# Self-avoiding walks and polygons crossing a domain on the square and hexagonal lattices.

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**Abstract.** We have analysed the recently extended series for the number of self-avoiding walks (SAWs)  $C_L(1)$  that cross an  $L \times L$  square between diagonally opposed corners. The number of such walks is known to grow as  $\lambda_S^{L^2}$ . We have made more precise the estimate of  $\lambda_S$ , based on additional series coefficients provided by several authors, and refined analysis techniques. We estimate that  $\lambda_S = 1.7445498 \pm 0.0000012$ . We have also studied the subdominant behaviour, and conjecture that

$$C_L(1) \sim \lambda_S^{L^2 + bL + c} \cdot L^g$$
,

where  $b = -0.04354 \pm 0.0001$ ,  $c = 0.5624 \pm 0.0005$ , and  $g = 0.000 \pm 0.005$ .

We implemented a very efficient algorithm for enumerating paths on the square and hexagonal lattices making use of a minimal perfect hash function and in-place memory updating of the arrays for the counts of the number of paths.

Using this algorithm we extended and then analysed series for SAWs spanning the square lattice and self-avoiding polygons (SAPs) crossing the square lattice. These are known to also grow as  $\lambda_S^{L^2}$ . The sub-dominant term  $\lambda^b$  is found to be the same as for SAWs crossing the square, while the exponent  $g = 1.75 \pm 0.01$  for spanning SAWs and  $g = -0.500 \pm 0.005$  for SAPs.

We have also studied the analogous problems on the hexagonal lattice, and generated series for a number of geometries. In particular, we study SAWs and SAPs crossing rhomboidal, triangular and square domains on the hexagonal lattice, as well as SAWs spanning a rhombus. We estimate that the analogous growth constant  $\lambda_H = 1.38724951 \pm 0.00000005$ , so an even more precise estimate than found for the square lattice. We also give estimates of the sub-dominant terms.

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## 1. Introduction

A *n-step self-avoiding walk* (SAW)  $\omega$  on a regular lattice is a sequence of *distinct* vertices  $\omega_0, \omega_1, \ldots, \omega_n$  such that each vertex is a nearest neighbour of its predecessor. SAWs are considered distinct up to translations of the starting point  $\omega_0$ . If  $\omega_0$  and  $\omega_n$  are nearest-neighbours we can form a closed (n+1)-step self-avoiding polygon (SAP) by adding an edge between the two end-points.

We consider SAWs on an  $L \times L$  square lattice, with the walks starting at the north-west corner (0, L) and finishing at the south-east corner (L, 0), and constrained within the square (see the first diagram in Figure 1). Clearly such walks vary in length from a minimum of 2L to a maximum of  $L^2 + 2L$  (if L is even). Guttmann and Whittington [1] computed the first 7 terms in 1990, then Bousquet-Mélou, Guttmann and Jensen [2] computed the terms up to L = 19. Iwashita et al. [3] computed the next two terms, L = 20 and 21, R. Spaans computed three more terms, L = 22 to 24, and Iwashita [4] computed the terms for L = 25 and 26. Details can be found in the On-line Encyclopaedia of Integer Sequences [5], OEIS A007764. Note that the listing in the OEIS runs from 1 to 27, which in our notation is L = 0 to 26.

Recall that the number of SAWs in the bulk,  $c_n$ , grows exponentially with length n as  $\mu^n$ , where  $\mu$  depends on the lattice. For the hexagonal lattice it is known [6] that  $\mu = \sqrt{2 + \sqrt{2}}$ , while for the square lattice the growth constant  $\mu$  has only been estimated numerically. The most precise estimate  $\mu = 2.63815853032790(3)$  was obtained by Jacobsen, Scullard and Guttmann [7].

We will be interested in the generating function  $C_L(x) = \sum_{n \geq 2L} c_n x^n$ , where  $c_n$  denotes the number of SAWs of length n crossing the square from (0, L) to (L, 0). Madras [8] proved that the limits  $\mu_1(x) := \lim_{L \to \infty} C_L(x)^{1/L}$  and  $\mu_2(x) := \lim_{L \to \infty} C_L(x)^{1/L^2}$  are well defined in  $\mathbb{R} \cup \{+\infty\}$ . More precisely, Madras proved (i)  $\mu_1(x)$  is finite for  $0 < x < 1/\mu$ , and is infinite for  $x > 1/\mu$ . Moreover,  $0 < \mu_1(x) < 1$  for  $0 < x < 1/\mu$  and  $\mu_1(1/\mu) = 1$ . (ii)  $\mu_2(x)$  is finite for all x > 0. Moreover,  $\mu_2(x) = 1$  for  $0 < x \leq 1/\mu$  and  $\mu_2(x) > 1$  for  $x > 1/\mu$ .

The existence of the limit

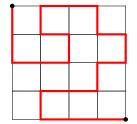
$$\lim_{L \to \infty} C_L(1)^{1/L^2} = \lambda_S \tag{1}$$

was proved in both [9] and [1] by different methods. In [2] we estimated  $\lambda_S = 1.744550 \pm 0.000005$ . Using the longer series now available, we have sharpened this to  $\lambda_S = 1.7445498 \pm 0.0000012$ . We have also estimated the sub-dominant terms by finding precise numerical evidence for the asymptotic behaviour

$$C_L(1) \sim \lambda_S^{L^2 + bL + c} \cdot L^g, \tag{2}$$

where  $b = -0.04354 \pm 0.0001$ ,  $c = 0.5624 \pm 0.0005$ , and  $g = 0.000 \pm 0.005$ , from which we conjecture that g = 0, exactly.

For SAPs crossing a square we calculated the coefficients up to L=26 and then analysed the data for the first time. The analysis clearly demonstrated that the two problems have the same growth constant. We conjecture that the subdominant term



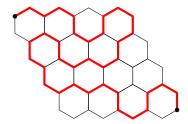




Figure 1: A square domain and rhomboidal and triangular domains of size n = 4 on the hexagonal lattice. Walks must extend between the points indicated by black circles as illustrated by the red walks.

 $\lambda^b$  is the same as for crossing SAWs, and that the corresponding exponent  $g=-\frac{1}{2}$ . For SAWs spanning a square we extended the known series up to L=26. This is a superset of SAWs crossing the square, as the origin can be any vertex on the left boundary and the end-point can be any vertex on the right boundary. In [8] it was proved that the two problems have the same growth constant and this is of course consistent with our analysis. We conjecture that the  $\lambda^b$  term is the same as for crossing SAWs and that  $g=\frac{7}{4}$ .

We have also studied the analogous problems on the hexagonal lattice. We initially considered the problem on a square domain of the hexagonal lattice (see the last two diagrams in Figure 15), but this was soon found to be a rather unnatural domain, as the paths changed according as the size L of the lattice was odd or even. A more natural domain is a rhombus, shown as the second diagram in Figure 1, or a triangular domain, shown as the third diagram in Figure 1. We studied both self-avoiding walks (SAWs) and self-avoiding polygons (SAPs) in these three domains. For the triangular domain, we studied two cases, according as the path is forced to include the top vertex of the triangle or not. We also studied SAWs which span a rhombus of width L.

In Section 2 we give a detailed description of the new and very efficient algorithm we used to calculate the series for SAWs crossing a rhombus and briefly mention how to amend the algorithm to enumerate other problems such as SAPs. In Section 3 we give a brief description of the methods we used in our analysis of the series. Further details can be found in Appendix A and Appendix B. Section 4.1 contains a detailed analysis of the extended series for SAWs crossing a square with SAPs and spanning SAWs given a more cursory treatment. This is followed in Section 4 by the results of our detailed asymptotic analysis of SAWs crossing rhomboidal and triangular domains with several other problems briefly mentioned. Section 6 contains our conclusions and gives a summary of the estimates we have obtained.

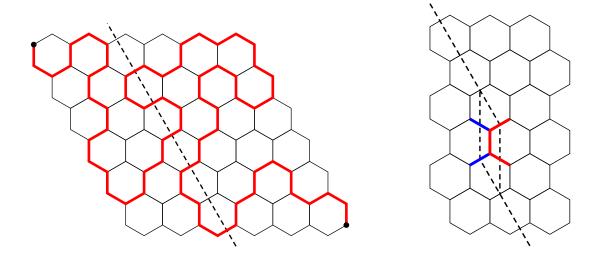


Figure 2: An example (left panel) of a SAW crossing a rhomboidal domain of the hexagonal lattice intersected by a TM line. The basic TM move (right panel) in which the intersection is moved so as to add another two vertices and three edges to the section of the domain already visited. The states of the two blue 'incoming' edges determines the type of update to apply while adding the three red edges to the visited section of the domain.

# 2. Algorithm to enumerate SAWs crossing a rhombus.

The algorithm we use to count the number SAWs on domains of the hexagonal lattice builds on the pioneering work of Enting [10] who enumerated square lattice self-avoiding polygons and extended by Conway, Enting and Guttmann [11] to enumerate square lattice SAWs. An algorithm for the enumeration of hexagonal SAWs was described in [12] and a detailed description of the general method can be found in [13].

## 2.1. Transfer matrix algorithm

If we take an example of a SAW crossing a rhombus and draw an line across the domain as shown in Figure 2 we observe that the partial SAW to the left of the intersection consists of arcs connecting two edges on the intersection (we shall refer to these as arc-ends), and a single edge that is not connected to any other edge on the intersection (we call this a free end). The free end is connected to the vertex in the upper left corner of the domain and the SAW must terminate in the lower right corner.

We are not allowed to form closed loops, so two arc ends can only be joined if they belong to different arcs. We must also ensure that the graphs we count have just a single component. To exclude arcs which close on themselves we label the occupied edges in such a way that we can easily determine whether or not two ends belong to the same arc. On two-dimensional lattices this can be done by relying on the fact that arcs can never intertwine. Each arc end is assigned a label depending on whether it is the lower

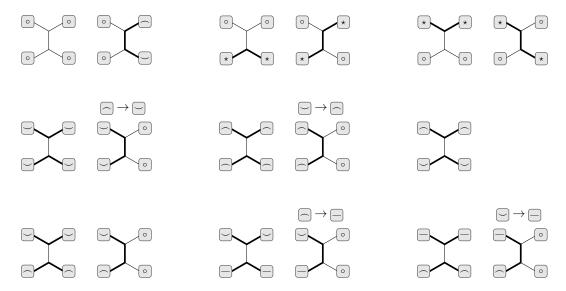


Figure 3: The possible updates in a TM move with thin edges empty and thick edges occupied by the walk. In the top row \*\infty refers to any type of occupied edge ||, ((), or )). Relabelling of arc ends are indicated above the update, i.e., for the second transition in the second row two lower arc ends are connected and the matching upper end is relabelled as a lower end.

or upper end of an arc and these labels can be viewed as balanced parenthesis. We shall refer to the configuration along the intersection as a *signature*, denoted by  $\Sigma$ , which can be represented by a string of edge states,  $\sigma_i$ , where

Take the SAW in Figure 2 and consider the configuration associated with the partial SAW to the left of the line. Reading from bottom to top we find the signature  $\Sigma = (0) + (0) = 0$ . Since all SAWs have to cross the rhombus it readily follows that any signature contains one free edge surrounded by a string of empty states and arc ends on either side (with the arc ends forming balanced parenthesis).

For each signature  $\Sigma$  we simply count the number of partial SAWs,  $\mathcal{C}(\Sigma)$ . SAWs in a given domain of the hexagonal lattice are counted by moving the intersection so as to add two vertices and three edges at a time, as illustrated in Figure 2. The updating of the counts  $\mathcal{C}(\Sigma)$  depends on the states of the edges to the left of the new vertices. In Figure 3 we display the possible local 'input' states and the 'output' states which arise as the kink in the boundary is propagated by one step. Not all the possible local input states are displayed since some are related by an obvious reflection symmetry with straightforward changes to the corresponding updating rules. We shall refer to the signature before the move as the *source*,  $\Sigma_{\rm S}$ , and a signature produced as a result of the

move as a target,  $\Sigma_{\rm T}$ . In all cases we see that the first update has the source appearing as a target as well. In the second row, last panel, we cannot connect the arc ends since this would result in the formation of a cycle. Most of the updates are local and involve only the two edges in the kink, but some of the updates involves a non-local transformation of the signature. This happens when we connect two lower (upper) arc ends or a free end to a lower (upper) arc end. In these cases we need to locate a matching arc end in the signature and relabel it accordingly. We illustrate these here:

```
()\circ)(()\circ()))))))) \rightarrow ()\circ(\circ\circ()\circ()))
```

Two consecutive blue tiles indicate the edges that are involved in the update as per Figure 2, while the isolated blue tile indicate the edge which has a change of state. In the first example we connect two lower arc ends (second row second update of Figure 3) and we then have to relabel the upper arc end of the inner arc to a lower arc end as indicated. How do we find the matching arc end? We start at the update position of the innermost arc and set a counter to 1, we then scan to the right and increase the counter by 1 for every () we encounter and decrease the counter by 1 for every (); once the counter records a value of 0 we have found the matching end. Similarly in the second example (illustrating the last update in row three of Figure 3) we connect an upper arc end to the free end and we then have to locate the matching lower end of the arc and change the state from () to ().

#### 2.2. Motzkin path representation of signatures

It is possible to represent the signatures as Motzkin paths, which are directed walks from (0,0) to (n,0) in the first quadrant of the square lattice with step-set  $\Omega = \{(1,0),(1,1),(1,-1)\}$ , see OEIS <u>A001006</u> for numerous references to this classical combinatorial problem. We shall refer to the steps as horizontal, up and down steps, respectively. The basic mapping from a signature to a Motzkin path is to map  $\odot$  to horizontal steps,  $\bigcap$  to up steps, and  $\bigcap$  to down steps.

What about the free end? Since the walk has to cross the domain the free end can never be enclosed inside an arc and therefore splits the signature into two 'halves' such that on either side of the free end one has a standard Motzkin path. Next we consider what happens to updates involving a free end and show that we don't need to explicitly keep track of the free end but can treat it as if it were an (excess) upper end arc, which we denote ).

- (or o) produces outputs of and o which is the same as for an input o with the free end remaining the excess o.
- $| \cdot | \cdot | \cdot | \cdot | \cdot |$  is clearly the same as  $| \cdot | \cdot | \cdot | \cdot |$ . In the case  $| \cdot | \cdot | \cdot | \cdot |$  we are connecting a free end to a lower arc end and relabelling the matching upper

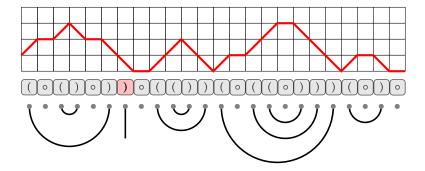


Figure 4: Illustration of the representation of an arc configuration along the TM intersection line (grey circles mark the intersection with an edge), the corresponding signature, and the corresponding Motzkin path starting at (0,1) and ending at (24,0). The free end can be represented by an excess upper arc end ) and located as the first return of the Motzkin path to level 0.

end as free. However, this is equivalent to ) ()  $\rightarrow$   $\circ$   $\circ$  with the incoming ) now moving to the position of the matching ) of the arc with no change of state required.

) | : | ) |  $\rightarrow$  | ) | is clearly the same as | ) |  $\rightarrow$  | ) |. In the case | ) |  $\rightarrow$   $\bigcirc$  | we are connecting a free end to an upper arc end and relabelling the matching lower end as free. However, this is equivalent to | ) |  $\rightarrow$   $\bigcirc$  | with the incoming | ) now moving to the position of the matching ( ) of the arc and being relabelled as | ).

() Not possible since free end would be enclosed inside an arc.

It now follows that the set of signatures can be represented as the set of Motzkin paths starting at height 1, i.e. at vertex (0,1), and ending at (L+1,0). This is another well known combinatorial problem as evidenced by its low sequence number, OEIS  $\underline{A002026}$ . Should we need to know the position of the free end (as it happens we don't for this problem) it is easy to find it as the excess  $\bigcirc$  when looking from the first state in the signature. In Motzkin parlance the position of the free end corresponds to the first return of the path to height 0. The representation is illustrated in Figure 4.

#### 2.3. Minimal perfect hashing

We implement the minimal perfect hashing scheme of Iwashita et al [4]. Let  $\mathcal{M}_{(m,h)}^{(0)}$  be the set of m-step Motzkin paths starting at height 0 and ending at height h. Similarly, let  $\mathcal{M}_{(n,h)}^{(1)}$  be the set of n-step Motzkin paths starting a height 1 and ending at height h. The total set of states is  $\mathcal{M}_{(L+1,0)}^{(1)}$  since there are L+1 edges along the TM intersection. So we seek to construct a mapping  $\Phi: \mathcal{M}_{(L+1,0)}^{(1)} \to \{1, \ldots, |\mathcal{M}_{(L+1,0)}^{(1)}|\}$ . We implement this as a sum of two functions

$$\Phi\left(\Sigma\right) = \Phi_{L}\left(\Sigma_{L}\right) + \Phi_{R}\left(\Sigma_{R}\right),\tag{4}$$

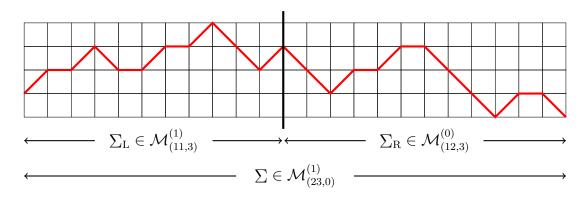


Figure 5: Illustration of the splitting of a Motzkin path representing a signature,  $\Sigma \in \mathcal{M}^{(1)}_{(23,0)}$  (L=22), into left and right halves of height h=3, with  $\Sigma_L \in \mathcal{M}^{(1)}_{(11,3)}$  and  $\Sigma_R \in \mathcal{M}^{(0)}_{(12,3)}$ . Note that the right Motzkin path has to be reversed, i.e., it starts at (23,0) and proceeds leftwards to (11,3).

where  $\Sigma_{\rm L}$  is the left part of the signature and  $\Sigma_{\rm R}$  the right part. We divide the signature at the halfway point so that  $\Sigma_{\rm L}$  contains the first  $m = \lfloor (L+1)/2 \rfloor$  states of  $\Sigma$ , and  $\Sigma_{\rm R}$  the remaining n = L+1-m states. We can view  $\Sigma$  as the concatenation of two Motzkin paths of height h with  $0 \le h \le m$ . We then have that  $\Sigma_{\rm L} \in \mathcal{M}^{(1)}_{(m,h)}$  and  $\Sigma_{\rm R} \in \mathcal{M}^{(0)}_{(n,h)}$ , though the Motzkin path from  $\mathcal{M}^{(0)}_{(n,h)}$  has to be reversed, see Figure 5.

We divide the storage array for the counts into sections based on the height h of the signatures  $\Sigma \in \mathcal{M}^{(1)}_{(L+1,0)}$ . Each  $\Sigma_{\mathrm{L}} \in \mathcal{M}^{(1)}_{(m,h)}$  can be concatenated with any reversed path from  $\mathcal{M}^{(0)}_{(n,h)}$ . The total number of paths of height h is therefore  $|\mathcal{M}^{(1)}_{(m,h)}| \cdot |\mathcal{M}^{(0)}_{(n,h)}|$ , and this is the size of the section of the storage array required to contain the counts for signatures of height h. Each main section of the storage array is divided into subsections of size  $|\mathcal{M}^{(0)}_{(n,h)}|$  containing the counts of the signatures with a particular left part  $\Sigma_{\mathrm{L}}$ . The paths in  $\mathcal{M}^{(1)}_{(m,h)}$  and  $\mathcal{M}^{(0)}_{(n,h)}$  are sorted in lexicographical order (using  $\odot < () < ()$ ) so that paths can be assigned unique indices  $I_{\mathrm{L}}$  and  $I_{\mathrm{R}}$  (note there is separate index function for each h and L). Define the number  $b_h$  as

$$b_0 := 0,$$
  
 $b_{h+1} := b_h + |\mathcal{M}_{(m,h)}^{(1)}| \cdot |\mathcal{M}_{(n,h)}^{(0)}|,$ 

then we define

$$\Phi_{\mathcal{L}}(\Sigma_{\mathcal{L}}) = b_h + (I_{\mathcal{L}} - 1) \cdot |\mathcal{M}_{(n,h)}^{(0)}|$$
  
$$\Phi_{\mathcal{R}}(\Sigma_{\mathcal{R}}) = I_{\mathcal{R}}.$$

 $\Phi_{L}$  tells us which subsection of the storage array to use and  $\Phi_{R}$  gives us the position within a given subsection.

## 2.4. Data representation and storage

The number of signatures  $|\mathcal{M}_{(L+1,0)}^{(1)}|$  can be expressed in terms of Motzkin numbers  $M_n = |\mathcal{M}_{(n,0)}^{(0)}|$  (OEIS <u>A001006</u>) since  $|\mathcal{M}_{(L+1,0)}^{(1)}| = M_{L+2} - M_{L+1}$  (OEIS <u>A002026</u>). The Motzkin numbers are given by the recurrence

$$M_0 = M_1 = 1, \quad (n+2)M_n = (2n+1)M_{n-1} + 3(n-1)M_{n-2}$$
 (5)

and have the generating function

$$\mathcal{M}(z) = \sum_{n=0}^{\infty} M_n z^n = (1 - z - \sqrt{1 - 2z - 3z^2})/(2z^2).$$
 (6)

From this it follows immediately that  $|\mathcal{M}_{(L+1,0)}^{(1)}| \sim 3^L$  and this gives the asymptotic growth in the storage required for the counts  $\mathcal{C}(\Sigma)$ . This growth in storage is the main limitation on the maximum size  $L_{\text{max}}$  that we can attain. We therefore perform all calculations of the walk counts modulo several prime numbers  $p_k$  which yields remainders of  $C_L(1)$  modulo  $p_k$ . The exact counts are then obtained from the remainders using the Chinese remainder theorem. We generally use primes of the form  $p_k = 2^{62} - r_k$ , such that  $p_k$  are the largest primes less than  $2^{62}$ . The counts  $\mathcal{C}(\Sigma)$  can therefore be stored in an array of 64-bit integers with  $\Phi(\Sigma)$  giving the position where  $\mathcal{C}(\Sigma)$  is stored.

The signatures are represented as 64-bit integers with 2 bits required for each state, with  $\odot = 00$ ,  $\bigcirc = 10$ , and  $\bigcirc = 01$ . The left and right parts of a signature can then be represented by a 32-bit integer and the hash functions  $\Phi_{\rm L}$  and  $\Phi_{\rm R}$  can be coded directly as simple arrays or look-up tables. The total size of these two arrays is about  $2^{L+2}$  so insignificant compared to the storage needed for the counts. The integer representation of signatures also means that transformations between a signature  $\Sigma$  and its left and right parts  $\Sigma_{\rm L}$ ,  $\Sigma_{\rm R}$  and from sources to targets can be done very efficiently using bit-wise manipulations.

#### 2.5. In-place memory updating

By controlling the order in which we access the signatures we can ensure that the counts can by updated in-place without the need for any temporary storage. The way we order the signatures is by height and for given height in lexicographical increasing order. Generally speaking, in-place updating is safe if a signature is updated only after it has been processed. The specific order of processing is controlled by the position at which we divide the signature into two halves. Importantly this dividing position need not be the same as the one used to construct the hash function and can be changed between iterations of the TM algorithm. The updates illustrated in Figure 3 shows that processing a given source signature  $\Sigma_{\rm S}$  always give rise to the same signature (as a target). A signature mapping to itself results is no change to its count and hence nothing needs to be done and in-place updating is trivially safe. We now consider the updates in detail and show how in-place updating can be done safely.

- $\Sigma_{\rm T} = \Sigma_{\rm L}$  oo  $\Sigma_{\rm R}$  leads to  $\Sigma_{\rm S}$  and the new signature  $\Sigma_{\rm T} = \Sigma_{\rm L}$   $\Sigma_{\rm R}$ . Updating the count for  $\Sigma_{\rm T}$  is safe since  $\Sigma_{\rm T}$  does not give rise to any new target signatures (apart from itself) when processed.
- $\bigcirc$  ()/( $\bigcirc$ : Processing  $\Sigma_1 = \Sigma_L \bigcirc$  ( $\Sigma_R$  leads to  $\Sigma_1$  and the signature  $\Sigma_2 = \Sigma_L \bigcirc$  ( $\Sigma_R$ , while similarly processing  $\Sigma_2$  gives rise to  $\Sigma_2$  and  $\Sigma_1$ . In-place updating of the counts for  $\Sigma_1$  and  $\Sigma_2$  is safe provided they are updated simultaneously, which is easily achieved.
- $\circ$ )/ $\circ$ : Same as above.
- ((): Processing the signature  $\Sigma_S = \Sigma_L$  (()))  $\Sigma_R$  leads to  $\Sigma_S$  and the new signature  $\Sigma_T = \Sigma_L$  (())  $\Sigma_R$ . Note that the matching upper arc ends () need not be consecutive or next to ((). We now look at the four sites involved in the update and consider how the height of the signature at the dividing position changes. We have

The possible dividing positions are indicated by vertical lines and the numbers below indicate the additional height of the signature from the height of  $\Sigma_{\rm L}$ .

We see that  $\Sigma_{\rm T}$  is never higher than  $\Sigma_{\rm S}$  and when they have the same height  $\Sigma_{\rm T}$  is lexicographically smaller than  $\Sigma_{\rm S}$ . Hence in all cases we process  $\Sigma_{\rm T}$  before updating its count and in-place updating is therefore safe.

))): Processing  $\Sigma_S = \Sigma_L((()))\Sigma_R$  leads to  $\Sigma_S$  and  $\Sigma_T = \Sigma_L(()) \circ \Sigma_R$ . We have

As for the case above in-place updating is safe.

() : No new signatures.

)(): Processing  $\Sigma_{\rm S} = \Sigma_{\rm L}$ )() $\Sigma_{\rm R}$  leads to  $\Sigma_{\rm S}$  and  $\Sigma_{\rm T} = \Sigma_{\rm L}$ )() $\Sigma_{\rm R}$ . We have

In-place updating is safe when the additional height is 0. There is a problem when the dividing position splits the signature between  $\bigcirc$  and  $\bigcirc$ . In that case  $\Sigma_T$  is higher than  $\Sigma_S$  and in-place updating is unsafe since the count of  $\Sigma_T$  is updated before  $\Sigma_T$  is processed.

The upshot of the above considerations is that in-place updating can be safely done provided the dividing position never splits the signature between two edges involved in an update. Thankfully we can easily avoid this from happening since we can change the dividing position so as to avoid such splits.

# Algorithm 1 Calculate the number of SAWs crossing a rhombus of size L

```
1: l_h \leftarrow |(L+1)/2|
 2: \Phi \leftarrow \text{ConstructHashFunction}(l_h)
 3: l_t \leftarrow l_h - 1
                                                                                         ▶ Upper signature divider
 4: l_b \leftarrow l_t - 1
                                                                                         ▶ Lower signature divider
 5: C[k] \leftarrow 0 for all 1 \le k \le |\mathcal{M}_{(L+1,0)}^{(1)}|
 6: \mathcal{C}[\Phi(\circ \circ \cdots \circ))] \leftarrow 1
 7: for Row = 0 to L do
                                                                           ▶ Build domain column-by-column
          m_h \leftarrow \min(l_t + 1, L + 1 - l_t)
\{\mathcal{M}_{(l_t,h)}^{(1)}, \mathcal{M}_{(L+1-l_t,h)}^{(0)}\} \leftarrow \text{ConstructSignatures}(l_t)
 8:
                                                                                              ▶ Max possible height
 9:
                                                                                                          \triangleright 0 \le h \le m_h
          for Col = L - 1 to l_t by -1 do
                                                                                        ▶ Build top half of column
10:
               for h=0 to m_h do
                                                                                              ▶ Height of signatures
11:
                    for all \Sigma_{\mathrm{L}} \in \mathcal{M}_{(l_t,h)}^{(1)} do
                                                                                                      ▶ Left signatures
12:
                         for all \Sigma_{\mathrm{R}} \in \mathcal{M}_{(L+1-l_t,h)}^{(0)} do
                                                                                                    ▶ Right signatures
13:
                              \Sigma_{\rm S} \leftarrow \Sigma_{\rm L} \Sigma_{\rm R}
                                                                                                    14:
                              UpdateCounts(\Sigma_s)
                                                                                        ▶ Process source signature
15:
                         end for
16:
17:
                    end for
               end for
18:
          end for
19:
20:
          m_h \leftarrow \min(l_b + 1, L + 1 - l_b)
                                                                                              \{\mathcal{M}_{(l_b,h)}^{(1)}, \mathcal{M}_{(L+1-l_b,h)}^{(0)}\} \leftarrow \text{ConstructSignatures}(l_b)
21:
          Col \leftarrow l_t - 1
22:
                                                                                         ▶ Add unit cell to column
          for h=0 to m_h do
                                                                                              ▶ Height of signatures
23:
               for all \Sigma_{\mathrm{L}} \in \mathcal{M}_{(l_b,h)}^{(1)} do
24:
                                                                                                      ▶ Left signatures
                    for all \Sigma_{\mathrm{R}} \in \mathcal{M}_{(L+1-l_b,h)}^{(0)} do
25:
                                                                                                    ▶ Right signatures
                         \Sigma_{\rm S} \leftarrow \Sigma_{\rm L} \Sigma_{\rm R}
                                                                                                    \triangleright Source signature
26:
                         UpdateCounts(\Sigma_s)
                                                                                        ▶ Process source signature
27:
                    end for
28:
               end for
29:
          end for
30:
          m_h \leftarrow \min(l_t + 1, L + 1 - l_t)
                                                                                              ▶ Max possible height
31:
          \{\mathcal{M}_{(l_t,h)}^{(1)},\mathcal{M}_{(L+1-l_t,h)}^{(0)}\} \leftarrow \text{ConstructSignatures}(l_t)
32:
          for Col = l_t - 2 to 0 by -1 do
                                                                                 ▶ Build bottom half of column
33:
34:
                                                                                               \triangleright Repeat lines 11:–18:
35:
          end for
36: end for
37: return \mathcal{C}[\Phi() \circ \cdots \circ \circ)
```

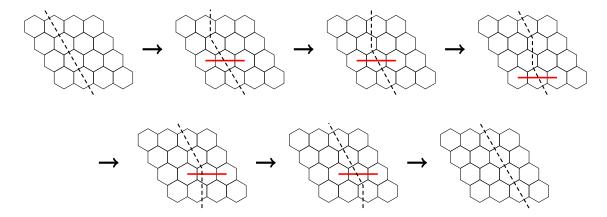


Figure 6: The TM moves used to add an extra column to a rhomboidal domain. The thick red line shows the dividing position during an update.

We are now ready to present our algorithm for counting SAWs crossing a rhombus. Algorithm 1 presents pseudo code for the main body of our algorithm. First up we divide signatures into two halves at position  $l_h$  and calculate the corresponding hash function  $\Phi$  or more specifically the two functions (look-up tables)  $\Phi_L$  and  $\Phi_R$ . The hash function  $\Phi$  remains fixed throughout the entire calculation. The value of  $l_h$  determines where counts are stored in memory. Then we define two parameters  $l_t$  and  $l_b$  which determine the order in which signatures are processed. Next we initialise the counts to zero except for the signature with a free end at the top vertex. After this comes the main body of the algorithm where we build the rhombus column-by-column up to size L with each column built cell-by-cell as illustrated in Figure 6. Note that the first move, from panel 1 to 2, and the final move are 'virtual'. These TM intersection moves add just a single edge and this means that all sources and targets are identical and hence no updating is actually required. These moves are included for illustrative purposes only.

We choose the parameters  $l_t$  and  $l_b < l_h$ , because memory access is crucial to the performance of the algorithm and we found that for L large these choices resulted in the best performance. The reason for breaking the column construction into three separate pieces is to allow in-place updating of the counts. During the first loop (at line 10) the position of the divider  $l_t$  is below the local states being changed by an update (see Figure 6) so in-place updating can be done safely. Next we change the divider to be at  $l_b = l_t - 1$ , since otherwise the divider would lie between the two local states in the TM kink and as explained above this would not be safe. We then change back to a divider at  $l_t$  (which now lies above the local states of the update) and complete the column. Note that we could have set  $l_b = l_t + 1 = l_h$  and completed the column with this divider, but as already stated using dividers strictly less than  $l_h$  is faster and the time taken by an extra call to Construct Signatures is insignificant. The routine Construct Signatures generates the sets of left and right signatures using a simple back-tracking algorithm.

The updating rules for the counts of the signatures are given in Algorithm 2.

Algorithm 2 Update the counts of signatures

```
1: procedure UPDATECOUNTS(\Sigma_{\rm S})
           S \leftarrow \text{InputState}(\Sigma_S)
 2:
                                                                                                      if S = \bigcirc \bigcirc then
 3:
                 \Sigma_{\rm T} \leftarrow {\rm CHANGESIGNATURE}(\Sigma_{\rm S}, (()))
 4:
                                                                                                                    ▶ Insert new arc
                 \mathcal{C}[\Phi(\Sigma_{\mathrm{T}})] \leftarrow \mathcal{C}[\Phi(\Sigma_{\mathrm{T}})] + \mathcal{C}[\Phi(\Sigma_{\mathrm{S}})]
                                                                                                      ▶ Update count of target
 5:
           else if S = \bigcirc then
 6:
                 \Sigma_{\rm T} \leftarrow {\rm CHANGESIGNATURE}(\Sigma_{\rm S}, \bullet)
 7:
                \mathcal{C}[\Phi(\Sigma_T)] \leftarrow \mathcal{C}[\Phi(\Sigma_T)] + \mathcal{C}[\Phi(\Sigma_S)]
 8:
                                                                                                      ▶ Update count of target
                 \mathcal{C}[\Phi(\Sigma_S)] \leftarrow \mathcal{C}[\Phi(\Sigma_T)]
                                                                                         ▶ Simultaneous update of source
 9:
           else if S = \bigcirc () then
10:
                                                                      ▷ Do nothing. Processed in previous update
                 Null
11:
12:
           else if S = \bigcirc then
                 \Sigma_{\rm T} \leftarrow {\rm CHANGESIGNATURE}(\Sigma_{\rm S}, \bullet)
13:
                \mathcal{C}[\Phi(\Sigma_{\mathrm{T}})] \leftarrow \mathcal{C}[\Phi(\Sigma_{\mathrm{T}})] + \mathcal{C}[\Phi(\Sigma_{\mathrm{S}})]
                                                                                                      ▶ Update count of target
14:
                \mathcal{C}[\Phi(\Sigma_S)] \leftarrow \mathcal{C}[\Phi(\Sigma_T)]
                                                                                         ▶ Simultaneous update of source
15:
           else if S = \bigcirc) then
16:
                 Null
                                                                      ▶ Do nothing. Processed in previous update
17:
           else if S = () then
18:
                 \Sigma_{\mathrm{T}} \leftarrow \mathrm{RelabelSignature}(\Sigma_{\mathrm{S}}, \circ \circ, ())
                                                                                           ▷ Connect arc ends and relabel
19:
                \mathcal{C}[\Phi(\Sigma_{\mathrm{T}})] \leftarrow \mathcal{C}[\Phi(\Sigma_{\mathrm{T}})] + \mathcal{C}[\Phi(\Sigma_{\mathrm{S}})]
20:
           else if S = (1) then
21:
                 Null
                                                                                        ▷ Do nothing. No new signatures
22:
           else if S = \bigcap ( then
23:
                 \Sigma_{\rm T} \leftarrow {\rm CHANGESIGNATURE}(\Sigma_{\rm S}, \bullet)
                                                                                                               ▷ Connect arc ends
24:
                 \mathcal{C}[\Phi(\Sigma_T)] \leftarrow \mathcal{C}[\Phi(\Sigma_T)] + \mathcal{C}[\Phi(\Sigma_S)]
25:
           else if S = (1) then
26:
                 \Sigma_{\rm T} \leftarrow {\rm RelabelSignature}(\Sigma_{\rm S}, \bullet, \bullet)
27:
                                                                                           ▶ Connect arc ends and relabel
                \mathcal{C}[\Phi(\Sigma_T)] \leftarrow \mathcal{C}[\Phi(\Sigma_T)] + \mathcal{C}[\Phi(\Sigma_S)]
28:
           end if
29:
30: end procedure
```

INPUTSTATE simply extracts the states of the two input edges involved in the update. Changesignature changes the states of the input edges to those indicated by the two blue tiles. Relabelsignature changes the input states to empty states and finds and relabels the matching arc end in those updates where two arc ends are connected in a TM update.

#### 2.6. Parallelisation

The transfer-matrix algorithm is very well suited to parallel computation. In previous work we implemented algorithms using the message passing interface (MPI) [14, 13]

suited for distributed memory systems. For this work we used shared memory computers and hence implemented the parallel algorithms using OpenMP which is somewhat simpler but relies on the same basic ideas. One of the main ways of achieving a good parallel algorithm using data decomposition is to try to find an invariant under the operation of the updating rules. That is we seek to find some property of the signature which does not alter in a single iteration. There is such an invariant since any edge not directly involved in the update cannot change from being empty to being occupied and vice versa (it may change, say, from state ) to (). That is only the kink edges can change their occupation status. This invariant allows us to parallelise the algorithm in such a way that we can do the calculation completely independently on each core. With the intersection straight (having no kinks) we distribute the data across cores so that signatures with the same occupation pattern along the *lower* half of the intersection are processed by the same core. We then do the TM updates inserting the top-half of a new column. This can be done *independently* by each core because the occupation pattern in the lower half remains unchanged. When reaching the half-way point we redistribute the data so that configurations with the same occupation pattern along the upper half of the intersection are processed by the same core and we then do the TM update inserting the bottom-half of a new column. This is then repeated column by column.

## 2.7. Changes needed to enumerate other hexagonal problems

The changes required to enumerate other types of configurations are mostly straightforward. To enumerate spanning SAWs we just need to change lines 6 and 37 in Algorithm 1. At 6 we need to initialise all signatures with just a single free end in some position (all other states empty) to have a count of one. This means a SAW can start in any position on the left side of the rhombus. Similarly at 37 we need to return the sum of the counts for signatures with just a single free end.

To enumerate SAPs crossing a rhombus the main change to note is that we no longer have a free end and any signature can therefore be represented by a standard Motzkin path from (0,0) to (L+1,0). So the total set of signatures for this problem is  $\mathcal{M}^{(0)}_{(L+1,0)}$ . Furthermore, we have that  $\sum_{L} \in \mathcal{M}^{(0)}_{(m,h)}$  and  $\sum_{R} \in \mathcal{M}^{(0)}_{(n,h)}$ . Again we need to change lines 6 and 37 of Algorithm 1. Line 6 is changed to:  $\mathcal{C}[\Phi(\text{Olo},\text{Olo})] \leftarrow 1$ . Line 37 is changed to:  $\text{return } \mathcal{C}[\Phi(\text{Olo},\text{Olo})]$ .

Enumerations in a triangular domain just requires us to change the way in which the transfer matrix intersection is moved, that is, the moves for the rhombus TM calculation shown in Figure 6 have to be changed appropriately.

## 2.8. Algorithm to enumerate square lattice problems

The algorithm for enumerating walks crossing a square has been described in [4], and for this work we implemented our own version which we won't describe here other than to say that the main body is identical to Algorithm 1, but of course the updating rules

are different, and Algorithm 2 must be amended accordingly. The interested reader can check out the actual code at our GitHub repository (see Section 7).

## 3. Series analysis

The method of series analysis has, for many years, been a powerful tool in the study of a variety of problems in statistical mechanics, combinatorics, and other fields. In essence, the problem is the following: Given the first N coefficients of the series expansion of some function, (where N is typically as low as 5 or 6, or as high as 100,000 or more), determine the asymptotic form of the coefficients, subject to some underlying assumptions, or equivalently the nature of the singularity of the function.

A typical example is the generating function of self-avoiding walks (SAWs) in dimension two or three. This is believed to behave as

$$F(z) = \sum_{n} c_n z^n \sim C \cdot (1 - z/z_c)^{-\gamma}. \tag{7}$$

In this case, among regular two-dimensional lattices, the value of  $z_c$  is only known for the hexagonal lattice [6], while  $\gamma = 43/32$  [15] is believed to be the correct exponent value for all two-dimensional lattices, but this has not been proved.

The method of series analysis is used when one or more of the critical parameters is not known. For example, for the three-dimensional version of the above problems, none of the quantities C,  $z_c$  or  $\gamma$  are known exactly. From the binomial theorem it follows from (7) that

$$c_n \sim \frac{C}{\Gamma(\gamma)} \cdot z_c^{-n} \cdot n^{\gamma - 1},$$
 (8)

where  $a_n \sim b_n$  means that  $\lim_{n\to\infty} a_n/b_n = 1$ . Here C,  $z_c$ , and  $\gamma$  are referred to as the critical amplitude, the critical point (usually the radius of convergence) and the critical exponent, respectively. In combinatorics one often refers to the growth constant  $\mu = 1/z_c$ , as the coefficients are dominated by the term  $\mu^n$ .

Obtaining these coefficients is typically a problem of exponential complexity, as is the case with our algorithm, described in Section 2. The consequence is that usually fewer than 50 terms are known (and in some cases far fewer).

The standard methods of series analysis include the ratio method, described in Appendix A, and the method of differential approximants, described in Appendix B. A relatively recent development has been the method of series extension [16], described in Appendix C, in which differential approximants based on the exactly known terms is used to obtain a significant number of additional approximate terms. These approximate terms, if of sufficient accuracy, can then be used in the ratio method and its extensions to obtain more precise estimates of the various critical parameters.

In our analysis below we make use of all of these methods, but will just refer to them under the assumption that the material in the appendices has been understood.

## 3.1. Methods of analysis

The existence of the limit (1) and the more detailed asymptotic form (2), which we shall take for granted and provide overwhelming numerical support for, suggests several methods of analysis that one can apply in order to estimate the growth constant  $\lambda$ . For the first method (M1), we look at the quantity

$$\lambda_L := C_L(1)^{1/L^2} \sim \lambda. \tag{9}$$

While it has not been proved that the ratios  $R_L := C_L(1)/C_{L-1}(1) \sim \lambda^{2L}$ , it is almost certainly true, and we will assume it to be so in our analysis. Given the expectation that  $R_L \sim \lambda^{2L}$ , for the second method (M2) we define the ratio-of-ratios

$$C_L := \frac{R_{L+1}}{R_L} = \frac{C_{L+1}(1)C_{L-1}(1)}{C_L(1)^2}.$$
(10)

From (2) it follows that

$$C_L = \lambda^2 \left( 1 - \frac{g}{L^2} + O(L^{-3}) \right). \tag{11}$$

All of the sequences defined above will be analysed using ratio methods.

Next we briefly describe three different methods that we have used to estimate the parameters b, c and g in the assumed asymptotic form (2). In the first method (P1) we use our best estimate of  $\lambda$  and form the sequence

$$d_L := C_L(1)/\lambda^{L^2} \sim \lambda^{bL+c} \cdot L^g. \tag{12}$$

This sequence, provided the assumed asymptotic form is correct, behaves as a typical power-law singularity, in which the coefficients grow as  $a_n \sim C \cdot \alpha^n \cdot n^g$ , and can be analysed as such. With that notation, the growth constant  $\alpha = \lambda^b$ , and the amplitude  $C = \lambda^c$ . Of course, we have to use our estimated value of  $\lambda$ .

For the second method (P2) we fit to the assumed form by writing

$$\log d_L \sim b \log(\lambda) L + c \log(\lambda) + g \log L. \tag{13}$$

We then use successive triples of data points  $(\log d_{k-1}, \log d_k, \log d_{k+1})$ , with  $k = 2, 3, \dots, L_{\max} - 1$ , to obtain estimates of the parameters  $b \log \lambda$ ,  $c \log \lambda$ , and g.

The third method (P3) makes use of the ratio  $C_L$  (10). According to its asymptotic form (11), we can fit the sequence  $\{C_L\}$  to  $c_0 + c_2/L^2 + c_3/L^3$ , so that  $c_0$  should give estimators of  $\lambda^2$ , and  $c_2$  give estimators of  $-g\lambda^2$ .

If  $C_L(1) \sim \lambda^{L^2}$ , then the ratios  $R_L = C_L(1)/C_{L-1}(1) \sim \lambda^{2L-1}$ , so the exponent  $\gamma$  in the canonical form (8) equals 1. It follows that the corresponding function,  $\mathcal{R}(z) := \sum_L R_L z^L$ , will have a simple pole at the critical point  $z_c = 1/\lambda^2$ . If we include sub-dominant terms, so that  $C_L(1) \sim \lambda^{L^2+bL+c}L^g$ , then  $R_L \sim \lambda^{2L-1+b}(1+O(1/L))$ , and all that has changed is the amplitude. The singularity is still a simple pole at  $z_c = 1/\lambda^2$ . The series  $\mathcal{R}(z)$  can therefore be analysed using differential approximants to obtain an estimate for  $\lambda$ .

Since  $\mathcal{R}(z)$  has a simple pole this suggests two other ways to estimate  $\lambda$ . Firstly, one can simply form Padé approximants, that is set  $P_{m,n}(z) := P_m(z)/Q_n(z)$ , where

 $P_m(z)$  and  $Q_n(z)$  are polynomials of degree m and n, respectively, chosen so the first n+m+1 terms in the Taylor expansion of  $P_{m,n}(z)$  coincide with those of  $\mathcal{R}(z)$ . The first real zero of  $Q_n(z)$  will then provide an estimate of  $1/\lambda^2$ .

The second method is a little more speculative and novel and we are not entirely sure of its validity. We force the differential approximants to have a singularity at a critical point  $\hat{z}_c$  close to the expected true value  $z_c = 1/\lambda^2$ . This is done by forming biased differential approximants as outlined in Appendix B.1 and the associated critical exponent is calculated. Many biased differential approximants are formed for each value of the biasing critical point  $\hat{z}_c$  and the average critical exponent calculated. One can then conjecture that the value of  $\hat{z}_c$  for which the average critical exponent attains the value -1 provides a reasonable estimate for  $1/\lambda^2$ .

## 4. Walks and polygons in a square.

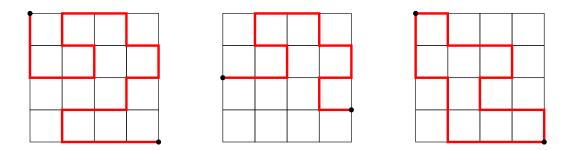


Figure 7: Classes of walk and polygon configurations investigated on the square lattice.

In this section we analyse walks and polygons crossing a square domain of the square lattice, using the techniques just discussed. We study three different variants of the problem, namely SAWs crossing or spanning a square and SAPs crossing a square. These are shown in Figure 7.

## 4.1. Walks crossing a square

Firstly, we apply method M1 (9) to the analysis of the series for walks crossing a square. For want of greater knowledge about the sub-dominant asymptotic terms we simply extrapolate  $\lambda_L$  against 1/L. Recall that we only have 27 terms. We therefore use the method of series extension, mentioned above and described in Appendix C, to extend the sequence of ratios  $R_L = C_L(1)/C_{L-1}(1)$ , and this sequence is then used to extend the  $C_L(1)$  series. In this way we obtained 20 additional approximate coefficients. These are given in Table 1.

We show a plot of  $\lambda_L$  against 1/L in the top-left panel of Figure 8, and it is seen to be quite well converged, and can visually be extrapolated to  $\lambda_S \approx 1.7442$ . It is reasonable to assume that the curvature is due to the presence of higher-order terms, such as  $1/L^2$ ,  $1/L^3$  etc. In the top-right panel of Figure 8 we show values of the estimator of

$\overline{L}$	$C_L(1)$ estimates.
27	$1.092762277820988255238897693624593273299\times 10^{176}$
28	$2.092263800732296637339584460940199207179\times 10^{189}$
29	$1.219188494943327773136239385657818116903\times 10^{203}$
30	$2.162167627691293760665426155350775028513\times 10^{217}$
31	$1.167003905184619653378731561256980927898\times 10^{232}$
32	$1.916990667670442255801047617746147903033\times 10^{247}$
33	$9.583688332141159129759056552823132225046\times 10^{262}$
34	$1.458178102419213554003374021702217439866 \times 10^{279}$
35	$6.752333021793147034314988105916341545574\times 10^{295}$
36	$9.516180772478135635389490590804240161152 \times 10^{312}$
37	$4.081663288146408412423849764027291063947 \times 10^{330}$
38	$5.328162506991801337436456805173755617688 \times 10^{348}$
39	$2.116818597440340726855200831821163701531 \times 10^{367}$
40	$2.559504109272639104369198989850180317538 \times 10^{386}$
41	$9.418767710224918432123841878087214586086 \times 10^{405}$
42	$1.054869066038373202559187284758968442477 \times 10^{426}$
43	$3.595581533556538000636173781717640795638 \times 10^{446}$
44	$3.729975451051537109220327069642553666508 \times 10^{467}$
45	$1.177630435162076031609822304850879989404 \times 10^{489}$
46	$1.131562339582151957192359485190854061339 \times 10^{511}$

Table 1: Estimated coefficients  $C_L(1)$ .

 $\lambda_S$  assuming  $\lambda_L$  converges to  $\lambda_S$  with correction term  $c_1/L + c_2/L^2$ , plotted against  $1/L^3$ , and we estimate  $\lambda_S \approx 1.74455$ . There is still considerable curvature in this plot and we therefore tried plotting against  $1/L^4$  instead, as shown in the bottom-left panel of Figure 8, and in this case the plot appears linear. The straight line is a simple linear fit to the data which intercepts the y-axis at  $\lambda_L = 1.74550025$  and we therefore conclude that  $\lambda_S \approx 1.744550$ . This analysis indicates that the  $1/L^3$  correction term is absent or at least has a very small amplitude. Finally in the bottom-right panel of Figure 8 we plot the estimator of  $\lambda_S$  assuming  $\lambda_L$  converges with correction terms  $c_1/L + c_2/L^2 + c_4/L^4$ , plotted against  $1/L^6$ . For this plot we have used only the first 4 of the 20 extra approximate coefficients, as using more than this produces some ripples in the plot, indicating that the approximate coefficients are insufficiently precise for such an extreme extrapolation. The linear fit has an intercept at  $\lambda_L = 1.745549827$  and hence we estimate  $\lambda_S \approx 1.7445498$ .

Next, we apply method M2 (10) to the analysis of  $C_L(1)$ . We show a plot of the ratios  $C_L \sim \lambda_L^2$  against  $1/L^2$  in the top-left panel of Figure 9. It is seen to display considerable curvature, but can be visually extrapolated to  $\lambda_S^2 \approx 3.04345$ . In fact the curvature in the plot is suggestive of quadratic behaviour which would mean that  $C_L$  depends on  $1/L^4$ . A plot of  $C_L$  against  $1/L^4$  is shown in the top-right panel of Figure 9

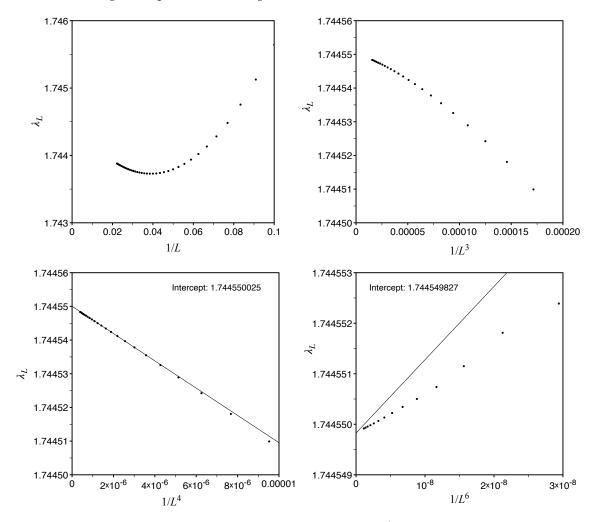


Figure 8: The first panel shows  $\lambda_L$  plotted against 1/L. The second and third panels show plots of the estimator  $\lambda_L$  using a quadratic correction term against  $1/L^3$  and  $1/L^4$ , respectively. The fourth panel is a plot of the estimator  $\lambda_L$  using the correction term  $c_1/L + c_2/L^2 + c_4/L^4$ .

and we do indeed see a nice linear plot. The linear fit has intercept at  $\lambda_L^2 = 3.043455344$  from which we estimate that  $\lambda_S \approx 1.744550$ . As above, we now directly include powers of 1/L in the extrapolation. In the bottom-left panel of Figure 9 we show the estimator of  $\lambda_S^2$  assuming  $\lambda_L^2$  converges with correction term  $c_4/L^4$ , plotted against  $1/L^6$  (intercept at 3.043454383). Then in the bottom-right panel of Figure 9 we show the estimator of  $\lambda_S^2$  assuming  $\lambda_L^2$  converges with correction terms  $c_2/L^2 + c_4/L^4$ , plotted against  $1/L^6$  (intercept at 3.043454164). For these plot we have used only the first 4 approximate coefficients, for similar reasons to those given above. The two extrapolated values of  $\lambda_L^2$  are in excellent agreement and we obtain the precise estimate  $\lambda_S \approx 1.74454985$ . The clear indication from this analysis is that the parameter g = 0. To further examine this point we plot in the last panel of Figure 11 the values of  $c_2 \sim -g\lambda_L^2$  from the analysis with correction terms  $c_2/L^2 + c_4/L^4$ . Clearly the value of this parameter is very small and entirely consistent with g = 0.

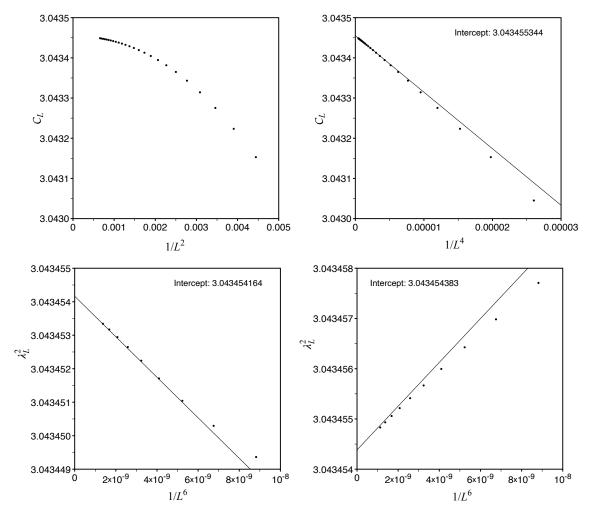


Figure 9: The top two panels show plots of  $C_L$  plotted against  $1/L^2$  and  $1/L^4$ , respectively. The bottom panels show plots of the estimators  $\lambda_L^2$  assuming correction terms  $c_4/L^4$  (left panel) and  $c_2/L^2 + c_4/L^4$  (right panel).

Next we will attempt to estimate the parameters b, c and g in the assumed asymptotic form (2) by the methods described in Section 3.1. Using our best estimate of  $\lambda_S = 1.7445498$ , we first form the sequence  $d_L = C_L(1)/\lambda_S^{L^2}$ . Usually, ratios are plotted against 1/L, and the gradient of the linear plot gives a measure of the exponent g. The ratio plot displays considerable curvature when plotted against 1/L, becoming approximately linear only when plotted against  $1/L^3$  as shown in the in the top-left panel of Figure 10. This suggests that the coefficient of 1/L in the asymptotic expansion of the expression for the ratios is zero, or at least very small, that is  $g \approx 0$ . We estimate from this plot that  $\alpha = 0.97605 \pm 0.00001$ , so that  $b = \log \alpha/\log \lambda_S = -0.04355 \pm 0.00001$ . From the plot it is clear that there is some residual curvature.

Next we performed a least-squares fit of the data to the form  $c_0+c_3/L^3+c_4/L^4+c_5/L^5$  using the data-points from L=20 up to L=35 (we display the data from L=15 to 41) and the resulting plot is shown in the top-right panel. We estimate from this plot that  $\alpha=0.976061\pm0.000005$ , so that  $b=\log\alpha/\log\lambda_S=-0.04354\pm0.00001$ . Finally,

we use this latter value of  $\alpha$  to estimate the value of the parameter c, or equivalently, the amplitude C, by observing that  $d_L/\lambda_S^{bL}\sim\lambda_S^c\cdot L^g$ . We have argued that  $g\approx 0$ , so that  $d_L/\lambda_S^{bL}\sim\lambda_S^c$ . In the bottom-left panel of Figure 10 we show a plot of the estimator of  $C=\lambda_S^c$  plotted against  $1/L^2$ , from which we estimate  $C=\lambda_S^c=1.3673\pm0.001$ , or  $c=0.5622\pm0.0005$ . As before we next did a least-squares fit of the data, but now to the form  $c_0+c_2/L^2+c_3/L^3+c_4/L^4$ , which we display in the bottom-right panel. We estimate  $C=\lambda_S^c=1.36723\pm0.0001$ , or  $c=0.56207\pm0.00005$ .

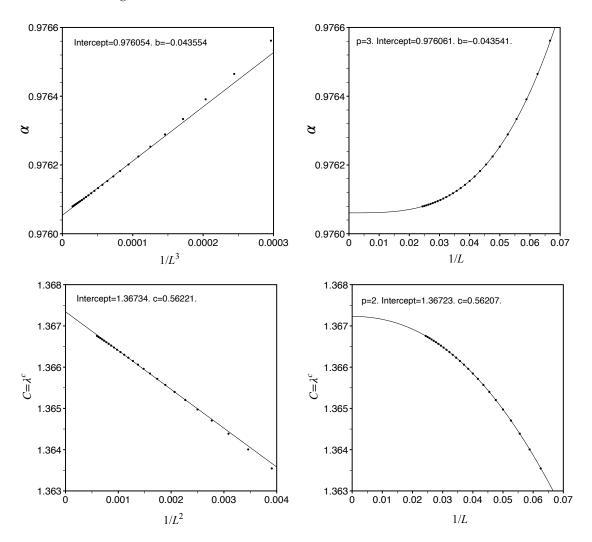


Figure 10: Ratios  $d_L/d_{L-1} \sim \alpha$  plotted against  $1/L^3$  (top-left) and the amplitude  $C = \lambda^c$  plotted against  $1/L^2$  (bottom-left). The panels on the right display the same data plotted against 1/L with the solid curve being a least-squares fit.

We now turn to the second method P2 to obtain estimates of the parameters  $b \log \lambda_S$ ,  $c \log \lambda_S$ , and g. As was the case above, these estimators have a lot of curvature when plotted against 1/L. Hence we plotted against integer powers p of 1/L until we found a value for which approximate linearity was achieved and we then performed a least-squares fit to the data to the form  $c_0 + c_p/L^p + c_{p+1}/L^{p+1} + c_{p+2}/L^{p+2}$ . Plots of these against 1/L are shown in the first three panels of Figure 11. From these plots, we estimate

 $b \log \lambda_S = -0.02422 \pm 0.00002$ , or  $b = -0.04353 \pm 0.00002$ ,  $c \log \lambda_S = 0.314 \pm 0.001$ , or  $c = 0.564 \pm 0.002$ , and  $g \approx 0$ . The agreement between the two methods is excellent and well within quoted confidence limits.

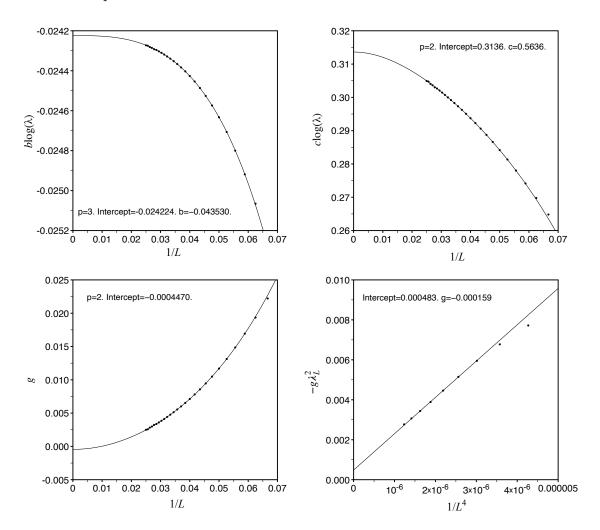


Figure 11: Estimators of  $b \log \lambda_S$ ,  $c \log \lambda_S$ , and g from method P2 plotted against 1/L, and the estimator  $-g\lambda_L^2$  from method P3 plotted against  $1/L^4$ .

Accordingly we conclude that our analysis has provided overwhelming numerical evidence that the conjectured asymptotic form

$$C_L(1) \sim \lambda_S^{L^2 + bL + c} \cdot L^g,$$

is correct. For SAWs crossing a square we estimate (conservatively) that the parameters take the values  $\lambda_S = 1.7445498 \pm 0.0000012$ ,  $b = -0.04354 \pm 0.0001$ ,  $c = 0.5624 \pm 0.001$ , and  $g = 0.000 \pm 0.005$ .

Next we use differential approximants to analyse the series  $\mathcal{R}(z) = \sum R_L z^L$ . We show the results of the analysis, using 3rd order differential approximants (and of course only the exactly known 27 terms) in Table 2. From this we estimate the radius of convergence as  $z_c = 1/\lambda_S^2 = 0.3285735 \pm 0.000001$ , which gives  $\lambda_S = 1.744551 \pm 0.000003$ .

L	Singularity	Exponent
0	0.3285739(14)	-0.99989(27)
1	0.32857478(95)	-1.00010(18)
2	0.32857481(99)	-1.00012(20)
3	0.3285745(10)	-1.00007(21)
4	0.3285746(24)	-1.00004(42)
5	0.3285730(49)	-0.9998(10)
6	0.3285745(18)	-1.00004(38)

Table 2: Estimates of the singularity and exponent of the sequence for the ratios of walks crossing a square series. The estimates are from third order differential approximants with various degrees L of the inhomogeneous polynomial.

(n,m)	Root	(n,m)	Root	(n,m)	Root
(8,8)	1.7445242860	(8,10)	1.7445439450	(8,12)	1.7445454270
(10,8)	1.7445415440	(10,10)	1.7445441380	(10,12)	1.7445488060
(12,10)	1.7445497750	(12,12)	1.7445487670	(12,14)	1.7445489710
(13,11)	1.7445488890	(13,13)	1.7445491150	(14,12)	1.7445491730

Table 3: Estimates of  $\lambda_S$  obtained from Padé approximants.

This is slightly less precise than the ratio methods. The estimates for the critical exponent are clearly supportive of  $\mathcal{R}(z)$  having a simple pole adding even more evidence to the validity of the assumed asymptotic form.

We also tried using Padé approximants to estimate  $\lambda_S$ . In Table 3 we list some estimates of  $\lambda_S$  obtained from  $P_{m,n}(z)$  Padé approximants to  $\mathcal{R}(z)$  by calculating the real roots of the denominator polynomial  $Q_n(z)$ , finding the smallest positive root to obtain an estimate of  $\lambda_S$ . It is clear that this method works just fine but it is, perhaps not surprisingly, at least an order of magnitude less accurate than differential approximants let alone the ratio methods. Hence we shall not consider this method or differential approximants any further.

Finally, we turn to the analysis of  $\mathcal{R}(z)$  using biased differential approximants (see Appendix B.1). We pick a biasing value  $\widehat{\lambda}_S$  and force the differential approximants to have a singularity of order 1 at  $z_c = 1/\widehat{\lambda}_S^2$ . We calculate many (> 100) 3rd order biased differential approximants with an inhomogeneous polynomial of degree K, such that the number of required terms of the approximants  $N \geq 22$ . Each approximant in turn provides us with an estimate of the critical exponent  $\gamma$ , which we confidently conjecture has the value -1. From all of these  $\gamma$  estimates we discard the outlying 10% on either side. The remaining estimates are used to calculate the mean and standard deviation. This procedure is then repeated for different values of  $\widehat{\lambda}_S$  so as to cover the full range of values within our estimated range  $\lambda_S = 1.7445498 \pm 0.0000012$ . In Figure 12 we show

a plot of the  $\gamma$  estimates (with error-bars) as a function of  $\hat{\lambda}_S$  for the two cases where the degree of the inhomogeneous polynomial is 0 and 4, respectively. We notice that the curve of exponent estimates intersects  $\gamma = -1$  over a very narrow range (smaller than the error estimate on  $\lambda_S$ ). Obviously it is very tempting to try and use this to provide an even more precise estimate of  $\lambda_S$ . One may say that  $\lambda_S$  could be estimated from the crossing with an error given by the width of the range over which error-bars on the exponent estimates overlap with  $\gamma = -1$  (or perhaps a factor of two or three times this range). However, this is a very new method and we are not yet confident that it is a valid method for obtaining more accurate estimates of critical points in cases where the exponent is known exactly. In particular we have no real idea of how to confidently estimate an error-bar. All we are willing to say at the moment is that it appears to be a promising method that warrants further detailed investigation.

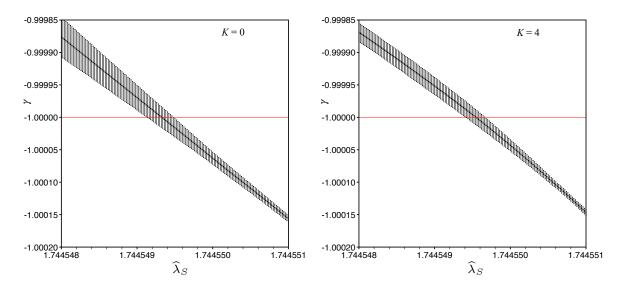


Figure 12: Biased estimators for the critical exponents  $\gamma$  of  $\mathcal{R}(z)$  plotted against the biasing value  $\widehat{\lambda}_S$ .

#### 4.2. Polygons crossing a square

Using the algorithm described in Section 2 we calculated  $P_L(1)$  to lattice size L = 26 and we then used the method of series extension to obtain a further 30 approximate terms.

We first estimated  $\lambda_S$  by method M1, that is extrapolating the sequence  $\lambda_L = P_L(1)^{1/L^2}$  against 1/L. There was some curvature in the plot, so we extrapolated against  $c_0 + c_1/L + c_2/L^2 + c_3/L^3$ . In this case the estimates appear fairly straight when plotted against  $1/L^2$  as shown in the left panel of Figure 13. From this plot we estimate that  $\lambda_S = 1.744550 \pm 0.000005$ . Next we used method M2, that is we looked at the ratio of ratios. We extrapolated against  $c_0 + c_2/L^2 + c_3/L^3$ , and plotted this against  $1/L^4$  as shown in the right panel of Figure 13. This allowed us to estimate  $\lambda_S^2 = 3.043454 \pm 0.000003$ , and hence  $\lambda_S = 1.7445498 \pm 0.0000008$ , in agreement with the previous analysis.

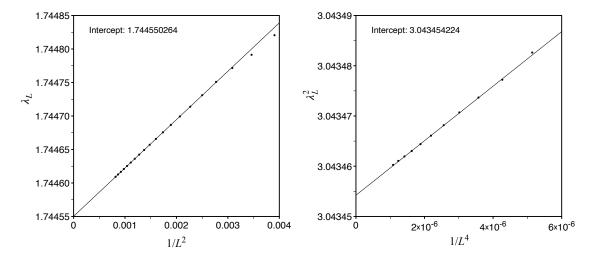


Figure 13: Estimators of  $\lambda_S$  from method M1 plotted against  $1/L^2$  and  $\lambda_S^2$  from method M2 plotted against  $1/L^4$ .

We estimated the values of the sub-dominant terms by method P2 and we also estimated g by method P3. The various plots are shown in Figure 14. In this way we estimate  $b \log \lambda_S \approx -0.02422$ , or  $b \approx -0.04352$ ,  $c \log \lambda_S \approx -0.6665$  or  $c \approx -1.195$ , and  $g \approx -0.5005$ . From the estimate  $-g\lambda_S^2 \approx 1.5235$  we get  $g \approx -0.5006$ . We conjecture with some confidence that  $g = -\frac{1}{2}$ , exactly. Using our best estimate for  $\lambda_S$  and our conjecture for the exact value of g we then turned to method P1. The plot of the estimator for  $\alpha$  is close to linear against 1/L, but to account for small correction we used a least-squares cubic fit in 1/L (solid curve) and found from the intercept that  $\alpha = \lambda_S^b \approx 0.9761$  and hence  $b \approx -0.04351$ . We next make use of the intercept value from the  $\alpha$ -plot to estimate c. We look at the quantity  $C = \lambda_S^c \sim d_L/(\alpha^L L^g)$ , plot it against 1/L, and use a cubic least-square fit to estimate the intercept  $C \approx 0.5130$  and hence  $c \approx -1.199$ . The parameter estimates from the various method are in good agreement and clearly it seems that b has the same value as for walks crossing a square.

#### 4.3. Walks spanning a square

We calculated  $C_L(1)$  to lattice size L=26 and we then used the method of series extension to obtain a further 30 approximate terms. The plots used to estimate the parameters of this model are shown in Appendix E Figure E1. We estimate  $\lambda_S \approx 1.74455$  from method M1 using a fourth degree estimator and  $\lambda_S^2 \approx 3.043455$  (and hence  $\lambda_S \approx 1.744550$ ) from method M2 fitting to a cubic polynomial. We estimated the values of the sub-dominant terms by method P2, and we also estimated g by method P3. We estimate  $b \approx -0.0435$ ,  $c \approx 0.603$ , and  $b \approx 1.74$ . From the estimate  $b \approx -0.0435$ , we get  $b \approx 1.75$ . It seems reasonable to conjecture that  $b \approx -0.0435$ , we finally used this value of  $b \approx -0.0435$  and  $b \approx -0.0435$  and

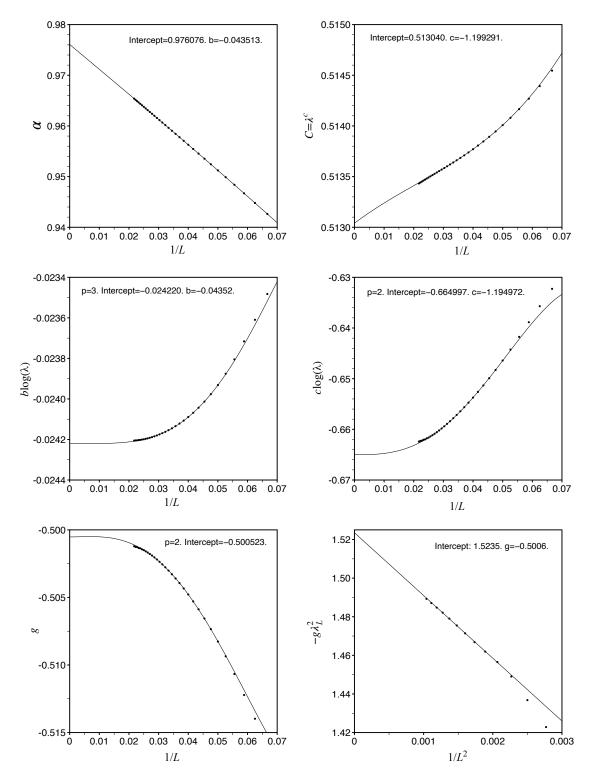


Figure 14: Plots of the estimators for parameters  $\alpha$  and  $C = \lambda_S^c$  from method P1,  $b \log \lambda_S$ ,  $c \log \lambda_S$ , and g from method P2 and the estimator  $-g\lambda_L^2$  from method P3 for polygons crossing a square.

# 5. Walks crossing a domain of the hexagonal lattice.

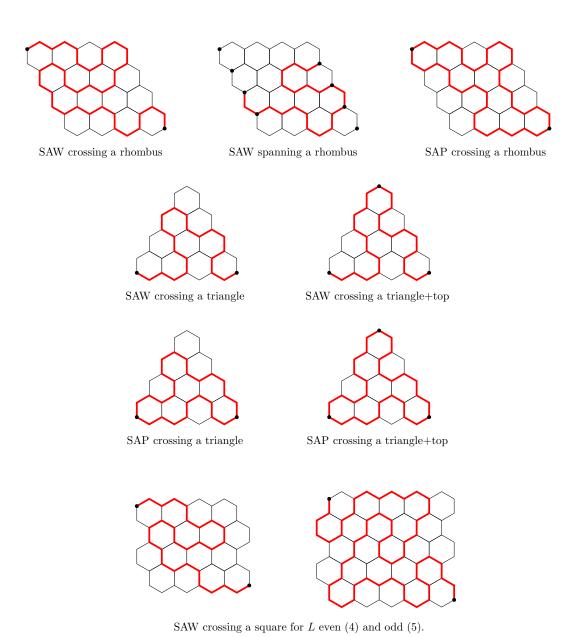


Figure 15: Classes of walk and polygon configurations investigated.

In this section we study walks and polygons crossing a specified domain of the hexagonal lattice. We study several different variants of the problem, including both SAWs and SAPs, on triangular, rhomboidal domains and and SAWs on square domains. These are illustrated in Figure 15. We expect that the number of walks  $C_L(1)$  for all these cases will have the asymptotic form (2). More specifically the number of walks should have dominant asymptotic growth determined by

 $C_L(1) \sim \kappa^{\#}$  vertices in domain.

The number vertices in a triangular domain of size L is  $L^2 + 4L + 1$ , while there are  $2L^2 + 4L - 1$  vertices in the rhomboidal and square domains. Hence, we expect there to be a common growth constant  $\lambda_H$  such that  $\kappa = \lambda_H$  for the triangular domain and  $\kappa = \lambda_H^2$  for the rhomboidal and square domains. The other parameters, b, c and g, in the asymptotic form (2) may differ from problem to problem.

In all cases we used the method of series extension to obtain further approximate terms from the known exact terms by using differential approximants to predict further coefficients, as described in Appendix C. How many further terms can be obtained varies from problem to problem and in each case we take all predicted coefficients whose spread among estimates (as measured by 1 standard deviation) is less then 1 part in  $10^5$ . In this way we expect the least accurate coefficients to be accurate to around 1 part in  $10^5$ . As a consequence, we expect that simple ratio plots will be smooth and indistinguishable from those obtained by the exact coefficients. However when we use more elaborate calculations, such as extrapolating against a polynomial in 1/L, that operation magnifies the errors. This is made manifest by smooth plots starting to display irregularities. Accordingly, we cut off such values, and don't use these less accurate coefficients in those plots. To be more specific, if we extend a series by, say, 60 terms, we will use them all in a ratio plot, but when fitting to say,  $c_0 + c_1/L + c_2/L^2 + c_3/L^3$ , we may only use the first 30 extra coefficients. Method M2 is particularly sensitive and we could often only make use of as few as 4 of the approximate terms.

## 5.1. Self-avoiding walks crossing a triangle.

The paths we are counting are shown in Figure 15. Using the algorithm described in Section 2 we calculated  $C_L(1)$  to lattice size L=27 and we then used the method of series extension to obtain a further 60 terms. We first estimated  $\lambda_H$  by method M1, that is extrapolating the sequence  $\lambda_L = C_L(1)^{1/L^2}$  against 1/L. There was some curvature in the plot, so we extrapolated against  $c_0 + c_1/L + \cdots + c_m/L^m$ , which allowed us to make a rather precise estimate,  $\lambda_H = 1.3872495 \pm 0.0000005$ . We show, in Figure 16, just how well-converged this data is. We next considered the sequence  $\{C_L\}$  which plotted against  $1/L^2$  is an almost straight line. We then fitted the sequence to  $c_0 + c_2/L^2 + c_3/L^3$ . This gave exceptionally good apparent precision, allowing for a very precise estimate. We estimate  $\lambda_H^2 = 1.9244612 \pm 0.0000002$ , or  $\lambda_H = 1.38724951 \pm 0.00000001$ . The plots are shown in Figure 17.

We estimated the values of the sub-dominant terms by method P2, fitting successive coefficients to

$$\log d_L \sim b \log(\lambda_H) L + c \log(\lambda_H) + g \log L$$

and we estimated  $-g\lambda_H^2$  from the cubic fit to the sequence  $\{C_L\}$ . The relevant plots are shown in Figure 18. In this way we estimate  $b \approx 0.4443$ ,  $c \approx 0.924$ ,  $g \approx 0.0834$ , and  $-g\lambda_H^2 \approx -0.1602$ , so  $g \approx 0.0832$ , which is suggestive of the exact fraction 1/12. This exponent value was then used in method P1 from which we estimate  $b \approx 0.4442$  and  $c \approx 0.9214$  in good agreement with the results of method P2.

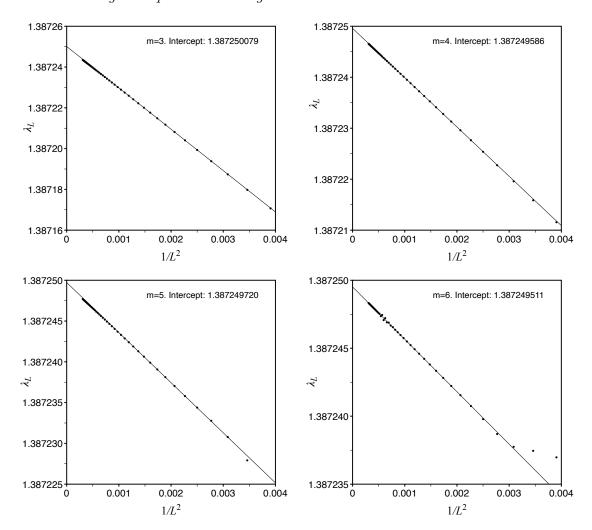


Figure 16: Estimators of  $\lambda_H$  from method M1 when fitting against polynomials in 1/L of degree m=3 to 6 for SAWs crossing a triangle.

Finally we display in Figure 19 the results from a biased differential approximant analysis of  $\mathcal{R}(z)$ . The biased estimates of  $\gamma$  cross the value -1 in a very narrow range very close to our estimate  $\lambda_H \approx 1.38724951$  from the previous analysis.

We therefore conclude that for SAWs crossing a triangular domain of the hexagonal lattice we have found very firm numerical evidence that the conjectured asymptotic form (2) is correct and we estimate that the parameters have the values  $\lambda_H = 1.38724951 \pm 0.00000005$ ,  $b = 0.4443 \pm 0.001$ ,  $c = 0.923 \pm 0.005$ , and  $g = 0.0833 \pm 0.0005$ , where possibly g = 1/12 exactly.

## 5.2. Self-avoiding walks crossing a rhombus.

The paths we are counting are shown in Figure 15. We calculated  $C_L(1)$  to lattice size L=26 and then extended this sequence by a further 50 terms. We first estimated  $\lambda_H^2$  using method M1 by extrapolating against  $c_0 + c_1/L + \cdots + c_m/L^m$ , as shown in Figure 20 for m=3 to 6. From this we estimate that  $\lambda_H^2 = 1.924461 \pm 0.000002$ , or

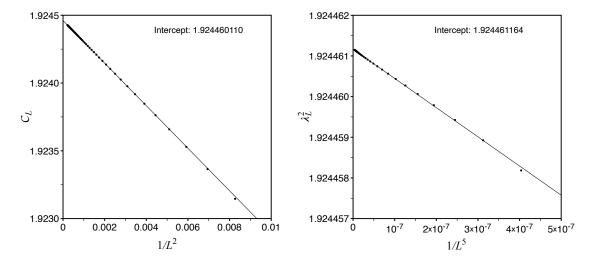


Figure 17:  $C_L$  plotted against  $1/L^2$  and the estimator  $\lambda_L^2$  from method M2 with a cubic fit plotted against  $1/L^5$  for SAWs crossing a triangle.

 $\lambda_H = 1.3872494 \pm 0.000008$ . Next we used method M2 to obtain an estimate for  $\lambda_H^4$ . In Figure 21 we show a plot of  $\mathcal{C}_L$  plotted against  $1/L^2$  and the estimates obtained by fitting the sequence  $\{\mathcal{C}_L\}$  to  $c_0 + c_2/L^2 + c_3/L^3$ . We estimate  $\lambda_H^4 = 3.7035506 \pm 0.0000006$ , or  $\lambda_H = 1.38724948 \pm 0.00000006$ . These estimates for  $\lambda_H$  are consistent with the estimate obtained above for the triangular domain.

We estimated the values of the sub-dominant terms by method P2, fitting successive coefficients to

$$\log d_L \sim 2b \log(\lambda_H) L + 2c \log(\lambda_H) + g \log L$$

and we estimated  $-g\lambda^2$  from the cubic fit to the sequence  $\{C_L\}$ . The relevant plots are shown in Figure 22. We estimate  $b \approx -0.3705$ ,  $c \approx 0.6258$ ,  $g \approx 0.167$ , and  $-g\lambda_H^2 \approx -0.615$ , so  $g \approx 0.166$ , which is suggestive of the exact fraction 1/6. From method P1 we then obtained the estimates  $b \approx -0.3707$  and  $c \approx 0.6266$ .

Finally we display in Figure 23 the results from a biased differential approximant analysis of  $\mathcal{R}(z)$ . Once again we see that the biased estimates of  $\gamma$  cross the value -1 in a narrow range contained within our best estimate  $\lambda_H \approx 1.38724951 \pm 0.00000005$ .

Hence, SAWs crossing a rhomboidal domain of the hexagonal lattice follows the conjectured asymptotic form (2) with growth constant  $\lambda_H^2$  and sub-dominant parameters  $b = -0.3706 \pm 0.0005$ ,  $c = 0.6262 \pm 0.001$ , and  $g = 0.167 \pm 0.002$ , where possibly g = 1/6.

#### 5.3. SAWs spanning a rhombus

We have series to lattice size L=26 and we managed to obtain a further 37 approximate terms. Method M1 with a degree six polynomial extrapolation allowed us to make the estimate  $\lambda_H^2=1.92446\pm0.00003$ , or  $\lambda_H=1.38725\pm0.00001$ . From method M2 and P3 with a cubic fit we estimate  $\lambda_H^4=3.703551\pm0.000005$ , or  $\lambda_H=1.3872495\pm0.0000005$ , and  $-g\lambda_H^4\approx-6.18$ , so  $g\approx1.67$ . From method P2 we estimate  $b\approx-0.3705$ ,  $c\approx1.44$ ,

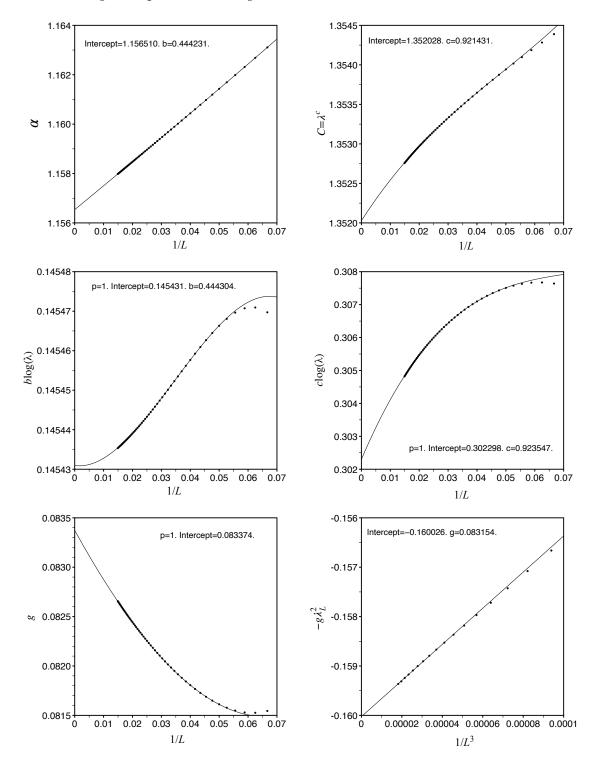


Figure 18: Estimators of  $\alpha$  and  $C = \lambda^c$  from method P1,  $b \log \lambda_S$ ,  $c \log \lambda_S$ , and g from method P2, and the estimator  $-g\lambda_L^2$  from method P3 for SAWs crossing a triangle.

and  $g \approx 1.675$ , in precise agreement with the estimate from method P3. We suggest that perhaps  $g = \frac{5}{3}$ . Method P1 then yielded the estimates  $b \approx -0.3706$  in agreement with the previous estimate and  $c \approx 1.56$  somewhat large but still consistent with the estimate

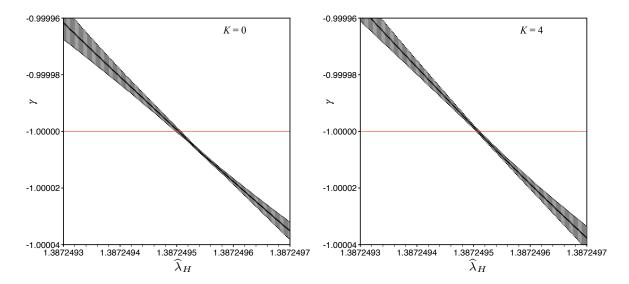


Figure 19: Biased estimates for the critical exponent  $\gamma$  of  $\mathcal{R}(z)$  plotted against the biasing value  $\widehat{\lambda}_H$  for SAWs crossing a triangle.

from method P2. The plots can be seen in Figure E2.

## 5.4. Polygons crossing a rhombus

We calculated series to lattice size L=26 and extended by 60 approximate terms. Method M1 yielded the estimate  $\lambda_H^2=1.92446\pm0.00001$  ( $\lambda_H=1.387249\pm0.000005$ ) and method M2 and P3  $\lambda_H^4=3.7035515\pm0.0000015$  ( $\lambda_H=1.38724956\pm0.00000015$ ) and  $-g\lambda_H^2\approx2.157$ , so  $g\approx-0.583$ , where in each method we used a cubic extrapolation of the sequence. This is clear evidence that the growth parameter  $\lambda_H$  for polygons is the same as for SAWs, which is to be expected. Method P2 was again used to estimate the values of the sub-dominant terms and we estimate  $b\approx-0.3705$ ,  $c\approx-1.0543$ , and  $g\approx-0.583$ , in agreement with the estimate from method P3. We hazard the guess that g=-7/12, exactly. From method P1 we then estimated  $b\approx-0.3705$  and  $c\approx-1.0529$ . The plots can be seen in Figure E3.

# 5.5. Self-avoiding walks crossing a triangle and passing through the top vertex.

We calculated  $C_L(1)$  to lattice size L=26 and extended the series by a further 60 approximate terms. We estimated  $\lambda_H$ , by method M1 where an extrapolation of degree six allowed us to estimate  $\lambda_H=1.3872495\pm0.0000005$ . Using method M2 we estimated  $\lambda_H^2=1.9244611\pm0.0000001$  ( $\lambda_H=1.38724947\pm0.00000005$ ) and  $-g\lambda_H^2\approx-0.160$ , so  $g\approx0.0831$ , in agreement with the value found for SAWs crossing a triangle. We estimated the values of the sub-dominant parameters by method P2 and we found  $b\approx0.4443$ ,  $c\approx-1.7861$ , and  $g\approx0.0833$ , in good agreement with the estimate given immediately above, and suggestive of an exact fraction 1/12. Method P1 yielded the estimates  $b\approx0.4442$  and  $c\approx-1.7891$  in agreement with the previous estimates. The

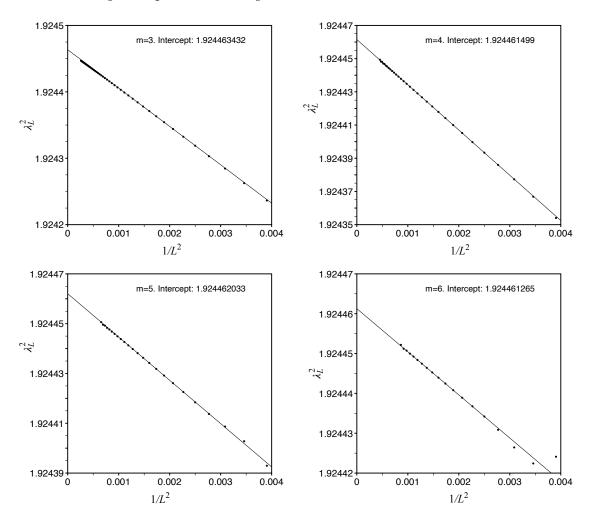


Figure 20: Estimators of  $\lambda_H$  from method M1 when fitting against polynomials in 1/L of degree m=3 to 6 for SAWs crossing a rhombus.

plots can be seen in Figure E4.

#### 5.6. Polygons in a triangle.

We define  $P_L(1)$  as the number of polygons in a triangular domain passing through two of the three corner vertices as shown in Figure 15. We calculated series to lattice size L=26 and obtained a further 50 approximate terms. We first estimated  $\lambda_H$ , by method M1 and found  $\lambda_H=1.387245\pm0.000002$ . Method M2 gave good apparent precision, giving us the precise estimate  $\lambda_H^2=1.924461\pm0.000001$  ( $\lambda_H=1.3872494\pm0.0000004$ ) and from method P3 we found  $-g\lambda_H^2\approx1.282$ , so  $g\approx-0.666$ , which is very suggestive of the exact fraction -2/3. We estimated the values of the sub-dominant terms from method P2 and found  $b\approx0.4443$ ,  $c\approx-1.394$ , and  $g\approx-0.666$ , in total agreement with the estimate of g given immediately above. Method P1 gave  $b\approx0.4443$  and  $c\approx-1.380$  in good agreement with the previous analysis. The plots can be seen in Figure E5.

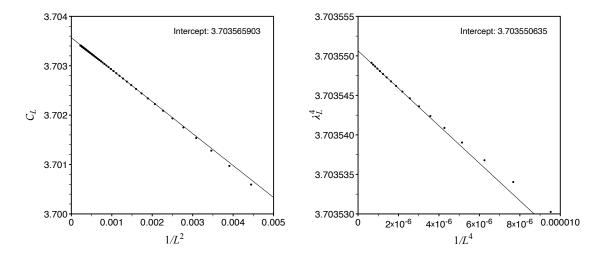


Figure 21:  $C_L$  plotted against  $1/L^2$  and the estimator  $\lambda_L^4$  from method M2 with a cubic fit plotted against  $1/L^4$  for SAWs crossing a rhombus.

# 5.7. Polygons in a triangle passing through the top vertex.

We define  $P_L(1)$  as the number of polygons in a triangular domain passing through all three corner vertices of the domain as illustrated in Figure 15. The series was calculated to lattice size L=26 and extended by 40 further approximate terms. Method M1 yielded the estimate  $\lambda_H=1.38725\pm0.00001$  and method M2 gave  $\lambda_H^2=1.924461\pm0.000002$  ( $\lambda_H=1.38725\pm0.00001$ ) while method P3 gave  $-g\lambda_H\approx1.284$ , so  $g\approx-0.667$ . Method P2 resulted in the estimates  $b\approx0.4443$ ,  $c\approx-4.106$ , and  $g\approx-0.667$ . We conjecture g=-2/3 exactly. Method P1 resulted in the estimates  $b\approx0.4443$  and  $c\approx-4.091$  in agreement with the previous results. The plots can be seen in Figure E6.

#### 5.8. SAWs crossing a square

The paths we are counting are shown in Figure 15. We calculated series to lattice size L=24 and we extended the series by a further 25 approximate terms. A consequence of the lattice geometry is that different paths had to be counted according as the lattice size L was odd or even, as shown in Figure 15. This induced a period-2 oscillation in the ratios and other parameters. To accommodate this we redefined the ratios as the square-root of the ratio of alternate terms. That is to say, the ratio  $r_L = \sqrt{C_L(1)/C_{L-2}(1)}$ . Similarly, when attempting to extrapolate the sequence  $\lambda_L = C_L(1)^{1/L^2}$  against a polynomial in 1/L we used tuples of alternate terms, rather than successive terms. Even after this adjustment the estimates of  $\lambda_L$  showed some parity effects. Hence we decided to look at the average of consecutive terms, that is,  $(\lambda_L + \lambda_{L-1})/2$ . Similar changes were made for all the other parameter estimators. The resulting plots are shown in Figure E7

This allowed us to make the precise estimate  $\lambda_H^2 = 1.924461 \pm 0.000005$  ( $\lambda_H = 1.387249 \pm 0.000003$ ) from a cubic fit to the sequence  $\{C_L(1)^{1/L^2}\}$ . When we fitted the sequence  $\{C_L\}$  to  $c_0 + c_2/L^2 + c_3/L^3$ , strong period-2 oscillations required a redefinition,

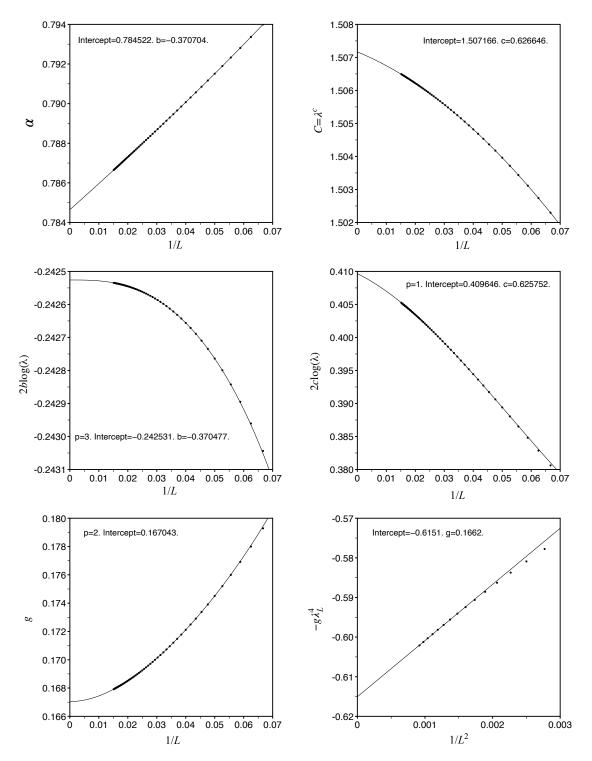


Figure 22: Estimators of  $\alpha$  and  $C = \lambda^c$  from method P1,  $b \log \lambda_S$ ,  $c \log \lambda_S$ , and g from method P2, and the estimator  $-g\lambda_L^2$  from method P3 for SAWs crossing a rhombus.

so we defined

$$\mathcal{C}_L^* := \left(\frac{C_{L-2}(1) \cdot C_{L+2}(1)}{C_L(1)^2}\right)^{1/4}.$$

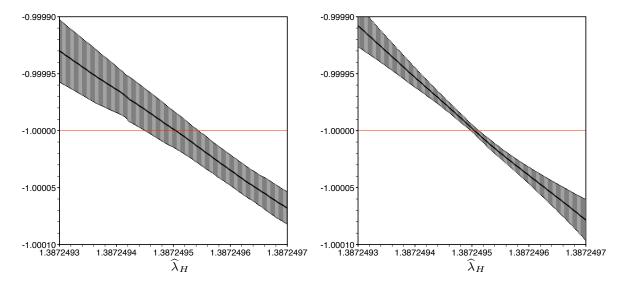


Figure 23: Biased estimates for the critical exponent  $\gamma$  of  $\mathcal{R}(z)$  plotted against the biasing value  $\widehat{\lambda}_H$  for SAWs crossing a rhombus.

As was the case for WCAS this redefined sequence of ratios showed linearity when plotted against  $1/L^4$  hence suggesting that g=0 in this case as well. We therefore extrapolated the new sequence against  $c_0 + c_4/L^4$ , and from a plot against  $1/L^7$  we made the estimate  $\lambda_H^4 = 3.7035505 \pm 0.0000015$ , or  $\lambda_H = 1.3872495 \pm 0.0000002$ .

We also estimated the values of the sub-dominant terms by method P2 appropriately altered to deal with parity effects. In that way we estimated  $b \approx -0.3765$ ,  $c \approx 0.736$ , and  $g \approx 0.003$ , in agreement with g = 0. Finally, we used also method P3 to estimate g. Here we fitted the sequence  $\{\mathcal{C}_L^*\}$  to  $c_0 + c_2/L^2 + c_4/L^4$ , so that  $c_2$  becomes an estimator for  $-g\lambda_H^4$  and from the plot we estimate that  $-g\lambda_H^4 \approx -0.00240$  which again is consistent with the conjecture that g = 0 exactly.

#### 6. Conclusion

For SAWs crossing a square on the square lattice, we conjecture that

$$C_L(1) \sim \lambda_S^{L^2 + bL + c} \cdot L^g$$
,

where  $\lambda_S = 1.7445498 \pm 0.0000012$ ,  $b = -0.04354 \pm 0.0001$ ,  $c = 0.5624 \pm 0.0005$ , and  $g = 0.000 \pm 0.005$ .

For SAWs crossing a closed, connected, convex region on the hexagonal lattice we similarly conjecture  $C_L(1) \sim \lambda_H^{L^2}$ , where our best estimate of  $\lambda_H = 1.38724951 \pm 0.00000005$ . For a number of combinatorial problems associated with SAWs on the hexagonal lattice, the growth constant is either known or conjectured. We have not been able to even guess a potential algebraic expression for  $\lambda_H$  that is remotely plausible.

We show in Table 4 our estimates of the parameters b, c, and g for the various geometries and path types we have studied, as well as the conjectured exact values of

Table 4: Estimates of the parameters b, c, and g when fitting to the assumed asymptotic form  $C_L(1) \sim \lambda_S^{L^2+bL+c} \cdot L^g$ , for the square lattice,  $C_L(1) \sim \lambda_H^{2(L^2+bL+c)} \cdot L^g$ , for the hexagonal lattice on non-triangular domains, and  $C_L(1) \sim \lambda_H^{L^2+bL+c} \cdot L^g$ , on triangular domains.

Geometry and lattice	b	c	g and conjecture	
Square lattice				
SAWs crossing a square	-0.04354	0.5624	0	0
SAWs spanning a square	-0.04354	0.5	1.75	7/4
SAPs crossing a square	-0.04354	-1.197	-0.5000	-1/2
Hexagonal lattice				
SAWs crossing a rhombus	-0.3706	0.626	0.167	1/6
SAWs spanning a rhombus	-0.3704	1.78	1.667	5/3
SAPS crossing a rhombus	-0.3705	-1.052	-0.583	-7/12
SAWs crossing a triangle	0.4443	0.923	0.0833	1/12
SAWs crossing a triangle + top vertex	0.4443	-1.787	0.0833	1/12
SAPs crossing a triangle	0.4444	-1.387	-0.666	-2/3
SAPs crossing a triangle + top vertex	0.4443	-4.10	-0.667	-2/3
SAWs crossing a square	-0.3765	0.736	0.003	0

the exponent g. For the hexagonal lattice, it is seen that the parameter b takes one of two values. The value  $b \approx -0.3705$  is associated with the rhomboidal geometry, while the value  $b \approx 0.4444$  is associated with the triangular geometry. All the exponents g appear to be multiples of 1/12.

#### 7. Resources

The enumeration data and extended series for all problems studied in this paper, some Maple worksheets used for the asymptotic analysis and some of the source code used to calculate the exact coefficients can be found at our GitHub repository https://github.com/IwanJensen/Self-avoiding-walks-and-polygons/tree/WCAS(H).

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## Appendix A. Ratio Method

The ratio method was perhaps the earliest systematic method of series analysis employed, and is still the most useful method when only a small number of terms are known. Given a series  $\sum c_n z^n$ , which behaves as in eqn. (7), it is assumed that  $\lim_{n\to\infty} c_n/c_{n-1}$  exists and is equal to the growth constant. For some combinatorial sequences such as classical pattern-avoiding permutations of length up to 5, this has been proved by Atapour and Madras [17].

From eqn. (8), it follows that the ratio of successive terms

$$r_n = \frac{c_n}{c_{n-1}} = \frac{1}{z_c} \left( 1 + \frac{\gamma - 1}{n} + o\left(\frac{1}{n}\right) \right).$$
 (A.1)

It is then natural to plot the successive ratios  $r_n$  against 1/n. If the correction terms  $o(\frac{1}{n})$  can be ignored<sup>‡</sup>, such a plot will be linear, with gradient  $\frac{\gamma-1}{z_c}$ , and intercept  $\mu=1/z_c$  at 1/n=0.

Linear intercepts  $l_n$  eliminate the  $O\left(\frac{1}{n}\right)$  term in eqn. (A.1), so in the case of a pure power-law singularity, one has

$$l_n := nr_n - (n-1)r_{n-1} = \mu \left(1 + \frac{c}{n^2} + O\left(\frac{1}{n^3}\right)\right).$$

Various refinements of the method can be readily derived. If the critical point is known exactly, it follows from eqn. (A.1) that estimators of the exponent  $\gamma$  are given by

$$\gamma_n := n(z_c \cdot r_n - 1) + 1 = \gamma + o(1).$$

If the critical point is not known exactly, one can still estimate the exponent  $\gamma$ . From eqn. (A.1) it follows that

$$\delta_n := 1 + n^2 \left( 1 - \frac{r_n}{r_{n-1}} \right) = \gamma + o(1).$$
 (A.2)

Similarly, if the exponent  $\gamma$  is known, estimators of the growth constant  $\mu$  are given by

$$\mu_n = \frac{nr_n}{n+\gamma-1} = \mu + o(1/n).$$

#### Appendix B. Differential approximants

The generating functions of some problems in enumerative combinatorics are sometimes algebraic, such as that for Av(1342) pattern-avoiding permutations, sometimes D-finite, such as with Av(12345) pattern-avoiding permutations, sometimes differentially algebraic, and sometimes transcendentally transcendental. The not infrequent occurrence of D-finite solutions was the origin of the method of differential approximants, a very successful method of series analysis for analysing power-law singularities [18].

‡ For a purely algebraic singularity eqn. (7), with no confluent terms, the correction term will be  $O(\frac{1}{n^2})$ .

The basic idea is to approximate a generating function F(z) by solutions of differential equations with polynomial coefficients. That is to say, by D-finite ODEs. The singular behaviour of such ODEs is well documented (see e.g. [19, 20]), and the singular points and exponents are readily calculated from the ODE.

The key point for series analysis is that even if *globally* the function is not describable by a solution of such a linear ODE (as is frequently the case) one expects that *locally*, in the vicinity of the (physical) critical points, the generating function is still well-approximated by a solution of a linear ODE, when the singularity is a generic power law (7).

An  $M^{th}$ -order differential approximant (DA) to a function F(z) is formed by matching the coefficients in the polynomials  $Q_k(z)$  and P(z) of degree  $N_k$  and K, respectively, so that the formal solution of the  $M^{th}$ -order inhomogeneous ordinary differential equation

$$\sum_{k=0}^{M} Q_k(z) \left( z \frac{\mathrm{d}}{\mathrm{d}z} \right)^k \tilde{F}(z) = P(z)$$
(B.1)

agrees with the first  $N = K + \sum_{k} (N_k + 1)$  series coefficients of F(z).

Constructing such ODEs only involves solving systems of linear equations. The function  $\tilde{F}(z)$  thus agrees with the power series expansion of the (generally unknown) function F(z) up to the first N series expansion coefficients. We normalise the DA by setting  $Q_M(0)=1$ , thus leaving us with N rather than N+1 unknown coefficients to find. The choice of the differential operator  $z\frac{\mathrm{d}}{\mathrm{d}z}$  in (B.1) forces the origin to be a regular singular point. The reason for this choice is that most lattice models with holonomic solutions, for example, the free-energy of the two-dimensional Ising model, possess this property. However this is not an essential choice.

From the theory of ODEs, the singularities of  $\tilde{F}(z)$  are approximated by zeros  $z_i$ ,  $i=1,\ldots,N_M$  of  $Q_M(z)$ , and the associated critical exponents  $\gamma_i$  are estimated from the indicial equation. If there is only a single root at  $z_i$  this is just

$$\gamma_i = M - 1 - \frac{Q_{M-1}(z_i)}{z_i Q'_M(z_i)}. (B.2)$$

Estimates of the critical amplitude C are rather more difficult to make, involving the integration of the differential approximant. For that reason the simple ratio method approach to estimating critical amplitudes is often used, whenever possible taking into account higher-order asymptotic terms [21].

Details as to which approximants should be used and how the estimates from many approximants are averaged to give a single estimate are given in [21]. Examples of the application of the method can be found in [22]. In that work, and in this, we reject so-called *defective* approximants, typically those that have a spurious singularity closer to the origin than the radius of convergence as estimated from the bulk of the approximants. Another method sometimes used is to reject outlying approximants, as judged from a histogram of the location of the critical point (i.e. the radius of convergence) given by the DAs. It is usually the case that such distributions are bell-shaped and rather

symmetrical, so rejecting approximants beyond two or three standard deviations is a fairly natural thing to do.

## Appendix B.1. Biased differential approximants

If the critical point  $z_c$  is known exactly (or very accurately) one may try to obtain improved numerical estimates for the exponents by forcing the differential equation (B.1) to have a singular point at  $z_c$ , that is one may look at biased differential approximants. In [23] we developed a new method in which we form biased approximants by multiplying the derivatives in (B.1) by appropriate "biasing polynomials". This allows us to bias in such a manner that the singularity at  $z_c$  is of order  $q \leq K$ . Let

$$F_k(z) = \left(z \frac{\mathrm{d}}{\mathrm{d}z}\right)^k F(z) \quad \text{and} \quad G_k(z) = (1 - z/z_c)^{q_k} F_k(z), \tag{B.3}$$

where  $q_k = \max(q + k - M, 0)$ . With this definition we have that  $G_k = (1 - z/z_c)^q F_k(z)$ , while subsequent lower order derivatives have "biasing polynomials" of degree decreasing in steps of 1 (until 0). Then we form biased differential approximants (BDA) such that

$$P(z) + \sum_{k=0}^{M} \widehat{Q}_k(z)G_k(z) = O(z^{N+1}).$$
(B.4)

For biased approximants the degree of the polynomial multiplying the k'th derivative still have degree  $N_k$  such that the degrees of  $\widehat{Q}_k(x) = N_k - q_k$  and the number of unknown coefficients is  $\widehat{N} = K + 1 + \sum_k (N_k - q_k + 1)$ .

#### Appendix C. Coefficient prediction

In [16] we showed that the ratio method and the method of differential approximants work serendipitously together in many cases, even when one has stretched exponential behaviour, in which case neither method works particularly well in unmodified form.

To be more precise, the method of differential approximants (DAs) produces ODEs which, by construction, have solutions whose series expansions agree term by term with the known coefficients used in their construction. Clearly, such ODEs implicitly define all coefficients in the generating function, but if N terms are used in the construction of the ODE, all terms of order  $z^N$  and beyond will be approximate, unless the exact ODE is discovered, in which case the problem is solved, without recourse to approximate methods.

It is useful to construct a number of DAs that use all available coefficients, and then use these to predict subsequent coefficients. Not surprisingly, if this is done for a large number of approximants, it is found that the predicted coefficients of the term of order  $z^n$ , where n > N, agree for the first k(n) digits, where k is a decreasing function of n. We take as the predicted coefficients the mean of those produced by the various DAs, with outliers excluded, and as a measure of accuracy we take the number of digits for

which the predicted coefficients agree, or the standard deviation. These two measures of uncertainty are usually in reasonable agreement.

Now it makes no logical sense to use the approximate coefficients as input to the method of differential approximants, as we have used the DAs to obtain these coefficients. However there is no logical objection to using the (approximate) predicted coefficients as input to the ratio method. Indeed, as the ratio method, in its most primitive form, looks at a graphical plot of the ratios, an accuracy of 1 part in  $10^4$  or  $10^5$  is sufficient, as errors of this magnitude are graphically unobservable.

Recall that, in the ratio method one looks at ratios of successive coefficients. We find that the ratios of the approximate coefficients are predicted with even greater precision than the coefficients themselves by the method of DAs. That is to say, while a particular coefficient and its successor might be predicted with an accuracy of 1 part in  $10^p$  for some value of p, the ratio of these successive coefficients is frequently predicted with significantly greater accuracy (the precision being typically improved by a factor varying between 2 and 20).

The DAs use all the information in the coefficients, and are sensitive to even quite small errors in the coefficients. As an example, in a recent study of some self-avoiding walk series, an error was detected in the eighteenth significant digit in a new coefficient, as the DAs were much better converged without the last, new, coefficient§. The DAs also require high numerical precision in their calculation. In favourable circumstances, they can give remarkably precise estimates of critical points and critical exponents, by which we mean up to or even beyond 20 significant digits in some cases. Surprisingly perhaps, this can be the case even when the underlying ODE is not D-finite. Of course, the singularity must be of the assumed power-law form.

Ratio methods, and direct fitting methods, by contrast are much more robust. The sort of small error that affects the convergence of DAs would not affect the behaviour of the ratios, or their extrapolants, and would thus be invisible to them. As a consequence, approximate coefficients are just as good as the correct coefficients in such applications, provided they are accurate enough. We re-emphasise that, in the generic situation (7), ratio type methods will rarely give the level of precision in estimating critical parameters that DAs can give. By contrast, the behaviour of ratios can more clearly reveal features of the asymptotics, such as the fact that a singularity is not of power-law type. This is revealed, for example, by curvature of the ratio plots [22].

As an example, consider the OGF for Av(12453) PAPs (see OEIS [5] A116485). This is known to order  $x^{38}$ . Let us take the coefficients to order  $x^{16}$  and use the method of series extension described above to predict the next 22 ratios, so that we can compare them to the exact ratios. The results, based on 3rd order differential approximants, are

 $<sup>\</sup>S$  Given 69 terms of the square-lattice self-avoiding walk series, the 70th term is predicted by 4th order ODEs to be  $4190893020903935057 \times 10^{12}$ . The actual coefficient is 4190893020903935054619120005916, which differs in the nineteenth digit. An error in the eighteenth digit was thus discovered during development. Several other less dramatic examples are known where lower-order errors have been discovered by this means.

shown in Table C1. For the first predicted ratio,  $r_{18}$ , the discrepancy is in the 10th significant digit. For the last predicted ratio,  $r_{39}$ , the error is in the 5th significant digit. This level of precision is perfectly adequate for ratio analysis.

Table C1: Ratios  $r_{18}$  to  $r_{39}$  actual and predicted from the coefficients of Av(12453), with percentage error shown.

Predicted ratios	Actual ratios	Percentage error
10.654655347	10.65465504	$4.78 \times 10^{-7}$
10.828226522	10.82822539	$1.04 \times 10^{-5}$
10.986854456	10.98685140	$2.79 \times 10^{-5}$
11.132386843	11.13238007	$4.78 \times 10^{-5}$
11.266382111	11.26636895	$6.08 \times 10^{-5}$
11.390163118	11.39013998	$2.03 \times 10^{-4}$
11.504857930	11.50482182	$3.14 \times 10^{-4}$
11.611441483	11.61138359	$4.99 \times 10^{-4}$
11.710743155	11.71066190	$6.94 \times 10^{-4}$
11.803496856	11.80338255	$9.68 \times 10^{-4}$
11.890333733	11.89017822	$1.31\times10^{-3}$
12.048402545	12.04814337	$2.15 \times 10^{-3}$
12.120553112	12.12022972	$2.67\times10^{-3}$
12.188650126	12.18824275	$3.34 \times 10^{-3}$
12.252994715	12.25252103	$3.87\times10^{-3}$
12.313939194	12.31336663	$4.65 \times 10^{-3}$
12.371707700	12.37104982	$5.32\times10^{-3}$
12.426619450	12.42581319	$6.49 \times 10^{-3}$
12.478784843	12.47787509	$7.29\times10^{-3}$
12.528486946	12.52743256	$8.41 \times 10^{-3}$

In practice we find that the more exact terms we know, the greater is the number of predicted terms, or ratios that can be predicted.

## Appendix D. Enumeration data

```
C_L(1)
95
2320
154259
30549774
17777600753
30283708455564
152480475641255213
2287842813828061810244
102744826737618542833764649
13848270995235582268846758977770\\
5613766870113075134552249300590982081\\
6856324633418315229580098999727214234534626\\
25264653780547704599613926971040640439380254497299\\
281194924965510769640501069703642937039678809002355743600\\
9461739046646537749639494171503923182753987897972167546351180871\\
13803603811254425104633152972993523761617439474917134222103400574517678544806707098426335287312812055811653812588064999045835964788\\
```

Table D1: Number of SAWs spanning a square.

```
P_L(1)
  3
  42
  1799
  232094
  92617031
  115156685746
  442641690778179
  5224287477491915786
  188825256606226776728029\\
11
  20879416139356164466643759334
  7057757437924198729598570424130207\\
  7287699030020917172151307665469211016474\\
  22973720258279267139936821063450448822110219653\\
  220999541336018343231658363621596453585823579325485544\\
  6485093759718494344865537501691711476194821918864090506157759
  580338710138214792049192419944468721379579881619954352303395183377868
  1281707896370751708653066922805265028882836851074044433082078379196572742914435468007626647333767206265847516495713522985546806840650483671342846200191630108286969\\
```

Table D2: Number of SAPs crossing a square.

```
14
25092
7374480
8029311942
32223151155864
476605408516689238
26016526700583361056456\\
5246595079903462547245876694
3911053741699230141571030313824664\\
10780907768757190963361134040036893772360\\
109919900687141309301630828947780890728732496678\\
4146148169372563020871034877194447551275644544417216784
578668580332775727107695799371628560927178835729875790606922120\\
4892542075116215747349775890169094456449789602921450060431267745393588411359934920766964621175270271453676206611892541512628195569791000
```

Table D3: Number of SAWs crossing a rhomboidal domain of the hexagonal lattice.

L	$C_L(1)$
1	2
2	50
3	2256
4	292006
5	124394172
6	182189852062
7	937116505296162
8	17167376550995687961
9	1130911800993488803731078
10	269650395624478266477331223678
11	233772496350603982679550385266064014
12	739330863241806743025423160490836132227125
13	8551000409049037000098287028025432585191736309022
14	362378501157171575915086740862352731989136965188978227480
15	5635516488885592354051749345529297798069126440063716209024866536
16	32200232301152973892060847293393239105831802930525492217459523426803019578
17	67665662468515970834966508500944029204762050650693102413477819738278462353187499568
18	523379303813002076273464810690096008845689319359263297454993915567005968614237413818526075604
19	14910759530495548949623554019916848509888902630562528597658761833271654794704911307455162596911430188758
20	1565552766529028680644951163416182891619237381422347104413735417263587336683056846547108571311383043144417391243521
21	606084065190103550545340197138093542241444175895240820944713102199472484407589836051473378294437969118344993099132112988392420
22	865520866516174852434302085316123704413013184383267585803777199791333436065070398518809805939514793360791184628285127191002600593996365368
23	4560992075553129850922927762995312993575376533697147813417446333497150777368818523814441027249715528879466279022480571252085818753070411144055018204876
$^{24}$	88718729299059562850997307819335993122314801341423818394843508516540284335394844383553598429174621313143090882997603769027577322213964104279601294505719147199388248
25	6371850587510704465849294714166605694358498327959158268953676175064720102088107446652503452442438514332661839511396033679344990952004170416492446601538481306356885482155377886638
26	169011336127263808956441260014789508545122612515869787545534380630798876325920139245100711076793585499650454967955439732124691809189880222254245883916081296002140478524588798551322551117645267311764526731176

Table D4: Number of SAWs spanning a rhomboidal domain of the hexagonal lattice.

```
3126
775842
727870836
2575728525240
34244061451559094
1703999058661009145746\\
316543880488539946466963896
219157996022284922702859434801868
564858713948847373563461482383973674774\\
5415142061627863782256892670635702203299498106\\
192965908859455255222444585453472066280402031983076676\\
25546198443752201604792021828520875111113011948793636471115986\\
12559327077982128401344048554297110314066721517873014754182697036556596\\
```

Table D5: Number of SAPs crossing a rhomboidal domain of the hexagonal lattice.

$\overline{L}$	$C_L(1)$
1	$\overline{2}$
2	7
3	44
4	515
5	11500
6	493704
7	40751496
8	6463642330
9	1970190022696
10	1154437344815284
11	1300686960810345198
12	2818300749120970598426
13	11745284697899678209887246
14	94153940687296424300453605522
15	1451915619132744566900848537333082
16	43072062058620235613855525243039798546
17	2458218787430131938141065342199631011888808
18	269917990612156037679955033913220231218482526540
19	57022048161016261704452967864058833682099233234074924
20	23177397882827812987656054354088621630193659021408496092114
21	18126208865601871898868235390674787298375068592505362074324218782
22	27275828087021466037231281803108531532614036012259410718518383677989994
23	78974101601865877096497572762267816542675600879070694217812459537275320667130
24	439980515324228439963646464930268543060978419686632840124513851873692354257184355418
25	4716606546189621488078969490297265985170243927748285792380749595975920915553704131199964610
26	972922226140204015288756543565253257355329959965239073010766134771324173294845790950440482582207169222261402040152887565435652532573553299599652390730107661347713241732948457909504404825822071692222614020401528875654356525325735532995996523907301076613477132417329484579095044048258220716922226140204015288756543565253257355329959965239073010766134771324173294845790950440482582207169222222222222222222222222222222222222
27	3861740982967126791934974463996504445993431827647538470677158069324943832308988274731817887045190314942500

Table D6: Number of SAWs crossing a triangular domain of the hexagonal lattice.

$\overline{L}$	$C_L(1)$
1	1
2	3
3	18
4	210
5	4716
6	203130
7	16781528
8	2661898722
9	811337884328
10	475395297020430
11	535618774376758222
12	1160567857061063474508
13	4836675324919658534327348
14	38772333263059858336182467950
15	597894854584620490267288203881970
16	17736956492510173648327596231133813426
17	1012287723222402775005385313973408357507928
18	111151484863070215708849728284201214059413569272
19	23481522343431693736560242087640111797935241906792060
20	9544388601505664173784379076794209212239937007395941459026
21	7464322880925069857683897811600948880215514557439627560911154272
22	11232110875321164747567467659828479928446150234247426811308149074039470
23	32521317511278850216940549112361104580618379635763819229016915699625133297104
24	181182764336015552734273130240200423605997687829676784582391379637383247087868602758
25	1942285584539983234933331010286728144642773519634047277599154248174196516886152824735901816912816161111111111111111111111111111
_26	40064669298138196682088095071796367265068180648770697785528635200087726423296089992305061500566756120064669298138196682088095071796367265068180648770697785528635200087726423296089992305061500566756120060606060606060606060606060606060606

Table D7: Number of SAWs crossing a triangular domain of the hexagonal lattice and including the top vertex.

$\overline{L}$	$P_L(1)$
1	1
2	2
3	9
4	85
5	1605
6	59896
7	4392639
8	629739138
9	175745776816
10	95207239875508
11	99934927799315359
12	202993550188918062298
13	797200289814680588454420
14	6048794511036987586252009778
15	88623124229469033988344357343229
16	2506168305598107863294101582119745559
17	136742066892485673488096591777101574684341
18	14391095306419863125025082539141317797920679808
19	2920637571762330449794165953013715565926946586966972
20	1142780121652579092442989213824129363529214905674607409456
21	861928813419640412952428304528142087056944927600343349249100770
22	1252960133060510490994725871202276919994651077934833437111933731780232
23	3509963453723621942826513300378279853247659026894598196945505524358307547596
24	18945984524072416973165104755335799616808372006565339168062614482119446796495592941
25	197032077332349626704638536077733550874900563736415557346148448949082140805149991012506724
26	3947507851539205775146388396017001015202508590957965919271768932077125446293950595857281240459716897168971689716897168971689716897168

Table D8: Number of SAPs crossing a triangular domain of the hexagonal lattice.

$\overline{L}$	$P_L(1)$
1	1
2	1
3	4
4	36
5	666
6	24696
7	1808820
8	259300148
9	72369408510
10	39205936157880
11	41152969216872016
12	83592236529606631688
13	328284931491454739745904
14	2490876950205850778116435156
15	36494758452603010620499864088198
16	1032033208911845667821292289616451218
17	56310006747344597198073248186075772148180
18	5926213428826485611611313527823854932071080074
19	1202710510511720770819662867223620040669484274841448
20	470593707331440145848250079430318880733169905225241510182
21	354939911811827613400027738254513445185773676790950877558157556
22	515965532286678291640886325718842923532551840839177342378988626653078
23	1445393283922054883637378235832608861381031003585207142018132021675532043232
24	7801904249270681046277482881424254681239226301915609070185058428520166740304455480
25	81137266805100512823257637730776600977600011085064069900554442194897045916216667639237206916216667639237206916916916916916916916916916916916916916
26	16255728614134316356915291073389786590743583483816545392748413268216354648804711850294400593468221635464880471185029440059346821600000000000000000000000000000000000

Table D9: Number of SAPs crossing a triangular domain of the hexagonal lattice and including top vertex.

```
14
264
21512
5663596
6478476233
23432328776346
365121393771314359
18039965927005597824652\\
3847346539490622663060402802
2604549807872636495439504536518768
7613280873970130888072912524910312775000\\
70659728324509466176595292882340210105184200002\\
2831956810062815172946024396329723966506233510418891138\\
360424703055912928274223706157781269084968015495478379832577374
198097258016637755765939369950089310341388296845374445597477414443215248
62392663751835087636515340004811611674555874089327041316405089409127243514061643853154930821350090724\\
13524071180124614895872809797043935746289243109268223573969721018213938124696255997302291968299972324147755286357424784253377182127831016598030435648\\
```

Table D10: Number of SAWs crossing a square domain of the hexagonal lattice.

# Appendix E. Supplementary numerical analysis

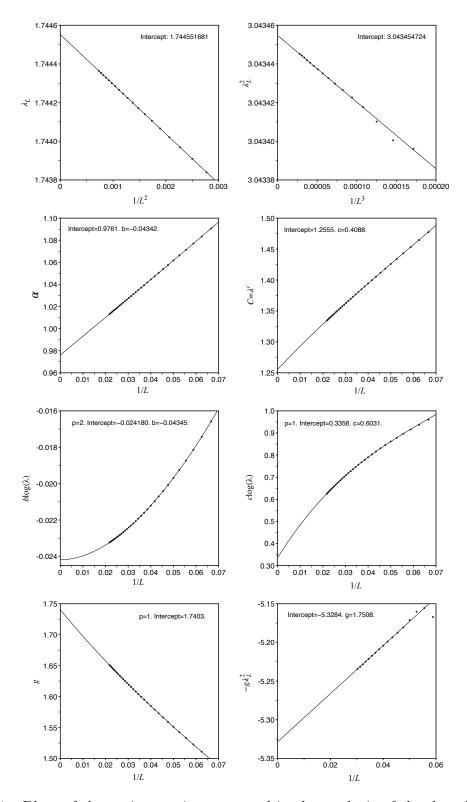


Figure E1: Plots of the various estimators used in the analysis of the data for SAWs spanning a square.

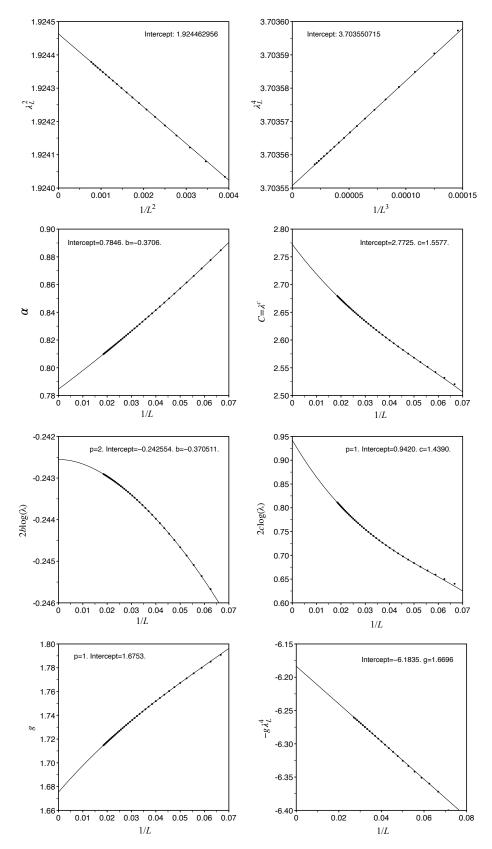


Figure E2: Plots of the various estimators used in the analysis of the data for SAWs spanning a rhomboidal domain of the hexagonal lattice.

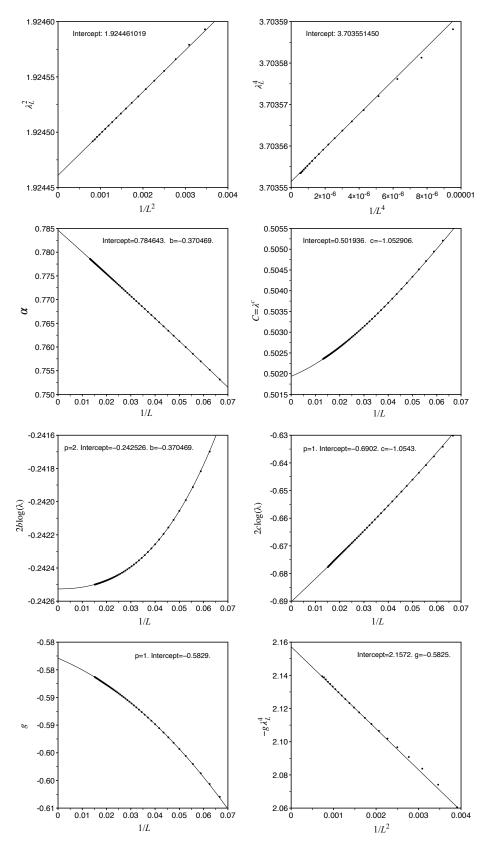


Figure E3: Plots of the various estimators used in the analysis of the data for SAPs crossing a rhomboidal domain of the hexagonal lattice.

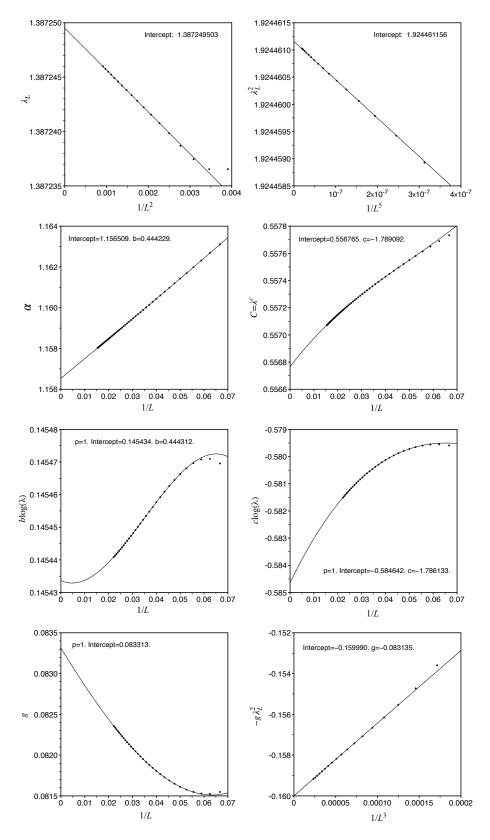


Figure E4: Plots of the various estimators used in the analysis of the data for SAWs crossing a triangular domain of the hexagonal lattice while passing through the topmost vertex.

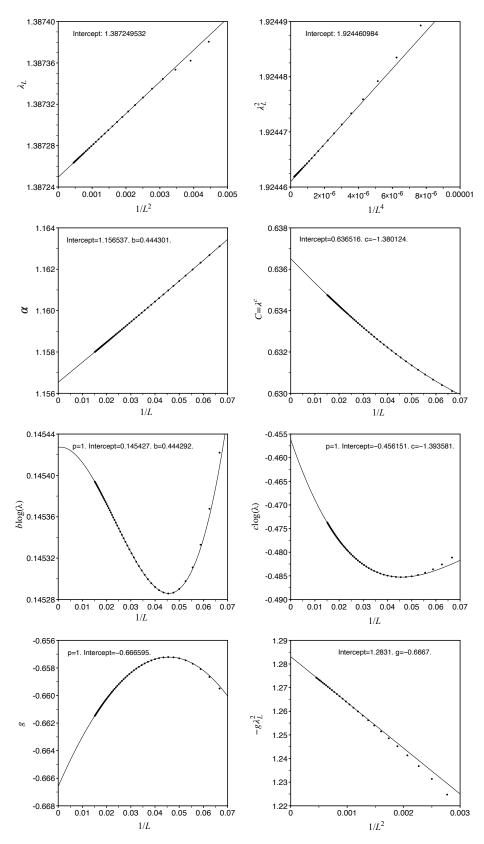


Figure E5: Plots of the various estimators used in the analysis of the data for SAPs crossing a triangular domain of the hexagonal lattice.

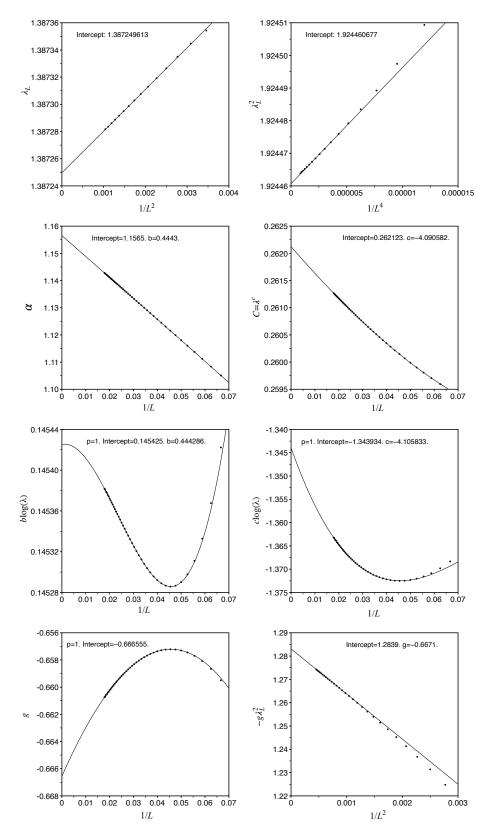


Figure E6: Plots of the various estimators used in the analysis of the data for SAPs crossing a triangular domain of the hexagonal lattice while passing through the topmost vertex.

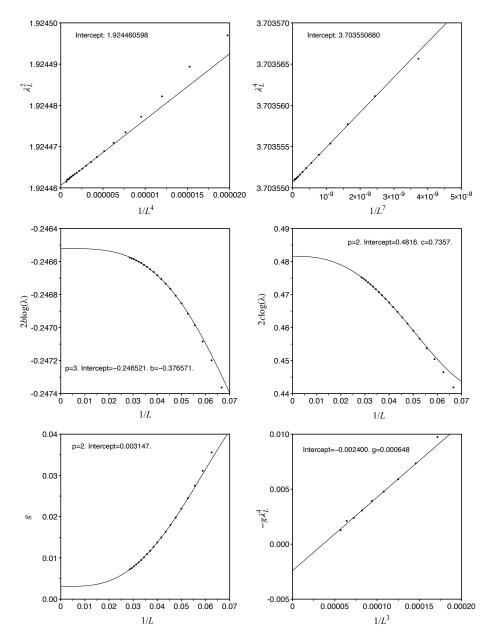


Figure E7: Plots of the various estimators used in the analysis of the data for SAWs crossing a square domain of the hexagonal lattice.