Convex Geometry of Arbitrage-Free Pricing
A Duality and Separation-Theoretic Framework for Financial Markets

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Abstract

We develop a convex-geometric formulation of arbitrage-free pricing in financial markets. Asset payoffs are modeled as elements of a real vector space endowed with a partial order induced by a positive cone. The set of attainable terminal wealth positions forms a convex cone whose geometric properties encode market completeness, arbitrage, and pricing bounds. Using separation theorems, dual cones, and support functions, we characterize equivalent martingale measures as supporting hyperplanes of the attainable set and derive sharp super- and sub-replication bounds as extremal points of convex hulls in payoff space. We extend the framework to infinite-dimensional spaces, transaction costs, and illiquid markets, and establish stability and sensitivity results for pricing under perturbations of the payoff geometry. Graphical representations of cones, convex hulls, and separating hyperplanes illustrate the geometric structure of arbitrage and pricing.

Contents

1 Introduction

The Fundamental Theorem of Asset Pricing (FTAP) asserts that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure. While traditionally formulated in probabilistic terms, the result admits a natural geometric interpretation: arbitrage corresponds to the existence of a strictly positive payoff in the attainable set, and pricing measures correspond to supporting hyperplanes that separate the attainable set from the positive orthant.

This paper develops a systematic convex-geometric framework for arbitrage-free pricing. We interpret asset payoffs as vectors in a real vector space, attainable portfolios as a convex cone, and prices as linear functionals in the dual space. Arbitrage-free prices arise as elements of the dual cone, and replication and pricing bounds are characterized by the geometry of convex hulls and their faces.

Our contributions are threefold. First, we provide a unified geometric formulation of arbitrage, replication, and pricing using separation theorems and duality. Second, we extend the framework to markets with transaction costs and illiquidity, where the attainable set becomes a non-polyhedral convex set. Third, we study the stability of pricing rules under perturbations of the payoff cone, establishing Lipschitz-type bounds for price sensitivity.

2 Mathematical Preliminaries

2.1 Vector Spaces and Cones

Let $(\mathcal{X}, \|\cdot\|)$ be a real Banach space representing terminal payoffs. Let $\mathcal{K} \subset \mathcal{X}$ be a closed, convex, pointed cone:

$$\mathcal{K} \cap (-\mathcal{K}) = \{0\}, \tag{1}$$

which induces a partial order $x \succeq y$ if and only if $x - y \in \mathcal{K}$.

Definition 1. *An element $x \in \mathcal{X}$ is* nonnegative *if $x \in \mathcal{K}$ and* strictly positive *if $x \in \operatorname{int}(\mathcal{K})$.*

2.2 Dual Cones and Linear Functionals

Let $\mathcal{X}^*$ denote the dual space of continuous linear functionals. The dual cone is defined by:

$$\mathcal{K}^* = \{\varphi \in \mathcal{X}^* \mid \varphi(x) \geq 0 \ \ \forall x \in \mathcal{K}\}. \tag{2}$$

Elements of $\mathcal{K}^*$ correspond to positive pricing rules.

3 Topology of the Attainable Set

Let τ denote a locally convex topology on $\mathcal{X}$ compatible with the dual pairing $\langle \mathcal{X}, \mathcal{X}^* \rangle$. The choice of topology is central for the validity of separation arguments and closedness of the attainable set.

Definition 2. *We say that the attainable set $\mathcal{A}$ satisfies the* No Free Lunch with Vanishing Risk (NFLVR) *condition if $\mathcal{A}$ is closed in the topology τ and*

$$\overline{\mathcal{A}}^{\tau} \cap \mathcal{K} = \{0\}.$$

Proposition 1. *If $\mathcal{A}$ is τ-closed and convex, then the polar cone $\mathcal{A}^\circ$ is weak-* compact in $\mathcal{X}^*$.*

Proof. By the Banach–Alaoglu theorem, the closed unit ball of $\mathcal{X}^*$ is weak-* compact. Since $\mathcal{A}^\circ$ is a closed subset of this ball under the polar topology, the result follows. □

This compactness property ensures the existence of extremal pricing rules attaining super- and sub-replication bounds.

4 The Attainable Set and Arbitrage

Let $\mathcal{A} \subset \mathcal{X}$ denote the set of attainable terminal payoffs generated by admissible trading strategies.

Definition 3. *The market is* arbitrage-free *if:*

$$\mathcal{A} \cap \operatorname{int}(\mathcal{K}) = \emptyset. \tag{3}$$

Proposition 2. *The attainable set $\mathcal{A}$ is a convex cone if trading strategies are scalable and additive.*

Proof. If $x, y \in \mathcal{A}$ and $\alpha, \beta \geq 0$, then the strategy $\alpha x + \beta y$ is admissible, implying closure under conic combinations. □

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FIGURE 1 — Attainable Cone and Arbitrage Region Shows arbitrage as intersection with the positive orthant

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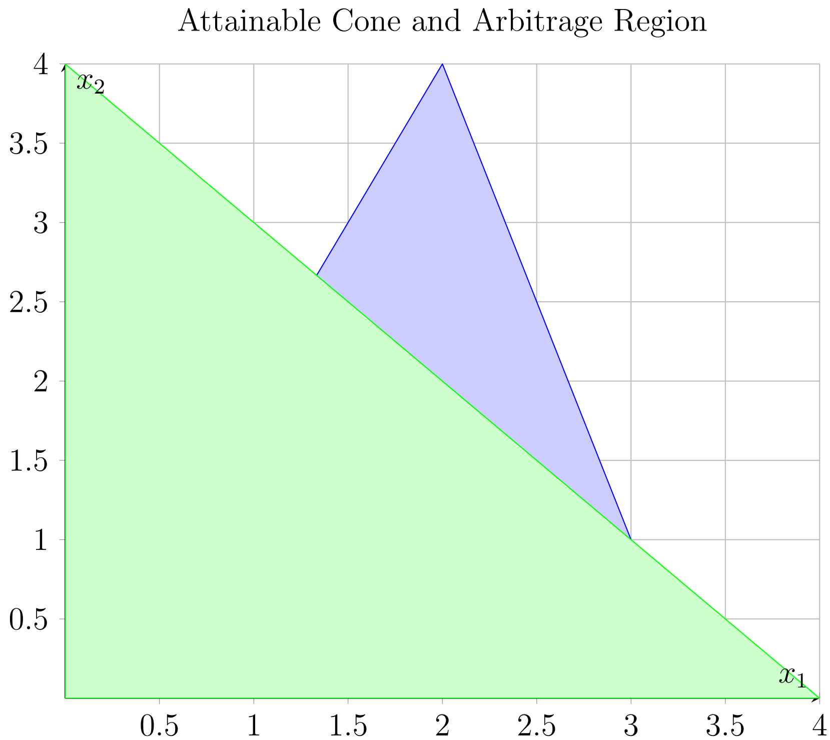

Figure 1: The attainable cone (blue) and the positive orthant (green). Arbitrage exists if their interiors intersect.

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5 Separation Theorems and the Fundamental Theorem

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Theorem 1 (Geometric FTAP)**.** *If $\mathcal{A} \cap \mathrm{int}(\mathcal{K}) = \emptyset$ and $\mathcal{A}$ is closed and convex, then there exists a nonzero $\varphi \in \mathcal{K}^*$ such that:*

$$\varphi(x) \leq 0 \quad \forall x \in \mathcal{A}. \tag{4}$$

Proof. By the Hahn–Banach separation theorem, there exists a continuous linear functional separating $\mathcal{A}$ and $\mathrm{int}(\mathcal{K})$. Positivity of φ follows from the ordering induced by $\mathcal{K}$. $\square$

This functional corresponds to an equivalent martingale measure in probabilistic formulations.

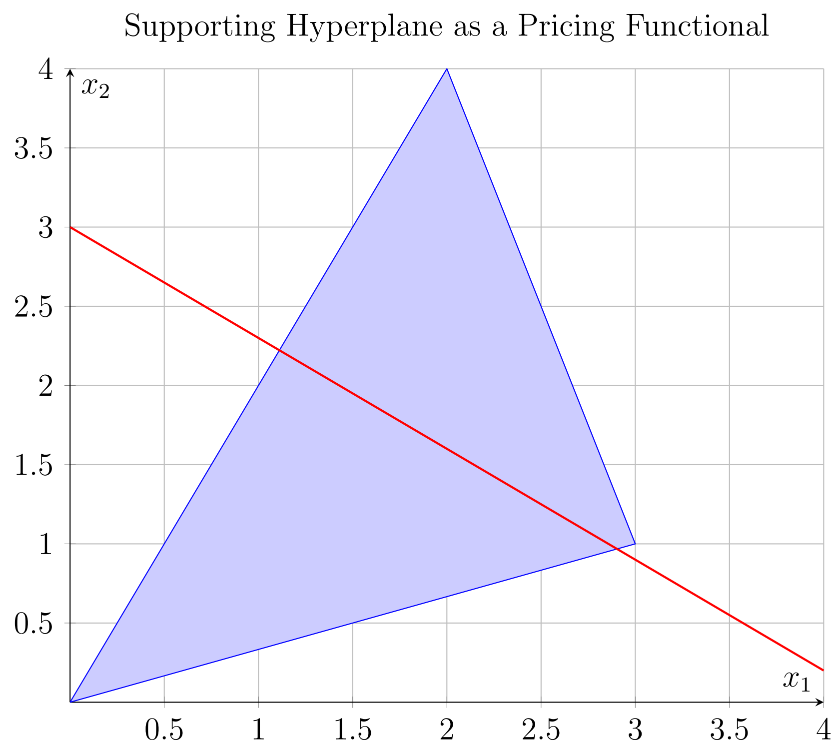

Figure 2: A supporting hyperplane (red) separates the attainable cone from the positive orthant. The normal vector defines a positive pricing functional.

6 Dual Representations and Polar Cones

Define the polar of a set $\mathcal{S} \subset \mathcal{X}$ by

$$\mathcal{S}^{\circ} = \{\varphi \in \mathcal{X}^{*} \mid \varphi(x) \leq 1 \;\; \forall x \in \mathcal{S}\}. \tag{5}$$

For a convex cone $\mathcal{A}$, the bipolar theorem implies

$$\mathcal{A}^{\circ\circ} = \overline{\mathrm{cone}(\mathcal{A})}^{\tau}. \tag{6}$$

Theorem 2 (Dual Representation of Super-Replication)**.** *Let $H \in \mathcal{X}$. Then the super-*

replication price admits the dual form

$$\pi^+(H) = \sup_{\varphi \in \mathcal{A}^\circ \cap \mathcal{K}^*} \varphi(H). \tag{7}$$

Proof. By definition, $\pi^+(H)$ is the support function of the set $\mathcal{A} + \mathcal{K}$ evaluated at H. The dual representation follows from Fenchel–Moreau duality and the bipolar theorem. □

7 Extreme Points and Market Completeness

Let $\mathcal{P} = \mathcal{A}^\circ \cap \mathcal{K}^*$ denote the set of admissible pricing functionals.

Definition 4. *A functional $\varphi \in \mathcal{P}$ is an* extreme point *if it cannot be written as a nontrivial convex combination of two distinct elements of $\mathcal{P}$.*

Theorem 3. *The market is complete if and only if $\mathcal{P}$ is a singleton.*

Proof. If $\mathcal{P} = \{\varphi\}$, then every payoff admits a unique price, implying replicability. Conversely, if the market is complete, replication implies uniqueness of supporting hyperplanes, and thus $\mathcal{P}$ contains exactly one element. □

By the Krein–Milman theorem, $\mathcal{P}$ is the closed convex hull of its extreme points. In incomplete markets, pricing rules correspond to mixtures of extremal states.

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8 Convex Hulls, Replication, and Pricing Bounds

Let $H \in \mathcal{X}$ be a contingent claim.

Definition 5. *The super-replication price is:*

$$\pi^+(H) = \inf \left\{ \varphi(x) \mid x \in \mathcal{A},\ x \succeq H \right\}. \tag{8}$$

Theorem 4. $\pi^+(H)$ *equals the support function of the attainable set evaluated at H:*

$$\pi^+(H) = \sup_{\varphi \in \mathcal{K}^*, ;\varphi|_{\mathcal{A}} \leq 0} \varphi(H). \tag{9}$$

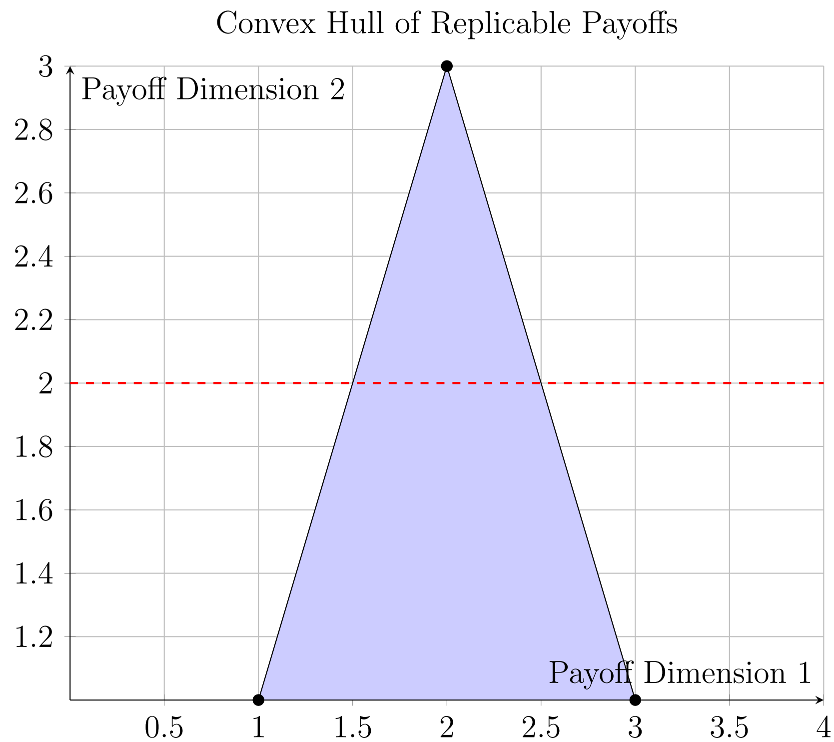

Figure 3: The convex hull of attainable payoffs (blue) and a supporting price functional (red). The intersection defines pricing bounds for a claim outside the hull.

9 Curvature and Second-Order Geometry of the Payoff Cone

While arbitrage and pricing bounds are determined by first-order separation, sensitivity and stability depend on second-order geometric properties of the attainable cone.

Let $\partial\mathcal{A}$ denote the boundary of $\mathcal{A}$ and let $x \in \partial\mathcal{A}$ admit a unique supporting functional $\varphi_x \in \mathcal{A}^\circ$.

Definition 6. *The* normal cone *at x is*

$$N_{\mathcal{A}}(x) = \{\varphi \in \mathcal{X}^* \mid \varphi(y - x) \leq 0 \ \ \forall y \in \mathcal{A}\}. \tag{10}$$

We define a generalized second fundamental form via the curvature of the support function $h_{\mathcal{A}}(u) = \sup_{x\in\mathcal{A}}\langle u, x\rangle$. If $h_{\mathcal{A}}$ is twice Gâteaux differentiable at $u = \varphi_x$, then the Hessian $\nabla^2 h_{\mathcal{A}}(\varphi_x)$ encodes the local curvature of the boundary $\partial\mathcal{A}$.

High curvature implies strong sensitivity of pricing bounds to perturbations of the payoff geometry, while flat faces correspond to robust replication regions.

10 Sensitivity and Directional Derivatives of Prices

Let $H \in \mathcal{X}$ and consider a perturbation $H_\varepsilon = H + \varepsilon \Delta H$.

Theorem 5. *If the super-replication price π^+ is Gâteaux differentiable at H, then*

$$\frac{d}{d\varepsilon} \pi^+(H_\varepsilon) \bigg|_{\varepsilon=0} = \varphi^*(\Delta H), \tag{11}$$

where φ^* is the unique supporting functional attaining the supremum in the dual representation.

If π^+ is not differentiable, the set of directional derivatives coincides with the image of ΔH under the normal cone $N_{\mathcal{A}}(x^*)$, where x^* is an optimal super-replicating portfolio.

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11 Infinite-Dimensional Markets

We extend the framework to $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 6. *If $\mathcal{A}$ is closed in the weak topology and $\mathcal{K}$ has nonempty interior, then the absence of arbitrage implies the existence of a strictly positive linear functional $\varphi \in L^q$ with $1/p + 1/q = 1$.*

12 Weak Topologies and Closedness of No-Arbitrage

In infinite-dimensional markets, closedness of $\mathcal{A}$ depends critically on the choice of topology.

Theorem 7. *Let $\mathcal{X} = L^p(\Omega, \mathcal{F}, \mathbb{P})$ for $1 \le p < \infty$. If $\mathcal{A}$ is closed in the weak topology $\sigma(L^p, L^q)$ and $\mathcal{K}$ has nonempty interior, then the absence of arbitrage implies the existence of a strictly positive pricing functional in L^q.*

This result fails in the norm topology for general semimartingale models, motivating the use of weak topologies in modern FTAP formulations.

13 Komlós-Type Compactness and Stability

Let $\{x_n\} \subset \mathcal{A}$ be a sequence of attainable payoffs with uniformly bounded negative parts.

Theorem 8 (Komlós)**.** *There exists a subsequence of convex combinations $\{\bar{x}_n\}$ that converges almost surely to some $x \in \mathcal{X}$.*

This compactness property ensures the existence of minimizing sequences for super-replication problems and provides stability of pricing bounds under approximation of strategies and discretization of trading.

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14 Transaction Costs and Non-Polyhedral Cones

With proportional transaction costs, the attainable set becomes a curved convex set rather than a polyhedral cone. Pricing functionals belong to a nonlinear dual cone characterized by:

$$\varphi(x+y) \leq \varphi(x) + \varphi(y), \quad \varphi(\lambda x) = \lambda \varphi(x), ; \lambda \geq 0. \tag{12}$$

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15 Stability of Arbitrage-Free Pricing

Let $\mathcal{A}_\varepsilon$ be a perturbation of the attainable cone.

Theorem 9. *If $d_H(\mathcal{A}, \mathcal{A}\varepsilon) \leq \varepsilon$ in the Hausdorff metric, then pricing bounds satisfy:*

$$|\pi^+\varepsilon(H) - \pi^+(H)| \leq C\varepsilon \tag{13}$$

for some constant C depending on the curvature of $\mathcal{A}$ at H.

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16 Geometric Interpretation of Market Completeness

Definition 7. *The market is* complete *if $\mathcal{A}$ spans $\mathcal{X}$.*

Geometrically, completeness corresponds to $\mathcal{A}$ having full dimension, implying that every payoff lies on a face of the attainable convex cone and admits a unique supporting hyperplane.

17 Discussion and Extensions

This convex-geometric framework extends naturally to multi-period markets, stochastic dominance constraints, and model uncertainty. Arbitrage-free pricing emerges as a property of the shape and curvature of the attainable payoff cone rather than of any particular probabilistic model.

18 Coherent Risk Measures as Support Functions

Let $\rho : \mathcal{X} \to \mathbb{R}$ be a coherent risk measure.

Theorem 10. *Every coherent risk measure admits a dual representation as a support function of a convex, weak-* compact set $\mathcal{Q} \subset \mathcal{K}^*$:*

$$\rho(H) = \sup_{\varphi \in \mathcal{Q}} \varphi(-H). \tag{14}$$

In this interpretation, risk measurement corresponds to pricing under worst-case supporting hyperplanes, and capital requirements correspond to shifts of the payoff cone.

19 Robust Pricing Under Model Uncertainty

Let $\{\mathcal{A}_\theta\}_{\theta \in \Theta}$ be a family of attainable sets indexed by model parameters.

Define the robust super-replication price:

$$\pi^+_{\text{rob}}(H) = \sup_{\theta \in \Theta} \pi^+_\theta(H). \tag{15}$$

Theorem 11. *If each $\mathcal{A}_\theta$ is convex and closed in τ, then π^+_{rob} is sublinear and admits the dual representation*

$$\pi^+_{\mathrm{rob}}(H) = \sup_{\varphi \in \bigcap_{\theta \in \Theta} \mathcal{A}^\circ_\theta} \varphi(H). \tag{16}$$

This formulation interprets Knightian uncertainty as intersection of dual cones and yields worst-case pricing rules as common supporting hyperplanes.

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20 Conclusion

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21 Conclusion and Open Problems

This paper develops a convex-geometric formulation of arbitrage-free pricing in which the primary mathematical objects are convex cones, their duals, and the geometry of supporting hyperplanes in payoff space. By recasting classical probabilistic results in the language of functional analysis and convex duality, we show that arbitrage, replication, and pricing bounds emerge as structural properties of the attainable set and its polar rather than as consequences of a specific stochastic model.

From a geometric perspective, the absence of arbitrage corresponds to strict separation between the attainable cone and the positive cone, while pricing functionals arise as extremal elements of the dual space. Super- and sub-replication prices are identified with support functions, and market completeness is characterized by the collapse of the dual set to a singleton. This unifies pricing, risk measurement, and model uncertainty within a single convex-analytic framework.

A central theme of the analysis is that first-order separation captures the existence of arbitrage-free prices, while second-order geometric structure governs stability and sensitivity. The curvature of the boundary of the attainable cone determines how pricing bounds respond to perturbations of payoffs, transaction costs, and model misspecification. Flat faces correspond to regions of robust replication, while highly curved regions signal fragility and large price sensitivity under small structural changes.

Several theoretical directions remain open. First, a systematic classification of attainable cones according to smoothness and curvature properties would clarify the relationship between market microstructure and the stability of pricing rules. In particular, characterizing when $\partial \mathcal{A}$ admits a well-defined second fundamental form in infinite-dimensional spaces remains an open problem in convex geometry.

Second, the extension of this framework to fully dynamic, multi-period markets raises questions about the geometry of *time-indexed families of cones* and their evolution under trading and information flow. Understanding whether no-arbitrage can be characterized by the existence of a continuous field of supporting functionals over time suggests connections to bundle theory and geometric control.

Third, robust pricing under model uncertainty naturally leads to the study of intersections of dual cones and their extremal structure. A deeper analysis of when such intersections remain weak-* compact, and how their extreme points behave under perturbations, would provide new insights into the geometry of Knightian uncertainty.

Finally, the geometric formulation presented here suggests links to broader areas of mathematics, including information geometry, where pricing functionals can be interpreted as coordinates on statistical manifolds, and optimal transport, where the support function representation connects naturally to Wasserstein duality. These connections point toward a unified geometric theory of markets in which prices, risks, and beliefs are embedded in a common convex and differential structure.

We view this work as a step toward such a theory, in which financial markets are studied not merely as stochastic systems, but as geometric objects whose shape, curvature, and dual structure encode the fundamental limits of pricing, hedging, and risk transfer.

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