Bid–Ask Spreads as Curvature in Price Space: An Analytic-Geometry Approach to Liquidity, Impact, and Stress

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Abstract

This paper develops an analytic-geometry framework that interprets bid–ask spreads and liquidity premia as geometric curvature of a price surface defined over order size and time. We model the observable quote pair (bid, ask) as a first-order discretization of a latent convex price surface $P(q, t)$, where q denotes signed trade size (positive for buy pressure, negative for sell pressure). In this representation, the bid–ask spread measures local nonlinearity of execution costs, while higher-order derivatives encode market depth, resilience, and adverse selection. We propose curvature-based liquidity measures that are (i) scale-consistent across assets, (ii) robust to microstructure noise, and (iii) predictive of future volatility, return reversals, and crash risk. We provide identification strategies and estimators using quote and trade data, and we outline empirical tests and falsification exercises. The geometry perspective unifies classic microstructure results on inventory risk, asymmetric information, and order-processing costs with modern convex analysis of limit order books.

Contents

1 Introduction

Bid–ask spreads are among the most widely used proxies for liquidity, trading costs, and market quality. Classic microstructure models explain spreads through combinations of order-processing costs, inventory risk, and adverse selection. Yet the spread is often treated as a scalar statistic rather than as a local geometric object that summarizes the shape of execution costs around the origin of the order book.

This paper proposes a geometric interpretation: *the bid–ask spread is a manifestation of curvature of a latent price surface*. Specifically, we posit an execution price surface $P(q, t)$ mapping signed order size q and time t into the average execution price per share (or per unit notional). Under mild regularity and convexity conditions, local curvature of this surface determines how marginal prices change with trade size and direction. In particular, the observed bid and ask are local one-sided evaluations of the surface near $q = 0$, and the spread is a discretized measure of the surface's local nonlinearity.

This analytic-geometry view yields three contributions:

1. **Geometry of trading costs.** We derive how the spread, depth, and price impact arise from the first and second derivatives of $P(q, t)$ and its dual (cost) function. This provides a coordinate-free interpretation of liquidity as curvature.

2. **Curvature-based liquidity measures.** We define curvature and "flatness" measures that generalize the spread and incorporate depth and resiliency in a unified way.

3. **Testable predictions.** The framework predicts that curvature rises in stress, forecasts volatility and reversals, and is amplified by information asymmetry. We propose empirical designs and placebo tests using high-frequency data.

2 Related Literature

The microstructure literature decomposes spreads into order-processing, inventory, and adverse-selection components. Canonical models include inventory-based market making and information-based spread formation, and empirical work documents the link between spreads, depth, and volatility. Our geometry perspective is consistent with these foundations but reframes the objects as derivatives of a convex surface, connecting to convex analysis and optimal execution.

3 A Price Surface and Its Geometry

3.1 Latent execution price surface

Let $q \in \mathbb{R}$ denote signed trade size (positive for buys, negative for sells). Define the *average execution price per unit* for a trade of size q at time t:

$$P(q,t) \;=\; \frac{1}{q}\int_0^q m(u,t)\,du \quad (q \neq 0), \tag{1}$$

where $m(u,t)$ is the *marginal execution price* (the price of the u-th marginal unit). For $q \to 0$, define $P(0,t) = m(0,t)$.

Define the *total execution cost* relative to a reference midprice $M(t)$:

$$C(q,t) \;=\; \int_0^q \big(m(u,t) - M(t)\big)\,du. \tag{2}$$

If the limit order book is shape-stable and no-arbitrage conditions hold locally, then $C(\cdot,t)$ is convex in q (marginal costs increase with size). Convexity is central: it implies that curvature is well-defined in the generalized sense (second derivative or subdifferential).

3.2 Bid, ask, and spread as local geometric objects

Let $A(t)$ and $B(t)$ be the best ask and best bid quotes. Under a small-trade approximation, interpret them as one-sided marginal prices:

$$A(t) \approx m(0^+,t), \qquad B(t) \approx m(0^-,t). \tag{3}$$

Define the mid and spread:

$$M(t) = \frac{A(t)+B(t)}{2}, \qquad S(t) = A(t) - B(t). \tag{4}$$

In our geometric view, $S(t)$ is a discretized measure of the kink/curvature of $C(q,t)$ at $q = 0$. If C is differentiable at 0, then $m(0^+,t) = m(0^-,t)$ and the spread collapses; in realistic markets, discrete ticks, adverse selection, and order-processing costs induce a kink (subgradient interval) at 0.

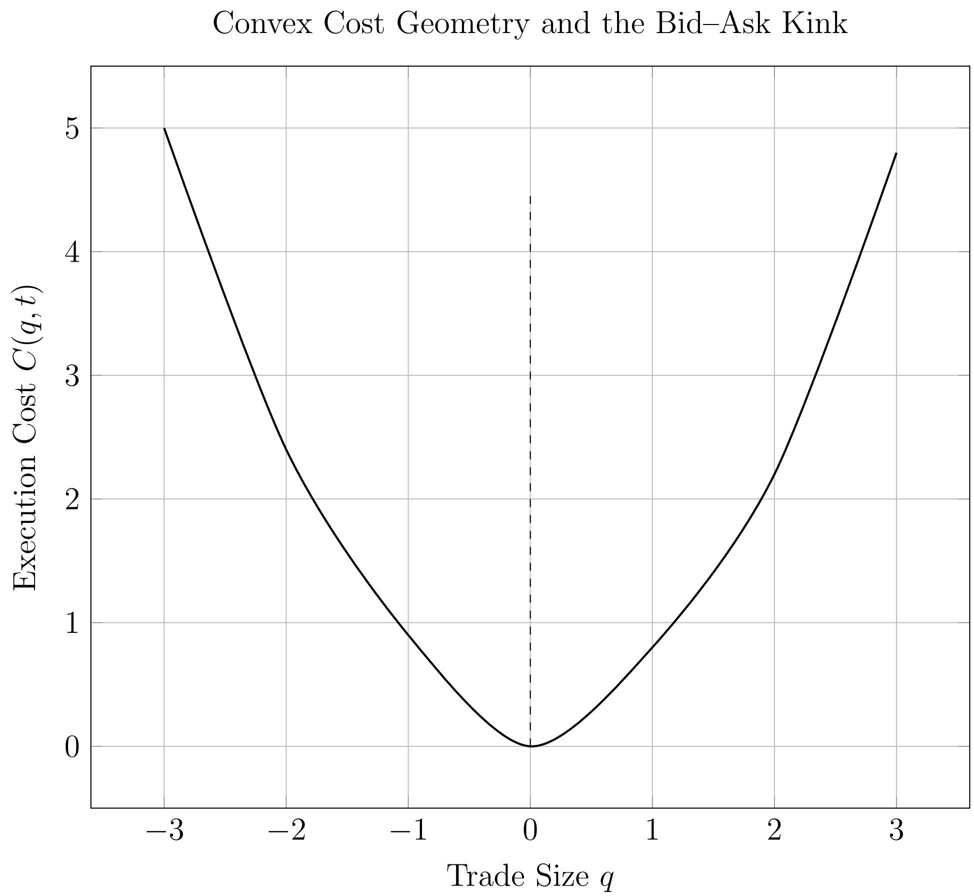

Figure 1: Convex execution cost function. The nondifferentiable point at $q = 0$ corresponds to the bid–ask spread. Curvature away from the origin captures market depth and nonlinear price impact.

3.3 Curvature, depth, and resilience

Assume $C(\cdot, t)$ is twice differentiable on $(0, \bar{q}]$ and $[-\bar{q}, 0)$ for some local window $\bar{q} > 0$. Then

$$m(q,t) = M(t) + \frac{\partial C}{\partial q}(q,t), \tag{5}$$

and *local depth* relates to the second derivative:

$$\kappa^{+}(t) := \frac{\partial^2 C}{\partial q^2}(0^{+}, t), \qquad \kappa^{-}(t) := \frac{\partial^2 C}{\partial q^2}(0^{-}, t). \tag{6}$$

Intuition: larger κ means marginal costs increase faster with trade size (shallower book / less depth).

To incorporate time and resiliency, consider the surface $C(q,t)$ in (q,t) coordinates. A simple resilience proxy is the *mean-reversion speed* of marginal impact:

$$\frac{\partial}{\partial t}\left(\frac{\partial C}{\partial q}(q,t)\right) \approx -\lambda(t)\frac{\partial C}{\partial q}(q,t) + \text{shock}, \tag{7}$$

where larger $\lambda(t)$ implies faster recovery (flatter effective surface over time).

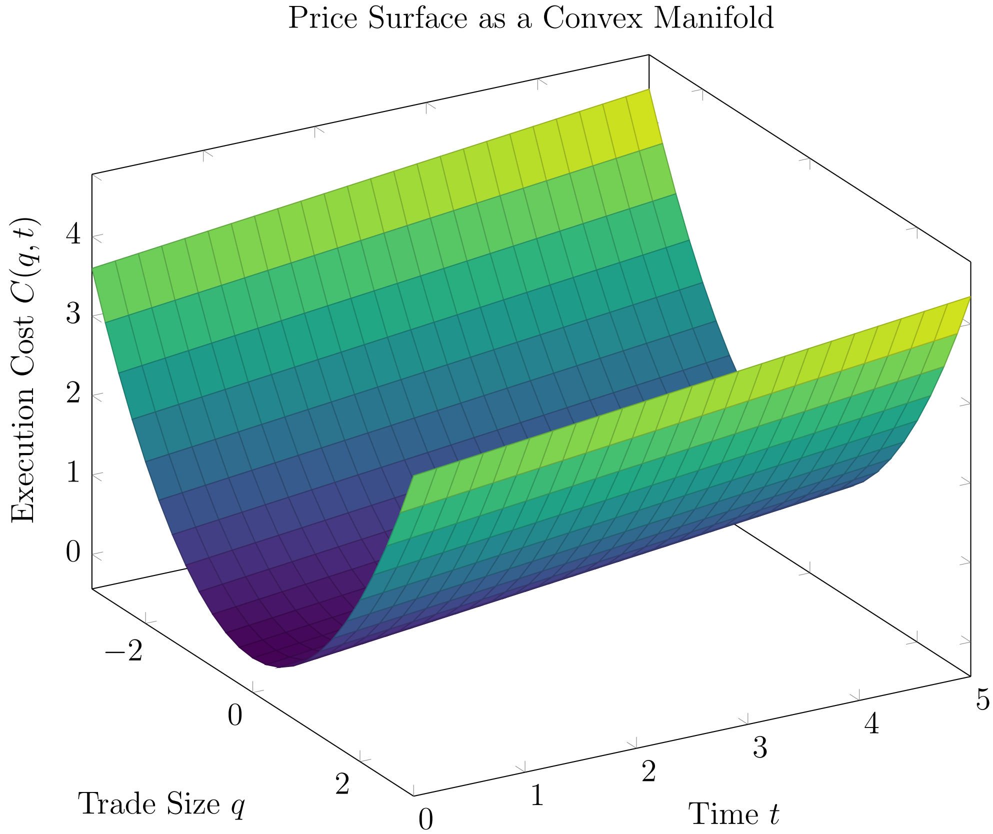

Figure 2: Execution cost surface viewed as a convex manifold. Local curvature in the q-direction corresponds to liquidity, while slope in the t-direction reflects resiliency and recovery dynamics.

3.4 A coordinate-free curvature measure

Define a local quadratic approximation of costs around $q = 0$:

$$C(q,t) \approx \begin{cases} \alpha^+(t)q + \frac{1}{2}\kappa^+(t)q^2, & q > 0, \\ \alpha^-(t)q + \frac{1}{2}\kappa^-(t)q^2, & q < 0, \end{cases} \tag{8}$$

where $\alpha^+(t) \approx \frac{S(t)}{2}$ and $\alpha^-(t) \approx -\frac{S(t)}{2}$ under symmetric order-processing costs.

A scale-consistent *dimensionless curvature* can be defined by normalizing by price and typical trade size q_0:

$$\mathrm{Curv}(t;q_0) := \frac{q_0}{M(t)} \cdot \frac{m(q_0,t) - m(0^+,t)}{q_0} = \frac{q_0}{M(t)} \cdot \frac{\partial^2 C}{\partial q^2}(\xi,t) \tag{9}$$

for some $\xi \in (0, q_0)$ (mean value theorem). This statistic compares incremental impact to level.

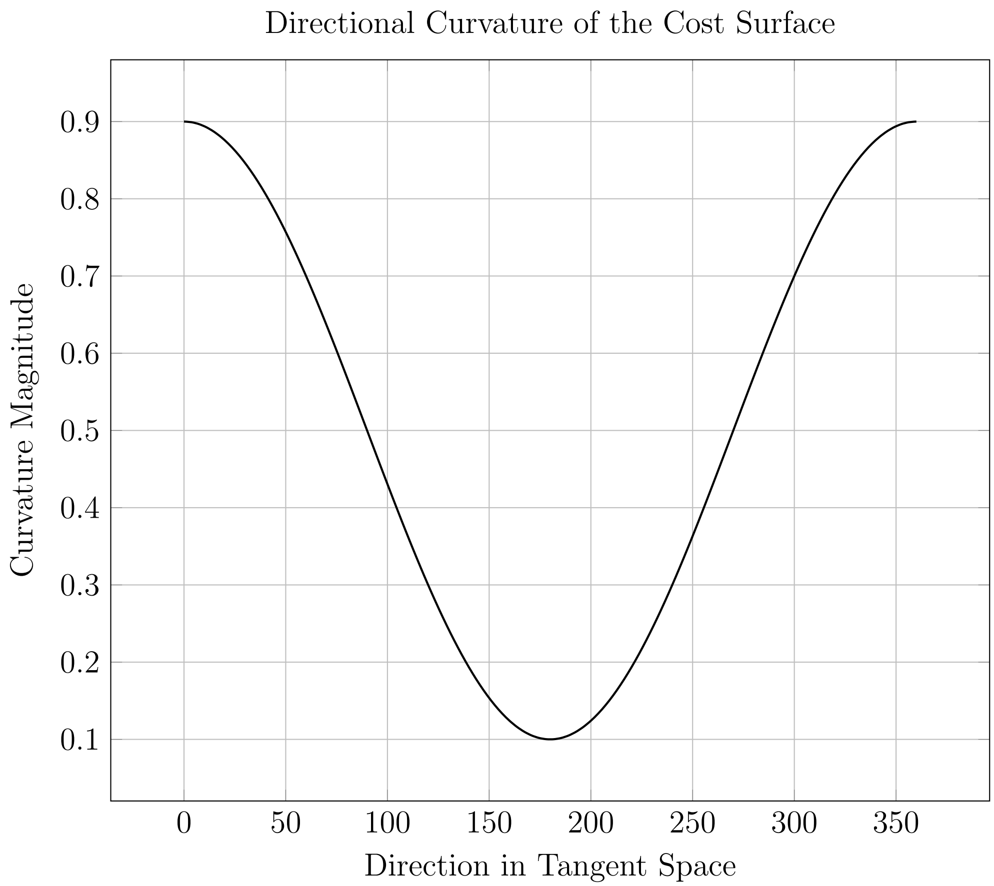

Figure 3: Directional curvature in the tangent space of the execution cost manifold. Asymmetry corresponds to differences in buy- and sell-side liquidity and informed trading pressure.

4 Manifold Structure of the Limit Order Book

We interpret the execution cost surface $C(q,t)$ as a two-dimensional differentiable manifold embedded in $\mathbb{R}^3$ with coordinates (q,t,C). Under regularity conditions, the surface admits a Riemannian metric induced by the ambient Euclidean space:

$$g = \begin{bmatrix} 1 + \left(\frac{\partial C}{\partial q}\right)^2 & \frac{\partial C}{\partial q}\frac{\partial C}{\partial t} \\ \frac{\partial C}{\partial q}\frac{\partial C}{\partial t} & 1 + \left(\frac{\partial C}{\partial t}\right)^2 \end{bmatrix}. \tag{10}$$

This metric defines local notions of distance, angle, and curvature on the surface. In this representation, execution paths correspond to curves $\gamma(s) = (q(s), t(s), C(q(s), t(s)))$ whose length measures cumulative execution cost under geometric distortion induced by liquidity conditions.

Liquidity can thus be interpreted as local flatness of the manifold, while illiquidity corresponds to regions of high metric distortion and curvature.

5 Microfoundation: Why Curvature Emerges

We connect curvature to standard economic mechanisms.

5.1 Order-processing and tick constraints

Fixed per-trade costs and discrete pricing produce a kink at the origin: the subgradient set of C at 0 has nonzero width, mapping directly to the spread. This yields a baseline curvature even absent information.

5.2 Inventory risk

A risk-averse liquidity supplier facing inventory I_t sets marginal prices that depend on inventory and risk:

$$m(q,t) = M(t) + \phi I_t + \psi q + \epsilon_t, \tag{11}$$

where $\psi > 0$ implies increasing marginal compensation for larger q (curvature), while ϕI_t shifts the surface asymmetrically.

5.3 Adverse selection

If buy orders are more likely when the fundamental value is high, the ask embeds an information premium. In a reduced form,

$$m(0^+, t) - M(t) = \text{order cost} + \text{info premium}, \tag{12}$$

and the info premium is increasing in order size for informed trades, implying larger $\kappa^+(t)$ during information events.

6 Gaussian and Mean Curvature of Execution Cost Surfaces

Let the execution cost surface be parameterized by $\mathbf{X}(q, t) = (q, t, C(q, t))$. The first fundamental form is given by:

$$E = 1 + C_q^2, \quad F = C_q C_t, \quad G = 1 + C_t^2, \tag{13}$$

where subscripts denote partial derivatives.

The second fundamental form is:

$$L = \frac{C_{qq}}{\sqrt{1 + C_q^2 + C_t^2}}, \quad M = \frac{C_{qt}}{\sqrt{1 + C_q^2 + C_t^2}}, \quad N = \frac{C_{tt}}{\sqrt{1 + C_q^2 + C_t^2}}. \tag{14}$$

The Gaussian curvature is:

$$K = \frac{LN - M^2}{EG - F^2}, \tag{15}$$

and the mean curvature is:

$$H = \frac{1}{2} \frac{EN - 2FM + GL}{EG - F^2}. \tag{16}$$

In this framework, local liquidity conditions correspond to regions where $|H|$ and $|K|$ are small, indicating near-flat geometry, while market stress corresponds to regions of large curvature, indicating rapid changes in marginal execution costs in both size and time dimensions.

7 Predictions and Testable Hypotheses

7.1 Curvature rises in stress

H1. Curvature measures $\kappa^{\pm}(t)$ and $\mathrm{Curv}(t; q_0)$ increase during market-wide stress, and the increase is stronger for assets with higher information sensitivity.

7.2 Curvature predicts volatility and reversals

H2. Higher curvature forecasts higher realized volatility over horizons $[t, t+h]$:

$$\mathrm{RV}_{t,t+h} = a + b\,\mathrm{Curv}(t; q_0) + \Gamma^{\top} X_t + u_t, \quad b > 0. \tag{17}$$

H3. Higher curvature predicts short-horizon return reversals (liquidity-driven mean reversion) conditional on order flow.

7.3 Geometry separates adverse selection from depth

H4. The *kink* at $q = 0$ (spread) and the *quadratic curvature* away from 0 (depth/impact) respond differently to information events: spreads jump at announcements; curvature responds to sustained informed trading.

8 Measurement and Estimation

8.1 Data requirements

At minimum:

- Best bid/ask quotes $(B(t), A(t))$ and quote sizes
- Trades (time, price, size, direction)
- Optional: full depth-of-book snapshots for better curvature estimates

8.2 Estimating the price surface from trades

Let p_k be trade price, q_k signed size for trade k executed at time t_k. Define a local window $\mathcal{W}(t)$ around time t. Estimate marginal impact as a function of size:

$$p_k - M(t_k) = \theta_0(t) + \theta_1(t)\, q_k + \theta_2(t)\, |q_k| + \theta_3(t)\, q_k^2 + \varepsilon_k, \tag{18}$$

using local regression (kernel weights) to allow time variation. Under symmetry and convexity, $\theta_3(t)$ proxies curvature.

If the full book is available, estimate $C(q,t)$ directly by integrating the book:

$$C(q,t) = \int_0^q \left(p_{\mathrm{LOB}}(u,t) - M(t)\right)\, du, \tag{19}$$

where $p_{\mathrm{LOB}}(u,t)$ is the marginal price obtained by walking the book.

8.3 Convexity-constrained estimation

Because $C(\cdot,t)$ is convex, we can estimate it nonparametrically under convexity constraints:

$$\min_{\widehat{C}} \sum_{k\in\mathcal{W}(t)} \left(\widehat{C}(q_k,t) - \widehat{C}(0,t) - (p_k - M(t_k))q_k\right)^2 \quad \text{s.t. } \widehat{C}(\cdot,t) \text{ convex.} \tag{20}$$

Convex regression yields stable curvature estimates and reduces microstructure noise.

8.4 From curvature to an execution-based spread

Define the *effective spread at size* q_0:

$$S_{\mathrm{eff}}(t;q_0) := \frac{C(q_0,t) - C(-q_0,t)}{q_0}. \tag{21}$$

Under the quadratic approximation,

$$S_{\mathrm{eff}}(t;q_0) \approx S(t) + \frac{1}{2}(\kappa^+(t) + \kappa^-(t))q_0, \tag{22}$$

which cleanly decomposes observed spread into a kink (quote spread) plus curvature (depth/impact).

9 Geodesics and Optimal Execution Paths

We define an execution strategy as a curve $\gamma(s) = (q(s), t(s))$ on the execution cost manifold. The realized trading cost corresponds to the surface length functional:

$$\mathcal{L}(\gamma) = \int_{s_0}^{s_1} \sqrt{E\dot{q}^2 + 2F\dot{q}\dot{t} + G\dot{t}^2}\, ds, \tag{23}$$

where E, F, G are the coefficients of the first fundamental form.

Optimal execution paths satisfy the Euler–Lagrange equations associated with $\mathcal{L}$ and correspond to geodesics on the surface. High-curvature regions deflect geodesics away from steep cost gradients, formalizing the intuition that traders optimally avoid executing large trades in periods of illiquidity or elevated market stress.

This formulation generalizes linear-quadratic execution models by embedding them in a geometric optimization framework.

10 Identification Strategies

10.1 Separating information from inventory

Use announcement windows (earnings, macro releases) as plausibly exogenous shocks to information asymmetry. The geometry prediction: spread (kink) reacts immediately; curvature responds to sustained informed volume.

10.2 Instrumenting order flow

Instrument signed order flow using lagged index flows, ETF creation/redemption shocks, or mechanical rebalancing events. Then estimate causal effects of informed flow on curvature.

10.3 Falsification

11 Convexity, Subgradients, and Arbitrage-Free Geometry

The convexity of $C(\cdot, t)$ implies the existence of a subdifferential $\partial C(0, t)$ at the origin, defined as the set of admissible marginal prices consistent with no-arbitrage:

$$\partial C(0, t) = \{p \in \mathbb{R} \mid C(q, t) \geq pq \ \ \forall q\} . \tag{24}$$

The bid and ask prices correspond to the lower and upper bounds of this set:

$$B(t) = \inf \partial C(0, t), \quad A(t) = \sup \partial C(0, t). \tag{25}$$

This interpretation links market microstructure directly to convex analysis: arbitrage-free pricing corresponds to supporting hyperplanes of the execution cost function, while violations of convexity imply the existence of profitable round-trip trades.

- **Placebo assets:** apply curvature measures to highly liquid instruments (e.g., on-the-run benchmarks) where curvature should be lower.
- **Time permutation:** permute intraday time labels; curvature should lose predictability for future volatility if the effect is not spurious.

12 Empirical Design

12.1 Baseline predictive regressions

For each asset i, estimate:

$$\mathrm{RV}_{i,t,t+h} = a_i + b \, \mathrm{Curv}_i(t; q_0) + c \, S_i(t) + \Gamma^\top X_{i,t} + u_{i,t}. \tag{26}$$

Key test: b remains significant controlling for $S_i(t)$, showing curvature adds information beyond the spread.

12.2 Cross-sectional tests

Test whether curvature is higher for:

- assets with higher information sensitivity (analyst coverage proxies, earnings volatility),
- assets with lower depth (book metrics),
- stress regimes (high VIX or market drawdowns).

13 Mechanism Checks

13.1 Decomposition into adverse selection vs depth

Compare:

1. Kink proxy: $S(t)$ or effective spread at tiny size
2. Curvature proxy: $\kappa(t)$ or $\mathrm{Curv}(t; q_0)$ for moderate q_0

during information vs non-information periods.

13.2 Resilience

Estimate $\lambda(t)$ from impact decay:

$$\Delta M(t + \Delta) - \Delta M(t) = -\lambda(t)\Delta M(t) + \epsilon, \tag{27}$$

and test whether high curvature coincides with low resilience.

14 Discussion: Geometry and Policy

Curvature-based measures can be used for market quality monitoring: a market can exhibit a narrow quoted spread but high curvature, implying that small trades are cheap while moderate trades are costly. This distinction matters for execution, price discovery, and financial stability.

15 Conclusion

We propose an analytic-geometry interpretation of liquidity: bid–ask spreads and execution costs emerge from local curvature of a latent convex price surface over size and time. This framework unifies microstructure components and produces measurable, testable curvature statistics that extend beyond the quoted spread. Future work can integrate this geometry into optimal execution, systemic liquidity stress testing, and cross-market contagion.

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