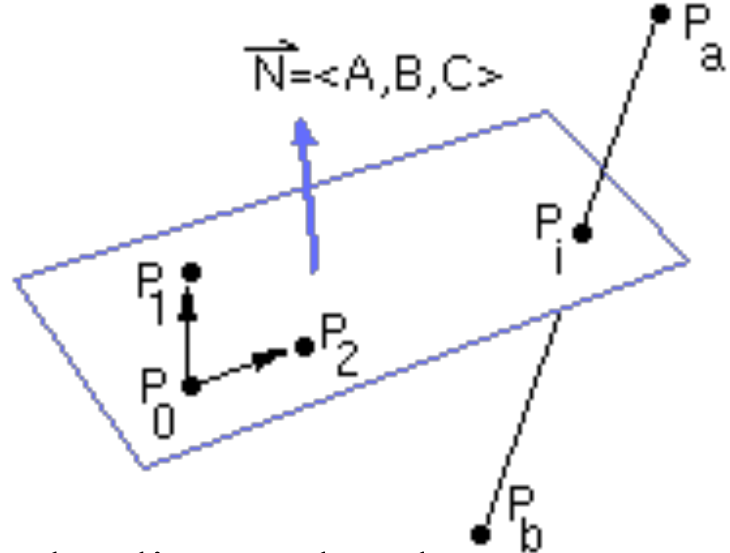
Explicit

z = - (A/C)x - (B/C)y - D/C, C $\neq$ 0



Implicit, Point normal form

Plane equation



$$F(x,y,z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$$

Points on Plane $F(x,y,z) = 0$

Observation : Let's take an arbitrary vector u that lies on the plane which can be defined by two points e.g. P1, P2 on the plane.

$$\mathbf{u} = P2 - P1$$

$$\left.\begin{array}{l} \mathbf{N} \bullet P1 + D = 0 \\ \mathbf{N} \bullet P2 + D = 0 \end{array}\right\} \Rightarrow \mathbf{N} \bullet (P2 - P1) = 0 \Rightarrow \mathbf{N} \bullet \mathbf{u} = 0 \Rightarrow \boxed{\mathbf{N} \perp \mathbf{u}}$$

Computing point normal form from 3 Points

$F(x, y, z) = Ax + By + Cz + D = \mathbf{N} \bullet P + D$

Points on Plane $F(x, y, z) = 0$

First way:

$\mathbf{N} \bullet P0 + D = 0$

$\mathbf{N} \bullet P1 + D = 0$

$\mathbf{N} \bullet P2 + D = 0$

$|\mathbf{N}| = 1$ (arbitrary choice)

Second way:

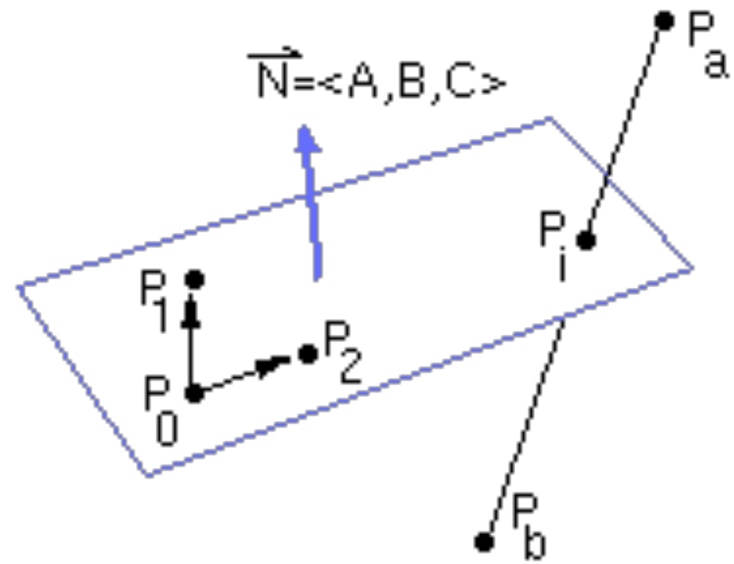
$\mathbf{N}$ is normal to F

Let's find a normal vector:

$\mathbf{N} = (P1 - P0) \times (P2 - P0)$

Compute D:

$D = -\mathbf{N} \bullet P0$



Intersection of line and plane

Implicit equation for the plane:

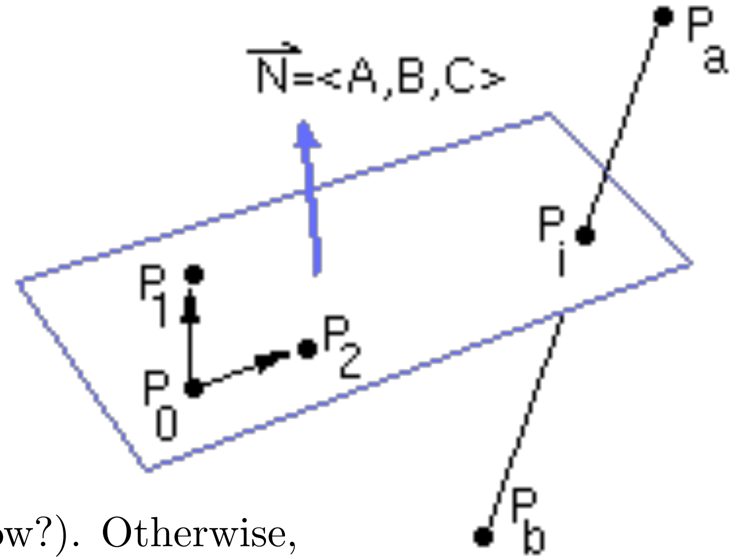
$$F(P) = \mathbf{N} \cdot P + D$$

Parametric equation for the line from P_a to P_b:

$$L(t) = P_a + t(P_b - P_a)$$

Plug $L(t)$ in $F(P)$ and solve for $t = t_i$:

$$\mathbf{N} \cdot [P_a + t_i(P_b - P_a)] = -D$$



If $N \cdot (P_a - P_b) = 0$ then zero or infinite solutions (how?). Otherwise,

$$t_i = \frac{-D - \mathbf{N} \cdot P_a}{\mathbf{N} \cdot P_b - \mathbf{N} \cdot P_a} = \frac{-F(P_a)}{F(P_b) - F(P_a)}$$

Finally, evaluate $L(t_i)$ for the intersection point P_i:

$$P_i = P_a + \frac{-F(P_a)}{F(P_b) - F(P_a)}(P_b - P_a) = \frac{P_a F(P_b) - P_b F(P_a)}{F(P_b) - F(P_a)}$$

Exercises

Orthogonal projection of a vector on another vector

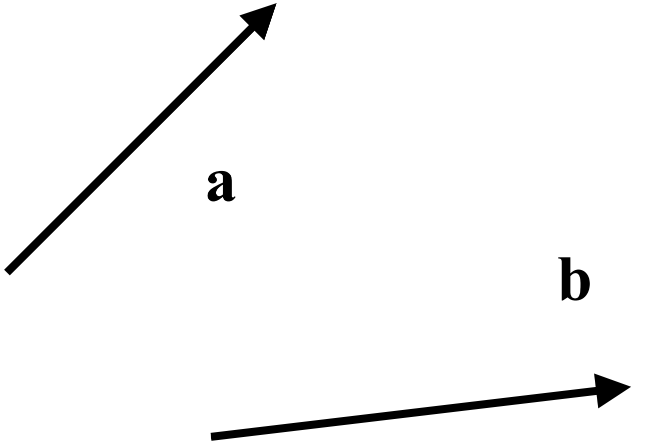
Orthogonal projection of a point on a plane

Intersect two lines in 2D, 3D

Intersect two line segments in 2D, 3D

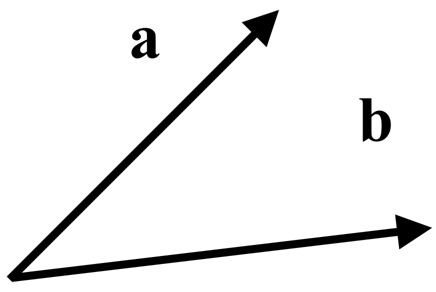
Intersect a line with a plane in 3D

Exercise 1: Project vector a on b



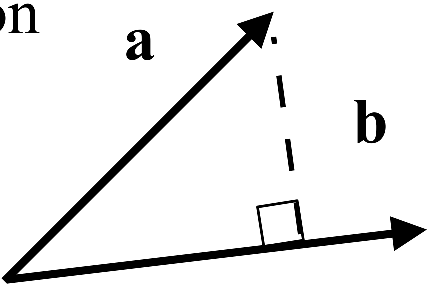
Exercise 1: Project vector a on b

For convenience



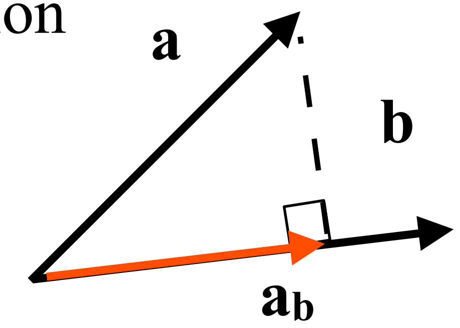
Exercise 1: Project vector a on b

Orthogonal projection



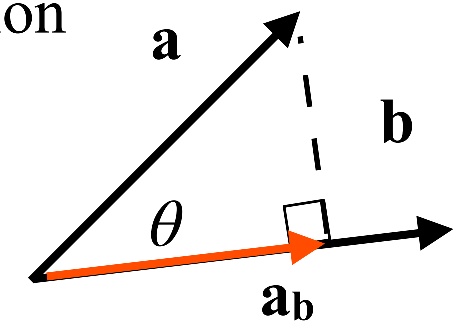
Exercise 1: Project vector a on b

Orthogonal projection



Exercise 1: Project vector a on b

Orthogonal projection

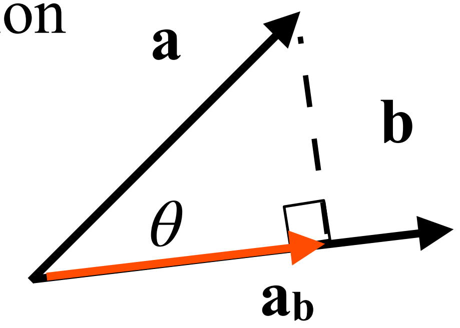


Magnitude of projection:

$$|\vec{a}_{\vec{b}}| = |\vec{a}| \cos(\theta)$$

Exercise 1: Project vector a on b

Orthogonal projection



Magnitude of projection:

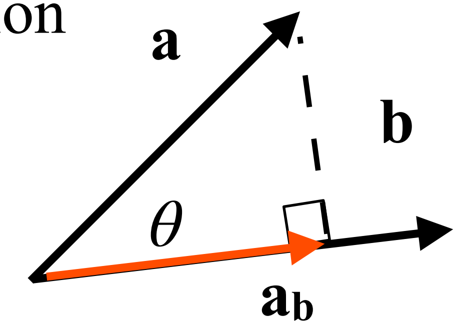
$$|\vec{a}_{\vec{b}}| = |\vec{a}| \cos(\theta)$$

Remember:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos(\theta)$$

Exercise 1: Project vector a on b

Orthogonal projection



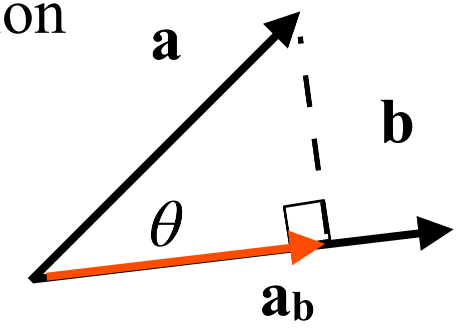
$$|\vec{a}_b| = |\vec{a}|\cos(\theta)$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$

So: $$|\vec{a}_b| = \vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}$$

Exercise 1: Project vector a on b

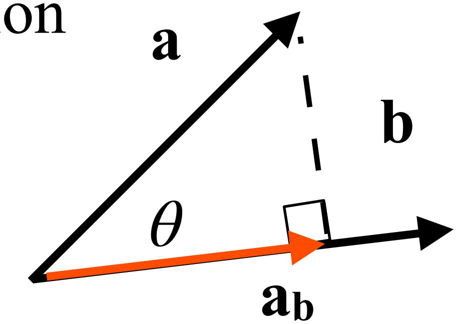
Orthogonal projection



$$|\vec{a}_b| = \vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}$$ Vector: $$\vec{a}_b = \left(\vec{a} \cdot \frac{\vec{b}}{|\vec{b}|}\right) \frac{\vec{b}}{|\vec{b}|}$$

Exercise 1: Project vector a on b

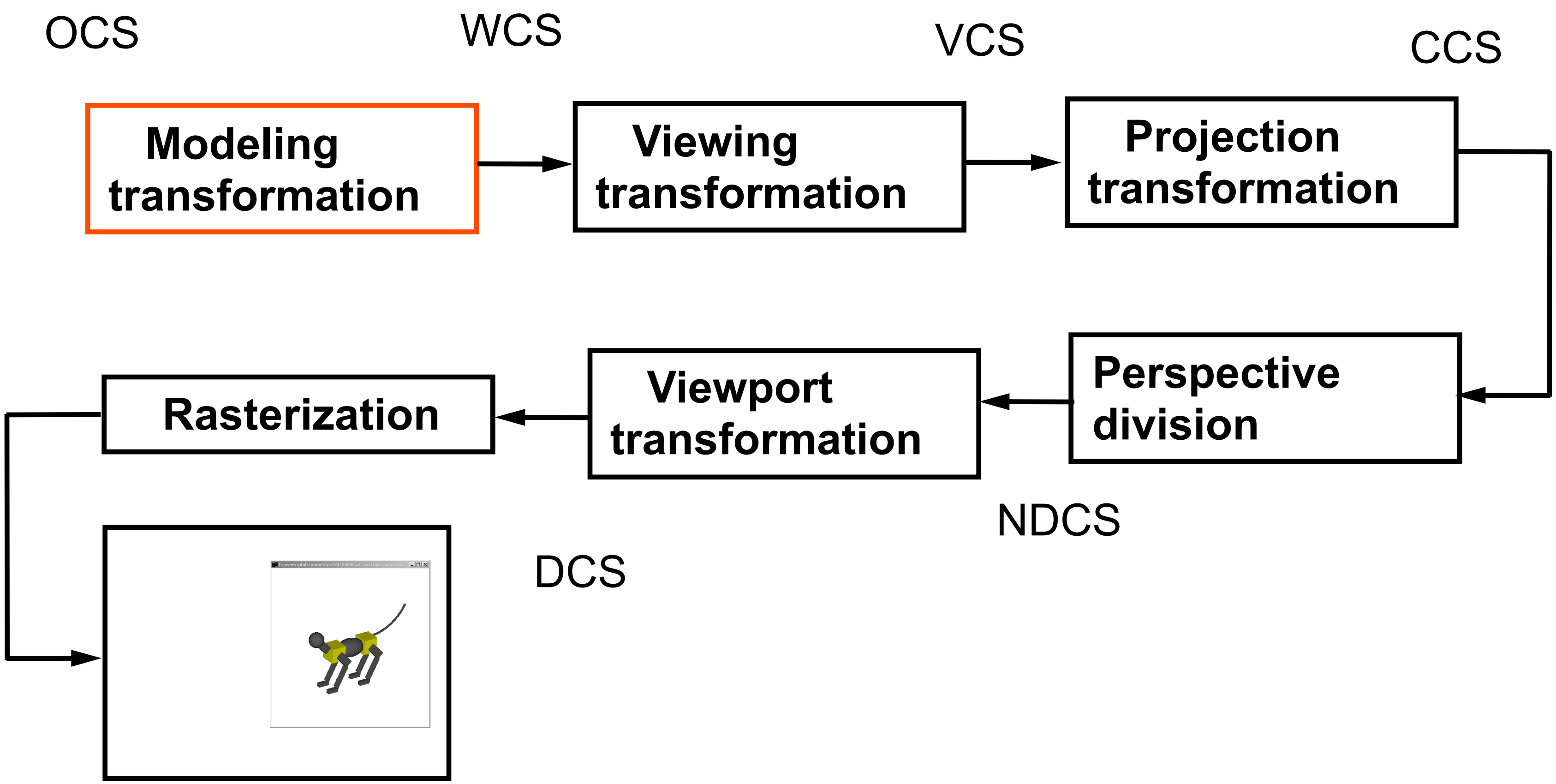
Orthogonal projection



Let unit direction $$\vec{n}_b = \frac{\vec{b}}{|\vec{b}|}$$

Then $$\vec{a}_b = (\vec{a} \cdot \vec{n}_b)\,\vec{n}_b$$

Z-buffer Graphics Pipeline

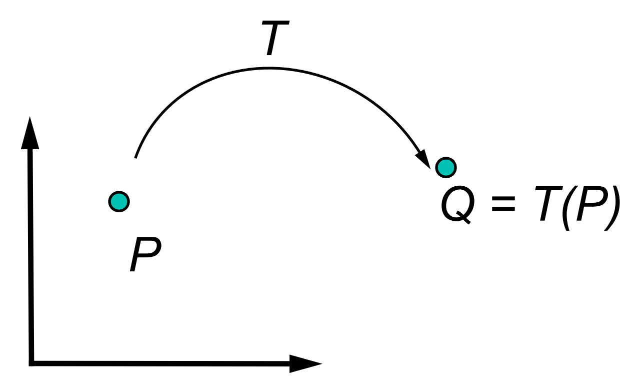


Transformations (2D)

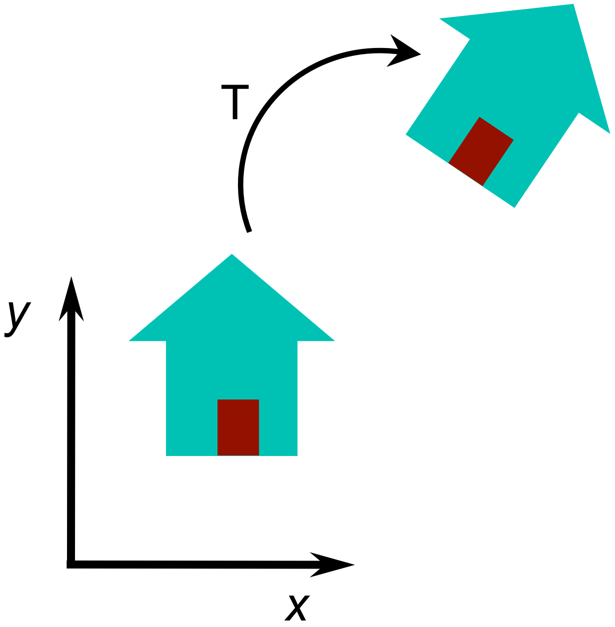
General Form: $Q = T(P), \quad P \in \Re^n, Q \in \Re^m$

If $n > m$ projection

Example: $(Q_x \;\; Q_y \;\; 1)^T = (cos(P_y)e^{-P_y} \;\; ln(P_x) \;\; 1)^T$



Why Transformations?



Affine Transformations (2D)

Linear in the coordinates

$$
\begin{aligned}
&Q = T(P) \\
&\begin{pmatrix} Q_x \\ Q_y \end{pmatrix} = \begin{pmatrix} m_{11}P_x + m_{12}P_y + m_{13} \\ m_{21}P_x + m_{22}P_y + m_{23} \end{pmatrix}, \\
&m_{11}, ..., m_{23} \in \Re
\end{aligned}
$$

In homogeneous coordinates:

$$
\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11}P_x + m_{12}P_y + m_{13} \\ m_{21}P_x + m_{22}P_y + m_{23} \\ 1 \end{pmatrix}
$$

Matrix Form of Affine Transformations

Transformation as a matrix multiplication

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

$$Q = MP$$

Transforming Points and Vectors

Points:

$$\begin{pmatrix} Q_x \\ Q_y \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ 1 \end{pmatrix}$$

Vectors:

$$\begin{pmatrix} W_x \\ W_y \\ 0 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix}$$