IBM Algorithms

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1 Given

1.1 Numerical Implementation

$$u^{n+1} - u^n = \int_{t_{n+1}}^{t_{n+1}} [-\nabla\phi_{non} - \nabla_h\phi_{hyd} - (u\cdot\nabla)u - f\times u + \nabla\cdot\tau + F_u]dt$$
$$u^{n+1} - u^n = \int_{t_{n+1}}^{t_{n+1}} [-\nabla\phi_{non} + G_u]dt$$

with projection pressure decomposition where $\nabla^2\phi_{non}^{n+1} = \frac{\nabla\cdot u^*}{\Delta t}$

$$u^* - u^n = \int_{t_{n+1}}^{t_{n+1}} [G_u]dt$$
$$u^{n+1} - u^* = -\Delta t\nabla\phi_{non}^{n+1}$$

1.2 Current Time Stepper (RK3)

Parameters for RK3

$$\gamma^1 = 8/15$$
$$\gamma^2 = 5/12$$
$$\gamma^3 = 3/4$$
$$\zeta^2 = -17/60$$
$$\zeta^3 = -5/12$$

Time steps for each stage

$$\texttt{first\_stage\_}\Delta\texttt{t} = \Delta t^1 = \gamma^1\Delta t = 8/15\Delta t$$
$$\texttt{second\_stage\_}\Delta\texttt{t} = \Delta t^2 = (\gamma^2 + \zeta^2)\Delta t = 2/15\Delta t$$
$$\texttt{third\_stage\_}\Delta\texttt{t} = \Delta t^3 = (\gamma^3 + \zeta^3) = 5/15$$

In reality the velocities are one set of updated values, not stored in separate places, as might be assumed from the following algorithms. Superscripts are included only for for clarity of steps.

Algorithm 1 Current RK3 Algorithm

INPUT: $U^n, \gamma_{1,2,3}, \zeta_{2,3}, \Delta t$, `grid`, `model`

for RK STAGE 1 **do**

Calculate tendencies: $G^1 = G^1_{u,v,w}$ and G^1_c (tracers)
$U^1 \leftarrow U_n + \gamma^1 \Delta t G^1$ (velocities)
$c^1 \leftarrow c_n + \gamma^1 \Delta t G^1_c$ (tracers)
Calculate pressure correction with `model`(G^1), `first_stage_`Δ`t`: $-\nabla\phi_{non}$
Update velocities: $U^{1p} \leftarrow U^1 - \nabla\phi_{non}\Delta t$
Update all other values in `model`

for RK STAGE 2 **do**

Calculate tendencies: G^2 and G^2_c (tracers)
$U^2 \leftarrow U^{1p} + \gamma^2 \Delta t G^2 + \zeta^2 \Delta t G^1$
$c^2 \leftarrow c^1 + \gamma^2 \Delta t G^2_c + \zeta^2 \Delta t G^1_c$
Calculate pressure correction with `model`(G^2), `second_stage_`Δ`t`
Update velocities: $U^{2p} \leftarrow U^2 - \nabla\phi_{non}\Delta t$
Update all other values in `model`

for RK STAGE 3 **do**

Calculate tendencies: G^3 and G^3_c (tracers)
$U^3 \leftarrow U^{2p} + \gamma^3 \Delta t G^3 + \zeta^3 \Delta t G^2$
$c^3 \leftarrow c^2 + \gamma^3 \Delta t G^3_c + \zeta^3 \Delta t G^2_c$
Calculate pressure correction with `model`(G^3), `third_stage_`Δ`t`
Update velocities: $U^{3p} \leftarrow U^3 - \nabla\phi_{non}\Delta t$
Update all other values in `model`

$U^{n+1} \leftarrow U^{3p}$ and $c^{n+1} \leftarrow c^3$

2 IBM

2.1 Coincidental boundaries

First, we might assume there are no tracers for simplicity. Also, assume some kind of Dirichlet boundary condition, V_b.

Algorithm 2 IBM-RK3 Algorithm for Coincidental Boundaries and Nodes

INPUT: $U^n, \gamma_{1,2,3}, \zeta_{2,3}, \Delta t, V_b$, `grid`, `model`

for RK STAGE 1 **do**
Calculate tendencies: $G^1 = G^1_{u,v,w}$
Update tendencies with correct forcing: $G^{1c} = G^1 + f^1$

if Fluid Node **then**
$G^{1c} = G^1$
if Solid Node **then**
$G^{1c} = \frac{-u_n}{\gamma^1 \Delta t}$
if Boundary Node **then**
$G^{1c} = \frac{V_b^1 - U_n}{\gamma^1 \Delta t}$

$U^1 \leftarrow U_n + \gamma^1 \Delta t G^{1c}$
Calculate pressure correction with `model`, `first_stage_`Δ`t`, $G^{1c} : -\nabla \phi_{non}$
Update velocities: $U^{1p} \leftarrow U^1 - \nabla \phi_{non} \Delta t$
Update all other values in `model`

for RK STAGE 2 **do**
Calculate tendencies: G^2
Update tendencies with correct forcing: $G^{2c} = G^2 + f^2$

if Fluid Node **then**
$G^{2c} = G^2$
if Solid Node **then**
$G^{2c} = \frac{-U^{1p} - \zeta^2 \Delta G^{1c}}{\gamma^2 \Delta t}$
if Boundary Node **then**
$G^{2c} = \frac{V_b^2 - U^{1p} - \zeta^2 \Delta G^{1c}}{\gamma^2 \Delta t}$

$U^2 \leftarrow U^{1p} + \gamma^2 \Delta t G^{2c} + \zeta^2 \Delta t G^{1c}$
Calculate pressure correction with `model`, `second_stage_`Δ`t`, G^{2c}
Update velocities: $U^{2p} \leftarrow U^2 - \nabla \phi_{non} \Delta t$
Update all other values in `model`

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for RK STAGE 3 **do**

Calculate tendencies: G^3_U

Update tendencies with correct forcing: $G^{3c} = G^3 + f^3$

if Fluid Node **then**

$G^{3c} = G^3$

if Solid Node **then**

$G^{3c} = \frac{-U^{2p} - \zeta^3 \Delta t G^{2c}}{\gamma^3 \Delta t}$

if Boundary Node **then**

$G^{3c} = \frac{V_b^3 - U^{2p} - \zeta^3 \Delta t G^{2c}}{\gamma^3 \Delta t}$

$U^3 \leftarrow U^{2p} + \gamma^3 \Delta t G^{3c} + \zeta^3 \Delta t G^{2c}$

Calculate pressure correction with `model`, `third_stage_`Δ`t`, G^{3c}

Update velocities: $U^{3p} \leftarrow U^3 - \nabla \phi_{non} \Delta t$

Update all other values in `model`

$U^{n+1} \leftarrow U^{3p}$

We could also consider even more simplified: assuming no slip boundary condition (ie. $V_b = 0$). Still with no tracers. (Case implemented 10-3-2020).

Algorithm 3 IBM-RK3 Algorithm for Coincidental Boundaries and Nodes with no slip BC

INPUT: $U^n, \gamma_{1,2,3}, \zeta_{2,3}, \Delta t, V_b$, `grid`, `model`

for RK STAGE 1 **do**

Calculate tendencies: $G^1 = G^1_{u,v,w}$

Update tendencies with correct forcing: $G^{1c} = G^1 + f^1$

if Fluid Node **then**

$G^{1c} = G^1$

if Immersed Node **then**

$G^{1c} = \frac{-U_n}{\gamma^1 \Delta t}$

$U^1 \leftarrow U_n + \gamma^1 \Delta t G^{1c}$

Calculate pressure correction with `model`, `first_stage_`Δ`t`, $G^{1c} : -\nabla \phi_{non}$

Update velocities: $U^{1p} \leftarrow U^1 - \nabla \phi_{non} \Delta t$

Update all other values in `model`

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for RK STAGE 2 **do**

Calculate tendencies: G^2

Update tendencies with correct forcing: $G^{2c} = G^2 + f^2$

if Fluid Node **then**

$G^{2c} = G^2$

if Immersed Node **then**

$G^{2c} = \frac{-U^{1p} - \zeta^2 \Delta G^{1c}}{\gamma^2 \Delta t}$

$U^2 \leftarrow U^{1p} + \gamma^2 \Delta t G^{2c} + \zeta^2 \Delta t G^{1c}$

Calculate pressure correction with `model`, `second_stage_`Δ`t`, G^{2c}

Update velocities: $U^{2p} \leftarrow U^2 - \nabla \phi_{non} \Delta t$

Update all other values in `model`

for RK STAGE 3 **do**

Calculate tendencies: G^3_U

Update tendencies with correct forcing: $G^{3c} = G^3 + f^3$

if Fluid Node **then**

$G^{3c} = G^3$

if Immersed Node **then**

$G^{3c} = \frac{-U^{2p} - \zeta^3 \Delta t G^{2c}}{\gamma^3 \Delta t}$

$U^3 \leftarrow U^{2p} + \gamma^3 \Delta t G^{3c} + \zeta^3 \Delta t G^{2c}$

Calculate pressure correction with `model`, `third_stage_`Δ`t`, G^{3c}

Update velocities: $U^{3p} \leftarrow U^3 - \nabla \phi_{non} \Delta t$

Update all other values in `model`

$U^{n+1} \leftarrow U^{3p}$