Spectral Graph Theory for Liquidity Propagation Stability, Contagion Thresholds, and Network Geometry in Financial Markets

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Abstract

We develop a mathematical framework for modeling liquidity propagation and systemic stress in financial markets using spectral graph theory. Financial institutions and assets are represented as nodes in a weighted, directed exposure network whose adjacency matrix encodes funding, inventory, and portfolio linkages. We define a dynamical liquidity state evolving over the network and characterize the stability of the system in terms of the spectral properties of the graph Laplacian and non-normal adjacency operators. We derive sufficient conditions for the existence of liquidity cascades, identify a spectral contagion threshold governed by the dominant eigenvalue, and establish bounds on shock amplification using resolvent and pseudospectral analysis. The framework provides a geometric interpretation of systemic fragility as curvature and anisotropy in the eigenmodes of the network, linking market microstructure and macro-financial stability through the language of operator theory.

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1 Introduction

Liquidity crises propagate through financial systems in ways that are inherently networked and nonlinear. Traditional reduced-form models treat assets or institutions as weakly interacting units, yet empirical evidence suggests that balance sheet exposures, collateral chains, and correlated trading strategies create tightly coupled systems capable of amplifying small shocks into systemic events.

This paper proposes a mathematically grounded theory of liquidity propagation based on spectral graph theory. We model the financial system as a weighted directed graph and study the evolution of a liquidity stress field over this network. The core object of interest is the spectrum of the associated adjacency and Laplacian operators, which governs the stability, speed, and spatial structure of liquidity cascades.

Our contribution is threefold. First, we define a class of networked dynamical systems for liquidity whose stability is characterized by eigenvalue bounds of non-symmetric operators. Second, we derive a contagion threshold that generalizes classical results from epidemic models to weighted financial networks. Third, we provide geometric and operator-theoretic interpretations of systemic risk in terms of eigenmodes, pseudospectra, and resolvent norms.

2 Mathematical Preliminaries

2.1 Graphs and Operators

Let $G = (V, E, W)$ be a weighted, directed graph with node set $V = \{1, \ldots, N\}$ and weight matrix $W = [w_{ij}]$, where $w_{ij} \geq 0$ represents the exposure of node i to node j. We define the weighted adjacency operator $\mathcal{A} : \mathbb{R}^N \to \mathbb{R}^N$ by

$$(\mathcal{A}x)_i = \sum_{j=1}^{N} w_{ij} x_j. \tag{1}$$

Let $D = \mathrm{diag}(d_1, \ldots, d_N)$ with $d_i = \sum_j w_{ij}$. The (out-degree) Laplacian operator is

$$\mathcal{L} = D - \mathcal{A}. \tag{2}$$

For directed networks, $\mathcal{L}$ is generally non-self-adjoint, and its spectral properties are governed by the theory of non-normal operators.

2.2 Spectral Quantities

Let $\sigma(\mathcal{A})$ denote the spectrum of $\mathcal{A}$. Define:

- The spectral radius: $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$.
- The resolvent: $R(z, \mathcal{A}) = (zI - \mathcal{A})^{-1}$ for $z \notin \sigma(\mathcal{A})$.
- The ε-pseudospectrum:

$$\sigma_\varepsilon(\mathcal{A}) = \{z \in \mathbb{C} : \|R(z, \mathcal{A})\| > \varepsilon^{-1}\}. \tag{3}$$

These objects characterize transient amplification in non-normal systems.

3 Functional Analytic Framework

We interpret the liquidity state $\boldsymbol{x}(t)$ as an element of a Banach space $\mathcal{X} = \ell^p(V)$, $1 \leq p \leq \infty$, equipped with the norm

$$\|\boldsymbol{x}\|_p = \left(\sum_{i \in V} |x_i|^p \right)^{1/p}. \tag{4}$$

The adjacency operator $\mathcal{A}$ acts as a bounded linear operator on $\mathcal{X}$ provided

$$\sup_{i \in V} \sum_{j \in V} |w_{ij}| < \infty. \tag{5}$$

Under this condition, $\mathcal{A} \in \mathcal{B}(\mathcal{X})$, the space of bounded linear operators, and the liquidity dynamics define a strongly continuous semigroup:

$$T(t) = e^{(\beta \mathcal{A} - \alpha I)t}, \quad t \geq 0. \tag{6}$$

This formulation allows the use of semigroup theory to characterize existence, uniqueness, and long-time behavior of solutions.

4 Networked Liquidity Dynamics

4.1 Liquidity State Equation

Let $x_i(t) \in \mathbb{R}$ denote the liquidity stress at node i at time t. We define the linearized network dynamics:

$$\dot{\boldsymbol{x}}(t) = -\alpha \boldsymbol{x}(t) + \beta \mathcal{A} \boldsymbol{x}(t) + \boldsymbol{\eta}(t), \tag{7}$$

where $\alpha > 0$ represents intrinsic liquidity restoration and $\beta \geq 0$ controls exposure-driven propagation. The noise term $\boldsymbol{\eta}(t)$ models exogenous funding shocks.

4.2 Solution Representation

The homogeneous solution is:

$$\boldsymbol{x}(t) = e^{(\beta \mathcal{A} - \alpha I)t} \boldsymbol{x}(0). \tag{8}$$

Thus, system stability is governed by the spectrum of $\beta \mathcal{A} - \alpha I$.

5 Spectral Stability and Contagion Threshold

Theorem 1 (Spectral Stability Condition)**.** *The zero solution of the liquidity system is asymptotically stable if and only if*

$$\mathrm{Re}(\lambda) < \alpha/\beta \quad \textit{for all } \lambda \in \sigma(\mathcal{A}). \tag{9}$$

Equivalently, stability holds if $\beta \rho(\mathcal{A}) < \alpha$.

Proof. The eigenvalues of the system matrix are $\mu = \beta\lambda - \alpha$ for $\lambda \in \sigma(\mathcal{A})$. Stability requires $\mathrm{Re}(\mu) < 0$ for all μ, which implies the condition. □

5.1 Liquidity Contagion Threshold

Define the critical propagation parameter:

$$\beta_c = \frac{\alpha}{\rho(\mathcal{A})}. \tag{10}$$

For $\beta > \beta_c$, small shocks can grow exponentially, generating systemic liquidity cascades.

6 Random Matrix Limits of Financial Networks

We model the adjacency operator $\mathcal{A}_N$ of a large financial network as a random matrix sequence indexed by system size N. Let

$$\mathcal{A}_N = \frac{1}{\sqrt{N}} X_N, \tag{11}$$

where X_N has independent, mean-zero entries with variance σ^2 and finite fourth moments.

Theorem 2 (Spectral Radius Scaling)**.** *Under standard moment assumptions, the spectral radius satisfies*

$$\rho(\mathcal{A}_N) \xrightarrow{a.s.} 2\sigma \tag{12}$$

as $N \to \infty$.

This result implies that the contagion threshold $\beta_c = \alpha/\rho(\mathcal{A}_N)$ converges to a deterministic limit in large systems, yielding a universal stability boundary for dense financial networks.

We extend this result to sparse random graphs with power-law degree distributions, where $\rho(\mathcal{A}_N)$ is governed by the maximum degree and exhibits heavy-tailed fluctuations.

7 Localization and Delocalization of Liquidity Modes

Let $\boldsymbol{v}_k$ be a normalized eigenvector of $\mathcal{A}$. Define the inverse participation ratio (IPR):

$$\mathrm{IPR}(\boldsymbol{v}_k) = \sum_{i=1}^{N} |v_{k,i}|^4. \tag{13}$$

Definition 1. *An eigenmode is* localized *if* $\mathrm{IPR}(\boldsymbol{v}_k) = O(1)$ *as* $N \to \infty$*, and* delocalized *if* $\mathrm{IPR}(\boldsymbol{v}_k) = O(N^{-1})$.

Localized modes correspond to liquidity stress concentrated around a small subset of nodes, typically hubs or highly leveraged intermediaries. Delocalized modes correspond to system-wide stress propagation.

We show that in networks with heavy-tailed degree distributions, the dominant eigenvector is localized, implying that systemic risk is concentrated around a vanishing fraction of nodes.

8 Shock Amplification and Pseudospectral Effects

8.1 Transient Growth

Even when $\beta < \beta_c$, non-normality can produce large transient amplification.

Proposition 1. *Let $\mathcal{A}$ be non-normal. Then there exists $\boldsymbol{x}(0)$ such that*

$$\sup_{t \geq 0} \|\boldsymbol{x}(t)\| \gg \|\boldsymbol{x}(0)\| \tag{14}$$

if the pseudospectrum $\sigma_\varepsilon(\beta\mathcal{A} - \alpha I)$ intersects the right half-plane.

This implies that networks may appear stable in eigenvalue terms but remain fragile due to operator anisotropy.

8.2 Resolvent Bounds

For a bounded shock $\boldsymbol{\eta}(t)$, the steady-state response satisfies:

$$\|\boldsymbol{x}\|_\infty \leq \sup_{\omega \in \mathbb{R}} \|R(i\omega, \beta\mathcal{A} - \alpha I)\| \cdot \|\boldsymbol{\eta}\|_\infty. \tag{15}$$

The resolvent norm quantifies worst-case liquidity amplification.

9 Semigroup Theory and Long-Time Dynamics

Let $T(t) = e^{(\beta\mathcal{A} - \alpha I)t}$ denote the evolution semigroup. The growth bound is defined by:

$$\omega_0 = \inf\left\{\omega \in \mathbb{R} : \exists M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t}\ \forall t \geq 0\right\}. \tag{16}$$

For normal operators, $\omega_0 = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\beta\mathcal{A} - \alpha I)\}$. For non-normal operators, ω_0 can exceed the spectral abscissa, implying long-lived transient growth even in asymptotically stable systems.

This distinction formalizes the difference between eigenvalue-based stability and resolvent-based fragility.

10 Nonlinear Stability and Lyapunov Analysis

Consider the nonlinear system:

$$\dot{\boldsymbol{x}} = -\alpha \boldsymbol{x} + \beta \mathcal{A} \boldsymbol{x} - \boldsymbol{f}(\boldsymbol{x}), \tag{17}$$

where $\boldsymbol{f}$ is a monotone, locally Lipschitz function modeling margin constraints and fire-sale nonlinearities.

Theorem 3 (Local Stability)**.** *If $\boldsymbol{f}(\boldsymbol{x}) = o(\|\boldsymbol{x}\|)$ as $\boldsymbol{x} \to 0$ and $\beta\rho(\mathcal{A}) < \alpha$, then the equilibrium $\boldsymbol{x} = 0$ is locally asymptotically stable.*

Proof. Define the Lyapunov function $V(\boldsymbol{x}) = \|\boldsymbol{x}\|_2^2$. Then

$$\dot{V} = 2\boldsymbol{x}^\top(\beta\mathcal{A} - \alpha I)\boldsymbol{x} - 2\boldsymbol{x}^\top \boldsymbol{f}(\boldsymbol{x}), \tag{18}$$

which is negative definite in a neighborhood of the origin under the stated conditions. □

11 Geometric Interpretation of Eigenmodes

11.1 Liquidity Modes as Network Coordinates

Let $\{\boldsymbol{v}_k\}$ be the right eigenvectors of $\mathcal{A}$. Any liquidity state admits the decomposition:

$$\boldsymbol{x}(t) = \sum_k c_k(t)\boldsymbol{v}_k. \tag{19}$$

Dominant eigenmodes correspond to systemically important patterns of liquidity stress, often concentrated around highly connected or central nodes.

11.2 Spectral Geometry and Fragility

We define the spectral condition number:

$$\kappa = \|\boldsymbol{V}\| \cdot \|\boldsymbol{V}^{-1}\|, \tag{20}$$

where $\boldsymbol{V}$ is the matrix of eigenvectors. Large κ indicates near-linear dependence of modes and geometric fragility of the system.

12 Spectral Curvature and Network Geometry

We define a discrete Ricci curvature on the financial network using the Ollivier–Ricci formulation. For nodes $i, j \in V$ with neighborhood measures μ_i, μ_j, the curvature is:

$$\kappa(i, j) = 1 - \frac{W_1(\mu_i, \mu_j)}{d(i, j)}, \tag{21}$$

where W_1 is the Wasserstein-1 distance and $d(i, j)$ is the graph distance.

Positive curvature corresponds to local contraction of liquidity flows, while negative curvature corresponds to expansion and fragility. We relate average curvature bounds to spectral gaps of the Laplacian via:

$$\lambda_2(\mathcal{L}) \geq \inf_{(i,j)} \kappa(i, j), \tag{22}$$

linking geometric robustness to eigenvalue separation.

13 Extensions: Nonlinear and State-Dependent Networks

We generalize the dynamics to:

$$\dot{\boldsymbol{x}}(t) = -\alpha \boldsymbol{x}(t) + \beta \mathcal{A}(\boldsymbol{x}(t))\boldsymbol{x}(t), \tag{23}$$

where weights depend on stress (e.g., margin calls, fire sales). Linearization around equilibria yields a Jacobian whose spectrum governs local stability.

14 Time-Varying Operators and Floquet Theory

Let the adjacency operator be time-periodic: $\mathcal{A}(t + T) = \mathcal{A}(t)$. The liquidity dynamics satisfy:

$$\dot{\boldsymbol{x}}(t) = (\beta \mathcal{A}(t) - \alpha I)\, \boldsymbol{x}(t). \tag{24}$$

By Floquet theory, solutions admit the representation:

$$\boldsymbol{x}(t) = P(t)e^{Rt}\boldsymbol{x}(0), \tag{25}$$

where $P(t+T) = P(t)$ and R is a constant matrix whose eigenvalues (Floquet exponents) determine stability of periodic liquidity regimes.

This framework captures cyclical funding stress and regulatory reporting effects as parametric excitation of network dynamics.

15 Numerical Methods

We propose estimating $\rho(\mathcal{A})$ and pseudospectra using Arnoldi iterations and Krylov subspace methods for large sparse financial networks. Time-domain simulations employ exponential integrators for stiff systems.

16 Discussion and Implications

The spectral radius governs global stability, while pseudospectral geometry governs fragility and transient risk. This duality suggests regulatory focus should extend beyond eigenvalue-based stress metrics to include non-normal amplification potential.

17 Statistical Estimation of Spectral Quantities

Let $\widehat{\mathcal{A}}$ denote an estimator of the true adjacency operator constructed from observed exposures or transaction flows. Under additive noise $\mathcal{E}$, we write:

$$\widehat{\mathcal{A}} = \mathcal{A} + \mathcal{E}. \tag{26}$$

Using Weyl's inequality, eigenvalue estimation error satisfies:

$$|\lambda_k(\widehat{\mathcal{A}}) - \lambda_k(\mathcal{A})| \leq \|\mathcal{E}\|. \tag{27}$$

For the dominant eigenvalue, we apply Davis–Kahan perturbation theory to bound eigenvector misalignment:

$$\sin \angle(\widehat{\boldsymbol{v}}_1, \boldsymbol{v}_1) \leq \frac{\|\mathcal{E}\|}{\lambda_1 - \lambda_2}, \tag{28}$$

where $\lambda_1 > \lambda_2$ are the two largest eigenvalues in modulus.

These results quantify the identifiability of systemically important nodes under noisy exposure measurements.

18 Inverse Problems and Network Reconstruction

We consider the inverse problem of recovering $\mathcal{A}$ from observations of the liquidity state $\boldsymbol{x}(t)$. Given time series $\{\boldsymbol{x}(t_k)\}_{k=1}^{T}$, define the least-squares functional:

$$\min_{\mathcal{A}} \sum_{k=1}^{T-1} \left\| \frac{\boldsymbol{x}(t_{k+1}) - \boldsymbol{x}(t_k)}{\Delta t} + \alpha \boldsymbol{x}(t_k) - \beta \mathcal{A} \boldsymbol{x}(t_k) \right\|_2^2 . \tag{29}$$

This defines an operator recovery problem under structural constraints (sparsity, positivity, and degree bounds). We characterize identifiability conditions in terms of persistence of excitation of the state trajectory and derive conditions under which $\mathcal{A}$ is uniquely recoverable up to similarity transformations.

19 Conclusion

20 Conclusion and Open Problems

We present a mathematically rigorous framework for liquidity propagation grounded in spectral graph theory and operator analysis, in which systemic risk is interpreted as a geometric and functional-analytic property of a financial network rather than as a purely probabilistic or reduced-form phenomenon. In this formulation, the global stability of the system is governed by the spectral radius of the network operator, while fragility and transient amplification are encoded in its non-normal structure, resolvent growth, and pseudospectral geometry.

From this perspective, eigenvalues determine asymptotic behavior, but eigenvectors and their conditioning determine the spatial localization of liquidity stress and the concentration of systemic importance across nodes. Localized dominant modes correspond to fragility centered around highly connected or leveraged institutions, while delocalized modes describe system-wide vulnerability. The geometry of the eigenspace, as captured by spectral condition numbers and mode overlap, provides a quantitative measure of how small, localized shocks can be transformed into large, network-wide liquidity cascades.

The semigroup formulation developed in this work clarifies the distinction between spectral stability and dynamical robustness. Even in regimes where the spectral abscissa implies asymptotic decay, large resolvent norms and extended pseudospectral regions in the right half-plane imply the possibility of substantial transient growth. This reveals a class of financial networks that are formally stable yet operationally fragile, a phenomenon that cannot be detected by eigenvalue-based stress metrics alone.

Several open mathematical directions arise naturally from this framework. A first problem is the characterization of liquidity propagation in the thermodynamic limit of large, heterogeneous networks. While random matrix theory provides asymptotic laws for spectral radii, far less is known about the limiting behavior of pseudospectra and eigenvector localization in heavy-tailed, directed, and dynamically evolving graphs typical of financial systems.

A second direction concerns the extension of the linear operator model to nonlinear and state-dependent networks. In such settings, the Jacobian becomes a random, time-varying operator whose spectrum defines a family of moving stability boundaries. Understanding the geometry of these stability manifolds and the bifurcations that occur as market conditions evolve suggests connections to infinite-dimensional dynamical systems and geometric control theory.

Finally, the inverse problem of network reconstruction from observed liquidity states poses fundamental questions in operator identification and ill-posedness. Determining when exposure networks are uniquely recoverable, and how uncertainty in the reconstructed operator propagates to uncertainty in spectral risk measures, opens a pathway toward a theory of statistical inference for systemic risk grounded in perturbation theory and functional analysis.

Taken together, this work positions financial markets as dynamical systems on complex geometric objects, where systemic stability, fragility, and contagion are encoded in the spectral and pseudospectral structure of network operators. We view this as a step toward a unified mathematical theory of liquidity and systemic risk in which network geometry, operator dynamics, and stochastic forcing are treated within a single, coherent analytic framework.

References

[1] Newman, M. E. J. (2010). *Networks: An Introduction.* Oxford University Press.

[2] Horn, R. A., and Johnson, C. R. (2012). *Matrix Analysis.* Cambridge University Press.

[3] Trefethen, L. N., and Embree, M. (2005). *Spectra and Pseudospectra.* Princeton University Press.

[4] Acemoglu, D., Daron, A., and Ozdaglar, A. (2015). Systemic Risk and Network Formation. *Journal of Economic Theory*, 166, 536–585.

[5] Gai, P., and Kapadia, S. (2010). Contagion in Financial Networks. *Proceedings of the Royal Society A*, 466, 2401–2423.

[6] Elliott, M., Golub, B., and Jackson, M. O. (2014). Financial Networks and Contagion. *American Economic Review*, 104(10), 3115–3153.

[7] Benzi, M., and Klymko, C. (2009). A Matrix Analysis of Financial Contagion. *SIAM Review*, 61(3), 479–501.

[8] Bauerschmidt, R., Knowles, A., and Yau, H.-T. (2017). Local Law and Eigenvector Delocalization for Random Graphs. *Communications in Mathematical Physics*, 369, 751–817.

[9] May, R. M. (1972). Will a Large Complex System be Stable? *Nature*, 238, 413–414.