IB DIPLOMA

MATHEMATICS

APPLICATIONS & INTERPRETATIONS

STANDARD LEVEL + HIGHER LEVEL

REVISION BOOK

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How to Use This Revision Book

This book is designed to help you revise for the IB DP Mathematics: Applications & Interpretation course. For each question:

- Work through the problem in the space provided.
- Use the lines and blank areas for your working.
- Write your **final answer** clearly on the "Final Answer" line.
- Once finished, check your work against the solutions section.

Remember to show all working where possible – partial credit is often awarded.

Contents

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IB DP Mathematics: Applications & Interpretation

Consolidated Topics 1–5

Questions and solutions organised by syllabus subtopics

This document presents a curated selection of questions from the five topics in the Applications and Interpretation course. The headings follow the IB syllabus order: for each Standard Level subtopic (e.g. SL 1.1) the corresponding Higher Level extension (if any) appears immediately after. All notation, macros and diagrams have been unified. Worked solutions are collected after the questions to encourage independent practice.

Name: ____________________ **Date:** ______________

Questions

Topic 1 — Number and Algebra (SL 1.1–1.8, AHL 1.9–1.16)

Overview (SL) Covers arithmetic, exponents and radicals, logarithms, sequences and series, financial mathematics, and basic algebraic manipulation. Includes solving linear, quadratic, and simple exponential equations.

Overview (HL) Extends SL content with complex numbers, more advanced series (e.g., sum to infinity for geometric sequences), rational exponents, and matrices including eigenvalues and eigenvectors.

Real-World Use

- Financial modeling and interest rate calculations
- Population growth and decay modeling
- Engineering calculations and coding algorithms
- Physics applications involving exponential and logarithmic relationships

Common Misconceptions

- Confusing laws of exponents (e.g., $a^m \times a^n = a^{m+n}$ but $(a^m)^n = a^{mn}$)
- Misinterpreting negative and fractional exponents

Advice

- Always check dimensional consistency in real-world applications.
- Write intermediate steps clearly to avoid sign and index errors.
- For sequences, understand the difference between arithmetic and geometric patterns.

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SL 1.1 Scientific Notation

The following exercises review conversion to and from scientific notation and simple operations on numbers expressed as powers of ten.

Q1 [Go to Solution p. 269] [Back to TOC]

Convert each number to scientific notation:

a) $0.000\,0426$

b) $85\,900\,000$

c) $\dfrac{7.2 \times 10^{-5}}{3 \times 10^{-2}}$

Final Answer: ______________________________

Q2 [Go to Solution p. 269] [Back to TOC]

Write $(3.5 \times 10^{-4})(8 \times 10^{6})$ in scientific notation.

Final Answer: ______________________________

Q3 [Go to Solution p. 269] [Back to TOC]

Perform each calculation and express your answer in scientific notation:

a) $(6 \times 10^{-3})(4 \times 10^{7})$

b) $\dfrac{9 \times 10^{5}}{3 \times 10^{-2}}$

c) $(2.4 \times 10^{-4}) + (3.1 \times 10^{-4})$

d) $(7.5 \times 10^{2}) - (2.50 \times 10^{1})$

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

Q4 [Go to Solution p. 269] [Back to TOC]

Between Earth's radius ($\approx 6.37 \times 10^6\,\mathrm{m}$) and the distance light travels in 0.02 s (speed of light $3.0 \times 10^8\,\mathrm{m\,s^{-1}}$), which is larger? Give your reasoning in scientific notation.

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

SL 1.2 Arithmetic Sequences and Series

Recall that an arithmetic sequence has general term $u_n = u_1 + (n-1)d$ and finite sum $S_n = \frac{n}{2}(2u_1 + (n-1)d)$.

Q5 [Go to Solution p. 270] [Back to TOC]

A sequence has first term $u_1 = 7$ and common difference $d = -3$. Find u_5 and u_{20}.

Final Answer: ______________________________

Q6 [Go to Solution p. 270] [Back to TOC]

Find the sum of the first 100 positive integers.

Final Answer: ______________________________

Q7 [Go to Solution p. 270] [Back to TOC]

For each arithmetic sequence below, determine the requested quantities.

a) For $u_n = 12 + 5(n-1)$, find u_1 and u_{10}.

b) If $u_3 = 14$ and $d = 4$, find u_1 and S_{20}.

c) The sequence $5, 9, 13, \ldots$ has sum $S_n = 1428$. Determine n (if it exists).

d) A sequence satisfies $u_4 = 11$ and $u_{12} = 43$. Determine d, u_1 and S_{50}.

Final Answer: ______________________________

Name: ____________________ **Date:** ______________

SL 1.3 Geometric Sequences and Series

For a geometric sequence, $u_n = u_1 r^{n-1}$ and the finite sum is $S_n = \dfrac{u_1(1-r^n)}{1-r}$ for $r \neq 1$.

Q8 [Go to Solution p. 271] [Back to TOC]

Given $u_1 = 3$ and $r = 2$, compute u_6 and S_6.

Final Answer: ____

Q9 [Go to Solution p. 271] [Back to TOC]

A salary starts at \$32 000 and increases by 5% each year. Write a formula for the total salary paid over five years (ignoring inflation) and evaluate it.

Final Answer: ____

Q10 [Go to Solution p. 271] [Back to TOC]

Answer the following geometric sequence problems:

a) For $u_1 = 9$ and $r = \frac{1}{3}$, find u_5 and S_5.

b) If $u_3 = 48$ and $u_6 = 384$, determine r and u_1.

c) A ball bounces to 80% of its previous height when dropped from 2.0000 m. Find the total vertical distance travelled (sum to infinity).

d) Solve for n given $S_n = 121$ when $u_1 = 1$ and $r = 0.1$, or explain why no such n exists.

Final Answer: ____

Name: ______________________ **Date:** ______________

SL 1.4 Financial Applications of Geometric Sequences

Interest and depreciation problems can be modelled as geometric sequences and series.

Key ideas

Financial applications of geometric sequences and series

- *Compound interest:* Interest is added to the principal, and future interest is calculated on the new total.
- *Annual depreciation:* A fixed percentage of the value is lost each year.
- Both processes follow geometric sequences:

$$\text{Term}_n = \text{Initial value} \times r^n$$

  where $r > 1$ for growth (compound interest) and $0 < r < 1$ for decay (depreciation).
- Technology such as the TI-Nspire's built-in `Finance` tools can quickly compute future or present values without manual calculation.
- Compound periods can be yearly, half-yearly, quarterly, monthly, or daily.
- The real value of an investment can be calculated by adjusting the interest rate for inflation.

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Key terms

Principal (P) The initial amount invested or borrowed.

Interest rate (i) The percentage increase per period (expressed as a decimal in formulas).

Number of periods (n) The total compounding periods.

Future value (FV) The value after compounding:

$$FV = P(1+i)^n$$

Present value (PV) The starting value given a future target:

$$PV = \frac{FV}{(1+i)^n}$$

Depreciation rate (d) The proportion lost each period.

Inflation rate (f) The annual percentage decrease in purchasing power.

Real interest rate Adjusted for inflation:

$$i_{\text{real}} \approx \frac{1 + i_{\text{nominal}}}{1+f} - 1$$

Worked example 1 — Compound interest You invest \$5000 at 4% annual interest, compounded quarterly, for 6 years. Find the future value.

Solution:

$$i = \frac{0.04}{4} = 0.01, \quad n = 6 \times 4 = 24$$

$$FV = 5000(1.01)^{24} \approx 5000 \times 1.26824 \approx \boxed{\$6348.67}$$

TI-Nspire method:

1. Press $\boxed{\text{Menu}} \to \boxed{\text{Finance}} \to \boxed{\text{Finance Solver}}$.
2. Set $N = 24$, $I\% = 4$, $PV = -5000$, $PMT = 0$, $FV =?$, $P/Y = 4$, $C/Y = 4$.
3. Scroll to FV and press $\boxed{\text{Enter}}$ to solve.

Worked example 2 — Annual depreciation A car bought for \$28,000 depreciates at 15% per year. Find its value after 5 years.

Solution:

$$r = 1 - 0.15 = 0.85, \quad n = 5$$

$$V_5 = 28000(0.85)^5 \approx 28000 \times 0.4437 \approx \boxed{\$12{,}423.70}$$

TI-Nspire method:

1. Open a Calculator page.

2. Define: `V(n) := 28000*(0.85)^n`
3. Evaluate: `V(5)` → Enter.

Worked example 3 — Real value after inflation An investment grows at 6% per year for 10 years, but inflation is 2% per year. Find the real growth factor over 10 years.

Solution:

$$i_{\text{real}} = \frac{1.06}{1.02} - 1 \approx 0.039216 = 3.9216\%$$

$$\text{Real factor} = (1.039216)^{10} \approx 1.466 \quad \Rightarrow \quad \text{Real increase} \approx 47.0\%.$$

TI-Nspire method:

1. In Calculator: `i_real := (1.06)/(1.02) - 1`
2. Then: `(1 + i_real)^10` → Enter.

Tips

- Always match N, P/Y (payments per year), and C/Y (compounding per year) on the TI-Nspire Finance Solver.
- For depreciation, the "interest rate" is negative in the Finance Solver.
- Use variable definitions (`P`, `i`, `n`) in Calculator to reduce repeated typing.

Q11 [Go to Solution p. 272] [Back to TOC]

You deposit $1,000 at 3.5% interest per annum, compounded annually. What is the value after four years?

Final Answer: ______________________________

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Q12 [Go to Solution p. 272] [Back to TOC]

A car is purchased for $24,000 and depreciates by 18% each year. Find its value after five years.

Final Answer: ______________________________

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Name: ________________ **Date:** ____________

Q13 [Go to Solution p. 272] [Back to TOC]

Compute the following:

a) \$6,500 invested at 4.2% per annum compounded annually for seven years.

b) A television is purchased for \$1,800 and depreciates by 25% annually; find its value after three years.

c) An investment grows by 6% per year while inflation is 2.5% per year. Compute the effective real growth factor and the real value of \$10,000 after ten years.

Final Answer: ________________________

Name: ________________ **Date:** ____________

SL 1.5 Integer Exponents and Logarithms

This section reviews exponent rules and introduces base 10 and natural logarithms.

Q14 [Go to Solution p. 273] [Back to TOC]

Simplify $\frac{2^3 \cdot 2^{-5}}{2^{-1}}$.

Final Answer: ________________

Q15 [Go to Solution p. 273] [Back to TOC]

Solve each of the following equations for x; use natural logarithms when appropriate:

a) $10^x = 4.2$

b) $e^{2x} = 7$

c) $3 \cdot 2^x = 40$

d) $5^{x-1} = 12$

Final Answer: ________________

Q16 [Go to Solution p. 273] [Back to TOC]

Use the laws of logarithms to expand each expression:

a) $\ln(\frac{9x^4}{\sqrt{y}})$

b) $\log_{10}(100x^3y)$

c) $\log(\frac{a^5}{b^2c})$

d) $\ln((e^{3t})^2)$

Final Answer: ________________

Name: ______________________ **Date:** ______________

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SL 1.6 Approximation, Bounds and Percentage Error

Problems in measurement often involve rounding and error bounds.

Q17 [Go to Solution p. 274] [Back to TOC]

Round 3.1462 to three significant figures and 0.004 981 to two decimal places.

Final Answer: ______________________________

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Q18 [Go to Solution p. 274] [Back to TOC]

Given $r = 2.5000\,\text{cm}$ measured to the nearest 0.1 cm, determine the bounds for r and hence bounds for the area $A = \pi r^2$.

Final Answer: ______________________________

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Name: ______________ **Date:** ______________

SL 1.7 Amortisation and Annuities

Financial calculations involving regular payments are solved using time-value of money formulas or technology.

Key ideas

What this covers.

- Use technology (e.g., TI-Nspire Finance Solver / spreadsheets) to work with *amortization* (paying off loans) and *annuities* (regular payments into or out of an account).
- In exams payments occur at the *end* of each period (annuity–immediate).
- Helpful (not examinable to derive) formulas for end-of-period payments with periodic rate i and n periods:

$$\mathrm{PV} = \mathrm{PMT}\,\frac{1-(1+i)^{-n}}{i}, \qquad \mathrm{FV} = \mathrm{PMT}\,\frac{(1+i)^{n}-1}{i}, \qquad \mathrm{PMT} = \mathrm{PV}\,\frac{i}{1-(1+i)^{-n}}.$$

- Outstanding balance after k payments: $\mathrm{OB}_k = \mathrm{PV}(1+i)^k - \mathrm{PMT}\,\dfrac{(1+i)^k-1}{i}$.
- In a schedule line t: $\text{interest}_t = i \times \text{balance}_{t-1}$, $\text{principal}_t = \mathrm{PMT} - \text{interest}_t$.

Worked example 1 — Mortgage payment and early schedule. Borrow \$200,000 at 4.8% p.a. compounded monthly for 25 years. Find the monthly payment, the first three lines of the amortization schedule, the total interest in year 1, and the balance after 5 years.

Parameters:

$$i = \frac{0.048}{12} = 0.004, \qquad n = 25 \times 12 = 300.$$

Payment (end of month):

$$\mathrm{PMT} = \$200{,}000 \cdot \frac{0.004}{1-(1.004)^{-300}} \approx \boxed{\$1{,}145.99}.$$

First three months (rounded to cents):

Month	Interest	Principal	New balance
1	\$800.00	\$345.99	\$199,654.01
2	\$798.62	\$347.38	\$199,306.63
3	\$797.23	\$348.77	\$198,957.86

Year 1 totals: interest ≈ \$9,507.43, principal repaid ≈ \$4,244.50, balance ≈ \$195,755.50. *Balance after* 5 *years (60 payments):* $\boxed{\$176{,}589.99}$. *Total interest over full term:* ≈ \$143,798.18.

TI-Nspire (Finance Solver).

1. Menu → Finance → Finance Solver.
2. Set **N** = 300, **I%** = 4.8, **PV** = 200000, **PMT** =?, **FV** = 0, **P/Y** = 12, **C/Y** = 12, **PMT: End**. (Cash-flow signs must differ; the solver will return a negative PMT; its magnitude

is the payment.)

3. To get year-1 totals: press Menu $\to$ Amortization, start $=1$, end $=12$. To see the first payment only, use start $=1$, end $=1$.
4. Balance after 5 years: use Amortization with start $=1$, end $=60$, then Ending Balance $\approx$ \$176,589.99.

Worked example 2 — Savings annuity (future value). Deposit \$300 at the end of each month for 10 years at 3.6% p.a. compounded monthly. Find the future value.

Parameters: $i = \frac{0.036}{12} = 0.003,\ n = 120.$

$$\text{FV} = \$300 \cdot \frac{(1.003)^{120} - 1}{0.003} \approx \$300 \cdot 144.1857 \approx \boxed{\$43{,}255.72}.$$

(Contributions total \$36,000; interest earned $\approx$ \$7,255.72.)

TI-Nspire.

1. Finance Solver: **N** $= 120$, **I%** $= 3.6$, **PV** $= 0$, **PMT** $= -300$ (payment out), **FV** $=?$, **P/Y** $= 12$, **C/Y** $= 12$, **PMT: End**.
2. Scroll to FV and press Enter $\Rightarrow FV \approx \$43{,}255.72$.

Worked example 3 — How long to clear a loan? A loan of \$15,000 at 8% p.a. compounded monthly is repaid by \$300 at the end of each month. How many months are required?

Parameters: $i = \frac{0.08}{12} \approx 0.006\bar{6}$. Solve from $\text{PMT} = \text{PV}\dfrac{i}{1-(1+i)^{-n}}$:

$$(1+i)^{-n} = 1 - \frac{\text{PV}\,i}{\text{PMT}} = 1 - \frac{15000 \cdot \frac{0.08}{12}}{300} = 1 - 0.333\bar{3} = 0.666\bar{6}.$$

$$n = \frac{-\ln(0.666\bar{6})}{\ln(1 + \frac{0.08}{12})} \approx \boxed{61.02 \text{ months}} \quad \text{(about 5.1 years)}.$$

TI-Nspire.

1. Finance Solver: **N** $=?$, **I%** $= 8$, **PV** $= 15000$, **PMT** $= -300$, **FV** $= 0$, **P/Y** $= 12$, **C/Y** $= 12$, **PMT: End**.
2. Solve for $N \Rightarrow N \approx 61.02$.

TI-Nspire tips

- **Signs matter:** At least one of PV, PMT, FV must have the opposite sign to the others. For loans, keep $PV > 0$ and let the computed PMT be negative.
- **End vs Begin:** Exams use end-of-period payments. Ensure PMT: End is selected.
- **P/Y and C/Y:** Set both to the compounding frequency (e.g., 12 for monthly).
- Use **Amortization** in the Finance Solver to get totals for any block of periods (interest paid,

principal repaid, ending balance).

Q19 [Go to Solution p. 275] [Back to TOC]

You borrow $9,000 at 6.0% per annum, compounded monthly, for three years. What is the monthly payment required to clear the loan? State the time-value inputs used.

Final Answer: ____

Q20 [Go to Solution p. 275] [Back to TOC]

An annuity pays $250 at the end of each month for four years at 4.8% per annum compounded monthly. Use a finance solver (or geometric sum formula) to find its present value.

Final Answer: ____

Name: ____________________ **Date:** ____________

SL 1.8 Systems of Equations and Polynomials (Technology)

Technology is used to solve linear systems up to three variables and to find roots of polynomials.

Key ideas

What this covers.

- Use technology (e.g., TI–Nspire) to solve:
  - Systems of up to 3 linear equations in up to 3 variables.
  - Polynomial equations (any degree supported by the calculator).
- In the syllabus, all systems will have a unique solution.
- Terminology:
  - *Root* or *zero* of a polynomial: a value of x for which $f(x) = 0$.
  - *Solution* of a system: an ordered pair/triple that satisfies all equations.
- No specific manual method is required in the exam — technology can be used directly.

Worked example 1 — Solving a system of linear equations. Solve:

$$\begin{cases} 2x + y - z = 3, \\ x - y + 2z = 9, \\ 3x + 2y + z = 10. \end{cases}$$

Using TI–Nspire (Menu Method):

1. Open a **Calculator** page.
2. Press [Menu] → [Algebra] → [Solve System of Linear Equations] → [Solve System of 3 equations].
3. Enter:
   $$2x + y - z = 3, \quad x - y + 2z = 9, \quad 3x + 2y + z = 10$$
   and specify variables $\{x, y, z\}$.
4. Press [Enter].

Result:

$$\boxed{x = 2, \quad y = -1, \quad z = 4}.$$

Check: Substituting into all equations confirms all are satisfied.

Worked example 2 — Solving a cubic equation. Find all real and complex roots of:

$$p(x) = x^3 - 6x^2 + 11x - 6.$$

By factoring manually: Test small integer roots (factor theorem). $x = 1$ is a root:

$$p(x) = (x - 1)(x^2 - 5x + 6) = (x - 1)(x - 2)(x - 3).$$

Roots: $\boxed{x = 1,\ x = 2,\ x = 3}$.

Using TI–Nspire (Menu Method):

1. Open a **Calculator** page.
2. Press Menu → Algebra → Polynomial Tools → Polynomial Root Finder.
3. Set Degree = 3 and Complex = Yes to allow all roots.
4. Enter coefficients: 1 (for x^3), -6 (for x^2), 11 (for x), -6 (constant term).
5. Press OK.

Result: Roots $\boxed{x = 1,\ x = 2,\ x = 3}$.

Worked example 3 — Complex roots of a quadratic. Solve:

$$x^2 + 4x + 13 = 0.$$

Using TI–Nspire:

1. **Calculator** page.
2. Type: `solve(x^2 + 4x + 13 = 0, x)` and press Enter.
3. Alternatively: Menu → Algebra → Polynomial Tools → Polynomial Root Finder, degree 2, complex=Yes.

Result:

$$\boxed{x = -2 + 3i, \quad x = -2 - 3i}.$$

TI–Nspire tips

- For `solve()`, always include the variable: `solve(eq, var)`.
- The *Polynomial Root Finder* quickly finds all roots and displays them in exact or decimal form.
- In *Solve System of Linear Equations*, choose the number of equations to match the problem (2 or 3).
- Ensure all equations are entered in the form $\cdots = \ldots$ before solving.

Q21 [Go to Solution p. 276] [Back to TOC]

Solve the system $\{2x + y = 11,\ x - y = 1\}$ by any method.

Final Answer: ____

Q22 [Go to Solution p. 276] [Back to TOC]

Use technology (GDC or CAS) to solve the system

$$\begin{cases} +x + 2y - 3z = 7 \\ +2x - y + z = 1 \\ -3x + 4y + 2z = 9 \end{cases}$$

and verify your solution.

Final Answer: ______________________________

Q23 [Go to Solution p. 276] [Back to TOC]

Find all real roots of the polynomial $p(x) = x^4 - 5x^2 + 4$.

Final Answer: ______________________________

Name: ____________________ **Date:** ____________

AHL 1.9 Laws of Logarithms

These exercises extend the rules of logarithms and practise solving logarithmic equations.

Q24 [Go to Solution p. 277] [Back to TOC]

Simplify $\log(50) + \log(20) - \log(5)$.

Final Answer: ____________________

Q25 [Go to Solution p. 277] [Back to TOC]

Solve for $x > 0$: $\log(3x) - \log(x - 2) = 1$.

Final Answer: ____________________

Q26 [Go to Solution p. 277] [Back to TOC]

Solve for x: $\log_5(x - 1) + \log_5(x + 1) = 2$.

Final Answer: ____________________

Name: ______________________ **Date:** ______________

AHL 1.10 Rational Exponents

Simplify expressions involving fractional exponents and radicals.

Q27 [Go to Solution p. 278] [Back to TOC]

Express $x^{\frac{3}{2}}$ and $x^{-\frac{2}{3}}$ using radicals.

Final Answer: ______________________________

Q28 [Go to Solution p. 278] [Back to TOC]

Simplify $\dfrac{25^{3/2} \cdot 10^{-1}}{5^{1/2}}$.

Final Answer: ______________________________

Q29 [Go to Solution p. 278] [Back to TOC]

Evaluate exactly $\left(27^{2/3}\right)\left(9^{3/2}\right)\left(3^{-1}\right)$.

Final Answer: ______________________________

Name: ____________________ **Date:** ______________

AHL 1.11 Infinite Geometric Series

Determine convergence and sums of infinite geometric series.

Q30 [Go to Solution p. 279] [Back to TOC]

Does the series $5 + 2.5 + 1.25 + \cdots$ converge? If so, to what value?

Final Answer: ______________________________________

Q31 [Go to Solution p. 279] [Back to TOC]

Find S_∞ (if it exists) for each series:

a) $3 + \frac{3}{4} + \frac{3}{16} + \cdots$

b) $7 - 3.5 + 1.75 - \cdots$

c) $10 + 8 + 6.4 + \cdots$

Final Answer: ______________________________________

Name: ______________________ **Date:** ______________

AHL 1.12 Complex Numbers (Cartesian Form)

Perform operations with complex numbers in Cartesian form.

Q32 [Go to Solution p. 280] [Back to TOC]

Compute $(2-3i)+(4+6i)$ and $(2-3i)(4+6i)$, expressing each result in $a+bi$ form.

Final Answer: ____

Q33 [Go to Solution p. 280] [Back to TOC]

Solve $z^2-6z+13=0$ and represent the roots on an Argand diagram.

Final Answer: ____

Name: ____________________ **Date:** ______________

AHL 1.13 Complex Numbers (Polar/Exponential)

Convert complex numbers between Cartesian and polar/exponential forms and evaluate powers in polar form.

Q34 [Go to Solution p. 281] [Back to TOC]

Express $z = 1 + i$ in polar form $r \operatorname{cis} \theta$.

Final Answer: ____________________________

Q35 [Go to Solution p. 281] [Back to TOC]

Compute $(\sqrt{3} - i)^5$ using polar form.

Final Answer: ____________________________

AHL 1.14 Matrices and Inverses

Matrices can be used to solve linear systems and to explore algebraic properties.

Q36 [Go to Solution p. 282] [Back to TOC]

Provide an example of two 2×2 matrices A and B such that $AB \neq BA$.

Final Answer: ____

Q37 [Go to Solution p. 282] [Back to TOC]

Solve the system $\{2x + y = 5,\ 3x - 2y = -4\}$ using matrix inversion.

Final Answer: ____

Name: ______________________ **Date:** ________________

AHL 1.15 Eigenvalues, eigenvectors, diagonalization, and applications

An eigenvector $\mathbf{v}$ *keeps the same direction after multiplication by* A*, but is scaled by the eigenvalue* λ.

Key concepts.

1. **Eigenvalue:** For a square matrix A, a scalar λ is an eigenvalue if there exists a non-zero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

   The vector $\mathbf{v}$ is called an **eigenvector** corresponding to λ.

2. **Characteristic equation:** $\det(A - \lambda I) = 0$ gives the characteristic polynomial. Roots are the eigenvalues.

3. **Eigenvector calculation:** For each eigenvalue λ, solve $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find eigenvectors.

4. **Diagonalization:** If A has n distinct real eigenvalues (here $n = 2$), it can be written as:

$$A = PDP^{-1},$$

   where P is the matrix whose columns are eigenvectors, and D is a diagonal matrix of eigenvalues.

5. **Powers of a matrix:** If $A = PDP^{-1}$, then

$$A^n = PD^nP^{-1},$$

   where D^n is easy to compute since D is diagonal.

6. **Applications:** - Movement of populations between two towns. - Predator–prey models. - Coupled linear recurrence relations. - Links to coupled differential equations (AHL 5.17).

- Only 2×2 cases are required for hand calculation.
- Diagonalization requires distinct real eigenvalues; complex or repeated eigenvalues are not in the syllabus for manual computation.
- The process:
  1. Find $\det(A - \lambda I) = 0$.
  2. Solve for λ_1 and λ_2.
  3. For each λ_i, solve $(A - \lambda_i I)\mathbf{v} = \mathbf{0}$.
  4. Form P and D.
- Powers of A are much easier once A is diagonalized.

Example 1 Finding eigenvalues and eigenvectors. Let

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}.$$

Step 1: Characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2.$$

$$= \lambda^2 - 7\lambda + 10 = 0.$$

Step 2: Solve:

$$\lambda^2 - 7\lambda + 10 = 0 \quad \Rightarrow \quad (\lambda - 5)(\lambda - 2) = 0.$$

Eigenvalues: $\lambda_1 = 5$, $\lambda_2 = 2$.

Step 3: Eigenvectors:

- For $\lambda = 5$:

$$(A - 5I)\mathbf{v} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad v_1 = v_2.$$

  Take $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- For $\lambda = 2$:

$$(A - 2I)\mathbf{v} = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad v_1 = -\frac{1}{2} v_2.$$

  Take $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Example 2 Diagonalization and powers of a matrix.

Using A from Example 1, form:

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}, \quad D = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Check:

$$P^{-1} = \frac{1}{(1)(2) - (-1)(1)} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

Indeed,

$$A = PDP^{-1}.$$

Powers:

$$A^n = PD^nP^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & 2^n \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}.$$

D^n is trivial to compute since it is diagonal.

Example 3 Population movement between two towns.

Name: ________________ **Date:** __________

Suppose the population distribution $\mathbf{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ evolves each year via:

$$\mathbf{x}_{n+1} = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix} \mathbf{x}_n.$$

Let $A = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}$. We diagonalize $A = PDP^{-1}$ and use:

$$\mathbf{x}_n = A^n \mathbf{x}_0 = PD^n P^{-1} \mathbf{x}_0.$$

As $n \to \infty$, $\mathbf{x}_n$ tends to the eigenvector corresponding to the largest eigenvalue, representing the steady-state population distribution.

Name: ______________________ **Date:** ______________

Q38 [Go to Solution p. 283] [Back to TOC]

For $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, list its eigenvalues and give a corresponding eigenvector for each eigenvalue.

Final Answer: ______________________________

Q39 [Go to Solution p. 283] [Back to TOC]

For $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, find its eigenvalues and eigenvectors. Then determine M^5 using diagonalisation.

Final Answer: ______________________________

Q40 [Go to Solution p. 283] [Back to TOC]

Finding eigenvalues and eigenvectors. Given

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix},$$

1. Find the characteristic polynomial of A.
2. Determine the eigenvalues of A.
3. Find a corresponding eigenvector for each eigenvalue.

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

Q41 [Go to Solution p. 283] [Back to TOC]

Diagonalization of a 2×2 matrix. Consider

$$B = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}.$$

1. Show that B has two distinct real eigenvalues.
2. Find a matrix P of eigenvectors and a diagonal matrix D such that $B = PDP^{-1}$.
3. Verify your diagonalization by computing PDP^{-1} explicitly.

Final Answer: ____

Q42 [Go to Solution p. 284] [Back to TOC]

Powers of a 2×2 matrix using diagonalization. Let

$$C = \begin{pmatrix} 7 & 0 \\ 2 & 3 \end{pmatrix}.$$

1. Diagonalize C as $C = PDP^{-1}$.
2. Use your result to compute C^6.

Final Answer: ____

Q43 [Go to Solution p. 284] [Back to TOC]

Application: population movement between two towns. The populations of two towns X and Y at the start of each year are related by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

1. Write the recurrence in the form $\mathbf{v}_{n+1} = M\mathbf{v}_n$.
2. Diagonalize M and hence find a formula for $\mathbf{v}_n$ in terms of n and the initial populations $\mathbf{v}_0$.
3. If $(x_0, y_0) = (5000, 3000)$, predict the populations after 10 years.

Final Answer: ____________________________________

Q44 [Go to Solution p. 285] [Back to TOC]

Eigenvalues and invertibility. Let

$$D = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}.$$

1. Find the eigenvalues of D.
2. Determine whether D is invertible using its eigenvalues.
3. If invertible, find D^{-1} using diagonalization.

Final Answer: ____________________________________

Q45 [Go to Solution p. 286] [Back to TOC]

Repeated eigenvalues case. Consider

$$E = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}.$$

1. Find the eigenvalues of E and their algebraic multiplicities.
2. Determine whether E is diagonalizable.
3. Explain why the result relates to the number of linearly independent eigenvectors.

Final Answer: ____________________________________

Q46 [Go to Solution p. 286] [Back to TOC]

Predator-prey model with matrices. The populations of prey P_n and predators Q_n at year n satisfy

$$\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} = \begin{pmatrix} 1.1 & -0.4 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} P_n \\ Q_n \end{pmatrix}.$$

1. Find the eigenvalues and eigenvectors of the transition matrix.
2. Diagonalize the matrix and find an expression for (P_n, Q_n).
3. Describe qualitatively the long-term behaviour of both populations.

Final Answer: ____________________

Q47 [Go to Solution p. 287] [Back to TOC]

Matrix powers in a recurrence. Let

$$F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

1. Diagonalize F and find F^n in terms of n.
2. Use your result to compute F^{20}.
3. Comment on the pattern in the entries of F^n.

Final Answer: ____________________

Q48 [Go to Solution p. 287] [Back to TOC]

Name: ______________________ **Date:** ______________

Eigen-decomposition in transformations. A transformation in $\mathbb{R}^2$ is represented by the matrix

$$G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

1. Find the eigenvalues of G.
2. Explain why G cannot be diagonalized over the real numbers.
3. Interpret geometrically the effect of repeatedly applying G to a vector.

Final Answer: ____

Name: ____________________ **Date:** ________________

Topic 2 — Functions (SL 2.1–2.6, AHL 2.7–2.10)

Overview (SL) Focuses on understanding and using function notation, identifying domains and ranges, and exploring straight lines, relations, and different families of functions (linear, quadratic, cubic, exponential, sinusoidal, and direct/inverse proportion). Covers key graph properties, curve sketching, points of intersection, and modelling skills in real contexts. Emphasizes the use of technology for graphing, solving equations, and interpreting results.

Overview (HL) Builds on SL content with composite and inverse functions, advanced graph transformations, rational and polynomial functions, and additional modelling families. Includes scaling graphs for large and small values, and using piecewise definitions to represent complex scenarios.

Real-World Use

- Predictive models in economics, biology, and environmental science
- Engineering and architecture design using constraints and optimisation
- Computer graphics transformations and animations
- Medical modelling of dosage-response relationships

Common Misconceptions

- Confusing $f^{-1}(x)$ (inverse function) with $\frac{1}{f(x)}$ (reciprocal)
- Ignoring domain restrictions after transformations or inversions
- Assuming symmetry without checking the specific function properties

Advice

- Always sketch a graph by hand first to understand its shape before using technology.
- Clearly state the domain and range in final answers.
- When finding inverses, swap x and y before solving and verify the result satisfies the original function's conditions.
- For modelling, think about the real-world limitations and whether your chosen function family makes sense in context.

Name: ____________________ **Date:** ______________

SL 2.1 Straight Lines

Q49 [Go to Solution p. 290] [Back to TOC]

Find the gradient of the line through $A(2, -1)$ and $B(8, 5)$. Then express $3x - 2y = 12$ in slope–intercept form and identify its gradient and intercepts.

Final Answer: ____

Q50 [Go to Solution p. 290] [Back to TOC]

Determine the equation of the line passing through $C(-3, 2)$ and $D(5, -6)$ in (i) point–slope form and (ii) $ax + by + d = 0$ form.

Final Answer: ____

Q51 [Go to Solution p. 290] [Back to TOC]

A line L has equation $y = 4x - 7$. Find the equations of the lines through $(2, 1)$ (a) parallel to L and (b) perpendicular to L.

Final Answer: ____

Q52 [Go to Solution p. 291] [Back to TOC]

Find the intersection point of the lines $3x + y = 10$ and $2x - 3y = 1$.

Name: ______________________ **Date:** ______________

Final Answer: ______________________________

____

Q53 [Go to Solution p. 291] [Back to TOC]

A road rises linearly from 120 m at $x = 0$ km to 420 m at $x = 15$ km. (i) Model the altitude $h(x)$ as a linear function. (ii) Estimate $h(8)$. (iii) For what x does $h = 300$ m?

Final Answer: ______________________________

____

Name: ______________ Date: ______________

SL 2.2 Relations and Functions

Q54 [Go to Solution p. 292] [Back to TOC]

State whether each relation is a function from $\mathbb{R}$ to $\mathbb{R}$: (i) $y = \sqrt{x}$; (ii) $x = y^2$. Justify your answers briefly.

Final Answer: ______________

____

Q55 [Go to Solution p. 292] [Back to TOC]

Let $f(x) = 2x - 5$. (a) Find the inverse function $f^{-1}(x)$. (b) Verify that $f(f^{-1}(x)) = x$.

Final Answer: ______________

____

Q56 [Go to Solution p. 292] [Back to TOC]

For $g(x) = \sqrt{9 - x^2}$, find the domain and range. For $h(x) = \frac{3}{x-2}$, state the domain and range; then determine the inverse $h^{-1}(x)$ along with its domain and range.

Final Answer: ______________

____

Q57 [Go to Solution p. 292] [Back to TOC]

Determine whether $p(x) = x^2 - 6x + 8$ is one-to-one on $\mathbb{R}$. If not, restrict the domain to make p invertible and find $p^{-1}(x)$ on that restricted domain.

Name: __________ **Date:** __________

Final Answer: __________

Name: ____________________ **Date:** ____________

SL 2.3 Properties and Families of Functions

Q58 [Go to Solution p. 292] [Back to TOC]

Classify each function by type: (i) $y = \frac{2x-1}{x+3}$; (ii) $y = 5 \cdot 2^x$; (iii) $y = |x-4|$.

Final Answer: ____________________

Q59 [Go to Solution p. 293] [Back to TOC]

For $g(x) = x^2 - 4x + 1$, determine the vertex, axis of symmetry, x- and y-intercepts, and range.

Final Answer: ____________________

Q60 [Go to Solution p. 293] [Back to TOC]

For $h(x) = 3\ln(x-1) - 2$, state the domain, identify any intercepts, and describe any horizontal or vertical asymptotes.

Final Answer: ____________________

Q61 [Go to Solution p. 293] [Back to TOC]

For $p(x) = 2\cos x - 1$, state the amplitude, period, range, and find all zeros of p in $[0, 2\pi]$.

Final Answer: ____________________

Name: ____________________ **Date:** ________________

Name: ______________________ **Date:** ______________

SL 2.4 Key properties of graphs, curve sketching and points of intersection

Q62 [Go to Solution p. 294] [Back to TOC]

A quadratic function $f(x) = -x^2 + 4x + 1$ is given. Using graphing technology, determine:

1. the axis of symmetry and the vertex;
2. the y-intercept and the x-intercepts (zeros), to 3 s.f. if necessary;
3. the maximum (or minimum) value of f and where it occurs;
4. whether the graph has any vertical or horizontal asymptotes, and whether f is even, odd, or neither (give a brief reason).

Final Answer: ____

____

Q63 [Go to Solution p. 294] [Back to TOC]

A rational function $g(x) = \dfrac{3x-6}{x+2}$ is given. Using graphing technology, find:

1. the domain of g;
2. the x- and y-intercepts;
3. the vertical and horizontal asymptotes and the end behaviour as $x \to \pm\infty$;
4. whether there are any holes in the graph and whether it has any symmetry.

Final Answer: ____

____

Q64 [Go to Solution p. 295] [Back to TOC]

Using graphing technology, find the point(s) of intersection of the curves

$$y = x^2 - 4x + 1 \quad \text{and} \quad y = 2x - 3.$$

Give exact values if possible; otherwise state coordinates correct to 3 d.p.

Name: ____________________ **Date:** ____________

Final Answer: ________________________________

Q65 [Go to Solution p. 295] [Back to TOC]

Using graphing technology, determine the point(s) of intersection of

$$y = 3^x \quad \text{and} \quad y = x + 2.$$

State each intersection coordinate to 3 d.p. and verify by substitution.

Final Answer: ________________________________

SL 2.5 Modelling linear, quadratics, exponential, cubic, sinusoidal and direct/inverse proportion

Q66 [Go to Solution p. 297] [Back to TOC]

(SL 2.5 – Linear models) Between 9:00 and 15:00 the temperature in a greenhouse changes linearly. At 9:00 it is 23°C and at 15:00 it is 17°C. Let t be the number of hours after 9:00 and $T(t) = mt + c$ the temperature model.

1. Find m and c.
2. Interpret m and c in context.
3. Predict the temperature at 12:00.

Final Answer: ____

____

Q67 [Go to Solution p. 297] [Back to TOC]

(SL 2.5 – Linear models; piecewise) A data plan charges \$0 for the first 100 MB, \$0.03 per MB for the next 900 MB, and \$0.01 per MB beyond 1000 MB. Let x be total MB used and $C(x)$ the cost in dollars.

1. Define $C(x)$ as a piecewise linear function.
2. Find $C(750)$ and $C(1400)$.
3. Sketch $C(x)$ for $0 \leq x \leq 1600$.

Final Answer: ____

____

Q68 [Go to Solution p. 297] [Back to TOC]

(SL 2.5 – Quadratic models) A ball is thrown and its height (m) after t seconds is modelled by a quadratic with vertex at (1.5, 6.1) and initial height 1.6 m.

1. Write $h(t)$ in vertex form and expand to standard form.
2. Find the axis of symmetry, intercepts, and the time the ball hits the ground.

Name: ______________________ **Date:** ______________

Final Answer: ____

____

Q69 [Go to Solution p. 298] [Back to TOC]

(SL 2.5 – Quadratic models) An arch is modelled by $y = ax^2 + bx + c$. It meets the ground at $x = -4$ and $x = 10$, and its height above the pier at $x = 0$ is 12 m.

1. Determine a, b, c.
2. State the axis of symmetry, the vertex, and the intercepts.

Final Answer: ____

____

Q70 [Go to Solution p. 299] [Back to TOC]

(SL 2.5 – Exponential growth/decay with horizontal asymptote) A lake's fish population is modelled by $P(t) = ke^{-0.4t} + 1200$ where t is in years. If $P(0) = 300$:

1. Find k and hence the model.
2. State the horizontal asymptote and its meaning.
3. Find t when $P(t) = 900$.

Final Answer: ____

____

Q71 [Go to Solution p. 300] [Back to TOC]

(SL 2.5 – Exponential decay; $f(x) = ka^{-x} + c$) A hot drink cools according to $T(t) = 22 + 48\,a^{-t}$ $(a > 0)$, where T is in °C and t in minutes. If $T(30) = 40$:

1. Find a.

2. Estimate the time when $T = 30°\text{C}$.
3. State the equation of the horizontal asymptote and interpret it.

Final Answer: ____

____

Q72 [Go to Solution p. 300] [Back to TOC]

(SL 2.5 – Direct variation $f(x) = ax^n$, $n \in \mathbb{Z}$) The mass M (g) of a solid cube varies directly with the cube of its side length s (cm). When $s = 2$, $M = 160$.

1. Find the model $M(s)$.
2. Calculate M when $s = 5$.

Final Answer: ____

____

Name: ______________________ **Date:** ______________

SL 2.6 Modelling skills, using, choosing and context

Q73 [Go to Solution p. 302] [Back to TOC]

Choose and justify a model. A storage tank is drained at a constant rate. At 9:00 the depth of water is 2.4 m and at 10:30 it is 1.5 m.

1. Choose a suitable model for the depth $h(t)$ (metres) t hours after 9:00. Determine its parameters.
2. State a reasonable domain for t and explain why.
3. Predict the time the tank will be empty and comment on whether this is interpolation or extrapolation.

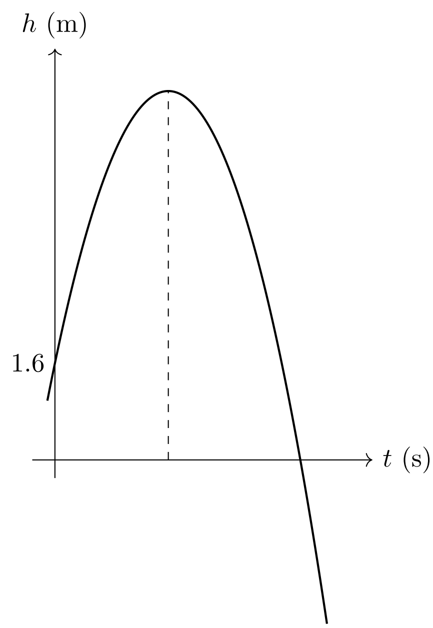
Final Answer: ____

____

Q74 [Go to Solution p. 302] [Back to TOC]

Quadratic (projectile). A ball is thrown and its height (m) $h(t)$ satisfies: the maximum height is 6.1 m at $t = 1.5$ s and $h(0) = 1.6$ m.

1. Find $h(t)$ in the form $a(t - 1.5)^2 + 6.1$ and then expand.
2. Find when the ball hits the ground.
3. Give a suitable domain for t and one limitation of the model.



Final Answer: ______________

Q75 [Go to Solution p. 302] [Back to TOC]

Exponential approach to an asymptote. A population approaches a limiting value of 1200. It is 300 at $t = 0$ years and 900 at $t = 3$ years. Model $P(t)$ with $P(t) = L - Ae^{-kt}$.

1. Find L, A and k.
2. State the horizontal asymptote and interpret it.
3. Predict $P(5)$ and explain why long-term forecasts may be unreliable.

Final Answer: ______________

Q76 [Go to Solution p. 302] [Back to TOC]

Direct variation (cube). The mass M (g) of a metal cube varies directly as the cube of its side length s (cm). When $s = 2$, $M = 160$.

1. Find the model $M(s)$.
2. Estimate the mass when $s = 5$.
3. State the domain for s and one modelling assumption.

Final Answer: ______________

Q77 [Go to Solution p. 303] [Back to TOC]

Inverse square law. Light intensity I at distance x (m) from a source satisfies $I(x) = k/x^2$. If $I(2) = 900$ (lux),

Name: ______________________ **Date:** ______________

1. Determine k and write $I(x)$.
2. Find the distance where $I = 100$.
3. Identify any vertical asymptote and give a realistic domain.

Final Answer: ____

____

Q78 [Go to Solution p. 303] [Back to TOC]

Cubic from intercept information. A cubic has x–intercepts at -2, 1 and 4 and y–intercept -8.

1. Find $f(x)$ and sketch its general shape (no scale required).
2. State its end behaviour as $x \to \pm\infty$.
3. Use the model to estimate $f(3)$ and comment on the reasonableness of the estimate.

Final Answer: ____

____

Q79 [Go to Solution p. 303] [Back to TOC]

Piecewise linear cost model. A data plan charges \$0 for the first 100 MB, \$0.03 per MB for the next 900 MB, and \$0.01 per MB thereafter.

1. Write a piecewise function $C(x)$ for cost (in dollars) for usage x MB.
2. Compute $C(750)$ and $C(1400)$.
3. Sketch $C(x)$ and state a suitable domain. Explain any kinks (non-differentiable points).

Final Answer: ____

____

Name: ____________________ **Date:** ______________

Q80 [Go to Solution p. 303] [Back to TOC]

Sinusoidal (seasonal daylight). In a city, the shortest daylight is 9 h and the longest is 15 h. Assume a period of 365 days with a maximum at day 172.

1. Build a model $H(t) = a\cos(b(t-c)) + d$ for daylight hours t days after Jan 1; determine a, b, c, d.
2. Estimate $H(20)$ and $H(250)$.
3. Explain why the model should not be used to predict over many years without adjustment.

Final Answer: ________________________________

Q81 [Go to Solution p. 303] [Back to TOC]

Model choice from data (technology allowed). A biologist measures nutrient concentration x and growth rate y:

x	1	2	3	4	5	6	7	8	9	10
y	0.8	1.1	1.5	2.2	2.7	3.3	4.0	4.7	5.3	5.9

1. Plot the data. Which family (linear / power / exponential) appears suitable? Justify from shape.
2. Fit your chosen model (parameters by technology). Report the equation and R^2.
3. Comment on the appropriateness and the dangers of extrapolation to $x = 15$.

Final Answer: ________________________________

Q82 [Go to Solution p. 303] [Back to TOC]

Testing and reflecting on a model. A linear model fitted to smartphone battery life vs. number of background apps gives

$$L(a) = 10.5 - 0.35\,a \quad \text{(hours for } a \text{ apps).}$$

1. Use the model to predict $L(8)$ and $L(40)$. Comment on reasonableness.
2. Suggest a more suitable model or domain restriction if the prediction is unrealistic for large a.
3. Describe one additional piece of data you would collect to improve the model.

Final Answer: ____

____

Q83 [Go to Solution p. 304] [Back to TOC]

Determine parameters from conditions. A rectangular pool is being filled; its depth (m) over time (min) is modelled by a quadratic $d(t) = at^2 + bt + c$. At $t = 0$, $d = 0$. At $t = 10$, $d = 0.3$ and the rate of rise is 0.06 m/min.

1. Find a, b, c by solving simultaneous equations.
2. For what times is $d(t)$ increasing? State the domain relevant to this context.
3. Estimate $d(25)$ and comment on whether this is interpolation or extrapolation.

Final Answer: ____

____

Q84 [Go to Solution p. 304] [Back to TOC]

Compare two plausible models. The number of views V of a new post after t hours is recorded:

t	0	1	2	3	4	6
V	120	220	360	520	660	900

1. Fit (i) a linear model $V = mt + c$ and (ii) an exponential $V = a\,e^{kt}$ (use technology).
2. Compare the two models using residuals and R^2. Which would you choose and why?
3. Use your chosen model to predict the time when $V = 1500$ and discuss reliability.

Final Answer: ____

Name: ______________________ **Date:** ______________

Name: ______________________ **Date:** ______________

AHL 2.7 Composite and Inverse Functions

Q85 [Go to Solution p. 305] [Back to TOC]

Given $f(x) = \dfrac{2x-3}{x+1}$ (domain $x \neq -1$), find $f^{-1}(x)$ and identify its domain and range. Restrict the domain of f to make it one-to-one and state the corresponding range.

Final Answer: ____

____

Q86 [Go to Solution p. 305] [Back to TOC]

For $h(x) = x^2 + 4x + 7$, determine a domain on which h is invertible and find $h^{-1}(x)$ on that domain.

Final Answer: ____

____

Name: ____________________ **Date:** ________________

AHL 2.8 Transformations of Graphs

Q87 [Go to Solution p. 305] [Back to TOC]

Describe the effect of each transformation applied to a base function $y = f(x)$: (i) $y = f(x) + 3$; (ii) $y = f(x - 2)$; (iii) $y = -f(x)$; (iv) $y = f(2x)$.

Final Answer: ____

____

Q88 [Go to Solution p. 306] [Back to TOC]

Starting with the function $y = \sqrt{x}$, apply the following transformations in order: (1) shift right by 3 units; (2) reflect in the x-axis; (3) apply a vertical stretch by factor 2. Write the equation after each step and the final equation.

Final Answer: ____

____

Q89 [Go to Solution p. 306] [Back to TOC]

For $f(x) = |x|$, write the equation obtained by (i) shifting left 4 units and up 2 units; (ii) reflecting in the y-axis and then applying a vertical stretch by factor 3.

Final Answer: ____

____

Q90 [Go to Solution p. 306] [Back to TOC]

Explain why performing a horizontal shift followed by a horizontal stretch is not the same as performing the stretch first and then the shift. Illustrate with a concrete example.

Name: ____________________ **Date:** ______________

Final Answer: ____

Q91 [Go to Solution p. 306] [Back to TOC]

(AHL 2.8 — Transformations & order) Starting from $y = x^2$, obtain $y = 3(x-2)^2 - 5$:

1. Write the sequence of elementary transformations (with equations after each step).
2. Show that swapping the order of "horizontal shift by 2" and "vertical stretch by 3" does not change the final curve, but explain (with a short example) why order *does* matter for combinations like $y = f(2x-4)$.

Final Answer: ____

AHL 2.9 Additional Modelling Families

Q92 [Go to Solution p. 308] [Back to TOC]

A radioactive sample has half-life 12 hours and initial quantity $N_0 = 500$. Write the function $N(t)$ describing the quantity remaining at time t and compute $N(30)$.

Final Answer: ____________________

Q93 [Go to Solution p. 309] [Back to TOC]

A logistic model for population $P(t)$ with carrying capacity $L = 120$ passes through $(0, 20)$ and $(6, 60)$. The model has the form $P(t) = \dfrac{L}{1 + Ce^{-kt}}$. Determine the constants C and k and state when $P = L/2$.

Final Answer: ____________________

Q94 [Go to Solution p. 309] [Back to TOC]

A daily tide height (in metres) can be modelled by $H(t) = a\sin(bt - c) + d$ with period 12.4 hours, maximum 5.8 m, minimum 0.6 m, and a high tide occurring at $t = 3.1$ h. Determine a, b, c, d.

Final Answer: ____________________

Q95 [Go to Solution p. 309] [Back to TOC]

Define the piecewise function

Name: ______________________________ **Date:** ____________________

$$f(x) = \begin{cases} mx+2, & x<1 \\ x^2+k, & x \geq 1 \end{cases}$$

and choose m, k so that f is continuous at $x = 1$.

Final Answer: ____

Name: ____________________ **Date:** ______________

AHL 2.10 — Scaling large and small numbers and graphs)

Q96 [Go to Solution p. 311] [Back to TOC]

Scaling large and small numbers using logarithms. A table shows the population of a certain bacteria culture over time:

Time t (hours)	Population P
0	1.5×10^2
1	3.0×10^2
2	6.0×10^2
3	1.2×10^3
4	2.4×10^3

1. Plot P against t on a standard (linear) scale.
2. Plot $\log_{10} P$ against t and describe the shape of the graph.
3. Explain why using a logarithmic scale for P may be more appropriate in this context.

Final Answer: ______________________________

____

Q97 [Go to Solution p. 312] [Back to TOC]

Linearizing exponential data. A radioactive isotope has activity $A(t) = A_0 e^{-kt}$, where t is in days.

1. Show that $\ln A$ is a linear function of t.
2. Given the data below, plot $\ln A$ against t and determine k from the gradient.

t (days)	A (counts/min)
0	850
2	623
4	456
6	333
8	243

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

Q98 [Go to Solution p. 312] [Back to TOC]

Linearizing power relationships. The table shows the period T and length L of a pendulum.

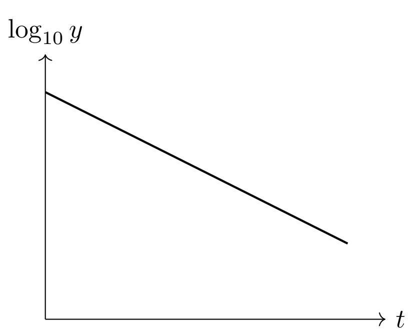
L (m)	T (s)
0.25	1.00
0.50	1.42
0.75	1.73
1.00	2.01
1.25	2.23

1. Theoretical models suggest $T = kL^n$. Show that $\log T$ is a linear function of $\log L$.
2. Plot $\log T$ against $\log L$ and estimate n and k.

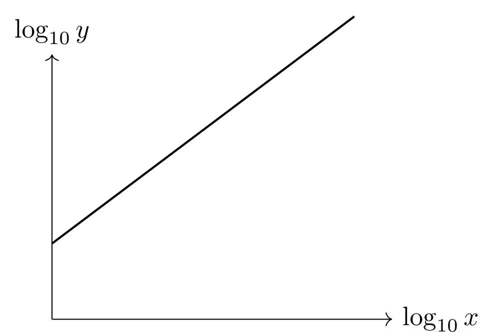
Final Answer: ______________________________

Q99 [Go to Solution p. 313] [Back to TOC]

Interpretation of semi-log and log-log graphs. The two graphs below show data for two different experiments:



Semi-log plot



Log-log plot

1. For the semi-log plot, explain what the straight-line relationship implies about y as a function of t.
2. For the log-log plot, explain what the straight-line relationship implies about y as a function of x.

Name: ______________________ **Date:** ______________

Final Answer: ____

Q100 [Go to Solution p. 313] [Back to TOC]

Comparing scales. Given the earthquake magnitudes and energy released:

Magnitude M	Energy (J)
5.0	2.0×10^{12}
6.0	6.3×10^{13}
7.0	2.0×10^{15}
8.0	6.3×10^{16}

1. Plot E against M using a logarithmic y-axis.
2. Describe the advantage of the logarithmic scale in representing this data.

Final Answer: ____

Name: ____________________ **Date:** ______________

Topic 3 — Geometry and Trigonometry (SL 3.1–3.6, AHL 3.7–3.16)

Overview (SL) Covers geometry in two and three dimensions, including 3D measurements, triangle trigonometry, applications of trigonometry, circle arcs and sectors, perpendicular bisectors, and Voronoi diagrams. Introduces trigonometric ratios, sine and cosine rules, unit circle basics, and radian measure.

Overview (HL) Extends SL content with advanced radian measure and circular sectors, unit circle applications, trigonometric equations, matrix transformations, vector arithmetic, vector equations of lines, vector kinematics, dot and cross products, and graph theory (including adjacency matrices and optimisation problems like the Chinese Postman and Travelling Salesman Problems).

Real-World Use

- Architecture, surveying, and structural engineering
- Navigation, GPS, and triangulation in mapping
- Computer graphics, animation, and simulation
- Network design, logistics, and route optimisation
- Robotics, motion planning, and kinematics

Common Misconceptions

- Mixing degrees and radians in calculations
- Confusing opposite, adjacent, and hypotenuse sides
- Using the wrong trigonometric rule for obtuse triangles
- Forgetting vector direction and magnitude distinction
- Misinterpreting graph theory diagrams and adjacency matrices
- Assuming all graphs are connected or planar

Advice for SL

- Draw and label diagrams before any calculation
- For non-right triangles, choose sine or cosine rule based on given data
- Check angle units before using trigonometric functions
- Practice converting between degrees and radians
- In Voronoi diagrams, identify seed points clearly before plotting regions

Advice for HL

- For vectors, track both magnitude and direction and check units
- In matrix transformations, understand geometric meaning before computation
- In graph theory:
  - Clearly define vertices, edges, and weights before starting
  - Use diagrams to visualise shortest paths, circuits, or connected components

  - Break complex problems (e.g., Chinese Postman, Travelling Salesman) into smaller steps
  - Know the difference between Eulerian and Hamiltonian paths
- For trigonometric equations, consider full domain and periodicity when finding solutions

Name: ______________________ **Date:** ______________

SL 3.1 3D Geometry and Measurements

Q101 [Go to Solution p. 317] [Back to TOC]

For $A(2, -1, 3)$ and $B(-4, 5, 1)$ in three–space, compute the distance $|AB|$ and the coordinates of the midpoint M.

Final Answer: ____

____

Q102 [Go to Solution p. 317] [Back to TOC]

A right circular cone has base radius $r = 6\,\text{cm}$ and slant height $\ell = 10\,\text{cm}$. (i) Find its height h. (ii) Determine its surface area (lateral plus base). (iii) Determine its volume.

Final Answer: ____

____

Q103 [Go to Solution p. 317] [Back to TOC]

In a right pyramid with square base side $a = 12\,\text{cm}$ and height $h = 15\,\text{cm}$, compute the volume and total surface area. Recall that the lateral faces are congruent isosceles triangles.

Final Answer: ____

____

Name: ______________________ Date: ______________

SL 3.2 Triangle Trigonometry

Q104 [Go to Solution p. 318] [Back to TOC]

In $\triangle ABC$, let $a = 8$, $b = 11$ and angle $C = 52°$. Use appropriate trigonometric rules to find (i) the area of the triangle, (ii) side c, and (iii) angle A.

Final Answer: ______________________

Q105 [Go to Solution p. 318] [Back to TOC]

A ladder of length 6.8 m leans against a vertical wall, making an angle of 68° with the horizontal ground. How high up the wall does the ladder reach? Give your answer to the nearest centimetre.

Final Answer: ______________________

Q106 [Go to Solution p. 318] [Back to TOC]

In $\triangle XYZ$, the sides have lengths $x = 12$, $y = 10$, and $z = 8$. Determine $\angle X$ to one decimal place.

Final Answer: ______________________

Name: ____________________ **Date:** ______________

SL 3.3 Applications of Trigonometry

Q107 [Go to Solution p. 319] [Back to TOC]

From a point P on level ground, the angle of elevation to the top of a tower is $28°$. If P is $65\,\text{m}$ from the base of the tower, find the height of the tower.

Final Answer: ______________________________

Q108 [Go to Solution p. 319] [Back to TOC]

Two points A and B lie on level ground separated by $400\,\text{m}$. The angle of elevation to the top of a hill is $14°$ from A and $21°$ from B (with B closer to the hill). Assuming A, B and the foot of the hill are collinear, find the height of the hill.

Final Answer: ______________________________

Q109 [Go to Solution p. 319] [Back to TOC]

(SL 3.3 — Bearings) A ship leaves harbour H and sails $18\,\text{km}$ on a bearing of $065°$ to point A, then changes course to a bearing of $145°$ and sails $12\,\text{km}$ to point B.

1. Draw a labelled bearing diagram from H showing A and B.
2. Calculate the straight-line distance HB.
3. Find the bearing of B from H (to the nearest degree).

Final Answer: ______________________________

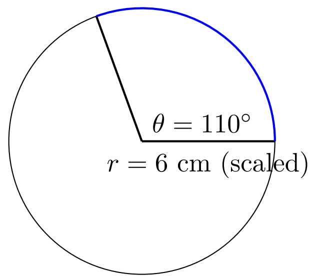
Name: ____________________ **Date:** ______________

SL 3.4 — Circle arc & sector

Q110 [Go to Solution p. 321] [Back to TOC]

In a circle of radius $r = 6$ cm, the central angle $\theta = 110°$.

1. Find the arc length s.
2. Find the sector area A.



Final Answer: ______________________________

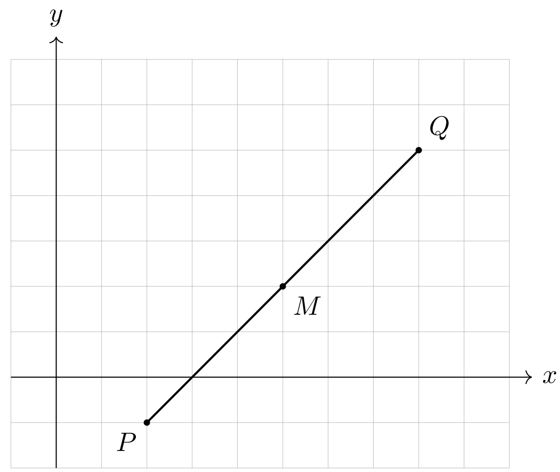
Name: ______________________ **Date:** ______________

AHL 3.5 Perpendicular bisector

Q111 [Go to Solution p. 322] [Back to TOC]

(SL 3.5 — Perpendicular bisector) Given $P(2, -1)$ and $Q(8, 5)$:

1. Find the midpoint M and the gradient of PQ.
2. Determine the equation of the perpendicular bisector of PQ in the form $ax + by + d = 0$.



Final Answer: ____

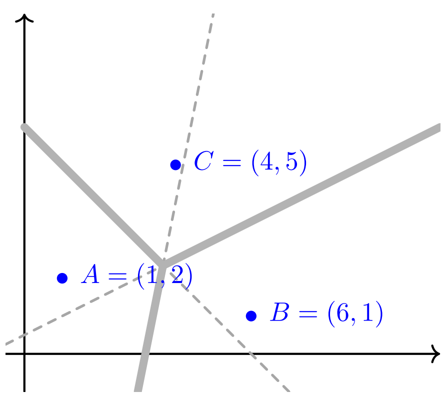
Name: ______________________ Date: ______________

SL 3.6 Voronoi diagrams: sites, vertices, edges, cells

Key ideas (what you must know).

1. A **Voronoi diagram** (for a set of sites/points) partitions the plane into **cells**; each cell contains all points *closer to its site* than to any other site.
2. A boundary between two cells lies on the **perpendicular bisector** of the segment joining the two corresponding sites.
3. **Edges** are pieces of these bisectors; **vertices** are intersection points where three (or more) edges meet.
4. **Nearest–neighbour interpolation:** any point in a cell is assigned the same value as that of its site (e.g., rainfall reading).
5. **"Toxic waste dump" point:** the location *maximising* distance from all sites is at a Voronoi **vertex** (equidistant from three sites in exam questions).

- In IB examinations, coordinates of sites are given; you may be asked to (i) find a boundary equation, (ii) decide which site is closest to a point, (iii) find/justify the dump location, or (iv) estimate area of a simple polygonal cell.
- Workflow to find a boundary between sites $S_1(x_1, y_1)$ and $S_2(x_2, y_2)$:
    1. Midpoint $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
    2. Slope $m = \frac{y_2-y_1}{x_2-x_1}$ (if $x_2 \neq x_1$); perpendicular slope $m_\perp = -\frac{1}{m}$.
    3. Use point–slope form through M: $y - y_M = m_\perp(x - x_M)$. Handle vertical/horizontal cases separately.



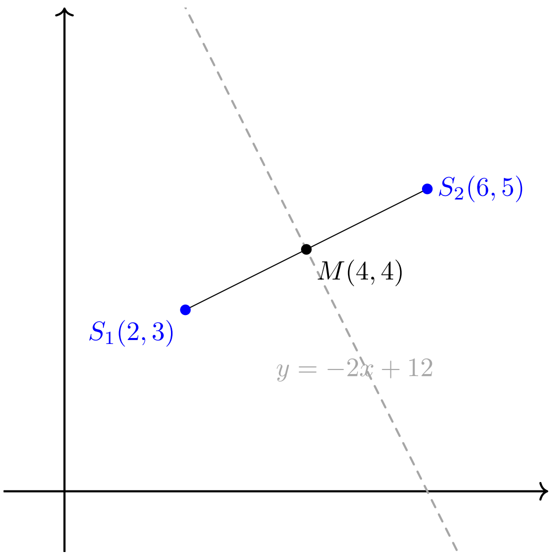
Worked example 1 — Boundary line between two cells. Sites $S_1(2, 3)$ and $S_2(6, 5)$. *Find the equation of the boundary between their cells.*

Midpoint: $M = \left(\frac{2+6}{2}, \frac{3+5}{2}\right) = (4, 4)$. *Slope of* $\overline{S_1S_2}$: $m = \frac{5-3}{6-2} = \frac{1}{2}$. *Perpendicular slope:* $m_\perp = -2$.

Through M with slope -2:

$$y - 4 = -2(x - 4) \quad \Rightarrow \quad y = -2x + 12.$$

Hence the boundary between the cells of S_1 and S_2 is $y = -2x + 12$.

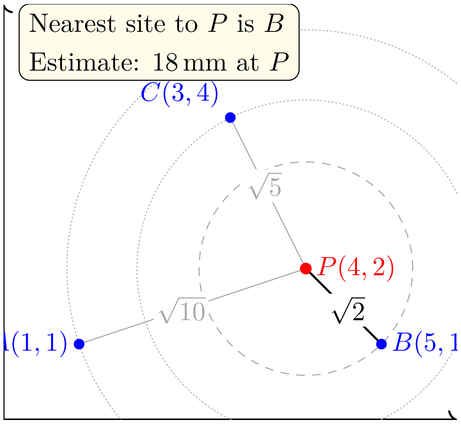


Worked example 2 — Nearest site and interpolation. Weather stations (sites) record rainfall: $A(1,1)$: 12 mm, $B(5,1)$: 18 mm, $C(3,4)$: 10 mm. Estimate rainfall at $P(4,2)$ using nearest–neighbour interpolation.

Compute distances:

$$AP = \sqrt{(4-1)^2 + (2-1)^2} = \sqrt{10},$$
$$BP = \sqrt{(4-5)^2 + (2-1)^2} = \sqrt{2},$$
$$CP = \sqrt{(4-3)^2 + (2-4)^2} = \sqrt{5}.$$

Smallest distance is to B. $\Rightarrow$ Estimated rainfall at P is 18 mm (the value at site B).



Worked example 3 — Toxic waste dump (intersection of three edges).

Sites $A(0,0)$, $B(6,0)$, $C(3,6)$. The optimal point is equidistant from all three sites (a Voronoi vertex). Find the intersection of the perpendicular bisectors of AB and BC.

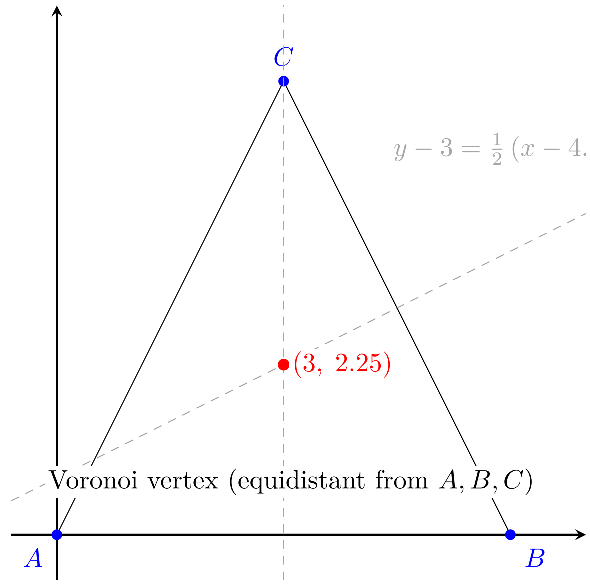
Bisector of AB: midpoint $(3,0)$; since AB is horizontal, the bisector is $x = 3$.

Bisector of BC: midpoint $(4.5, 3)$, slope of BC is -2, hence perpendicular slope $m_\perp = \frac{1}{2}$:

$$y - 3 = \tfrac{1}{2}\,(x - 4.5).$$

Intersection with $x = 3$: $y - 3 = \frac{1}{2}(3 - 4.5) = -\frac{3}{4} \Rightarrow y = 2.25$.

Therefore the dump location is $\boxed{(3,\ 2.25)}$.



Q112 [Go to Solution p. 323] [Back to TOC]

Given sites $A(0,0)$, $B(4,0)$ and $C(2,3)$: (i) Write equations of the perpendicular bisectors of $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$. (ii) Sketch the Voronoi diagram determined by A, B, C. (iii) Decide to which region the point $P(3,2)$ belongs.

Final Answer: ____

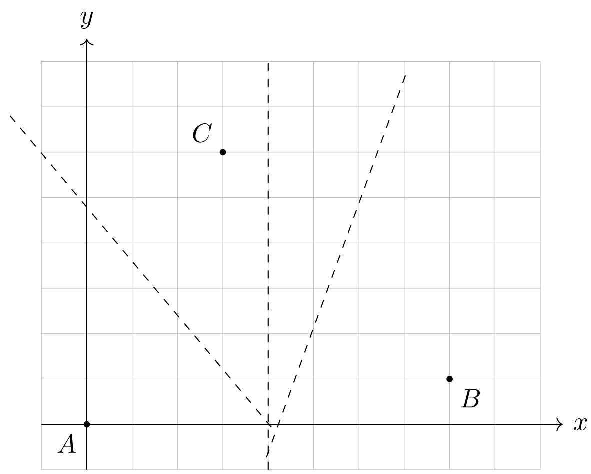
____

Q113 [Go to Solution p. 323] [Back to TOC]

The three facilities are located at $A(0,0)$, $B(8,1)$, $C(3,6)$.

1. Construct (by reasoning/sketch) the Voronoi diagram for $\{A, B, C\}$.
2. A "toxic waste dump" must be located to maximize the minimum distance to the facilities. Mark the candidate site on your diagram and justify.

Name: ______________________ **Date:** ______________



Final Answer: ____

Name: ______________________ **Date:** ______________

AHL 3.7 Radian Measure and Circular Sectors

Q114 [Go to Solution p. 325] [Back to TOC]

Convert $126°$ to radians and $\frac{7\pi}{9}$ radians to degrees.

Final Answer: ______________________________

Q115 [Go to Solution p. 325] [Back to TOC]

In a circle of radius 9 cm, an arc has length 14.4 cm. Find the central angle in (i) radians and (ii) degrees. Then determine the area of the corresponding sector.

Final Answer: ______________________________

Q116 [Go to Solution p. 325] [Back to TOC]

A sector of area $75\,\text{cm}^2$ has central angle 1.5 rad. Find the radius of the circle and the length of the arc bounding the sector.

Final Answer: ______________________________

Name: ______________________ Date: ______________

AHL 3.8 Unit Circle and Trigonometric Equations

Q117 [Go to Solution p. 326] [Back to TOC]

On the unit circle, mark the coordinates corresponding to $\theta = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$. State the exact coordinates of each point.

Final Answer: ______________________

Q118 [Go to Solution p. 326] [Back to TOC]

Solve on $0 \leq \theta < 2\pi$ the equation $2\sin\theta\cos\theta = \sin\theta$. List all solutions in radians.

Final Answer: ______________________

Q119 [Go to Solution p. 327] [Back to TOC]

In $\triangle ABC$, let $a = 8$, $A = 40°$ and $b = 10$. Determine all possible values of angle B (if any) and find the corresponding values of C and c. Explain why there may be two solutions.

Final Answer: ______________________

Name: ______________________ **Date:** ______________

AHL 3.9 Matrix Transformations

Q120 [Go to Solution p. 328] [Back to TOC]

Give the 2×2 matrix that reflects points in the x–axis and state its determinant. Then find the matrix that reflects points in the line $y = x$ and state its determinant.

Final Answer: ____

____

Q121 [Go to Solution p. 328] [Back to TOC]

Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (a rotation by $90°$ anticlockwise). A point $P(1, 3)$ is mapped to P' by the transformation $A\mathbf{x} + \mathbf{t}$ with translation $\mathbf{t} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Determine P' and the area scaling factor associated with A.

Final Answer: ____

____

Q122 [Go to Solution p. 328] [Back to TOC]

Consider the matrix $M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ acting on a unit square in the plane. Compute $|\det M|$ and interpret its value in terms of area. If the unit square has vertices $(0, 0), (1, 0), (1, 1), (0, 1)$, sketch or describe qualitatively the image of the square under M.

Final Answer: ____

____

Name: ____________________ **Date:** ______________

AHL3.10 — Vector arithmetic

Q123 [Go to Solution p. 330] [Back to TOC]

Scalar or vector? For each quantity, state whether it is a scalar or a vector: mass, displacement, temperature, force, velocity, speed, electric current.

Final Answer: ____

Q124 [Go to Solution p. 330] [Back to TOC]

Directed line segment and component forms. Let $A(2, -1)$ and $B(-3, 4)$.

1. Write $\overrightarrow{AB}$ as a column vector.
2. Write $\overrightarrow{AB}$ in $\mathbf{i}, \mathbf{j}$ form.
3. What is the magnitude $|\overrightarrow{AB}|$?

Final Answer: ____

Q125 [Go to Solution p. 330] [Back to TOC]

Base vectors in 3D. Express $\mathbf{v} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$ in terms of the base vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Hence find $|\mathbf{v}|$.

Final Answer: ____

Q126 [Go to Solution p. 330] [Back to TOC]

Name: ______________________ **Date:** ______________

Zero and negative vectors. Given $\mathbf{u} = \langle a, b \rangle$ with $a, b \in \mathbb{R}$:

1. Write $-\mathbf{u}$ and $|-\mathbf{u}|$.
2. For which values of a, b is $\mathbf{u} = \mathbf{0}$?

Final Answer: ______________________________

____

Q127 [Go to Solution p. 330] [Back to TOC]

Sum and difference (algebraic). Let $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$.

1. Compute $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$.
2. Find their magnitudes.

Final Answer: ______________________________

____

Q128 [Go to Solution p. 330] [Back to TOC]

Resultant of multiple vectors. A particle experiences forces $\mathbf{F}_1 = 4\mathbf{i} + 3\mathbf{j}$ N, $\mathbf{F}_2 = -5\mathbf{i} + 2\mathbf{j}$ N and $\mathbf{F}_3 = 2\mathbf{i} - 6\mathbf{j}$ N.

1. Find the resultant force $\mathbf{R}$.
2. Determine $|\mathbf{R}|$.

Final Answer: ______________________________

____

Q129 [Go to Solution p. 331] [Back to TOC]

Name: ________________ **Date:** ____________

Parallel vectors and scalar multiples.

1. For what value(s) of k is $(6, -9)$ parallel to $(2k, -3k)$?
2. If $\mathbf{p} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, find k so that $k\mathbf{p}$ has magnitude 10.

Final Answer: ________________

Q130 [Go to Solution p. 331] [Back to TOC]

Position vectors. Let O be the origin. The position vectors of A and B are $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

1. Express $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
2. If $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, find $\overrightarrow{AB}$ and its magnitude.

Final Answer: ________________

Q131 [Go to Solution p. 331] [Back to TOC]

Displacement by successive moves. A robot moves by $\mathbf{d}_1 = 5\mathbf{i} - 2\mathbf{j}$, then $\mathbf{d}_2 = -3\mathbf{i} + 4\mathbf{j}$, then $\mathbf{d}_3 = 2\mathbf{j}$.

1. Find the total displacement.
2. How far is the robot from its start point?

Final Answer: ________________

Q132 [Go to Solution p. 331] [Back to TOC]

Normalizing (unit vector).

1. Find the unit vector in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$.
2. A particle has speed $7\,\mathrm{m\,s^{-1}}$ in the direction of $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$. Find its velocity vector.

Final Answer: ______________________________

Q133 [Go to Solution p. 331] [Back to TOC]

Unit vector in 3D. Let $\mathbf{w} = \langle -2, 1, 2 \rangle$.

1. Compute the unit vector $\hat{\mathbf{w}}$.
2. Give a vector of length 15 in the same direction as $\mathbf{w}$.

Final Answer: ______________________________

Q134 [Go to Solution p. 331] [Back to TOC]

Unknown component from magnitude. Let $\mathbf{u} = k\mathbf{i} - 4\mathbf{j}$ with $|\mathbf{u}| = 10$.

1. Find the possible values of k.
2. For each k, write the unit vector in the direction of $\mathbf{u}$.

Final Answer: ______________________________

Q135 [Go to Solution p. 332] [Back to TOC]

Name: ____________________ **Date:** ______________

Geometric description from components. Vector $\mathbf{r} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$.

1. State the direction (as a bearing from the positive x-axis, in degrees, to the nearest degree).
2. Write a different vector parallel to $\mathbf{r}$ with magnitude 5.

Final Answer: ____________________

Q136 [Go to Solution p. 332] [Back to TOC]

Midpoint and median using position vectors. In triangle OAB, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$, let M be the midpoint of AB.

1. Show that $\overrightarrow{OM} = \dfrac{\mathbf{a}+\mathbf{b}}{2}$.
2. Hence find the vector of the median from O to AB.

Final Answer: ____________________

Q137 [Go to Solution p. 332] [Back to TOC]

Column $\leftrightarrow$ $\mathbf{i}, \mathbf{j}, \mathbf{k}$ conversion. Convert each vector into the other form:

1. $\begin{pmatrix} 4 \\ -7 \end{pmatrix}$,
2. $-2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$,
3. $\begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$.

Final Answer: ____________________

Name: ____________________ **Date:** ______________

Q138 [Go to Solution p. 332] [Back to TOC]

Resultant as sum of given directions. Two hikers pull a sled. Hiker A exerts 40 N due east; Hiker B exerts 30 N at 60° north of east.

1. Write each force as a vector in $\mathbf{i}, \mathbf{j}$.
2. Find the resultant force and its magnitude.

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

AHL3.11 — Vector equation of a line

Q139 [Go to Solution p. 333] [Back to TOC]

2D: Vector to parametric (and points). Given the line

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

(a) write its parametric equations for x and y; (b) find the points on the line corresponding to $\lambda = 0$ and $\lambda = 2$; (c) determine whether $A(7, 0)$ lies on the line.

Final Answer: ____

____

Q140 [Go to Solution p. 333] [Back to TOC]

3D: Parametric to vector. The line is given parametrically by

$$x = 1 + 2\lambda, \qquad y = -3 + \lambda, \qquad z = 4 - 5\lambda.$$

(a) Write the vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. (b) State a point on the line and a direction vector.

Final Answer: ____

____

Q141 [Go to Solution p. 333] [Back to TOC]

2D: Line through two points. Find a vector equation and parametric equations for the line through $P(4, -1)$ and $Q(-2, 5)$. Hence find where the line meets the x–axis.

Final Answer: ____

Q142 [Go to Solution p. 333] [Back to TOC]

3D: Line through two points. Let $A(2,-1,3)$ and $B(5,0,-2)$. (a) Find a direction vector for $\overline{AB}$. (b) Write the vector and parametric equations of the line through A and B. (c) Give the coordinates of the point on this line corresponding to $\lambda = -2$.

Final Answer: ____________________

Q143 [Go to Solution p. 333] [Back to TOC]

2D: Parallel lines and intersection. Consider

$$\ell_1 : \ \mathbf{r} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \qquad \ell_2 : \ \mathbf{r} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 8 \end{pmatrix}.$$

(a) State the relationship between ℓ_1 and ℓ_2 (same, parallel distinct, or intersecting). (b) If they intersect, find the point of intersection and the corresponding λ, μ.

Final Answer: ____________________

Q144 [Go to Solution p. 333] [Back to TOC]

3D: Intersecting or skew? Let

$$\ell_1 : \ x = 3 + \lambda, \ y = 1 + 2\lambda, \ z = -2 + 3\lambda, \qquad \ell_2 : \ x = 6 - 2\mu, \ y = -1 + \mu, \ z = 1 + \mu.$$

Determine whether the lines are parallel, intersecting, or skew. If they intersect, find the point of intersection.

Final Answer: ____________________

Name: ______________________ **Date:** ______________

Q145 [Go to Solution p. 334] [Back to TOC]

2D: From Cartesian to vector. The line has equation $y = \frac{1}{2}x - 3$. (a) Write a vector equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ for this line. (b) Give one possible choice of $\mathbf{a}$ and $\mathbf{b}$.

Final Answer: ______________________________

Q146 [Go to Solution p. 334] [Back to TOC]

3D: Point on a line? For

$$\mathbf{r} = \begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 6 \end{pmatrix},$$

decide whether $C(1, 2, 7)$ lies on the line. If so, find the corresponding value of λ.

Final Answer: ______________________________

Q147 [Go to Solution p. 334] [Back to TOC]

2D: Line through a point parallel to a given line. Find the vector and parametric equations of the line through $S(-5, 2)$ that is parallel to

$$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 7 \end{pmatrix}.$$

Final Answer: ______________________________

Name: ____________ **Date:** ____________

Q148 [Go to Solution p. 334] [Back to TOC]

3D: Fix a coordinate value. On the line

$$\mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix},$$

find the point where $z = 1$. State the corresponding value of λ.

Final Answer: ____________

Q149 [Go to Solution p. 334] [Back to TOC]

Mixed forms. A line passes through $P(1, 4, 0)$ and has direction vector proportional to $\langle 2, -1, 3 \rangle$. (a) Write the vector equation using parameter λ. (b) Convert to parametric form. (c) Find the value of λ at which the x–coordinate equals 7.

Final Answer: ____________

Q150 [Go to Solution p. 334] [Back to TOC]

2D: Intersection with another form. Let

$$\ell: \ \mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 1 \end{pmatrix}.$$

Find the intersection point (if any) of ℓ with the line $2x + y = 1$.

Final Answer: ____________

Name: ____________________ **Date:** ____________________

Name: ______________________ **Date:** ______________

AHL3.12 — Vector applications to kinematics

Q151 [Go to Solution p. 335] [Back to TOC]

2D constant velocity: position and path. A particle moves with constant velocity $\mathbf{v} = \langle 3, -2 \rangle$ m/s from initial position $\mathbf{r}_0 = \langle -4, 5 \rangle$ m at $t = 0$.

1. Write $\mathbf{r}(t)$ in the form $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$.
2. Find the position at $t = 6$ s and the displacement from $t = 2$ s to $t = 10$ s.
3. Eliminate t to obtain the Cartesian equation of the path.

Final Answer: ____

____

Q152 [Go to Solution p. 335] [Back to TOC]

3D constant velocity: meeting or not. Two particles move in $\mathbb{R}^3$:

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 7 \\ -1 \\ -2 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix},$$

with t in seconds and positions in metres.

1. Determine whether the particles ever occupy the same point at the same time.
2. If so, find the collision time and position; if not, explain why not.

Final Answer: ____

____

Q153 [Go to Solution p. 335] [Back to TOC]

Relative position and closest approach (2D). Two cars move on a plane:

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ 6 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \end{pmatrix}, \qquad \mathbf{r}_2 = \begin{pmatrix} 10 \\ -2 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

1. Write the relative position of car 2 from car 1, $\overrightarrow{12}(t) = \mathbf{r}_2 - \mathbf{r}_1$.
2. Find the time $t \geq 0$ when the cars are closest, and the minimum distance between them.

Final Answer: ____

____

Q154 [Go to Solution p. 335] [Back to TOC]

Ship safety check (constant velocities). A ship S_1 starts at $(2, 9)$ km and sails with velocity $\langle -8,\, 3\rangle$ km/h. Another ship S_2 starts at $(15, -3)$ km and sails with velocity $\langle -5,\, 1\rangle$ km/h.

1. Will the ships meet? If yes, find the meeting time and point.
2. Otherwise, find the minimum distance between them and the time it occurs.
3. State whether a 2 km safety radius is violated.

Final Answer: ____

____

Q155 [Go to Solution p. 335] [Back to TOC]

3D: crossing tracks vs. collision. Aircraft A and B fly with

$$\mathbf{r}_A = \begin{pmatrix} 30 \\ -20 \\ 2 \end{pmatrix} + t \begin{pmatrix} 6 \\ 4 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 0 \\ 40 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix},$$

where coordinates are km, t in minutes, and the third coordinate is altitude in km (constant).

1. Do their ground tracks (projections to the xy-plane) intersect? If so, at what ground point and time for each?
2. Do the aircraft collide? Justify your answer using the full 3D motion.

Final Answer: ____

Name: ______________________ Date: ______________

Q156 [Go to Solution p. 336] [Back to TOC]

Variable velocity given as components. A particle has velocity components (m/s)

$$v_x(t) = 7, \qquad v_y(t) = 6 - 4t,$$

and at $t = 0$ is at $(x, y) = (1, 2)$ m.

1. Find $x(t), y(t)$ and hence $\mathbf{r}(t)$.
2. Eliminate t to obtain the path y as a function of x.
3. Find the time when the speed is minimum and state that minimum speed.

Final Answer: ______________________________

Q157 [Go to Solution p. 336] [Back to TOC]

Projectile motion (special case of variable velocity). A ball is fired from the origin with initial speed $u = 20$ m/s at angle $\theta = 40^\circ$ above the horizontal. Ignore air resistance and take $g = 9.8$ m/s^2.

1. Write $v_x(t), v_y(t)$ and $x(t), y(t)$.
2. Find the time of flight, the range, and the maximum height.
3. Determine the equation of the trajectory $y(x)$.

Final Answer: ______________________________

Q158 [Go to Solution p. 336] [Back to TOC]

Projectile with a time shift. Another ball follows the same motion as in the previous question but is launched $a = 0.6$ s later. Model its position as $\mathbf{r}_2(t) = \mathbf{r}_1(t - a)$ for $t \geq a$.

1. Write the explicit parametric form of $\mathbf{r}_2(t)$.

2. Find all times (if any) when the two balls are at the same height y.
3. Do they ever have the same position at the same time? Justify.

Final Answer: ____

Q159 [Go to Solution p. 336] [Back to TOC]

Uniform circular motion (variable velocity with constant speed). A particle moves on the circle of radius 5 m centred at the origin with

$$\mathbf{r}(t) = \begin{pmatrix} 5\cos(\omega t) \\ 5\sin(\omega t) \end{pmatrix}, \qquad \omega > 0.$$

1. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$, and show the speed is constant.
2. State the direction of $\mathbf{a}(t)$ relative to $\mathbf{r}(t)$.
3. If the period is $T = 4\pi$ s, find ω and the numerical value of the speed.

Final Answer: ____

Q160 [Go to Solution p. 336] [Back to TOC]

Mixed: recover velocity from position. A particle's position is $\mathbf{r}(t) = \langle 2t - 1,\ 4 - 3e^{-t} \rangle$ m.

1. Find $\mathbf{v}(t)$ and $\mathbf{a}(t)$.
2. Determine the time when the velocity is horizontal.
3. Find the total distance travelled from $t = 0$ to $t = 3$ (state a definite integral; exact evaluation not required).

Final Answer: ____

Name: ________________ **Date:** ________________

Q161 [Go to Solution p. 337] [Back to TOC]

Chasing problem (relative motion). Runner A starts at $(0, 0)$ and runs east at 5 m/s. Runner B starts at $(60, 80)$ m and runs with constant velocity $\langle -3, -4\rangle$ m/s.

1. Write $\mathbf{r}_A(t)$ and $\mathbf{r}_B(t)$.
2. Find the time and minimum distance between A and B.
3. Decide whether B ever catches A.

Final Answer: ________________

Q162 [Go to Solution p. 337] [Back to TOC]

Reconstructing initial data from two sightings. A drone moves with constant velocity in 3D. At $t = 2$ s it is at $(4, -1, 7)$ m and at $t = 9$ s it is at $(18, 5, -8)$ m.

1. Find its constant velocity vector.
2. Determine its initial position $\mathbf{r}(0)$.
3. At what time is it closest to the point $(10, 0, 0)$?

Final Answer: ________________

Name: ____________________ **Date:** ______________

AHL3.13 — Vector dot and cross products

Q163 [Go to Solution p. 338] [Back to TOC]

Dot product and angle (3D). Let $\mathbf{u} = \langle 3, -1, 2\rangle$ and $\mathbf{v} = \langle 1, 4, -2\rangle$.

1. Compute $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u}|, |\mathbf{v}|$.
2. Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$ (in radians, to 3 s.f.).
3. State whether $\mathbf{u}$ and $\mathbf{v}$ are perpendicular.

Final Answer: ____________________________________

Q164 [Go to Solution p. 338] [Back to TOC]

Acute angle between two lines (3D).

$$\ell_1 : \ \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad \ell_2 : \ \mathbf{r} = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}.$$

Find the *acute* angle between the lines.

Final Answer: ____________________________________

Q165 [Go to Solution p. 338] [Back to TOC]

Cross product and right-hand rule. Let $\mathbf{a} = \langle 2, 1, 3\rangle$ and $\mathbf{b} = \langle -1, 4, 2\rangle$.

1. Compute $\mathbf{a} \times \mathbf{b}$ and its magnitude.
2. Find the unit vector $\mathbf{n}$ perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ given by the right-hand rule.

Final Answer: ____________________________________

Q166 [Go to Solution p. 338] [Back to TOC]

Area of a parallelogram and a triangle. Vectors $\mathbf{p} = \langle 3, 1, 0\rangle$ and $\mathbf{q} = \langle 1, 2, 0\rangle$ lie in the xy-plane.

1. Find the area of the parallelogram spanned by $\mathbf{p}$ and $\mathbf{q}$.
2. Hence find the area of the triangle with sides $\mathbf{p}$ and $\mathbf{q}$.

Final Answer: ______________________________

Q167 [Go to Solution p. 338] [Back to TOC]

Area of a triangle from three points (3D). For $P(1, 2, 3)$, $Q(3, -1, 4)$, $R(0, 2, 1)$, compute the area of $\triangle PQR$.

Final Answer: ______________________________

Q168 [Go to Solution p. 339] [Back to TOC]

Projection and component along a direction. Let $\mathbf{a} = \langle 3, 4, 0\rangle$ and $\mathbf{b} = \langle 1, 2, 2\rangle$.

1. Find the *scalar component* of $\mathbf{a}$ in the direction of $\mathbf{b}$.
2. Find the *vector projection* of $\mathbf{a}$ onto $\mathbf{b}$.

Final Answer: ______________________________

Q169 [Go to Solution p. 339] [Back to TOC]

Perpendicular component magnitude. For the vectors in the previous question, find the magnitude of the component of $\mathbf{a}$ perpendicular to $\mathbf{b}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$.

Final Answer: ______________________________

Q170 [Go to Solution p. 339] [Back to TOC]

Resolve a vector into parallel and perpendicular parts. Let $\mathbf{u} = \langle -2, 5, 1\rangle$ and $\mathbf{b} = \langle 4, -1, 2\rangle$. Write $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$ with $\mathbf{u}_{\parallel}$ parallel to $\mathbf{b}$ and $\mathbf{u}_{\perp}$ perpendicular to $\mathbf{b}$. Determine both vectors.

Final Answer: ______________________________

Q171 [Go to Solution p. 339] [Back to TOC]

Work done (dot product application). A constant force $\mathbf{F} = \langle 6, -2, 5\rangle$ N moves a particle through the displacement $\mathbf{d} = \langle 3, 4, -1\rangle$ m. Find the work done W in joules.

Final Answer: ______________________________

Q172 [Go to Solution p. 339] [Back to TOC]

Angle in 2D via dot product. Given $\mathbf{p} = \langle 5, 2\rangle$ and $\mathbf{q} = \langle -1, 4\rangle$,

1. find the angle θ between $\mathbf{p}$ and $\mathbf{q}$;
2. state whether the vectors are acute, right, or obtuse to each other.

Name: ______________________ **Date:** ______________

Final Answer: ____

Q173 [Go to Solution p. 340] [Back to TOC]

Acute angle between lines in the plane. Lines ℓ_1 and ℓ_2 have direction vectors $\langle 2, 3\rangle$ and $\langle -1, 4\rangle$, respectively. Find the acute angle between ℓ_1 and ℓ_2.

Final Answer: ____

Q174 [Go to Solution p. 340] [Back to TOC]

Mixed: show perpendicular via dot, area via cross. Vectors $\mathbf{u} = \langle 1, 2, 3\rangle$, $\mathbf{v} = \langle -2, 1, 0\rangle$, $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

1. Verify that $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
2. Find the area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.

Final Answer: ____

Name: ____________________ **Date:** ______________

AHL3.14 — Graph theory

AHL 3.14 — Graph theory: Key terms

Graph: A set of *vertices* (or *nodes*) connected by *edges*.

Vertex (node): A fundamental unit represented by a point in the graph.

Edge: A line connecting two vertices. Can be *weighted* or *unweighted*.

Adjacent vertices: Two vertices connected directly by an edge.

Adjacent edges: Two edges that share a common vertex.

Degree of a vertex: The number of edges incident to the vertex.

Simple graph: A graph with no loops and no multiple edges between the same pair of vertices.

Complete graph: A simple graph in which every pair of distinct vertices is connected by an edge.

Weighted graph: A graph where each edge has an associated numerical value (weight).

Connected graph: A graph where there is a path between any two vertices.

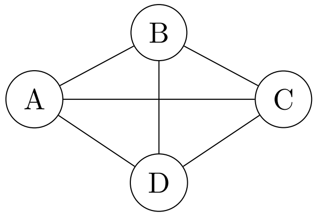
Strongly connected graph: In a directed graph, there is a directed path from every vertex to every other vertex.

Directed graph (digraph): A graph where edges have a direction, shown by arrows.

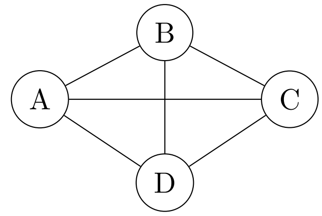
In-degree / Out-degree: In a directed graph, the in-degree is the number of incoming edges to a vertex; the out-degree is the number of edges leaving the vertex.

Subgraph: A graph whose vertices and edges are subsets of another graph.

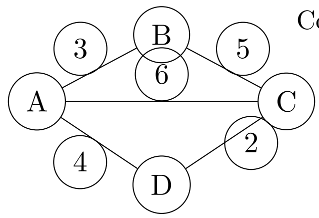
Tree: A connected graph with no cycles.



Simple graph



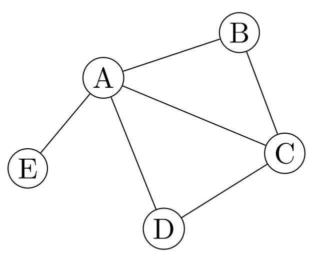
Complete graph K_4



Weighted graph

Q175 [Go to Solution p. 341] [Back to TOC]

Basic terms; degree of a vertex. Consider the undirected graph G:



(a) List the vertices and edges. (b) Which pairs of vertices are adjacent? (c) Find the degree of each vertex and write the degree sequence (non-increasing order).

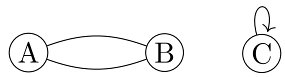
Final Answer: ____

____

Q176 [Go to Solution p. 341] [Back to TOC]

Simple vs. non-simple. For each of the following small graphs, state whether it is *simple*. If not, explain why (loop and/or multiple edge):

Name: ______________________ **Date:** ______________



Final Answer: ____

____

Q177 [Go to Solution p. 341] [Back to TOC]

Complete graphs. (a) For the complete graph K_5, state the degree of each vertex and the total number of edges. (b) In general, prove or state a formula for the number of edges in K_n and the degree of each vertex.

Final Answer: ____

____

Q178 [Go to Solution p. 341] [Back to TOC]

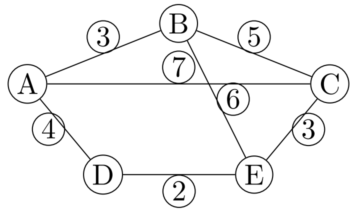
Adjacency matrix (undirected). For the graph in Question 1, write the adjacency matrix using the vertex order (A, B, C, D, E). Hence verify that the sum of the entries of the matrix equals $2|E|$.

Final Answer: ____

____

Q179 [Go to Solution p. 341] [Back to TOC]

Weighted graph: shortest path. In the weighted graph below, edge labels are distances in km.



(a) Find a shortest path from A to C and its total weight. (b) What is the length of the minimum A–E path?

Final Answer: ____

Q180 [Go to Solution p. 341] [Back to TOC]

Connectedness. For the undirected graph in Question 1, is G connected? If a vertex is removed to make it disconnected, give one possible choice and justify.

Final Answer: ____

Q181 [Go to Solution p. 341] [Back to TOC]

Directed graphs: in-degree and out-degree. A directed graph D has adjacency matrix (rows = sources, columns = targets) in the order (A, B, C, D):

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

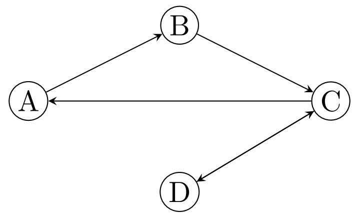
(a) For each vertex, find its out-degree and in-degree. (b) Is D strongly connected? Explain briefly.

Final Answer: ____

Name: ______ Date: ______

Q182 [Go to Solution p. 342] [Back to TOC]

Directed graph: strongly connected or not. Consider the digraph



Decide whether the digraph is *strongly connected*. If it is, give a directed path from D to A and from A to D; if not, explain which condition fails.

Final Answer: ______

Q183 [Go to Solution p. 342] [Back to TOC]

Model a real situation as a graph. A small metro network has stations $\{S_1, S_2, S_3, S_4, S_5\}$. Direct connections exist between S_1–S_2 (4 min), S_2–S_3 (2), S_3–S_5 (6), S_2–S_4 (5), S_4–S_5 (3), S_1–S_4 (8). (a) Represent this as a weighted graph. (b) What is the quickest travel time from S_1 to S_5? State the route.

Final Answer: ______

Q184 [Go to Solution p. 342] [Back to TOC]

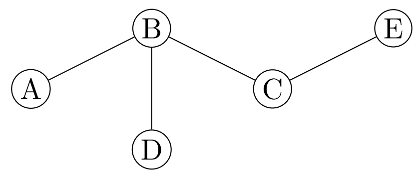
Subgraphs. Using the graph in Question 1, choose a non-trivial subgraph H by specifying a subset of vertices and edges. (a) Is H connected? (b) Does H contain a cycle? (c) State $|V(H)|$ and $|E(H)|$.

Final Answer: ______

Name: ____________________ **Date:** ____________

Q185 [Go to Solution p. 342] [Back to TOC]

Trees. Consider the following simple graph:



(a) Is this graph a tree? Justify using the defining properties. (b) List the leaves (vertices of degree 1). (c) If a new edge AD is added, is the resulting graph still a tree? Explain.

Final Answer: ____________________

Q186 [Go to Solution p. 342] [Back to TOC]

Counting edges via degrees (handshake). An undirected simple graph has degree sequence $(4, 4, 3, 3, 2, 2, 2, 2, 2)$. (a) How many vertices and edges does the graph have? (b) Could such a graph be complete? Why or why not?

Final Answer: ____________________

Q187 [Go to Solution p. 342] [Back to TOC]

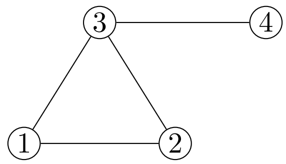
Complete/weighted hybrid. On K_4 with vertices $\{A, B, C, D\}$, assign symmetric weights $w(AB) = 1,\ w(AC) = 4,\ w(AD) = 3,\ w(BC) = 2,\ w(BD) = 5,\ w(CD) = 1$. (a) Draw the weighted complete graph. (b) Find the minimum-weight Hamiltonian path starting at A (list all ties if any).

Final Answer: ____________________

Name: ______________________ Date: ______________

Q188 [Go to Solution p. 342] [Back to TOC]

From graph to matrix and back. For the undirected graph below, write its adjacency matrix using order $(1, 2, 3, 4)$ and list all vertices adjacent to 3.



Final Answer: ______________________________

Name: ________________________ **Date:** ________________

AHL3.15 — Adjacency matrices

Q189 [Go to Solution p. 344] [Back to TOC]

Adjacency matrix from a graph (undirected) and 2-step walks. Consider the graph G with vertices $V = \{1, 2, 3, 4\}$ and edges $\{12, 23, 34, 41, 13\}$.

1. Write the 4×4 adjacency matrix A in the order $(1, 2, 3, 4)$.
2. Compute A^2 and interpret the entry $(A^2)_{14}$.
3. How many walks of length 2 are there from vertex 2 to vertex 4?

Final Answer: ____

Q190 [Go to Solution p. 344] [Back to TOC]

Walk counts from powers of A. A graph has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

1. Find $(A^3)_{12}$ and explain its meaning.
2. Compute the total number of walks of length ≤ 3 from 1 to 2. (State a matrix expression and evaluate the $(1, 2)$ entry.)

Final Answer: ____

Q191 [Go to Solution p. 344] [Back to TOC]

Closed walks. For the matrix A in Question 2:

1. Find the number of closed walks of length 3 starting and ending at vertex 3.
2. Determine whether the graph contains a triangle (3-cycle). Explain briefly using a matrix

entry.

Final Answer: ____

____

Q192 [Go to Solution p. 344] [Back to TOC]

From directed graph to adjacency matrix and reachability. A digraph D has arcs $A \to B$, $B \to C$, $C \to A$, $C \to D$, $D \to B$.

1. Write its adjacency matrix A in the order (A, B, C, D) (rows = sources, columns = targets).
2. Compute A^2 and A^3. Using these, decide whether there is a walk from D to A, and if so, give a possible length.

Final Answer: ____

____

Q193 [Go to Solution p. 345] [Back to TOC]

Weighted adjacency table. The undirected weighted graph has vertices $\{P, Q, R, S\}$ and edge weights $w(PQ) = 4$, $w(QR) = 3$, $w(RS) = 5$, $w(PS) = 7$, $w(PR) = 2$. Missing pairs are non-adjacent.

1. Construct the 4×4 *weighted adjacency matrix* W (use 0 for non-edges).
2. What is the total weight of the specific walk $P \to R \to S \to P$?
3. List all walks of length 2 from Q to S and their total weights.

Final Answer: ____

____

Q194 [Go to Solution p. 345] [Back to TOC]

Transition matrix of a simple random walk (undirected). Let G be the simple graph with edges $\{12, 13, 23, 24\}$ on $V = \{1, 2, 3, 4\}$.

1. Construct the transition matrix P for the simple random walk on G, where from each vertex you choose uniformly among its neighbors.
2. Verify that each row of P sums to 1.
3. Compute $(P^2)_{14}$ and interpret it as a probability.

Final Answer: ____________________

____

Q195 [Go to Solution p. 345] [Back to TOC]

Transition matrix of a directed random walk (uniform over out-edges). A digraph has arcs $1 \to 2$, $1 \to 3$, $2 \to 3$, $2 \to 4$, $3 \to 1$, $4 \to 3$.

1. Build the row-stochastic transition matrix P by assigning equal probability to each out-edge from a vertex.
2. Is the digraph strongly connected? Justify briefly from the graph or using powers of P.

Final Answer: ____________________

____

Q196 [Go to Solution p. 346] [Back to TOC]

Weighted random walk (probability proportional to weight). On the weighted digraph with outgoing weights from each vertex:

$$W = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix} \quad \text{(rows = sources, columns = targets)},$$

define a random walk that moves from i to j with probability proportional to W_{ij}.

1. Construct the transition matrix P by normalizing each row of W.
2. From state 1, what is the probability of being at state 3 after two steps?

Final Answer: ____

____

Q197 [Go to Solution p. 346] [Back to TOC]

Counting at most k-step walks. Let A be the adjacency matrix of a graph. Show that the matrix

$$S_k = I + A + A^2 + \cdots + A^k$$

has the property that $(S_k)_{ij}$ equals the number of walks of length $\leq k$ from i to j. Evaluate $(S_3)_{12}$ explicitly for the A given in Question 2.

Final Answer: ____

____

Q198 [Go to Solution p. 346] [Back to TOC]

Stationarity check (link to Markov chains). For the transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

1. Compute P^2 and P^3.
2. Find a probability row vector $\pi = [\pi_1, \pi_2, \pi_3]$ with $\pi P = \pi$ (any solution will do; you may solve linear equations).
3. Briefly interpret π in the context of long-run behavior.

Final Answer: ____

Name: ______________________ **Date:** ______________

Q199 [Go to Solution p. 346] [Back to TOC]

PageRank-style transition with damping (small web). A small web has directed links: $A \to B, C$; $B \to C$; $C \to A$; $D \to C$ (page D has a single out-link to C). Let P be the row-stochastic transition matrix for the uniform random surfer on links. Let $G = \alpha P + (1-\alpha)\frac{1}{4}J$ be the damped transition matrix with $\alpha = 0.85$ and J the 4×4 all-ones matrix.

1. Construct P and G.
2. Starting from the uniform distribution $v^{(0)} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$, compute one iteration $v^{(1)} = v^{(0)}G$.

Final Answer: ______________________________

Q200 [Go to Solution p. 347] [Back to TOC]

Using powers to test strong connectivity. A digraph on vertices $\{1, 2, 3, 4\}$ has adjacency matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

1. Compute $R = I + A + A^2 + A^3$.
2. From R, decide which ordered pairs (i, j) are mutually reachable.
3. Is the digraph strongly connected? If not, identify a strongly connected component.

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

AHL 3.16 — Chinese Postman Problem, Travelling Salesman Problem and more graph theory

*AHL 3.16 — Graph algorithms and optimisation problems

*Key terms

Walk: A sequence of vertices connected by edges; vertices and edges may repeat.

Trail: A walk with no repeated edges.

Path: A trail with no repeated vertices.

Circuit: A trail that starts and ends at the same vertex.

Cycle: A path that starts and ends at the same vertex.

Eulerian trail/circuit: Uses every edge once (circuit returns to the start).

Hamiltonian path/cycle: Visits every vertex once (cycle returns to the start).

Minimum spanning tree (MST): A spanning tree of minimum total weight.

Kruskal: Sort edges by weight and add if no cycle; stop at $n-1$ edges.

Prim: Grow one tree by repeatedly adding the lightest edge leaving the tree.

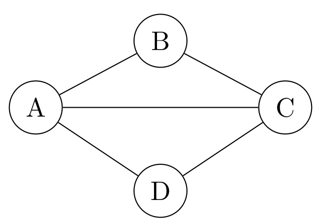
CPP: (Chinese Postman) Shortest closed trail covering every edge at least once.

TSP: (Travelling Salesman) Lightest Hamiltonian cycle in a weighted complete graph.

Nearest neighbour: Greedy TSP upper-bound heuristic: always go to the nearest unvisited vertex.

Deleted vertex: Lower bound for TSP: delete a vertex, take MST of the rest, add two lightest incident edges to the deleted vertex.

*Walks, trails, paths, circuits, cycles



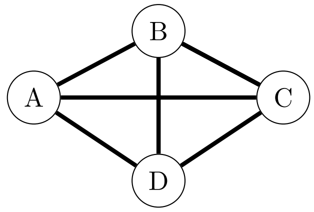
Example graph

Worked example. On this graph:

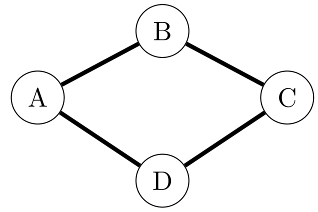
- $A \to B \to C \to D \to A$ is a *trail* and a *circuit* (no repeated edges, returns to A).
- $A \to B \to C \to A$ is a *cycle* (no repeated vertices, closes at A).
- $A \to B \to C \to A \to D$ is a *walk* (repeats vertex A), but not a trail.

*Eulerian and Hamiltonian examples

Name: ______________________ **Date:** ______________



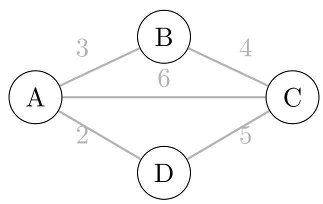
Eulerian circuit



Hamiltonian cycle

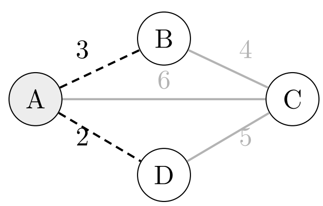
Worked example. *Eulerian:* all four vertices have even degree (2 or 3 with duplicated edge in path drawn), so an Eulerian circuit exists (every edge used once), e.g. $A-B-C-D-A-C-B-D-A$. *Hamiltonian:* $A-B-C-D-A$ visits each vertex once and returns to A.

*Kruskal's algorithm (MST)



Worked example. Sort edges: $2, 3, 4, 5, 6$. Add $AD(2)$, $AB(3)$, $BC(4)$. Stop at 3 edges ($n-1$ for $n=4$). MST weight $= 2+3+4 = 9$. Reject $DC(5)$ and $AC(6)$ as they would create a cycle.

*Prim's algorithm (MST)



Worked example. Start at A. Choose lightest edge leaving the tree: $AD(2)$, then from $\{A, D\}$ choose $AB(3)$, then $BC(4)$. MST edges $\{AD, AB, BC\}$ with total 9.

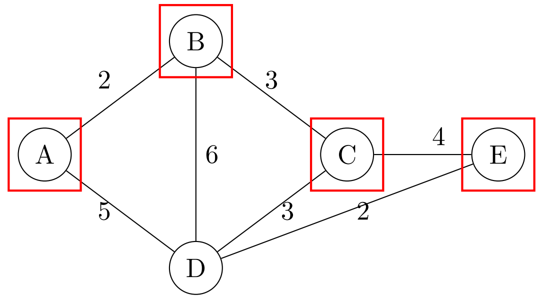
Matrix (weight) method for Prim.

$$W = \begin{pmatrix} 0 & 3 & 6 & 2 \\ 3 & 0 & 4 & \infty \\ 6 & 4 & 0 & 5 \\ 2 & \infty & 5 & 0 \end{pmatrix}$$

Worked example. From row/col A pick $AD(2)$; with $\{A, D\}$ the smallest connection is $AB(3)$; with $\{A, B, D\}$ the next is $BC(4)$. Same MST as above, weight 9.

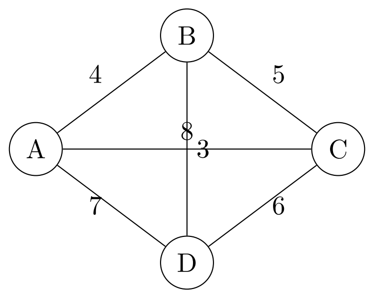
*Chinese Postman Problem (CPP): 2 odd vertices **Worked example (text).** If a connected graph has exactly two odd vertices u, v, duplicate a shortest u–v path. Now all degrees are even, so an Eulerian circuit exists. Its length equals (sum of original edge weights) + (length of duplicated path).

*Chinese Postman Problem (CPP): 4 odd vertices



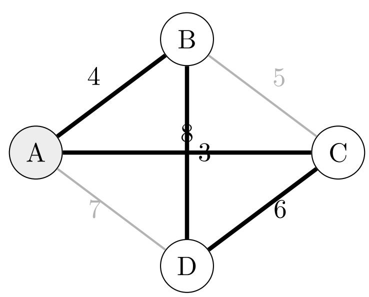
Worked example. Odd vertices are $\{A, B, C, E\}$. Pairings to test: $(A, B) + (C, E)$ vs. $(A, C) + (B, E)$. Compute the shortest path cost for each pair and duplicate those paths. Choose the pairing with the smaller added cost; the resulting multigraph is all-even, so an Euler circuit gives the minimum closed route.

*Travelling Salesman Problem (TSP)



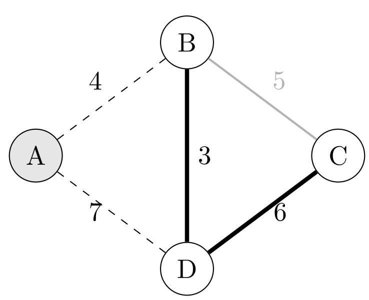
Worked example. All pairs are connected (complete weighted graph). A Hamiltonian cycle is, for example, $A - B - C - D - A$ with weight $4 + 5 + 6 + 7 = 22$; however, there may be lighter cycles. We now estimate bounds using the heuristics below.

*Nearest Neighbour (upper bound for TSP)



Worked example. Start at A, nearest is $B(4)$; from B nearest unvisited is $D(3)$; from D nearest unvisited is $C(6)$; return $C \to A(8)$. Tour $A - B - D - C - A$ gives an *upper bound* $4 + 3 + 6 + 8 = 21$.

*Deleted Vertex (lower bound for TSP)



Worked example. Delete A. MST on $\{B, C, D\}$ has weight $3 + 6 = 9$. Add the two lightest edges incident to A: $AB(4)$ and $AD(7)$. *Lower bound* $= 9 + 4 + 7 = 20$. Combining with the nearest neighbour upper bound 21, the optimal TSP tour weight lies in [20, 21].

Q201 [Go to Solution p. 348] [Back to TOC]

Walks, trails, paths, circuits, cycles. In the undirected graph with edges $\{AB, BC, CD, DA, AE, EC\}$ on $V = \{A, B, C, D, E\}$:

1. Classify the vertex sequence $A \to B \to C \to D \to A$ as a walk/trail/path/circuit/cycle (tick all that apply).
2. Do the same for $A \to E \to C \to D \to A \to B$ and for $A \to B \to C \to A$.
3. State the definitions of *trail*, *path*, *circuit*, and *cycle* in your own words.

Final Answer: ____________________

Q202 [Go to Solution p. 348] [Back to TOC]

Eulerian trails and circuits (existence and construction). For the graph from the previous question:

1. Determine whether an Eulerian circuit exists. If not, determine whether an Eulerian trail exists.
2. If one exists, construct an explicit sequence of edges in order.
3. Justify your answer using vertex degrees.

Final Answer: ____________________

Q203 [Go to Solution p. 348] [Back to TOC]

Hamiltonian paths and cycles. Consider the graph with edges $\{AB, BC, CD, DE, EA, AC\}$ on $V = \{A, B, C, D, E\}$.

1. Decide whether a Hamiltonian cycle exists; if so, write one.
2. Decide whether a Hamiltonian path exists that is not a cycle; if so, give one.
3. Explain briefly why your answers are correct.

Final Answer: ____

Q204 [Go to Solution p. 349] [Back to TOC]

Tree vs. cycle detection (undirected). A graph G on n vertices has m edges and is connected.

1. Show that if $m = n - 1$ then G is a tree.
2. For the graph with $V = \{1, 2, 3, 4, 5\}$ and edges $\{12, 23, 34, 45, 15, 25\}$, use a cycle-detection method (e.g. DFS tree/back edges) to find a cycle, or explain why none exists.

Final Answer: ____

Q205 [Go to Solution p. 349] [Back to TOC]

Minimum spanning tree (Kruskal). For the weighted undirected graph below, use *Kruskal's algorithm* to find a minimum spanning tree (MST). List edges in the order selected and give the total weight.

	A	B	C	D	E
A	–	4	2	7	9
B	4	–	1	3	6
C	2	1	–	5	8
D	7	3	5	–	4
E	9	6	8	4	–

(Blank/dash indicates symmetry; use the upper triangle as weights.)

Final Answer: ____

Q206 [Go to Solution p. 349] [Back to TOC]

Minimum spanning tree (Prim, matrix method). Using the same graph as the previous question, apply *Prim's algorithm (matrix method)* starting at vertex A.

1. Show the candidate row/column minima at each step and the edge chosen.
2. State the final MST and its total weight. Confirm it matches Question 5.

Final Answer: ____

____

Q207 [Go to Solution p. 349] [Back to TOC]

Chinese postman: two odd vertices. For the graph with edges and weights $\{AB{:}3,\ BC{:}4,\ CA{:}5,\ AD{:}1\}$ on $V = \{A, B, C, D\}$:

1. Identify the vertices of odd degree.
2. Compute the length of a shortest route that traverses each edge at least once and returns to the start (Chinese postman length).
3. Write one optimal route.

Final Answer: ____

____

Q208 [Go to Solution p. 350] [Back to TOC]

Chinese postman: four odd vertices and pairings. Consider the undirected weighted graph on $V = \{P, Q, R, S\}$ with edges $PQ{:}2,\ QR{:}2,\ RS{:}3,\ SP{:}3,\ PR{:}4,\ QS{:}1$.

1. List the odd-degree vertices and compute the total weight of all edges.
2. Compute the minimum additional weight needed by optimally pairing the odd vertices (show all possible pairings).
3. Hence find the Chinese postman length and name the duplicated edges.

Final Answer: ____

Q209 [Go to Solution p. 350] [Back to TOC]

Why the Chinese postman algorithm works (explain). Explain why pairing up the odd-degree vertices with minimum total added distance always produces an Eulerian multigraph of minimum added weight. Your explanation can refer to the Handshaking Lemma and parity constraints at vertices.

Final Answer: ____________

Q210 [Go to Solution p. 350] [Back to TOC]

Travelling Salesman Problem (TSP): exact on a small instance. A complete weighted graph on $V = \{A, B, C, D, E\}$ has distance matrix (symmetric):

	A	B	C	D	E
A	0	7	9	8	7
B	7	0	4	2	6
C	9	4	0	3	5
D	8	2	3	0	4
E	7	6	5	4	0

1. Determine a Hamiltonian cycle of least total weight and state its length.
2. Briefly justify optimality (e.g. by comparison with close alternatives or structured enumeration).

Final Answer: ____________

Q211 [Go to Solution p. 350] [Back to TOC]

Nearest neighbour heuristic (upper bound for TSP). Using the matrix in Question 10:

1. Run the nearest neighbour algorithm starting at A. State the tour and its length.
2. Repeat starting at B. Which start gives the better upper bound?
3. Compare your best upper bound with the exact optimum (if known from Question 10).

Final Answer: ____

____

Q212 [Go to Solution p. 350] [Back to TOC]

Deleted-vertex lower bound for TSP. Using the matrix in Question 10, apply the *deleted-vertex bound* with vertex A:

1. Compute an MST on the subgraph induced by $\{B, C, D, E\}$ and state its weight.
2. Add the two smallest edges incident with A to obtain a lower bound. State the bound and compare it to your upper bound from Question 11.

Final Answer: ____

____

Q213 [Go to Solution p. 350] [Back to TOC]

From practical to classical TSP via least-distance table. A delivery network has road lengths (undirected, not complete): $U-V:2,\ V-W:2,\ U-W:5,\ W-X:1,\ X-Y:1,\ V-Y:7,\ U-Y:10$.

1. Construct the table of *least distances* between all pairs of vertices (fill in missing entries via shortest paths).
2. Using this completed table, apply the nearest neighbour algorithm from U to obtain a feasible TSP tour and its length (upper bound).
3. Use a deleted-vertex bound to obtain a lower bound. Comment on the gap.

Final Answer: ____

____

Q214 [Go to Solution p. 351] [Back to TOC]

Cycle edges vs. tree edges (algorithmic reasoning). Run a depth-first search (DFS) on the graph with $V = \{1, 2, 3, 4, 5, 6\}$ and edges $\{12, 23, 34, 45, 15, 26, 36\}$, starting at 1 and exploring smaller-numbered neighbours first.

Name: ____________________ **Date:** ______________

1. List the DFS tree edges in discovery order.
2. Identify one back edge and state a simple cycle containing it.

Final Answer:

Q215 [Go to Solution p. 351] [Back to TOC]

Euler vs. Hamilton in practice. Give a small real-world example where an *Eulerian* route is the appropriate model and another where a *Hamiltonian* tour is appropriate. For each, state the graph representation and what the vertices and edges represent.

Final Answer:

Name: ____________________ **Date:** ____________________

Topic 4 — Statistics and Probability (SL 4.1–4.11, AHL 4.12–4.19)

Overview (SL) Covers populations, samples, and sampling methods; measures of central tendency and dispersion; data presentation and bivariate statistics; probability rules; conditional probability and trees; discrete and continuous distributions; normal distribution; and correlation and significance testing using Spearman's rank and t-tests.

Overview (HL) Extends SL content with advanced topics such as designing investigations and sampling techniques, regression with non-linear functions, linear combinations and expectations/variance, the central limit theorem, confidence intervals, Poisson distribution, hypothesis testing with Type I and II errors, and Markov chains.

Real-World Use

- Medical research, drug trials, and public health studies
- Business forecasting and risk assessment
- Sports analytics and performance optimisation
- Quality control in manufacturing and engineering
- Financial modelling and actuarial science
- Reliability testing in engineering systems

Common Misconceptions

- Confusing mutually exclusive with independent events
- Misinterpreting correlation as causation
- Misreading or overgeneralising from small samples
- Treating probability as certainty or impossibility
- Ignoring underlying assumptions of statistical tests
- Over-reliance on p-values without considering effect size or context

Advice for SL

- Identify the type of data (categorical, discrete, continuous) before selecting statistical methods
- Draw probability trees or Venn diagrams for multi-step or combined event problems
- Check that probabilities sum to 1 for discrete distributions and integrate to 1 for continuous distributions
- Use technology to confirm calculations but understand the method
- Remember correlation does not imply causation

Advice for HL

- In hypothesis testing, define null and alternative hypotheses clearly and interpret in context
- Use the central limit theorem to justify normal approximations for large samples
- For regression, check residual plots to assess model fit
- Understand the difference between Type I and Type II errors and the trade-off between significance level and power
- In Markov chains, define states and transition probabilities before analysing long-term behaviour
- For Poisson and normal approximations, verify conditions (e.g., large λ for Poisson, $np \geq 5$ and $n(1-p) \geq 5$ for binomial)

SL 4.1 Populations, Samples and Sampling Methods

Q216 [Go to Solution p. 353] [Back to TOC]

Define a population and a sample in the context of a statistical study. Give one advantage and one disadvantage of using a simple random sample.

Final Answer: ____

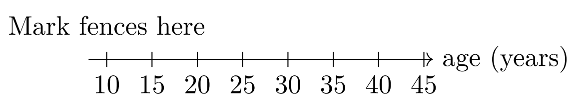
____

Q217 [Go to Solution p. 353] [Back to TOC]

Identify outliers with fences (number line diagram). The ages (in years) of a small club are

$$12, \ 14, \ 15, \ 16, \ 18, \ 19, \ 20, \ 22, \ 23, \ 25, \ 40.$$

1. Find Q_1, Q_2 (median), Q_3 and the interquartile range (IQR).
2. Compute the lower and upper fences: $Q_1 - 1.5\text{IQR}$ and $Q_3 + 1.5\text{IQR}$.
3. State which values are outliers by this rule.
4. *On the number line below*, lightly mark the fences and circle any outliers.



Final Answer: ____

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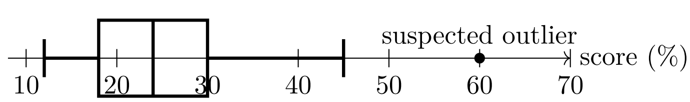
Q218 [Go to Solution p. 353] [Back to TOC]

Reading a box-and-whisker plot (with a suspected high outlier). A class test has quartiles $Q_1 = 18$, median $Q_2 = 24$, $Q_3 = 30$. The smallest non-outlier is 12. The diagram shows the distribution and one very high score.



1. Compute the IQR and the upper fence $Q_3 + 1.5\text{IQR}$.
2. Using the fence, decide whether 60 is an outlier and justify.

Name: ____________________ **Date:** ______________

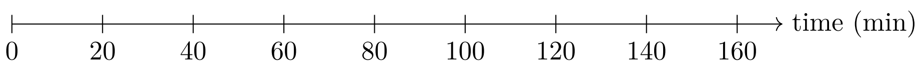
3. Give two possible reasons (in context) for such a high score and whether it should be kept or removed in analysis.

Final Answer: ____

Q219 [Go to Solution p. 353] [Back to TOC]

Commuting times in a city (context + mini diagram). A survey records daily commuting times (minutes). Summary: $Q_1 = 25$, $Q_3 = 50$. Two reported values are 4 and 150 minutes.

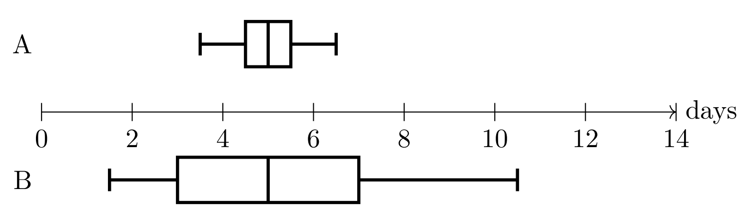
1. Compute the IQR and both fences.
2. Decide whether 4 and 150 are outliers. Explain briefly.
3. *On the line* place the fences and mark where 4 and 150 lie.



Final Answer: ____

Q220 [Go to Solution p. 353] [Back to TOC]

Comparing spread and the chance of outliers. Two factories, A and B, make the same product. Their delivery times (days) have the same median = 5, but Factory A has IQR = 1 and Factory B has IQR = 3. The schematic boxplots are shown.



1. Which factory is *more likely* to have values flagged as outliers by the $1.5 \times$ IQR rule? Explain.
2. Give one advantage and one disadvantage of removing outliers before comparing the factories.

Final Answer: ____

Name: ____________________ **Date:** ____________

Q221 [Go to Solution p. 354] [Back to TOC]

Effect of an outlier on mean and median. Twenty scores have mean 72 and median 73. One recorded value is later found to be 5, far lower than the rest.

1. If the outlier 5 is removed, compute the new mean.
2. Would the median change after removing 5? Explain clearly.
3. In reporting to parents, which measure (mean/median) is more robust to such an outlier? Justify.

Final Answer: ____________________

Q222 [Go to Solution p. 354] [Back to TOC]

Outlier or data-entry mistake? Reason from context. A wildlife team records daily numbers of birds at a feeder for 60 days. The IQR method flags the value $x = 0$ as an outlier on one very rainy day.

1. Give two realistic explanations why $x = 0$ could be a *valid* observation.
2. Give two reasons it might be a *recording error*.
3. State one clear rule the team could follow to decide whether to keep or remove outliers in future studies.

Final Answer: ____________________

Name: ____________________ **Date:** ____________________

SL 4.2 Measures of Central Tendency

Q223 [Go to Solution p. 355] [Back to TOC]

For the data set $\{3, 7, 8, 10, 12, 12, 16, 20\}$, compute the mean, median and mode. Comment on the suitability of each measure for summarizing these data.

Final Answer: ____

Name: ______________________ **Date:** ______________

SL 4.3 Measures of Dispersion

Key ideas

Key terms:

- **Mean** — the sum of all data values divided by the number of values.
- **Median** — the middle value when the data is ordered.
- **Mode** — the most frequent value in the data set.
- **Grouped data** — data presented in intervals; mean is estimated using midpoints.
- **Modal class** — the class interval with the highest frequency.
- **Range** — maximum value minus minimum value.
- **Interquartile range (IQR)** — $Q_3 - Q_1$; measures spread of the middle 50% of data.
- **Variance** — the average of the squared deviations from the mean.
- **Standard deviation (SD)** — the square root of the variance.
- **Effect of transformations:**
  - Adding/subtracting a constant: changes mean, SD unchanged.
  - Multiplying/dividing by a constant: both mean and SD scaled by the constant.

Worked Example 1 — Mean and Standard Deviation from raw data

A class of 6 students has the following test scores:

$$12,\ 15,\ 14,\ 10,\ 9,\ 20$$

Step 1: Mean

$$\bar{x} = \frac{12+15+14+10+9+20}{6} = \frac{80}{6} \approx 13.33$$

Step 2: Variance and Standard Deviation (population formula)

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = \frac{(12-13.33)^2 + (15-13.33)^2 + \cdots + (20-13.33)^2}{6}$$

$$\sigma^2 \approx 12.22 \quad \Rightarrow \quad \sigma \approx 3.50$$

Interpretation: The mean score is 13.33 and the standard deviation is about 3.50 marks.

Worked Example 2 — Mean from grouped data

A frequency table is given below:

Class interval	Frequency (f)	Midpoint (x)
$0 \le x < 10$	5	5
$10 \le x < 20$	8	15
$20 \le x < 30$	7	25
$30 \le x < 40$	4	35

Name: ______________________ Date: ______________

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(5)(5) + (8)(15) + (7)(25) + (4)(35)}{5 + 8 + 7 + 4} = \frac{25 + 120 + 175 + 140}{24} = \frac{460}{24} \approx 19.17$$

TI-Nspire instructions:

1. Press `home` → `Add Lists & Spreadsheet`.
2. Enter data values in column A (or class midpoints for grouped data) and frequencies (if applicable) in column B.
3. Press `menu` → `4:Statistics` → `1:Stat Calculations` → `1:One-Variable Statistics`.
4. Set `X List` to your data column and `Freq` to the frequency column (or 1 if raw data).
5. The mean ($\bar{x}$), standard deviation (s or σ), quartiles (Q_1, Q_3), and other statistics will be displayed.
6. For quartiles: Use the output or plot a boxplot via `menu` → `4:Statistics` → `1:Stat Calculations` → `Box Plot`.

Worked Example 3 — Effect of transformations

Original data: $[5, 7, 9]$

$$\bar{x} = 7, \quad \sigma \approx 1.63$$

Adding 3 to each value: Data becomes $[8, 10, 12]$; $\bar{x} = 10$, σ unchanged.

Doubling each value: Data becomes $[10, 14, 18]$; $\bar{x} = 14$, $\sigma \approx 3.27$ (double the original).

Q224 [Go to Solution p. 356] [Back to TOC]

For the data set $\{3, 7, 8, 10, 12, 12, 16, 20\}$, calculate the range, interquartile range (IQR) and sample standard deviation. Interpret these statistics.

Final Answer: ____

____

Name: ________________________ **Date:** ________________

SL 4.4 Data Presentation and Bivariate Statistics

Q225 [Go to Solution p. 357] [Back to TOC]

A class of 10 students recorded the number of hours they studied (x) and their corresponding test scores (y):

$$(2, 68), (3, 75), (4, 78), (4, 80), (5, 85), (6, 88), (6, 90), (7, 92), (8, 94), (9, 96).$$

(i) Plot the scatter diagram. (ii) Compute the Pearson correlation coefficient r. (iii) Determine the least–squares regression line $y = mx + c$ for predicting score from hours studied.

Final Answer: ____

Name: ______________________ **Date:** ______________

SL 4.5 — Probability basics

Q226 [Go to Solution p. 358] [Back to TOC]

A bag contains 5 red balls, 3 blue balls, and 2 green balls.

1. List the sample space if one ball is chosen at random.
2. Find the probability of choosing:
   (a) a red ball
   (b) a blue or green ball
3. State the probability of *not* choosing a red ball.

Final Answer: ____

Q227 [Go to Solution p. 358] [Back to TOC]

A fair six-sided die is rolled once.

1. Write the sample space.
2. Find the probability of rolling:
   (a) a number greater than 4
   (b) an even number
3. Find the probability of the complement of event $A = \{\text{rolling a } 6\}$.

Final Answer: ____

Q228 [Go to Solution p. 358] [Back to TOC]

In a survey of 200 students, 60 say they walk to school.

1. Estimate the probability that a randomly chosen student walks to school.
2. If the school has 1200 students, estimate the expected number who walk to school.

Final Answer: ____

Q229 [Go to Solution p. 358] [Back to TOC]

A box contains 4 white, 5 black, and 1 red marble.

1. An experiment is conducted where one marble is selected and replaced each time. Estimate the relative frequency of selecting a white marble after 500 trials.
2. Compare the result from (a) with the theoretical probability.

Final Answer: ____

Q230 [Go to Solution p. 358] [Back to TOC]

A coin is tossed 100 times, and heads appears 54 times.

1. Find the relative frequency of getting heads.
2. Compare this with the theoretical probability for a fair coin.
3. Explain why the results are not identical.

Final Answer: ____

Q231 [Go to Solution p. 359] [Back to TOC]

A school basketball team wins 75% of its games.

1. If they play 32 games in a season, find the expected number of games they win.
2. Find the expected number of games they lose.

Name: ____________________ **Date:** ________________

Final Answer: ____

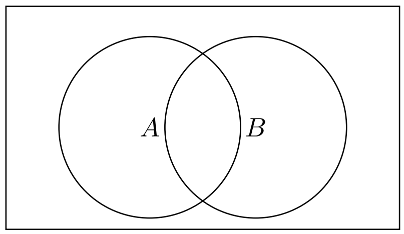
Name: ______________________ **Date:** ______________

SL 4.6 Probability Rules

Q232 [Go to Solution p. 360] [Back to TOC]

In a survey, $P(A) = 0.55$, $P(B) = 0.40$, and $P(A \cap B) = 0.22$.

1. Using a Venn diagram, find $P(A \cup B)$ and $P(A^c \cap B)$.
2. Determine whether A and B are independent.
3. If $P(B) = 0.40$, compute $P(A \mid B)$.



Final Answer: ____

Q233 [Go to Solution p. 360] [Back to TOC]

Suppose $P(A) = 0.6$, $P(B) = 0.5$ and $P(A \cap B) = 0.3$. Find (i) $P(A \cup B)$; (ii) $P(A^c)$; (iii) determine whether A and B are independent and justify your answer.

Final Answer: ____

Name: ____________________ **Date:** ____________

SL 4.7 Conditional Probability and Trees

Q234 [Go to Solution p. 361] [Back to TOC]

A factory has two machines, M_1 and M_2, producing 60% and 40% of the items, respectively. Machine M_1 produces 2% defective items and M_2 produces 5% defective items. (i) Draw a probability tree diagram for machines and defect status. (ii) Compute the overall probability that a randomly selected item is defective. (iii) Given that an item is defective, find the probability that it was produced by M_1.

Final Answer: ____________________

Q235 [Go to Solution p. 361] [Back to TOC]

(SL 4.7 — Discrete RV & $E[X]$) A game pays $x with probabilities:

$$x:\ 0,\ 1,\ 2,\ 4,\qquad P(X=x):\ 0.25,\ 0.30,\ 0.20,\ 0.25.$$

1. Compute $E[X]$ and interpret whether the game is fair for the player.
2. If the organiser adds an entry fee c, find c that makes the game fair.

Final Answer: ____________________

Name: ______________________ **Date:** ______________

SL 4.8 Discrete and Continuous Distributions

Q236 [Go to Solution p. 362] [Back to TOC]

(SL 4.8 — Binomial) Defects occur independently with probability $p = 0.08$ per item.

1. Let $X \sim \text{Bin}(n, 0.08)$ with $n = 15$. Find $P(X = 2)$ and $P(X \geq 3)$.
2. State the mean and variance of X.
3. Explain briefly why a binomial model is appropriate here.

Final Answer: ______________________

Q237 [Go to Solution p. 362] [Back to TOC]

Let $X \sim \text{B}(n = 10, p = 0.3)$.

(i) Compute $P(X = 4)$.

(ii) Compute $P(X \geq 6)$.

Final Answer: ______________________

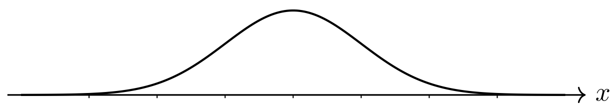
Name: ______________________ **Date:** ______________

SL 4.9 Normal distribution

Q238 [Go to Solution p. 363] [Back to TOC]

Properties & diagram. The random variable X is normally distributed with mean μ and standard deviation σ.

1. On the diagram below, label μ, $\mu \pm \sigma$, $\mu \pm 2\sigma$, and $\mu \pm 3\sigma$ on the x-axis.
2. Shade the region corresponding to approximately 68% of the data and write this percentage on the diagram.
3. Using the 68–95–99.7 rule, estimate the percentage of values lying between $\mu - 2\sigma$ and $\mu + 3\sigma$.

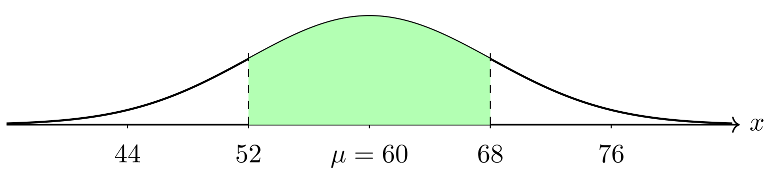


Final Answer: ____

Q239 [Go to Solution p. 363] [Back to TOC]

Normal probability (technology). Let $X \sim \mathcal{N}(\mu = 60,\ \sigma = 8)$. Use technology to find:

1. $P(52 \le X \le 68)$
2. $P(X \ge 76)$
3. $P(X \le 44)$



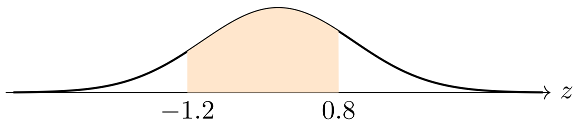
Final Answer: ____

Q240 [Go to Solution p. 363] [Back to TOC]

Standard normal interval. Let $Z \sim \mathcal{N}(0, 1)$. Use technology (or a table) to compute:

$$P(-1.2 < Z < 0.8), \qquad P(Z \le -1.5), \qquad P(Z \ge 1.96).$$

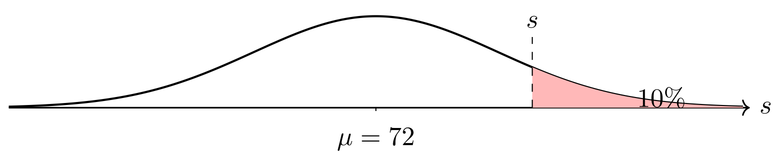
Name: ______________________ **Date:** ______________



Final Answer: ______________________________

Q241 [Go to Solution p. 363] [Back to TOC]

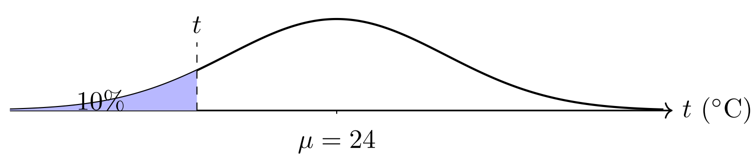
Inverse normal (percentile). A retailer classifies the top 10% of weekly sales as "excellent". If $S \sim \mathcal{N}(72, 9^2)$ (units in thousands of \$), find the minimum sales value s that qualifies as "excellent", i.e. $P(S \geq s) = 0.10$.



Final Answer: ______________________________

Q242 [Go to Solution p. 364] [Back to TOC]

Cut-off for the lowest decile. Daily maximum temperatures in a city follow $T \sim \mathcal{N}(24, 6^2)$ (in °C). Find the temperature t such that $P(T \leq t) = 0.10$. Illustrate on the diagram.



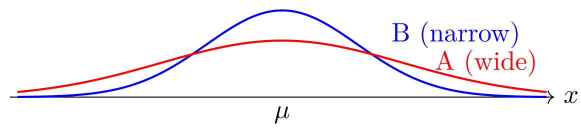
Final Answer: ______________________________

Name: ______________________________ **Date:** ____________________

Q243 [Go to Solution p. 364] [Back to TOC]

Two normals, same mean, different spread. Curves A and B below have the same mean but different standard deviations.

1. Which curve has the larger standard deviation? Explain using a property of the normal curve.
2. For the wider curve, estimate the proportion within one standard deviation of the mean.

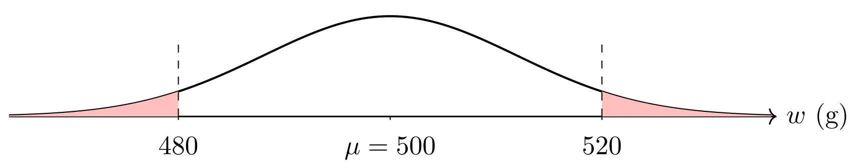


Final Answer: ____

Q244 [Go to Solution p. 364] [Back to TOC]

Quality control tails (technology). Mass of packaged rice $W \sim \mathcal{N}(500,\ 12^2)$ grams. A pack is rejected if $W \notin [480, 520]$.

1. Using technology, find the probability that a randomly selected pack is rejected.
2. If 10 000 packs are produced, how many do you expect to reject?



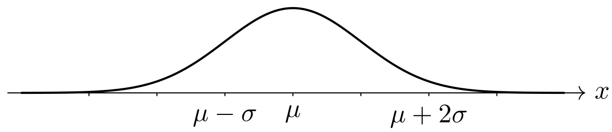
Final Answer: ____

Q245 [Go to Solution p. 364] [Back to TOC]

Diagram reading. The curve below shows a normal distribution with mean μ and standard deviation σ.

1. Shade and label the region representing $P(\mu - \sigma \le X \le \mu + 2\sigma)$.
2. Without technology, use the empirical rule to estimate this probability.
3. Then use technology to compute the exact value (to four decimal places).

Name: ______________________ **Date:** ______________



Final Answer: ____

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Q246 [Go to Solution p. 365] [Back to TOC]

The lifetimes (in hours) of a certain type of light bulb follow a normal distribution with mean $\mu = 1200$ and standard deviation $\sigma = 100$.

(i) Find the probability that a bulb lasts at least 1250 hours.

(ii) Find the lifetime which marks the 90th percentile.

Final Answer: ____

____

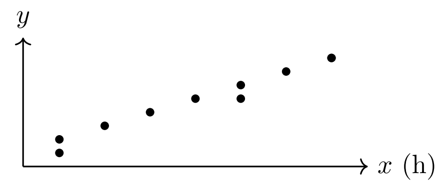
SL 4.10 Spearmans Rank Correlation Coefficient

Q247 [Go to Solution p. 366] [Back to TOC]

Spearman's r_s with ties (use technology). The table shows the "study time" x (hours) and "quiz score" y (out of 20) for $n = 10$ students. (Ties are present.)

x	1.0	1.0	1.5	2.0	2.5	3.0	3.0	3.5	4.0	4.0
y	10	11	12	13	14	14	15	16	17	17

1. Rank x and y (average the ranks for any ties) and write the two rank rows.
2. Using technology, compute Spearman's rank correlation coefficient r_s.
3. Interpret the direction and strength of the association.



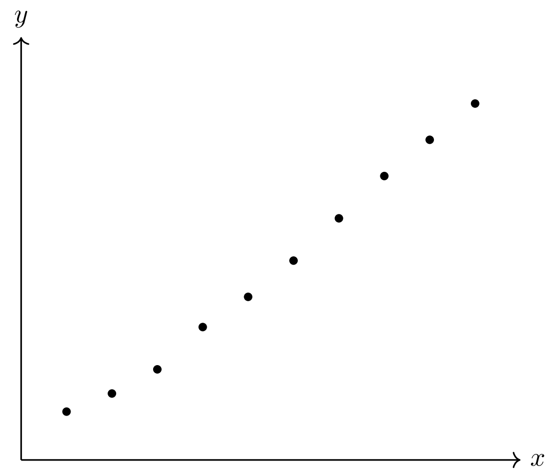
Final Answer: ______________________________

Q248 [Go to Solution p. 366] [Back to TOC]

Monotonic but not linear: compare Pearson r and Spearman r_s (use technology). A biologist measures nutrient concentration x and plant growth rate y. Data (monotone increasing but curved):

x	1	2	3	4	5	6	7	8	9	10
y	0.8	1.1	1.5	2.2	2.7	3.3	4.0	4.7	5.3	5.9

1. Enter the data and compute Pearson's correlation r and Spearman's r_s.
2. Which coefficient is more appropriate here? Justify briefly.
3. Use the more appropriate coefficient to comment on the association.



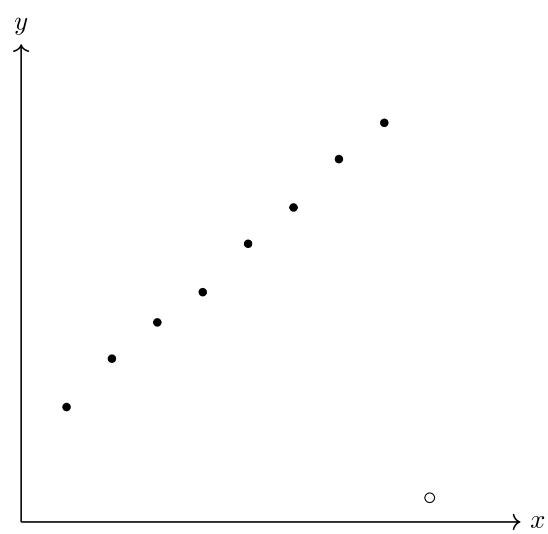
Final Answer: ____

Q249 [Go to Solution p. 366] [Back to TOC]

Effect of an outlier (use technology). Dataset A (filled points) and an additional potential outlier (open circle):

x	1	2	3	4	5	6	7	8	9
y	1.3	2.1	2.7	3.2	4.0	4.6	5.4	6.0	-0.2

1. Using only the first eight points (x, y) with $x = 1, \ldots, 8$, compute Pearson's r and Spearman's r_s.
2. Now include the point $(9, -0.2)$ and recompute r and r_s.
3. Which coefficient is more affected by the outlier? Explain briefly.



Name: ______________________ **Date:** ______________

Final Answer: ____

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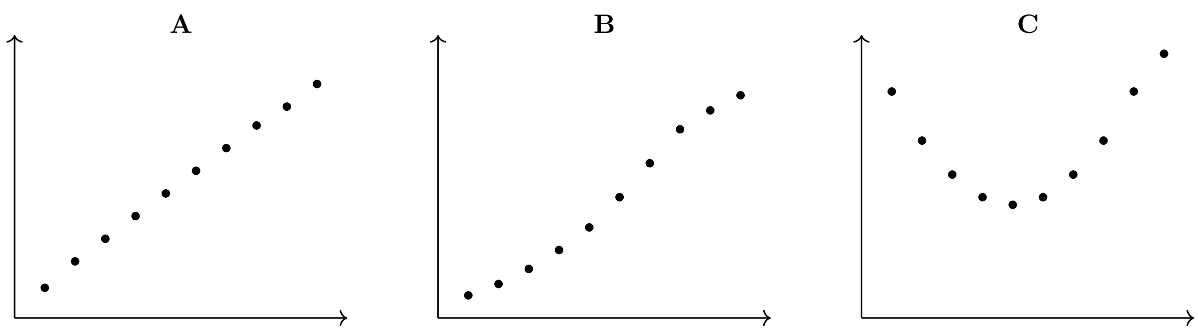
Q250 [Go to Solution p. 367] [Back to TOC]

Choosing a correlation measure from diagrams (use data provided). Three datasets with the *same* x range display different patterns.

1. For each panel, state whether Pearson's r, Spearman's r_s, or "neither" is most appropriate, and why.
2. Without calculation, rank the three panels from largest to smallest $|r|$ (absolute Pearson correlation).
3. For panel B, would r_s be closer to 0, to 0.5, or to 1? Explain.



Final Answer: ____

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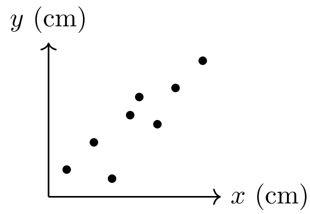
Q251 [Go to Solution p. 367] [Back to TOC]

Compute r and r_s and compare (use technology). Eight athletes are measured for leg length x (cm) and vertical jump height y (cm):

x	88	91	93	95	96	98	100	103
y	49	52	48	55	57	54	58	61

1. Compute Pearson's correlation r and Spearman's rank correlation r_s.
2. Which coefficient better describes the association? Refer to linearity/monotonicity.



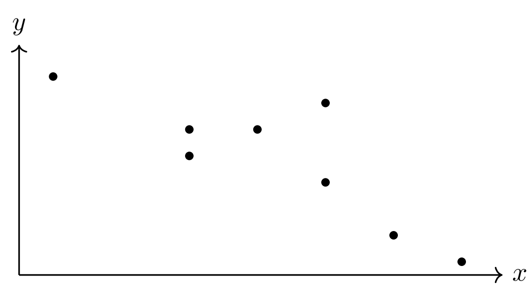
Final Answer: ____

Q252 [Go to Solution p. 367] [Back to TOC]

Ties in ranks (use technology). The pairs (x, y) include repeated values. In Spearman's method, equal values receive the *average* of their rank positions.

x	3	5	5	6	7	7	8	9
y	9	6	7	7	8	5	3	2

1. Write the rank for each x and for each y (averaging ties). Show both rank rows.
2. Using technology, compute r_s directly from the (x, y) data (do *not* type the ranks).
3. State whether the association is positive or negative and whether it appears strong or weak.



Final Answer: ____

Name: ____________________ **Date:** ______________

SL 4.11 Chi-squared and t-tests

Key Terms and Uses

- **Null hypothesis (H_0)** — A statement that there is no effect, relationship, or difference; any observed variation is due to chance.
- **Alternative hypothesis (H_1)** — A statement that there is an effect, relationship, or difference.
- **Significance level (α)** — Probability of rejecting H_0 when it is actually true (Type I error). Common levels: 1%, 5%, 10%.
- **p-value** — Probability, assuming H_0 is true, of obtaining a result at least as extreme as the observed one.
- **Chi-square test of independence** — Used to determine whether two categorical variables are independent (e.g., gender vs. preference for a product).
- **Chi-square goodness-of-fit test** — Used to test whether a categorical distribution fits a given expected distribution (e.g., colour of sweets in a packet matches advertised proportions).
- **Degrees of freedom** — For goodness-of-fit: $n - 1$. For contingency tables: $(r - 1)(c - 1)$.
- **t-test** — Used to compare the means of two groups to check if any observed difference is statistically significant.
- **One-tailed test** — Tests for a difference in one specific direction only.
- **Two-tailed test** — Tests for a difference in either direction.

When to use:

- Use **Chi-square goodness-of-fit** when comparing one categorical variable against an expected proportion.
- Use **Chi-square test of independence** when comparing two categorical variables to check for association.
- Use a **two-sample t-test** when comparing the means of two independent groups.
- Use **one-tailed t-test** when testing in a specific direction, otherwise use a **two-tailed t-test**.

Chi-square test steps (independence or goodness-of-fit)

1. State H_0 and H_1.
2. Calculate expected frequencies:
$$E = \frac{(\text{row total}) \times (\text{column total})}{\text{grand total}}$$
3. Compute:
$$\chi^2 = \sum \frac{(O - E)^2}{E}$$
4. Find degrees of freedom.
5. Use technology to find the p-value.
6. Compare p with α, or compare χ^2 with the critical value.

t-test steps (two-sample pooled)

Name: ______________________ **Date:** ______________

1. State H_0 and H_1 (two-tailed or one-tailed).
2. Check assumptions: normal distribution, equal variances.
3. Use technology to calculate t and p.
4. Compare p with α to decide whether to reject H_0.

Worked Example 1 — Chi-square goodness-of-fit

A company advertises that their mixed fruit bags contain equal numbers of four fruit types: apple, banana, cherry, and grape. A student buys a bag and counts: apple (28), banana (22), cherry (25), grape (25). Test at the 5% significance level whether the bag's contents match the advertised proportions.

Solution:

$$H_0 : \text{The proportions are equal} \quad H_1 : \text{The proportions are not equal}$$

Total $= 28 + 22 + 25 + 25 = 100$; expected for each $= 100/4 = 25$.

$$\chi^2 = \frac{(28-25)^2}{25} + \frac{(22-25)^2}{25} + \frac{(25-25)^2}{25} + \frac{(25-25)^2}{25} = \frac{9}{25} + \frac{9}{25} + 0 + 0 = 0.72$$

df $= 4 - 1 = 3$. Using TI-Nspire, $p \approx 0.868$.

Since $p > 0.05$, fail to reject H_0: no evidence the proportions differ from those advertised.

TI-Nspire instructions:

1. `menu` → `Statistics` → `Stat Tests` → `Chi-Square GOF Test`.
2. Observed list: {28,22,25,25}.
3. Expected list: {25,25,25,25}.
4. df = 3 → `Calculate`.

Worked Example 2 — Chi-square test of independence

A school investigates whether there is an association between gender and preference for a new school lunch menu. The results from a survey of 120 students are:

	Like	Dislike	Total
Male	30	25	55
Female	50	15	65
Total	80	40	120

Test at the 5% significance level whether preference is independent of gender.

Solution:

$$H_0 : \text{Preference is independent of gender} \quad H_1 : \text{Preference is not independent of gender}$$

Expected frequencies:

$$E_{\text{Male, Like}} = \frac{55 \times 80}{120} \approx 36.67$$

$$E_{\text{Male, Dislike}} = \frac{55 \times 40}{120} \approx 18.33$$

$$E_{\text{Female, Like}} = \frac{65 \times 80}{120} \approx 43.33$$

$$E_{\text{Female, Dislike}} = \frac{65 \times 40}{120} \approx 21.67$$

$$\chi^2 = \frac{(30 - 36.67)^2}{36.67} + \frac{(25 - 18.33)^2}{18.33} + \frac{(50 - 43.33)^2}{43.33} + \frac{(15 - 21.67)^2}{21.67}$$
$$\approx 1.213 + 2.424 + 1.026 + 2.051 = 6.714$$

df $= (2 - 1)(2 - 1) = 1$. Using TI-Nspire, $p \approx 0.0096$.

Since $p < 0.05$, reject H_0: there is evidence of an association between gender and preference.

TI-Nspire instructions:

1. Enter the 2×2 table into a spreadsheet.
2. `menu` → `Statistics` → `Stat Tests` → `Chi-Square 2-way Test`.
3. Select the data range.
4. `Calculate`.

Worked Example 3 — Two-sample t-test

> A sports scientist wants to know if a new training program changes average sprint times. Group A (traditional training): mean = 12.4s, $s = 0.6$, $n = 10$. Group B (new training): mean = 11.9s, $s = 0.5$, $n = 8$. Test at the 5% significance level if the mean sprint times are different.

Solution:

$$H_0 : \mu_A = \mu_B \quad H_1 : \mu_A \neq \mu_B$$

Pooled t-test:

$$s_p^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2} = \frac{9(0.6^2) + 7(0.5^2)}{16} = \frac{3.24 + 1.75}{16} = 0.312$$

$$s_p = 0.558$$

$$t = \frac{12.4 - 11.9}{0.558\sqrt{\frac{1}{10} + \frac{1}{8}}} = \frac{0.5}{0.558 \times 0.471} \approx 1.89$$

df $= 10 + 8 - 2 = 16$. Using TI-Nspire, $p \approx 0.077$ (two-tailed).

Since $p > 0.05$, fail to reject H_0: no significant evidence of a difference in mean sprint times.

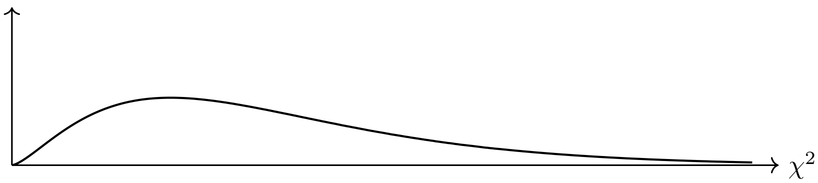
TI-Nspire instructions:

1. `menu` → `Statistics` → `Stat Tests` → `2-Sample t Test`.
2. Select `Stats`.
3. Enter means, standard deviations, and sample sizes.
4. Choose $\neq$ for two-tailed test.
5. Pooled = `Yes` → `Calculate`.

Q253 [Go to Solution p. 369] [Back to TOC]

Null/alternative, significance and p-value (concept). A test statistic χ^2 follows a χ^2 distribution (with degrees of freedom ν appropriate to context).

1. State suitable H_0 and H_1 for a χ^2 test (in words or symbols), e.g. "no association between variables" vs "there is an association".
2. On the diagram, *shade* the **critical region** for a 5% *upper-tail* test and mark the critical value $x^2_{0.05,\ \nu}$.
3. If a calculation gives some observed value x^2_{obs}, indicate on the same diagram the *p-value* region.



Final Answer: ____

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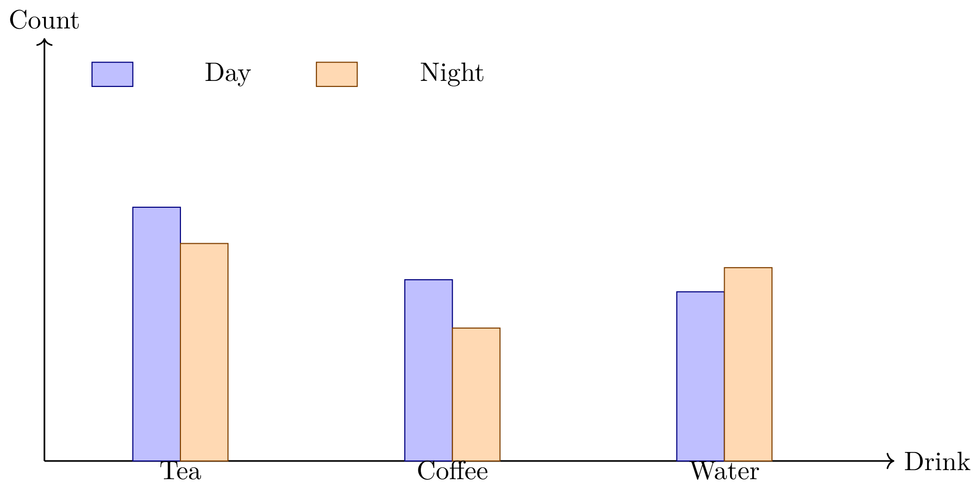
Q254 [Go to Solution p. 369] [Back to TOC]

χ^2 **test for independence (contingency table).** A cafeteria records customers' preferred drink by shift.

	Tea	Coffee	Water	Row total
Day	42	30	28	100
Night	36	22	32	90
Column total	78	52	60	190

1. Write H_0 and H_1 in context.
2. Compute all expected frequencies $E_{ij} = \dfrac{(\text{row total})(\text{column total})}{190}$ and verify that each expected value exceeds 5.
3. Determine the *degrees of freedom*. Using technology, calculate the test statistic χ^2 and the p-value.
4. Test at the 5% level and state a conclusion in context.



Name: ______________________ **Date:** ______________

Final Answer: ____

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Q255 [Go to Solution p. 370] [Back to TOC]

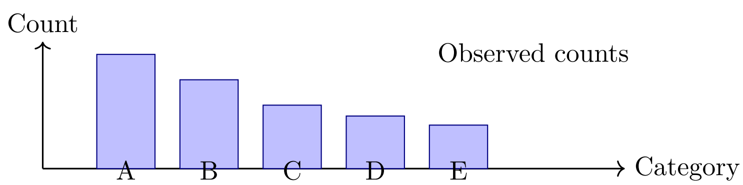
χ^2 **goodness of fit (given proportions).** A manufacturer claims the colours of a candy are distributed as

$$(30\%, 25\%, 20\%, 15\%, 10\%).$$

From a sample of $n = 400$ candies, the observed counts are

$$(126,\ 98,\ 70,\ 58,\ 48).$$

1. State H_0 and H_1.
2. Compute the *expected* counts and verify the usual conditions for a χ^2 test.
3. Using technology, calculate $\chi^2 = \sum \frac{(O-E)^2}{E}$, determine the degrees of freedom, and find the p-value.
4. Test at the 5% level and give a conclusion in context.



Final Answer: ____

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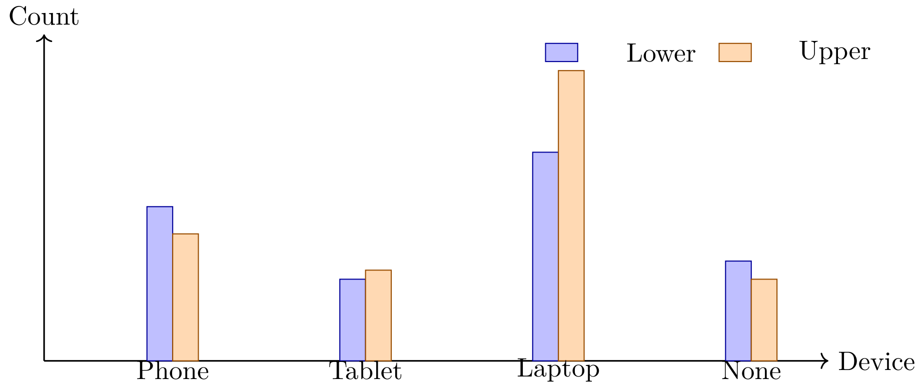
Q256 [Go to Solution p. 371] [Back to TOC]

χ^2 **test for independence (second layout).** A survey records device type used in class (Phone, Tablet, Laptop, None) by grade (Lower vs Upper).

	Phone	Tablet	Laptop	None	Row total
Lower	34	18	46	22	120
Upper	28	20	64	18	130
Column total	62	38	110	40	250

1. Write H_0 and H_1.
2. Compute all expected counts and the degrees of freedom.

3. Use technology to obtain χ^2 and the p-value. Decide at the 1% significance level.

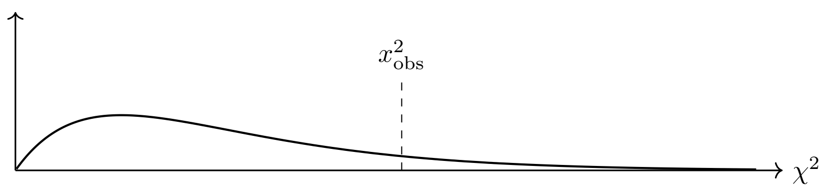


Final Answer: ____

Q257 [Go to Solution p. 372] [Back to TOC]

Reading a χ^2 curve. For a test with $\nu = 4$ degrees of freedom:

1. On the diagram, shade the critical region for a 10% upper-tail test and label the critical value $x^2_{0.10,4}$.
2. If $x^2_{\text{obs}} = 7.3$, indicate the p-value region and state whether H_0 would be rejected at the 10% level and at the 5% level (without calculating the exact p-value).



Final Answer: ____

Q258 [Go to Solution p. 372] [Back to TOC]

Two-tailed test with summary statistics (use technology). Battery life (hours) was measured for two brands using independent random samples. Assume both populations are approximately normal with equal variances.

Brand	n	$\bar{x}$	s
A	15	8.2	1.1
B	17	7.6	1.3

1. State H_0 and H_1 to test whether the mean lifetimes are different.
2. Using a pooled two-sample t-test, find the test statistic, degrees of freedom and the p-value.
3. At the 5% level, state your conclusion in context.
4. Find a 95% confidence interval for $\mu_A - \mu_B$ and interpret it.

Final Answer: ______________________________

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Q259 [Go to Solution p. 373] [Back to TOC]

One-tailed test with summary statistics (use technology). A grower compares a new fertilizer (N) to the current fertilizer (C) for plant height (cm) after 6 weeks. Independent random samples, normal populations, equal variances assumed.

	n	$\bar{x}$	s
N	12	42.1	5.6
C	10	38.5	6.1

1. Test $H_0 : \mu_N = \mu_C$ vs $H_1 : \mu_N > \mu_C$ at $\alpha = 0.05$.
2. Report the p-value and your conclusion.
3. Give a 95% confidence interval for $\mu_N - \mu_C$ (use technology) and comment on whether it supports the same decision.

Final Answer: ______________________________

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Q260 [Go to Solution p. 373] [Back to TOC]

Interpreting calculator output. A calculator reports for a two-sample pooled t-test: $t = -1.87$, df $= 26$, two-tailed $p = 0.073$.

1. What decision would you make at the 10% level? at the 5% level?

2. Which sample appears to have the larger mean? Explain from the sign of t.
3. If the calculator also gave a 90% CI of $(-0.3,\ 6.1)$ for $\mu_1 - \mu_2$, explain how it matches your 10% decision.

Final Answer: ______________________________

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Q261 [Go to Solution p. 373] [Back to TOC]

Write hypotheses and choose one- vs two-tailed. For each research question, write suitable H_0 and H_1 (in symbols) and state whether a one- or two-tailed test is appropriate.

1. A coach believes a new warm-up *reduces* mean 100 m times compared with the usual warm-up.
2. A nutritionist wants to know if the mean daily calcium intake differs between two schools.
3. A manufacturer claims a new process *increases* mean tensile strength relative to the current process.

Assume independent random samples, normality, and equal variances when a t-test is used.

Final Answer: ______________________________

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Q262 [Go to Solution p. 373] [Back to TOC]

Using raw data (two-tailed, use technology). Times to complete a puzzle (minutes) for two independent groups:

Group A: 12, 10, 9, 11, 13, 12, 8, 10 Group B: 7, 9, 11, 10, 8, 6, 9, 7

Assume normal populations with equal variances.

1. Compute $\bar{x}_A,\ s_A,\ \bar{x}_B,\ s_B$.
2. Perform a *two-tailed* pooled two-sample t-test for $\mu_A = \mu_B$ vs $\mu_A \neq \mu_B$. Report t, df and the p-value.
3. State your conclusion at $\alpha = 0.05$ and interpret the difference in context.

Final Answer: ______________________________

Q263 [Go to Solution p. 374] [Back to TOC]

Checking assumptions conceptually. For each statement, say whether it supports using the pooled two-sample t-test, and give a brief reason.

1. Histograms for both groups are roughly symmetric with no strong outliers; the sample SDs are similar.
2. The samples are two classes from the same school year where several students appear in both groups.
3. Normal probability plots are approximately linear for both groups; side-by-side boxplots show similar spreads.
4. Sample sizes are $n_1 = 8$ and $n_2 = 9$; both SDs are quite different ($s_1 = 2.0$, $s_2 = 6.0$).

Final Answer: ____

Q264 [Go to Solution p. 374] [Back to TOC]

One- vs two-tailed decision via a confidence interval. For two independent normal samples (equal variances assumed) a calculator gives the 95% CI for $\mu_1 - \mu_2$ as $(-1.4,\ 3.8)$.

1. What is the outcome of the two-tailed test at $\alpha = 0.05$? Explain.
2. Would the one-tailed test $H_0 : \mu_1 \leq \mu_2$ vs $H_1 : \mu_1 > \mu_2$ be significant at $\alpha = 0.05$? Explain briefly.

Final Answer: ____

AHL 4.12 Designing investigations, categories and sampling techniques

Q265 [Go to Solution p. 375] [Back to TOC]

Questionnaire design (identify and fix bias). A student drafts the following survey items to study screen time and sleep. For each item: (i) name any problem (leading/loaded, double-barrelled, ambiguous, social-desirability, poor scale, etc.); (ii) rewrite it to be precise, neutral, and answerable; (iii) specify the response format (options/numeric units).

1. "Do you agree that *excessive* screen time hurts grades?" *(Yes/No)*
2. "How many hours do you *usually* sleep and how many are *deep* sleep?" *(one box)*
3. "You don't look at your phone after midnight, right?" *(Yes/No)*
4. "Rate your health." *(bad / OK / good)*
5. "How often do you use social media for *fun or study*?" *(never/rarely/sometimes/often)*
6. "What is your GPA?" *(open response)* and "What year are you?" *(open response)*

Add one demographic item with a *prefer-not-to-say* option.

Final Answer: ____

____

Q266 [Go to Solution p. 375] [Back to TOC]

Sampling plan and data to analyse. You want to estimate the mean daily screen time of students at a school of 1200 students.

1. Define the target population and a sampling frame.
2. Propose a probability sampling method (simple random / stratified / cluster) and justify.
3. Describe how you will handle non-response, missing data, and outliers before analysis.
4. List the variables you will collect (with measurement units and type: numerical/ordinal/nominal) and explain which are *relevant* to the research question.

Final Answer: ____

____

Q267 [Go to Solution p. 375] [Back to TOC]

Selecting relevant variables from many. To predict final exam score Y, a spreadsheet contains: prior GPA, hours studied, attendance %, average sleep (h), class size, teacher ID, practice tests taken, phone unlocks/day, caffeine drinks/day.

1. Choose a subset of *relevant* explanatory variables with justification (domain knowledge, measurability, confounding, causality).
2. Describe two checks you would make before modelling (e.g. multicollinearity, transformations, influential points).
3. State which variables you would *ignore* and why.

Final Answer: ____

____

Q268 [Go to Solution p. 376] [Back to TOC]

Categorizing numerical data for a χ^2 goodness-of-fit test. Defect counts per item are believed to follow Poisson($\lambda = 2.4$). A random sample of $n = 200$ items produced the observed frequencies below.

Defects k	0	1	2	3	4	5	6+
Observed O_k	22	54	60	38	17	7	2

1. Compute $E_k = 200\,P(X = k)$ for $k = 0, 1, 2, \ldots$ and decide how to *recombine* categories so all expected counts are > 5; justify your grouping.
2. State the final categories and their O and E values in a table suitable for technology input.
3. Using the grouped table, perform the χ^2 goodness-of-fit test, report χ^2, the p-value, and your conclusion.

Final Answer: ____

____

Q269 [Go to Solution p. 376] [Back to TOC]

Degrees of freedom when parameters are estimated. For each scenario, the data are grouped into k categories with a fully specified model family; some parameters are estimated from the sample before computing expected counts. Give the *degrees of freedom* used for the χ^2 goodness-of-fit test and explain.

1. $k = 7$, model Binomial($n = 6, p$) with p estimated from the data.
2. $k = 8$, model Poisson(λ) with λ estimated from the data.

3. $k = 10$, model $\mathcal{N}(\mu, \sigma^2)$ with both μ and σ estimated from the data.

Final Answer: ____________

Q270 [Go to Solution p. 376] [Back to TOC]

Test–retest reliability (technology). A 40-point motivation scale was given to the same 12 students twice, two weeks apart.

ID	1	2	3	4	5	6	7	8	9	10	11	12
Time 1	34	28	40	31	25	37	29	33	35	27	30	32
Time 2	36	27	41	30	26	36	30	32	36	26	31	33

1. Compute the test–retest reliability as the Pearson correlation r between the two administrations (use technology).
2. Make a scatterplot and comment on linearity and any obvious outliers.
3. Interpret r in context (strength and direction).

Final Answer: ____________

Q271 [Go to Solution p. 376] [Back to TOC]

Parallel-forms reliability (technology). Ten students sat Form A and Form B of a vocabulary test (scores out of 20):

ID	1	2	3	4	5	6	7	8	9	10
Form A	16	12	14	18	10	15	13	17	11	16
Form B	15	11	14	17	12	14	13	16	10	15

1. Compute the parallel-forms reliability (correlation between A and B).
2. Check for systematic bias by finding the mean of (A−B); give a 95% CI or run a paired t-test (name your choice).
3. Comment on whether Forms A and B appear interchangeable.

Final Answer: ____

____

Q272 [Go to Solution p. 377] [Back to TOC]

Criterion-related validity (technology). A short anxiety scale S (0–40) is compared with an established long scale L (T-scores) for $n = 15$ participants:

ID	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
S	12	18	25	30	22	35	28	15	40	10	27	32	21	24	16
L	38	45	53	60	50	72	58	42	78	35	56	68	49	54	41

1. Compute the correlation between S and L and draw the regression of L on S.
2. Interpret the strength of evidence for *criterion-related validity*.
3. If a cut-score of $S \geq 30$ is proposed to flag "high anxiety", estimate the proportion flagged and comment on possible false positives/negatives relative to L.

Final Answer: ____

____

Q273 [Go to Solution p. 377] [Back to TOC]

Content validity (blueprint/mapping). An end-of-unit test has 8 items. The unit learning objectives (LO) are:

LO1: Definitions LO2: Procedures LO3: Applications LO4: Interpretation.

The teacher's draft blueprint is:

Item	LO1	LO2	LO3	LO4
1	✓			
2		✓		
3		✓		
4			✓	
5			✓	
6				✓
7				✓
8		✓		

1. Compute the coverage proportion for each LO and identify any imbalances.
2. Propose a revised blueprint that improves *content validity* without changing the total number of items.

3. Suggest one additional item stem that targets an under-represented LO.

Final Answer: ____

____

Q274 [Go to Solution p. 377] [Back to TOC]

Choosing relevant and appropriate data to analyse (cleaning rules). A CSV file contains survey responses with variables: ID, age, program, hours_sleep, weekday_screen_h, weekend_screen_h, GPA, missing_items.

1. Write reproducible inclusion/exclusion rules (e.g. plausible ranges, handling of missing_items).
2. Specify how you will create a single "average daily screen time" variable from weekday/weekend values (state the weighting).
3. Describe how you would document all cleaning steps so another analyst can reproduce your final dataset.

Final Answer: ____

____

Name: ____________________ **Date:** ________________

AHL 4.13 Regression with non-linear functions

Key Terms and Concepts

- **Regression** — A statistical method to model the relationship between two variables.
- **Non-linear regression** — Regression involving curves such as quadratic, cubic, exponential, power, or sine functions instead of straight lines.
- **Least squares regression** — A method that minimises the sum of the squared residuals between observed and predicted values.
- **Residual** (e) — The difference between the observed value y and the predicted value $\hat{y}$.

$$e_i = y_i - \hat{y}_i$$

- **Sum of squared residuals** (SS_{res}) — A measure of how well a model fits the data:

$$SS_{\text{res}} = \sum (y_i - \hat{y}_i)^2$$

- **Total sum of squares** (SS_{tot}) — Total variation in y from its mean:

$$SS_{\text{tot}} = \sum (y_i - \bar{y})^2$$

- **Coefficient of determination** (R^2) — Proportion of variation in the dependent variable explained by the model:

$$R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}}$$

  For linear regression, R^2 is the square of Pearson's correlation coefficient r.

When to use:

- **Linear regression** — When data points roughly follow a straight-line pattern.
- **Quadratic, cubic regression** — When curvature is evident in the data.
- **Exponential regression** — When data grows or decays at a constant percentage rate.
- **Power regression** — When y varies as a power of x.
- **Sine regression** — When data is periodic (e.g., seasonal trends).

Worked Example 1 — Quadratic regression (contextual)

A ball is thrown into the air and its height (m) is recorded at various times (s):

t (s)	0	0.5	1.0	1.5	2.0	2.5
h (m)	0	4.2	7.5	9.0	9.2	7.8

Fit a quadratic regression model to the data and use it to predict the maximum height.

Solution:

1. Enter t into `List1`, h into `List2` on the TI-Nspire.
2. `menu` → `Statistics` → `Stat Calculations` → `Quadratic Regression`.

3. x list: `List1`, y list: `List2`.
4. Choose `Calculate`.

Suppose the TI-Nspire gives:

$$h = -1.96t^2 + 5.81t + 0.02$$

The vertex occurs at:

$$t = -\frac{b}{2a} = -\frac{5.81}{2(-1.96)} \approx 1.48 \text{ s}$$

Max height $\approx h(1.48) \approx 9.3$ m.

Worked Example 2 — Exponential regression (contextual)

A bacteria culture grows over time. The recorded population size is:

Time (hours)	0	2	4	6	8
Population	100	180	325	590	1070

Fit an exponential model and estimate the population at 10 hours.

Solution:

1. Enter data into `List1` (time), `List2` (population).
2. `menu` → `Statistics` → `Stat Calculations` → `ExpReg`.
3. Choose `Calculate`.

Suppose the TI-Nspire outputs:

$$P = 100.4(1.317)^t$$

At $t = 10$:

$$P \approx 100.4(1.317)^{10} \approx 1928$$

Worked Example 3 — R^2 interpretation

A cubic regression model is fitted to data and the TI-Nspire reports $R^2 = 0.982$. Interpret this value.

Solution: $R^2 = 0.982$ means that 98.2% of the variation in the dependent variable is explained by the cubic model. Only 1.8% of variation is due to factors not captured by the model or random error.

General TI-Nspire Regression Instructions:

1. Enter x values in `List1`, y values in `List2`.
2. `menu` → `Statistics` → `Stat Calculations`.
3. Choose regression type: `LinReg`, `QuadReg`, `CubicReg`, `ExpReg`, `PwrReg`, `SinReg`.
4. Specify x list and y list.
5. Select "Save RegEqn to..." to store equation in a function variable (e.g., $f1(x)$).
6. View R^2 and parameters. To see residuals, store them to a list and plot.

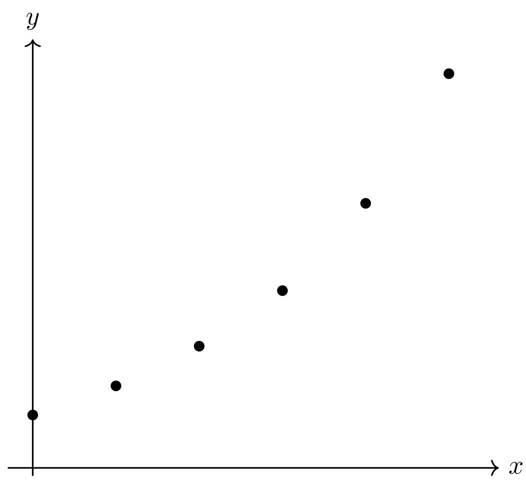
Name: ______________________ **Date:** ______________

Q275 [Go to Solution p. 378] [Back to TOC]

Choosing a model (exponential vs linear; use technology). A substrate grows over time. The measurements are:

x (days)	0	1	2	3	4	5
y	2.0	3.1	4.6	6.7	10.0	14.9

1. Plot the data and use technology to fit (i) a linear model $y = mx + c$ and (ii) an exponential model $y = a\,e^{bx}$.
2. For each model, report the parameters, R^2, and the sum of squared residuals SS_{res}.
3. Which model fits *better*? Justify with R^2, SS_{res} and the residual plot.
4. Using the better model, estimate y at $x = 6$. Comment on whether this is extrapolation.



Final Answer: ______________________

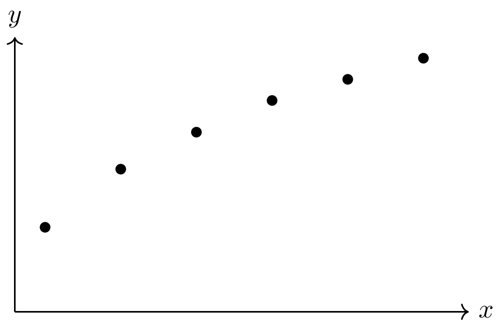
Q276 [Go to Solution p. 378] [Back to TOC]

Power model vs linear (use technology). A biomechanics study records

x	1	2	3	4	5	6
y	2.2	3.3	4.0	4.6	5.0	5.4

1. Fit a power model $y = a\,x^{\,b}$ and a linear model $y = mx + c$.
2. Give R^2 and SS_{res} for each; include a brief comment on which is more appropriate *and why* (shape, residuals).
3. Use the chosen model to predict y when $x = 8$.

Name: ______________________ **Date:** ______________



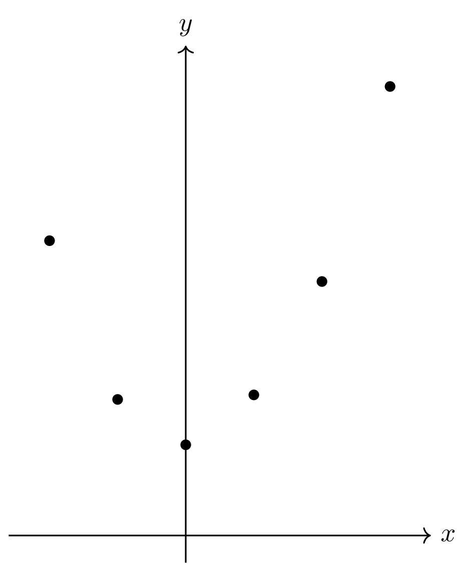
Final Answer: ____

Q277 [Go to Solution p. 378] [Back to TOC]

Quadratic or cubic? (use technology). For a process with a turning point, data were collected:

x	-2	-1	0	1	2	3
y	6.5	3.0	2.0	3.1	5.6	9.9

1. Fit a quadratic model $y = ax^2 + bx + c$ using least squares and state the vertex.
2. Fit a cubic model $y = px^3 + qx^2 + rx + s$.
3. Compare SS_{res} and R^2 for the two fits; check residual plots.
4. Which model would you report? Explain why a slightly smaller SS_{res} is not always better.



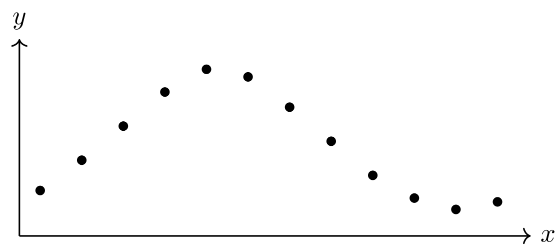
Final Answer: ______________________________

Q278 [Go to Solution p. 379] [Back to TOC]

Sinusoidal regression (seasonality; use technology). Monthly demand index (Jan=1,. . . ,Dec=12):

x	1	2	3	4	5	6	7	8	9	10	11	12
y	12.0	12.8	13.7	14.6	15.2	15.0	14.2	13.3	12.4	11.8	11.5	11.7

1. Fit a sine model $y = A\sin(B(x - C)) + D$ using technology.
2. Report A, B, C, D, the period $\frac{2\pi}{B}$, and R^2.
3. Interpret A and D (amplitude and mean level) in context.
4. Use the model to forecast y for month $x = 15$.



Final Answer: ______________________________

Q279 [Go to Solution p. 379] [Back to TOC]

Compute SS_{res} and R^2 from small data. Observed outcomes: $y = (3.2,\ 4.1,\ 5.0,\ 6.0)$. Two competing models give predictions

$$\hat{y}^{(1)} = (3.0, 4.3, 4.8, 6.2), \qquad \hat{y}^{(2)} = (3.4, 3.9, 5.2, 6.1).$$

1. Compute $SS^{(1)}_{\text{res}} = \sum(y - \hat{y}^{(1)})^2$ and $SS^{(2)}_{\text{res}}$.
2. Compute $SS_{\text{tot}} = \sum(y - \bar{y})^2$ and hence $R_1^2 = 1 - SS^{(1)}_{\text{res}}/SS_{\text{tot}}$ and R_2^2.
3. Which model fits better by these criteria? Are the differences practically important?

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

Q280 [Go to Solution p. 380] [Back to TOC]

R^2 **from a correlation (linear models).** In a linear regression the Pearson correlation between x and y is $r = -0.84$.

1. Find R^2 and interpret it as a percentage of variability explained.
2. Explain why the *sign* of r does not affect R^2.

Final Answer: ______________________

Q281 [Go to Solution p. 380] [Back to TOC]

Deciding between models (beyond R^2). Two models fitted to the same dataset give $R^2 = 0.982$ (Model A) and $R^2 = 0.988$ (Model B). Model B has two more parameters.

1. Explain why choosing Model B solely because it has the larger R^2 can be misleading.
2. Describe two other pieces of evidence you would examine (e.g. residual patterns, plausibility of form, overfitting or validation error).
3. State which model you would report if residuals for Model A are pattern-free but Model B shows curvature left in the residuals.

Final Answer: ______________________

AHL 4.14 Linear combinations, expectations/variance

Q282 [Go to Solution p. 381] [Back to TOC]

Linear transformation of a random variable. Suppose a random variable X has $\mathbb{E}(X) = 50$ and $\text{Var}(X) = 9$. Let $Y = 2X - 7$.

1. Find $\mathbb{E}(Y)$ and $\text{Var}(Y)$.
2. Hence find the standard deviation of Y.
3. Briefly explain why adding a constant does not change the variance.

Final Answer: ______________

Q283 [Go to Solution p. 381] [Back to TOC]

Unit conversion (linear transformation). Daily maximum temperature in degrees Celsius is modelled by a random variable C with $\mathbb{E}(C) = 21.4$ and $\text{SD}(C) = 3.2$. Let $F = 1.8C + 32$ be the temperature in Fahrenheit.

1. Find $\mathbb{E}(F)$ and $\text{Var}(F)$.
2. Interpret the effect of the scale factor 1.8 on the variance.

Final Answer: ______________

Q284 [Go to Solution p. 381] [Back to TOC]

Expectation of a linear combination (independence *not* required). Let random variables X_1 and X_2 have means $\mu_1 = 8$ and $\mu_2 = 3$ (no assumption about independence). Compute $\mathbb{E}(2X_1 - 3X_2 + 5)$ and state the rule you used.

Final Answer: ______________

Q285 [Go to Solution p. 381] [Back to TOC]

Variance of a linear combination (independent variables). Let X_1, X_2, X_3 be independent with

$$\mathbb{E}(X_1) = 4,\ \mathrm{Var}(X_1) = 1.2, \qquad \mathbb{E}(X_2) = 5,\ \mathrm{Var}(X_2) = 2.0, \qquad \mathbb{E}(X_3) = 2,\ \mathrm{Var}(X_3) = 0.5.$$

For $S = 3X_1 - 2X_2 + X_3$:

1. Find $\mathbb{E}(S)$ and $\mathrm{Var}(S)$.
2. Hence find the mean and variance of the average $A = \dfrac{S}{2}$.

Final Answer: ________________________________

Q286 [Go to Solution p. 381] [Back to TOC]

Sample mean of i.i.d. variables. Let $X_1, \ldots, X_n$ be independent, identically distributed with $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Let $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} X_i$.

1. Using the linearity rules, show that $\mathbb{E}(\bar{X}) = \mu$.
2. Show that $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$.
3. Explain how increasing n affects the standard deviation of $\bar{X}$.

Final Answer: ________________________________

Q287 [Go to Solution p. 381] [Back to TOC]

Unbiasedness in words. Explain what it means to say that $\bar{X}$ is an *unbiased* estimator of μ. Give a short, concrete example (one sentence) to illustrate the meaning of “unbiased” in context.

Final Answer: ________________________________

Q288 [Go to Solution p. 382] [Back to TOC]

Compute $\bar{x}$ and s_{n-1}^2 from raw data (use technology). A sample of $n = 8$ observations is

$$12,\ 10,\ 9,\ 11,\ 13,\ 12,\ 8,\ 10.$$

1. Compute the sample mean $\bar{x}$.
2. Compute the unbiased sample variance

$$s_{n-1}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2,$$

   and the corresponding sample standard deviation s_{n-1}.
3. If all measurements were accidentally recorded in metres instead of centimetres (i.e. each value divided by 100), what happens to s_{n-1}^2?

Final Answer: ______________________________

Q289 [Go to Solution p. 382] [Back to TOC]

Mean and unbiased variance from grouped (frequency) data. The discrete values x_i occur with frequencies f_i as follows:

x_i	2	4	5	7	9
f_i	3	6	5	4	2

(total $n = \sum f_i$).

1. Compute the sample mean $\bar{x} = \dfrac{1}{n}\sum f_i x_i$.
2. Compute the unbiased sample variance using $s_{n-1}^2 = \dfrac{\sum f_i(x_i - \bar{x})^2}{n-1}$.

Final Answer: ______________________________

Q290 [Go to Solution p. 382] [Back to TOC]

Name: ______________________ **Date:** ______________

Variance of a weighted combination of *independent* sample means. Two independent random samples are taken:

$\bar{X}_A$ from population A with variance σ_A^2, size n_A; $\bar{X}_B$ from population B with variance σ_B^2, size n_B.

Consider $W = 0.4\,\bar{X}_A + 0.6\,\bar{X}_B$.

1. Find $\mathbb{E}(W)$ in terms of the population means μ_A, μ_B.
2. Using independence, find $\mathrm{Var}(W)$ in terms of $\sigma_A^2, \sigma_B^2, n_A, n_B$.
3. Evaluate $\mathrm{Var}(W)$ when $\sigma_A = 6$, $n_A = 25$, $\sigma_B = 10$, $n_B = 36$.

Final Answer: ______________________

Q291 [Go to Solution p. 382] [Back to TOC]

Effect of linear rescaling on sample variance (units). A height dataset has sample variance $s_{cm}^2 = 64$ when measured in centimetres. Heights are converted to metres by $H_m = \dfrac{1}{100} H_{cm}$.

1. Without re-computing from raw data, find the variance in metres, s_m^2.
2. Explain the general rule for how variance changes under $Y = aX + b$.

Final Answer: ______________________

AHL 4.15 Central limit theorem, and combinations of normal distributions

Q292 [Go to Solution p. 384] [Back to TOC]

Sampling mean from a normal population. Suppose $X \sim \mathcal{N}(\mu = 72,\ \sigma^2 = 16)$. A random sample of size $n = 25$ is taken.

1. State the distribution of the sample mean $\bar{X}$ (give its mean and variance).
2. Using technology, find $P(\bar{X} > 74)$.
3. Let $S = \sum_{i=1}^{25} X_i$. State the distribution of S and compute $P(S < 1770)$.

Final Answer: ____________

Q293 [Go to Solution p. 384] [Back to TOC]

Linear combination of independent normals. Let $X \sim \mathcal{N}(10, 3^2)$ and $Y \sim \mathcal{N}(16, 4^2)$ be independent. Define $A = 0.3X + 0.7Y$.

1. Find $\mathbb{E}(A)$ and $\text{Var}(A)$.
2. State the distribution of A.
3. Using technology, evaluate $P(14 \le A \le 17)$.

Final Answer: ____________

Q294 [Go to Solution p. 384] [Back to TOC]

Weighted sum of several normals. Independent variables $X_1 \sim \mathcal{N}(20, 5^2)$, $X_2 \sim \mathcal{N}(15, 2^2)$, $X_3 \sim \mathcal{N}(12, 3^2)$. Let $W = 2X_1 - X_2 + \frac{1}{2}X_3$.

1. Find $\mathbb{E}(W)$ and $\text{Var}(W)$.
2. State the distribution of W.
3. Compute $P(W > 35)$ using technology.

Final Answer: ____________

Q295 [Go to Solution p. 384] [Back to TOC]

CLT with a non-normal population (mean of waiting times). Waiting time T (minutes) has an exponential distribution with mean 5 minutes (variance 25). A simple random sample of $n = 40$ customers is taken.

1. Using the central limit theorem, give the approximate distribution of $\bar{T}$.
2. Estimate $P(4.5 < \bar{T} < 5.5)$.
3. If the manager wants $\text{SD}(\bar{T}) \le 0.4$ minutes, what sample size n is required (use the CLT)?

Final Answer: ____

Q296 [Go to Solution p. 384] [Back to TOC]

Sample proportion as a sample mean (CLT). Each trial results in *success* with probability $p = 0.3$ independently of the others. In $n = 200$ trials let $\hat{p}$ be the sample proportion of successes.

1. Treating $\hat{p}$ as the mean of 0–1 variables, give its approximate distribution (mean and variance).
2. Estimate $P(\hat{p} \ge 0.35)$ using a normal approximation.
3. A report claims the success rate exceeds 0.33. What is $P(\hat{p} > 0.33)$ under $p = 0.30$?

Final Answer: ____

Q297 [Go to Solution p. 385] [Back to TOC]

Sum vs mean. Independent $X_i \sim \mathcal{N}(\mu = 50,\ \sigma^2 = 100)$, $i = 1, \ldots, n$.

1. Express the distribution of the sum $S_n = \sum_{i=1}^{n} X_i$.
2. Express the distribution of the mean $\bar{X} = S_n/n$.
3. For $n = 36$, compute $P(48 < \bar{X} < 52)$.

Final Answer: ________________

Q298 [Go to Solution p. 385] [Back to TOC]

Mixture of two normal samples (independent). Group A: $n_A = 20$ scores from $\mathcal{N}(70, 9^2)$. Group B: $n_B = 30$ scores from $\mathcal{N}(75, 10^2)$. Let $\bar{X}_A$ and $\bar{X}_B$ be the sample means (independent).

1. Find the distribution of the difference $\bar{X}_B - \bar{X}_A$.
2. Compute $P(\bar{X}_B - \bar{X}_A \geq 3)$ using technology.
3. If both groups are doubled in size, state how the variance of $\bar{X}_B - \bar{X}_A$ changes.

Final Answer: ________________

Q299 [Go to Solution p. 385] [Back to TOC]

Interpreting the CLT. Answer in concise sentences.

1. State the central limit theorem in words as it applies to the sample mean.
2. Give one example where $n = 25$ might still be inadequate for normal approximation, and one where even $n = 10$ might be adequate.
3. Explain the difference between the distribution of X and the distribution of $\bar{X}$.

Final Answer: ________________

Name: ____________________ **Date:** ______________

AHL 4.16 Confidence intervarls

Q300 [Go to Solution p. 386] [Back to TOC]

Known σ: compute and interpret a CI. A machine fills cereal boxes. The fill weights (g) are normally distributed with known standard deviation $\sigma = 12$. A random sample of $n = 40$ boxes has sample mean $\bar{x} = 83.5$.

1. Find a 95% confidence interval for the population mean μ (use z).
2. Interpret the interval in context.

Final Answer: ______________________________

Q301 [Go to Solution p. 386] [Back to TOC]

Unknown σ: t interval. Times to assemble a device (min) are normal. For $n = 12$ workers, $\bar{x} = 6.2$ and sample standard deviation $s = 1.1$.

1. Construct a 90% confidence interval for μ (use t with $n - 1$ df).
2. Explain why t is used even though n is small.

Final Answer: ______________________________

Q302 [Go to Solution p. 386] [Back to TOC]

Raw data, use technology. A sample of $n = 8$ lifetimes (hours) is

$$12,\ 10,\ 9,\ 11,\ 13,\ 12,\ 8,\ 10.$$

Assuming a normal population, use technology to compute a 99% confidence interval for μ. State n, $\bar{x}$, s, df, and the critical value t^* you used.

Final Answer: ______________________________

Q303 [Go to Solution p. 386] [Back to TOC]

Planning sample size (known σ). The population standard deviation is believed to be $\sigma = 3.4$ units. What sample size n guarantees a 95% margin of error at most $E = 0.50$ for a z-based CI for μ? (Show the formula you use and round up to the next integer.)

Final Answer: ______________________________

Q304 [Go to Solution p. 386] [Back to TOC]

Planning with an s estimate. A pilot study gives $s \approx 4.8$ minutes for time-on-task. How large must n be so that a 95% CI for the mean has margin of error at most 1.0 minute? (Use the planning rule with $z_{0.975} = 1.96$.)

Final Answer: ______________________________

Q305 [Go to Solution p. 387] [Back to TOC]

Effect of confidence level. Two independent teams report CIs for the same mean: Team A gives $(18.3,\ 21.7)$ and Team B gives $(17.5,\ 22.5)$ using the same data.

1. Which team likely used the higher confidence level? Explain.
2. Which interval has the larger margin of error? Compute both margins.

Final Answer: ______________________________

Q306 [Go to Solution p. 387] [Back to TOC]

Paired data: mean difference CI (use technology). Ten participants completed a task *before* and *after* training (times in seconds):

Before	52	48	60	55	50	62	58	57	54	59
After	47	44	53	50	48	58	55	51	49	54

Let $D = \text{Before} - \text{After}$. Assuming D is normal, find a 95% CI for μ_D (the mean change). Interpret.

Final Answer: ______________________________

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Q307 [Go to Solution p. 387] [Back to TOC]

Identify the confidence level from an interval. A lab reports μ for a normal population with known $\sigma = 9$ using $n = 36$ and $\bar{x} = 74.3$. Their reported CI is $(71.4,\ 77.2)$.

1. Compute the margin of error and the corresponding z^*.
2. What confidence level (approximately) did they use?

Final Answer: ______________________________

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Q308 [Go to Solution p. 387] [Back to TOC]

Interpretation check (concept). A student writes: "There is a 95% probability that μ lies in our 95% CI." Is this correct? If not, rewrite a correct, clear interpretation of a 95% CI for μ.

Final Answer: ______________________________

____

Q309 [Go to Solution p. 387] [Back to TOC]

Which distribution: z or t? For each scenario, circle z or t and justify briefly.

Name: ______________________ **Date:** ______________

1. Normally distributed outcome, σ known, $n = 12$.
2. Right-skewed outcome, σ unknown, $n = 60$ (use CLT).
3. Normally distributed outcome, σ unknown, $n = 9$.

Final Answer: ____

AHL 4.17 Poisson Distribution

Q310 [Go to Solution p. 388] [Back to TOC]

Customers arrive at a shop according to a Poisson process at an average rate of 3 per hour. Let Y be the number of customers in a two–hour interval.

(i) State the distribution of Y.

(ii) Compute $P(Y = 5)$.

(iii) Compute $P(Y \geq 7)$.

Final Answer: ______________________________

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Q311 [Go to Solution p. 388] [Back to TOC]

Customers arrive at a kiosk at a mean rate of 3.2 per 10 minutes, independently.

1. Define a suitable Poisson model for the number N of arrivals in 30 minutes and state its mean and variance.
2. Compute $P(N = 12)$ and $P(N \geq 15)$.
3. Briefly justify why the Poisson model is appropriate (conditions).

Final Answer: ______________________________

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Name: ________________ Date: ____________

AHL 4.18 Hypothesis testing and Type errors

Key ideas

Hypothesis testing (overview).

1. State H_0 (null hypothesis) and H_1 (alternative hypothesis); choose significance level α.
2. Select the correct test (distribution and assumptions).
3. Find the *critical region* for the given α (or compute a test statistic and a p-value).
4. Make a decision: reject H_0 if the test statistic lies in the critical region (p-value $\leq \alpha$); otherwise fail to reject H_0.

Errors. Type I error (false positive) occurs with probability α (by design). Type II error has probability β (false negative); power $= 1 - \beta$.

Test for a population mean (normal), σ known — z test

Worked example

Example. A manufacturer claims $\mu = 50$. A sample of $n = 36$ gives $\bar{x} = 52.5$, $\sigma = 8$. Test $H_0 : \mu = 50$ vs $H_1 : \mu > 50$ at $\alpha = 0.05$.
Critical value: $z_{0.95} = 1.6449$.
Critical sample mean:

$$\bar{x}_c = 50 + 1.6449 \cdot \frac{8}{\sqrt{36}} = \mathbf{52.1931}.$$

Since $\bar{x} = 52.5 > \bar{x}_c$, reject H_0.

Type I/II focus

Type II error. Given that the true mean is $\mu = 51$:

$$\beta = P(\bar{X} < 52.1931 \mid \mu = 51) = P\left(Z < \frac{52.1931 - 51}{8/\sqrt{36}}\right) = P(Z < 0.8899) = \mathbf{0.8146}.$$

Power $= 1 - \beta = \mathbf{0.1854}$ (low for such a small difference from 50).

Calculator tip (TI-Nspire)

z Test: `Menu` $\to$ `Statistics` $\to$ `Stat Tests` $\to$ `z Test`. Input: $\mu_0 = 50$, $\sigma = 8$, $\bar{x} = 52.5$, $n = 36$, Alt: $>$.

Test for a proportion (binomial), one-tailed

Name: ____________________ **Date:** ____________

Worked example

Example. A company claims 20% of calls are complaints. From $n = 25$ calls, $x = 8$ are complaints. Test $H_0 : p = 0.20$ vs $H_1 : p > 0.20$ at $\alpha = 0.05$.
Find smallest c with $P(X \geq c \mid p = 0.20) \leq 0.05$:
$P(X \geq 9) = \mathbf{0.04677}$, so *critical region* is $\{9, 10, \ldots, 25\}$.
Since $x = 8$ is not in the critical region, fail to reject H_0.

Type I/II focus

Type II error. Given that the true proportion is $p = 0.25$:

$$\beta = P(X \leq 8 \mid n = 25, p = 0.25) = \mathbf{0.85056}.$$

Power $= 1 - \beta = \mathbf{0.14944}$.

Calculator tip (TI-Nspire)

Binomial CDF: `Menu` $\rightarrow$ `Statistics` $\rightarrow$ `Distributions` $\rightarrow$ `Binomial Cdf`. For right tail $P(X \geq c)$, compute $1 - P(X \leq c - 1)$.

Test for a mean count (Poisson), one-tailed

Worked example

Example. Historic mean breakdown rate is $\lambda = 1.5$/week. Over 10 weeks, test $H_0 : \lambda = 15$ vs $H_1 : \lambda > 15$ (total counts) at $\alpha = 0.05$.
Find smallest k with $P(X \geq k \mid \lambda = 15) \leq 0.05$:
$P(X \geq 23) = \mathbf{0.03274}$, so *critical region* is $\{23, 24, \ldots\}$.
If observed $X = 25$, reject H_0.

Type I/II focus

Type II error. Given that the true mean is $\lambda = 18$:

$$\beta = P(X \leq 22 \mid \text{Pois}(18)) = \mathbf{0.85509}.$$

Power $= 1 - \beta = \mathbf{0.14491}$.

Calculator tip (TI-Nspire)

Poisson CDF: `Menu` $\rightarrow$ `Statistics` $\rightarrow$ `Distributions` $\rightarrow$ `Poisson Cdf`. For right tail $P(X \geq k)$, compute $1 - P(X \leq k - 1)$.

Name: ____________________ **Date:** ______________

Type I and Type II errors — one-page summary

Key ideas

Definitions.

- **Type I error** (false positive): reject H_0 when it is true. Probability $= \alpha$.
- **Type II error** (false negative): fail to reject H_0 when H_1 is true. Probability $= \beta$.
- **Power** $= 1 - \beta$: probability of correctly rejecting H_0.

IB note: In exams, for discrete distributions (binomial/Poisson) and normal with known σ, you may be given a specific alternative value to compute β.

Test	Critical region (right-tailed)	Type I/II example (given alt.)
Normal mean, σ known	$\bar{X} \geq 52.1931$	Type I $= 0.0500$. Given $\mu = 51$: $\beta = 0.8146$, Power $= 0.1854$.
Binomial proportion ($n = 25$)	$X \geq 9$	Type I $= 0.04677$. Given $p = 0.25$: $\beta = 0.85056$, Power $= 0.14944$.
Poisson mean (λ per 10 wks)	$X \geq 23$	Type I $= 0.03274$. Given $\lambda = 18$: $\beta = 0.85509$, Power $= 0.14491$.

Calculator tip (TI-Nspire)

IB reminder: For β calculations, the alternative parameter value will be given in the question. First find the critical region from α under H_0, then evaluate β as the probability of the non-rejection region under the alternative.

Name: ______________________ **Date:** ______________

IB-style example: Binomial test with Type II error

Worked example

A company claims that 20% of products are defective. A quality inspector takes a random sample of $n = 25$ products.

(a) At the $\alpha = 0.05$ significance level, find the smallest critical value c for a right-tailed test of

$$H_0 : p = 0.20 \quad \text{vs} \quad H_1 : p > 0.20.$$

(b) The inspector finds $x = 8$ defectives. State whether H_0 should be rejected and justify your answer.

(c) Given that the true proportion is $p = 0.25$, calculate the Type II error β and the power of the test.

Solution:

(a) We seek the smallest c with $P(X \geq c \mid p = 0.20) \leq 0.05$.
Using binomial cumulative probabilities:

$$P(X \geq 9) = 0.04677 \quad \Rightarrow \quad \text{critical region} = \{9, 10, \ldots, 25\}.$$

(b) Since $x = 8 \notin \{9, \ldots, 25\}$, it is not in the critical region. **Decision:** Fail to reject H_0. There is insufficient evidence at the 5% level to suggest $p > 0.20$.

(c) Given $p = 0.25$,

$$\beta = P(X \leq 8 \mid n = 25, p = 0.25) = 0.85056.$$

Power $= 1 - \beta = 0.14944$.

Calculator tip (TI-Nspire)

On TI-Nspire: `Menu` → `Statistics` → `Distributions` → `Binomial Cdf`.
(a) Use $1 - P(X \leq c - 1)$ to find smallest c with probability ≤ 0.05 under $p = 0.20$. (c) For β, compute $P(X \leq c - 1)$ under $p = 0.25$.

Q312 [Go to Solution p. 388] [Back to TOC]

Critical values and critical regions. A manufacturer claims that the mean lifetime of its light bulbs is 2000 hours. A sample of $n = 36$ bulbs is tested and the sample mean is found to be 1960 hours with a known population standard deviation $\sigma = 120$ hours.

1. State the null and alternative hypotheses for a one-tailed test at the 5% significance level.
2. Determine the critical value of $\bar{x}$ that defines the rejection region.
3. Decide whether the manufacturer's claim should be rejected based on the sample.

Final Answer: ________________

Q313 [Go to Solution p. 388] [Back to TOC]

Test for population mean (normal distribution). A machine fills bottles with orange juice. The fill volumes are normally distributed with unknown standard deviation. A sample of $n = 15$ bottles has mean 250.4 ml and standard deviation 1.2 ml.

1. Test, at the 1% significance level, whether the machine is filling the bottles with the nominal mean volume of 250 ml.
2. State the null and alternative hypotheses clearly.
3. Explain whether a z-test or t-test is appropriate here.

Final Answer: ________________

Q314 [Go to Solution p. 389] [Back to TOC]

Matched pairs t-test (normal distribution). A group of 10 students takes a mathematics test before and after attending a revision course. Their scores (out of 50) are recorded.

1. State appropriate null and alternative hypotheses to test whether the course improved scores.
2. Explain why a matched pairs test is appropriate in this case.
3. Perform the test at the 5% significance level.

Final Answer: ____

____

Q315 [Go to Solution p. 389] [Back to TOC]

Test for proportion (binomial distribution). A factory claims that only 2% of its products are defective. A customer tests a sample of 80 products and finds 5 defective ones.

1. Carry out a one-tailed hypothesis test at the 5% significance level to determine whether the proportion of defective products is greater than 2%.
2. State the null and alternative hypotheses.
3. Identify the critical region and conclude.

Final Answer: ____

____

Q316 [Go to Solution p. 389] [Back to TOC]

Test for population mean (Poisson distribution). A website records the number of hits per minute. Historically, the mean rate is 5 hits per minute. A sample of 60 one-minute intervals shows a total of 330 hits.

1. Carry out a one-tailed hypothesis test at the 5% significance level to determine whether the mean rate has increased.
2. State the null and alternative hypotheses.
3. Identify the critical region and conclude.

Final Answer: ____

____

Q317 [Go to Solution p. 389] [Back to TOC]

Test for correlation coefficient. A study investigates the relationship between students' hours of study and their exam scores. A sample of 12 students produces a sample product moment correlation coefficient $r = 0.65$.

1. Test, at the 5% significance level, the hypothesis that the population correlation coefficient $\rho = 0$.
2. State the null and alternative hypotheses.
3. Explain why the t-test for correlation is appropriate here.

Final Answer: ____

____

Q318 [Go to Solution p. 390] [Back to TOC]

Type I and Type II errors. A manufacturer claims that their batteries last 500 hours on average. A hypothesis test is conducted with $H_0 : \mu = 500$ against $H_1 : \mu < 500$, at the 5% significance level.

1. Explain what is meant by a Type I error in this context.
2. Explain what is meant by a Type II error in this context.
3. Given that $\sigma = 20$ hours and $n = 25$, calculate the probability of a Type II error if the true mean is $\mu = 495$ hours.

Final Answer: ____

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Name: ______________________ **Date:** ______________

AHL 4.19 Markov Chains

Q319 [Go to Solution p. 391] [Back to TOC]

A simple weather model has two states: sunny (S) and rainy (R). Each morning, the weather transitions according to the matrix

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix},$$

where the first row/column correspond to S and the second to R. If today is sunny, (i) find the probability that it will be rainy two days hence; (ii) find the long–term steady–state distribution of the chain.

Final Answer: ____

Name: ______________________ Date: ______________

Topic 5 — Calculus (SL 5.1–5.8, AHL 5.9–5.18)

Overview (SL) Introduces limits and the concept of a derivative, increasing and decreasing functions, and basic differentiation of polynomial, exponential, and trigonometric functions. Covers tangents and normals, basic integration, finding local minima and maxima, optimisation problems, and numerical integration using the trapezium rule.

Overview (HL) Extends SL content to include differentiation of more complex functions (implicit and parametric), second derivatives, integration by substitution, areas and volumes of revolution, kinematics, modelling with differential equations (including separation of variables), slope fields, Euler's method, phase portraits, and second-order differential equations.

Real-World Use

- Modelling change in economics, biology, and resource use
- Physics applications such as motion, forces, and energy
- Engineering design optimisation and structural analysis
- Epidemiology for modelling infection and decay rates
- Predictive modelling in data science and machine learning

Common Misconceptions

- Confusing the derivative function $f'(x)$ with the slope at a single point
- Forgetting the constant of integration when finding antiderivatives
- Misapplying the chain, product, or quotient rules
- Assuming all stationary points are maxima or minima without further testing
- Confusing definite integrals (as areas) with indefinite integrals (as families of functions)

Advice for SL

- Practice differentiating and integrating common functions without a calculator to build fluency
- Interpret results in the context of the problem
- Use diagrams to link geometric concepts (slopes, areas) with algebraic methods
- In optimisation, check endpoints as well as stationary points
- For trapezium rule problems, sketch the graph to see over- or under-estimation

Advice for HL

- In implicit and parametric differentiation, clearly show steps and variables
- For substitution in definite integrals, change the limits appropriately
- When modelling with differential equations, define variables and parameters first
- Use slope fields and Euler's method for qualitative insights before analytic solutions
- For second-order differential equations, identify whether solutions are oscillatory, exponential, or mixed, and interpret them in context
- In kinematics, be explicit about whether derivatives are with respect to time or another variable

Name: ____________________ **Date:** ______________

SL 5.1 Introduction to concept of limits

Q320 [Go to Solution p. 393] [Back to TOC]

Limit from a table (removable discontinuity). Consider $f(x) = \dfrac{x^2 - 9}{x - 3}$ for $x \neq 3$ and $f(3)$ is undefined.

1. Complete the table (use a calculator) and then estimate $\lim_{x \to 3} f(x)$.
2. Does the limit equal $f(3)$? Explain briefly.

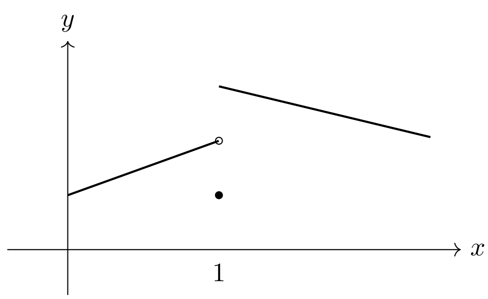
x	2.8	2.9	2.99	3.01	3.1	3.2
$f(x)$						

Final Answer: ____________________

Q321 [Go to Solution p. 393] [Back to TOC]

Left- and right-hand limits from a graph. The graph of a function $y = f(x)$ near $x = 1$ is sketched below. Use it to answer the questions.

1. Estimate $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$.
2. State whether $\lim_{x \to 1} f(x)$ exists.
3. What is the value of $f(1)$?



Final Answer: ____________________

Q322 [Go to Solution p. 393] [Back to TOC]

Name: ____________________ **Date:** ______________

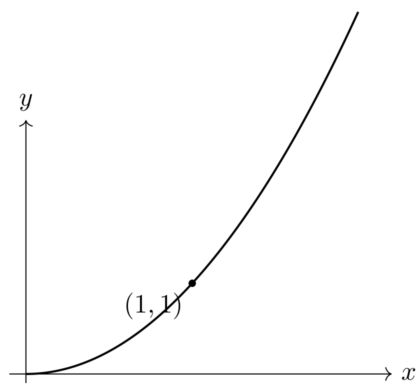
Average rate of change and instantaneous rate (velocity idea). A particle's position (m) is $s(t) = 3t^2$ with t in seconds.

1. Compute the average rate of change of s on $[2, 2.1]$ and on $[2, 2.01]$.
2. Use these to estimate the instantaneous velocity at $t = 2$ s.
3. Give appropriate units for your answers.

Final Answer: ____

Q323 [Go to Solution p. 393] [Back to TOC]

Secant slopes approaching a tangent slope (graphical estimate). For $y = x^2$, estimate the slope of the tangent at $x = 1$ by computing slopes of secants between $x = 1$ and $x = 1.5,\ 1.2,\ 1.1,\ 1.01$. Explain what number the slopes appear to be approaching.



Final Answer: ____

Q324 [Go to Solution p. 393] [Back to TOC]

Recognising derivative notation and variables. Match each derivative to its independent/dependent variables and a possible context.

1. $\dfrac{dy}{dx}$,
2. $f'(3)$,

3. $\dfrac{dV}{dr}$,
4. $\dfrac{ds}{dt}$.

State, for each, (i) the independent variable, (ii) the dependent variable, and (iii) an appropriate unit if y is in metres and x is in seconds (adapt as needed).

Final Answer: ____

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Q325 [Go to Solution p. 393] [Back to TOC]

Estimating a limit numerically (no algebraic manipulation). Estimate $\lim_{h\to 0} \dfrac{\sin(\frac{\pi}{4}+h)-\sin(\frac{\pi}{4})}{h}$ by evaluating the expression for $h = 0.1, 0.01, 0.001$ (radians). What value does it appear to approach?

Final Answer: ____

____

Q326 [Go to Solution p. 394] [Back to TOC]

Instantaneous rate from a time series (table). Water temperature T (in °C) is recorded every 10 min:

t (min)	0	10	20	30	40	50
T (°C)	20.0	22.1	23.8	24.7	25.1	25.3

1. Compute the average rate of change on $[20, 30]$ and $[30, 40]$.
2. Use a symmetric difference to estimate dT/dt at $t = 30$ min and give units.
3. Is the water warming faster or slower at $t = 30$ compared with earlier? Explain.

Final Answer: ____

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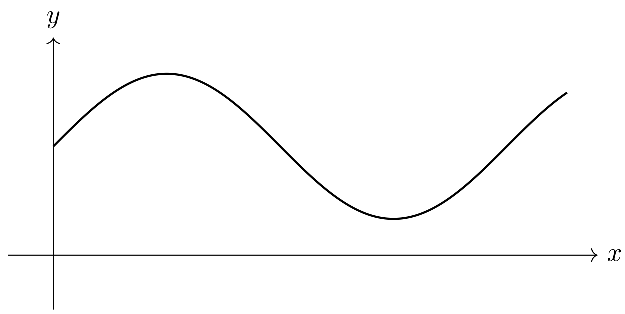
Name: ______________________ **Date:** ______________

Q327 [Go to Solution p. 394] [Back to TOC]

Reading slope sign from a curve. In the sketch below, decide where the slope (derivative) of f is positive, zero, or negative. Mark approximate x-values where the slope is zero.



Final Answer: ______________________________

Q328 [Go to Solution p. 394] [Back to TOC]

Limit versus function value (hole in the graph). The function g is defined by $g(x) = \dfrac{(x-2)(x+1)}{x-2}$ for $x \neq 2$ and $g(2) = 5$.

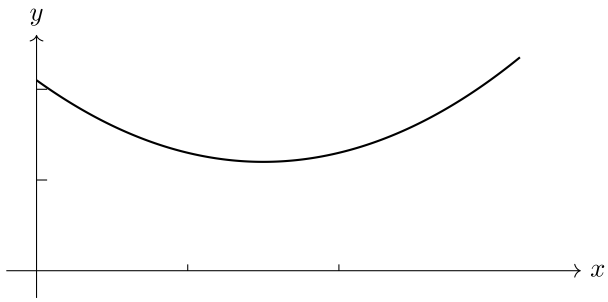
1. From the formula, what is $\lim_{x\to 2} g(x)$?
2. Compare your limit with $g(2)$. What kind of discontinuity occurs?

Final Answer: ______________________________

Q329 [Go to Solution p. 394] [Back to TOC]

Tangent slope from a picture (draw your own tangent). The curve $y = f(x)$ is drawn below. By drawing a tangent at $x = 1$, estimate its slope using two points on the tangent. Interpret your slope as a rate of change of y with respect to x.

Name: ______________________ **Date:** ______________



Final Answer: ____

Name: ____________________ **Date:** ____________

SL 5.2 Increasing and decreasing functions

Q330 [Go to Solution p. 395] [Back to TOC]

Analytic: polynomial. For $f(x) = x^4 - 4x^2 + 1$:

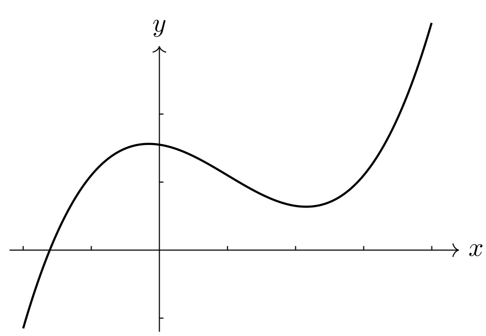
1. find $f'(x)$ and all real solutions of $f'(x) = 0$;
2. determine the intervals where f is increasing and where it is decreasing;
3. state the x–coordinates of any local maxima/minima (justify using the sign of f').

Final Answer: ____________________

Q331 [Go to Solution p. 395] [Back to TOC]

Read from a graph of f. The graph of a function $y = f(x)$ is shown. Use it to answer the questions (give approximate values if needed).

1. On which intervals is f increasing? decreasing?
2. Estimate the x–coordinates of any local maxima and minima.
3. Mark where $f'(x)$ appears to be 0, positive, or negative.

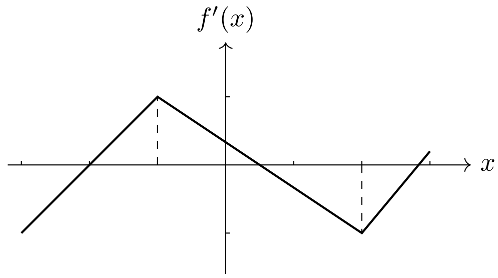


Final Answer: ____________________

Q332 [Go to Solution p. 395] [Back to TOC]

Given the graph of f'. The graph below is $y = f'(x)$ for a differentiable function f.

1. For which x is f increasing? decreasing?
2. At which x does f have a local maximum? a local minimum?
3. Sketch a possible shape of $y = f(x)$ on the same interval.



Final Answer: ______________________________

Q333 [Go to Solution p. 395] [Back to TOC]

Rational function (state the domain!). For $f(x) = \dfrac{x+1}{x-2}$:

1. find $f'(x)$ and solve $f'(x) = 0$;
2. determine where f is increasing/decreasing (give intervals in the correct domain);
3. indicate the role of the vertical asymptote in your answer.

Final Answer: ______________________________

Q334 [Go to Solution p. 395] [Back to TOC]

Trigonometric on a closed interval (technology allowed). For $f(x) = \sin x + \frac{1}{2}\cos(2x)$ on $[0, 2\pi]$:

1. find $f'(x)$ and solve $f'(x) = 0$ on the interval;
2. list the subintervals where f is increasing and where it is decreasing;
3. identify any local maxima/minima (as x–values) within $[0, 2\pi]$.

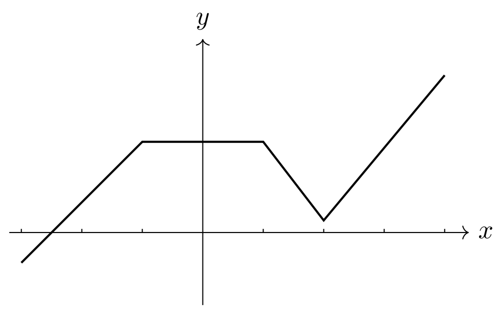
Name: ______________________ **Date:** ______________

Final Answer: ______________________

Q335 [Go to Solution p. 396] [Back to TOC]

Piecewise linear graph of f. A function f is defined for $-3 \le x \le 4$ and its graph is shown.

1. State the intervals where f is increasing, decreasing, and constant.
2. At which x is $f'(x)$ undefined? Explain using the graph.



Final Answer: ______________________

Q336 [Go to Solution p. 396] [Back to TOC]

Sign chart from a factored derivative. Suppose $f'(x) = (x-1)^2(x+2)(3-x)$.

1. Without expanding, determine the sign of $f'(x)$ on each interval determined by the critical points.
2. State the intervals where f is increasing/decreasing.
3. Classify the stationary points of f at $x = -2, 1, 3$ (max/min/flat/none).

Final Answer: ______________________

Name: ______________________ **Date:** ______________

Q337 [Go to Solution p. 396] [Back to TOC]

Table of derivative values. The table gives approximate values of $f'(x)$.

x	-3	-2	-1	0	1	2	3	4
$f'(x)$	-1.2	-0.5	0	0.8	0.4	0	-0.6	-0.2

1. On which subintervals of $[-3, 4]$ is f increasing? decreasing?
2. Estimate the x–values of any local maxima or minima of f suggested by the data.

Final Answer: ____

Q338 [Go to Solution p. 396] [Back to TOC]

From monotonicity of f to f'. A differentiable function f satisfies: f is increasing on $(-\infty, -1)$, decreasing on $(-1, 2)$, and increasing on $(2, \infty)$. Sketch a possible graph of $f'(x)$ consistent with this information, indicating the likely zeros and the sign of f' on each interval.

Final Answer: ____

Name: ____________________ **Date:** ______________

SL5,3 Basic differentiation

Q339 [Go to Solution p. 398] [Back to TOC]

Differentiate basic powers. Use the power rule to find $f'(x)$ for each:

1. $f(x) = 7x^6$
2. $f(x) = -4x^{-3}$
3. $f(x) = \dfrac{5}{x} = 5x^{-1}$
4. $f(x) = 12$

Final Answer: ____________________

Q340 [Go to Solution p. 398] [Back to TOC]

Polynomials with integer exponents. Differentiate and simplify:

$$g(x) = 3x^7 - 5x^4 + 2x^3 - 9x + 6 - \frac{8}{x^2}.$$

Write your final answer with integer powers of x.

Final Answer: ____________________

Q341 [Go to Solution p. 398] [Back to TOC]

Slope at a point. For $h(x) = 2x^5 - x^2 + 3x - 4$,

1. find $h'(x)$;
2. find the slope of the graph at $x = -1$;
3. hence write the tangent line at $x = -1$.

Final Answer: ____________________

Q342 [Go to Solution p. 398] [Back to TOC]

Tangent and normal. Let $y = x^4 - 2x^2 + 1$.

1. Find the equation of the tangent at $x = 2$.
2. Find the equation of the normal at $x = 2$.

Final Answer: ________________

Q343 [Go to Solution p. 398] [Back to TOC]

Stationary points of a cubic. For $f(x) = x^3 - 6x^2 + 9x + 1$:

1. compute $f'(x)$ and solve $f'(x) = 0$;
2. state whether each stationary point is a local maximum or minimum (use the sign of f' or f'');
3. give the coordinates of the stationary points.

Final Answer: ________________

Q344 [Go to Solution p. 399] [Back to TOC]

Increasing/decreasing via a factored derivative. A function has derivative $f'(x) = x(x-3)^2(x+1)$.

1. Without expanding, determine the sign of $f'(x)$ on the intervals determined by the roots.
2. State where f is increasing and where it is decreasing.
3. Classify the stationary point at $x = 3$.

Final Answer: ________________

Name: ______________________ Date: ______________

Q345 [Go to Solution p. 399] [Back to TOC]

Find unknown coefficients from derivative data. Let $p(x) = ax^3 + bx^2 + cx + 4$. Given $p'(1) = 0$ and $p'(2) = 6$, and the tangent at $x = 0$ is horizontal,

1. find a, b, c;
2. write $p(x)$ explicitly.

Final Answer: ______________________

Q346 [Go to Solution p. 399] [Back to TOC]

Parallel/perpendicular tangents. For $y = 2x^3 - x$, find all points on the curve where the tangent is

1. parallel to the line $y = 5x - 1$;
2. perpendicular to the line $y = \frac{1}{2}x + 3$.

Give the equations of the required tangents.

Final Answer: ______________________

Q347 [Go to Solution p. 399] [Back to TOC]

Applied rate of change (units). The displacement of a car is modelled by $s(t) = 4t^3 - 3t^2 + 2$ (metres, t in seconds).

1. Find the velocity $v(t)$ and acceleration $a(t)$.
2. Evaluate $v(2)$ and $a(2)$ with correct units.

Final Answer: ______________________

Name: ______________________ **Date:** ______________

Q348 [Go to Solution p. 399] [Back to TOC]

Rational with integer exponents. Rewrite $r(x) = \dfrac{3x^2 - 1}{x^3}$ using integer powers of x only, then differentiate.

Final Answer: ______________________

Q349 [Go to Solution p. 399] [Back to TOC]

Optimisation with a polynomial (power rule only). A box without a lid is made from a square sheet of side 20 cm by cutting out equal squares of side x cm from the corners and folding up the sides. The volume is

$$V(x) = x(20 - 2x)^2, \qquad 0 < x < 10.$$

1. Differentiate V and find critical values.
2. Determine the value of x that maximises V (justify with the sign of V' or V'').

Final Answer: ______________________

Q350 [Go to Solution p. 400] [Back to TOC]

Graph–based derivative estimate (power rule check). Consider $f(x) = x^4 - 4x^2 + 1$. Use algebra to compute $f'(x)$ and evaluate $f'(1)$. Then, using nearby values $x = 0.9$ and $x = 1.1$, estimate the slope numerically via a secant and compare with your exact derivative value.

Final Answer: ______________________

Name: ______________________ Date: ______________

SL 5.4 Tangents and normals

Tangents and Normals (at a given point).

- The *tangent* to a curve at P touches the curve at P and shares its instantaneous gradient there.
- The *normal* at P is the line through P perpendicular to the tangent.
- For $y = f(x)$ and $P = (a, f(a))$:

$$m_{\tan} = f'(a), \qquad m_{\text{norm}} = -\frac{1}{m_{\tan}} \quad (m_{\tan} \neq 0).$$

- Point–slope forms:

$$\text{Tangent: } y - f(a) = f'(a)\,(x-a), \qquad \text{Normal: } y - f(a) = -\frac{1}{f'(a)}(x-a).$$

- Use both *analytic methods* (differentiate, substitute) and *technology* (graphing/CAS) to verify gradients and intersections.

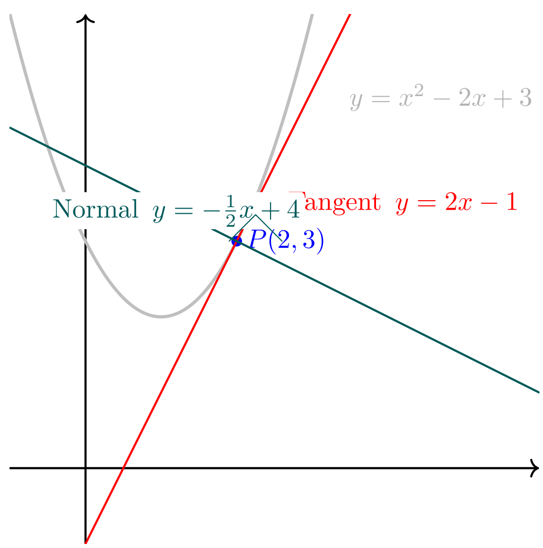
Worked example — Tangent and normal at a point. For $f(x) = x^2 - 2x + 3$ find the equations of the tangent and normal at $x = 2$.

Step 1 (point). $P = (2, f(2)) = (2, 3)$.

Step 2 (gradient). $f'(x) = 2x - 2 \Rightarrow f'(2) = 2$. Hence $m_{\tan} = 2$ and $m_{\text{norm}} = -\frac{1}{2}$.

Step 3 (equations).

$$\boxed{y - 3 = 2(x-2) \;\Rightarrow\; y = 2x - 1}, \qquad \boxed{y - 3 = -\tfrac{1}{2}(x-2) \;\Rightarrow\; y = -\tfrac{1}{2}x + 4}.$$



Check with technology. On a graphing tool, plot $y = x^2 - 2x + 3$, the tangent $y = 2x - 1$, and the normal $y = -\frac{1}{2}x + 4$. Confirm all meet at $P(2, 3)$ and that the tangent's slope is 2 (the derivative value at $x = 2$).

Name: ______________________ **Date:** ______________

Tangent Line on TI-Nspire (Non-CAS): Numeric + Graphical Methods

Find the equation of the tangent to $y = x^3 - 2x + 1$ at $x = 2$.

Method A — Calculator app (numeric derivative)

1. Open Doc → Add Calculator.
2. Define the function and evaluate the point:
   - Type `f(x) := x^3 - 2x + 1` → Enter
   - Type `x0 := 2` → Enter
   - Type `y0 := f(x0)` → Enter
3. Compute the numeric slope using `nDeriv`:

   $$m := \texttt{nDeriv}(f(x), x, x0)$$

   Type exactly: `m := nDeriv(f(x), x, x0)` → Enter
4. Write the tangent in point–slope form:

   $$y - y_0 = m(x - x_0) \quad \text{so} \quad y = m(x - x0) + y0.$$

   You can store the explicit form as:
   - `t(x) := m*(x - x0) + y0` → Enter
5. (Optional) Check numerically at a few x-values or graph it (see Method B).

Method B — Graphs app (graphical tangent tool)

1. Ctrl + Doc → Add Graphs.
2. In the function entry line, type `f(x) = x^3 - 2x + 1` → Enter.
3. Use the tangent tool:
   - Menu → Analyze Graph → Tangent at Point.
   - Move the cursor near $x = 2$ (or type 2 and Enter) to place the tangent.
4. The calculator displays the tangent line on the graph (often in $y = mx + b$ form). If needed, convert to point–slope form $y - f(2) = m(x - 2)$.
5. (Optional) Also graph the line from Method A by typing `t(x)` in a new entry line to verify.

Worked numbers (for this example)

$$f(2) = 2^3 - 2 \cdot 2 + 1 = 5,$$
$$m = \texttt{nDeriv}(x^3 - 2x + 1,\ x,\ 2) \approx 10 \quad (\text{since } f'(x) = 3x^2 - 2,\ f'(2) = 10),$$
$$\text{Tangent:} \quad y = 10(x - 2) + 5 = 10x - 15.$$

Tips

- `nDeriv(expr, var, value)` computes a numeric derivative at a point on non-CAS models.
- Defining `f(x)` first makes both `nDeriv` and graphing easier and less error-prone.
- In Graphs, you can also type the line directly: `y = 10x - 15` to compare with $f(x)$.

Q351 [Go to Solution p. 401] [Back to TOC]

Tangent and normal at a given x-value. For $f(x) = x^3 - 2x^2 + 5x - 7$,

1. find $f'(x)$;
2. find the equation of the tangent to $y = f(x)$ at $x = 2$;
3. find the equation of the normal there.

Final Answer: ______________________________

Q352 [Go to Solution p. 401] [Back to TOC]

Tangent through a given point on the curve. The point $P(1, \ln 3)$ lies on $y = \ln(3x)$. Find the equation of the tangent line at P and write it in the form $y = mx + c$.

Final Answer: ______________________________

Q353 [Go to Solution p. 401] [Back to TOC]

Normal line. For $y = e^{2x}$, find the equation of the normal at $x = \ln 2$. Give your answer in the form $ax + by + c = 0$ with integer coefficients.

Final Answer: ______________________________

Name: ____________________ **Date:** ____________

Q354 [Go to Solution p. 401] [Back to TOC]

Tangent parallel to a given line. Let $y = x^3 - 3x$. Find all points on the curve where the tangent is parallel to the line $y = 6x - 4$.

Final Answer: ____________________

Q355 [Go to Solution p. 401] [Back to TOC]

Tangent perpendicular to a given line. For $y = \sqrt{x}$ (domain $x > 0$), find the point(s) where the tangent is perpendicular to the line $3x + y = 0$. Then write the normal at that point.

Final Answer: ____________________

Q356 [Go to Solution p. 402] [Back to TOC]

Horizontal and vertical tangents. Consider $y = x^{2/3}(x - 3)$ for $x \in \mathbb{R}$.

1. Find all x where the curve has a horizontal tangent.
2. Determine whether the curve has a vertical tangent or a cusp at $x = 0$, and justify briefly.

Final Answer: ____________________

Q357 [Go to Solution p. 402] [Back to TOC]

Normal passing through a fixed point. For the curve $y = x^2 + 1$, find the point(s) on the curve whose *normal* passes through the point $(0, 2)$.

Name: ____________________ **Date:** ______________

Final Answer: ____

Q358 [Go to Solution p. 402] [Back to TOC]

Tangent to a circle (analytic). The circle $x^2 + y^2 = 25$ and the line $\ell : y = mx + 1$ are tangent. Find the possible values of m and the point(s) of tangency.

Final Answer: ____

Q359 [Go to Solution p. 402] [Back to TOC]

Exponential model; technology may help. Let $f(x) = 5e^{-0.4x} + 1$.

1. Find the tangent at $x = 2$.
2. Use technology to *numerically* find $x > 0$ such that the normal at $(x, f(x))$ passes through the origin. Give x to two decimal places and the corresponding line equation.

Final Answer: ____

Q360 [Go to Solution p. 403] [Back to TOC]

Where is the tangent of given slope? For $y = \sin x + \frac{x}{2}$ (radians),

1. show that $y' = \cos x + \frac{1}{2}$;
2. find all $x \in [0, 2\pi]$ at which the tangent has slope 1;
3. write the tangent line equation for one such x.

Final Answer: ____

Name: ______________________ **Date:** ______________

Q361 [Go to Solution p. 403] [Back to TOC]

Normal of minimal distance to a point. For $y = x^2 - 4x + 7$, find the point on the curve where the *normal* line is closest to the point $(0, 0)$ (i.e., the normal passes through $(0, 0)$). Then find the corresponding tangent equation.

Final Answer: ______________________________

Q362 [Go to Solution p. 403] [Back to TOC]

Graph-and-verify (technology). A function f is given by $f(x) = \ln(x + 2) - \frac{x}{3}$ for $x > -2$.

1. Compute the tangent at $x = 1$.
2. Use graphing technology to draw the curve and this tangent on the same axes and verify visually that your line touches the curve only at the computed point.

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

SL 5.5 Integration

Key ideas

Core concepts.

- **Anti-differentiation** (indefinite integration) is the reverse of differentiation.
- For $f(x) = ax^n$ where $n \neq -1$:

$$\int ax^n \, dx = \frac{a}{n+1} x^{n+1} + C.$$

- The **constant of integration** C can be found using a given point (boundary condition).
- **Definite integral** $\int_a^b f(x)\, dx$ represents the net area between the curve $y = f(x)$ and the x-axis from $x = a$ to $x = b$.
- If $f(x) > 0$ on $[a, b]$, the definite integral gives the area directly.
- The link between **anti-derivatives** and **area** is given by the **Fundamental Theorem of Calculus**:

$$\int_a^b f(x)\, dx = F(b) - F(a),$$

where $F'(x) = f(x)$.

Name: ______________________ **Date:** ______________

Key terms

Anti-derivative A function whose derivative is the given function.

Constant of integration (C) Arbitrary constant added when finding an indefinite integral.

Definite integral The numerical value of $\int_a^b f(x)\,dx$ over a closed interval.

Boundary condition A point (x_0, y_0) used to determine C.

Area under a curve The definite integral of $f(x)$ over the desired interval when $f(x) \geq 0$.

Worked example 1 — Find the original function. Given:

$$\frac{dy}{dx} = 3x^2 + x, \quad y = 10 \text{ when } x = 1.$$

Solution:

$$y = \int (3x^2 + x)\,dx = x^3 + \frac{1}{2}x^2 + C.$$

Using $y(1) = 10$:

$$10 = 1 + \frac{1}{2} + C \quad \Rightarrow \quad C = 8.5.$$

$\boxed{y = x^3 + \frac{1}{2}x^2 + 8.5}$.

Worked example 2 — Area under a curve. Find:

$$\int_2^6 (3x^2 + 4)\,dx.$$

By hand:

$$\int (3x^2 + 4)\,dx = x^3 + 4x + C.$$

Evaluate:

$$[x^3 + 4x]_2^6 = (216 + 24) - (8 + 8) = 240 - 16 = \boxed{224}.$$

Using TI–Nspire:

1. Open a **Calculator** page.
   (a) Open a **Calculator** page.
   (b) Type (ASCII entry): `integral(3*x^2 + 4, x, 2, 6)` then press $\boxed{\text{Enter}}$.
   (c) Or use the template: insert $\int$ from the **Template** key and fill $\int_2^6 (3x^2 + 4)\,dx$.
   btnTemplate key).
2. Press $\boxed{\text{Enter}}$ to get 224.
3. Alternatively: Graph $y = 3x^2 + 4$, then $\boxed{\text{Menu}} \to \boxed{\text{Analyze Graph}} \to \boxed{\text{Integral}}$, select lower bound $x = 2$, upper bound $x = 6$.

Worked example 3 — Region enclosed by a curve and x-axis. Find the area bounded by $y = -x^2 + 4x - 3$ and the x-axis.

Step 1: Find limits of integration. Solve $-x^2 + 4x - 3 = 0 \Rightarrow x = 1$ and $x = 3$.

Step 2: Integrate.

$$\int_1^3 (-x^2+4x-3)\,dx = \left[-\frac{x^3}{3}+2x^2-3x\right]_1^3$$

At $x=3$: $-9+18-9=0$. At $x=1$: $-\frac{1}{3}+2-3=-\frac{4}{3}$. Area $=0-(-\frac{4}{3})=\boxed{\frac{4}{3}}$ units^2.

Using TI–Nspire (graph method):

1. Graph $y=-x^2+4x-3$.
2. $\boxed{\text{Menu}} \to \boxed{\text{Analyze Graph}} \to \boxed{\text{Zero}}$ to find x-intercepts ($x=1, x=3$).
3. $\boxed{\text{Menu}} \to \boxed{\text{Analyze Graph}} \to \boxed{\text{Integral}}$, lower bound $x=1$, upper bound $x=3$.
4. TI–Nspire displays the shaded area and the value $\frac{4}{3}$.

TI–Nspire tips

- The integral template $\int f(x)\,dx$ can be accessed via the $\boxed{\text{Template}}$ key ($\boxed{x}$ with fraction and root symbols).
- Always specify the integration variable.
- In Graphs, use *Analyze Graph* → *Integral* to compute and shade the area.
- Check whether the curve dips below the x-axis — if so, split the integral or take absolute values for total area.

Q363 [Go to Solution p. 405] [Back to TOC]

Indefinite integrals (power rule). Find $\displaystyle\int f(x)\,dx$ and simplify. Include $+C$.

1. $f(x)=7x^5-3x^2+4$
2. $f(x)=2x^{-3}-5x^{-1}+9x$
3. $f(x)=-6x^7+x-8$

Final Answer: ____________________

Q364 [Go to Solution p. 405] [Back to TOC]

Determine the constant from a boundary condition. Given $\dfrac{dy}{dx}=3x^2+x$ and $y=10$ when $x=1$, find the particular solution $y(x)$.

Final Answer: ____________________

Name: ______________________ **Date:** ______________

Q365 [Go to Solution p. 405] [Back to TOC]

Initial value problem (velocity → displacement). A particle moves on a line with velocity $v(t) = 4t - 3$ m s^{-1}. When $t = 0$ s its position is $s(0) = 2$ m.

1. Find the displacement $s(t)$.
2. How far is the particle from the origin at $t = 5$ s?

Final Answer: ______________________________

Q366 [Go to Solution p. 405] [Back to TOC]

Evaluate a definite integral. Compute exactly:

$$\int_2^6 (3x^2 + 4)\, dx.$$

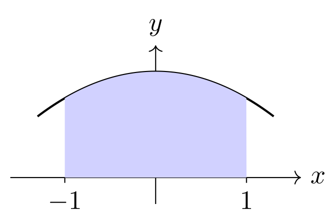
(You may check with technology.)

Final Answer: ______________________________

Q367 [Go to Solution p. 405] [Back to TOC]

Area under a curve above the x–axis. Let $f(x) = 4 - x^2$. On the interval $[-1, 1]$ the curve lies above the x–axis.

1. Write a definite integral for the shaded area and evaluate it.
2. Give the area to two decimal places.



Final Answer: ____

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Q368 [Go to Solution p. 405] [Back to TOC]

Area where the function changes sign. For $g(x) = x^2 - 4x$, the curve meets the x–axis at $x = 0$ and $x = 4$. On $[0, 4]$, part of the curve is below the axis.

1. Sketch and shade the region between the curve and the x–axis on $[0, 4]$.
2. Compute the *total* area between the curve and the x–axis by splitting at the zeros.

Final Answer: ____

____

Q369 [Go to Solution p. 405] [Back to TOC]

Recover a function from its derivative (boundary value). A function $y = f(x)$ satisfies $f'(x) = 5x^4 - 2x$ and $f(2) = 7$.

1. Find $f(x)$.
2. Determine $f(0)$.

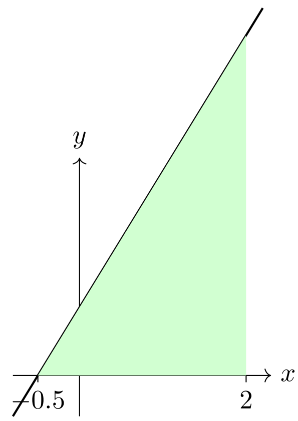
Final Answer: ____

____

Q370 [Go to Solution p. 406] [Back to TOC]

Area interpretation (set up the integral first). The graph of $y = 2x + 1$ and the x–axis enclose a region for $x \in [-0.5, 2]$ where the line is above the axis.

1. Write a definite integral representing this area.
2. Evaluate the area.



Final Answer: ____________________________________

Q371 [Go to Solution p. 406] [Back to TOC]

Average value (optional extension, can be checked with technology). The average value of f on $[a, b]$ is $\dfrac{1}{b-a}\int_a^b f(x)\,dx$. Find the average value of $f(x) = 3x^2 - x$ on $[1, 4]$.

Final Answer: ____________________________________

Q372 [Go to Solution p. 406] [Back to TOC]

From acceleration to position (two integrations). A moving object has acceleration $a(t) = 6t$ m s^{-2}. At $t = 0$ s the velocity is $v(0) = 2$ m s^{-1} and the position is $s(0) = 5$ m.

1. Find $v(t)$.
2. Find $s(t)$.
3. How far has the object travelled between $t = 0$ and $t = 3$ s? (Use a definite integral of $v(t)$ if needed.)

Final Answer: ____________________________________

Name: ____________________ **Date:** ________________

Name: ________________________ **Date:** ________________

SL 5.6 Local minimums and maximums

Q373 [Go to Solution p. 407] [Back to TOC]

Solve $f'(x) = 0$ and classify. For $f(x) = x^3 - 6x^2 + 9x + 2$:

1. Find the x–values where the gradient is zero.
2. Determine whether each is a local maximum, a local minimum, or neither (use f'' or a sign chart of f').
3. Give the coordinates of the stationary points.

Final Answer: ____

Q374 [Go to Solution p. 407] [Back to TOC]

Closed interval: local vs global. Let $g(x) = x^4 - 4x^2$ on the domain $-3 \le x \le 3$.

1. Solve $g'(x) = 0$ and classify the stationary points.
2. Find the greatest and least *values* of $g(x)$ on $[-3, 3]$ (justify by checking endpoints).

Final Answer: ____

Q375 [Go to Solution p. 407] [Back to TOC]

Classifying from a factored derivative. A differentiable function has derivative

$$h'(x) = (x - 2)^2(x + 1).$$

1. State all stationary x–values.
2. Without finding $h(x)$, decide the nature of the stationary point(s) at each value (local max/min or stationary point of inflection). Explain using the sign of h'.

Final Answer: ____

Q376 [Go to Solution p. 407] [Back to TOC]

Technology: locate a turning point. On $0 \leq x \leq 10$, let $p(x) = x\,e^{-0.3x}$.

1. Use technology to solve $p'(x) = 0$. Give the x–value correct to three decimal places.
2. Verify that this x gives a local maximum and find the corresponding $p(x)$.

Final Answer: ______________________________

Q377 [Go to Solution p. 407] [Back to TOC]

Applied maximum (horizontal tangent). The revenue (in \$) from selling an item at price p dollars is

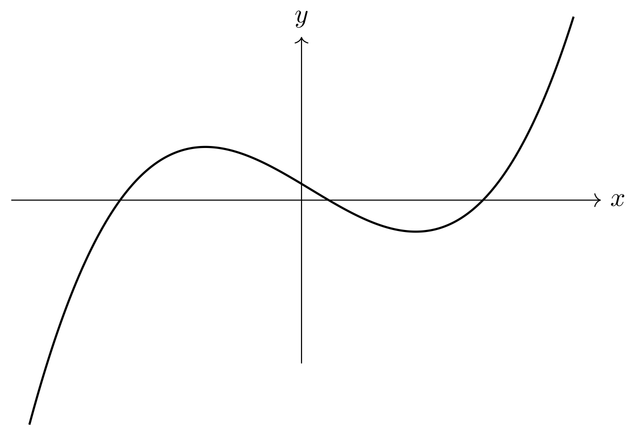
$$R(p) = -200p^2 + 5200p, \qquad p > 0.$$

1. Find the price that maximizes revenue (solve $R'(p) = 0$).
2. State the maximum revenue.

Final Answer: ______________________________

Q378 [Go to Solution p. 408] [Back to TOC]

Sketch-based estimation. The curve below is $y = f(x)$.



1. Estimate the x–values where $f'(x) = 0$.
2. Classify each stationary point as a local maximum or local minimum from the shape.

Final Answer: ______________________________

Q379 [Go to Solution p. 408] [Back to TOC]

Rational function. For $q(x) = \dfrac{x^3 - 3x}{x^2 + 1}$:

1. Compute $q'(x)$ and solve $q'(x) = 0$ (you may use technology for solving the resulting equation).
2. Classify the stationary points using a sign chart of q'.

Final Answer: ______________________________

Q380 [Go to Solution p. 408] [Back to TOC]

Count of stationary points from f'. Suppose $r'(x) = x^3 - 4x$ and r is defined for all real x.

1. Find all real roots of $r'(x)$.
2. Determine the intervals where r is increasing and decreasing.
3. Classify the stationary points of r.

Name: ____________________ **Date:** ______________

Final Answer: ____

Q381 [Go to Solution p. 408] [Back to TOC]

Horizontal tangents for a sinusoid with trend. Let $s(x) = \sin x + 0.2x$ for real x.

1. Solve $s'(x) = 0$ on $[-3\pi, 3\pi]$.
2. Identify which of these correspond to local minima of s (use $s''(x)$ or the sign of s').

Final Answer: ____

Q382 [Go to Solution p. 409] [Back to TOC]

Local does not mean global. A function t has stationary points at $x = -2$ (local max), $x = 1$ (local min), and no others. On the domain $[-5, 4]$ it also satisfies $t(-5) = 20$ and $t(4) = 18$.

1. Explain why the *global* maximum on $[-5, 4]$ may occur at an endpoint.
2. Which x–values are candidates for the global maximum and minimum on $[-5, 4]$? (No computation of t at interior points required.)

Final Answer: ____

Name: ______________________ Date: ______________

SL 5.7 Optimisation

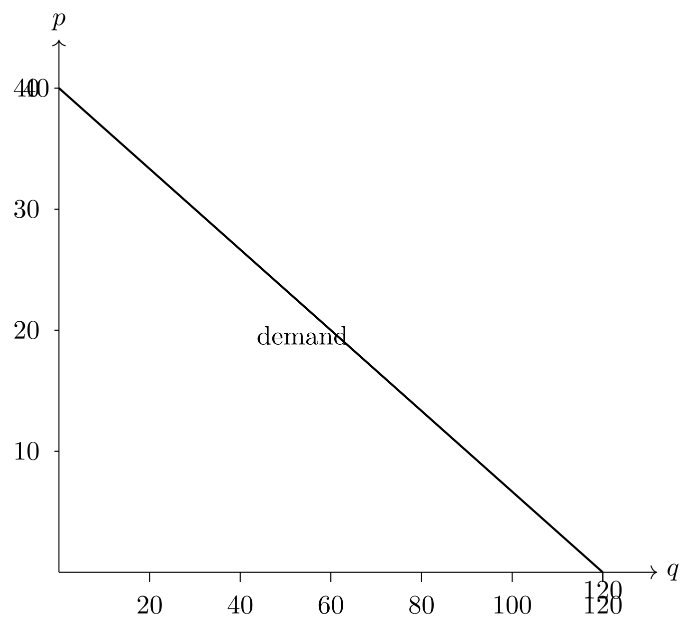
Q383 [Go to Solution p. 410] [Back to TOC]

Price to maximise profit (linear demand). A shop estimates that weekly demand for a product is

$$q = 120 - 3p,$$

where p is the selling price (in \$) and q the number sold. The weekly cost to produce q items is $C(q) = 420 + 8q$ dollars.

1. Write the revenue $R(q)$, the profit $P(q) = R(q) - C(q)$ and then P as a function of p only.
2. Find the price p that maximises the profit and the corresponding maximum profit.
3. What price gives zero profit (breakeven)?



Final Answer: ____

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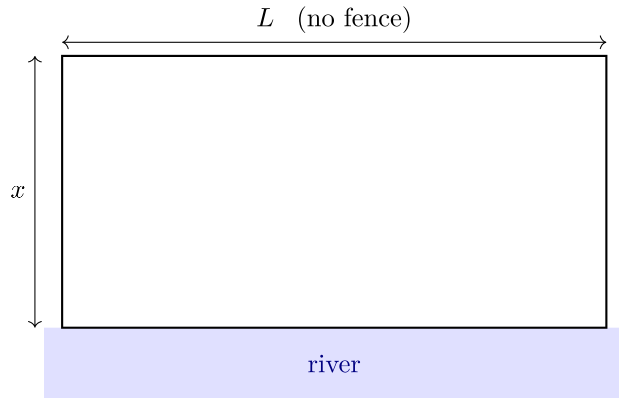
Q384 [Go to Solution p. 410] [Back to TOC]

Rectangular paddock beside a river. A farmer has L metres of fencing to make a rectangular paddock beside a straight river (no fence is needed along the river side).

1. Let x be the distance perpendicular to the river. Write the area A as a function of x and L.
2. Find the dimensions that maximise the area.
3. State the maximum area in terms of L.

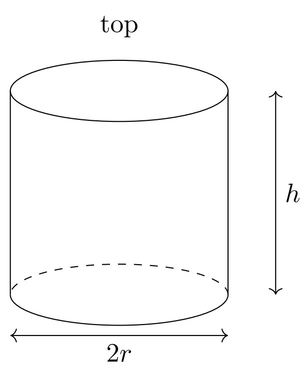


Final Answer: ____________________

Q385 [Go to Solution p. 410] [Back to TOC]

Cylindrical can: minimum surface for fixed volume. A can must hold 500 cm^3 of liquid. It has a circular top and base and a curved side (ignore seams).

1. Express the surface area S in terms of the radius r only.
2. Find the values of r and h that minimise S.
3. State the minimum surface area to the nearest cm^2.



Final Answer: ____________________

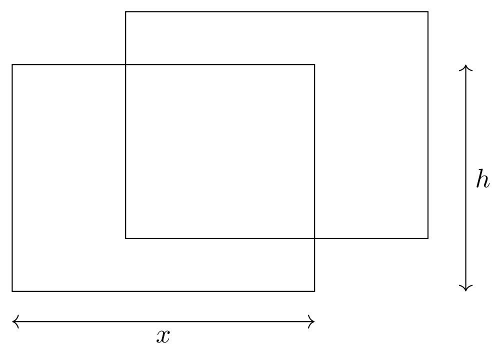
Q386 [Go to Solution p. 411] [Back to TOC]

Packaging with different material costs. A closed rectangular box with square base of side x cm and height h cm has volume 2000 cm^3. The base material costs \$0.06 per cm^2, the lid \$0.03 per cm^2, and the sides \$0.04 per cm^2.

1. Show that the total cost can be expressed as

$$C(x) = 0.09x^2 + \frac{320}{x} \quad \text{dollars.}$$

2. Find x and h that minimise the cost and state the minimum cost.



Final Answer: ____

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Q387 [Go to Solution p. 411] [Back to TOC]

Maximise the volume of a cylinder inside a sphere. A right circular cylinder is inscribed in a sphere of radius $R = 5$ cm (the cylinder's axis passes through the centre of the sphere).

1. Show that the cylinder volume can be written as

$$V(r) = \pi r^2 \left(2\sqrt{R^2 - r^2}\right) \qquad (0 < r < R).$$

2. Find the radius r and height h that maximise the cylinder's volume, and the maximum volume.

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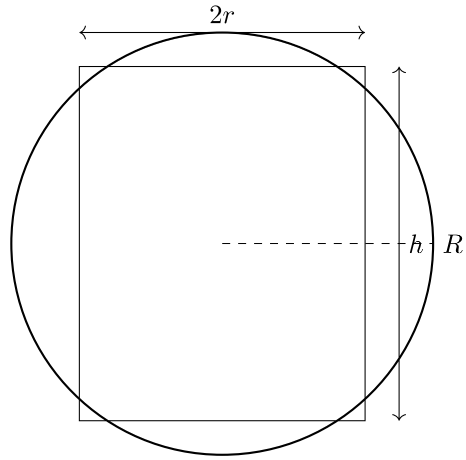


Final Answer: ____

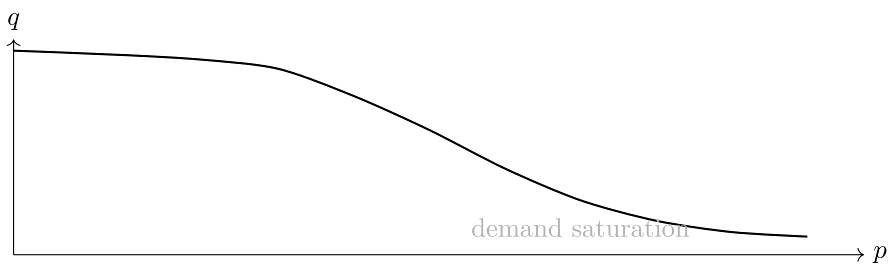
Q388 [Go to Solution p. 411] [Back to TOC]

Maximise profit with a saturation model. A company models weekly sales by

$$q(p) = \frac{900}{1 + e^{0.4(p-18)}} \quad \text{(units if the price is } p \text{ dollars).}$$

The weekly cost is $C(q) = 2000 + 6q$ dollars.

1. Express the profit P as a function of p only.
2. Use calculus (and technology to solve numerically) to find the price that maximises profit and the corresponding weekly profit (nearest dollar).
3. Give a brief reason why very low or very high prices reduce profit in this model.



Final Answer: ____

Name: ______________________ **Date:** ______________

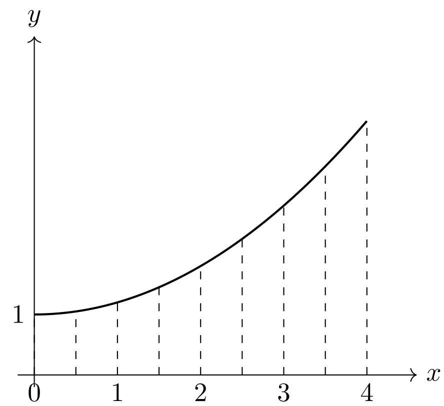
SL 5.8 Numerical methods - Trapezium rule

Q389 [Go to Solution p. 413] [Back to TOC]

From a function (equal subintervals). Let $f(x) = 0.2x^2 + 1$. Use the composite trapezoidal rule with $n = 8$ equal subintervals to estimate

$$\int_0^4 f(x)\,dx.$$

Show your step size h and the working you use to combine the ordinates.



Final Answer: ______________________________

Q390 [Go to Solution p. 413] [Back to TOC]

Velocity table to distance. The velocity v (m s^{-1}) of a vehicle was recorded every 5 s.

t (s)	0	5	10	15	20	25	30
v (m s^{-1})	0	12	21	27	30	29	26

1. Use the trapezoidal rule to estimate the distance travelled in the first 30 s.
2. Hence estimate the average velocity over this time.

Final Answer: ______________________________

Q391 [Go to Solution p. 413] [Back to TOC]

Name: ____________________ Date: ____________

Cross-sectional area from equally spaced measurements. At equally spaced positions $x = 0, 2, 4, 6, 8, 10, 12$ (m), the depth y (m) of a channel was measured.

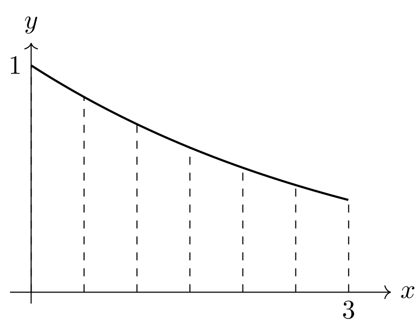
x (m)	0	2	4	6	8	10	12
y (m)	0	1.8	2.5	3.1	2.7	2.0	0

Use the trapezoidal rule to estimate the cross-sectional area of the channel (in m^2). Comment briefly on why the estimate is reasonable in this context.

Final Answer: ____________________

Q392 [Go to Solution p. 413] [Back to TOC]

Overestimate or underestimate? Consider $f(x) = e^{-0.3x}$ on $0 \le x \le 3$. Use $n = 6$ equal subintervals to obtain the trapezoidal estimate of $\int_0^3 f(x)\,dx$. Then decide, with a reason based on the concavity of f, whether the trapezoidal estimate is an overestimate or an underestimate of the true area.



Final Answer: ____________________

Q393 [Go to Solution p. 413] [Back to TOC]

Sine curve and comparison. Estimate $\int_0^{\pi} \sin x\,dx$ using the trapezoidal rule with step size $h = \frac{\pi}{6}$ (so $n = 6$ subintervals). State clearly the ordinates you use.

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Final Answer: __________________________________

Name: ______________________ **Date:** ______________

AHL 5.9 Differentiation of further functions

Q394 [Go to Solution p. 415] [Back to TOC]

Basic derivatives. Find $\dfrac{d}{dx}$ of each function.

1. $y = \sin x$
2. $y = \cos x$
3. $y = \tan x$
4. $y = e^x$
5. $y = \ln x$
6. $y = x^{5/3}$

Final Answer: ______________________

Q395 [Go to Solution p. 415] [Back to TOC]

Chain rule (composites). Differentiate the following.

1. $y = \sin(3x^2)$
2. $y = e^{2x-1}$
3. $y = \ln\big(\sqrt{x^2+1}\big)$
4. $y = (5-2x)^7$
5. $y = (x^2+x+1)^{3/2}$

Final Answer: ______________________

Q396 [Go to Solution p. 415] [Back to TOC]

Product rule. Compute y'.

1. $y = x^2 e^{3x}$
2. $y = (x+1)\ln x$

3. $y = x\sin(2x)$

Final Answer: ______

Q397 [Go to Solution p. 415] [Back to TOC]

Quotient rule. Differentiate the following.

1. $y = \dfrac{x^2+1}{x-1}$
2. $y = \dfrac{\tan x}{x}$
3. $y = \dfrac{e^x}{x^2}$

Final Answer: ______

Q398 [Go to Solution p. 415] [Back to TOC]

Mixed rules.

1. $y = e^x \cos x$.
2. $y = \ln(x^2+1)\,\sin(3x)$.
3. $y = (x^2+1)\,e^{-x^2}$.

Find y' in each case and then evaluate y' at $x = 0$.

Final Answer: ______

Q399 [Go to Solution p. 415] [Back to TOC]

Tangent and normal. For $y = x\,e^{-x^2}$:

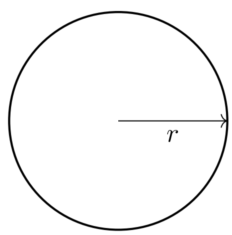
1. Find $\dfrac{dy}{dx}$.
2. Find the equation of the tangent line at $x = 1$.
3. Hence find the equation of the normal line at $x = 1$.

Final Answer: ____

Q400 [Go to Solution p. 416] [Back to TOC]

Related rates: expanding circle. A circular oil slick expands so that its radius r (m) increases at a constant rate $\dfrac{dr}{dt} = 0.30$ m min^{-1}.

1. Find the rate of change of the area $A = \pi r^2$ when $r = 20$ m.
2. Find the rate of change of the circumference $C = 2\pi r$ when $r = 20$ m.



expanding circle

Final Answer: ____

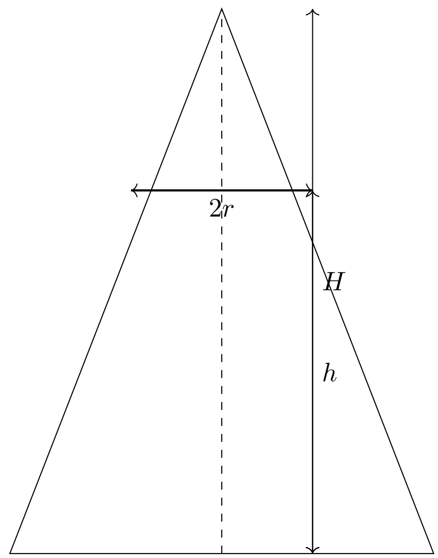
Q401 [Go to Solution p. 416] [Back to TOC]

Related rates: water in a cone. Water is poured into a right circular cone standing on its tip. The cone has height $H = 30$ cm and top radius $R = 10$ cm. Let h be the depth of water and r its surface radius at time t seconds. Because the triangles are similar, $r = \dfrac{R}{H}h = \dfrac{1}{3}h$.

1. Express the volume V of water as a function of h only.

Name: ______________________ **Date:** ______________

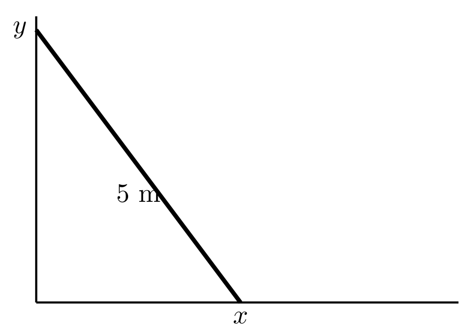
2. If the volume increases at $\dfrac{dV}{dt} = 15$ cm^3 s^{-1}, find $\dfrac{dh}{dt}$ when $h = 12$ cm.



Final Answer: ______________________________

Q402 [Go to Solution p. 416] [Back to TOC]

Related rates: sliding ladder. A 5 m ladder leans against a vertical wall. The bottom slides away from the wall at 0.8 m s^{-1}. When the bottom is 3 m from the wall, how fast is the top sliding down?



Final Answer: ______________________________

Name: ____________________ **Date:** ______________

Q403 [Go to Solution p. 416] [Back to TOC]

Log and trig composite.

1. Differentiate $y = \ln(\cos x)$.
2. Hence find the slope of the tangent to $y = \ln(\cos x)$ at $x = \dfrac{\pi}{4}$.

Final Answer: ____________________

Name: ________________ Date: __________

AHL5.10 Second derivative

Q404 [Go to Solution p. 418] [Back to TOC]

Compute first and second derivatives. Find $f'(x)$ and $f''(x)$ for each function.

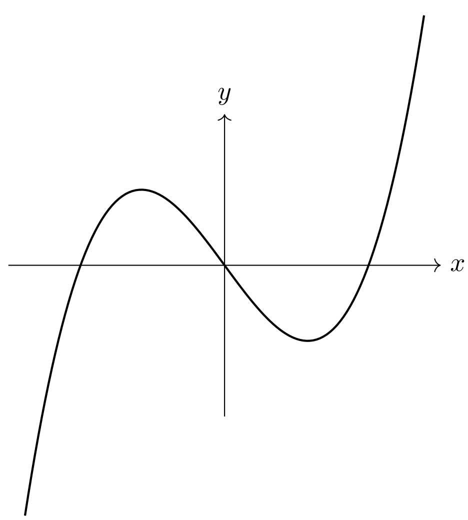
1. $f(x) = 3x^4 - 5x^2 + 7$
2. $f(x) = \dfrac{x^2+1}{x-2}$ (simplify the result)
3. $f(x) = e^{2x}\sin x$
4. $f(x) = \ln(x^2+1)$

Final Answer: ________________

Q405 [Go to Solution p. 418] [Back to TOC]

Second derivative test (polynomial). Let $f(x) = x^3 - 3x$.

1. Find all critical points.
2. Use the second derivative test to classify each critical point as a local maximum or minimum.
3. State the x–intervals where f is increasing and decreasing.



Final Answer: ________________

Name: ____________________ **Date:** ____________

Q406 [Go to Solution p. 418] [Back to TOC]

Point of inflection (sign change test). For $g(x) = x^3 - 6x^2 + 9x$:

1. Find $g''(x)$ and solve $g''(x) = 0$.
2. Show that the concavity changes at this x–value and hence identify the point of inflection (give coordinates).

Final Answer: ____________________

Q407 [Go to Solution p. 418] [Back to TOC]

Concavity intervals from f''. Consider $h(x) = \ln x$ on $(0, \infty)$.

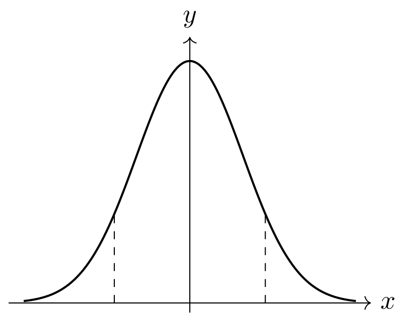
1. Compute $h''(x)$.
2. Decide for which x the graph is concave-up or concave-down.
3. Explain why there are no points of inflection.

Final Answer: ____________________

Q408 [Go to Solution p. 419] [Back to TOC]

Inflection in a bell-shaped curve. Let $y = e^{-x^2}$.

1. Show that $y''(x) = (4x^2 - 2)e^{-x^2}$.
2. Find all inflection points and the intervals on which the curve is concave-up or concave-down.



Final Answer: ____

Q409 [Go to Solution p. 419] [Back to TOC]

Second derivative test may be inconclusive. Let $p(x) = x^4$.

1. Find $p'(x)$ and $p''(x)$, and determine all critical points.
2. Apply the second derivative test at each critical point and comment on why it is inconclusive or conclusive.
3. Use another method (e.g. sign of p' or the graph) to classify the critical point(s).

Final Answer: ____

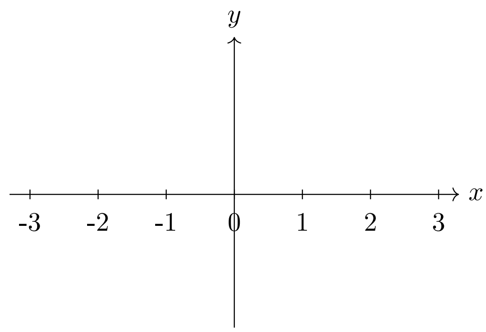
Q410 [Go to Solution p. 419] [Back to TOC]

Concavity and sketch from derivative information. A function f is twice differentiable on $[-3, 3]$ and satisfies

$$f'(x) > 0 \text{ on } (-3,-1), \quad f'(x) < 0 \text{ on } (-1,1), \quad f'(x) > 0 \text{ on } (1,3),$$
$$f''(x) < 0 \text{ on } (-3,0), \qquad f''(x) > 0 \text{ on } (0,3).$$

1. Mark on the x–axis the likely locations of a local max/min.
2. Decide where f is concave-up and concave-down and locate any point of inflection.
3. Produce a neat qualitative sketch consistent with this information (no exact scale required).

Name: ____________________ Date: ____________



Final Answer: ____________________

Q411 [Go to Solution p. 419] [Back to TOC]

Applied context (kinematics). A particle moves along a line with position $s(t) = t^3 - 6t^2 + 9t$ metres (t in seconds).

1. Find the velocity $v(t)$ and acceleration $a(t)$.
2. Determine the time intervals when the velocity is increasing and when it is decreasing. (Hint: relate this to the sign of $a(t)$.)
3. State the time and position of any point of inflection of $s(t)$ and interpret it physically.

Final Answer: ____________________

AHL 5.11 - Integration by substitution

Q412 [Go to Solution p. 420] [Back to TOC]

Indefinite integral: power rule. Find

$$\int \left(3x^{5/2} - 4x^{-3} + 7\right) dx$$

and simplify. State the condition on n for $\int x^n \, dx$ to be valid.

Final Answer: ____________________

Q413 [Go to Solution p. 420] [Back to TOC]

Basic trig and exponential. Evaluate the indefinite integrals:

$$\int \sin x \, dx, \qquad \int \cos(3x) \, dx, \qquad \int e^{2x-5} \, dx.$$

Final Answer: ____________________

Q414 [Go to Solution p. 420] [Back to TOC]

Secant squared. Find $\displaystyle\int \sec^2(4x - \tfrac{\pi}{6}) \, dx$.

Final Answer: ____________________

Q415 [Go to Solution p. 420] [Back to TOC]

Definite integral: powers. Compute

$$\int_1^4 \left(3x^{1/2} + \frac{2}{x^2}\right) dx.$$

Final Answer: ______________________

Q416 [Go to Solution p. 420] [Back to TOC]

Definite integral: sine and cosine. Evaluate

$$\int_0^{\pi/3} \cos x \, dx \qquad \text{and} \qquad \int_0^{\pi} \sin x \, dx.$$

Final Answer: ______________________

Q417 [Go to Solution p. 420] [Back to TOC]

Substitution (inspection). Find $\displaystyle\int \sin(2x+5)\, dx$.

Final Answer: ______________________

Q418 [Go to Solution p. 420] [Back to TOC]

Substitution (linear). Evaluate $\displaystyle\int \frac{1}{3x+2}\, dx$.

Final Answer: ____

Q419 [Go to Solution p. 420] [Back to TOC]

Substitution with chain rule reverse. Find $\int 4x\,\sin(x^2)\,dx$.

Final Answer: ____

Q420 [Go to Solution p. 421] [Back to TOC]

Quotient in derivative form. Evaluate $\int \frac{\cos(5x)}{1+\sin(5x)}\,dx$.

Final Answer: ____

Q421 [Go to Solution p. 421] [Back to TOC]

Definite integral via substitution. Compute $\int_0^1 2x\,e^{x^2}\,dx$ exactly.

Final Answer: ____

Q422 [Go to Solution p. 421] [Back to TOC]

Name: ____________________ **Date:** ______________

Mixed practice (indefinite). Find

$$\int \left(\frac{2x}{1+x^2} + e^x - 5\cos x \right) dx.$$

Final Answer: ____

Q423 [Go to Solution p. 421] [Back to TOC]

Initial value problem. A function F satisfies

$$F'(x) = \frac{2x}{1+x^2} + e^x, \qquad F(0) = 1.$$

Find the explicit formula for $F(x)$.

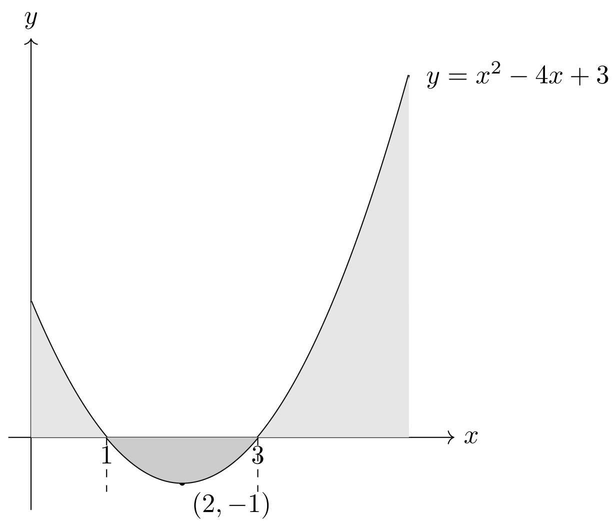
Final Answer: ____

AHL 5.12 - Area and volumes of revolution

Q424 [Go to Solution p. 422] [Back to TOC]

Area with sign and total area. For the curve $y = x^2 - 4x + 3$ on $0 \leq x \leq 5$:

1. Sketch the curve and find the *signed* area $\displaystyle\int_0^5 (x^2 - 4x + 3)\,dx$.
2. Hence find the *total geometric area* enclosed between the curve and the x-axis on this interval.

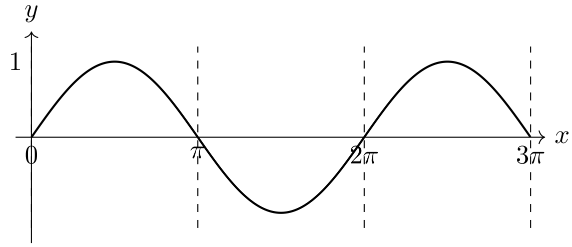


Final Answer: ____

Q425 [Go to Solution p. 422] [Back to TOC]

Sine: signed vs total area. Let $f(x) = \sin x$ on $0 \leq x \leq 3\pi$.

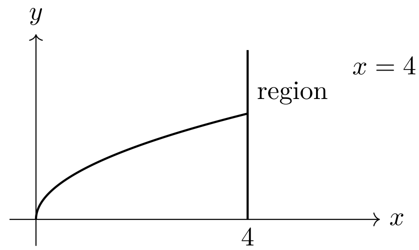
1. Compute the signed area $\displaystyle\int_0^{3\pi} \sin x\,dx$.
2. Compute the total geometric area between $y = \sin x$ and the x-axis on $0 \leq x \leq 3\pi$.



Final Answer: ____

Q426 [Go to Solution p. 422] [Back to TOC]

Area with respect to y. Find the area of the region bounded by the parabola $x = y^2$, the vertical line $x = 4$, and the x-axis. (Integrate with respect to y.)

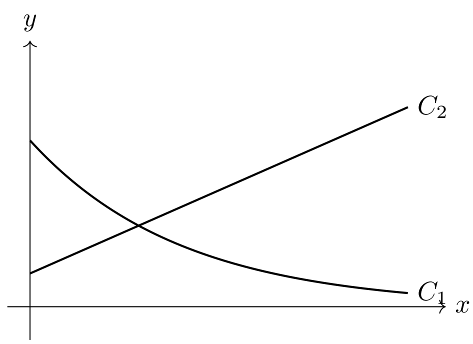


Final Answer: ____

Q427 [Go to Solution p. 422] [Back to TOC]

Area between two curves. Let $C_1: \ y = e^{-x/2}$ and $C_2: \ y = 0.2x + 0.2$.

1. Find the points of intersection of C_1 and C_2.
2. Compute the area of the region enclosed by the two curves.

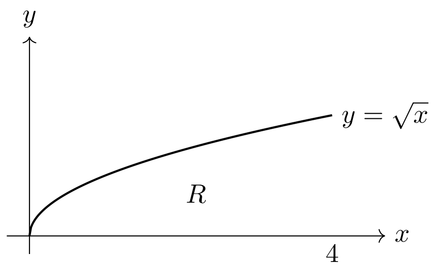


Final Answer: ____

Name: ______________________ **Date:** ______________

Q428 [Go to Solution p. 422] [Back to TOC]

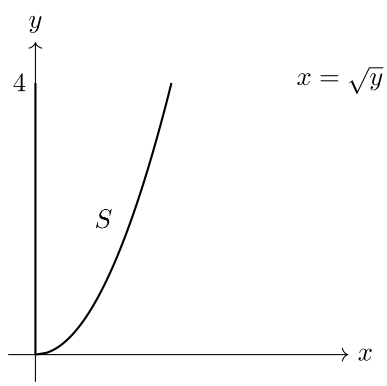
Volume of revolution about the x-axis (discs). Let R be the region under the curve $y = \sqrt{x}$ from $x = 0$ to $x = 4$, above the x-axis. Find the volume when R is revolved about the x-axis. (Use $V = \int_a^b \pi y^2 \, dx$.)



Final Answer: ____

Q429 [Go to Solution p. 423] [Back to TOC]

Volume of revolution about the y-axis (discs in y). Let S be the region bounded by the curve $x = \sqrt{y}$, the y-axis, and the lines $y = 0$ and $y = 4$. Find the volume when S is revolved about the y-axis. (Use $V = \int_a^b \pi x^2 \, dy$.)

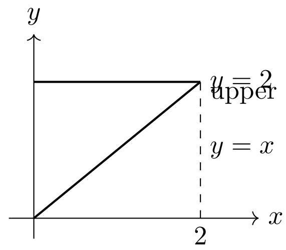


Final Answer: ____

Name: ______________________ **Date:** ______________

Q430 [Go to Solution p. 423] [Back to TOC]

Washers about the x-axis. Between $x = 0$ and $x = 2$, consider the region bounded above by $y = 2$ and below by $y = x$. Find the volume obtained by revolving this region about the x-axis. (Use washers: $V = \int \pi\left(R^2 - r^2\right) dx$.)

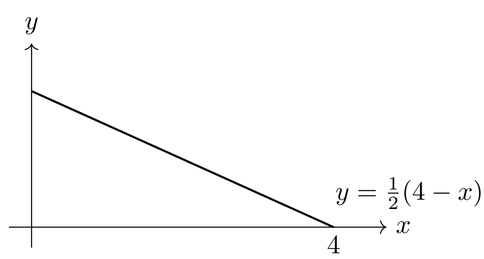


Final Answer: ______________________________

Q431 [Go to Solution p. 423] [Back to TOC]

Set up two integrals (do not evaluate). Let T be the region bounded by $y = \frac{1}{2}(4 - x)$, the x-axis, and $x = 0$.

1. Write an integral in x for the volume when T is revolved about the x-axis.
2. Express x as a function of y and write an integral in y for the volume when T is revolved about the y-axis.



Final Answer: ______________________________

Name: ______________________ **Date:** ______________

AHL 5.13 - Kinematics

Q432 [Go to Solution p. 424] [Back to TOC]

Displacement, velocity and acceleration from $s(t)$. A particle moves on a straight line with displacement (in metres)

$$s(t) = t^3 - 6t^2 + 9t - 2, \qquad t \geq 0.$$

1. Find the velocity $v(t)$ and acceleration $a(t)$.
2. Find the time(s) when the particle is at rest.
3. Find the displacement between $t = 0$ and $t = 5$.
4. Find the total distance travelled on $0 \leq t \leq 5$.

Final Answer: ____

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Q433 [Go to Solution p. 424] [Back to TOC]

Signed displacement vs total distance from $v(t)$. The velocity (m s^{-1}) of a car is $v(t) = 3t - 6$, for $0 \leq t \leq 5$ (time in s).

1. Compute the signed displacement $\int_0^5 v(t)\,dt$.
2. Compute the total distance travelled $\int_0^5 |v(t)|\,dt$.
3. State the time intervals when the car is moving in the positive direction.

Final Answer: ____

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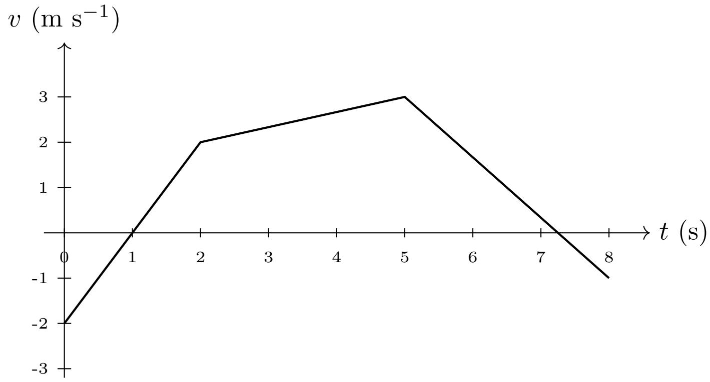
Q434 [Go to Solution p. 424] [Back to TOC]

Reading displacement and distance from a v–t graph. The velocity $v(t)$ (m s^{-1}) of a runner is shown below for $0 \leq t \leq 8$ s.

1. Determine the signed displacement over $0 \leq t \leq 8$.
2. Determine the total distance travelled over $0 \leq t \leq 8$.

3. Find all times when the runner changes direction.



Final Answer: ____

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Q435 [Go to Solution p. 424] [Back to TOC]

Recovering $s(t)$ from $a(t)$ with initial conditions. A particle has acceleration $a(t) = 6t - 4$ m s^{-2}. At $t = 0$ s, its velocity is $v(0) = 2$ m s^{-1} and its displacement is $s(0) = -3$ m.

1. Find the velocity $v(t)$ and displacement $s(t)$.
2. At what time does the particle come to instantaneous rest?
3. How far has it travelled by $t = 5$ s? (Give total distance.)

Final Answer: ____

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Q436 [Go to Solution p. 425] [Back to TOC]

Using $a = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = v\,\dfrac{dv}{ds}$. A car coasts along a straight road; its acceleration is proportional to its velocity and opposes the motion:

$$a = -k\,v, \qquad k > 0 \text{ constant.}$$

At the point $s = 0$ the speed is v_0.

1. Using $a = v\,\dfrac{dv}{ds}$, show that $v(s) = v_0 e^{-ks}$.

2. How far does the car travel while its speed drops from v_0 to $\frac{1}{2}v_0$?
3. How long does this take?

Final Answer: ____

____

Q437 [Go to Solution p. 425] [Back to TOC]

Braking with speed-squared drag. A sled experiences a resistive acceleration $a = -c\,v^2$ (with $c > 0$ constant). At $s = 0$ its speed is u.

1. Use $a = v\,\dfrac{dv}{ds}$ to obtain $v(s)$.
2. Show that the sled never actually stops in finite distance, but find the distance to reduce the speed to $u/3$.
3. Find the time needed to reduce the speed from u to $u/3$.

Final Answer: ____

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Q438 [Go to Solution p. 425] [Back to TOC]

Dot notation. Let $x(t)$ be the position of a particle moving along a line and suppose

$$\dot{x}(t) = 4e^{-t} - 2\sin t, \qquad x(0) = 1.$$

1. Find $\ddot{x}(t)$.
2. Find $x(t)$.
3. Determine the total distance travelled on $0 \le t \le 2\pi$.

Final Answer: ____

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Name: ____________________ **Date:** ______________

Q439 [Go to Solution p. 426] [Back to TOC]

Displacement from speed data (magnitude of velocity). A cyclist's speed (the magnitude of velocity) is

$$|v(t)| = \begin{cases} 2t, & 0 \le t < 3, \\ 6 - t, & 3 \le t \le 6, \end{cases} \qquad (\text{m s}^{-1}\text{; time in s}).$$

Assume the motion is always in the positive direction.

1. Sketch $|v(t)|$ and compute the distance travelled on $0 \le t \le 6$.
2. If instead the cyclist reverses direction instantaneously at $t = 4$ s (keeping the same *speed*), find the signed displacement and the total distance on $0 \le t \le 6$.

Final Answer: ________________________________

Name: ____________________ **Date:** ______________

AHL 5.14 - Modelling with differential equations and solving by separation of variables

Q440 [Go to Solution p. 427] [Back to TOC]

Proportional growth to the square root. An algal bloom has mass $G(t)$ (g) at time t (days). The instantaneous growth rate is proportional to $\sqrt{G}$.

1. Form a differential equation for G and state the constant(s).
2. Solve for the *general solution*.
3. If $G(0) = 9$ g and $G(4) = 25$ g, find $G(t)$.

Final Answer: ______________________________

Q441 [Go to Solution p. 427] [Back to TOC]

Exponential model. A culture of bacteria has population $P(t)$. The rate of change of P is proportional to P.

1. Write the differential equation and solve it to obtain the general solution.
2. If $P(0) = 1200$ and the doubling time is 8 hours, find $P(t)$ and $P(20)$.

Final Answer: ______________________________

Q442 [Go to Solution p. 427] [Back to TOC]

Radioactive decay (half-life). A substance has mass $m(t)$ (mg) and decays at a rate proportional to its current mass. The half-life is 3 years.

1. Set up and solve the differential equation for $m(t)$.
2. If $m(0) = 40$ mg, find the time for the mass to fall to 5 mg.

Final Answer: ______________________________

Name: ______________________ **Date:** ______________

Q443 [Go to Solution p. 427] [Back to TOC]

Newton's law of cooling. A hot drink at temperature $T(t)$ (°C) cools in a room at constant ambient temperature $T_a = 22$°C with rate proportional to the temperature difference from the room.

1. Form the differential equation and solve for the general solution.
2. If $T(0) = 82$°C and $T(10) = 52$°C (minutes), find $T(t)$ and the time it reaches 30°C.

Final Answer: ______________________________

Q444 [Go to Solution p. 428] [Back to TOC]

Logistic growth with carrying capacity. A fish population $N(t)$ (hundreds) follows a logistic model with carrying capacity K and intrinsic growth rate r:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right).$$

1. Solve the differential equation to obtain the general solution.
2. Given $K = 500$, $r = 0.6$ (yr^{-1}) and $N(0) = 50$, find $N(t)$ and the time for the population to reach 250.

Final Answer: ______________________________

Q445 [Go to Solution p. 428] [Back to TOC]

Mixing problem (separable). A 100 L tank initially contains pure water. Brine with concentration 0.3 kg/L flows in at 2 L/min and the well-stirred mixture is drained at the same rate.

1. Let $y(t)$ (kg) be the amount of salt in the tank at time t (min). Set up the differential equation for y.
2. Solve for $y(t)$ and find the time when the concentration reaches 0.2 kg/L.

Final Answer: ______________________________

Q446 [Go to Solution p. 428] [Back to TOC]

Falling object with linear drag. A ball of mass m falls vertically under gravity and air resistance proportional to its velocity. Let $v(t)$ be the downward velocity.

1. Model the motion with a differential equation and state the constants used.
2. Solve for $v(t)$ given $v(0) = 0$.
3. Find the terminal speed.

Final Answer: ____

Q447 [Go to Solution p. 428] [Back to TOC]

Tank draining (Torricelli's law). Water drains from a vertical tank; the height $h(t)$ satisfies $\dfrac{dh}{dt} = -k\sqrt{h}$, where $k > 0$ is constant.

1. Solve for $h(t)$ (general solution).
2. If $h(0) = 1.6$ m and $k = 0.25$ $\mathrm{m}^{1/2}\mathrm{s}^{-1}$, find the time for the tank to empty.

Final Answer: ____

Q448 [Go to Solution p. 429] [Back to TOC]

Inverse-square heating. A heated wire loses heat at a rate inversely proportional to the square of its temperature $H(t)$ (in appropriate units): $\dfrac{dH}{dt} = -k/H^2$.

1. Solve for the general solution $H(t)$.
2. With $H(0) = 6$ and $k = 3$, find $H(t)$ and the time when $H = 3$.

Final Answer: ____

Name: ____________________ **Date:** ______________

Q449 [Go to Solution p. 429] [Back to TOC]

Fitting a separable model from data. A yeast culture satisfies $\frac{dY}{dt} = aY^{\frac{2}{3}}$ with $a > 0$. At $t = 0$, $Y = 8$; at $t = 9$, $Y = 27$.

1. Solve the differential equation for the general solution.
2. Determine a and write the particular solution.
3. Predict Y at $t = 16$.

Final Answer: ____________________

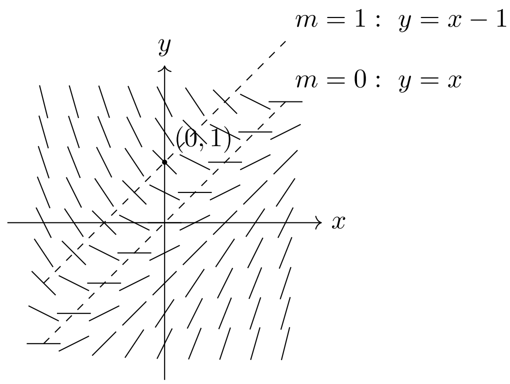
Name: ______________________ Date: ______________

AHL 5.15- Slope fields and their diagrams

Q450 [Go to Solution p. 430] [Back to TOC]

Direction field and integral curve. Consider the differential equation $\dfrac{dy}{dx} = x - y$.

1. Sketch the slope field on $-2 \le x, y \le 2$.
2. On the same axes, sketch the solution curve through $(0, 1)$.
3. Draw the *isoclines* for slopes $m = 0$ and $m = 1$, and explain how they help your sketch.

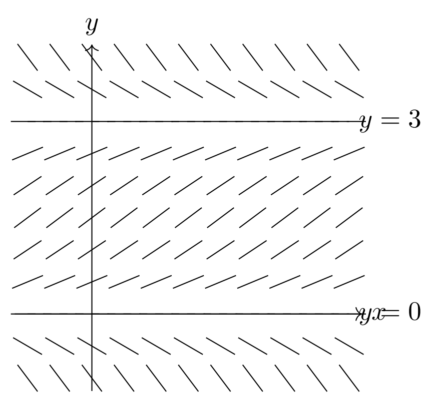


Final Answer: ______________________

Q451 [Go to Solution p. 430] [Back to TOC]

Logistic slope field; equilibria and stability. Consider $\dfrac{dy}{dx} = y\left(1 - \dfrac{y}{3}\right)$.

1. Sketch the slope field on $-1 \le x \le 4$, $-1 \le y \le 4$.
2. Identify the equilibrium solutions and classify each as stable/unstable.
3. Sketch solution curves for $y(0) = 0.5$, $y(0) = 2$, and $y(0) = 4$.



Name: ______________________ **Date:** ______________

Final Answer: ____

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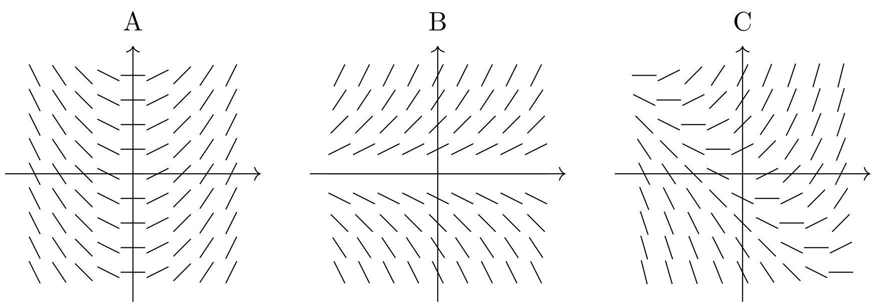
Q452 [Go to Solution p. 431] [Back to TOC]

Match the equation to the slope field. Three slope fields (A, B, C) are shown. Match each with one of

$$\text{(i)}\ \frac{dy}{dx} = x, \qquad \text{(ii)}\ \frac{dy}{dx} = y, \qquad \text{(iii)}\ \frac{dy}{dx} = x + y.$$

Explain your reasoning (use isoclines or symmetry).



Final Answer: ____

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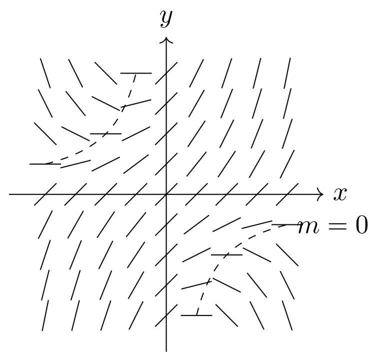
Q453 [Go to Solution p. 431] [Back to TOC]

Isoclines from a slope field. For $\frac{dy}{dx} = 1 + xy$:

1. Sketch the slope field on $-2 \le x, y \le 2$.
2. Find the isoclines (curves along which the slope is constant m).
3. Mark where the field has zero slope and describe regions of positive/negative slope.

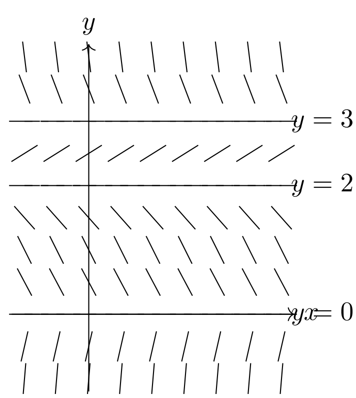


Final Answer: ______________________________

Q454 [Go to Solution p. 432] [Back to TOC]

Equilibria from a cubic in y. Consider $\dfrac{dy}{dx} = y(y-2)(3-y)$.

1. Sketch the slope field on $-1 \le x \le 3$, $-1 \le y \le 4$.
2. Identify all equilibrium solutions and classify each as stable/unstable/semi-stable.
3. Sketch solution curves for initial values $y(0) = -0.5,\ 1,\ 2.5,\ 3.5$.



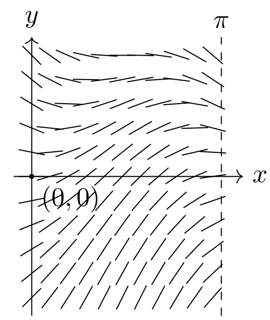
Final Answer: ______________________________

Name: ____________________ **Date:** ______________

Q455 [Go to Solution p. 432] [Back to TOC]

Estimating a value from a slope field. The slope field for $\frac{dy}{dx} = \sin x - \frac{y}{2}$ is given below.

1. Sketch the solution through $(0, 0)$ and use the field to estimate $y(\pi)$.
2. Explain how the sign and magnitude of the short line segments guide your estimate.



Final Answer: ____

Name: ____________________ Date: ____________

AHL 5.16- Euler's method

Key ideas

- **Initial value problem (IVP).** Solve $\dfrac{dy}{dx} = f(x,y)$ with $y(x_0) = y_0$.
- **Euler's method (single equation).**

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h\,f(x_n, y_n).$$

  h is the *step size*. Smaller $h \Rightarrow$ better accuracy (more work).

- **Euler's method (coupled system).** For $\dfrac{dx}{dt} = f_1(x,y,t)$ and $\dfrac{dy}{dt} = f_2(x,y,t)$:

$$\begin{aligned} t_{n+1} &= t_n + h, \\ x_{n+1} &= x_n + h\,f_1(x_n, y_n, t_n), \\ y_{n+1} &= y_n + h\,f_2(x_n, y_n, t_n). \end{aligned}$$

- **Interpretation.** Each step moves along the tangent with slope $f(x_n, y_n)$ for a short distance h.
- **Error.** Local truncation error is $O(h^2)$; global error is $O(h)$.

Worked example (single equation) Approximate the solution of $y' = x - y$ with $y(0) = 1$ at $x = 1$ using $h = 0.2$.

At (x_n, y_n) the slope is $f(x_n, y_n) = x_n - y_n$. Apply $y_{n+1} = y_n + h\,f(x_n, y_n)$ with $h = 0.2$.

n	x_n	y_n	$f(x_n, y_n)$
0	0.0	1.0000	-1.0000
1	0.2	0.8000	-0.6000
2	0.4	0.6800	-0.2800
3	0.6	0.6240	-0.0240
4	0.8	0.6192	0.1808
5	1.0	0.6554	—

Hence $y(1) \approx 0.6554$. (Exact solution $y = x - 1 + 2e^{-x}$ gives $y(1) \approx 0.7358$.)

TI-Nspire (IB-permissible) — single equation

1. **Home** → **Lists & Spreadsheet**.
2. Headers: `x` in A1, `y` in B1, `f` in C1.
3. Enter initial values: A2 `0`, B2 `1`.
4. In C2 type `=a2 - b2` ($f(x,y) = x - y$).
5. In A3 type `=a2 + 0.2` (step $h = 0.2$).
6. In B3 type `=b2 + 0.2*c2`.
7. In C3 type `=a3 - b3`.
8. Select A3:C3, then **Menu** → **Data** → **Fill Down** until $x = 1.0$ appears.
9. Read the last value in column `y` (this is the Euler estimate at $x = 1$).

Name: ______________________ **Date:** ______________

Coupled systems example (predator–prey) Lotka–Volterra model:

$$\frac{dx}{dt} = 0.1x - 0.02xy, \qquad \frac{dy}{dt} = -0.1y + 0.01xy, \qquad x(0) = 40,\ y(0) = 9,\ h = 1.$$

Euler updates give the first few steps:

n	t_n	x_n	y_n	$f_1(x_n, y_n)$	$f_2(x_n, y_n)$
0	0	40.000	9.000	−3.200	2.700
1	1	36.800	11.700	−4.931	3.136
2	2	31.869	14.836	−6.269	3.244

So $(x(3), y(3)) \approx (25.600,\ 18.080)$.

TI-Nspire (IB-permissible) — coupled system

1. **Home** → **Lists & Spreadsheet**.
2. Headers: `t` (A1), `x` (B1), `y` (C1), `f1` (D1), `f2` (E1). Optional: put `h` in H1 and its value in H2.
3. Initial row: A2 `0`, B2 `40`, C2 `9`.
4. D2: `=0.1*B2 - 0.02*B2*C2` (f_1).
5. E2: `=-0.1*C2 + 0.01*B2*C2` (f_2).
6. A3: `=A2 + 1` (or `=A2 + $H$2` if you used H2).
7. B3: `=B2 + 1*D2` (or `=$H$2*D2`).
8. C3: `=C2 + 1*E2` (or `=$H$2*E2`).
9. Fill down A3:E3 using **Menu** → **Data** → **Fill Down** for as many steps as required.
10. Read the (x, y) values row by row to trace the phase trajectory.

Checklist / exam tips

- Include *one full Euler step* by hand to show the method, then use your GDC to complete the table.
- Clearly state the step size h and the update formula you used.
- Smaller h generally improves accuracy; if asked, comment on error by comparing with an exact solution when available.

Q456 [Go to Solution p. 434] [Back to TOC]

Forward Euler: table and value. Use Euler's method with step size $h = 0.2$ to approximate the solution of

$$y' = x + y, \qquad y(0) = 1,$$

at $x = 1.0$.

1. Write the Euler update $y_{n+1} = y_n + h\,f(x_n, y_n)$ for this IVP.
2. Complete a table of (x_n, y_n) for $n = 0, 1, \ldots, 5$.
3. The exact solution is $y(x) = 2e^x - x - 1$. Compute the absolute error at $x = 1.0$.

Final Answer: ____________________

Q457 [Go to Solution p. 434] [Back to TOC]

Step-size comparison. Apply Euler's method to

$$y' = y\left(1 - \frac{y}{3}\right), \qquad y(0) = 0.6,$$

to approximate $y(2)$ using:

1. step size $h = 0.5$,
2. step size $h = 0.25$.

Compare the two approximations and comment on how halving h affects the result.

Final Answer: ____________________

Q458 [Go to Solution p. 434] [Back to TOC]

Threshold time by Euler stepping. A cooling model satisfies $y' = -0.7\,y + 0.3$ with $y(0) = 0$. Using Euler's method with $h = 0.2$, iterate forward until $y_n \geq 0.25$.

1. List the first few (x_n, y_n) values.
2. Report the smallest $t = x_n$ such that $y_n \geq 0.25$.
3. Use linear interpolation between the last two steps to refine the hitting time.

Final Answer: ____________________

Q459 [Go to Solution p. 435] [Back to TOC]

Spreadsheet setup. For the IVP $y' = \sin x - \frac{1}{2}y$, $y(0) = 1$, suppose a spreadsheet has x_0 in cell `A2`, y_0 in `B2`, and the step h in `D1` $= 0.1$.

1. Write the formulas for cells `A3` and `B3` that implement one Euler step.
2. Copy down to approximate $y(1)$.
3. Briefly explain how you would change the sheet to try a different step size easily.

Final Answer: ______________________________

Q460 [Go to Solution p. 435] [Back to TOC]

Euler polygon vs exact curve. Consider $y' = x - y$, $y(0) = 1$.

1. Perform three Euler steps with $h = 0.5$ to approximate $y(1.5)$.
2. The exact solution is $y(x) = x - 1 + 2e^{-x}$. Compute the true value at $x = 1.5$ and the percentage error of the Euler approximation.
3. On axes, sketch the Euler polygon alongside the exact curve, and label all step points.

Final Answer: ______________________________

Q461 [Go to Solution p. 436] [Back to TOC]

Global error estimate by halving h. For $y' = y\cos x$, $y(0) = 1$, approximate $y(\frac{\pi}{2})$ using Euler's method with

1. $h = \frac{\pi}{8}$,
2. $h = \frac{\pi}{16}$.

Assuming Euler's global error is $O(h)$, use Richardson extrapolation

$$y^* \approx y_{h/2} + \big(y_{h/2} - y_h\big)$$

to produce a refined estimate y^* for $y(\frac{\pi}{2})$.

Final Answer: ______________________________

Name: ____________________ **Date:** ______________

Q462 [Go to Solution p. 436] [Back to TOC]

Stability intuition for $y' = \lambda y$. Let $\lambda = -5$ and consider Euler's update $y_{n+1} = (1 + h\lambda)y_n$.

1. For $h \in \{0.05, 0.2, 0.5\}$, compute $1 + h\lambda$.
2. Which step sizes lead to monotone decay in the iterates (no sign flip)? Which lead to oscillatory decay?
3. Explain why large h can give qualitatively wrong behaviour for stiff decay problems.

Final Answer: ______________________________

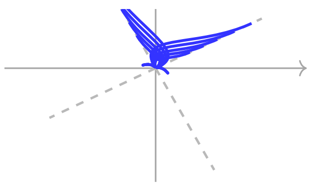
Name: ______________________ Date: ______________

AHL 5.17- Phase portraits

How to sketch a phase portrait (2×2 linear system) Given $\dot{x} = ax + by,\ \dot{y} = cx + dy$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $\mathrm{tr}A = a + d$, $\det A = ad - bc$, $\Delta = (\mathrm{tr}A)^2 - 4\det A$.

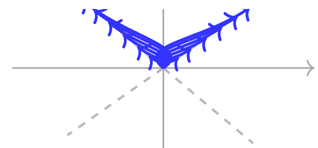
Stable node

$\det > 0,\ \Delta > 0,\ \mathrm{tr} < 0$



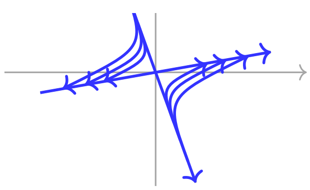
Unstable node

$\det > 0,\ \Delta > 0,\ \mathrm{tr} > 0$



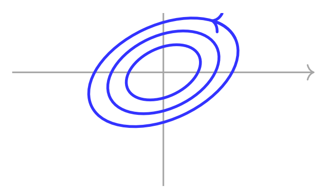
Saddle

$\det < 0$



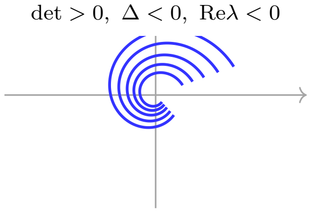
Centre (ellipses; anticlockwise)

$\mathrm{tr} = 0,\ \det > 0$



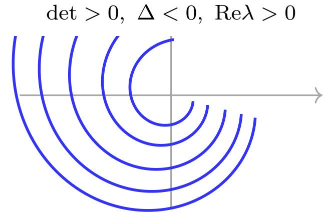
Stable spiral (anticlockwise)

$\det > 0,\ \Delta < 0,\ \mathrm{Re}\lambda < 0$



Unstable spiral (clockwise)

$\det > 0,\ \Delta < 0,\ \mathrm{Re}\lambda > 0$



Choose type: $\det < 0 \Rightarrow$ saddle; $\det > 0,\ \Delta > 0 \Rightarrow$ node; $\det > 0,\ \Delta < 0 \Rightarrow$ spiral/centre. **Stability:** by $\mathrm{tr}A$ (negative = stable, positive = unstable). **Rotation test:** $b = \dot{x}(0,1) > 0 \Rightarrow$ anticlockwise, $b < 0 \Rightarrow$ clockwise (or use $c = \dot{y}(1,0)$). **Exact solution (real, distinct):** $\mathbf{x}(t) = C_1\mathbf{v}_1 e^{\lambda_1 t} + C_2\mathbf{v}_2 e^{\lambda_2 t}$.

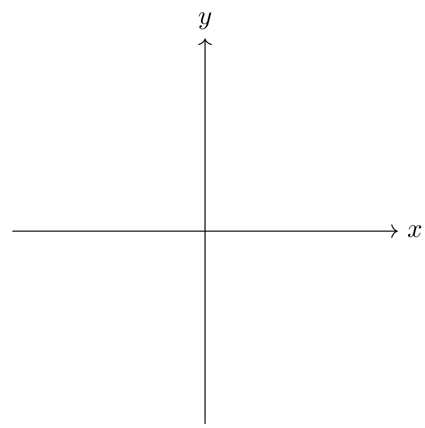
Name: ______________________ **Date:** ______________

Q463 [Go to Solution p. 437] [Back to TOC]

Centre (purely imaginary): sketch only. For

$$\frac{dx}{dt} = x + 3y, \qquad \frac{dy}{dt} = -2x - y,$$

1. Compute $\operatorname{tr} A$ and $\det A$, and determine the eigenvalue type (no exact solutions).
2. Sketch the phase portrait (show several closed trajectories) and indicate the direction of motion.



Final Answer: ____

Q464 [Go to Solution p. 437] [Back to TOC]

Spiral sink: sketch only. Consider

$$\frac{dx}{dt} = -2x - 5y, \qquad \frac{dy}{dt} = 2x - 3y.$$

1. Classify the equilibrium at the origin using trace–determinant (no exact solution).
2. Sketch a phase portrait showing the spiral behaviour and arrows.

Final Answer: ____

Q465 [Go to Solution p. 437] [Back to TOC]

Name: ______________________ **Date:** ______________

Spiral source: sketch only. For

$$\frac{dx}{dt} = x - 4y, \qquad \frac{dy}{dt} = x + y,$$

1. Use $\operatorname{tr} A$ and $\det A$ to justify the classification.
2. Sketch the phase portrait (spiral out from the origin) and mark the direction of rotation.

Final Answer: ____

Q466 [Go to Solution p. 438] [Back to TOC]

Saddle: eigenlines and qualitative sketch.

$$\frac{dx}{dt} = 3x, \qquad \frac{dy}{dt} = -2y.$$

1. Find the eigenvalues and eigenvectors.
2. Sketch the phase portrait, clearly drawing the stable and unstable eigenlines and several trajectories approaching/departing them.

Final Answer: ____

Q467 [Go to Solution p. 438] [Back to TOC]

Stable node (real negative eigenvalues): sketch; no exact solution needed.

$$\frac{dx}{dt} = -3x + y, \qquad \frac{dy}{dt} = -2y.$$

1. Compute eigenvalues and a basis of eigenvectors.
2. Classify the origin and sketch a representative family of trajectories with arrows.

Final Answer: ____

Name: ____________________ **Date:** ______________

Q468 [Go to Solution p. 439] [Back to TOC]

Exact solution (allowed case: real, distinct eigenvalues).

$$\frac{dx}{dt} = 3x + y, \qquad \frac{dy}{dt} = x + 3y, \qquad (x(0), y(0)) = (1, 0).$$

1. Find the eigenvalues and eigenvectors.
2. Hence find the exact solution $(x(t), y(t))$.
3. Classify the origin and sketch the phase portrait, superimposing the trajectory of the given initial condition.

Final Answer: ____________________

Q469 [Go to Solution p. 439] [Back to TOC]

Trace–determinant classification: sketch only. Without solving for eigenvectors, classify the equilibrium at the origin for each matrix, and state "sink/source/centre/spiral/saddle". Sketch a small, labelled portrait for each.

$$\text{(i) } A = \begin{pmatrix} -2 & 0 \\ 3 & -1 \end{pmatrix}, \quad \text{(ii) } A = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}, \quad \text{(iii) } A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}, \quad \text{(iv) } A = \begin{pmatrix} 2 & 1 \\ 3 & -4 \end{pmatrix}.$$

Final Answer: ____________________

Q470 [Go to Solution p. 440] [Back to TOC]

Nullclines and rotation direction: sketch only. For

$$\frac{dx}{dt} = -2y, \qquad \frac{dy}{dt} = x,$$

1. Draw the x'– and y'–nullclines in the phase plane.
2. Using a test point (e.g. $(1, 0)$), decide whether trajectories rotate clockwise or counterclockwise.

3. Sketch several closed orbits with arrows.

Final Answer: ____

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Q471 [Go to Solution p. 440] [Back to TOC]

Long–time behaviour near a saddle: sketch only.

$$\frac{dx}{dt} = x + 2y, \qquad \frac{dy}{dt} = 3x - y.$$

1. Show that the origin is a saddle by signs of $\det A$ and $\operatorname{tr} A$.
2. Find the eigenvectors and identify the stable and unstable manifolds.
3. Describe the $t \to +\infty$ behaviour of trajectories starting off each side of the stable line, and sketch.

Final Answer: ____

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Q472 [Go to Solution p. 441] [Back to TOC]

Exact solution (allowed: real, distinct eigenvalues) and interpretation.

$$\frac{dx}{dt} = -x + 2y, \qquad \frac{dy}{dt} = 2x - y, \qquad (x(0), y(0)) = (0, 1).$$

1. Find eigenvalues/eigenvectors and determine the type (e.g. saddle, node).
2. Obtain the exact solution $(x(t), y(t))$.
3. Using your solution, decide whether the trajectory approaches or moves away from the origin as $t \to \infty$ and sketch it on a phase portrait.

Final Answer: ____

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Name: ____________________ **Date:** ____________

AHL 5.18- Second order differential equations

1. Reduction of a Second-Order ODE to First-Order Form

A general second-order ODE can be written as:

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right)$$

or, for constant coefficients:

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0$$

To solve numerically or link with a first-order framework, introduce the substitution:

$$y = \frac{dx}{dt}$$

This gives the system:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = f(x, y, t)$$

For example, for

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0,$$

we obtain:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -ay - bx$$

Example – Reduction Reduce

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$$

Let $y = \frac{dx}{dt}$. Then:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -3y - 2x$$

This is now a coupled first-order system.

---

2. Euler's Method Algorithm for Second-Order ODEs

Given the first-order system:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = f(x, y, t),$$

Euler's method updates are:

$$x_{n+1} = x_n + h\,y_n,$$
$$y_{n+1} = y_n + h\,f(x_n, y_n, t_n),$$
$$t_{n+1} = t_n + h,$$

where h is the step size.

Example – Euler's Method Solve numerically:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0, \quad x(0) = 1, \quad \frac{dx}{dt}(0) = 0$$

Step size $h = 0.1$, find x and y for the first three steps.

Name: ______________________ **Date:** ______________

We have:

$$f(x, y, t) = -3y - 2x$$

Initial values: $t_0 = 0,\ x_0 = 1,\ y_0 = 0.$

n	t_n	x_n	y_n
0	0.0	1.0000	0.0000
1	0.1	$1.0000 + 0.1(0.0000) = 1.0000$	$0.0000 + 0.1(-3(0.0000) - 2(1.0000)) = -0.2000$
2	0.2	$1.0000 + 0.1(-0.2000) = 0.9800$	$-0.2000 + 0.1(-3(-0.2000) - 2(0.9800)) = -0.3920$
3	0.3	$0.9800 + 0.1(-0.3920) = 0.9408$	$-0.3920 + 0.1(-3(-0.3920) - 2(0.9408)) = -0.57456$

Thus:

$$(t, x, y) \approx (0.0, 1.0000, 0.0000),\ (0.1, 1.0000, -0.2000),\ (0.2, 0.9800, -0.3920),\ (0.3, 0.9408, -0.5746)$$

---

3. Notes for IB Examinations

- Always write the substitution $y = \frac{dx}{dt}$ clearly before reducing to a system.
- Organise Euler's method steps in a table with t, x, y.
- A calculator or spreadsheet may be needed for many steps.
- Interpret results numerically and graphically.

Name: ______________________ **Date:** ______________

Q473 [Go to Solution p. 442] [Back to TOC]

Rewrite as a first–order system. Let

$$\frac{d^2x}{dt^2} = f(x, \tfrac{dx}{dt}, t), \qquad x(0) = x_0,\ \tfrac{dx}{dt}(0) = v_0.$$

(a) Introduce $y = \frac{dx}{dt}$. Write the equivalent coupled system for (x, y). (b) Do this explicitly for $f(x, \dot{x}, t) = -\sin x - 0.3\,\dot{x} + 2\cos t$. (c) State the phase–plane axes and the equilibrium condition in terms of f.

Final Answer: ____

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Q474 [Go to Solution p. 442] [Back to TOC]

Euler scheme for second–order ODEs (general formula). Starting from the system $x' = y,\ y' = f(x, y, t)$, derive the forward Euler updates

$$x_{n+1} = x_n + h\,y_n, \qquad y_{n+1} = y_n + h\,f(x_n, y_n, t_n),$$

with $t_{n+1} = t_n + h$. Explain in one sentence how the local truncation error scales with h.

Final Answer: ____

____

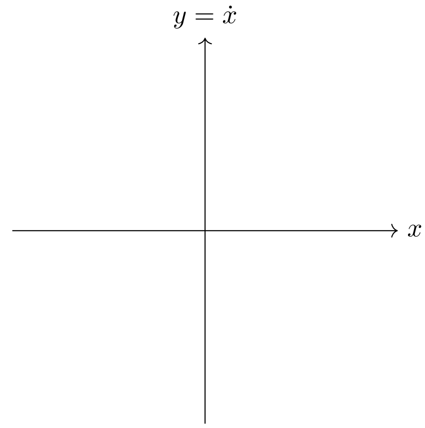
Q475 [Go to Solution p. 442] [Back to TOC]

Euler steps on a nonlinear oscillator. Consider

$$\frac{d^2x}{dt^2} = -\sin x - 0.2\,\frac{dx}{dt}, \qquad x(0) = 1.0,\ \frac{dx}{dt}(0) = 0.$$

(a) Write the system $x' = y,\ y' = -\sin x - 0.2y$. (b) Using Euler with step $h = 0.1$, compute (x_1, y_1) and (x_2, y_2). (c) On axes labelled x (horizontal) and $y = \dot{x}$ (vertical), sketch the two Euler points and indicate the direction of progression.

Name: ______________________ **Date:** ______________



Final Answer: ____

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Q476 [Go to Solution p. 442] [Back to TOC]

Linear constant–coefficients; exact solution allowed (real distinct).

$$\frac{d^2x}{dt^2} - 5\frac{dx}{dt} + 6x = 0, \qquad x(0) = 1,\ \frac{dx}{dt}(0) = 0.$$

(a) Write the system $x' = y,\ y' = 5y - 6x$ and its matrix A. (b) Find the eigenvalues/eigenvectors of A. (c) Hence find the exact solution $x(t)$ and $y(t)$. (d) Classify the origin and sketch a small phase portrait with several trajectories and arrows.

Final Answer: ____

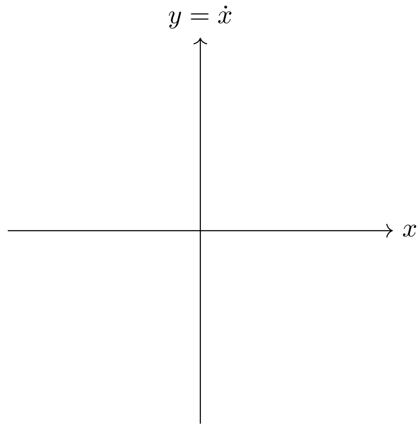
____

Q477 [Go to Solution p. 443] [Back to TOC]

Critically damped case; sketch only (no exact form required).

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0.$$

(a) Write the first–order system and matrix A. (b) Using trace–determinant, classify the equilibrium (note the repeated eigenvalue). (c) On the phase plane, sketch the node structure and typical trajectories with arrows, indicating the *slow* direction.



Final Answer: ____

Q478 [Go to Solution p. 443] [Back to TOC]

Underdamped oscillator; phase portrait only.

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0.$$

(a) Form the system and compute $\operatorname{tr} A$ and $\det A$. (b) Classify the origin using trace–determinant and state the rotation direction at $(1, 0)$. (c) Sketch a spiral portrait (sink) with several trajectories and direction arrows.

Final Answer: ____

Q479 [Go to Solution p. 443] [Back to TOC]

Driven system; one Euler step.

$$\frac{d^2x}{dt^2} = -x - 0.4\,\frac{dx}{dt} + 3\cos t, \qquad x(0) = 0,\ \frac{dx}{dt}(0) = 1.$$

(a) Write the coupled system. (b) With $h = 0.1$, compute one Euler step to obtain (x_1, y_1) at $t_1 = 0.1$. (c) Briefly explain why Euler's method can mis-estimate amplitude/phase for oscillatory forcing.

Name: ______________________ **Date:** ______________

Final Answer: ______________________________

Q480 [Go to Solution p. 443] [Back to TOC]

Mass–spring–damper model and parameters. For

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0, \qquad m, k > 0,\ c \geq 0,$$

(a) Non-dimensionalize to obtain $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ and define ω_n, ζ. (b) Write the system, the matrix A, and give $\operatorname{tr} A$ and $\det A$ in terms of ζ, ω_n. (c) For $\zeta < 1,\ \zeta = 1,\ \zeta > 1$, state the phase–portrait type (centre/spiral/node) and whether solutions decay or not.

Final Answer: ______________________________

Q481 [Go to Solution p. 444] [Back to TOC]

Conservative oscillator; energy and phase curves.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

(a) Show that $E = \frac{1}{2}y^2 + \frac{1}{2}\omega^2 x^2$ is constant along trajectories, where $y = \dot{x}$. (b) Deduce the shape of phase curves and sketch three distinct closed orbits with arrows indicating direction. (c) Using your sketch, explain why the period is independent of amplitude.

Final Answer: ______________________________

Q482 [Go to Solution p. 444] [Back to TOC]

Compare Euler with exact for a real–distinct case.

$$\frac{d^2x}{dt^2} - x = 0, \qquad x(0) = 1,\ \frac{dx}{dt}(0) = 0.$$

Name: ______________________ **Date:** ______________

(a) Write the system and matrix A, and find eigenvalues/eigenvectors. (b) Find the exact solution for $x(t)$. (c) Using Euler with $h = 0.1$, compute (x_1, y_1), (x_2, y_2), (x_3, y_3). (d) Compare $x(0.3)$ from parts (b) and (c) and comment on the sign of the error.

Final Answer: ____

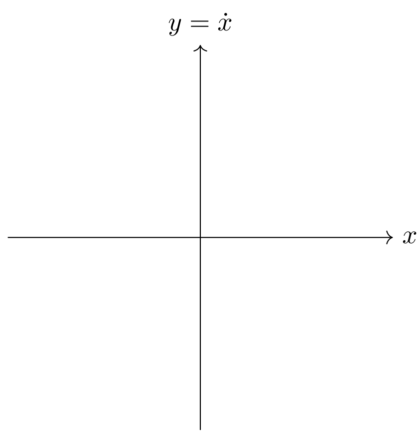
____

Q483 [Go to Solution p. 445] [Back to TOC]

Nonlinear physical example (pendulum with damping); Euler steps and qualitative picture.

$$\frac{d^2x}{dt^2} + 0.1\,\frac{dx}{dt} + \sin x = 0, \qquad x(0) = \frac{\pi}{2},\ \frac{dx}{dt}(0) = 0.$$

(a) Give the coupled system. (b) Perform two Euler steps with $h = 0.05$. (c) On blank axes, sketch a qualitative phase portrait near the origin and indicate the expected long-time behaviour.



Final Answer: ____

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Q484 [Go to Solution p. 445] [Back to TOC]

Matrix-to-second-order translation. You are given the planar linear system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a, b > 0.$$

(a) Show that x satisfies $\ddot{x} + a\dot{x} + bx = 0$. (b) Using trace–determinant, give conditions on a, b for an overdamped node, critical damping, and underdamped spiral. (c) For the overdamped case only (real distinct eigenvalues), give the exact solution for $x(t)$ with $x(0) = x_0,\ \dot{x}(0) = v_0$.

Final Answer: ____

Solutions

Topic 1 Solutions

SL 1.1 Standard form

Solution to Q1. [Back to Question p. 8] [Back to TOC]

(a) $0.0000426 = 4.26 \times 10^{-5}$.

(b) $85\,900\,000 = 8.59 \times 10^{7}$.

(c) $\dfrac{7.2 \times 10^{-5}}{3 \times 10^{-2}} = \dfrac{7.2}{3} \times 10^{-5-(-2)} = 2.4 \times 10^{-3}$.

Solution to Q2. [Back to Question p. 8] [Back to TOC]

$(3.5 \times 10^{-4})(8 \times 10^{6}) = (3.5 \cdot 8) \times 10^{-4+6} = 28 \times 10^{2} = 2.8 \times 10^{3}$.

Solution to Q3. [Back to Question p. 8] [Back to TOC]

(a) $(6 \times 10^{-3})(4 \times 10^{7}) = 24 \times 10^{4} = 2.4 \times 10^{5}$.

(b) $\dfrac{9 \times 10^{5}}{3 \times 10^{-2}} = 3 \times 10^{7}$.

(c) $2.4 \times 10^{-4} + 3.1 \times 10^{-4} = 5.5 \times 10^{-4}$.

(d) $7.5 \times 10^{2} - 2.50 \times 10^{1} = 750 - 25 = 725 = 7.25 \times 10^{2}$.

Solution to Q4. [Back to Question p. 9] [Back to TOC]

Distance travelled by light: $d = (3.0 \times 10^{8}\,\mathrm{m\,s^{-1}})(0.02\,\mathrm{s}) = 6.0 \times 10^{6}\,\mathrm{m}$. Earth's radius is $6.37 \times 10^{6}\,\mathrm{m}$. Hence Earth's radius is larger (by $0.37 \times 10^{6} = 3.7 \times 10^{5}\,\mathrm{m}$).

SL 1.2 Arithmetic sequences and series

Solution to Q5. [Back to Question p. 10] [Back to TOC]

Using $u_n = u_1 + (n-1)d$ with $u_1 = 7$ and $d = -3$:

$$u_5 = 7 + 4(-3) = -5, \quad u_{20} = 7 + 19(-3) = -50.$$

Solution to Q6. [Back to Question p. 10] [Back to TOC]

The sum of the first n positive integers is $S_n = \frac{n(n+1)}{2}$. For $n = 100$,

$$S_{100} = \frac{100 \times 101}{2} = 5050.$$

Solution to Q7. [Back to Question p. 10] [Back to TOC]

(a) From $u_n = 12 + 5(n-1)$ we have $u_1 = 12$ and $u_{10} = 12 + 5 \cdot 9 = 57$.

(b) Given $u_3 = 14$ and $d = 4$, we find $u_1 = u_3 - 2d = 6$. The sum of the first 20 terms is $S_{20} = \frac{20}{2}(2 \cdot 6 + 19 \cdot 4) = 880$.

(c) For $u_1 = 5$ and $d = 4$, the sum formula gives $S_n = \frac{n}{2}(2 \cdot 5 + (n-1) \cdot 4) = 2n^2 + 3n$. The equation $2n^2 + 3n = 1428$ has no positive integer solution, so there is no such n.

(d) From $u_4 = u_1 + 3d = 11$ and $u_{12} = u_1 + 11d = 43$, subtracting gives $8d = 32$, so $d = 4$ and $u_1 = -1$. Then $S_{50} = \frac{50}{2}(2(-1) + (50-1) \cdot 4) = 4850$.

SL 1.3 Geometric sequences and series

Solution to Q8. [Back to Question p. 11] [Back to TOC]

For a geometric sequence $u_n = u_1 r^{n-1}$ with $u_1 = 3$ and $r = 2$,

$$u_6 = 3 \cdot 2^5 = 96, \qquad S_6 = 3\frac{2^6 - 1}{2 - 1} = 3 \times 63 = 189.$$

Solution to Q9. [Back to Question p. 11] [Back to TOC]

The total salary paid over five years is

$$S = 32{,}000\left(1 + 1.05 + 1.05^2 + 1.05^3 + 1.05^4\right) = 32{,}000\,\frac{1.05^5 - 1}{1.05 - 1}.$$

Using $1.05^5 \approx 1.27628$, we get $S \approx 176{,}819.84$.

Solution to Q10. [Back to Question p. 11] [Back to TOC]

(a) With $u_1 = 9$ and $r = \frac{1}{3}$, $u_5 = 9 \cdot (1/3)^4 = \frac{1}{9}$ and

$$S_5 = 9\frac{1 - (1/3)^5}{1 - 1/3} = \frac{121}{9} \approx 13.44.$$

(b) Given $u_3 = 48$ and $u_6 = 384$, we have $r = 2$ and $u_1 = 12$.

(c) A ball dropped from height 2 m rebounds to 80% of the previous height. The total vertical distance travelled is

$$D = 2 + 4\sum_{n=1}^{\infty} 0.8^n = 18\,\text{m}.$$

(d) For $u_1 = 1$ and $r = 0.1$, the sum $S_n = \frac{1-0.1^n}{0.9}$ remains below 1.11 for all n, so it never equals 121.

SL 1.4 Financial Applications of Geometric sequences

Solution to Q11. [Back to Question p. 14] [Back to TOC]

The value after four years is

$$A = 1000(1 + 0.035)^4 = 1000(1.035)^4 \approx 1{,}147.52.$$

Solution to Q12. [Back to Question p. 14] [Back to TOC]

Each year the value is multiplied by $1 - 0.18 = 0.82$. After five years,

$$V = 24{,}000 \times 0.82^5 \approx 8{,}897.76.$$

Solution to Q13. [Back to Question p. 15] [Back to TOC]

(a) The future value is $A = 6{,}500(1.042)^7 \approx 8{,}669.37$.

(b) The depreciated value is $V = 1{,}800(0.75)^3 = 759.38$.

(c) The effective real growth factor per year is $\frac{1+0.06}{1+0.025} = 1.034146\ldots$. Over ten years the real value is $10{,}000 \times 1.034146^{10} \approx 13{,}990.07$.

SL 1.5 Integer Exponents and Logarithms

Solution to Q14. [Back to Question p. 16] [Back to TOC]

Using exponent rules,

$$\frac{2^3 \cdot 2^{-5}}{2^{-1}} = 2^{3-5+1} = 2^{-1} = \frac{1}{2}.$$

Solution to Q15. [Back to Question p. 16] [Back to TOC]

(a) $10^x = 4.2$ implies $x = \log_{10}(4.2) \approx 0.623$.

(b) $e^{2x} = 7$ implies $2x = \ln 7$, so $x = \frac{\ln 7}{2} \approx 0.973$.

(c) $3 \cdot 2^x = 40$ implies $2^x = \frac{40}{3}$, so $x = \log_2\left(\frac{40}{3}\right) \approx 3.737$.

(d) $5^{x-1} = 12$ implies $x - 1 = \log_5(12)$, so $x = 1 + \log_5(12) \approx 2.544$.

Solution to Q16. [Back to Question p. 16] [Back to TOC]

(a) $\ln\left(\dfrac{9x^4}{\sqrt{y}}\right) = \ln 9 + 4\ln x - \frac{1}{2}\ln y$.

(b) $\log_{10}(100x^3y) = \log_{10} 100 + 3\log_{10} x + \log_{10} y = 2 + 3\log_{10} x + \log_{10} y$.

(c) $\log\left(\dfrac{a^5}{b^2c}\right) = 5\log a - 2\log b - \log c$.

(d) $\ln\big((e^{3t})^2\big) = \ln(e^{6t}) = 6t$.

SL 1.6 Approximation, Bounds and Percentage Errors

Solution to Q17. [Back to Question p. 18] [Back to TOC]

3.1462 rounded to three significant figures is 3.15.
0.004 981 rounded to two decimal places is 0.00.

Solution to Q18. [Back to Question p. 18] [Back to TOC]

Since $r = 2.5\,\text{cm}$ to the nearest $0.1\,\text{cm}$, r lies between 2.45 and $2.55\,\text{cm}$.
Hence $A = \pi r^2$ lies between $\pi(2.45)^2 \approx 18.86\,\text{cm}^2$ and $\pi(2.55)^2 \approx 20.43\,\text{cm}^2$.

SL 1.7 Amortization and Annuities

Solution to Q19. [Back to Question p. 21] [Back to TOC]

Loan payment (amortization). Let $P = 9{,}000$, monthly rate $i = \frac{0.06}{12} = 0.005$, number of months $n = 36$.

By formula:

$$M = \frac{P\,i}{1-(1+i)^{-n}} = \frac{9{,}000 \times 0.005}{1-(1.005)^{-36}} \approx \boxed{273.80}.$$

TI–Nspire (Finance Solver):

1. **Menu** $\to$ **Finance** $\to$ **Finance Solver**.
2. Enter **N** $= 36$, **I%** $= 6$, **PV** $= 9000$, **PMT** $=?$, **FV** $= 0$, **P/Y** $= 12$, **C/Y** $= 12$, **PMT** : **End**.
3. Solve for **PMT**. (Calculator returns a negative cash outflow: $|\text{PMT}| \approx \boxed{273.80}$ per month.)

Solution to Q20. [Back to Question p. 21] [Back to TOC]

Present value of an annuity. End–of–month payments 250 for $n = 48$ months at monthly rate $i = \frac{0.048}{12} = 0.004$.

By formula:

$$PV = 250\,\frac{1-(1+i)^{-n}}{i} = 250\,\frac{1-(1.004)^{-48}}{0.004} \approx \boxed{10{,}898.56}.$$

TI–Nspire (Finance Solver):

1. **Menu** $\to$ **Finance** $\to$ **Finance Solver**.
2. Enter **N** $= 48$, **I%** $= 4.8$, **PV** $=?$, **PMT** $= -250$ (payment out), **FV** $= 0$, **P/Y** $= 12$, **C/Y** $= 12$, **PMT** : **End**.
3. Solve for **PV** $\Rightarrow \boxed{10{,}898.56}$ (to cents).

SL 1.8 Systems of Equations and Polynomials (Technology)

Solution to Q21. [Back to Question p. 23] [Back to TOC]

From the system

$$\begin{cases} 2x + y = 11, \\ x - y = 1, \end{cases}$$

solve by elimination or substitution. From the second equation $y = x - 1$, substitute into the first: $2x + (x - 1) = 11$ gives $3x = 12$ so $x = 4$. Then $y = 4 - 1 = 3$. Hence the solution is $(x, y) = (4, 3)$.

Solution to Q22. [Back to Question p. 24] [Back to TOC]

Solve

$$\begin{cases} x + 2y - 3z = 7, \\ 2x - y + z = 1, \\ -3x + 4y + 2z = 9. \end{cases}$$

Use row reduction on the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 7 \\ 2 & -1 & 1 & 1 \\ -3 & 4 & 2 & 9 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 + 3R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 7 \\ 0 & -5 & 7 & -13 \\ 0 & 10 & -7 & 30 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{5} R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 7 \\ 0 & 1 & -\frac{7}{5} & \frac{13}{5} \\ 0 & 10 & -7 & 30 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 10R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 7 \\ 0 & 1 & -\frac{7}{5} & \frac{13}{5} \\ 0 & 0 & 7 & 4 \end{array}\right].$$

Hence $z = \frac{4}{7}$. From the second row, $y - \frac{7}{5}z = \frac{13}{5} \Rightarrow y = \frac{13}{5} + \frac{7}{5} \cdot \frac{4}{7} = \frac{17}{5}$. From the first row,

$$x = 7 - 2y + 3z = 7 - 2 \cdot \frac{17}{5} + 3 \cdot \frac{4}{7} = \frac{67}{35}.$$

Therefore

$$\boxed{x = \tfrac{67}{35}, \quad y = \tfrac{17}{5}, \quad z = \tfrac{4}{7}}.$$

(Quick check: $x + 2y - 3z = 7$, $2x - y + z = 1$, and $-3x + 4y + 2z = 9$.)

Solution to Q23. [Back to Question p. 24] [Back to TOC]

We are given:

$$p(x) = x^4 - 5x^2 + 4.$$

Let $y = x^2$, so the equation becomes:

$$y^2 - 5y + 4 = 0.$$

Factor:

$$(y - 1)(y - 4) = 0 \quad \Rightarrow \quad y = 1 \text{ or } y = 4.$$

Returning to x:

$$x^2 = 1 \quad \Rightarrow \quad x = \pm 1,$$

$$x^2 = 4 \quad \Rightarrow \quad x = \pm 2.$$

Thus, the real roots are:

$$x = -2,\ -1,\ 1,\ 2.$$

AHL 1.9 Law of logarithms

Solution to Q24. [Back to Question p. 25] [Back to TOC]

We simplify:

$$\log(50) + \log(20) - \log(5) = \log\left(\frac{50 \times 20}{5}\right) = \log\left(\frac{1000}{5}\right) = \log(200).$$

Solution to Q25. [Back to Question p. 25] [Back to TOC]

We have:

$$\log(3x) - \log(x-2) = 1.$$

Using the quotient rule:

$$\log\left(\frac{3x}{x-2}\right) = 1.$$

This means:

$$\frac{3x}{x-2} = 10.$$

Multiply through:

$$3x = 10(x-2) \quad \Rightarrow \quad 3x = 10x - 20 \quad \Rightarrow \quad 7x = 20 \quad \Rightarrow \quad x = \frac{20}{7}.$$

Since $x > 2$ is required, $x = \frac{20}{7}$ is valid.

Solution to Q26. [Back to Question p. 25] [Back to TOC]

We start by applying the logarithmic product rule:

$$\log_5(x-1) + \log_5(x+1) = \log_5\left[(x-1)(x+1)\right].$$

This simplifies to:

$$\log_5(x^2 - 1) = 2.$$

Rewriting in exponential form:

$$x^2 - 1 = 5^2 = 25.$$

Thus:

$$x^2 = 26 \quad \Rightarrow \quad x = \pm\sqrt{26}.$$

From the original domain restrictions:

$$x - 1 > 0 \quad \Rightarrow \quad x > 1,$$

so we discard the negative root.

Therefore:

$$\boxed{x = \sqrt{26}}$$

is the solution.

AHL 1.10 Rational exponents

Solution to Q27. [Back to Question p. 26] [Back to TOC]

We rewrite fractional exponents as radicals:

$$x^{3/2} = \left(\sqrt{x}\right)^3 = \sqrt{x^3}, \quad x^{-2/3} = \frac{1}{x^{2/3}} = \frac{1}{\sqrt[3]{x^2}}.$$

Solution to Q28. [Back to Question p. 26] [Back to TOC]

We have:

$$\frac{25^{3/2} \cdot 10^{-1}}{5^{1/2}}.$$

First, $25^{3/2} = (\sqrt{25})^3 = 5^3 = 125$. So:

$$\frac{125 \cdot 10^{-1}}{5^{1/2}} = \frac{125 \cdot \frac{1}{10}}{\sqrt{5}} = \frac{12.5}{\sqrt{5}} = \frac{25}{2\sqrt{5}}.$$

Rationalizing:

$$\frac{25}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{25\sqrt{5}}{10} = \frac{5\sqrt{5}}{2}.$$

Solution to Q29. [Back to Question p. 26] [Back to TOC]

We have:

$$(27^{2/3})(9^{3/2})(3^{-1}).$$

First, $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$. Also, $9^{3/2} = (\sqrt{9})^3 = 3^3 = 27$. Thus:

$$9 \cdot 27 \cdot 3^{-1} = 243 \cdot \frac{1}{3} = 81.$$

AHL 1.11 Infinite Geometric Series

Solution to Q30. [Back to Question p. 27] [Back to TOC]

This is a geometric series with first term $a = 5$ and ratio $r = \frac{1}{2}$. Since $|r| < 1$, it converges and

$$S_\infty = \frac{a}{1-r} = \frac{5}{1-\frac{1}{2}} = 10.$$

Solution to Q31. [Back to Question p. 27] [Back to TOC]

Each series is geometric with first term a and ratio r. It converges iff $|r| < 1$, and then $S_\infty = \dfrac{a}{1-r}$.

(a) $3 + \frac{3}{4} + \frac{3}{16} + \cdots$: $a = 3,\ r = \frac{1}{4}$.

$$S_\infty = \frac{3}{1-\frac{1}{4}} = \frac{3}{\frac{3}{4}} = 4.$$

(b) $7 - 3.5 + 1.75 - \cdots$: $a = 7,\ r = -\frac{1}{2}$.

$$S_\infty = \frac{7}{1-(-\frac{1}{2})} = \frac{7}{\frac{3}{2}} = \frac{14}{3} \approx 4.67.$$

(c) $10 + 8 + 6.4 + \cdots$: $a = 10,\ r = 0.8$.

$$S_\infty = \frac{10}{1-0.8} = \frac{10}{0.2} = 50.$$

AHL 1.12 Complex Numbers (Cartesian Form)

Solution to Q32. [Back to Question p. 28] [Back to TOC]

Sum: $(2-3i)+(4+6i)=(2+4)+(-3i+6i)=6+3i$.

Product:

$$(2-3i)(4+6i)=2\cdot 4+2\cdot 6i-3i\cdot 4-3i\cdot 6i=8+12i-12i-18i^2=8+0i-18(-1)=26.$$

Solution to Q33. [Back to Question p. 28] [Back to TOC]

Solve $z^2-6z+13=0$.

$$z=\frac{6\pm\sqrt{(-6)^2-4\cdot 1\cdot 13}}{2}=\frac{6\pm\sqrt{36-52}}{2}=\frac{6\pm\sqrt{-16}}{2}=\frac{6\pm 4i}{2}=3\pm 2i.$$

Thus the roots are $z_1=3+2i$ and $z_2=3-2i$, which plot on the Argand diagram at $(3,2)$ and $(3,-2)$, symmetric about the real axis.

AHL 1.13 Complex Numbers (Polar/Exponential)

Solution to Q34. [Back to Question p. 29] [Back to TOC]

For $z = 1 + i$, the modulus and argument are

$$r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \theta = \arg z = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

Hence z in polar form is

$$z = r \operatorname{cis} \theta = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right) = \sqrt{2}\,(\cos \tfrac{\pi}{4} + i \sin \tfrac{\pi}{4}).$$

Solution to Q35. [Back to Question p. 29] [Back to TOC]

Let $z = \sqrt{3} - i$. Then

$$r = |z| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{3 + 1} = 2, \qquad \theta = \arg z = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}.$$

Hence $z = 2 \operatorname{cis}\left(-\frac{\pi}{6}\right)$. By De Moivre's theorem,

$$z^5 = 2^5 \operatorname{cis}\left(-\frac{5\pi}{6}\right) = 32\left(\cos(-\tfrac{5\pi}{6}) + i \sin(-\tfrac{5\pi}{6})\right) = 32\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -16\sqrt{3} - 16i.$$

AHL 1.14 Matrices and Inverses

Solution to Q36. [Back to Question p. 30] [Back to TOC]

Example: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but $BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so $AB \neq BA$.

Solution to Q37. [Back to Question p. 30] [Back to TOC]

$A = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. $A^{-1} = \dfrac{1}{(2)(-2)-(1)(3)} \begin{bmatrix} -2 & -1 \\ -3 & 2 \end{bmatrix} = \dfrac{1}{-7} \begin{bmatrix} -2 & -1 \\ -3 & 2 \end{bmatrix}$. Then $\mathbf{x} = A^{-1} \begin{bmatrix} 5 \\ -4 \end{bmatrix} = \dfrac{1}{-7} \begin{bmatrix} -10+4 \\ -15-8 \end{bmatrix} = \dfrac{1}{-7} \begin{bmatrix} -6 \\ -23 \end{bmatrix} = \begin{bmatrix} \frac{6}{7} \\ \frac{23}{7} \end{bmatrix}$, so $x = \frac{6}{7}$, $y = \frac{23}{7}$.

AHL 1.15 Eigenvalues and Diagonalisation

Solution to Q38. [Back to Question p. 34] [Back to TOC]

For $M = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$ with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Solution to Q39. [Back to Question p. 34] [Back to TOC]

For $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\lambda = 3, -1$ with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively. Thus $M = PDP^{-1}$ with $D = \operatorname{diag}(3, -1)$, so $M^5 = PD^5P^{-1} = \begin{bmatrix} 121 & 122 \\ 122 & 121 \end{bmatrix}$.

Solution to Q40. [Back to Question p. 34] [Back to TOC]

Finding eigenvalues and eigenvectors. For

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}:$$

1. *Characteristic polynomial.*

$$\chi_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 = \lambda^2 - 7\lambda + 10.$$

2. *Eigenvalues.* Solve $\lambda^2 - 7\lambda + 10 = 0 \Rightarrow (\lambda - 5)(\lambda - 2) = 0$, so

$$\lambda_1 = 5, \qquad \lambda_2 = 2.$$

3. *Eigenvectors.*

   - For $\lambda = 5$: $(A - 5I) = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$, so $-x + y = 0 \Rightarrow y = x$. A corresponding eigenvector is $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
   - For $\lambda = 2$: $(A - 2I) = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$, so $2x + y = 0 \Rightarrow y = -2x$. A corresponding eigenvector is $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Solution to Q41. [Back to Question p. 34] [Back to TOC]

Diagonalization of a 2×2 matrix. For

$$B = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}:$$

1. *Distinct real eigenvalues.*

$$\chi_B(\lambda) = \det(B - \lambda I) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 4 = \lambda^2 - 10\lambda + 21 = (\lambda - 7)(\lambda - 3).$$

Hence $\lambda_1 = 7$, $\lambda_2 = 3$ (distinct, real).

2. *Eigenvectors and diagonalization.*

   For $\lambda_1 = 7$: $(B - 7I) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ gives $-2x + 2y = 0 \Rightarrow y = x$, so $\mathbf{p}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   For $\lambda_2 = 3$: $(B - 3I) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ gives $2x + 2y = 0 \Rightarrow y = -x$, so $\mathbf{p}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

   Take
   $$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}.$$

3. *Verification.* The inverse is
   $$P^{-1} = \frac{1}{\det P} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

   Compute
   $$PDP^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 10 & 4 \\ 4 & 10 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix} = B.$$

Solution to Q42. [Back to Question p. 35] [Back to TOC]

Powers of a 2×2 matrix using diagonalization. For
$$C = \begin{pmatrix} 7 & 0 \\ 2 & 3 \end{pmatrix}:$$

1. *Diagonalize C.* The eigenvalues are the diagonal entries (triangular matrix): $\lambda_1 = 7$, $\lambda_2 = 3$.

   Eigenvector for $\lambda_1 = 7$: $(C - 7I) = \begin{pmatrix} 0 & 0 \\ 2 & -4 \end{pmatrix}$ gives $2x - 4y = 0 \Rightarrow x = 2y$, so $\mathbf{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

   Eigenvector for $\lambda_2 = 3$: $(C - 3I) = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix}$ gives $x = 0$, so $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

   Take
   $$P = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 7 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

   Thus $C = PDP^{-1}$.

2. *Compute C^6.* Using $C^n = PD^nP^{-1}$,
   $$D^6 = \begin{pmatrix} 7^6 & 0 \\ 0 & 3^6 \end{pmatrix} \quad \Rightarrow \quad C^6 = \begin{pmatrix} 7^6 & 0 \\ \dfrac{7^6 - 3^6}{2} & 3^6 \end{pmatrix}.$$

   (You can check this by induction or by carrying out PD^6P^{-1}.)

Solution to Q43. [Back to Question p. 35] [Back to TOC]

Application: population movement between two towns.

$$\mathbf{v}_{n+1} = M\mathbf{v}_n, \qquad M = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}.$$

1. The recurrence is already in the form $\mathbf{v}_{n+1} = M\mathbf{v}_n$ with M as above.
2. *Diagonalize M and find $\mathbf{v}_n$.*

$$\chi_M(\lambda) = \det(M - \lambda I) = (0.9 - \lambda)(0.8 - \lambda) - 0.02 = \lambda^2 - 1.7\lambda + 0.70.$$

Discriminant $= 1.7^2 - 4(0.70) = 0.09$, so

$$\lambda_1 = 1, \qquad \lambda_2 = 0.7.$$

Eigenvector for $\lambda_1 = 1$: $(M - I) = \begin{pmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{pmatrix}$ gives $-x + 2y = 0 \Rightarrow x = 2y$, so $\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Eigenvector for $\lambda_2 = 0.7$: $(M - 0.7I) = \begin{pmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{pmatrix}$ gives $x = -y$, so $\mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Set

$$P = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.7 \end{pmatrix}, \quad P^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

Then $M = PDP^{-1}$ and

$$\mathbf{v}_n = M^n\mathbf{v}_0 = PD^nP^{-1}\mathbf{v}_0 = P\begin{pmatrix} 1 & 0 \\ 0 & 0.7^n \end{pmatrix} P^{-1}\mathbf{v}_0.$$

3. *Numerical prediction for $(x_0, y_0) = (5000, 3000)$.* First

$$P^{-1}\mathbf{v}_0 = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5000 \\ 3000 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 8000 \\ -1000 \end{pmatrix} = \begin{pmatrix} \frac{8000}{3} \\ -\frac{1000}{3} \end{pmatrix}.$$

Hence

$$\mathbf{v}_{10} = P\begin{pmatrix} 1 & 0 \\ 0 & 0.7^{10} \end{pmatrix}\begin{pmatrix} \frac{8000}{3} \\ -\frac{1000}{3} \end{pmatrix} = \begin{pmatrix} \frac{16000}{3} - \frac{1000}{3}\,0.7^{10} \\ \frac{8000}{3} + \frac{1000}{3}\,0.7^{10} \end{pmatrix}.$$

Since $0.7^{10} \approx 0.0282475$, this gives

$$(x_{10}, y_{10}) \approx (5323.9,\ 2676.1) \quad (\text{total} \approx 8000 \text{ conserved}).$$

Solution to Q44. [Back to Question p. 36] [Back to TOC]

Eigenvalues and invertibility. For

$$D = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}:$$

1. *Eigenvalues.*

$$\chi_D(\lambda) = \det(D - \lambda I) = \begin{vmatrix} 6 - \lambda & 2 \\ 3 & 1 - \lambda \end{vmatrix} = (6 - \lambda)(1 - \lambda) - 6 = \lambda^2 - 7\lambda = \lambda(\lambda - 7).$$

Thus $\lambda_1 = 0$, $\lambda_2 = 7$.

2. *Invertibility.* A matrix is invertible iff 0 is *not* an eigenvalue. Since $\lambda = 0$ is an eigenvalue of D, it is *not* invertible (singular).

3. *Inverse via diagonalization.* Not applicable (no inverse exists).

Solution to Q45. [Back to Question p. 36] [Back to TOC]

Repeated eigenvalues case. For

$$E = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}:$$

1. *Eigenvalues and multiplicities.*

$$\chi_E(\lambda) = \det(E - \lambda I) = (4 - \lambda)^2,$$

so $\lambda = 4$ with algebraic multiplicity 2.

2. *Diagonalizability.* Solve $(E - 4I)\mathbf{x} = \mathbf{0}$ with $(E - 4I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This gives $y = 0$ and x free, so the eigenspace is $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, of dimension 1. Since the geometric multiplicity (1) is < 2, E is *not* diagonalizable.

3. *Explanation.* A 2×2 matrix is diagonalizable iff there are 2 linearly independent eigenvectors. Here there is only one (up to scale), so no basis of eigenvectors exists.

Solution to Q46. [Back to Question p. 37] [Back to TOC]

Predator–prey model with matrices.

$$M = \begin{pmatrix} 1.1 & -0.4 \\ 0.3 & 0.8 \end{pmatrix}.$$

1. *Eigenvalues/eigenvectors.*

$$\chi_M(\lambda) = \det(M - \lambda I) = (1.1 - \lambda)(0.8 - \lambda) + 0.12 = \lambda^2 - 1.9\lambda + 1.$$

Discriminant $= 1.9^2 - 4 = -0.39 < 0$, so

$$\lambda_{1,2} = 0.95 \pm 0.312\mathrm{i} \quad \big(|\lambda| = \sqrt{\det M} = 1\big).$$

An eigenvector for λ_1 may be taken as

$$\mathbf{z} = \begin{pmatrix} 1 \\ \dfrac{1.1 - \lambda_1}{0.4} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.375 - 0.78\,\mathrm{i} \end{pmatrix}.$$

2. *Expression for (P_n, Q_n).* Over $\mathbb{C}$, $M = PDP^{-1}$ with $D = \operatorname{diag}(\lambda_1, \lambda_2)$ and $P = [\mathbf{z}\ \ \overline{\mathbf{z}}]$, so

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = M^n \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = PD^nP^{-1} \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix} = \Re\!\big(\alpha\, \lambda_1^n \mathbf{z}\big),$$

for a complex constant α determined by (P_0, Q_0). Equivalently (real form), since $\lambda_{1,2} = e^{\pm i\theta}$ with $\cos\theta = \frac{\text{tr}M}{2} = 0.95$, there exists real S with $M = SR_\theta S^{-1}$, $R_\theta = \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)$, hence

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = SR_\theta^n S^{-1} \begin{pmatrix} P_0 \\ Q_0 \end{pmatrix}.$$

3. *Long–term behaviour.* Since $|\lambda_{1,2}| = 1$, the linear model predicts *bounded oscillations of constant amplitude* (quasi-periodic with angle $\theta = \arccos(0.95) \approx 0.318$ rad).

Solution to Q47. [Back to Question p. 37] [Back to TOC]

Matrix powers in a recurrence. For

$$F = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}:$$

1. *Diagonalize and find F^n.*

$$\chi_F(\lambda) = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-3)(\lambda-1).$$

Eigenvectors: for $\lambda = 3$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; for $\lambda = 1$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Let

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then

$$F^n = PD^nP^{-1} = \frac{1}{2}\begin{pmatrix} 3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1 \end{pmatrix}.$$

2. *Compute F^{20}.* Since $3^{20} = 3{,}486{,}784{,}401$,

$$F^{20} = \begin{pmatrix} \dfrac{3^{20}+1}{2} & \dfrac{3^{20}-1}{2} \\ \dfrac{3^{20}-1}{2} & \dfrac{3^{20}+1}{2} \end{pmatrix} = \begin{pmatrix} 1{,}743{,}392{,}201 & 1{,}743{,}392{,}200 \\ 1{,}743{,}392{,}200 & 1{,}743{,}392{,}201 \end{pmatrix}.$$

3. *Pattern.* F^n has equal diagonal entries and equal off–diagonal entries, with closed form $\frac{1}{2}\left(\begin{smallmatrix}3^n+1 & 3^n-1 \\ 3^n-1 & 3^n+1\end{smallmatrix}\right)$.

Solution to Q48. [Back to Question p. 37] [Back to TOC]

Eigen-decomposition in transformations. For

$$G = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}:$$

1. *Eigenvalues.* $\chi_G(\lambda) = \det(G - \lambda I) = \lambda^2 + 1$, so

$$\lambda_{1,2} = \pm i.$$

2. *Diagonalization over $\mathbb{R}$.* The eigenvalues are non-real; hence there are no real eigenvectors and G cannot be diagonalized over $\mathbb{R}$ (it can over $\mathbb{C}$).

3. *Geometric interpretation.* G is rotation by $90°$ counterclockwise. Repeated application rotates a vector by $90°$ each time: $G^0 = I$, G^1 rotates by $90°$, $G^2 = -I$ ($180°$), $G^3 = -G$ ($270°$), $G^4 = I$, and so on. Norms are preserved.

Topic 2 Solutions

SL 2.1 Straight Lines

Solution to Q49. [Back to Question p. 40] [Back to TOC]

Gradient between $A(2,-1)$ and $B(8,5)$:

$$m = \frac{5-(-1)}{8-2} = \frac{6}{6} = 1$$

Convert $3x - 2y = 12$ to slope–intercept form:

$$3x - 2y = 12 \quad \Rightarrow \quad -2y = -3x + 12$$

$$y = \frac{3}{2}x - 6$$

Gradient: $m = \frac{3}{2}$ **y-intercept:** $(0,-6)$ **x-intercept:** set $y = 0$:

$$0 = \frac{3}{2}x - 6 \quad \Rightarrow \quad x = 4 \quad \Rightarrow \quad (4,0)$$

Final Answer: Gradient between A and B: 1 Slope–intercept form: $y = \frac{3}{2}x - 6$ Intercepts: x-int $(4,0)$, y-int $(0,-6)$

Solution to Q50. [Back to Question p. 40] [Back to TOC]

Q43:

Gradient between $C(-3,2)$ and $D(5,-6)$:

$$m = \frac{-6-2}{5-(-3)} = \frac{-8}{8} = -1$$

(i) Point–slope form: Using point $C(-3,2)$:

$$y - 2 = -1(x+3)$$

(ii) General form $ax + by + d = 0$: From $y - 2 = -x - 3$:

$$x + y + 1 = 0$$

Final Answer: (i) $y - 2 = -1(x+3)$ (ii) $x + y + 1 = 0$

Solution to Q51. [Back to Question p. 40] [Back to TOC]

Q44. Given $L : y = 4x - 7$.

(a) A line parallel to L has the same gradient $m = 4$ and passes through $(2,1)$:

$$y - 1 = 4(x-2) \;\Rightarrow\; y = 4x - 7.$$

(So the required line coincides with L.)

(b) A line perpendicular to L has gradient $m_{\perp} = -\frac{1}{4}$ and passes through $(2,1)$:

$$y - 1 = -\frac{1}{4}(x-2) \;\Rightarrow\; y = -\frac{1}{4}x + \frac{3}{2}.$$

Final Answer: $\boxed{y = 4x - 7}$ (parallel through $(2,1)$); $\boxed{y = -\frac{1}{4}x + \frac{3}{2}}$ (perpendicular through $(2,1)$).

Solution to Q52. [Back to Question p. 40] [Back to TOC]

Q45. Solve the system

$$\begin{cases} 3x + y = 10, \\ 2x - 3y = 1. \end{cases}$$

From the first equation $y = 10 - 3x$. Substitute into the second:

$$2x - 3(10 - 3x) = 1 \;\Rightarrow\; 2x - 30 + 9x = 1 \;\Rightarrow\; 11x = 31 \;\Rightarrow\; x = \frac{31}{11}.$$

Then

$$y = 10 - 3\left(\frac{31}{11}\right) = \frac{110 - 93}{11} = \frac{17}{11}.$$

Final Answer: $\boxed{\left(\frac{31}{11}, \frac{17}{11}\right)}$.

Solution to Q53. [Back to Question p. 41] [Back to TOC]

(i) The slope is

$$m = \frac{420 - 120}{15 - 0} = \frac{300}{15} = 20 \text{ m/km}.$$

Since $h(0) = 120$, the linear model is

$$h(x) = 20x + 120.$$

(ii) Estimate $h(8)$:

$$h(8) = 20(8) + 120 = 160 + 120 = 280 \text{ m}.$$

(iii) Solve $h(x) = 300$:

$$20x + 120 = 300 \quad \Rightarrow \quad 20x = 180 \quad \Rightarrow \quad x = 9 \text{ km}.$$

Final Answer: (i) $\boxed{h(x) = 20x + 120}$ (ii) $\boxed{h(8) = 280 \text{ m}}$ (iii) $\boxed{x = 9 \text{ km}}$

SL 2.2 Relations and Functions

Solution to Q54. [Back to Question p. 42] [Back to TOC]

(i) $y = \sqrt{x}$ is *not* a function $\mathbb{R} \to \mathbb{R}$ (undefined for $x < 0$); with domain restricted to $[0, \infty)$ it is a function.
(ii) $x = y^2$ is not a function $y = y(x)$ since most $x > 0$ correspond to two y–values.

Solution to Q55. [Back to Question p. 42] [Back to TOC]

$f^{-1}(x) = \dfrac{x+5}{2}$. Check: $f(f^{-1}(x)) = 2 \cdot \dfrac{x+5}{2} - 5 = x$.

Solution to Q56. [Back to Question p. 42] [Back to TOC]

$g(x) = \sqrt{9 - x^2}$: domain $[-3, 3]$, range $[0, 3]$. $h(x) = \dfrac{3}{x-2}$: domain $\mathbb{R} \setminus \{2\}$, range $\mathbb{R} \setminus \{0\}$. $h^{-1}(x) = \dfrac{x+3}{2-x}$ with domain $\mathbb{R} \setminus \{0\}$ and range $\mathbb{R} \setminus \{2\}$.

Solution to Q57. [Back to Question p. 42] [Back to TOC]

$p(x) = x^2 - 6x + 8 = (x-3)^2 - 1$ is not one-to-one on $\mathbb{R}$. On $[3, \infty)$, $p^{-1}(x) = 3 + \sqrt{x+1}$ (domain $x \geq -1$). On $(-\infty, 3]$, $p^{-1}(x) = 3 - \sqrt{x+1}$.

Solution to Q58. [Back to Question p. 44] [Back to TOC]

(i) Rational function; (ii) Exponential function; (iii) Absolute–value (piecewise linear) function.

SL 2.3 Properties and Families of Functions

Solution to Q59. [Back to Question p. 44] [Back to TOC]

$g(x) = x^2 - 4x + 1 = (x-2)^2 - 3$. Vertex $(2, -3)$; axis $x = 2$. x–intercepts $2 \pm \sqrt{3}$; y–intercept $(0, 1)$; range $y \geq -3$.

Solution to Q60. [Back to Question p. 44] [Back to TOC]

Domain $x > 1$. No y–intercept. x–intercept from $3\ln(x-1) - 2 = 0 \Rightarrow x = 1 + e^{2/3}$. Vertical asymptote $x = 1$. No horizontal asymptote (logarithmic growth).

Solution to Q61. [Back to Question p. 44] [Back to TOC]

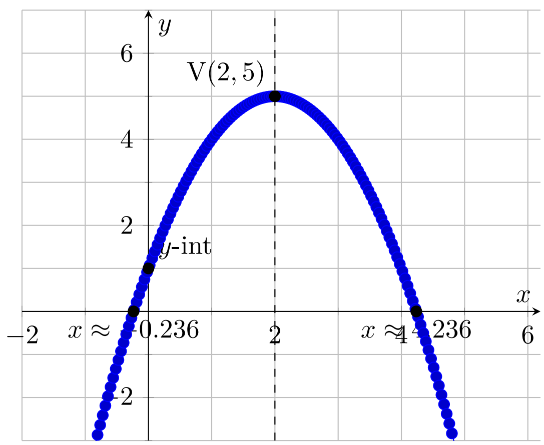
For $p(x) = 2\cos x - 1$: amplitude 2, period 2π, range $[-3, 1]$. Zeros when $\cos x = \frac{1}{2}$, i.e. $x = \frac{\pi}{3}$, $\frac{5\pi}{3}$ in $[0, 2\pi]$.

SL 2.4 Key properties of graphs, curve sketching and points of intersection

Solution to Q62. [Back to Question p. 46] [Back to TOC]

Given $f(x) = -x^2 + 4x + 1$.

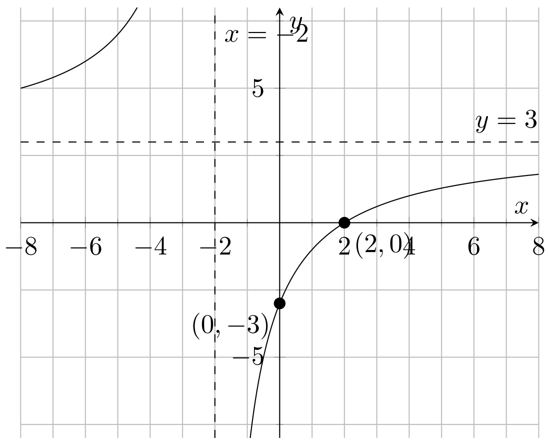
1. **Axis of symmetry & vertex.** $x = \dfrac{-b}{2a} = \dfrac{-4}{2(-1)} = 2.$ $f(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5.$ Vertex $(2, 5)$; axis $x = 2$.
2. **Intercepts.** y–intercept: $f(0) = 1 \Rightarrow (0, 1)$. x–intercepts: solve $-x^2+4x+1 = 0 \iff x^2-4x-1 = 0$,
$$x = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5} \approx -0.236,\ 4.236.$$
3. **Extremum.** Because $a = -1 < 0$, the parabola opens downward. Maximum value 5 occurs at $x = 2$.
4. **Asymptotes, parity.** No vertical or horizontal asymptotes for a quadratic. $f(-x) = -x^2 - 4x + 1 \neq f(x)$ and $\neq -f(x)$, so f is *neither* even nor odd.



Solution to Q63. [Back to Question p. 46] [Back to TOC]

Given $g(x) = \dfrac{3x-6}{x+2}$.

1. **Domain.** Denominator $\neq 0 \Rightarrow x \neq -2$. Domain $= \mathbb{R} \setminus \{-2\}$.
2. **Intercepts.** x–intercept: $3x - 6 = 0 \Rightarrow x = 2 \Rightarrow (2, 0)$. y–intercept: $g(0) = \frac{-6}{2} = -3 \Rightarrow (0, -3)$.
3. **Asymptotes.** Vertical: $x = -2$. Degrees equal $\Rightarrow$ horizontal $y = \dfrac{3}{1} = 3$.
4. **Holes & symmetry.** $3x - 6 = 3(x - 2)$ shares no factor with $(x + 2)$, so *no holes*. Graph has no even/odd symmetry.



Solution to Q64. [Back to Question p. 46] [Back to TOC]

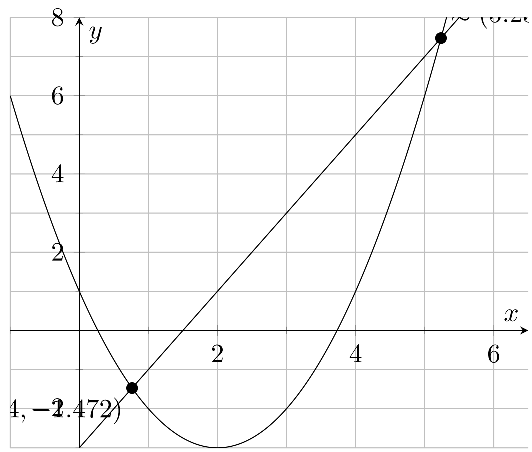
Intersections of $y = x^2 - 4x + 1$ **and** $y = 2x - 3$.

Set $x^2 - 4x + 1 = 2x - 3 \Rightarrow x^2 - 6x + 4 = 0$.

$$x = \frac{6 \pm \sqrt{36 - 16}}{2} = 3 \pm \sqrt{5}.$$

Then $y = 2x - 3 = 2(3 \pm \sqrt{5}) - 3 = 3 \pm 2\sqrt{5}$.

$$(3 - \sqrt{5},\ 3 - 2\sqrt{5}) \approx (0.764,\ -1.472), \qquad (3 + \sqrt{5},\ 3 + 2\sqrt{5}) \approx (5.236,\ 7.472).$$



Solution to Q65. [Back to Question p. 47] [Back to TOC]

Intersections of $y = 3^x$ **and** $y = x + 2$.

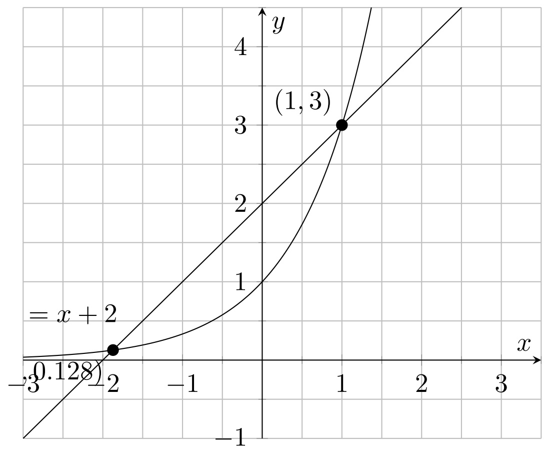
Solve $3^x = x + 2$. One exact solution is $x = 1$ (since $3^1 = 3 = 1 + 2$). Using technology/Newton's method gives a second solution

$$x \approx -1.872130575, \qquad y = x + 2 \approx 0.127869425.$$

Thus the intersection points are

$$(-1.872,\ 0.128) \quad \text{and} \quad (1,\ 3) \quad \text{(to 3 d.p.)}.$$

Verification: For $x = -1.872$, $3^x \approx 0.127869 \approx x + 2$; for $x = 1$, $3^x = 3 = x + 2$.



y
4
$(1, 3)$
3
2
1
$= x + 2$
x
$0.128)$
-3
-2
-1
1
2
3
-1

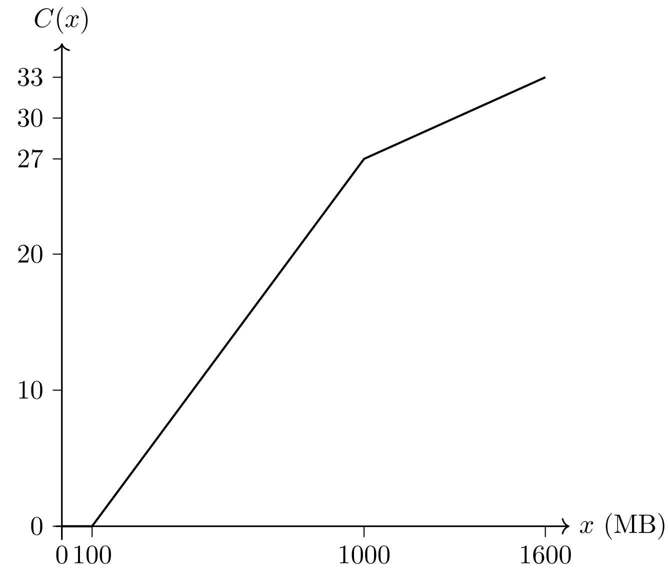
SL 2.5 Modelling linear, quadratics, exponential, cubic, sinusoidal and direct/inverse proportion

Solution to Q66. [Back to Question p. 48] [Back to TOC]

Two points $(0, 23)$ and $(6, 17)$ give slope $m = \frac{17-23}{6} = -1$; hence $T(t) = -t + 23$. At $t = 3$, $T(3) = 20°\text{C}$.

Solution to Q67. [Back to Question p. 48] [Back to TOC]

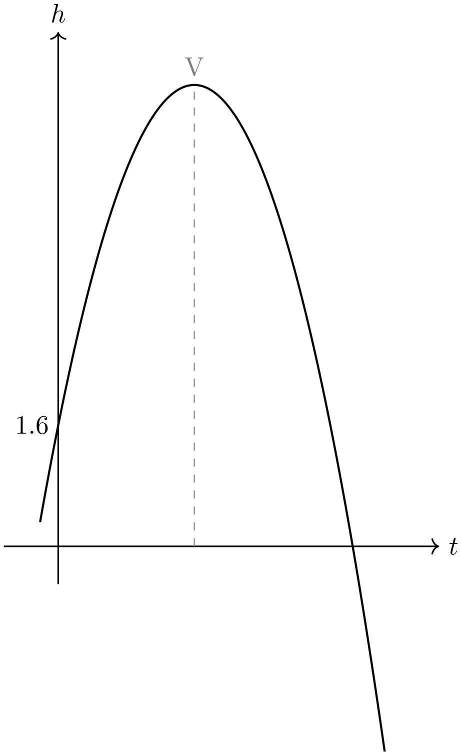
$$C(x) = \begin{cases} 0, & 0 \le x \le 100, \\ 0.03(x - 100), & 100 < x \le 1000, \\ 27 + 0.01(x - 1000), & x > 1000. \end{cases}$$



$C(750) = \$19.50$, $C(1400) = \$31$.

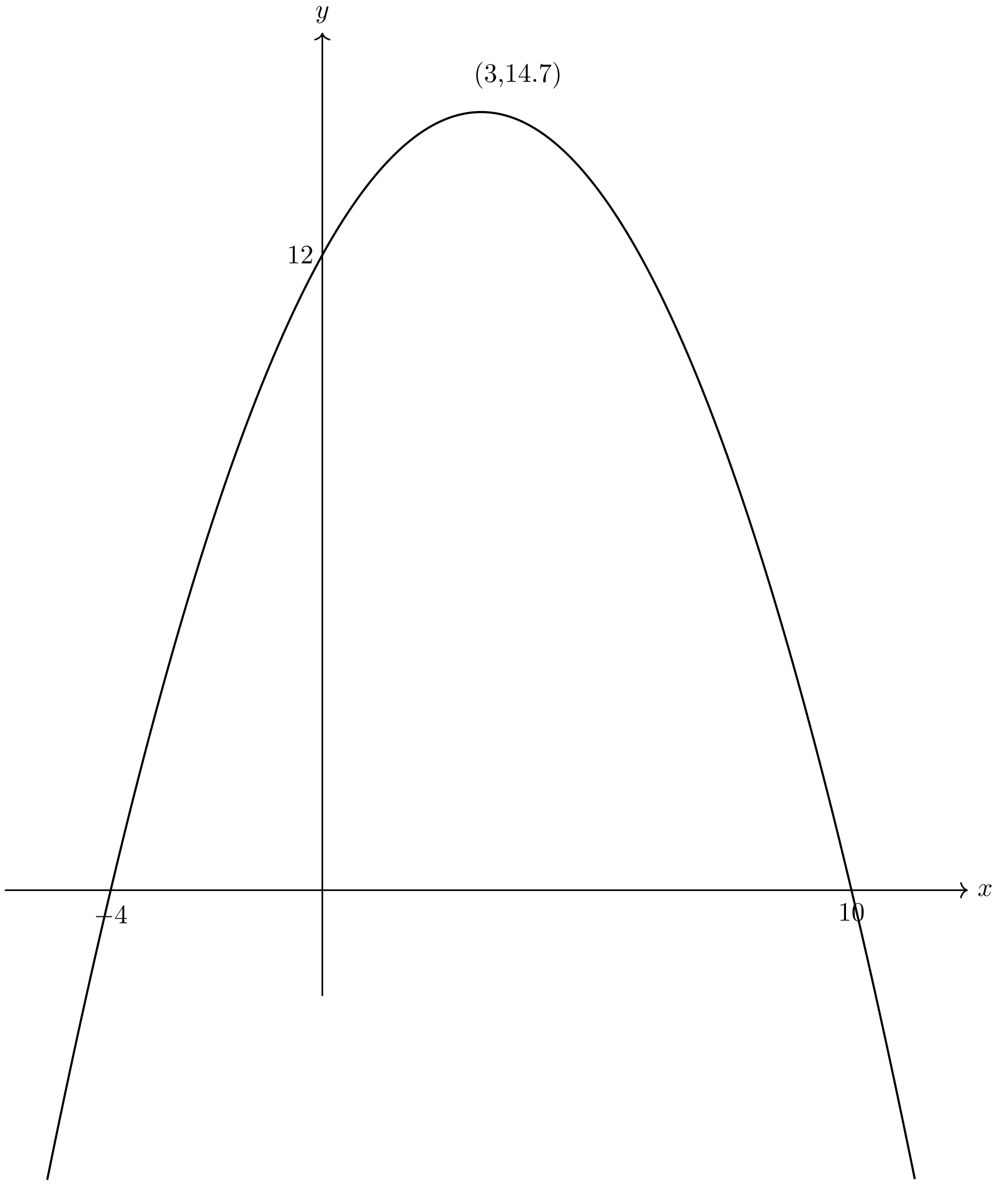
Solution to Q68. [Back to Question p. 48] [Back to TOC]

Vertex form $h(t) = a(t-1.5)^2+6.1$. Using $h(0) = 1.6$ gives $a = -2$, so $h(t) = -2(t-1.5)^2+6.1 = -2t^2+6t+1.6$. Axis $t = 1.5$; intercepts at $t \approx -0.246$, 3.246.



Solution to Q69. [Back to Question p. 49] [Back to TOC]

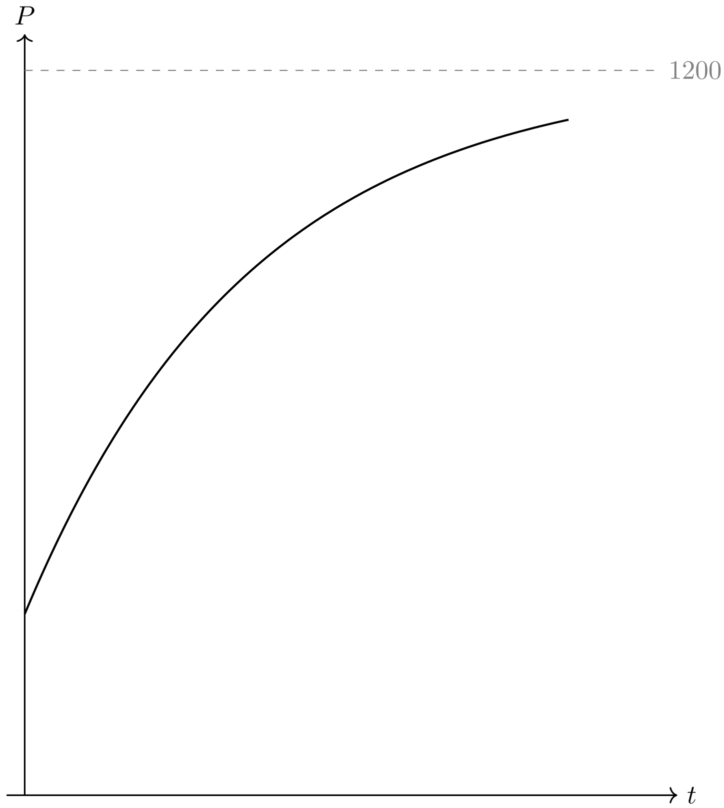
Zeros -4 and 10 plus $y(0) = 12$ give $y = -\frac{3}{10}(x+4)(x-10) = -0.3x^2 + 1.8x + 12$. Axis $x = 3$, vertex



$(3, 14.7)$.

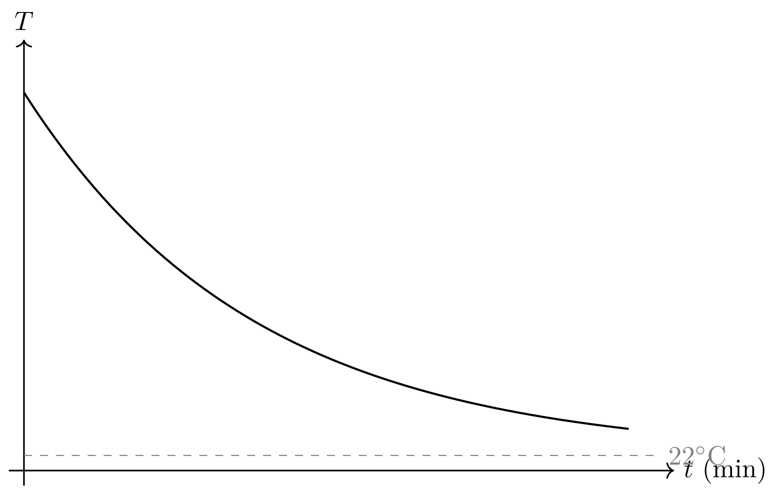
Solution to Q70. [Back to Question p. 49] [Back to TOC]

$P(t) = 1200 - 900e^{-0.4t}$. Asymptote $P = 1200$. Solve $P = 900 \Rightarrow t = \dfrac{\ln 3}{0.4} \approx 2.747$ years.



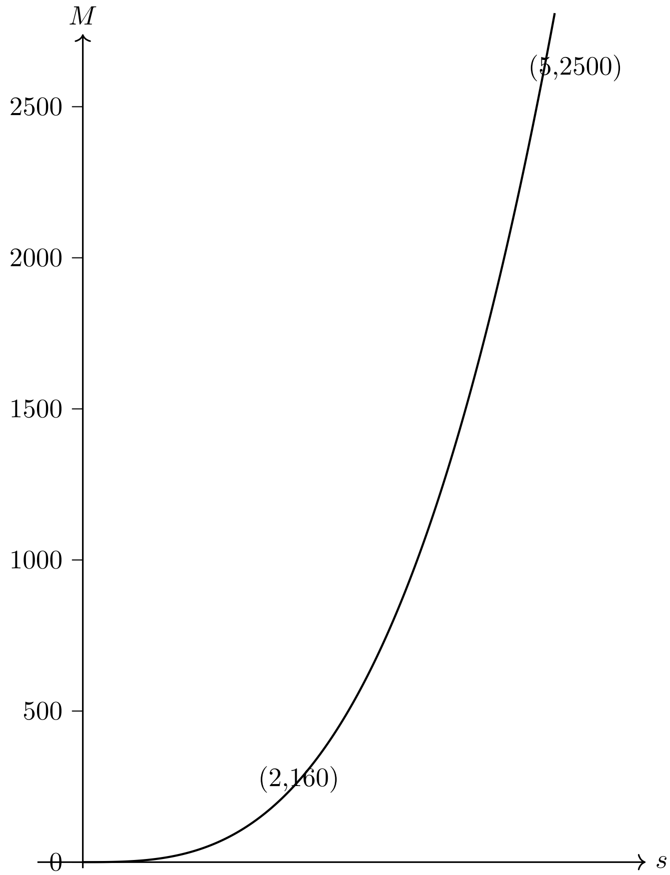
Solution to Q71. [Back to Question p. 49] [Back to TOC]

$T(t) = 22 + 48a^{-t}$ with $a = (8/3)^{1/30} \approx 1.0332$. Time to 30°C: $t = \dfrac{\ln 6}{\ln a} \approx 54.8$ min. Asymptote $T = 22$°C.



Solution to Q72. [Back to Question p. 50] [Back to TOC]

$M = 20s^3$. With $s = 2 \Rightarrow M = 160$; with $s = 5 \Rightarrow M = 2500$ g.



M
2500
2000
1500
1000
500
0
s
(5,2500)
(2,160)

SL 2.6 Modelling skills, using, choosing and context

Solution to Q73. [Back to Question p. 51] [Back to TOC]

A constant draining rate suggests a *linear* model $h(t) = mt + c$ for depth (m) t hours after 9:00.

Using $(t, h) = (0, 2.4)$ and $(1.5, 1.5)$,

$$m = \frac{1.5 - 2.4}{1.5} = -0.6, \qquad c = 2.4,$$

so

$$\boxed{h(t) = 2.4 - 0.6t}.$$

A reasonable domain is $0 \le t \le 4$ (from the reading start until empty), since negative time is impossible and the model is only valid while water remains.

Empty when $h(t) = 0 \Rightarrow 2.4 - 0.6t = 0 \Rightarrow t = \dfrac{2.4}{0.6} = 4$ hours, i.e. at **1:00 pm**. This is *extrapolation* beyond the observed interval $0 \le t \le 1.5$.

Solution to Q74. [Back to Question p. 51] [Back to TOC]

Vertex form $h(t) = a(t - 1.5)^2 + 6.1$. Using $h(0) = 1.6$,

$$1.6 = a(2.25) + 6.1 \quad \Rightarrow \quad a = -2.$$

Hence

$$\boxed{h(t) = -2(t - 1.5)^2 + 6.1 = -2t^2 + 6t + 1.6}.$$

Ground hit when $h(t) = 0$:

$$-2(t - 1.5)^2 + 6.1 = 0 \;\Rightarrow\; (t - 1.5)^2 = 3.05 \;\Rightarrow\; t = 1.5 \pm \sqrt{3.05}.$$

The physical root is $t \approx 1.5 + 1.746 = \boxed{3.246 \text{ s}}$.

Suitable domain: $0 \le t \le 3.246$ (launch to landing). Limitation: ignores air resistance and assumes level ground and constant acceleration.

Solution to Q75. [Back to Question p. 52] [Back to TOC]

Take $P(t) = L - Ae^{-kt}$ with limiting value $L = 1200$. At $t = 0$, $P(0) = 300 \Rightarrow A = L - 300 = 900$. At $t = 3$, $900 = 1200 - 900e^{-3k} \Rightarrow e^{-3k} = 1/3 \Rightarrow \boxed{k = \dfrac{\ln 3}{3} \approx 0.3662}$.

Thus

$$\boxed{P(t) = 1200 - 900e^{-0.3662\,t}}.$$

Horizontal asymptote $P = 1200$: long–term population size.

$P(5) = 1200 - 900e^{-0.3662\cdot 5} \approx 1200 - 900(0.1605) \approx \boxed{1056}$. Long-term forecasts can be unreliable if growth parameters change (resources, environment) or if the model form ceases to hold.

Solution to Q76. [Back to Question p. 52] [Back to TOC]

$M = ks^3$. Given $160 = k(2)^3 = 8k \Rightarrow k = 20$, so

$$\boxed{M(s) = 20s^3 \text{ g}}.$$

At $s = 5$, $M = 20(125) = \boxed{2500 \text{ g}} = 2.5$ kg. Domain $s > 0$. Assumes constant density (same material) and perfect cube.

Solution to Q77. [Back to Question p. 52] [Back to TOC]

$I(x) = k/x^2$ and $I(2) = 900 \Rightarrow 900 = k/4 \Rightarrow \boxed{k = 3600}$, so $\boxed{I(x) = \dfrac{3600}{x^2}}$. For $I = 100$, $100 = 3600/x^2 \Rightarrow x^2 = 36 \Rightarrow \boxed{x = 6 \text{ m}}$ (distance > 0). Vertical asymptote at $x = 0$; realistic domain $x > 0$.

Solution to Q78. [Back to Question p. 53] [Back to TOC]

Let $f(x) = a(x+2)(x-1)(x-4)$. Since $f(0) = -8$,

$$-8 = a(2)(-1)(-4) = 8a \Rightarrow a = -1.$$

Thus

$$\boxed{f(x) = -(x+2)(x-1)(x-4)}.$$

End behaviour: leading term $-x^3 \Rightarrow f(x) \to -\infty$ as $x \to +\infty$ and $f(x) \to +\infty$ as $x \to -\infty$. Estimate $f(3) = -(5)(2)(-1) = \boxed{10}$, reasonable as it lies between the roots $x = 1$ and $x = 4$ with the correct sign.

Solution to Q79. [Back to Question p. 53] [Back to TOC]

$$\boxed{C(x) = \begin{cases} 0, & 0 \le x \le 100, \\ 0.03(x - 100), & 100 < x \le 1000, \\ 0.03 \cdot 900 + 0.01(x - 1000), & x > 1000. \end{cases}}$$

Hence $C(750) = 0.03(650) = \boxed{\$19.50}$, and $C(1400) = 0.03(900) + 0.01(400) = 27 + 4 = \boxed{\$31.00}$. Domain $x \ge 0$. Kinks at $x = 100$ and $x = 1000$ where the rate (slope) changes, so $C'(x)$ is discontinuous there.

Solution to Q80. [Back to Question p. 54] [Back to TOC]

Amplitude $a = \frac{15-9}{2} = 3$, midline $d = \frac{15+9}{2} = 12$, period $T = 365 \Rightarrow b = 2\pi/365$, maximum at $t = 172$ suggests a cosine shift $c = 172$:

$$\boxed{H(t) = 3\cos\Big(\tfrac{2\pi}{365}(t - 172)\Big) + 12}.$$

$H(20) \approx 3\cos\big(2\pi \cdot (-152/365)\big) + 12 \approx 3(-0.8660) + 12 = \boxed{9.40 \text{ h}}$,

$$H(250) \approx 3\cos\big(2\pi \cdot (78/365)\big) + 12 \approx 3(0.224) + 12 = \boxed{12.67 \text{ h}}.$$

Over many years parameters (amplitude/phase) drift, so using the same model without recalibration can mislead.

Solution to Q81. [Back to Question p. 54] [Back to TOC]

The scatter is increasing and nearly straight; a *linear* model is appropriate.

Least-squares fit (to 3 s.f.):

$$\boxed{\hat{y} = 0.590\,x - 0.0933}, \qquad R^2 \approx \boxed{0.993}.$$

(For comparison: power and exponential fits give smaller R^2.) Extrapolating to $x = 15$ gives $\hat{y} \approx 0.590(15) - 0.093 \approx 8.65$. This is far outside the observed range ($x \le 10$), so caution is required.

Solution to Q82. [Back to Question p. 54] [Back to TOC]

Given $L(a) = 10.5 - 0.35a$,

$$L(8) = 10.5 - 0.35(8) = \boxed{7.7\text{ h}}, \qquad L(40) = 10.5 - 0.35(40) = \boxed{-3.5\text{ h (nonsense)}}.$$

For many apps, battery life cannot be negative; the linear model breaks down. Either restrict the domain (e.g. $a \leq 25$), or choose a model that levels off near 0 (e.g. exponential decay $L(a) = \alpha + \beta e^{-\gamma a}$ with $\alpha \approx 0$). Additional useful data: measurements across a wider range of a and possibly phone/usage covariates.

Solution to Q83. [Back to Question p. 55] [Back to TOC]

$d(t) = at^2 + bt + c$, $d(0) = 0 \Rightarrow c = 0$. Also $d(10) = 0.3 \Rightarrow 100a + 10b = 0.3$. Rate $d'(t) = 2at + b$ and $d'(10) = 0.06 \Rightarrow 20a + b = 0.06$. Solve:

$$b = 0.06 - 20a, \quad 100a + 10(0.06 - 20a) = 0.3 \Rightarrow -100a + 0.6 = 0.3 \Rightarrow a = 0.003,\ b = 0.$$

Thus

$$\boxed{d(t) = 0.003\,t^2}.$$

$d'(t) = 0.006t > 0$ for $t > 0$, so increasing for $t > 0$. Relevant domain: $t \geq 0$ until the pool is full. $d(25) = 0.003(625) = \boxed{1.875\text{ m}}$ (an extrapolation beyond $t = 10$).

Solution to Q84. [Back to Question p. 55] [Back to TOC]

Using least squares:

- Linear: $\boxed{\hat{V} = 134\,t + 106}$ with $R^2 \approx \boxed{0.997}$.
- Exponential: $\hat{V} = ae^{kt}$ (fit gives, e.g., $a \approx 156.3$, $k \approx 0.332$) with $R^2 \approx \boxed{0.815}$.

The linear model fits substantially better (larger R^2, smaller residuals). Using the linear model, $1500 = 134t + 106 \Rightarrow t = \dfrac{1394}{134} \approx \boxed{10.4\text{ h}}$. This is well beyond the observed range ($t \leq 6$), so the prediction is an extrapolation and should be treated cautiously.

AHL 2.7 Composite and Inverse Functions

Solution to Q85. [Back to Question p. 57] [Back to TOC]

For $f(x) = \dfrac{2x-3}{x+1}$, solve $y = \dfrac{2x-3}{x+1}$: $x = \dfrac{y+3}{2-y}$, hence $f^{-1}(x) = \dfrac{x+3}{2-x}$. Domain of f: $x \neq -1$; range of f: $y \neq 2$. Domain of f^{-1}: $x \neq 0$; range of f^{-1}: $y \neq -1$.

Solution to Q86. [Back to Question p. 57] [Back to TOC]

$h(x) = x^2 + 4x + 7 = (x+2)^2 + 3$. On $x \geq -2$, h is one-to-one and $h^{-1}(x) = -2 + \sqrt{x-3}$ (domain $x \geq 3$). On $x \leq -2$, $h^{-1}(x) = -2 - \sqrt{x-3}$.

Solution to Q87. [Back to Question p. 58] [Back to TOC]

(i) $y = f(x) + 3$ shifts up by 3. (ii) $y = f(x-2)$ shifts right by 2. (iii) $y = -f(x)$ reflects in the x–axis. (iv) $y = f(2x)$ compresses horizontally by factor $\frac{1}{2}$.

SL 2.8 Transformation of Graphs

Solution to Q88. [Back to Question p. 58] [Back to TOC]

Start $y = \sqrt{x}$. After shift right 3: $y = \sqrt{x-3}$. Reflect in x–axis: $y = -\sqrt{x-3}$. Vertical stretch by 2: $y = -2\sqrt{x-3}$.

Solution to Q89. [Back to Question p. 58] [Back to TOC]

Let $f(x) = |x|$.

(i) *Shift left 4 and up 2.* A horizontal shift left by 4 replaces x with $x+4$; a vertical shift up by 2 adds $+2$:

$$y = |x+4| + 2.$$

(ii) *Reflect in the y-axis, then apply a vertical stretch by factor 3.* Reflection in the y-axis sends $x \mapsto -x$: $y = |-x| = |x|$ (no change since $|x|$ is even). A vertical stretch by factor 3 multiplies the output by 3:

$$y = 3|x|.$$

Final Answer: $\boxed{y = |x+4| + 2}$ and $\boxed{y = 3|x|}$.

Solution to Q90. [Back to Question p. 58] [Back to TOC]

A horizontal shift and a horizontal stretch do *not* commute.

Write a right shift by a as $x \mapsto x - a$ (so $y = f(x-a)$), and a horizontal stretch about the y-axis by factor $k > 0$ as $x \mapsto x/k$ (so $y = f(x/k)$).

Shift then stretch:

$$y = f(x-a) \xrightarrow{\text{stretch } k} y = f\Big(\frac{x}{k} - a\Big).$$

Stretch then shift:

$$y = f\Big(\frac{x}{k}\Big) \xrightarrow{\text{shift } a} y = f\Big(\frac{x-a}{k}\Big) = f\Big(\frac{x}{k} - \frac{a}{k}\Big).$$

Since $\frac{x}{k} - a \neq \frac{x}{k} - \frac{a}{k}$ in general (unless $a = 0$ or $k = 1$), the two results differ.

Concrete example: take $f(x) = x^2$, $a = 2$, $k = 3$.

$$\text{Shift then stretch: } y = \Big(\frac{x}{3} - 2\Big)^2 \quad (\text{vertex at } x = 6).$$

$$\text{Stretch then shift: } y = \Big(\frac{x-2}{3}\Big)^2 \quad (\text{vertex at } x = 2).$$

The graphs are different, so the operations do not commute.

Final Answer: $\boxed{\text{In general } f(\frac{x}{k} - a) \neq f(\frac{x-a}{k})}$, e.g. with $f(x) = x^2, a = 2, k = 3$.

Solution to Q91. [Back to Question p. 59] [Back to TOC]

From $y = x^2$ to $y = 3(x-2)^2 - 5$:

$$\begin{aligned}
&\text{Shift right by } 2: \quad y = (x-2)^2;\\
&\text{Vertical stretch by } 3: \quad y = 3(x-2)^2;\\
&\text{Shift down by } 5: \quad y = 3(x-2)^2 - 5.
\end{aligned}$$

Swapping the order of "shift right by 2" and "vertical stretch by 3" here gives the same final equation because

$$3\big((x-2)^2\big) = (\ 3x^2 - 12x + 12\ ) = 3x^2 - 12x + 12,$$

and shifting/skewing in y commutes with vertical scaling for base x^2. However, order *does* matter in general for horizontal operations embedded inside $f(2x-4)$:

$$f(2x-4) = f\big(2(x-2)\big)$$

corresponds to first a horizontal compression by factor $\frac{1}{2}$ (replace x by $2x$), then a shift right by 2. If you shift first and then compress, the image of a given x is different (e.g., test with $f(x) = x^2$ and compare images of $x = 0$).

AHL 2.8 Transformations and order

Solution to Q92. [Back to Question p. 60] [Back to TOC]

Exponential decay with half-life 12 h and $N_0 = 500$:

$$N(t) = N_0\left(\tfrac{1}{2}\right)^{t/12} = 500\left(\tfrac{1}{2}\right)^{t/12}.$$

At $t = 30$ h,

$$N(30) = 500\left(\tfrac{1}{2}\right)^{30/12} = 500\left(\tfrac{1}{2}\right)^{5/2} = \frac{500}{4\sqrt{2}} = \frac{125}{\sqrt{2}} \approx 88.4.$$

Final Answer: $\boxed{N(t) = 500\left(\tfrac{1}{2}\right)^{t/12}}$, $\boxed{N(30) \approx 88.4}$.

SL 2.9 Additional Modelling Families

Solution to Q93. [Back to Question p. 60] [Back to TOC]

Logistic model $P(t) = \dfrac{L}{1 + Ce^{-kt}}$ with $L = 120$. Using $(0, 20)$:

$$20 = \frac{120}{1 + C} \;\Rightarrow\; 1 + C = 6 \;\Rightarrow\; \boxed{C = 5}.$$

Using $(6, 60)$:

$$60 = \frac{120}{1 + 5e^{-6k}} \;\Rightarrow\; 1 + 5e^{-6k} = 2 \Rightarrow\; 5e^{-6k} = 1 \Rightarrow e^{-6k} = \tfrac{1}{5} \Rightarrow \boxed{k = \tfrac{\ln 5}{6}}.$$

When $P = L/2$ we have $1 + Ce^{-kt} = 2 \Rightarrow e^{-kt} = \frac{1}{C}$, hence

$$t = \frac{\ln C}{k} = \frac{\ln 5}{\ln 5/6} = \boxed{6 \text{ (hours)}}.$$

Final Answer: $\boxed{C = 5,\ k = \dfrac{\ln 5}{6}}$, $P = L/2$ at $\boxed{t = 6}$.

Solution to Q94. [Back to Question p. 60] [Back to TOC]

Let $H(t) = a\sin(bt - c) + d$. Period $T = 12.4 = \frac{62}{5}$ h, so

$$b = \frac{2\pi}{T} = \frac{2\pi}{62/5} = \boxed{\tfrac{5\pi}{31}} \text{ rad/h}.$$

From max 5.8 m and min 0.6 m,

$$a = \frac{5.8 - 0.6}{2} = \boxed{2.6}, \qquad d = \frac{5.8 + 0.6}{2} = \boxed{3.2}.$$

A high tide occurs at $t = 3.1 = \frac{31}{10}$ h. For a maximum, $bt - c = \frac{\pi}{2} + 2\pi n$. With $b = \frac{5\pi}{31}$ and $t = \frac{31}{10}$,

$$bt = \frac{5\pi}{31} \cdot \frac{31}{10} = \frac{\pi}{2} \;\Rightarrow\; c = \boxed{0}.$$

Final Answer: $\boxed{a = 2.6,\ b = \tfrac{5\pi}{31},\ c = 0,\ d = 3.2}$, so

$$\boxed{H(t) = 2.6\,\sin\!\big(\tfrac{5\pi}{31}\,t\big) + 3.2}.$$

Solution to Q95. [Back to Question p. 60] [Back to TOC]

The function is

$$f(x) = \begin{cases} mx + 2, & x < 1, \\ x^2 + k, & x \geq 1. \end{cases}$$

For continuity at $x = 1$, the left-hand limit must equal the right-hand limit:

$$\lim_{x \to 1^-} f(x) = m(1) + 2 = m + 2,$$

$$\lim_{x \to 1^+} f(x) = (1)^2 + k = 1 + k.$$

Setting these equal:

$$m + 2 = 1 + k \quad \Rightarrow \quad m - k = -1.$$

Thus m and k must satisfy $\boxed{m - k = -1}$ for f to be continuous at $x = 1$.

Final Answer: Any (m, k) such that $m - k = -1$, e.g. $\boxed{m = 2,\ k = 3}$.

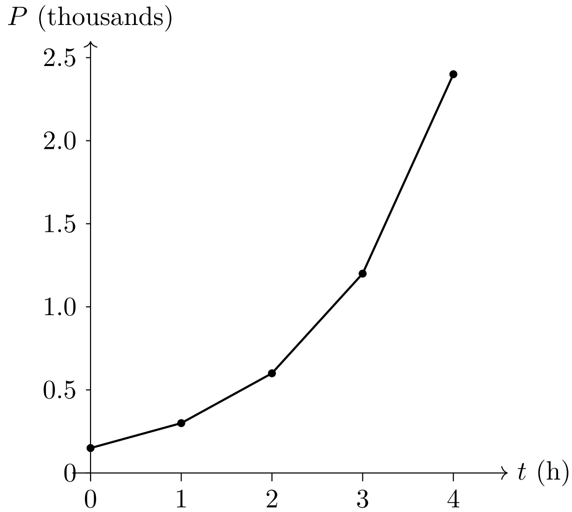
AHL 2.10 — Scaling large and small numbers and graphs)

Solution to Q96. [Back to Question p. 62] [Back to TOC]

Scaling large and small numbers using logarithms.

Given data: $P(t) = \{150, 300, 600, 1200, 2400\}$ for $t = 0, 1, 2, 3, 4$ hours.

(a) Linear (ordinary) plot of P vs t. The population doubles each hour. A linear-scale plot is steep and rapidly leaves the lower part of the y–axis unused. (A simple sketch is shown; y–axis is in thousands.)

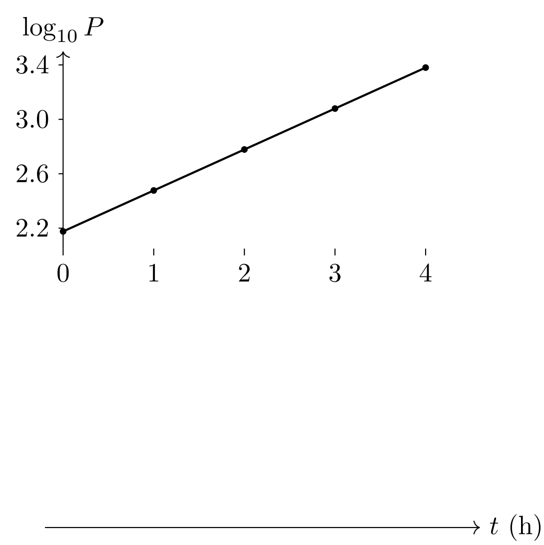


(b) Semi-log plot of $\log_{10} P$ vs t.

$$\log_{10} P = \log_{10}(150 \cdot 2^t) = \underbrace{\log_{10} 150}_{2.17609} + \underbrace{\log_{10} 2}_{0.30103}\, t,$$

so points lie on a straight line of slope 0.30103. Values (to 5 s.f.):

t	0	1	2	3	4
$\log_{10} P$	2.17609	2.47712	2.77815	3.07918	3.38021



Description: A straight line on the semi-log plot indicates exponential growth/decay. Here, slope 0.3010 means P multiplies by $10^{0.3010} = 2$ each hour (doubling).

(c) Why a log scale? The logarithmic y–scale compresses the wide range of P (hundreds to thousands), makes equal percentage changes appear equal distances, and reveals the exponential pattern as a straight line, aiding parameter estimation.

Solution to Q97. [Back to Question p. 62] [Back to TOC]

Linearizing exponential data.

(a) If $A(t) = A_0 e^{-kt}$, then

$$\ln A = \ln A_0 - kt,$$

which is linear in t with intercept $\ln A_0$ and gradient $-k$.

(b) Compute $\ln A$:

t (days)	0	2	4	6	8
A (counts/min)	850	623	456	333	243
$\ln A$	6.745	6.435	6.122	5.808	5.493

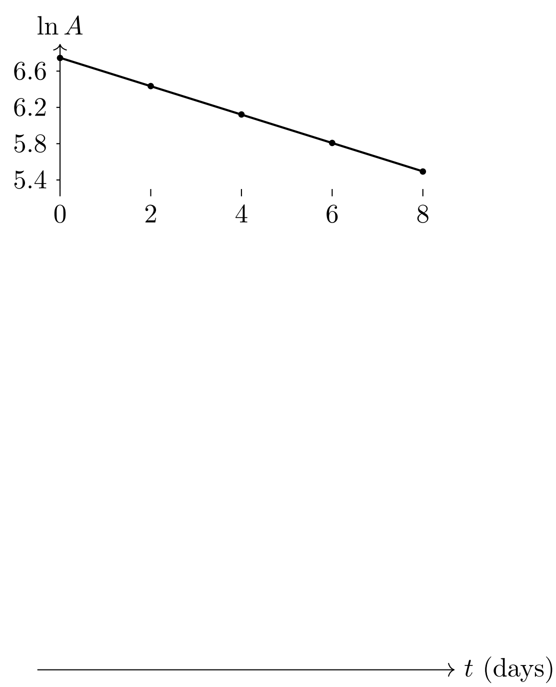
A straight-line fit to $(t, \ln A)$ gives (least squares):

$$\ln A \approx 6.7469 \; - \; 0.15654\, t,$$

so the decay constant is

$$k \approx 0.15654 \text{ day}^{-1}.$$

Hence $A_0 \approx e^{6.7469} \approx 850$, and the model is $A(t) \approx 850\, e^{-0.15654\, t}$. (Consistently, the half-life is $t_{1/2} = \ln 2/k \approx 0.6931/0.15654 \approx 4.43$ days.)



Solution to Q98. [Back to Question p. 63] [Back to TOC]

Linearizing power relationships.

(a) If $T = kL^n$ with $k > 0$, take common logarithms:

$$\log T = \log k + n \log L,$$

which is linear in $\log L$ with gradient n and intercept $\log k$.

(b) Using base-10 logs:

L (m)	0.25	0.50	0.75	1.00	1.25
T (s)	1.00	1.42	1.73	2.01	2.23
$\log L$	-0.60206	-0.30103	-0.12494	0	0.09691
$\log T$	0	0.15229	0.23805	0.30320	0.34830

Least-squares fit of $(x, y) = (\log L, \log T)$ yields

$$y \approx 0.30138 + 0.49946\, x.$$

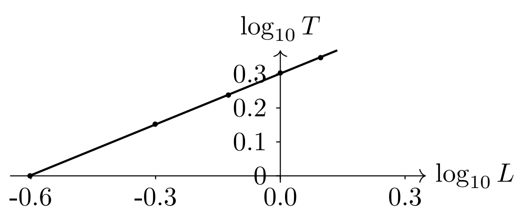
Therefore

$$n \approx 0.499 \approx \tfrac{1}{2}, \qquad k = 10^{0.30138} \approx 2.002.$$

So an appropriate model is

$$\boxed{T \approx 2.00\, L^{1/2}}$$

(in very close agreement with $T = 2\pi\sqrt{L/g}$ for $g \approx 9.8$ m s^{-2}).



Solution to Q99. [Back to Question p. 63] [Back to TOC]

Interpretation of semi-log and log-log graphs.

(a) Semi-log plot. The given straight line is

$$\log_{10} y = 3 - 0.5\, t.$$

Hence

$$y = 10^{3-0.5t} = 1000 \cdot 10^{-0.5t} = 1000\, e^{-(0.5 \ln 10)\, t} \approx 1000\, e^{-1.1513\, t}.$$

Conclusion: y decays exponentially with t; each unit increase in t multiplies y by $10^{-0.5} = 1/\sqrt{10} \approx 0.316$.

(b) Log-log plot. The straight line shown is

$$\log_{10} y = 1 + 0.75\, \log_{10} x \quad \Longrightarrow \quad y = 10^1\, x^{0.75} = 10\, x^{3/4}.$$

Conclusion: y follows a power law in x with exponent 0.75.

Solution to Q100. [Back to Question p. 64] [Back to TOC]

Comparing scales (earthquake energy vs magnitude).

On a semi-log graph ($\log_{10} E$ on the y–axis), the points are

$$(5,\ 12.301),\ (6,\ 13.799),\ (7,\ 15.301),\ (8,\ 16.799).$$

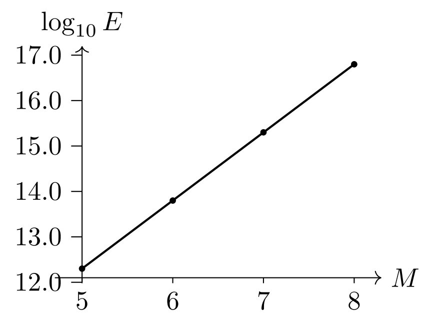
These are nearly collinear. Using endpoints,

$$\text{slope } m \approx \frac{16.799 - 12.301}{8 - 5} \approx 1.4994, \quad \text{intercept } b \approx 12.301 - 1.4994 \cdot 5 \approx 4.8038.$$

Thus

$$\log_{10} E \approx 4.804 + 1.499\, M \quad \Longleftrightarrow \quad E \approx (6.37 \times 10^4)\ 10^{\,1.499M}.$$

Interpretation: energy scales exponentially with magnitude; increasing M by 1 multiplies E by approximately $10^{1.5} \approx 31.6$.



Why a log scale? The energies span 10^{12}–10^{17} J. A logarithmic y–axis compresses this range, avoids crowding near the origin, and converts the exponential relationship into an (approximately) straight line, making trends and parameter estimates clear.

Topic 3 — Geometry and Trigonometry (SL 3.1–3.6, AHL 3.7–3.16)

Topic 3 Solutions

SL 3.1 3D Geometry and Measurements

Solution to Q101. [Back to Question p. 67] [Back to TOC]

$|AB| = \sqrt{(-4-2)^2 + (5+1)^2 + (1-3)^2} = \sqrt{76} = 2\sqrt{19}$. The midpoint is $M = (-1, 2, 2)$.

Solution to Q102. [Back to Question p. 67] [Back to TOC]

Right circular cone with radius $r = 6$ cm and slant height $\ell = 10$ cm.

(i) Height:

$$h = \sqrt{\ell^2 - r^2} = \sqrt{10^2 - 6^2} = \sqrt{64} = \boxed{8 \text{ cm}}.$$

(ii) Surface area (lateral + base):

$$S = \pi r\ell + \pi r^2 = \pi(6)(10) + \pi(6^2) = 60\pi + 36\pi = \boxed{96\pi \text{ cm}^2}.$$

(iii) Volume:

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi(36)(8) = \boxed{96\pi \text{ cm}^3}.$$

Final Answer: $h = 8$ cm, $S = 96\pi$ cm^2, $V = 96\pi$ cm^3.

Solution to Q103. [Back to Question p. 67] [Back to TOC]

Right square pyramid with base side $a = 12$ cm and height $h = 15$ cm.

Volume:

$$V = \tfrac{1}{3}a^2 h = \tfrac{1}{3}(12^2)(15) = \boxed{720 \text{ cm}^3}.$$

Slant height of each lateral face (altitude of the triangular face):

$$s = \sqrt{h^2 + \left(\frac{a}{2}\right)^2} = \sqrt{15^2 + 6^2} = \sqrt{261} = \boxed{3\sqrt{29} \text{ cm}}.$$

Lateral area (4 congruent triangles):

$$A_{\text{lat}} = 4\left(\tfrac{1}{2}as\right) = 4\left(\tfrac{1}{2} \cdot 12 \cdot s\right) = 24s = 24(3\sqrt{29}) = \boxed{72\sqrt{29} \text{ cm}^2}.$$

Total surface area:

$$A_{\text{tot}} = a^2 + A_{\text{lat}} = 12^2 + 72\sqrt{29} = \boxed{144 + 72\sqrt{29} \text{ cm}^2}.$$

Final Answer: $V = 720$ cm^3, $A_{\text{tot}} = 144 + 72\sqrt{29}$ cm^2.

SL 3.2 Triangle trigonometry

Solution to Q104. [Back to Question p. 68] [Back to TOC]

Area $= \frac{1}{2}ab\sin C = \frac{1}{2} \cdot 8 \cdot 11 \sin 52° \approx 34.7$. Using the cosine rule gives $c \approx 9.77$ and the sine rule gives $A \approx 41.3°$.

Solution to Q105. [Back to Question p. 68] [Back to TOC]

The ladder, ground, and wall form a right triangle with hypotenuse 6.8 m and angle 68° to the horizontal. The vertical height reached is

$$h = 6.8\sin(68°) \approx 6.30485 \text{ m}.$$

To the nearest centimetre,

$$\boxed{h \approx 6.30 \text{ m } (= 630 \text{ cm}).}$$

Solution to Q106. [Back to Question p. 68] [Back to TOC]

In $\triangle XYZ$, we are given $x = 12,\ y = 10,\ z = 8$, where x is opposite $\angle X$. By the cosine rule:

$$\cos X = \frac{y^2 + z^2 - x^2}{2yz} = \frac{10^2 + 8^2 - 12^2}{2(10)(8)} = \frac{100 + 64 - 144}{160} = \frac{20}{160} = 0.125.$$

Thus:

$$X = \cos^{-1}(0.125) \approx \boxed{82.8°}.$$

SL 3.3 Applications of Trigonometry

Solution to Q107. [Back to Question p. 69] [Back to TOC]

Let the height of the tower be h, the distance from P to the base of the tower be 65 m, and the angle of elevation be $28°$. From $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$:

$$\tan 28° = \frac{h}{65} \quad \Rightarrow \quad h = 65\tan 28°.$$

Evaluating:

$$h \approx 65 \times 0.531709 \approx \boxed{34.6 \text{ m}}.$$

Solution to Q108. [Back to Question p. 69] [Back to TOC]

Let F be the foot of the hill, B be the point closer to the hill, and set

$$BF = x \quad \Rightarrow \quad AF = x + 400 \text{ (m)}.$$

If the height of the hill is h, then from right triangles:

$$\tan 14° = \frac{h}{x+400}, \qquad \tan 21° = \frac{h}{x}.$$

Hence

$$x\tan 21° = (x+400)\tan 14° \;\Rightarrow\; x\big(\tan 21° - \tan 14°\big) = 400\tan 14°$$
$$\Rightarrow \quad x = \frac{400\tan 14°}{\tan 21° - \tan 14°}.$$

Then

$$h = x\tan 21° = \frac{400\tan 14°\,\tan 21°}{\tan 21° - \tan 14°} \approx \frac{400(0.2493)(0.3839)}{0.3839 - 0.2493} \approx 2.85 \times 10^2 \text{ m}.$$

Final Answer: $\boxed{h \approx 285 \text{ m}}$.

Solution to Q109. [Back to Question p. 69] [Back to TOC]

Let due North be the positive y-axis and East the positive x-axis. For a bearing β (clockwise from North), the displacement of length L has components

$$(\,E,\, N\,) = \big(L\sin\beta,\; L\cos\beta\big).$$

From H to A: $L_1 = 18$, $\beta_1 = 65°$,

$$(E_1, N_1) = \big(18\sin 65°,\; 18\cos 65°\big) \approx (16.3135,\; 7.6071).$$

From A to B: $L_2 = 12$, $\beta_2 = 145°$,

$$(E_2, N_2) = \big(12\sin 145°,\; 12\cos 145°\big) \approx (6.8829,\; -9.8298).$$

Hence $H \to B$ has components

$$(E, N) = (E_1 + E_2,\; N_1 + N_2) \approx (23.1964,\; -2.2227).$$

Distance $HB = \sqrt{E^2 + N^2} \approx \sqrt{23.196^2 + (-2.223)^2} \approx 23.30$ km.

Bearing of B from H is $\theta = \operatorname{atan2}(E, N)$ in degrees. With $E > 0$, $N < 0$ (SE quadrant),

$$\theta \approx \operatorname{atan2}(23.1964, -2.2227) \approx 95.5^\circ.$$

So $HB \approx 23.30\,\text{km}$ on a bearing $\approx 096^\circ$ (nearest degree).

SL 3.4 Circle arc and sector

Solution to Q110. [Back to Question p. 70] [Back to TOC]

Convert $\theta = 110°$ to radians: $\theta = \frac{110\pi}{180} = \frac{11\pi}{18}$. With $r = 6$ cm,

$$s = r\theta = 6 \cdot \frac{11\pi}{18} = \frac{11\pi}{3} \approx 11.52 \text{ cm}, \qquad A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2} \cdot 36 \cdot \frac{11\pi}{18} = 11\pi \approx 34.56 \text{ cm}^2.$$

SL 3.5 Perpendicular bisector

Solution to Q111. [Back to Question p. 71] [Back to TOC]

$P(2,-1)$, $Q(8,5)$. Midpoint $M\left(\frac{2+8}{2}, \frac{-1+5}{2}\right) = (5,2)$. Slope of PQ:

$$m_{PQ} = \frac{5-(-1)}{8-2} = \frac{6}{6} = 1 \Rightarrow m_{\perp} = -1.$$

Perpendicular bisector through M: $y - 2 = -1(x-5) \iff y = -x + 7 \iff x + y - 7 = 0$.

SL 3.6 Voronoi diagrams

Solution to Q112. [Back to Question p. 74] [Back to TOC]

Let $A(0,0),\ B(4,0),\ C(2,3)$.

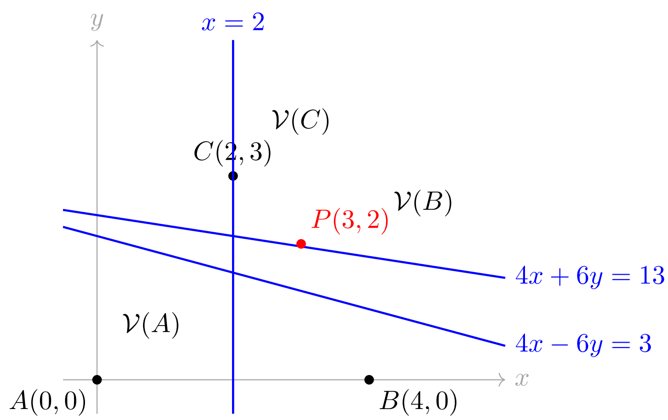
(i) Perpendicular bisectors.

$$\overline{AB}: \text{midpoint } (2,0),\ \text{AB is horizontal} \Rightarrow \boxed{x=2}.$$

$$\overline{AC}: \text{midpoint } (1,\ \tfrac{3}{2}),\ \text{slope}(AC)=\tfrac{3}{2} \Rightarrow m_\perp = -\tfrac{2}{3},$$
$$y-\tfrac{3}{2} = -\tfrac{2}{3}(x-1) \implies \boxed{4x+6y=13}.$$

$$\overline{BC}: \text{midpoint } (3,\ \tfrac{3}{2}),\ \text{slope}(BC)=-\tfrac{3}{2} \Rightarrow m_\perp = \tfrac{2}{3},$$
$$y-\tfrac{3}{2} = \tfrac{2}{3}(x-3) \implies \boxed{4x-6y=3}.$$

(ii) Voronoi diagram. The Voronoi edges are precisely the three bisectors above. They split the plane into three convex regions, each consisting of the points closer to one site than the other two.



(iii) Region of $P(3,2)$. Check against the bisectors:

$$\begin{aligned}
&x=2: \quad 3>2 \Rightarrow \text{closer to } B \text{ than } A;\\
&4x-6y=3: \quad 4(3)-6(2)=0<3 \Rightarrow \text{closer to } C \text{ than } B;\\
&4x+6y=13: \quad 4(3)+6(2)=24>13 \Rightarrow \text{closer to } C \text{ than } A.
\end{aligned}$$

Therefore P is closest to C; i.e., $P \in \mathcal{V}(C)$.

Final Answer: Perpendicular bisectors: $\boxed{x=2}$, $\boxed{4x+6y=13}$, $\boxed{4x-6y=3}$. The Voronoi regions are determined by these three lines as sketched; the point $P(3,2)$ lies in the region of C.

Solution to Q113. [Back to Question p. 74] [Back to TOC]

Voronoi edges are the perpendicular bisectors of the segments joining sites. With $A(0,0)$, $B(8,1)$, $C(3,6)$:

- Midpoint $AB=(4,0.5)$; slope $AB=\frac{1}{8}$ so the bisector has slope -8.
- Midpoint $AC=(1.5,3)$; slope $AC=2$ so the bisector has slope $-\frac{1}{2}$.

Intersecting these two bisectors gives the common Voronoi vertex (also the circumcenter of $\triangle ABC$):

$$y-0.5=-8(x-4), \qquad y-3=-\tfrac{1}{2}(x-1.5) \ \Rightarrow\ (x,y)=\left(\tfrac{23}{6},\ \tfrac{11}{6}\right) \approx (3.833,\ 1.833).$$

This point is equidistant from A, B, C (distance ≈ 4.249). The "toxic waste dump" location that maximizes the minimum distance to the facilities is this circumcenter.

SL 3.7 Radian Measure and Circular Sectors

Solution to Q114. [Back to Question p. 76] [Back to TOC]

To convert degrees to radians, multiply by $\frac{\pi}{180}$:

$$126^\circ = 126 \times \frac{\pi}{180} = \frac{126\pi}{180} = \frac{7\pi}{10} \text{ radians.}$$

To convert radians to degrees, multiply by $\frac{180}{\pi}$:

$$\frac{7\pi}{9} \text{ radians} = \frac{7\pi}{9} \times \frac{180}{\pi} = \frac{7 \times 180}{9} = 140^\circ.$$

Final Answer: $\boxed{\frac{7\pi}{10} \text{ rad}, \ 140^\circ}$.

Solution to Q115. [Back to Question p. 76] [Back to TOC]

$126^\circ = \frac{7\pi}{10}$ radians. Arc length 14.4 in a circle of radius 9 corresponds to angle $\theta = 14.4/9 = 1.6$ rad and sector area $\frac{1}{2}r^2\theta = 64.8\,\text{cm}^2$.

Solution to Q116. [Back to Question p. 76] [Back to TOC]

Given $A = 75$ cm^2 and $\theta = 1.5$ rad, the formula for the area of a sector is

$$A = \frac{1}{2}r^2\theta.$$

Thus,

$$75 = \frac{1}{2}r^2(1.5) \quad \Rightarrow \quad 75 = 0.75r^2 \quad \Rightarrow \quad r^2 = 100 \quad \Rightarrow \quad r = \boxed{10 \text{ cm}}.$$

The arc length is

$$s = r\theta = 10 \times 1.5 = \boxed{15 \text{ cm}}.$$

Final Answer: Radius = 10 cm, Arc length = 15 cm.

AHL 3.8 Unit circle and Trigonometric Equations

Solution to Q117. [Back to Question p. 77] [Back to TOC]

On the unit circle, each point has coordinates $(\cos\theta, \sin\theta)$.

For $\theta = \frac{\pi}{6}$:

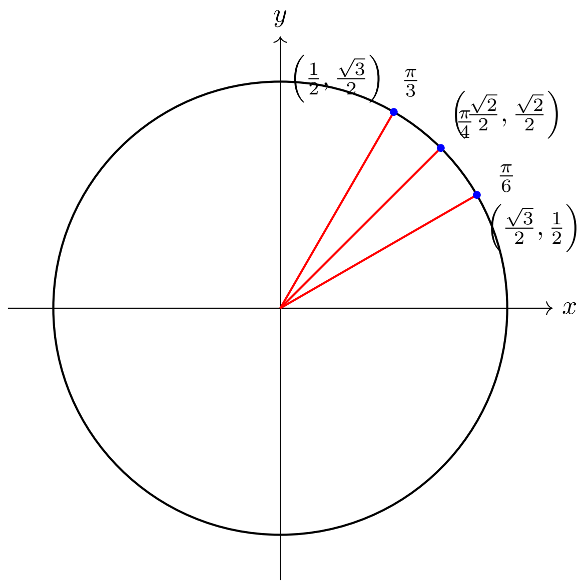
$$(\cos\tfrac{\pi}{6}, \sin\tfrac{\pi}{6}) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

For $\theta = \frac{\pi}{4}$:

$$(\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

For $\theta = \frac{\pi}{3}$:

$$(\cos\tfrac{\pi}{3}, \sin\tfrac{\pi}{3}) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$



Final Answer:

$$\frac{\pi}{6} : \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad \frac{\pi}{4} : \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \quad \frac{\pi}{3} : \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

Solution to Q118. [Back to Question p. 77] [Back to TOC]

Solve $2\sin\theta\cos\theta = \sin\theta$ for $0 \le \theta < 2\pi$.

Factor:

$$2\sin\theta\cos\theta - \sin\theta = \sin\theta(2\cos\theta - 1) = 0.$$

Hence either

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = \tfrac{1}{2}.$$

On $0 \le \theta < 2\pi$:

$$\sin\theta = 0 \ \Rightarrow \ \theta = 0,\ \pi; \qquad \cos\theta = \tfrac{1}{2} \ \Rightarrow \ \theta = \frac{\pi}{3},\ \frac{5\pi}{3}.$$

Final Answer: $\boxed{\theta \in \left\{0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}\right\}.}$

Solution to Q119. [Back to Question p. 77] [Back to TOC]

Given $a = 8$ (opposite A), $A = 40°$, and $b = 10$ (opposite B). By the Sine Rule,

$$\frac{\sin B}{b} = \frac{\sin A}{a} \quad \Longrightarrow \quad \sin B = \frac{b \sin A}{a} = \frac{10 \sin 40°}{8} \approx 0.80348.$$

Hence

$$B_1 = \sin^{-1}(0.80348) \approx 53.46°, \qquad B_2 = 180° - B_1 \approx 126.54°.$$

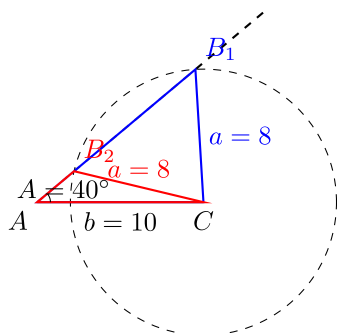
The corresponding third angles are

$$C_1 = 180° - A - B_1 \approx 86.54°, \qquad C_2 = 180° - A - B_2 \approx 13.46°.$$

Using the Sine Rule again ($\frac{c}{\sin C} = \frac{a}{\sin A}$),

$$c_1 = \frac{a \sin C_1}{\sin A} = \frac{8 \sin 86.54°}{\sin 40°} \approx 12.42, \qquad c_2 = \frac{a \sin C_2}{\sin A} = \frac{8 \sin 13.46°}{\sin 40°} \approx 2.90.$$

Why two solutions? This is the SSA *ambiguous case.* With A and the two sides a (opposite A) and b given, the ray making angle A at A can intersect the circle of radius a centered at C in two points, giving one acute and one obtuse angle B.



Final Answers:

Case	B	C	c
1	53.46°	86.54°	12.42
2	126.54°	13.46°	2.90

AHL 3.9 Matrix Transformations

Solution to Q120. [Back to Question p. 78] [Back to TOC]

Reflection in the x–axis is represented by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with determinant -1.

Solution to Q121. [Back to Question p. 78] [Back to TOC]

Given

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{t} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

The image of P under the affine map $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{t}$ is

$$P' = AP + \mathbf{t} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}.$$

The area–scaling factor of the linear part is $|\det A|$:

$$\det A = 0 \cdot 0 - (-1) \cdot 1 = 1 \quad \Rightarrow \quad \text{area factor} = \boxed{1}.$$

(So the transformation preserves area and orientation.)

Solution to Q122. [Back to Question p. 78] [Back to TOC]

Let

$$M = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

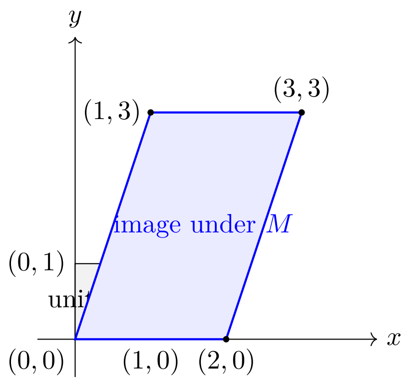
$$\det M = 2 \cdot 3 - 0 \cdot 1 = \boxed{6}.$$

For any region in the plane, the (signed) area is scaled by $\det M$; hence the unit square (area 1) is mapped to a parallelogram of area $\boxed{6}$. Because $\det M > 0$, the orientation is preserved.

Acting on the unit square's vertices:

$$\begin{aligned} (0,0) &\mapsto (0,0), \\ (1,0) &\mapsto M(1,0) = (2,0), \\ (0,1) &\mapsto M(0,1) = (1,3), \\ (1,1) &\mapsto M(1,1) = (3,3). \end{aligned}$$

Thus the image is the parallelogram spanned by vectors $(2,0)$ and $(1,3)$ with vertices $(0,0), (2,0), (3,3), (1,3)$. Geometrically: scale by 2 in the x-direction, by 3 in the y-direction, and shear in the x-direction by $+y$ (since $x' = 2x + y,\ y' = 3y$).



Final Answer: $|\det M| = 6$; the unit square maps to the parallelogram with vertices $(0,0), (2,0), (3,3), (1,3)$ and area 6.

Topic AHL3.10 — Vector arithmetic

Solution to Q123. [Back to Question p. 79] [Back to TOC]

Scalar or vector?

- mass — *scalar*
- displacement — *vector*
- temperature — *scalar*
- force — *vector*
- velocity — *vector*
- speed — *scalar*
- electric current — *scalar* (direction is treated via sign/convention)

Solution to Q124. [Back to Question p. 79] [Back to TOC]

Directed line segment and component forms. $A(2,-1),\ B(-3,4)$. Then $\overrightarrow{AB} = \mathbf{B} - \mathbf{A} = (-3-2,\ 4-(-1)) = (-5,5)$.

$$\text{(i) } \overrightarrow{AB} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}, \qquad \text{(ii) } \overrightarrow{AB} \qquad = -5\mathbf{i} + 5\mathbf{j},$$
$$\text{(iii) } |\overrightarrow{AB}| = \sqrt{(-5)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}.$$

Solution to Q125. [Back to Question p. 79] [Back to TOC]

Base vectors in 3D.

$$\mathbf{v} = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix} = 2\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}, \qquad |\mathbf{v}| = \sqrt{2^2 + (-5)^2 + 7^2} = \sqrt{78}.$$

Solution to Q126. [Back to Question p. 79] [Back to TOC]

Zero and negative vectors. For $\mathbf{u} = \langle a, b \rangle$:

$$-\mathbf{u} = \langle -a, -b \rangle, \qquad |-\mathbf{u}| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |\mathbf{u}|.$$

$\mathbf{u} = \mathbf{0}$ iff $a = 0$ and $b = 0$.

Solution to Q127. [Back to Question p. 80] [Back to TOC]

Sum and difference (algebraic). $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{b} = -2\mathbf{i} + 5\mathbf{j}$.

$$\mathbf{a} + \mathbf{b} = (3-2)\mathbf{i} + (-4+5)\mathbf{j} = \mathbf{i} + \mathbf{j}, \qquad |\mathbf{a} + \mathbf{b}| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$
$$\mathbf{a} - \mathbf{b} = (3-(-2))\mathbf{i} + (-4-5)\mathbf{j} = 5\mathbf{i} - 9\mathbf{j}, \qquad |\mathbf{a} - \mathbf{b}| = \sqrt{5^2 + (-9)^2} = \sqrt{106}.$$

Solution to Q128. [Back to Question p. 80] [Back to TOC]

Resultant of multiple vectors.

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = (4-5+2)\mathbf{i} + (3+2-6)\mathbf{j} = \mathbf{i} - \mathbf{j}.$$

$$|\mathbf{R}| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \text{ N}.$$

Solution to Q129. [Back to Question p. 80] [Back to TOC]

Parallel vectors and scalar multiples.

1. $(6, -9) = 3(2, -3)$ and $(2k, -3k) = k(2, -3)$, so they are parallel for any real k (the zero vector $k = 0$ has no direction).
2. $|\mathbf{p}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$. We need $|k|\,|\mathbf{p}| = 10 \Rightarrow |k| = \dfrac{10}{\sqrt{6}} = \dfrac{5\sqrt{6}}{3}$, so $k = \pm\dfrac{5\sqrt{6}}{3}$.

Solution to Q130. [Back to Question p. 81] [Back to TOC]

Position vectors.

1. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \mathbf{b} - \mathbf{a}$.
2. With $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$:

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix}, \qquad |\overrightarrow{AB}| = \sqrt{(-5)^2 + 4^2} = \sqrt{41}.$$

Solution to Q131. [Back to Question p. 81] [Back to TOC]

Displacement by successive moves.

$$\mathbf{d}_{\text{tot}} = (5 - 3 + 0)\mathbf{i} + (-2 + 4 + 2)\mathbf{j} = 2\mathbf{i} + 4\mathbf{j}, \qquad |\mathbf{d}_{\text{tot}}| = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5}.$$

Solution to Q132. [Back to Question p. 82] [Back to TOC]

Normalizing (unit vector). For $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$, $|\mathbf{v}| = 5$.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

With speed $7\,\mathrm{m\,s^{-1}}$, the velocity vector is

$$\mathbf{v}_{\text{phys}} = 7\,\hat{\mathbf{v}} = \frac{21}{5}\mathbf{i} + \frac{28}{5}\mathbf{j}\ \mathrm{m\,s^{-1}}.$$

Solution to Q133. [Back to Question p. 82] [Back to TOC]

Unit vector in 3D. $\mathbf{w} = \langle -2, 1, 2 \rangle$, $|\mathbf{w}| = \sqrt{(-2)^2 + 1^2 + 2^2} = 3$.

$$\hat{\mathbf{w}} = \frac{1}{3}(-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

A vector of length 15 in the same direction is $15\hat{\mathbf{w}} = 5(-2, 1, 2) = (-10, 5, 10)$.

Solution to Q134. [Back to Question p. 82] [Back to TOC]

Unknown component from magnitude. $|\mathbf{u}| = \sqrt{k^2 + (-4)^2} = 10 \Rightarrow k^2 + 16 = 100 \Rightarrow k = \pm\sqrt{84} =$

$\pm 2\sqrt{21}$. Since $|\mathbf{u}| = 10$, the unit vector is

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{10} = \frac{k}{10}\mathbf{i} - \frac{2}{5}\mathbf{j} = \left(\pm\frac{\sqrt{21}}{5}\right)\mathbf{i} - \frac{2}{5}\mathbf{j}.$$

Solution to Q135. [Back to Question p. 82] [Back to TOC]

Geometric description from components. $\mathbf{r} = (-6, 8)$ has $|\mathbf{r}| = \sqrt{(-6)^2 + 8^2} = 10$. Direction angle from $+x$-axis:

$$\theta = \pi - \arctan\left(\frac{8}{6}\right) = \pi - \arctan\left(\frac{4}{3}\right) \approx 126.9^\circ \text{ (so } 127^\circ \text{ to nearest degree).}$$

A parallel vector of magnitude 5 (same direction) is $\frac{5}{10}\mathbf{r} = (-3, 4)$.

Solution to Q136. [Back to Question p. 83] [Back to TOC]

Midpoint and median using position vectors. Let M be the midpoint of AB. In position vectors,

$$\overrightarrow{OM} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} = \frac{\mathbf{a} + \mathbf{b}}{2}.$$

Thus the median from O to AB has vector $\overrightarrow{OM} = \frac{\mathbf{a} + \mathbf{b}}{2}$.

Solution to Q137. [Back to Question p. 83] [Back to TOC]

Column $\leftrightarrow$ $\mathbf{i}, \mathbf{j}, \mathbf{k}$ conversion.

$$\text{(i)} \quad \begin{pmatrix} 4 \\ -7 \end{pmatrix} = 4\mathbf{i} - 7\mathbf{j},$$

$$\text{(ii)} \quad -2\mathbf{i} + 3\mathbf{j} - \mathbf{k} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix},$$

$$\text{(iii)} \quad \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = 5\mathbf{k}.$$

Solution to Q138. [Back to Question p. 84] [Back to TOC]

Resultant as sum of given directions. Take $\mathbf{i}$ east, $\mathbf{j}$ north.
Hiker A: $40\mathbf{i}$. Hiker B: $30(\cos 60^\circ\,\mathbf{i} + \sin 60^\circ\,\mathbf{j}) = (15)\mathbf{i} + (15\sqrt{3})\mathbf{j}$.

$$\mathbf{R} = (40 + 15)\mathbf{i} + (0 + 15\sqrt{3})\mathbf{j} = 55\mathbf{i} + 15\sqrt{3}\,\mathbf{j}.$$

$$|\mathbf{R}| = \sqrt{55^2 + (15\sqrt{3})^2} = \sqrt{3025 + 675} = \sqrt{3700} = 10\sqrt{37} \text{ N } (\approx 60.8 \text{ N}).$$

Topic AHL3.11 — Vector equation of a line

Solution to Q139. [Back to Question p. 85] [Back to TOC]

2D: Vector to parametric (and points).
$\mathbf{r} = \begin{pmatrix}1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\end{pmatrix}$. Hence

$$x = 1 + 3\lambda, \qquad y = 2 - \lambda.$$

At $\lambda = 0$: $(x, y) = (1, 2)$. At $\lambda = 2$: $(x, y) = (1 + 6,\ 2 - 2) = (7, 0)$. Since $A(7, 0)$ is obtained when $\lambda = 2$, A lies on the line.

Solution to Q140. [Back to Question p. 85] [Back to TOC]

3D: Parametric to vector.
Given

$$x = 1 + 2\lambda, \quad y = -3 + \lambda, \quad z = 4 - 5\lambda,$$

take $\mathbf{a} = \begin{pmatrix}1\\-3\\4\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}2\\1\\-5\end{pmatrix}$. Then $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. A point on the line is $(1, -3, 4)$ (when $\lambda = 0$); a direction vector is $\langle 2, 1, -5\rangle$.

Solution to Q141. [Back to Question p. 85] [Back to TOC]

2D: Line through two points.
$\overrightarrow{PQ} = Q - P = (-2 - 4,\ 5 - (-1)) = (-6, 6)$. Vector form: $\mathbf{r} = \begin{pmatrix}4\\-1\end{pmatrix} + \lambda\begin{pmatrix}-6\\6\end{pmatrix}$. Parametric:

$$x = 4 - 6\lambda, \qquad y = -1 + 6\lambda.$$

x–axis: $y = 0 \Rightarrow -1 + 6\lambda = 0 \Rightarrow \lambda = \frac{1}{6}$. Thus $x = 4 - 6 \cdot \frac{1}{6} = 3$ and the intercept is $(3, 0)$.

Solution to Q142. [Back to Question p. 86] [Back to TOC]

3D: Line through two points.
(a) $\overrightarrow{AB} = B - A = (3, 1, -5)$.
(b) $\mathbf{r} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-5\end{pmatrix}$. Parametric: $x = 2 + 3\lambda,\ y = -1 + \lambda,\ z = 3 - 5\lambda$.
(c) For $\lambda = -2$: $(x, y, z) = \big(2 - 6,\ -1 - 2,\ 3 + 10\big) = (-4, -3, 13)$.

Solution to Q143. [Back to Question p. 86] [Back to TOC]

2D: Parallel lines and intersection.
Direction vectors: $\mathbf{b}_1 = \langle 1, 4\rangle$, $\mathbf{b}_2 = \langle 2, 8\rangle = 2\mathbf{b}_1$. Hence the lines are parallel. Test if coincident: does $(-1, 5)$ satisfy ℓ_1? Solve $(2, -3) + \lambda(1, 4) = (-1, 5) \Rightarrow \lambda = -3$ from x, but then $y = -3 + 4(-3) = -15 \neq 5$. Therefore the lines are *distinct parallel* and do not intersect (so part (b) has no solution).

Solution to Q144. [Back to Question p. 86] [Back to TOC]

3D: Intersecting or skew?
Solve

$$\begin{cases} 3 + \lambda = 6 - 2\mu \\ 1 + 2\lambda = -1 + \mu \\ -2 + 3\lambda = 1 + \mu \end{cases} \Rightarrow \begin{cases} \lambda + 2\mu = 3 \\ 2\lambda - \mu = -2 \\ 3\lambda - \mu = 3 \end{cases}$$

From the first two, $\lambda = -\frac{1}{5}$, $\mu = \frac{8}{5}$. Check in the third: $3(-\frac{1}{5}) - \frac{8}{5} = -\frac{11}{5} \neq 3$. No common solution and the direction vectors $\langle 1, 2, 3\rangle$ and $\langle -2, 1, 1\rangle$ are not parallel, so the lines are *skew*.

Solution to Q145. [Back to Question p. 87] [Back to TOC]

2D: From Cartesian to vector.
For $y = \frac{1}{2}x - 3$, a point is $(0, -3)$ and a direction vector is $\langle 2, 1\rangle$ (slope 1/2). Thus $\mathbf{r} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. One valid choice is $\mathbf{a} = (0, -3)$, $\mathbf{b} = (2, 1)$ (many others are possible).

Solution to Q146. [Back to Question p. 87] [Back to TOC]

3D: Point on a line?
Solve for λ using each coordinate: $-2 + 3\lambda = 1 \Rightarrow \lambda = 1$. Then $y = 4 - 2(1) = 2$ and $z = 1 + 6(1) = 7$. All coordinates match $C(1, 2, 7)$; hence C lies on the line, with $\lambda = 1$.

Solution to Q147. [Back to Question p. 87] [Back to TOC]

2D: Line through a point parallel to a given line.
Keep the direction vector $\langle -3, 7\rangle$ and pass through $S(-5, 2)$:

$$\mathbf{r} = \begin{pmatrix} -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 7 \end{pmatrix}, \qquad x = -5 - 3\lambda, \quad y = 2 + 7\lambda.$$

Solution to Q148. [Back to Question p. 88] [Back to TOC]

3D: Fix a coordinate value.
$z = 5 - 4\lambda = 1 \Rightarrow \lambda = 1$. Then $x = 0 + 2(1) = 2$, $y = -3 + 1 = -2$. Point: $(2, -2, 1)$ with $\lambda = 1$.

Solution to Q149. [Back to Question p. 88] [Back to TOC]

Mixed forms.
Direction $\propto \langle 2, -1, 3\rangle$. Take exactly $\langle 2, -1, 3\rangle$. (a) $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.
(b) Parametric: $x = 1 + 2\lambda$, $y = 4 - \lambda$, $z = 3\lambda$.
(c) $x = 7 \Rightarrow 1 + 2\lambda = 7 \Rightarrow \lambda = 3$.

Solution to Q150. [Back to Question p. 88] [Back to TOC]

2D: Intersection with another form.
$\ell : \; x = 3 - 4\lambda, \; y = -2 + \lambda$. Impose $2x + y = 1$:

$$2(3 - 4\lambda) + (-2 + \lambda) = 1 \;\Rightarrow\; 4 - 7\lambda = 1 \;\Rightarrow\; \lambda = \tfrac{3}{7}.$$

Hence

$$x = 3 - 4 \cdot \tfrac{3}{7} = \tfrac{9}{7}, \qquad y = -2 + \tfrac{3}{7} = -\tfrac{11}{7}.$$

Intersection point: $\left(\frac{9}{7}, -\frac{11}{7}\right)$.

Topic AHL3.12 — Vector applications to kinematics

Solution to Q151. [Back to Question p. 90] [Back to TOC]

2D constant velocity: position and path.
$\mathbf{v} = \langle 3, -2 \rangle$, $\mathbf{r}_0 = \langle -4, 5 \rangle$.

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}t = \begin{pmatrix} -4+3t \\ 5-2t \end{pmatrix}.$$

At $t = 6$: $(x, y) = (14, -7)$. Displacement $t : 2 \to 10$: $\Delta\mathbf{r} = \mathbf{v}(10-2) = \langle 24, -16 \rangle$. Eliminate t: $x = -4 + 3t \Rightarrow t = \frac{x+4}{3}$, so

$$y = 5 - 2\left(\frac{x+4}{3}\right) = \frac{7-2x}{3} \quad \Leftrightarrow \quad 2x + 3y = 7.$$

Solution to Q152. [Back to Question p. 90] [Back to TOC]

3D constant velocity: meeting or not.
$\mathbf{r}_A = \langle 1, -2, 4 \rangle + t\langle 2, 1, -3 \rangle$, $\mathbf{r}_B = \langle 7, -1, -2 \rangle + t\langle -1, 0, 2 \rangle$. Collision requires same position at the same t. From x: $1 + 2t = 7 - t \Rightarrow t = 2$. Then $y_A(2) = 0 \neq -1 = y_B(2)$. Hence no collision. (Indeed, solving $\mathbf{r}_A(t) = \mathbf{r}_B(s)$ gives no common (t, s), so the tracks do not meet in space.)

Solution to Q153. [Back to Question p. 90] [Back to TOC]

Relative position and closest approach (2D).
$\overrightarrow{12}(t) = \mathbf{r}_2 - \mathbf{r}_1 = \langle 10 - 6t, -8 + 4t \rangle$. Minimise $D^2(t) = (10-6t)^2 + (-8+4t)^2$:

$$\frac{d}{dt}D^2 = -12(10-6t) + 8(-8+4t) = 104t - 184 = 0 \Rightarrow t = \frac{23}{13}.$$

Then $\overrightarrow{12} = \left\langle -\frac{8}{13}, -\frac{12}{13} \right\rangle$ and

$$D_{\min} = \sqrt{\frac{16}{13}} = \frac{4}{\sqrt{13}} \text{ units.}$$

Solution to Q154. [Back to Question p. 91] [Back to TOC]

Ship safety check (constant velocities).
Relative position $\mathbf{d}(t) = \mathbf{r}_2 - \mathbf{r}_1 = \langle 13 + 3t, -12 - 2t \rangle$. Minimise $D^2 = (13+3t)^2 + (-12-2t)^2$:

$$\frac{d}{dt}D^2 = 126 + 26t = 0 \Rightarrow t = -\frac{63}{13} < 0.$$

Thus for $t \geq 0$, $D(t)$ increases; the minimum future distance is at $t = 0$:

$$D(0) = \sqrt{13^2 + 12^2} = \sqrt{313} \approx 17.7 \text{ km}.$$

They do not meet (solution of $2 - 8t = 15 - 5t$ gives $t = -13/3$), and the 2 km safety radius is never violated.

Solution to Q155. [Back to Question p. 91] [Back to TOC]

3D: crossing tracks vs. collision.
Ground tracks intersect when

$$\begin{cases} 30 + 6p = 3q \\ -20 + 4p = 40 - 8q \end{cases} \quad \Rightarrow p = -1,\ q = 8,$$

giving ground point $(24, -24)$. But altitudes are constant and distinct: $z_A = 2$ km, $z_B = 5$ km; moreover the ground intersection occurs at different times. Hence no collision.

Solution to Q156. [Back to Question p. 92] [Back to TOC]

Variable velocity given as components.
$v_x = 7,\ v_y = 6 - 4t,\ (x, y)(0) = (1, 2)$.

$$x(t) = 1 + \int_0^t 7\,ds = 1 + 7t, \qquad y(t) = 2 + \int_0^t (6 - 4s)\,ds = 2 + 6t - 2t^2.$$

Eliminate t: $t = \frac{x-1}{7} \Rightarrow y = 2 + \frac{6}{7}(x-1) - \frac{2}{49}(x-1)^2$. Speed squared $= 49 + (6-4t)^2$ is minimised when $6 - 4t = 0 \Rightarrow t = \frac{3}{2}$, giving $v_{\min} = \sqrt{49} = 7$ m/s.

Solution to Q157. [Back to Question p. 92] [Back to TOC]

Projectile motion.
$v_x = u\cos\theta,\ v_y = u\sin\theta - gt$; $x = u\cos\theta\, t,\ y = u\sin\theta\, t - \frac{1}{2}gt^2$. Time of flight $T = \dfrac{2u\sin\theta}{g} \approx \dfrac{2\cdot 20\sin 40^\circ}{9.8} \approx$ 2.626 s. Range $R = \dfrac{u^2\sin 2\theta}{g} \approx \dfrac{400\sin 80^\circ}{9.8} \approx 40.20$ m. Max height $H = \dfrac{(u\sin\theta)^2}{2g} \approx 8.43$ m. Trajectory:

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}\,x^2.$$

Solution to Q158. [Back to Question p. 92] [Back to TOC]

Projectile with a time shift.
With $a = 0.6$ s and $t \ge a$,

$$\mathbf{r}_2(t) = \big\langle u\cos\theta\,(t-a),\ u\sin\theta\,(t-a) - \tfrac{1}{2}g(t-a)^2\big\rangle.$$

Same height when $y_1(t) = y_2(t)$. Cancelling terms gives

$$0 = -u\sin\theta\, a + ga\,t - \tfrac{1}{2}ga^2 \ \Rightarrow\ t = \frac{u\sin\theta}{g} + \frac{a}{2} \approx 1.313 + 0.300 \approx 1.613 \text{ s}.$$

For same position we would need $x_1(t) = x_2(t) \Rightarrow t = t - a$, impossible for $a \neq 0$. Hence they never coincide in position at the same time.

Solution to Q159. [Back to Question p. 93] [Back to TOC]

Uniform circular motion.
$\mathbf{r}(t) = \langle 5\cos\omega t,\ 5\sin\omega t\rangle$.

$$\mathbf{v} = \mathbf{r}' = \langle -5\omega\sin\omega t,\ 5\omega\cos\omega t\rangle, \qquad |\mathbf{v}| = 5\omega \text{ (constant)}.$$

$$\mathbf{a} = \mathbf{v}' = \langle -5\omega^2\cos\omega t,\ -5\omega^2\sin\omega t\rangle = -\omega^2\mathbf{r}(t),$$

so $\mathbf{a}$ is inward (radial) and perpendicular to $\mathbf{v}$. If $T = 4\pi$ s, then $\omega = \dfrac{2\pi}{T} = \frac{1}{2}$ rad/s and the speed is $5\omega = 2.5$ m/s.

Solution to Q160. [Back to Question p. 93] [Back to TOC]

Mixed: recover velocity from position.
$\mathbf{r}(t) = \langle 2t - 1,\ 4 - 3e^{-t}\rangle \Rightarrow \mathbf{v}(t) = \langle 2,\ 3e^{-t}\rangle$, $\mathbf{a}(t) = \langle 0,\ -3e^{-t}\rangle$. Velocity is horizontal when $v_y = 0$, but $3e^{-t} > 0$ for all t; hence there is *no finite* time (it becomes asymptotically horizontal as $t \to \infty$). Total

distance on $0 \le t \le 3$:

$$\text{Distance} = \int_0^3 \sqrt{2^2 + (3e^{-t})^2}\, dt = \int_0^3 \sqrt{4 + 9e^{-2t}}\, dt.$$

Solution to Q161. [Back to Question p. 94] [Back to TOC]

Chasing problem (relative motion).
$\mathbf{r}_A = \langle 5t, 0\rangle$, $\mathbf{r}_B = \langle 60 - 3t,\, 80 - 4t\rangle$. Relative vector $\mathbf{d}(t) = \mathbf{r}_B - \mathbf{r}_A = \langle 60 - 8t,\, 80 - 4t\rangle$. Minimise $D^2 = (60 - 8t)^2 + (80 - 4t)^2$:

$$\frac{d}{dt} D^2 = -16(60 - 8t) - 8(80 - 4t) = 160t - 1600 = 0 \Rightarrow t = 10 \text{ s}.$$

Then $\mathbf{d}(10) = \langle -20, 40\rangle$ and

$$D_{\min} = \sqrt{(-20)^2 + 40^2} = 20\sqrt{5} \text{ m } (\approx 44.7 \text{ m}).$$

Since $D_{\min} > 0$, B never catches A.

Solution to Q162. [Back to Question p. 94] [Back to TOC]

Reconstructing initial data from two sightings.
With constant velocity,

$$\mathbf{v} = \frac{\mathbf{r}(9) - \mathbf{r}(2)}{9 - 2} = \frac{\langle 18, 5, -8\rangle - \langle 4, -1, 7\rangle}{7} = \left\langle 2,\ \frac{6}{7},\ -\frac{15}{7} \right\rangle.$$

Initial position $\mathbf{r}(0) = \mathbf{r}(2) - 2\mathbf{v} = \left\langle 0,\ -\frac{19}{7},\ \frac{79}{7} \right\rangle$. Closest approach to $P = (10, 0, 0)$ occurs when $(\mathbf{r}(t) - P)\cdot\mathbf{v} = 0$:

$$t^* = -\frac{(\mathbf{r}(0) - P)\cdot\mathbf{v}}{\|\mathbf{v}\|^2} = -\frac{\langle -10,\ -\frac{19}{7},\ \frac{79}{7}\rangle \cdot \langle 2, \frac{6}{7}, -\frac{15}{7}\rangle}{4 + \frac{36}{49} + \frac{225}{49}} = \frac{2279}{457} \text{ s } \approx 4.99 \text{ s}.$$

Topic AHL3.13 — Vector dot and cross products

Solution to Q163. [Back to Question p. 95] [Back to TOC]

Dot product and angle (3D).
$\mathbf{u} = \langle 3, -1, 2\rangle,\ \mathbf{v} = \langle 1, 4, -2\rangle.$

$$\mathbf{u}\cdot\mathbf{v} = 3(1) + (-1)(4) + 2(-2) = 3 - 4 - 4 = -5.$$

$|\mathbf{u}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}, \quad |\mathbf{v}| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{21}.$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{-5}{\sqrt{14}\sqrt{21}} = \frac{-5}{\sqrt{294}} \ \Rightarrow\ \theta = \cos^{-1}\left(\frac{-5}{\sqrt{294}}\right) \approx 1.87 \text{ rad (3 s.f.)}.$$

Since $\mathbf{u}\cdot\mathbf{v} \neq 0$, they are *not* perpendicular.

Solution to Q164. [Back to Question p. 95] [Back to TOC]

Acute angle between two lines (3D).
Direction vectors: $\mathbf{a} = \langle 1, 2, 2\rangle$, $\mathbf{b} = \langle 2, -1, 2\rangle$.

$$\mathbf{a}\cdot\mathbf{b} = 1\cdot 2 + 2(-1) + 2\cdot 2 = 4, \quad |\mathbf{a}| = |\mathbf{b}| = 3.$$

Acute angle:

$$\cos\theta = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{a}||\mathbf{b}|} = \frac{4}{9} \ \Rightarrow\ \theta = \cos^{-1}\left(\tfrac{4}{9}\right) \approx 1.11 \text{ rad}.$$

Solution to Q165. [Back to Question p. 95] [Back to TOC]

Cross product and right-hand rule.
$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ -1 & 4 & 2 \end{vmatrix} = (-10, -7, 9).$

$$|\mathbf{a}\times\mathbf{b}| = \sqrt{(-10)^2 + (-7)^2 + 9^2} = \sqrt{230}.$$

Unit normal (right-hand rule):

$$\mathbf{n} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = \frac{1}{\sqrt{230}}\langle -10, -7, 9\rangle.$$

Solution to Q166. [Back to Question p. 96] [Back to TOC]

Area of a parallelogram and a triangle.
$\mathbf{p}\times\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix} = \langle 0, 0, 5\rangle$. Area(parallelogram) $= |\mathbf{p}\times\mathbf{q}| = 5$. Area(triangle) $= \frac{1}{2}|\mathbf{p}\times\mathbf{q}| = \frac{5}{2}$.

Solution to Q167. [Back to Question p. 96] [Back to TOC]

Area of a triangle from three points (3D).
$\overrightarrow{PQ} = \langle 2, -3, 1\rangle,\ \overrightarrow{PR} = \langle -1, 0, -2\rangle.$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -1 & 0 & -2 \end{vmatrix} = \langle 6, 3, -3\rangle.$$

$|\overrightarrow{PQ} \times \overrightarrow{PR}| = \sqrt{36+9+9} = \sqrt{54} = 3\sqrt{6}$. Area$(\triangle PQR) = \frac{1}{2} \cdot 3\sqrt{6} = \frac{3}{2}\sqrt{6}$.

Solution to Q168. [Back to Question p. 96] [Back to TOC]

Projection and component along a direction.
$\mathbf{a} = \langle 3, 4, 0\rangle$, $\mathbf{b} = \langle 1, 2, 2\rangle$.

$$\mathbf{a} \cdot \mathbf{b} = 3(1) + 4(2) + 0(2) = 11, \quad |\mathbf{b}| = 3.$$

Scalar component (along $\mathbf{b}$):

$$\mathrm{comp}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{11}{3}.$$

Vector projection:

$$\mathrm{proj}_{\mathbf{b}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{11}{9}\langle 1, 2, 2\rangle = \left\langle \frac{11}{9}, \frac{22}{9}, \frac{22}{9} \right\rangle.$$

Solution to Q169. [Back to Question p. 96] [Back to TOC]

Perpendicular component magnitude.
Magnitude of component of $\mathbf{a}$ perpendicular to $\mathbf{b}$ in their plane:

$$|\mathbf{a}_\perp| = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}.$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 0 \\ 1 & 2 & 2 \end{vmatrix} = \langle 8, -6, 2\rangle, \qquad |\mathbf{a} \times \mathbf{b}| = \sqrt{8^2 + (-6)^2 + 2^2} = \sqrt{104} = 2\sqrt{26}.$$

Thus $|\mathbf{a}_\perp| = \dfrac{2\sqrt{26}}{3}$.

Solution to Q170. [Back to Question p. 97] [Back to TOC]

Resolve a vector into parallel and perpendicular parts.
$\mathbf{u} = \langle -2, 5, 1\rangle$, $\mathbf{b} = \langle 4, -1, 2\rangle$. $\mathbf{u}_\parallel = \dfrac{\mathbf{u} \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$, with $\mathbf{u} \cdot \mathbf{b} = -8 - 5 + 2 = -11$ and $|\mathbf{b}|^2 = 21$.

$$\mathbf{u}_\parallel = \frac{-11}{21}\langle 4, -1, 2\rangle = \left\langle -\frac{44}{21}, \frac{11}{21}, -\frac{22}{21} \right\rangle.$$

$$\mathbf{u}_\perp = \mathbf{u} - \mathbf{u}_\parallel = \left\langle \frac{2}{21}, \frac{94}{21}, \frac{43}{21} \right\rangle,$$

and $\mathbf{u}_\perp \cdot \mathbf{b} = 0$ (check: $\frac{8-94+86}{21} = 0$).

Solution to Q171. [Back to Question p. 97] [Back to TOC]

Work done (dot product application).
$W = \mathbf{F} \cdot \mathbf{d} = 6 \cdot 3 + (-2) \cdot 4 + 5 \cdot (-1) = 18 - 8 - 5 = 5$ J.

Solution to Q172. [Back to Question p. 97] [Back to TOC]

Angle in 2D via dot product.
$\mathbf{p} \cdot \mathbf{q} = 5(-1) + 2(4) = 3$. $|\mathbf{p}| = \sqrt{29}$, $|\mathbf{q}| = \sqrt{17}$.

$$\cos\theta = \frac{3}{\sqrt{29}\sqrt{17}} = \frac{3}{\sqrt{493}} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{3}{\sqrt{493}}\right) \approx 1.44 \text{ rad } (82.3^\circ).$$

Since $0 < \theta < \frac{\pi}{2}$, the vectors are *acute*.

Solution to Q173. [Back to Question p. 98] [Back to TOC]

Acute angle between lines in the plane.
$\mathbf{d}_1 = \langle 2, 3\rangle,\ \mathbf{d}_2 = \langle -1, 4\rangle.$

$$\cos\theta = \frac{|\mathbf{d}_1 \cdot \mathbf{d}_2|}{|\mathbf{d}_1||\mathbf{d}_2|} = \frac{|2(-1)+3(4)|}{\sqrt{13}\sqrt{17}} = \frac{10}{\sqrt{221}} \Rightarrow \theta = \cos^{-1}\left(\frac{10}{\sqrt{221}}\right) \approx 0.841 \text{ rad } (48.2^\circ).$$

Solution to Q174. [Back to Question p. 98] [Back to TOC]

Mixed: show perpendicular via dot, area via cross.
$\mathbf{u} = \langle 1, 2, 3\rangle,\ \mathbf{v} = \langle -2, 1, 0\rangle.$

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 1 & 0 \end{vmatrix} = \langle -3, -6, 5\rangle.$$

$\mathbf{u} \cdot \mathbf{w} = -3 - 12 + 15 = 0,\quad \mathbf{v} \cdot \mathbf{w} = 6 - 6 + 0 = 0$ so $\mathbf{w} \perp \mathbf{u}$ and $\mathbf{w} \perp \mathbf{v}$. Area of parallelogram $= |\mathbf{u} \times \mathbf{v}| = \sqrt{(-3)^2 + (-6)^2 + 5^2} = \sqrt{70}$.

Topic AHL3.14 — Graph theory

Solution to Q175. [Back to Question p. 100] [Back to TOC]

Basic terms; degree of a vertex.
Vertices: $V(G) = \{A, B, C, D, E\}$. Edges (undirected): $E(G) = \{AB, BC, CD, DA, AC, AE\}$ (6 edges).
Adjacency:

$$N(A) = \{B, C, D, E\}, \quad N(B) = \{A, C\},$$
$$N(C) = \{A, B, D\}, \quad N(D) = \{A, C\}, \quad N(E) = \{A\}.$$

Degrees: $\deg A = 4$, $\deg B = 2$, $\deg C = 3$, $\deg D = 2$, $\deg E = 1$. Degree sequence (non-increasing): $(4, 3, 2, 2, 1)$.

Solution to Q176. [Back to Question p. 100] [Back to TOC]

Simple vs. non-simple.
First graph: *not* simple — there are two parallel edges between A and B.
Second graph: *not* simple — vertex C has a loop. (A simple graph has no loops and no multiple edges.)

Solution to Q177. [Back to Question p. 101] [Back to TOC]

Complete graphs.
(a) In K_5, each vertex is adjacent to the other 4 vertices, so deg $= 4$. Number of edges: $\binom{5}{2} = 10$.
(b) In general, in K_n each vertex has degree $n-1$; the number of edges is $|E| = \binom{n}{2} = \dfrac{n(n-1)}{2}$.

Solution to Q178. [Back to Question p. 101] [Back to TOC]

Adjacency matrix (undirected).
With order (A, B, C, D, E),

$$\mathrm{Adj}(G) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Sum of all entries $= 12$. Since the graph is undirected, this equals $2|E|$, so $|E| = 12/2 = 6$, agreeing with the edge list.

Solution to Q179. [Back to Question p. 101] [Back to TOC]

Weighted graph: shortest path.
Candidate $A \to C$ paths and weights: $A{-}C : 7$, $A{-}B{-}C : 3 + 5 = 8$, $A{-}D{-}E{-}C : 4 + 2 + 3 = 9$, $A{-}B{-}E{-}C : 3 + 6 + 3 = 12$. Hence a shortest $A \to C$ path is the direct edge with total weight $\boxed{7}$.
For $A \to E$: $A{-}D{-}E : 4 + 2 = 6$ (shortest), $A{-}B{-}E : 3 + 6 = 9$, $A{-}C{-}E : 7 + 3 = 10$. Thus the minimum A–E path length is $\boxed{6}$ via $A{-}D{-}E$.

Solution to Q180. [Back to Question p. 102] [Back to TOC]

Connectedness.
Yes, G is connected (every vertex is reachable from A, and E is attached to A). Removing vertex A disconnects the graph (vertex E becomes isolated), so A is a cut-vertex.

Solution to Q181. [Back to Question p. 102] [Back to TOC]

Directed graphs: in-degree and out-degree.

Row sums give out-degrees; column sums give in-degrees.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{array}{c|cccc} & A & B & C & D \\ \hline \text{outdeg} & 2 & 2 & 1 & 1 \\ \text{indeg} & 1 & 1 & 2 & 2 \end{array}$$

(b) $D \to A$ (from last row/first column), and from A we reach B and C; from B we reach C and D; from C we reach D. Thus every vertex can reach every other; D is *strongly connected.*

Solution to Q182. [Back to Question p. 103] [Back to TOC]

Directed graph: strongly connected or not.
There is a directed 3-cycle $A \to B \to C \to A$ and a 2-cycle $C \leftrightarrow D$. Hence the digraph is *strongly connected.* Example paths: $D \to C \to A$ and $A \to B \to C \to D$.

Solution to Q183. [Back to Question p. 103] [Back to TOC]

Model a real situation as a graph.
(a) Weighted, undirected graph with edges: $S_1S_2(4)$, $S_2S_3(2)$, $S_3S_5(6)$, $S_2S_4(5)$, $S_4S_5(3)$, $S_1S_4(8)$.
(b) Shortest $S_1 \to S_5$: compare S_1–S_4–S_5 : 8+3 = 11, S_1–S_2–S_3–S_5 : 4+2+6 = 12, S_1–S_2–S_4–S_5 : 4+5+3 = 12. Minimum time $\boxed{11 \text{ min}}$ via $S_1 - S_4 - S_5$.

Solution to Q184. [Back to Question p. 103] [Back to TOC]

Subgraphs.
Example choice: H with $V(H) = \{A, B, C\}$, $E(H) = \{AB, BC, AC\}$. (a) H is connected. (b) H contains a cycle (triangle ABC). (c) $|V(H)| = 3$, $|E(H)| = 3$.

Solution to Q185. [Back to Question p. 104] [Back to TOC]

Trees.
(a) The graph is connected and has 5 vertices and 4 edges with no cycle, so it is a tree. (b) Leaves: A, D, E (degree 1). (c) Adding edge AD creates a cycle $A-B-D-A$; the graph would no longer be a tree (also 5 vertices, 5 edges).

Solution to Q186. [Back to Question p. 104] [Back to TOC]

Counting edges via degrees (handshake).
Sum of degrees $= 4+4+3+3+2+2+2+2 = 22$. (a) There are 8 vertices and $|E| = \frac{22}{2} = 11$ edges. (b) A complete graph on 8 vertices would have degree 7 at each vertex, so this sequence cannot be complete.

Solution to Q187. [Back to Question p. 104] [Back to TOC]

Complete/weighted hybrid.
Hamiltonian paths starting at A (weights in parentheses):
$A-B-C-D$ $(1+2+1=4)$, $A-B-D-C$ $(1+5+1=7)$, $A-C-B-D$ $(4+2+5=11)$,
$A-C-D-B$ $(4+1+5=10)$, $A-D-B-C$ $(3+5+2=10)$, $A-D-C-B$ $(3+1+2=6)$.
Minimum is $\boxed{A-B-C-D}$ with total weight $\boxed{4}$ (unique).

Solution to Q188. [Back to Question p. 105] [Back to TOC]

From graph to matrix and back.

Edges: $12, 23, 31, 34$. With order $(1, 2, 3, 4)$,

$$\mathrm{Adj} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Vertices adjacent to 3: $\{1, 2, 4\}$.

AHL3.15 — Adjacency matrices

Solution to Q189. [Back to Question p. 106] [Back to TOC]

Adjacency matrix from a graph (undirected) and 2-step walks. Edges: $\{12, 23, 34, 41, 13\}$ on $V = \{1, 2, 3, 4\}$ (order $(1, 2, 3, 4)$).

1. The adjacency matrix is
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$
2. A^2 (counts of 2-step walks) is
$$A^2 = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}.$$
The entry $(A^2)_{14} = 1$ means there is exactly one walk of length 2 from 1 to 4 (namely $1 \to 3 \to 4$).
3. The number of 2-step walks from 2 to 4 is $(A^2)_{24} = 2$ (the walks $2 \to 1 \to 4$ and $2 \to 3 \to 4$).

Solution to Q190. [Back to Question p. 106] [Back to TOC]

Walk counts from powers of A.

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \\ 4 & 4 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{pmatrix}.$$

1. $(A^3)_{12} = 3$: there are 3 distinct walks of length 3 from vertex 1 to vertex 2.
2. The total number of walks of length ≤ 3 from 1 to 2 is the $(1, 2)$ entry of
$$S_3 = I + A + A^2 + A^3,$$
which equals $0 + 1 + 1 + 3 = 5$.

Solution to Q191. [Back to Question p. 106] [Back to TOC]

Closed walks. Using the A from Question 2:

1. The number of closed walks of length 3 at vertex 3 is $(A^3)_{33} = 2$.
2. Yes, the graph contains a triangle (the 3-cycle on $\{1, 2, 3\}$). One way to see this is that $(A^3)_{ii} > 0$ for $i \in \{1, 2, 3\}$; for a simple undirected graph, each triangle through i contributes 2 to $(A^3)_{ii}$ (clockwise/counterclockwise).

Solution to Q192. [Back to Question p. 107] [Back to TOC]

From directed graph to adjacency matrix and reachability. Order (A, B, C, D), rows=sources, cols=targets.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

From $(A^3)_{DA} = (A^3)_{41} = 1$, there is a walk from D to A of length 3 (e.g. $D \to B \to C \to A$).

Solution to Q193. [Back to Question p. 107] [Back to TOC]

Weighted adjacency table. Order (P, Q, R, S). Missing pairs have weight 0; undirected means symmetry.

$$W = \begin{pmatrix} 0 & 4 & 2 & 7 \\ 4 & 0 & 3 & 0 \\ 2 & 3 & 0 & 5 \\ 7 & 0 & 5 & 0 \end{pmatrix}.$$

1. (As above.)
2. Total weight of $P \to R \to S \to P$ is $w(PR) + w(RS) + w(SP) = 2 + 5 + 7 = 14$.
3. Walks of length 2 from Q to S:

$$Q \to P \to S \text{ (weight } 4 + 7 = 11), \qquad Q \to R \to S \text{ (weight } 3 + 5 = 8).$$

Solution to Q194. [Back to Question p. 107] [Back to TOC]

Transition matrix of a simple random walk (undirected). Edges $\{12, 13, 23, 24\}$ with degrees $\deg(1) = 2,\ \deg(2) = 3,\ \deg(3) = 2,\ \deg(4) = 1$.

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Each row sums to 1 by construction (uniform over neighbors). Moreover,

$$(P^2)_{14} = \tfrac{1}{2} \cdot \tfrac{1}{3} = \tfrac{1}{6},$$

coming from the only 2-step route $1 \to 2 \to 4$.

Solution to Q195. [Back to Question p. 108] [Back to TOC]

Transition matrix of a directed random walk (uniform over out-edges). Arcs $1 \to 2, 1 \to 3, 2 \to 3, 2 \to 4, 3 \to 1, 4 \to 3$ give

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The digraph is strongly connected: for example

$$1 \to 2 \to 4, \quad 4 \to 3 \to 1 \to 2, \quad 2 \to 3 \to 1, \quad 3 \to 1 \to 2 \to 4,$$

so every vertex can reach every other (equivalently, suitable powers of P have positive entries in all positions).

Solution to Q196. [Back to Question p. 108] [Back to TOC]

Weighted random walk (probability proportional to weight). Normalize each row of

$$W = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 3 \\ 4 & 0 & 0 \end{pmatrix} \Rightarrow P = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ 1 & 0 & 0 \end{pmatrix}.$$

From state 1, the probability to be at state 3 after two steps is

$$(P^2)_{13} = \tfrac{2}{3} \cdot \tfrac{3}{4} + \tfrac{1}{3} \cdot 0 = \tfrac{1}{2}.$$

Solution to Q197. [Back to Question p. 109] [Back to TOC]

Counting at most k-step walks. By induction on m, $(A^m)_{ij}$ counts walks of length m from i to j: for $m = 1$ this is the definition of A; if true for m, then

$$(A^{m+1})_{ij} = \sum_{\ell} (A^m)_{i\ell} \, A_{\ell j}$$

sums, over all intermediate vertices ℓ, the number of m-walks $i \to \ell$ times the indicator of an edge $\ell \to j$, i.e. the number of $(m{+}1)$-walks $i \to j$. Hence $S_k = I + A + \cdots + A^k$ has $(S_k)_{ij}$ equal to the number of walks of length $\leq k$. For the A in Question 2,

$$(S_3)_{12} = (I)_{12} + (A)_{12} + (A^2)_{12} + (A^3)_{12} = 0 + 1 + 1 + 3 = 5.$$

Solution to Q198. [Back to Question p. 109] [Back to TOC]

Stationarity check (link to Markov chains).

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{pmatrix}, \quad P^3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

A stationary distribution $\pi = [\pi_1, \pi_2, \pi_3]$ satisfies $\pi P = \pi$ and $\sum_i \pi_i = 1$. Solving gives

$$\pi = \left[\tfrac{2}{5}, \, \tfrac{2}{5}, \, \tfrac{1}{5}\right].$$

Interpretation: π gives the long-run proportion of time spent in each state (and here the chain is irreducible and aperiodic, so $X_n \Rightarrow \pi$).

Solution to Q199. [Back to Question p. 110] [Back to TOC]

PageRank-style transition with damping (small web). Order (A, B, C, D). The uniform link-following matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

With $\alpha = 0.85$ and J the 4×4 all-ones matrix,

$$G = \alpha P + (1-\alpha)\tfrac{1}{4}\,J.$$

Starting from $v^{(0)} = \left[\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right]$,

$$v^{(1)} = v^{(0)}G = 0.85\left(v^{(0)}P\right) + 0.15 \cdot \tfrac{1}{4}\,[1,1,1,1] = \left[\tfrac{1}{4},\ \tfrac{23}{160},\ \tfrac{91}{160},\ \tfrac{3}{80}\right] \approx [0.25,\ 0.14375,\ 0.56875,\ 0.0375].$$

Solution to Q200. [Back to Question p. 110] [Back to TOC]

Using powers to test strong connectivity.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus

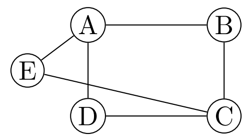
$$R = I + A + A^2 + A^3 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From R, vertices $1, 2, 3$ are mutually reachable (both directions have positive entries), but vertex 4 cannot reach others (row 4 has zeros off the diagonal). Hence the digraph is *not* strongly connected. A strongly connected component is $\{1, 2, 3\}$; $\{4\}$ is another (sink) component.

AHL3.16 — Chinese Postman Problem, Travelling Salesman Problem and more graph theory

Solution to Q201. [Back to Question p. 114] [Back to TOC]

Walks, trails, paths, circuits, cycles. The graph has edges $\{AB, BC, CD, DA, AE, EC\}$.



1. $A \to B \to C \to D \to A$ uses AB, BC, CD, DA with no edge repeated and returns to the start.

   Classification: walk, trail, *circuit*, *cycle*; not a path (start/end repeat).

2. $A \to E \to C \to D \to A \to B$ uses AE, EC, CD, DA, AB with no edge repeated and distinct start/end.

   Classification: walk, *trail*; not a path (vertex A repeats), not a circuit/cycle (open).

3. $A \to B \to C \to A$ would require edge CA, which does *not* exist. Hence it is not a walk in this graph.

Definitions (brief).

- *Trail:* a walk with no repeated edges.
- *Path:* a walk with no repeated vertices.
- *Circuit:* a closed trail (starts = ends, no edge repeats).
- *Cycle:* a closed path (starts = ends, no vertex repeats except start/end).

Solution to Q202. [Back to Question p. 114] [Back to TOC]

Eulerian trails and circuits. Degrees: $\deg(A) = 3,\ \deg(B) = 2,\ \deg(C) = 3,\ \deg(D) = 2,\ \deg(E) = 2.$ Only A and C are odd.

1. **No Eulerian circuit** (not all degrees even). **Eulerian trail exists** (connected with exactly two odd vertices).
2. One Eulerian trail is
   $$A \to B \to C \to D \to A \to E \to C,$$
   which uses each edge exactly once, starting at A and ending at C.
3. Justification: In any undirected graph, an Eulerian circuit exists iff every vertex has even degree; an Eulerian trail exists iff there are exactly two odd-degree vertices (the trail starts/ends at them).

Solution to Q203. [Back to Question p. 114] [Back to TOC]

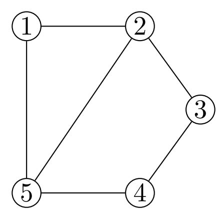
Hamiltonian paths and cycles. Edges $\{AB, BC, CD, DE, EA, AC\}$ on $\{A, B, C, D, E\}$.

1. **Hamiltonian cycle exists:** $A - B - C - D - E - A$ (all listed edges).
2. **Hamiltonian path (not a cycle):** $B - C - D - E - A$.
3. Each visits every vertex exactly once; the first returns to the start (cycle), the second does not (path).

Solution to Q204. [Back to Question p. 115] [Back to TOC]

Tree vs. cycle detection.

1. If a connected graph on n vertices has $n-1$ edges, it must be acyclic (otherwise removing one edge from a cycle would keep it connected with at most $n-2$ edges, contradiction). Thus it is a tree. Conversely, every tree has exactly $n-1$ edges.
2. For $V = \{1, 2, 3, 4, 5\}$, $E = \{12, 23, 34, 45, 15, 25\}$: $1-2-3-4-5-1$ is a 5-cycle; moreover $1-2-5-1$ (using $12, 25, 15$) is a 3-cycle. A DFS from 1 (taking smaller neighbours first) reveals back edge $5-1$ closing the cycle.



Solution to Q205. [Back to Question p. 115] [Back to TOC]

Minimum spanning tree (Kruskal). List (upper-triangle) edges in increasing order:

$$BC(1),\ AC(2),\ BD(3),\ AB(4),\ DE(4),\ CD(5),\ BE(6),\ AD(7),\ CE(8),\ AE(9).$$

Kruskal picks: BC, AC, BD (skip AB as it closes a cycle), then DE to connect E.

$$\text{MST edges } = \{BC(1),\ AC(2),\ BD(3),\ DE(4)\}, \quad \text{weight } 1+2+3+4 = 10.$$

Solution to Q206. [Back to Question p. 115] [Back to TOC]

Minimum spanning tree (Prim, matrix method) from A.

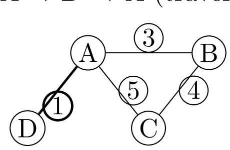
Tree vertices	Chosen edge & reason
$\{A\}$	$AC(2)$ is the smallest from A (vs. 4,7,9)
$\{A, C\}$	$CB(1)$ is the smallest crossing edge (1 vs. 3,4,5,7,8,9)
$\{A, B, C\}$	$BD(3)$ is the smallest to bring in D or E (3 vs. 4,5,7,8,9)
$\{A, B, C, D\}$	$DE(4)$ is the smallest to bring in E (4 vs. 6,8,9)

Result: same MST $\{BC, AC, BD, DE\}$ with total weight 10.

Solution to Q207. [Back to Question p. 116] [Back to TOC]

Chinese postman: two odd vertices. Edges and weights: $AB : 3,\ BC : 4,\ CA : 5,\ AD : 1$. Degrees: $\deg(A) = 3,\ \deg(B) = 2,\ \deg(C) = 2,\ \deg(D) = 1$. Odd vertices: A and D.

1. Odd vertices: A, D.
2. Duplicate the shortest A–D path (edge AD of weight 1). Total original weight $= 3+4+5+1 = 13$, so Chinese postman length $= 13 + 1 = 14$.
3. One optimal circuit: $A \to B \to C \to A \to D \to A$ (traverses AD twice).



Solution to Q208. [Back to Question p. 116] [Back to TOC]

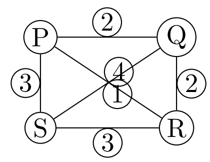
Chinese postman: four odd vertices and pairings. All four vertices P, Q, R, S are odd (it is K_4). Total weight:

$$2+2+3+3+4+1=15.$$

Possible pairings of odd vertices and added weights:

$$(PQ, RS): 2+3=5, \quad (PR, QS): 4+1=5, \quad (PS, QR): 3+2=5.$$

Minimum added weight is 5, so Chinese postman length $= 15+5=20$. One choice: duplicate PR and QS.



Solution to Q209. [Back to Question p. 117] [Back to TOC]

Why the Chinese postman algorithm works. In an undirected graph, an Eulerian circuit exists iff all degrees are even. Any closed route that uses each edge at least once induces a multigraph where the degree of every vertex is even (each arrival must be matched by a departure). Therefore we must add edges so that all currently odd-degree vertices become even. This forces the odd vertices to be *paired* and joined by added paths. The added total weight equals the sum of the chosen pairwise path lengths, so minimizing the total added weight is exactly the problem of finding a minimum-weight perfect matching on the odd vertices with edge weights equal to shortest-path distances. Adding these paths produces an Eulerian multigraph of minimum added weight, hence a shortest postman tour.

Solution to Q210. [Back to Question p. 117] [Back to TOC]

TSP: exact on a small instance. Testing tours (fixing start A) gives, for example,

$$A-B-D-C-E-A: \ 7+2+3+5+7=24, \qquad A-E-C-D-B-A: \ 7+5+3+2+7=24.$$

A check over the $4! = 24$ A-anchored tours shows 24 is minimal.

$$\boxed{\text{Optimal length } = 24, \text{ one optimal cycle } A-B-D-C-E-A.}$$

Solution to Q211. [Back to Question p. 117] [Back to TOC]

Nearest neighbour heuristic (upper bound).

1. Start at A: choose $B(7) \to D(2) \to C(3) \to E(5) \to A(7)$, tour $A-B-D-C-E-A$ of length 24. (If the first tie had chosen E, the length would be 26; we take the shorter.)
2. Start at B: $B \to D(2) \to C(3) \to E(5) \to A(7) \to B(7)$ gives length 24.
3. Both starts yield the same (and optimal) upper bound 24.

Solution to Q212. [Back to Question p. 118] [Back to TOC]

Deleted-vertex lower bound for TSP (delete A). On $\{B, C, D, E\}$ the MST has edges $BD(2)$, $CD(3)$, $DE(4)$, weight 9. Add the two smallest A-incident edges ($AB = 7$, $AE = 7$), giving lower bound $9+7+7=23$. Compared to the upper bound 24, the gap is 1.

Solution to Q213. [Back to Question p. 118] [Back to TOC]

From practical to classical TSP via least-distance table. *Least distances* (by shortest paths) for U, V, W, X, Y:

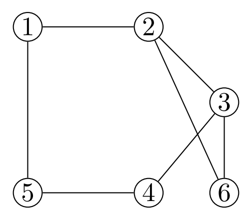
	U	V	W	X	Y
U	0	2	4	5	6
V	2	0	2	3	4
W	4	2	0	1	2
X	5	3	1	0	1
Y	6	4	2	1	0

Nearest neighbour from U: $U \to V(2) \to W(2) \to X(1) \to Y(1) \to U(6)$ gives a feasible tour of length $2+2+1+1+6 = 12$ (an *upper bound* for the practical instance).

Deleted-vertex bound (delete U): MST on $\{V, W, X, Y\}$ uses $WX(1)$, $XY(1)$, $WV(2)$, weight 4; add two smallest U-links $(2, 4)$ to get a *lower bound* $4 + 2 + 4 = 10$. Gap: $12 - 10 = 2$.

Solution to Q214. [Back to Question p. 118] [Back to TOC]

Cycle edges vs. tree edges (DFS). Run DFS from 1 exploring smaller-numbered neighbours first.



Tree edges in discovery order:

$$(1, 2),\ (2, 3),\ (3, 4),\ (4, 5),\ (3, 6).$$

One back edge is $(5, 1)$, which lies on the simple cycle $1-2-3-4-5-1$.

Solution to Q215. [Back to Question p. 119] [Back to TOC]

Euler vs. Hamilton in practice.

- *Eulerian (edge-focused):* Street-sweeping or postal delivery where the objective is to traverse each *street* at least once and return to base. Model: vertices = intersections; edges = streets; seek an Eulerian circuit (Chinese postman).
- *Hamiltonian (vertex-focused):* Visiting sales calls or inspection sites once each with minimal travel. Model: vertices = cities/sites; edge weights = travel costs; seek a minimum Hamiltonian cycle (TSP).

Topic 4 Solutions

SL 4.1 Populations, Samples and Sampling Methods

Solution to Q216. [Back to Question p. 121] [Back to TOC]

A population is the entire set of individuals or objects of interest in a study (for example, all students in a school). A sample is a subset of the population selected for analysis. A simple random sample gives each member of the population an equal chance of selection. This reduces selection bias and allows for generalization, but it may be impractical for large or inaccessible populations and does not guarantee that all subgroups are represented.

Solution to Q217. [Back to Question p. 121] [Back to TOC]

Identify outliers with fences (number line diagram). Data (sorted):

$$12,\ 14,\ 15,\ 16,\ 18,\ 19,\ 20,\ 22,\ 23,\ 25,\ 40.$$

1. With $n = 11$, Q_2 is the 6th value $\Rightarrow Q_2 = 19$. Lower half $= \{12, 14, 15, 16, 18\}$ so $Q_1 = 15$. Upper half $= \{20, 22, 23, 25, 40\}$ so $Q_3 = 23$. Hence $\text{IQR} = Q_3 - Q_1 = 23 - 15 = 8$.
2. Lower fence $= Q_1 - 1.5\text{IQR} = 15 - 1.5(8) = 15 - 12 = 3$.
   Upper fence $= Q_3 + 1.5\text{IQR} = 23 + 12 = 35$.
3. Any value < 3 or > 35 is an outlier. The only outlier is $\boxed{40}$.
4. On the number line, mark fences at 3 and 35; circle 40 as the outlier.

Solution to Q218. [Back to Question p. 121] [Back to TOC]

Reading a box-and-whisker plot (with a suspected high outlier). Given $Q_1 = 18$, $Q_2 = 24$, $Q_3 = 30$, smallest non-outlier $= 12$, and a point at 60.

1. $\text{IQR} = 30 - 18 = 12$; upper fence $= Q_3 + 1.5\text{IQR} = 30 + 1.5(12) = 30 + 18 = 48$.
2. Since $60 > 48$, the 60% score is **an outlier** by the $1.5 \times \text{IQR}$ rule.
3. Possible reasons: (i) exceptionally strong student or bonus questions legitimately earned; (ii) data-entry issue (e.g. typed 60 instead of 50) or academic integrity problem. Keep/remove decision: keep if verified as genuine performance; remove (or analyze separately) if it is an error or violates the study's assumptions.

Solution to Q219. [Back to Question p. 122] [Back to TOC]

Commuting times in a city (context + mini diagram). $Q_1 = 25$, $Q_3 = 50$.

1. $\text{IQR} = 50 - 25 = 25$. Lower fence $= 25 - 1.5(25) = 25 - 37.5 = -12.5$. Upper fence $= 50 + 1.5(25) = 50 + 37.5 = 87.5$.
2. $4 > -12.5$ so 4 is *not* an outlier by this rule. $150 > 87.5$ so 150 *is* an outlier.
3. On the axis, place fences at -12.5 and 87.5; mark 4 inside the fences and 150 beyond the upper fence.

Solution to Q220. [Back to Question p. 122] [Back to TOC]

Comparing spread and the chance of outliers. Both medians are 5; Factory A has $\text{IQR} = 1$, Factory B has $\text{IQR} = 3$.

1. **Factory A** is *more likely* to have values flagged as outliers by the $1.5 \times \text{IQR}$ rule because its fences ($Q_1 - 1.5\,\text{IQR}$ and $Q_3 + 1.5\,\text{IQR}$) lie much closer to the quartiles when the IQR is small. With the wider IQR of Factory B, fences are farther out, so fewer points are flagged for similar tail behaviour.

2. Advantage of removing outliers: comparisons focus on the typical performance and reduce distortion of the mean/scale by extreme values. Disadvantage: you may discard *genuine* extreme delays, biasing conclusions and hiding important reliability issues; sample size also decreases.

Solution to Q221. [Back to Question p. 123] [Back to TOC]

Effect of an outlier on mean and median. There are $n = 20$ scores with mean 72 $\Rightarrow$ total $= 20 \times 72 = 1440$.

1. Removing the outlier 5 leaves sum $1440 - 5 = 1435$ over 19 scores, so

$$\text{new mean} = \frac{1435}{19} = 75 + \frac{10}{19} \approx \boxed{75.53}.$$

2. If 5 was the smallest value (very likely), it was not one of the middle two positions used to compute the median when $n = 20$. After removal, the median becomes the 10th value of the remaining 19, which will typically be the same as before. Hence the median *does not change* (stays about 73).

3. The **median** is more robust to such a low outlier; the mean is pulled downward strongly by extreme values.

Solution to Q222. [Back to Question p. 123] [Back to TOC]

Outlier or data-entry mistake? Reason from context.

1. Valid reasons for $x = 0$: extremely heavy rain kept birds away; feeder temporarily empty or a predator/deterrent present that day.

2. Possible recording errors: observer missed the count or started the log at the wrong time; device/app recorded a default zero; day mislabeled.

3. Example rule: *Flag any point outside the* $1.5 \times \text{IQR}$ *fences and investigate the field notes; retain if a documented, plausible condition explains it, otherwise treat as an error and exclude with justification.* Apply the rule consistently and report both analyses (with and without flagged points) when conclusions could change.

SL 4.2 Measures of Central Tendency

Solution to Q223. [Back to Question p. 124] [Back to TOC]

The mean is $\bar{x} = \frac{3+7+8+10+12+12+16+20}{8} = \frac{88}{8} = 11$. Ordered data: $3, 7, 8, 10, 12, 12, 16, 20$. The median is the average of the 4th and 5th terms: $\frac{10+12}{2} = 11$. The mode is 12 (occurs twice). In this case the mean and median coincide, indicating a symmetric distribution. The mode highlights the most frequent value but is less informative here.

SL 4.3 Measures of Dispersion

Solution to Q224. [Back to Question p. 126] [Back to TOC]

The range is $20 - 3 = 17$. To find quartiles, the median splits the data into $3, 7, 8, 10$ and $12, 12, 16, 20$. The lower quartile Q_1 is the median of the first half: $\frac{7+8}{2} = 7.5$. The upper quartile Q_3 is $\frac{12+16}{2} = 14$. Thus $\text{IQR} = Q_3 - Q_1 = 14 - 7.5 = 6.5$. For the sample standard deviation,

$$s = \sqrt{\frac{\sum(x_i - \bar{x})^2}{n-1}} = \sqrt{\frac{(3-11)^2 + (7-11)^2 + \cdots + (20-11)^2}{7}} \approx 5.66.$$

Together these statistics indicate moderate spread with a few values (16,20) pulling the upper tail.

SL 4.4 Data Presentation and Bivariate Statistics

Solution to Q225. [Back to Question p. 127] [Back to TOC]

(i) A scatter plot should show a positive association between study hours and scores. (ii) Compute the correlation coefficient using $r = \frac{n\sum xy - \sum x \sum y}{\sqrt{n\sum x^2 - (\sum x)^2}\sqrt{n\sum y^2 - (\sum y)^2}}$. Here $n = 10$, $\sum x = 50$, $\sum y = 846$, $\sum x^2 = \sum x^2$, and $\sum xy = 2 \cdot 68 + 3 \cdot 75 + \cdots + 9 \cdot 96$; substituting yields $r \approx 0.987$, indicating a very strong positive correlation. (iii) The regression line has slope $m = r\frac{s_y}{s_x}$ and intercept $c = \bar{y} - m\bar{x}$, where $(\bar{x}, \bar{y}) = (5, 84.6)$ approximately. One finds $m \approx 3.4$ and $c \approx 67.6$, giving $\hat{y} \approx 3.4x + 67.6$. This line can be used to predict a student's score based on study hours.

SL 4.5 Probability basics

Solution to Q226. [Back to Question p. 128] [Back to TOC]

Q1. Bag with 5 red, 3 blue, 2 green (total 10).

1. Sample space (by colour): $\{R, B, G\}$.
2. Probabilities:
   (a) $P(R) = \dfrac{5}{10} = \dfrac{1}{2}$.
   (b) $P(B \text{ or } G) = \dfrac{3+2}{10} = \dfrac{5}{10} = \dfrac{1}{2}$.
3. $P(\text{not } R) = 1 - P(R) = 1 - \frac{1}{2} = \frac{1}{2}$ (equivalently $\frac{3+2}{10}$).

Solution to Q227. [Back to Question p. 128] [Back to TOC]

Q2. One fair die.

1. Sample space: $\{1, 2, 3, 4, 5, 6\}$.
2. (a) $P(\text{number} > 4) = \dfrac{2}{6} = \dfrac{1}{3}$ (outcomes $5, 6$).
   (b) $P(\text{even}) = \dfrac{3}{6} = \dfrac{1}{2}$ (outcomes $2, 4, 6$).
3. $A = \{\text{roll a } 6\}$, so $P(A^c) = 1 - P(A) = 1 - \dfrac{1}{6} = \dfrac{5}{6}$.

Solution to Q228. [Back to Question p. 128] [Back to TOC]

Q3. Survey: 60 of 200 walk.

1. $\hat{P}(\text{walk}) = \dfrac{60}{200} = 0.30$.
2. Expected number in a school of 1200: $0.30 \times 1200 = \boxed{360}$.

Solution to Q229. [Back to Question p. 129] [Back to TOC]

Q4. Marbles: 4 white, 5 black, 1 red (total 10).

1. With replacement, the long-run relative frequency $\approx P(\text{white}) = \dfrac{4}{10} = 0.4$, so in 500 trials we expect about $0.4 \times 500 = \boxed{200}$ whites.
2. Theoretical probability is 0.4; experimental results should be close to 0.4 but may vary due to randomness.

Solution to Q230. [Back to Question p. 129] [Back to TOC]

Q5. 100 coin tosses; 54 heads.

1. Relative frequency of heads $= \dfrac{54}{100} = 0.54$.
2. Theoretical probability for a fair coin is 0.5.
3. Difference arises from sampling variability (random fluctuation in a finite sample). A biased coin or non-random procedure could also cause discrepancy.

Solution to Q231. [Back to Question p. 129] [Back to TOC]

Q6. Team wins 75% of games; 32 games.

1. Expected wins $= 0.75 \times 32 = \boxed{24}$.
2. Expected losses $= 32 - 24 = \boxed{8}$.

SL 4.6 Probability Rules

Solution to Q232. [Back to Question p. 131] [Back to TOC]

Given $P(A) = 0.55$, $P(B) = 0.40$, $P(A \cap B) = 0.22$:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.55 + 0.40 - 0.22 = 0.73.$$

$$P(A^c \cap B) = P(B) - P(A \cap B) = 0.40 - 0.22 = 0.18.$$

Check independence: $P(A)P(B) = 0.55 \cdot 0.40 = 0.22 = P(A \cap B)$, so A and B are independent. Then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.22}{0.40} = 0.55.$$

Solution to Q233. [Back to Question p. 131] [Back to TOC]

By the addition rule, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.5 - 0.3 = 0.8$. The complement of A has probability $P(A^c) = 1 - 0.6 = 0.4$. To test independence, compare $P(A \cap B)$ with $P(A)P(B)$: $P(A)P(B) = 0.6 \times 0.5 = 0.3$. Since this equals $P(A \cap B)$, A and B are independent events.

SL 4.7 Conditional Probability, Trees and DRV

Solution to Q234. [Back to Question p. 132] [Back to TOC]

(i) Draw branches for M_1 (probability 0.6) and M_2 (0.4). From each, draw branches for "good" and "defective" with respective probabilities 0.98/0.02 for M_1 and 0.95/0.05 for M_2. (ii) The overall probability of defect is $P(D) = 0.6 \times 0.02 + 0.4 \times 0.05 = 0.012 + 0.02 = 0.032$ (3.2%). (iii) By Bayes' theorem,

$$P(M_1 \mid D) = \frac{P(D \mid M_1)P(M_1)}{P(D)} = \frac{0.02 \times 0.6}{0.032} = \frac{0.012}{0.032} = 0.375.$$

So there is a 37.5% chance a defective item came from M_1.

Solution to Q235. [Back to Question p. 132] [Back to TOC]

X takes $0, 1, 2, 4$ with probabilities $0.25, 0.30, 0.20, 0.25$.

$$E[X] = 0 \cdot 0.25 + 1 \cdot 0.30 + 2 \cdot 0.20 + 4 \cdot 0.25 = 0 + 0.30 + 0.40 + 1.00 = 1.70.$$

A fair game (zero expected gain to the player) would charge entry fee $c = E[X] = \$1.70$, so the expected net is $E[X] - c = 0$.

SL 4.8 Discrete and Continuous Distributions

Solution to Q236. [Back to Question p. 133] [Back to TOC]

$X \sim \text{Bin}(n = 15, p = 0.08)$. Then

$$P(X = 2) = \binom{15}{2}(0.08)^2(0.92)^{13} \approx 0.2273, \qquad P(X \geq 3) = 1 - \sum_{k=0}^{2}\binom{15}{k}(0.08)^k(0.92)^{15-k} \approx 0.1130.$$

Mean $E[X] = np = 15(0.08) = 1.2$, variance $\text{Var}(X) = np(1-p) = 1.2(0.92) = 1.104$. The binomial model is appropriate because we have a fixed n, independent trials, two outcomes per trial, and constant success probability $p = 0.08$.

Solution to Q237. [Back to Question p. 133] [Back to TOC]

For $X \sim \text{Bin}(10, 0.3)$,

$$P(X = 4) = \binom{10}{4}0.3^4 0.7^6 = 210 \times 0.3^4 \times 0.7^6 \approx 0.200.$$

Similarly,

$$P(X \geq 6) = 1 - \big[P(X \leq 5)\big],$$

which can be computed from the cumulative distribution or by summing $P(X = k)$ for $k = 6$ to 10. A calculator yields approximately 0.047.

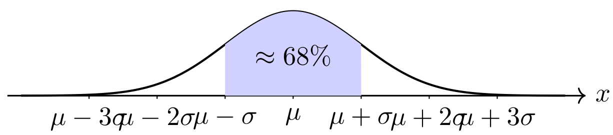
ASL 4.9 Normal distribution

Solution to Q238. [Back to Question p. 134] [Back to TOC]

Properties & diagram.

1. The mean and $\pm k\sigma$ points are marked below.
2. The central region $[\mu - \sigma, \mu + \sigma]$ contains about 68% of the data (shaded).
3. Using the 68–95–99.7 rule:

$$P(\mu - 2\sigma \le X \le \mu + 3\sigma) = P(|X - \mu| \le 2\sigma) + P(\mu + 2\sigma < X \le \mu + 3\sigma) \approx 95\% + \frac{99.7 - 95}{2}\% = 97.35\%.$$

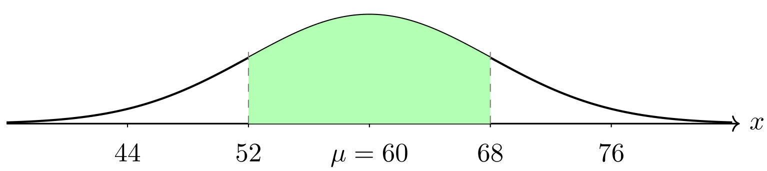


Solution to Q239. [Back to Question p. 134] [Back to TOC]

Normal probability (technology). $X \sim \mathcal{N}(60, 8)$.

$$z = \frac{x - \mu}{\sigma}.$$

1. $P(52 \le X \le 68) = P(-1 \le Z \le 1) = \Phi(1) - \Phi(-1) \approx 0.6827$.
2. $P(X \ge 76) = P(Z \ge 2) = 1 - \Phi(2) \approx 0.0228$.
3. $P(X \le 44) = P(Z \le -2) = \Phi(-2) \approx 0.0228$.



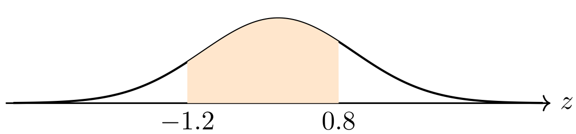
Solution to Q240. [Back to Question p. 134] [Back to TOC]

Standard normal interval. With $Z \sim \mathcal{N}(0, 1)$:

$$P(-1.2 < Z < 0.8) = \Phi(0.8) - \Phi(-1.2) \approx 0.7881 - 0.1151 = \boxed{0.6731},$$

$$P(Z \le -1.5) = \Phi(-1.5) = \boxed{0.0668},$$

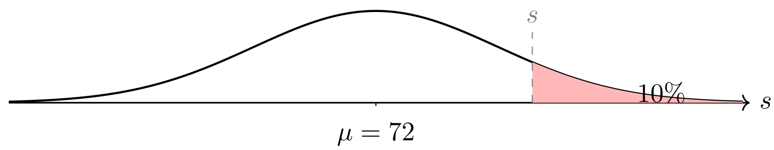
$$P(Z \ge 1.96) = 1 - \Phi(1.96) = \boxed{0.0250}.$$



Solution to Q241. [Back to Question p. 135] [Back to TOC]

Inverse normal (percentile). $S \sim \mathcal{N}(72, 9^2)$ and $P(S \ge s) = 0.10$. The 90th percentile of the standard normal is $z_{0.90} \approx 1.2816$. Hence

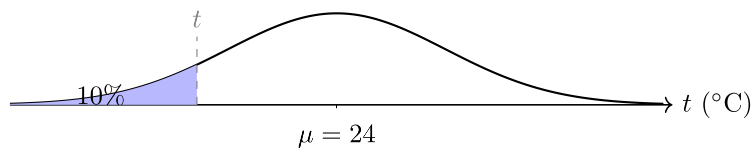
$$s = \mu + z_{0.90}\sigma = 72 + 1.2816(9) \approx \boxed{83.5} \text{ (thousand \$)}.$$



Solution to Q242. [Back to Question p. 135] [Back to TOC]

Cut-off for the lowest decile. $T \sim \mathcal{N}(24, 6^2)$ and $P(T \le t) = 0.10$. Here $z_{0.10} = -1.2816$, so

$$t = \mu + z_{0.10}\sigma = 24 + (-1.2816)(6) \approx \boxed{16.3°\text{C}}.$$



Solution to Q243. [Back to Question p. 136] [Back to TOC]

Two normals, same mean, different spread.

1. Curve A (the flatter/wider one) has the larger standard deviation. For a normal curve, larger $\sigma \Rightarrow$ lower peak and heavier spread about the same mean.
2. For *any* normal distribution, the proportion within one standard deviation of the mean is

$$P(\mu - \sigma \le X \le \mu + \sigma) \approx \boxed{68\% \text{ (more precisely } 68.27\%)}.$$

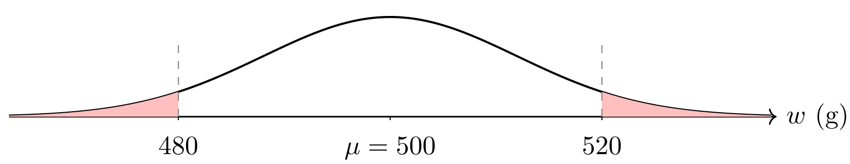
Solution to Q244. [Back to Question p. 136] [Back to TOC]

Quality control tails (technology). $W \sim \mathcal{N}(500, 12^2)$, reject if $W \notin [480, 520]$.

$$z = \frac{20}{12} = 1.\overline{6}.$$

$$P(\text{reject}) = P(|Z| \ge 1.6667) = 2\big(1 - \Phi(1.6667)\big) \approx \boxed{0.0956}.$$

Out of 10 000 packs, expect $10\,000 \times 0.0956 \approx \boxed{956}$ rejects.



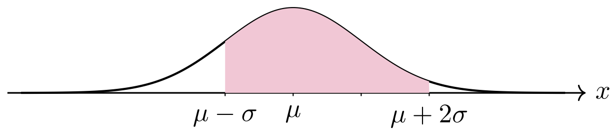
Solution to Q245. [Back to Question p. 136] [Back to TOC]

Diagram reading. Empirical estimate:

$$P(\mu - \sigma \le X \le \mu + 2\sigma) \approx 68\% + \frac{95 - 68}{2}\% = 68\% + 13.5\% = \boxed{81.5\%}.$$

Exact (standardize to Z):

$$P(-1 \le Z \le 2) = \Phi(2) - \Phi(-1) \approx 0.97725 - 0.15866 = \boxed{0.8186}.$$



Solution to Q246. [Back to Question p. 137] [Back to TOC]

Standardize: $Z = \frac{X-\mu}{\sigma}$. (i) For $P(X \geq 1250)$, $Z = \frac{1250-1200}{100} = 0.5$. Thus $P(Z \geq 0.5) = 1 - \Phi(0.5) \approx 1 - 0.6915 = 0.3085$. (ii) The 90th percentile corresponds to $z_{0.90} \approx 1.281$. Thus $x = \mu + z\sigma = 1200 + 1.281 \times 100 \approx 1328.1$ hours.

SL 4.10 Spearmans Rank Correlation Coefficient

Solution to Q247. [Back to Question p. 138] [Back to TOC]

Spearman's r_s with ties.

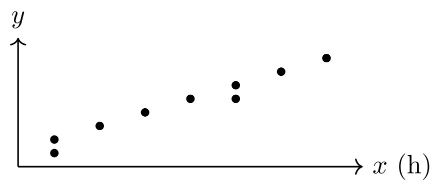
Ranks (average ties).

x	1.0	1.0	1.5	2.0	2.5	3.0	3.0	3.5	4.0	4.0
rank(x)	1.5	1.5	3.0	4.0	5.0	6.5	6.5	8.0	9.5	9.5
y	10	11	12	13	14	14	15	16	17	17
rank(y)	1.0	2.0	3.0	4.0	5.5	5.5	7.0	8.0	9.5	9.5

Since ties occur, compute r_s as the *Pearson correlation of the rank variables*. Using technology,

$$\boxed{r_s \approx 0.988}$$

(very strong positive monotonic association).



Solution to Q248. [Back to Question p. 138] [Back to TOC]

Monotonic but not linear. For the given data:

$$\boxed{r \approx 0.9966, \qquad r_s = 1.0000}$$

All y-values strictly increase with x, so the rank orders match exactly, giving $r_s = 1$.

Which coefficient? Spearman's r_s is more appropriate for a curved *monotonic* relationship; Pearson's r measures linearity and slightly under-represents the strength here.

Solution to Q249. [Back to Question p. 139] [Back to TOC]

Effect of an outlier.

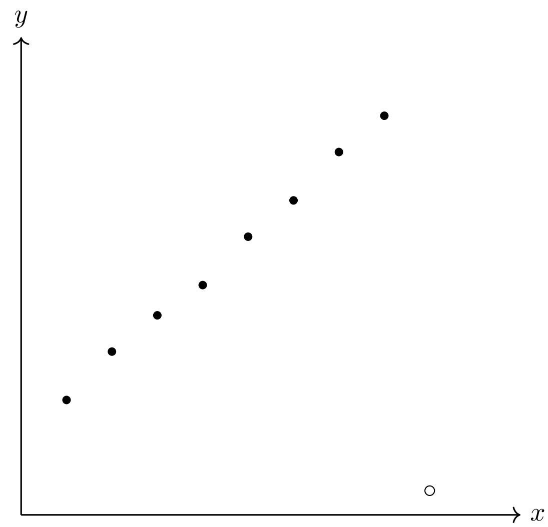
Using the first 8 points only:

$$\boxed{r \approx 0.9990, \qquad r_s = 1.0000}$$

Including the outlier $(9, -0.2)$:

$$\boxed{r \approx 0.2857, \qquad r_s \approx 0.4000}$$

Conclusion. Both coefficients decrease, but *Pearson's r is affected much more* by the outlier because it depends on distances; Spearman's r_s depends only on the rank order.



Solution to Q250. [Back to Question p. 140] [Back to TOC]

Choosing a correlation measure from diagrams.

Appropriate measure.

- **Panel A (roughly linear):** Pearson's r (linear association).
- **Panel B (monotone curved):** Spearman's r_s (monotonic but not linear).
- **Panel C (U-shape):** Neither r nor r_s alone is suitable (not monotonic; a curved model is needed).

Ranking by $|r|$ (using the provided numbers):

$$|r|_{\mathrm{A}} \approx 0.9997 \;>\; |r|_{\mathrm{B}} \approx 0.9912 \;>\; |r|_{\mathrm{C}} \approx 0.3244.$$

Panel B: size of r_s. Since the data are strictly increasing with no ties, the rank orders match, so

$$\boxed{r_s \text{ is very close to 1 (in fact 1.0000).}}$$

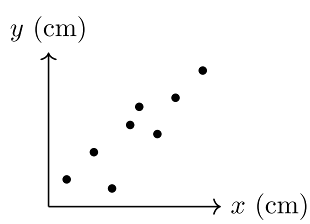
Solution to Q251. [Back to Question p. 140] [Back to TOC]

Compute r and r_s and compare.

From the data for $n = 8$ athletes:

$$\boxed{r \approx 0.8740, \qquad r_s \approx 0.8571}$$

There is a clear positive association; the scatter looks reasonably linear, so *Pearson's* r is an appropriate summary (both measures agree on a strong/moderate–strong positive relationship).



Solution to Q252. [Back to Question p. 141] [Back to TOC]

Ties in ranks.

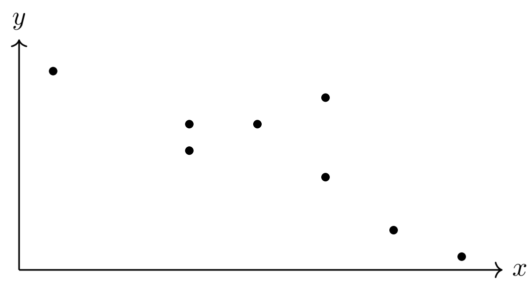
Ranks (average ties).

x	3	5	5	6	7	7	8	9
rank(x)	1.0	2.5	2.5	4.0	5.5	5.5	7.0	8.0
y	9	6	7	7	8	5	3	2
rank(y)	8.0	4.0	5.5	5.5	7.0	3.0	2.0	1.0

Compute Spearman's coefficient as the Pearson correlation of the rank variables:

$$\boxed{r_s \approx -0.758}$$

which indicates a *moderately strong negative* monotonic relationship.



SL 4.11 Hypothesis, significance, p-value

Solution to Q253. [Back to Question p. 144] [Back to TOC]

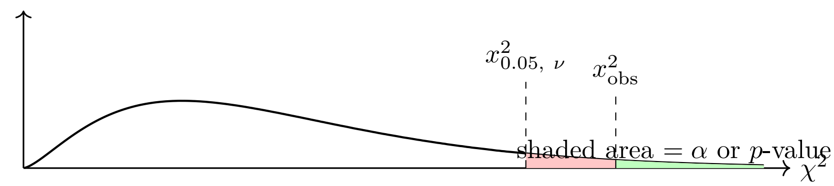
Null/alternative, significance and p-value.

(i) Hypotheses. For a χ^2 test:

$$H_0: \text{ model holds (e.g., variables are independent / distribution matches claim)}, \qquad H_1: \text{ model does not hold.}$$

(ii) Critical region. For an upper-tail test at $\alpha = 0.05$, the critical region is the area to the right of the critical value $x^2_{0.05,\ \nu}$ (depends on the degrees of freedom ν).

(iii) p-value. Given an observed statistic x^2_{obs}, the p-value is the upper-tail area $P(\chi^2_\nu \geq x^2_{\text{obs}})$.



Solution to Q254. [Back to Question p. 145] [Back to TOC]

χ^2 test for independence (contingency table).

Expected frequencies. With $N = 190$:

	Tea	Coffee	Water
Day (100)	$100 \cdot \frac{78}{190} = 41.053$	$100 \cdot \frac{52}{190} = 27.368$	$100 \cdot \frac{60}{190} = 31.579$
Night (90)	$90 \cdot \frac{78}{190} = 36.947$	$90 \cdot \frac{52}{190} = 24.632$	$90 \cdot \frac{60}{190} = 28.421$

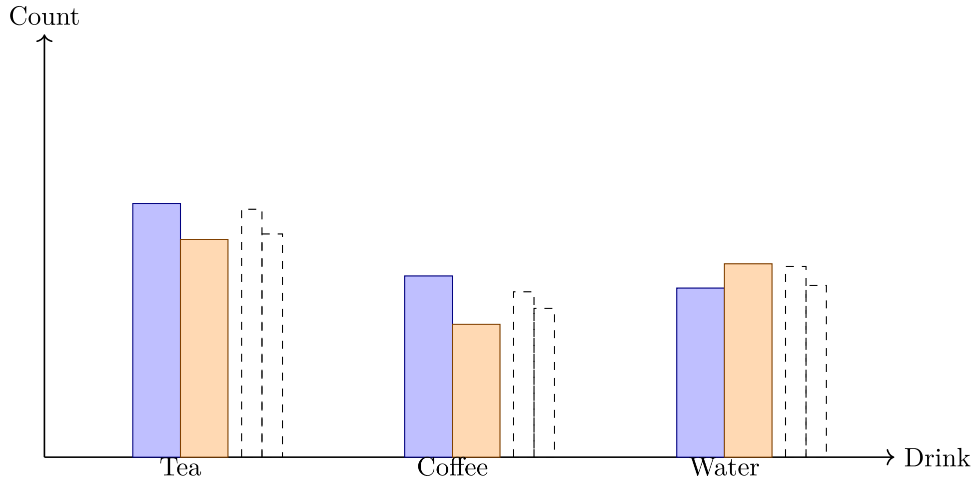
Test statistic. Using $\chi^2 = \sum \frac{(O-E)^2}{E}$:

$$\chi^2 \approx \frac{(42-41.053)^2}{41.053} + \frac{(30-27.368)^2}{27.368} + \frac{(28-31.579)^2}{31.579} + \frac{(36-36.947)^2}{36.947} + \frac{(22-24.632)^2}{24.632} + \frac{(32-28.421)^2}{28.421} \approx \boxed{1.437}.$$

Degrees of freedom: $(r-1)(c-1) = (2-1)(3-1) = \boxed{2}$. Upper-tail p-value (for $\nu = 2$) is

$$p = P(\chi^2_2 \geq 1.437) = e^{-1.437/2} \approx \boxed{0.488}.$$

Decision (5%): $p > 0.05$; do not reject H_0. There is no evidence of association between shift and drink.



Solution to Q255. [Back to Question p. 146] [Back to TOC]

χ^2 **goodness of fit.**

Expected counts under H_0:

$$E = (120,\ 100,\ 80,\ 60,\ 40).$$

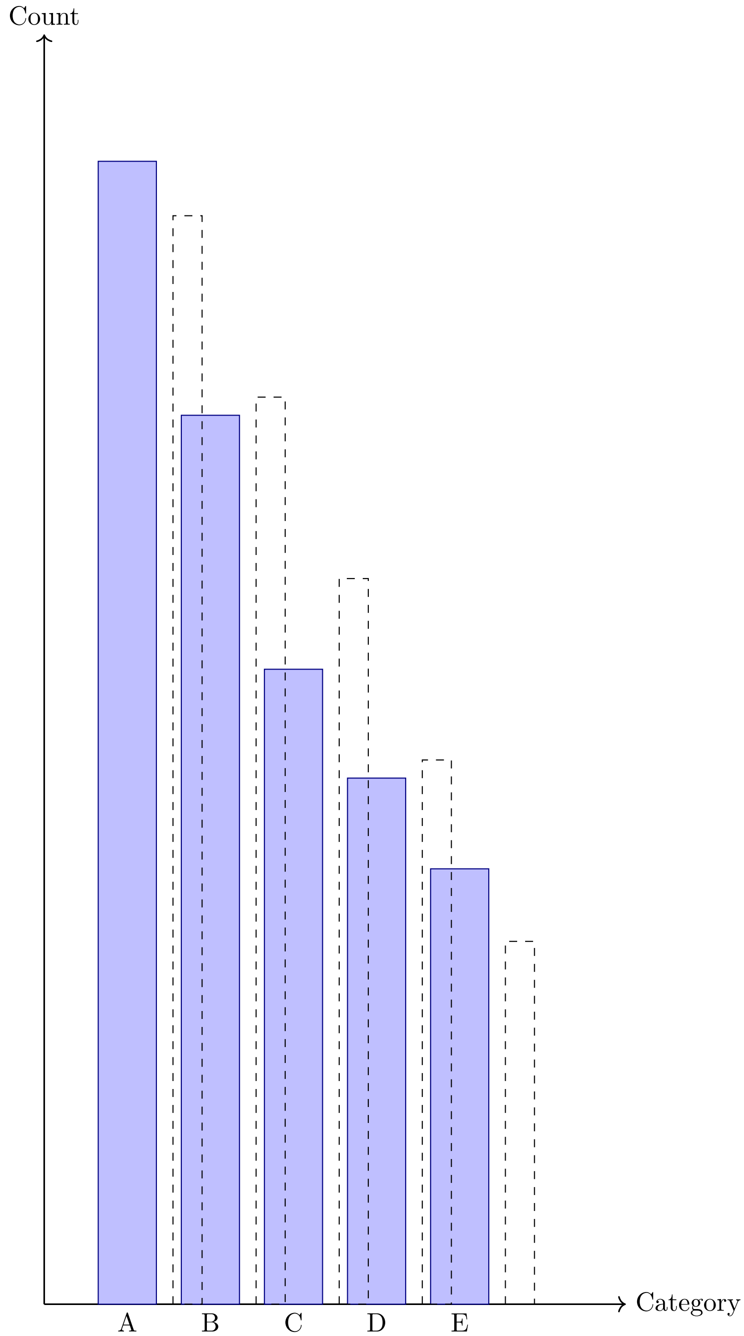
Test statistic ($df = k - 1 = 4$):

$$\chi^2 = \frac{(126-120)^2}{120} + \frac{(98-100)^2}{100} + \frac{(70-80)^2}{80} + \frac{(58-60)^2}{60} + \frac{(48-40)^2}{40} = \boxed{3.257}.$$

Upper-tail p-value for $\nu = 4$:

$$p = P(\chi^2_4 \geq 3.257) \approx \boxed{0.515}.$$

Decision (5%): $p > 0.05$; do not reject H_0. The sample is consistent with the claimed colour proportions.



Solution to Q256. [Back to Question p. 146] [Back to TOC]

χ^2 test for independence (second layout).

Expected counts (totals: Lower 120, Upper 130, $N = 250$):

	Phone	Tablet	Laptop	None
Lower	29.76	18.24	52.80	19.20
Upper	32.24	19.76	57.20	20.80

Degrees of freedom: $(2-1)(4-1) = \boxed{3}$.

Test statistic:

$$\chi^2 \approx \frac{(34-29.76)^2}{29.76} + \frac{(18-18.24)^2}{18.24} + \frac{(46-52.80)^2}{52.80} + \frac{(22-19.20)^2}{19.20} + \frac{(28-32.24)^2}{32.24} + \frac{(20-19.76)^2}{19.76} + \frac{(64-57.20)^2}{57.20} + \frac{(18-20.8}{20.80}$$

Upper-tail p-value (df $= 3$):

$$p = P(\chi^2_3 \geq 3.637) \approx \boxed{0.307}.$$

Decision (1%): $p > 0.01$; do not reject H_0. There is no evidence of association between grade and device at the 1% level.

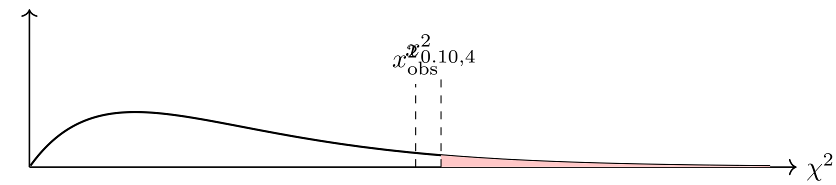
Solution to Q257. [Back to Question p. 147] [Back to TOC]

Reading a χ^2 curve. For $\nu = 4$:

- The 10% upper-tail critical value is $x^2_{0.10,4} \approx \boxed{7.779}$. Shade the region to the right of this value.
- With $x^2_{\text{obs}} = 7.3$, the p-value is

$$p = P(\chi^2_4 \geq 7.3) = e^{-7.3/2}\left(1 + \tfrac{7.3}{2}\right) \approx \boxed{0.121}.$$

Since $p > 0.10$, do *not* reject H_0 at the 10% level; likewise do not reject at the 5% level.



Solution to Q258. [Back to Question p. 147] [Back to TOC]

Two-tailed test with summary statistics.

Given $n_A = 15$, $\bar{x}_A = 8.2$, $s_A = 1.1$ and $n_B = 17$, $\bar{x}_B = 7.6$, $s_B = 1.3$. Assume independent normal populations with equal variances.

Hypotheses:

$$H_0 : \mu_A = \mu_B \qquad \text{vs} \qquad H_1 : \mu_A \neq \mu_B.$$

Pooled variance:

$$s_p^2 = \frac{(n_A-1)s_A^2 + (n_B-1)s_B^2}{n_A + n_B - 2} = \frac{14(1.21) + 16(1.69)}{30} = \frac{43.98}{30} = 1.466, \quad s_p = \sqrt{1.466} \approx 1.211.$$

Standard error:

$$\text{SE} = s_p\sqrt{\frac{1}{n_A} + \frac{1}{n_B}} = 1.211\sqrt{\tfrac{1}{15} + \tfrac{1}{17}} \approx 1.211 \cdot 0.3543 \approx 0.429.$$

Test statistic and df:

$$t = \frac{\bar{x}_A - \bar{x}_B}{\text{SE}} = \frac{0.6}{0.429} \approx 1.40, \qquad \text{df} = n_A + n_B - 2 = 30.$$

p-value (two-tailed): $p \approx 0.17$. Since $p > 0.05$, **do not reject** H_0 at the 5% level; there is no clear evidence that the mean lifetimes differ.

95% CI for $\mu_A - \mu_B$:

$$(\bar{x}_A - \bar{x}_B) \pm t_{0.975,30}\,\text{SE} = 0.6 \pm 2.042(0.429) = 0.6 \pm 0.875,$$

so $\boxed{(-0.28,\ 1.48)}$. The interval contains 0, agreeing with the test.

Solution to Q259. [Back to Question p. 148] [Back to TOC]

One-tailed test with summary statistics.

Data: $n_N = 12,\ \bar{x}_N = 42.1,\ s_N = 5.6$ and $n_C = 10,\ \bar{x}_C = 38.5,\ s_C = 6.1$. Assume independent normal populations and equal variances.

Hypotheses:

$$H_0 : \mu_N = \mu_C \qquad \text{vs} \qquad H_1 : \mu_N > \mu_C.$$

Pooled variance:

$$s_p^2 = \frac{11(5.6)^2 + 9(6.1)^2}{20} = \frac{344.96 + 334.89}{20} = 33.993, \qquad s_p \approx 5.830.$$

SE: $\text{SE} = s_p\sqrt{\frac{1}{12} + \frac{1}{10}} \approx 5.830 \cdot \sqrt{0.1833} \approx 2.496.$

Test statistic and df:

$$t = \frac{\bar{x}_N - \bar{x}_C}{\text{SE}} = \frac{3.6}{2.496} \approx 1.44, \qquad \text{df} = 20.$$

p-value (one-tailed): $p \approx 0.08$. Since $p > 0.05$, **do not reject** H_0 at 5%; the data do not show a significant increase.

95% CI for $\mu_N - \mu_C$:

$$3.6 \pm t_{0.975,20}\,\text{SE} = 3.6 \pm 2.086(2.496) = 3.6 \pm 5.21,$$

so $\boxed{(-1.61,\ 8.81)}$, consistent with the decision.

Solution to Q260. [Back to Question p. 148] [Back to TOC]

Interpreting calculator output.

Given $t = -1.87$, df $= 26$, and (two-tailed) $p = 0.073$:

(a) At 10%: $p < 0.10 \Rightarrow$ **reject** H_0. At 5%: $p > 0.05 \Rightarrow$ **do not reject** H_0.

(b) $t < 0$ means $\bar{x}_1 - \bar{x}_2 < 0$, so sample 1 has the *smaller* mean (evidence that $\mu_1 < \mu_2$).

(c) A two-tailed test at 10% is equivalent to checking whether the **90%** CI for $\mu_1 - \mu_2$ excludes 0. If a reported 90% CI were $(-0.3, 6.1)$, it *includes* 0 and would indicate *no* rejection at 10%, which conflicts with $p = 0.073$. The consistent interval (given $t < 0$ and $p = 0.073$) would exclude 0, e.g. something like $\boxed{(-6.1,\ -0.3)}$.

Solution to Q261. [Back to Question p. 149] [Back to TOC]

Write hypotheses and choose tailedness.

(a) "Reduces mean 100 m time." $H_0 : \mu_{\text{new}} = \mu_{\text{usual}}$ (or $\mu_{\text{new}} \geq \mu_{\text{usual}}$), $H_1 : \mu_{\text{new}} < \mu_{\text{usual}}$. *One-tailed (left).*

(b) "Means differ between schools." $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \neq \mu_2$. *Two-tailed.*

(c) "Increases mean tensile strength." $H_0 : \mu_{\text{new}} = \mu_{\text{current}}$ (or $\mu_{\text{new}} \leq \mu_{\text{current}}$), $H_1 : \mu_{\text{new}} > \mu_{\text{current}}$. *One-tailed (right).*

Solution to Q262. [Back to Question p. 149] [Back to TOC]

Using raw data (two-tailed).

Group A: $12, 10, 9, 11, 13, 12, 8, 10$.

$$n_A = 8, \quad \bar{x}_A = \frac{85}{8} = 10.625, \quad s_A^2 = \frac{\sum x^2 - n_A \bar{x}_A^2}{n_A - 1} = \frac{923 - 8(10.625)^2}{7} \approx 2.839, \quad s_A \approx 1.685.$$

Group B: $7, 9, 11, 10, 8, 6, 9, 7$.

$$n_B = 8, \quad \bar{x}_B = \frac{67}{8} = 8.375, \quad s_B \approx 1.685.$$

Assume equal variances $\Rightarrow s_p \approx 1.685$.

$$\mathrm{SE} = s_p \sqrt{\tfrac{1}{n_A} + \tfrac{1}{n_B}} = 1.685 \sqrt{\tfrac{1}{8} + \tfrac{1}{8}} = 1.685 \cdot 0.5 = 0.8425.$$

$$t = \frac{\bar{x}_A - \bar{x}_B}{\mathrm{SE}} = \frac{2.25}{0.8425} \approx 2.67, \qquad \mathrm{df} = n_A + n_B - 2 = 14.$$

p-value (two-tailed): $p \approx 0.018$. At $\alpha = 0.05$ we **reject** H_0; the mean completion times differ (Group A larger, i.e. slower on average).

A 95% CI: $(\bar{x}_A - \bar{x}_B) \pm t_{0.975,14}\mathrm{SE} \approx 2.25 \pm 2.145(0.8425) = 2.25 \pm 1.81$, so $\boxed{(0.44,\ 4.06)}$, which excludes 0.

Solution to Q263. [Back to Question p. 150] [Back to TOC]

Checking assumptions conceptually.

(1) *Supports* using pooled two-sample t: approximate normality, no outliers, and similar spreads.
(2) *Does not support*: the same students in both groups violate independence.
(3) *Supports*: near-normal and similar variability from the boxplots.
(4) *Does not support* pooled t (equal variances dubious); consider Welch's two-sample t instead.

Solution to Q264. [Back to Question p. 150] [Back to TOC]

One- vs two-tailed decision via a confidence interval.

Given the 95% CI for $\mu_1 - \mu_2$ is $(-1.4,\ 3.8)$:

- Two-tailed test at $\alpha = 0.05$: the interval contains 0, so **do not reject** H_0.
- One-tailed test $H_0 : \mu_1 \le \mu_2$ vs $H_1 : \mu_1 > \mu_2$ at $\alpha = 0.05$: the entire CI is not above 0, so there is **insufficient evidence** to claim $\mu_1 > \mu_2$ at 5%.

AHL 4.12 Designing investigations, categories and sampling techniques

Solution to Q265. [Back to Question p. 151] [Back to TOC]

Questionnaire design (identify and fix bias).

For each item we give the problem, a fix, and a response format.

1. *Problem:* Leading/loaded ("*excessive* ... hurts").
   Fix: "In a typical weekday, how many hours do you spend on screens *outside of classes*?"
   Format: Numeric hours to nearest 0.5.
2. *Problem:* Double-barrelled and ambiguous ("usually"; two quantities in one box).
   Fix: (i) "On a typical night, how many hours do you sleep?" (numeric)
   (ii) "On a typical night, how many hours are *deep* sleep?" (numeric)
3. *Problem:* Leading/social desirability ("You don't ..., right?").
   Fix: "After what time do you *stop* using your phone on weeknights?"
   Format: Multiple choice: before 9pm / 9–10pm / 10–11pm / 11pm–12am / after 12am.
4. *Problem:* Poor scale (uneven, vague labels).
   Fix: 5-point Likert: very poor / poor / fair / good / excellent.
5. *Problem:* Double concept ("fun *or* study").
   Fix: Split into two items: frequency for fun; frequency for study.
   Format: never / monthly / weekly / few times a week / daily.
6. *Problem:* Open responses increase entry errors and may be intrusive.
   Fix: "What is your GPA on the school's 0–4 scale?" (numeric, one decimal). "What is your year level?" (9 / 10 / 11 / 12 / other / prefer not to say).
7. *Add a demographic item:* "Which of the following best describes you?" (male / female / another term ☐ / prefer not to say).

Solution to Q266. [Back to Question p. 151] [Back to TOC]

Sampling plan and data to analyse.

Population & frame: All current students at the school (1200). Frame: the school enrolment list.

Sampling method: **Stratified random sampling** by grade (and optionally gender) to ensure representation; take proportional samples from each stratum (e.g. 10% of each grade).

Handling non-response/outliers: Send two reminders; record response indicator; compare respondents vs frame on grade/gender and apply post-stratification weights if needed. Screen numeric fields with plausibility ranges (e.g. sleep 0–14 h, screen time 0–18 h). Winsorize extreme outliers or justify removal with a pre-registered rule.

Variables (type/units): screen_weekday (h, numerical), screen_weekend (h, numerical), sleep (h, numerical), grade (nominal), gender (nominal), GPA (ratio, 0–4), extracurricular hours (numerical). These are relevant to the mean daily screen time and potential confounders.

Outcome construction: average daily screen time

$$\text{avg} = \frac{5 \cdot \text{screen\_weekday} + 2 \cdot \text{screen\_weekend}}{7} \quad \text{(h/day)}.$$

Document all steps in a reproducible log (date, rule, counts affected).

Solution to Q267. [Back to Question p. 151] [Back to TOC]

Selecting relevant variables from many.

Good predictors of final score Y: prior GPA, hours studied, attendance %, practice tests, average sleep (possible nonlinear), phone unlocks/day (proxy for distraction).
Check: pairwise plots and VIF for multicollinearity (e.g. hours studied vs practice tests), transform skewed counts (log for unlocks), and examine leverage/influence (Cook's D).
Less relevant: teacher ID (categorical with many levels; confounded with class), raw class size (weak direct causal link). Clearly describe inclusions/exclusions with reasons.

Solution to Q268. [Back to Question p. 152] [Back to TOC]

Categorizing numerical data for χ^2 GOF (Poisson($\lambda = 2.4$), $n = 200$).

Expected counts $E_k = 200\, P(X = k)$:

k	0	1	2	3	4	5	6+
E_k	18.14	43.54	52.25	41.80	25.08	12.04	7.13

All $E > 5$ once we combine $k \geq 6$. With the given observations $O = (22, 54, 60, 38, 17, 7, 2)$, the test statistic is

$$\chi^2 = \sum \frac{(O - E)^2}{E} \approx \boxed{13.233}.$$

Here the model parameter λ is *given*, so df $= 7 - 1 = \boxed{6}$ and the upper-tail p-value is $\boxed{p \approx 0.0395}$. At 5% we reject H_0: the sample shows some departure from Poisson(2.4).

Solution to Q269. [Back to Question p. 152] [Back to TOC]

Degrees of freedom when parameters are estimated.

General rule for GOF: df $= k - 1 - m$ where k = number of categories and m = number of parameters estimated from the data.

1. Binomial($n = 6, p$), $k = 7$, estimate p ($m = 1$): df $= 7 - 1 - 1 = \boxed{5}$.
2. Poisson(λ), $k = 8$, estimate λ ($m = 1$): df $= 8 - 1 - 1 = \boxed{6}$.
3. Normal $\mathcal{N}(\mu, \sigma^2)$, $k = 10$, estimate μ, σ ($m = 2$): df $= 10 - 1 - 2 = \boxed{7}$.

Solution to Q270. [Back to Question p. 153] [Back to TOC]

Test–retest reliability.

Pearson correlation between Time 1 and Time 2 scores:

$$r \approx \boxed{0.970}.$$

Scatterplot is near linear with no strong outliers; reliability is *very high* and positive, indicating excellent stability over two weeks.

Solution to Q271. [Back to Question p. 153] [Back to TOC]

Parallel-forms reliability and bias.

Correlation between Form A and Form B:

$$r_{AB} \approx \boxed{0.937} \quad \text{(very strong)}.$$

Paired differences D =A−B have

$$\bar{D} = 0.50, \quad s_D = 0.972, \quad n = 10, \quad \text{SE} = 0.307, \quad t = \frac{\bar{D}}{\text{SE}} \approx 1.63 \ \ (\text{df} = 9),$$

giving two-tailed $p \approx \boxed{0.14}$. The 95% CI for the mean difference is

$$\bar{D} \pm t_{0.975,9}\text{SE} = 0.50 \pm 2.262(0.307) = \boxed{(-0.20,\ 1.20)}.$$

Conclusion: no evidence of systematic score shift; the two forms appear interchangeable.

Solution to Q272. [Back to Question p. 154] [Back to TOC]

Criterion-related validity for short scale S vs long scale L.

Correlation:

$$r(S, L) \approx \boxed{0.991}, \qquad R^2 \approx 0.982.$$

Regression of L on S:

$$\widehat{L} = \boxed{19.108 + 1.443\, S}.$$

This very strong, linear relationship supports criterion validity.

With cut-score $S \geq 30$: 4 of 15 participants are flagged by S (proportion $\boxed{26.7\%}$). Using a conventional criterion $L \geq 60$ (T-score "elevated"), the confusion table is perfect for this sample: $\text{TP} = 4,\ \text{FP} = 0,\ \text{FN} = 0,\ \text{TN} = 11$. Hence, in this sample S's rule coincides with L's rule; in practice one would evaluate this on a larger validation set.

Solution to Q273. [Back to Question p. 154] [Back to TOC]

Content validity (blueprint/mapping).

Coverage in the draft (ticks counted per LO):

$$\text{LO1: } 1/8 = 12.5\%, \quad \text{LO2: } 3/8 = 37.5\%, \quad \text{LO3: } 2/8 = 25\%, \quad \text{LO4: } 2/8 = 25\%.$$

LO1 is under-represented; LO2 is heavy.

Revised blueprint (balanced): allocate 2 items per LO (total 8). For example: Items 1–2 → LO1, 3–4 → LO2, 5–6 → LO3, 7–8 → LO4.

Example additional stem for LO1 (Definitions): "Define 'margin of error' in the context of a 95% confidence interval, and state two factors that affect its size."

Solution to Q274. [Back to Question p. 155] [Back to TOC]

Choosing relevant/appropriate data (cleaning rules).

Inclusion/exclusion (documented before looking at outcomes):

- Keep respondents in the enrolment frame; remove duplicates by stable ID.
- Valid ranges: age $[10, 20]$ (adjust to setting), sleep $[0, 14]$ h, screen times $[0, 18]$ h, GPA $[0, 4]$ (or school scale).
- If `missing_items` $> 20\%$ (or > 2 out of 10 key items), exclude from analysis; otherwise impute single missing numeric values by grade-level median.

Outcome construction:

$$\text{avg\_screen} = \frac{5 \cdot \text{weekday\_screen\_h} + 2 \cdot \text{weekend\_screen\_h}}{7}.$$

Outliers: flag if z-score $|z| > 3$ or outside $[Q_1 - 1.5\,\text{IQR}, Q_3 + 1.5\,\text{IQR}]$; inspect and decide (typo vs true extreme).

Reproducibility: keep a change log (rule, date, rows affected), version raw/clean files, and provide code/notebook used to clean and derive variables.

tocsubsectionAHL 4.13 Non-linear regression

Solution to Q275. [Back to Question p. 158] [Back to TOC]

Choosing a model (exponential vs linear).

Data: $(x, y) = (0, 2.0), (1, 3.1), (2, 4.6), (3, 6.7), (4, 10.0), (5, 14.9)$.

Linear fit ($y = mx + c$ by least squares):

$$m = 2.4943, \qquad c = 0.6476.$$

With $\hat{y} = mx + c$,

$$SS_{\text{res}} = \sum (y - \hat{y})^2 = 8.5128, \qquad R^2 = 1 - \frac{SS_{\text{res}}}{SS_{\text{tot}}} = 0.9275.$$

Exponential fit ($y = a\,e^{bx}$ via $\ln y = \ln a + bx$):

$$a = 2.04295, \qquad b = 0.39802.$$

With $\hat{y} = ae^{bx}$,

$$SS_{\text{res}} = 0.01585, \qquad R^2 = 0.999865.$$

Conclusion. Exponential fits far better (much smaller SS_{res}, larger R^2) and residuals are negligible.

Prediction at $x = 6$:

$$\hat{y}(6) = 2.04295\, e^{0.39802 \cdot 6} \approx \boxed{22.254}.$$

This is a one-step *extrapolation* beyond the observed range.

Solution to Q276. [Back to Question p. 158] [Back to TOC]

Power model vs linear.

Data: $x = 1, \ldots, 6$ and $y = (2.2, 3.3, 4.0, 4.6, 5.0, 5.4)$.

Linear $y = mx + c$:

$$m = 0.6200, \quad c = 1.9133, \quad SS_{\text{res}} = 0.2813, \quad R^2 = 0.9599.$$

Power $y = a\,x^{b}$ (use $\ln y = \ln a + b \ln x$):

$$a = 2.26443, \quad b = 0.49954, \quad SS_{\text{res}} = 0.04951, \quad R^2 = 0.99294.$$

Conclusion. The power model is clearly better (higher R^2, smaller SS_{res} and more appropriate gently-curving shape).

Prediction at $x = 8$:

$$\hat{y}(8) = 2.26443 \cdot 8^{\,0.49954} \approx \boxed{6.399}.$$

Solution to Q277. [Back to Question p. 159] [Back to TOC]

Quadratic or cubic?

Data: $(-2, 6.5), (-1, 3.0), (0, 2.0), (1, 3.1), (2, 5.6), (3, 9.9)$.

Quadratic $y = ax^2 + bx + c$:

$$a = 0.94643, \quad b = -0.20643, \quad c = 2.12286.$$

Vertex at

$$x_v = -\frac{b}{2a} = 0.10906, \qquad y_v = 2.1116.$$

Fit statistics: $SS_{\text{res}} = 0.20486$, $R^2 = 0.99526$.

Cubic $y = px^3 + qx^2 + rx + s$:

$$p = -0.051852, \quad q = 1.02421, \quad r = 0.016534, \quad s = 1.99841,$$

with $SS_{\text{res}} = 0.030635$, $R^2 = 0.99929$.

Report. Both models fit extremely well; the cubic has slightly smaller SS_{res}, but the quadratic already leaves pattern-free residuals and is more *parsimonious*. Unless domain knowledge suggests asymmetric behaviour, report the **quadratic**. A marginally smaller SS_{res} is not always worth the extra parameter(s).

Solution to Q278. [Back to Question p. 160] [Back to TOC]

Sinusoidal regression (seasonality).

Fit $y = A\sin(B(x - C)) + D$ to the monthly data (technology, non-linear least squares) gives

$$A = 1.8049, \quad B = 0.55642, \quad C = 2.4639, \quad D = 13.2670.$$

Hence the period is

$$T = \frac{2\pi}{B} \approx \boxed{11.29 \text{ months}},$$

$$R^2 = 0.99621, \qquad SS_{\text{res}} = 0.0750.$$

Interpretation. $A \approx 1.80$ is the amplitude (typical swing ± 1.8 about the mean), $D \approx 13.27$ is the mean level.

Forecast for month $x = 15$:

$$\hat{y}(15) = 13.2670 + 1.8049 \sin\big(0.55642(15 - 2.4639)\big) \approx \boxed{14.419}.$$

Solution to Q279. [Back to Question p. 160] [Back to TOC]

Compute SS_{res} and R^2 from small data.

With $y = (3.2, 4.1, 5.0, 6.0)$ and $\bar{y} = 4.575$,

$$SS_{\text{tot}} = \sum (y - \bar{y})^2 = 4.3275.$$

Model 1: $SS_{\text{res}}^{(1)} = \sum (y - \hat{y}^{(1)})^2 = 0.1600$,

$$R_1^2 = 1 - \frac{0.1600}{4.3275} = \boxed{0.9630}.$$

Model 2: $SS_{\text{res}}^{(2)} = 0.1300$,

$$R_2^2 = 1 - \frac{0.1300}{4.3275} = \boxed{0.9700}.$$

Model 2 fits slightly better, but differences are small; practical significance should be considered.

Solution to Q280. [Back to Question p. 161] [Back to TOC]

R^2 from a correlation (linear models).

With $r = -0.84$,

$$R^2 = r^2 = (-0.84)^2 = \boxed{0.7056} \ (\approx 70.6\% \text{ of the variability explained}).$$

The sign of r indicates direction of linear association, but $R^2 = r^2$ is non-negative and depends only on the *magnitude*, not the sign.

Solution to Q281. [Back to Question p. 161] [Back to TOC]

Deciding between models (beyond R^2).

(a) A higher R^2 (0.988 vs 0.982) can result simply from adding parameters; R^2 never decreases as complexity increases, so it can favour overfitting.

(b) Examine:

- Residual diagnostics (random scatter, constant variance, no structure).
- Out-of-sample performance (validation/test error, cross-validation), or adjusted R^2/AIC/BIC which penalise complexity.
- Plausibility/interpretability of the model form and parameters.

(c) If Model A's residuals are pattern-free but Model B shows curvature, report **Model A** despite its slightly smaller R^2; validity and assumptions outweigh a marginal R^2 gain.

AHL 4.14 Linear combinations, expectations/variance

Solution to Q282. [Back to Question p. 162] [Back to TOC]

Linear transformation of a random variable.

Given $\mathbb{E}(X) = 50$ and $\operatorname{Var}(X) = 9$ and $Y = 2X - 7$,

$$\mathbb{E}(Y) = 2\,\mathbb{E}(X) - 7 = 2(50) - 7 = \boxed{93}, \qquad \operatorname{Var}(Y) = 2^2 \operatorname{Var}(X) = 4(9) = \boxed{36}.$$

Hence $\operatorname{SD}(Y) = \sqrt{36} = \boxed{6}$. Adding a constant (-7) shifts all outcomes equally and does not change spread, so the variance is unaffected.

Solution to Q283. [Back to Question p. 162] [Back to TOC]

Unit conversion (linear transformation).

$\mathbb{E}(C) = 21.4$, $\operatorname{SD}(C) = 3.2 \Rightarrow \operatorname{Var}(C) = 10.24$ and $F = 1.8C + 32$.

$$\mathbb{E}(F) = 1.8 \cdot 21.4 + 32 = \boxed{70.52}, \qquad \operatorname{Var}(F) = 1.8^2 \cdot 10.24 = 3.24 \cdot 10.24 = \boxed{33.1776}.$$

Multiplying by 1.8 multiplies the variance by 1.8^2; the $+32$ shift leaves the variance unchanged.

Solution to Q284. [Back to Question p. 162] [Back to TOC]

Expectation of a linear combination (independence not needed).

By linearity of expectation,

$$\mathbb{E}(2X_1 - 3X_2 + 5) = 2\mu_1 - 3\mu_2 + 5 = 2(8) - 3(3) + 5 = \boxed{12}.$$

Solution to Q285. [Back to Question p. 163] [Back to TOC]

Variance of a linear combination (independent variables).

For $S = 3X_1 - 2X_2 + X_3$ with independence,

$$\mathbb{E}(S) = 3 \cdot 4 - 2 \cdot 5 + 1 \cdot 2 = \boxed{4},$$

$$\operatorname{Var}(S) = 3^2(1.2) + (-2)^2(2.0) + 1^2(0.5) = 10.8 + 8 + 0.5 = \boxed{19.3}.$$

For $A = S/2$: $\mathbb{E}(A) = \mathbb{E}(S)/2 = \boxed{2}$, $\operatorname{Var}(A) = \operatorname{Var}(S)/4 = \boxed{4.825}$.

Solution to Q286. [Back to Question p. 163] [Back to TOC]

Sample mean of i.i.d. variables.

With $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$,

$$\mathbb{E}(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{1}{n}(n\mu) = \boxed{\mu}.$$

If X_i are independent with variance σ^2,

$$\operatorname{Var}(\bar{X}) = \operatorname{Var}\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\sum \operatorname{Var}(X_i) = \frac{1}{n^2}(n\sigma^2) = \boxed{\frac{\sigma^2}{n}}.$$

Thus $\operatorname{SD}(\bar{X}) = \sigma/\sqrt{n}$, so increasing n reduces the spread at rate $1/\sqrt{n}$.

Solution to Q287. [Back to Question p. 163] [Back to TOC]

Unbiasedness in words.

"Unbiased" means the estimator's expected value equals the parameter. Here $\mathbb{E}(\bar{X}) = \mu$, so over many random samples the *average* of the sample means equals the true population mean. Example: repeatedly sample 40 students from a school with true mean height 170 cm; the long–run average of the 40-student sample means will be 170 cm.

Solution to Q288. [Back to Question p. 164] [Back to TOC]

Compute $\bar{x}$ and s^2_{n-1} from raw data.

Data: $12, 10, 9, 11, 13, 12, 8, 10$ ($n = 8$). Sum $= 85 \Rightarrow \bar{x} = \frac{85}{8} = \boxed{10.625}$.

Unbiased variance

$$s^2_{n-1} = \frac{1}{7}\sum(x_i - \bar{x})^2 = \frac{1}{7}(19.875) = \boxed{2.8393} \text{ (SD} \approx \boxed{1.685}\text{)}.$$

If each value is divided by 100 (cm $\to$ m), the variance is multiplied by $(\frac{1}{100})^2$, so $s^2_m = \boxed{2.8393 \times 10^{-4}}$ $(= 0.00028393)$.

Solution to Q289. [Back to Question p. 164] [Back to TOC]

Mean and unbiased variance from grouped (frequency) data.

Totals: $n = \sum f_i = 3 + 6 + 5 + 4 + 2 = \boxed{20}$, $\sum f_i x_i = 6 + 24 + 25 + 28 + 18 = \boxed{101}$, hence

$$\bar{x} = \frac{101}{20} = \boxed{5.05}.$$

Unbiased variance:

$$\sum f_i(x_i - \bar{x})^2 = 3(2 - 5.05)^2 + 6(4 - 5.05)^2 + 5(5 - 5.05)^2 + 4(7 - 5.05)^2 + 2(9 - 5.05)^2 = \boxed{80.95}.$$

Therefore

$$s^2_{n-1} = \frac{80.95}{20 - 1} = \boxed{4.2605} \quad (\text{SD} \approx 2.064).$$

Solution to Q290. [Back to Question p. 164] [Back to TOC]

Variance of a weighted combination of independent sample means.

$$\mathbb{E}(W) = 0.4\,\mu_A + 0.6\,\mu_B.$$

Using $\text{Var}(\bar{X}_A) = \sigma^2_A/n_A$, $\text{Var}(\bar{X}_B) = \sigma^2_B/n_B$, independence:

$$\text{Var}(W) = 0.4^2\frac{\sigma^2_A}{n_A} + 0.6^2\frac{\sigma^2_B}{n_B}.$$

With $\sigma_A = 6$, $n_A = 25$, $\sigma_B = 10$, $n_B = 36$:

$$\text{Var}(W) = 0.16 \cdot \frac{36}{25} + 0.36 \cdot \frac{100}{36} = 0.16 \cdot 1.44 + 0.36 \cdot 2.777\overline{7} = 0.2304 + 1.0000 = \boxed{1.2304},$$

so $\text{SD}(W) = \boxed{1.109}$.

Solution to Q291. [Back to Question p. 165] [Back to TOC]

Effect of linear rescaling on sample variance (units).

Given $s_{cm}^2 = 64$ and $H_m = \frac{1}{100} H_{cm}$,

$$s_m^2 = \left(\frac{1}{100}\right)^2 s_{cm}^2 = \boxed{0.0064}.$$

General rule: for $Y = aX + b$, $\operatorname{Var}(Y) = a^2 \operatorname{Var}(X)$ (the shift b does not affect variance).

AHL 4.15 Central limit theorem, and combinations of normal distributions

Solution to Q292. [Back to Question p. 166] [Back to TOC]

Sampling mean from a normal population.

Given $X \sim \mathcal{N}(72, 16)$ and $n = 25$:

1. $\bar{X} \sim \mathcal{N}\left(\mu = 72,\ \dfrac{\sigma^2}{n} = \dfrac{16}{25}\right)$, so $\mathrm{SD}(\bar{X}) = \frac{4}{5} = 0.8$.
2. $P(\bar{X} > 74) = P\big(Z > \frac{74-72}{0.8}\big) = P(Z > 2.5) \approx \boxed{0.00621}$.
3. $S = \sum_{i=1}^{25} X_i \sim \mathcal{N}(25 \cdot 72,\ 25 \cdot 16) = \mathcal{N}(1800, 400)$, so $P(S < 1770) = P\big(Z < \frac{1770-1800}{20}\big) = P(Z < -1.5) \approx \boxed{0.06681}$.

Solution to Q293. [Back to Question p. 166] [Back to TOC]

Linear combination of independent normals.

$X \sim \mathcal{N}(10, 3^2)$, $Y \sim \mathcal{N}(16, 4^2)$, independent; $A = 0.3X + 0.7Y$.

$$\mathbb{E}(A) = 0.3(10) + 0.7(16) = \boxed{14.2}, \qquad \mathrm{Var}(A) = 0.3^2(9) + 0.7^2(16) = \boxed{8.65}.$$

Hence $A \sim \mathcal{N}(14.2,\ 8.65)$ and

$$P(14 \le A \le 17) = \Phi\left(\frac{17 - 14.2}{\sqrt{8.65}}\right) - \Phi\left(\frac{14 - 14.2}{\sqrt{8.65}}\right) \approx \boxed{0.3566}.$$

Solution to Q294. [Back to Question p. 166] [Back to TOC]

Weighted sum of several normals.

$X_1 \sim \mathcal{N}(20, 5^2)$, $X_2 \sim \mathcal{N}(15, 2^2)$, $X_3 \sim \mathcal{N}(12, 3^2)$ independent; $W = 2X_1 - X_2 + \frac{1}{2}X_3$.

$$\mathbb{E}(W) = 2(20) - 1(15) + \tfrac{1}{2}(12) = \boxed{31}, \qquad \mathrm{Var}(W) = 2^2(25) + (-1)^2(4) + \left(\tfrac{1}{2}\right)^2(9) = \boxed{106.25}.$$

Thus $W \sim \mathcal{N}(31,\ 106.25)$ and

$$P(W > 35) = P\left(Z > \frac{35 - 31}{\sqrt{106.25}}\right) \approx \boxed{0.3490}.$$

Solution to Q295. [Back to Question p. 167] [Back to TOC]

CLT with a non-normal population.

Exponential mean 5 implies variance 25.

1. By the CLT, $\bar{T} \approx \mathcal{N}\left(5,\ \dfrac{25}{40}\right)$ with $\mathrm{SD}(\bar{T}) = \sqrt{25/40} \approx 0.7906$.
2. $P(4.5 < \bar{T} < 5.5) = \Phi\big(\frac{0.5}{0.7906}\big) - \Phi\big(\frac{-0.5}{0.7906}\big) \approx \boxed{0.4729}$.
3. Require $\mathrm{SD}(\bar{T}) = \sqrt{25/n} \le 0.4 \Rightarrow \sqrt{n} \ge 12.5 \Rightarrow n \ge \boxed{157}$.

Solution to Q296. [Back to Question p. 167] [Back to TOC]

Sample proportion as a sample mean (CLT).

With $p = 0.3$ and $n = 200$, $\hat{p}$ is approximately normal with

$$\mathbb{E}(\hat{p}) = 0.3, \qquad \mathrm{Var}(\hat{p}) = \frac{p(1-p)}{n} = \frac{0.21}{200} = 0.00105, \quad \mathrm{SD}(\hat{p}) \approx 0.03240.$$

$$P(\hat{p} \geq 0.35) = P\left(Z \geq \frac{0.05}{0.03240}\right) \approx \boxed{0.0614}, \qquad P(\hat{p} > 0.33) = P\left(Z > \frac{0.03}{0.03240}\right) \approx \boxed{0.1773}.$$

Solution to Q297. [Back to Question p. 167] [Back to TOC]

Sum vs mean.

If $X_i \sim \mathcal{N}(50, 100)$ independent,

$$S_n = \sum_{i=1}^{n} X_i \sim \mathcal{N}(50n,\ 100n), \qquad \bar{X} = \frac{S_n}{n} \sim \mathcal{N}\left(50,\ \frac{100}{n}\right).$$

For $n = 36$, $\mathrm{SD}(\bar{X}) = \sqrt{100/36} = 1.6667$ and

$$P(48 < \bar{X} < 52) = \Phi\left(\frac{2}{1.6667}\right) - \Phi\left(\frac{-2}{1.6667}\right) = \Phi(1.2) - \Phi(-1.2) \approx \boxed{0.7699}.$$

Solution to Q298. [Back to Question p. 168] [Back to TOC]

Mixture of two normal samples (independent).

$\bar{X}_A \sim \mathcal{N}\left(70, \frac{9^2}{20}\right)$, $\bar{X}_B \sim \mathcal{N}\left(75, \frac{10^2}{30}\right)$, independent. Thus

$$\bar{X}_B - \bar{X}_A \sim \mathcal{N}\left(5,\ \frac{81}{20} + \frac{100}{30}\right) = \mathcal{N}\left(5,\ 7.383\bar{3}\right),$$

SD ≈ 2.717 and

$$P(\bar{X}_B - \bar{X}_A \geq 3) = 1 - \Phi\left(\frac{3-5}{2.717}\right) = \Phi(0.736) \approx \boxed{0.7691}.$$

Doubling both sample sizes halves the variance (each term's denominator doubles), so Var becomes $\frac{1}{2}$ as large and SD decreases by a factor $\sqrt{1/2}$.

Solution to Q299. [Back to Question p. 168] [Back to TOC]

Interpreting the CLT.

1. For i.i.d. observations with mean μ and variance σ^2, the sample mean $\bar{X}$ is approximately $\mathcal{N}(\mu, \sigma^2/n)$ for large n, regardless of the parent distribution.
2. Inadequate at $n = 25$: very heavy-tailed or extremely skewed populations (e.g. Pareto with infinite variance). Adequate at $n = 10$: roughly symmetric, light-tailed populations (e.g. uniform or normal-like).
3. X is a single observation from the population; $\bar{X}$ is the average of n observations. $\bar{X}$ is less variable, with $\mathrm{Var}(\bar{X}) = \sigma^2/n$, and (by CLT) is approximately normal even when X is not.

AHL 4.16 Confidence intervals

Solution to Q300. [Back to Question p. 169] [Back to TOC]

Known σ: compute and interpret a CI.

Given $\sigma = 12$, $n = 40$, $\bar{x} = 83.5$. For a 95% z–interval,

$$\mathrm{SE} = \frac{\sigma}{\sqrt{n}} = \frac{12}{\sqrt{40}} \approx 1.897, \qquad z^* = 1.96, \qquad E = z^*\,\mathrm{SE} \approx 1.96(1.897) = 3.72.$$

Hence

$$\mu \in \bar{x} \pm E = (83.5 \pm 3.72) = \boxed{(79.78,\ 87.22)} \text{ (to 2 d.p.).}$$

Interpretation: Using this method, 95% of such intervals would capture the true mean fill weight. For this sample we are 95% confident that the mean fill is between 79.78 g and 87.22 g.

Solution to Q301. [Back to Question p. 169] [Back to TOC]

Unknown σ: t interval.

$n = 12$, $\bar{x} = 6.2$, $s = 1.1$, $\mathrm{df} = 11$. For a 90% CI,

$$t^* = t_{0.95,11} \approx 1.796, \qquad \mathrm{SE} = \frac{s}{\sqrt{n}} = \frac{1.1}{\sqrt{12}} \approx 0.317, \qquad E = t^*\,\mathrm{SE} \approx 1.796(0.317) = 0.57.$$

Thus

$$\boxed{(6.20 \pm 0.57) = (5.63,\ 6.77)}.$$

Why t? σ is unknown; for a normal population we use the t–distribution with $n-1$ degrees of freedom, regardless of sample size.

Solution to Q302. [Back to Question p. 169] [Back to TOC]

Raw data, use technology.

Data: $12, 10, 9, 11, 13, 12, 8, 10$. Here

$$n = 8, \quad \bar{x} = \frac{85}{8} = 10.625, \quad s = \sqrt{\tfrac{1}{7}\sum(x_i - \bar{x})^2} \approx 1.685, \quad \mathrm{df} = 7.$$

For a 99% CI, $t^* = t_{0.995,7} \approx 3.499$, $\mathrm{SE} = s/\sqrt{n} \approx 1.685/\sqrt{8} = 0.596$, so

$$E = t^*\,\mathrm{SE} \approx 3.499(0.596) = 2.08, \qquad \boxed{(10.625 \pm 2.08) = (8.54,\ 12.71)}.$$

Values stated: $n = 8$, $\bar{x} = 10.625$, $s \approx 1.685$, $\mathrm{df} = 7$, $t^* \approx 3.499$.

Solution to Q303. [Back to Question p. 170] [Back to TOC]

Planning sample size (known σ).

For a z–interval, $E = z^*\sigma/\sqrt{n}$. With $E = 0.50$, $\sigma = 3.4$, $z^* = 1.96$,

$$n \ \geq \ \left(\frac{z^*\sigma}{E}\right)^2 = \left(\frac{1.96 \times 3.4}{0.50}\right)^2 = (13.328)^2 \approx 177.7.$$

Round up: $\boxed{n = 178}$.

Solution to Q304. [Back to Question p. 170] [Back to TOC]

Planning with an s estimate.

Use $n \approx (z^* s/E)^2$ with $z^* = 1.96$, $s \approx 4.8$, $E = 1.0$:

$$n \ \geq \ \left(\frac{1.96 \times 4.8}{1.0}\right)^2 = (9.408)^2 \approx 88.5 \ \Rightarrow \ \boxed{n = 89}.$$

Solution to Q305. [Back to Question p. 170] [Back to TOC]

Effect of confidence level.

Team B's interval $(17.5, 22.5)$ is wider, so B likely used the higher confidence level. Margins: Team A: $E_A = (21.7 - 18.3)/2 = \boxed{1.70}$; Team B: $E_B = (22.5 - 17.5)/2 = \boxed{2.50}$.

Solution to Q306. [Back to Question p. 170] [Back to TOC]

Paired data: mean difference CI.

Differences $D = \text{Before} - \text{After}$: $5, 4, 7, 5, 2, 4, 3, 6, 5, 5$.

$$\bar{D} = \frac{46}{10} = 4.6, \qquad s_D = \sqrt{\tfrac{1}{9}\sum (D_i - \bar{D})^2} \approx 1.431, \qquad \text{SE} = \frac{s_D}{\sqrt{10}} \approx 0.453.$$

With $\text{df} = 9$, $t^* = t_{0.975,9} \approx 2.262$, so

$$E = t^*\,\text{SE} \approx 2.262(0.453) = 1.03, \qquad \boxed{\mu_D \in (4.6 \pm 1.03) = (3.58,\ 5.63)}.$$

Interpretation: The mean time decreased by about 3.6 to 5.6 seconds after training (positive D indicates improvement).

Solution to Q307. [Back to Question p. 171] [Back to TOC]

Identify the confidence level from an interval.

Known $\sigma = 9$, $n = 36 \Rightarrow \text{SE} = 9/\sqrt{36} = 1.5$. Reported CI $(71.4, 77.2)$ has margin $E = \frac{77.2 - 71.4}{2} = 2.9$. Thus

$$z^* = \frac{E}{\text{SE}} = \frac{2.9}{1.5} = 1.933 \ \Rightarrow \ \text{CL} \approx 2\Phi(1.933) - 1 \approx 2(0.9733) - 1 = \boxed{0.946} \text{ (about 94.6\%).}$$

Solution to Q308. [Back to Question p. 171] [Back to TOC]

Interpretation check (concept).

The statement is not correct: μ is a fixed (not random) value. A correct interpretation is: "If we repeatedly take random samples and compute a 95% CI each time, then about 95% of those intervals will contain μ. For this sample, we are 95% confident that our interval contains μ."

Solution to Q309. [Back to Question p. 171] [Back to TOC]

Which distribution: z or t?

1. z (normal outcome, σ known).
2. t (σ unknown; with $n = 60$ the CLT makes $\bar{X}$ near normal, but we still use t for the mean when σ is unknown).
3. t (normal and σ unknown, small n).

AHL 4.17 Poisson Distribution

Solution to Q310. [Back to Question p. 173] [Back to TOC]

In two hours, the expected number of arrivals is $\lambda = 3\times 2 = 6$, so $Y \sim \text{Pois}(6)$. (ii) $P(Y=5) = e^{-6}\frac{6^5}{5!} \approx 0.1606$. (iii) $P(Y \geq 7) = 1 - \sum_{k=0}^{6} e^{-6}\frac{6^k}{k!} \approx 0.3931$.

Solution to Q311. [Back to Question p. 173] [Back to TOC]

Rate $= 3.2$ per 10 minutes $\Rightarrow \lambda = 0.32$ per minute. Over 30 minutes, mean $\Lambda = 0.32 \cdot 30 = 9.6$. Model $N \sim \text{Poisson}(9.6)$ with $E[N] = \text{Var}(N) = 9.6$.

$$P(N=12) = e^{-9.6}\frac{9.6^{12}}{12!} \approx 0.08663, \qquad P(N \geq 15) = 1 - \sum_{k=0}^{14} e^{-9.6}\frac{9.6^k}{k!} \approx 0.06428.$$

Poisson is appropriate: independent arrivals, events rare relative to time scale, constant average rate.

Solution to Q312. [Back to Question p. 178] [Back to TOC]

Critical values and critical regions.

Let $X_1, \ldots, X_{36} \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma = 120$ known and $\bar{X} = 1960$.

(i) Hypotheses (one–tailed):

$$H_0: \mu = 2000 \qquad \text{vs} \qquad H_1: \mu < 2000.$$

(ii) Critical value for $\bar{X}$ at $\alpha = 0.05$. Use the z-statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}, \qquad \sigma/\sqrt{n} = \frac{120}{\sqrt{36}} = 20.$$

Lower–tail critical $z_{0.05} = -1.6449$. Hence the critical value for $\bar{X}$ is

$$\bar{x}_{\text{crit}} = \mu_0 + z_{0.05}\frac{\sigma}{\sqrt{n}} = 2000 - 1.6449(20) = \boxed{1967.10}.$$

Reject H_0 if $\bar{X} < 1967.10$.

(iii) Decision. Observed $z = \dfrac{1960 - 2000}{20} = -2.00$ with $p = \Phi(-2.00) = 0.02275 < 0.05$, so $\boxed{\text{reject } H_0}$. There is evidence the mean lifetime is < 2000 h.

Solution to Q313. [Back to Question p. 178] [Back to TOC]

Test for population mean (normal, σ unknown).

Data: $n = 15$, $\bar{x} = 250.4$ ml, $s = 1.2$ ml.

Hypotheses (two–tailed):

$$H_0: \mu = 250 \qquad \text{vs} \qquad H_1: \mu \neq 250.$$

Test statistic (Student t):

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{0.4}{1.2/\sqrt{15}} = \frac{0.4}{0.30984} = 1.290, \qquad \text{df} = 14.$$

Critical value at $\alpha = 0.01$: $t_{0.995,14} \approx 2.977$. Since $|t| = 1.290 < 2.977$, do not reject H_0. (Approx.

two–tailed $p \approx 0.218$.)

Conclusion: At the 1% level there is *no* evidence the mean fill differs from 250 ml. A 99% CI is $\bar{x} \pm t_{0.995,14}\, s/\sqrt{n} = 250.4 \pm 0.923 = (249.48,\ 251.32)$, which contains 250.

Solution to Q314. [Back to Question p. 178] [Back to TOC]

Matched pairs t-test (method).

Let $d_i = (\text{after})_i - (\text{before})_i$ for $i = 1, \dots, 10$.

$$H_0 : \mu_d = 0 \qquad \text{vs} \qquad H_1 : \mu_d > 0 \quad \text{(improvement)}.$$

Compute $\bar{d} = \frac{1}{n} \sum d_i$, $s_d^2 = \frac{1}{n-1} \sum (d_i - \bar{d})^2$, and

$$t = \frac{\bar{d}}{s_d/\sqrt{n}}, \qquad \text{df} = n - 1 = 9.$$

Reject H_0 at level $\alpha = 0.05$ if $t > t_{0.95,9} \approx 1.833$. (*Numerical evaluation requires the actual paired scores; the steps above show the full procedure.*)

Solution to Q315. [Back to Question p. 179] [Back to TOC]

Test for proportion (binomial, one–tailed).

Let $X \sim \text{Bin}(n = 80, p)$ count defectives; observed $x = 5$.

$$H_0 : p = 0.02 \qquad \text{vs} \qquad H_1 : p > 0.02.$$

Exact binomial tail (critical region). Find smallest c with $\Pr_{H_0}(X \geq c) \leq 0.05$.

$$\Pr_{H_0}(X \geq 5) = \sum_{k=5}^{80} \binom{80}{k} 0.02^k (0.98)^{80-k} \approx \boxed{0.02236} < 0.05,$$

whereas $\Pr_{H_0}(X \geq 4) \approx 0.07685 > 0.05$. Hence the critical region is $X \geq 5$.

Decision. Since $x = 5 \in$ critical region and p-value $\approx 0.0224 < 0.05$, $\boxed{\text{reject } H_0}$. There is evidence the defect rate exceeds 2%.

Solution to Q316. [Back to Question p. 179] [Back to TOC]

Test for population mean (Poisson, one–tailed).

Under H_0, the rate is $\lambda_0 = 5$ hits/min. Over $n = 60$ minutes the total count Y is Poisson with mean $\Lambda_0 = n\lambda_0 = 300$. Observed total $y = 330$; test

$$H_0 : \Lambda = 300 \qquad \text{vs} \qquad H_1 : \Lambda > 300.$$

Exact Poisson tail:

$$p = \Pr(Y \geq 330 \mid \Lambda_0 = 300) = \sum_{k=330}^{\infty} e^{-300} \frac{300^k}{k!} \approx \boxed{0.0459}.$$

(For comparison, a normal approximation with continuity correction gives ≈ 0.041.)

Decision: $p \approx 0.046 < 0.05 \Rightarrow \boxed{\text{reject } H_0}$. There is evidence the mean rate has increased.

Solution to Q317. [Back to Question p. 179] [Back to TOC]

Test for correlation coefficient.

$n = 12$ pairs; sample correlation $r = 0.65$. Hypotheses:

$$H_0 : \rho = 0 \qquad \text{vs} \qquad H_1 : \rho \neq 0.$$

Use the t-statistic with $\text{df} = n - 2 = 10$:

$$t = r\sqrt{\frac{n-2}{1-r^2}} = 0.65\sqrt{\frac{10}{1-0.65^2}} = 0.65\sqrt{\frac{10}{0.5775}} \approx 2.705.$$

Two-tailed $p \approx 0.0221$. Since $p < 0.05$ (and $|t| = 2.705 > t_{0.975,10} \approx 2.228$),

$$\boxed{\text{Reject } H_0}$$

and conclude a significant positive correlation.

Solution to Q318. [Back to Question p. 180] [Back to TOC]

Type I and Type II errors.

Test $H_0 : \mu = 500$ vs $H_1 : \mu < 500$ at $\alpha = 0.05$, with $\sigma = 20$, $n = 25$ (so $\text{SE} = 20/\sqrt{25} = 4$).

(i) Type I error. Rejecting H_0 when $\mu = 500$ is true (false positive).

(ii) Type II error. Failing to reject H_0 when the true mean differs (here, is smaller).

(iii) Probability of Type II error at $\mu = 495$. Lower–tail critical $z_{0.05} = -1.6449$ gives the critical sample mean

$$\bar{x}_{\text{crit}} = \mu_0 + z_{0.05} \cdot \text{SE} = 500 - 1.6449 \times 4 = \boxed{493.42}.$$

We fail to reject if $\bar{X} \geq 493.42$. Under $\mu = 495$,

$$\beta = \Pr_{\mu=495}(\bar{X} \geq 493.42) = \Pr\left(Z \geq \frac{493.42 - 495}{4}\right) = \Pr(Z \geq -0.395) = \Phi(0.395) \approx \boxed{0.654}.$$

Thus the test has power $1 - \beta \approx \boxed{0.346}$ at $\mu = 495$.

AHL 4.19 Markov Chains

Solution to Q319. [Back to Question p. 181] [Back to TOC]

(i) The two–step transition matrix is P^2.

$$P^2 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}^2 = \begin{pmatrix} 0.7 \cdot 0.7 + 0.3 \cdot 0.4 & 0.7 \cdot 0.3 + 0.3 \cdot 0.6 \\ 0.4 \cdot 0.7 + 0.6 \cdot 0.4 & 0.4 \cdot 0.3 + 0.6 \cdot 0.6 \end{pmatrix} = \begin{pmatrix} 0.61 & 0.39 \\ 0.46 & 0.54 \end{pmatrix}.$$

Starting from state S, the probability of being in state R after two days is the $(1, 2)$–entry of P^2, namely 0.39. (ii) The steady–state vector $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi}P = \boldsymbol{\pi}$ and $\pi_S + \pi_R = 1$. Solving $\boldsymbol{\pi} = \begin{pmatrix} \pi_S & \pi_R \end{pmatrix}$ yields

$$\pi_S = \frac{4}{7} \approx 0.571, \quad \pi_R = \frac{3}{7} \approx 0.429.$$

Topic 5 Calculus

SL 5.1 Introduction to the concept of limits

Solution to Q320. [Back to Question p. 183] [Back to TOC]

Q1. Limit from a table.

$$f(x) = \frac{x^2-9}{x-3} = x+3 \quad (x \neq 3).$$

Table values (using $x+3$):

x	2.8	2.9	2.99	3.01	3.1	3.2
$f(x)$	5.8	5.9	5.99	6.01	6.1	6.2

Hence $\lim_{x\to 3} f(x) = \boxed{6}$. But $f(3)$ is undefined, so the limit does *not* equal $f(3)$ (removable hole).

Solution to Q321. [Back to Question p. 183] [Back to TOC]

Q2. One–sided limits from the graph. From the sketch:

$$\lim_{x\to 1^-} f(x) \approx \boxed{1.2}, \qquad \lim_{x\to 1^+} f(x) \approx \boxed{1.8}.$$

Since the one–sided limits differ, $\lim_{x\to 1} f(x)$ *does not exist*. The filled dot shows $f(1) = \boxed{0.6}$.

Solution to Q322. [Back to Question p. 183] [Back to TOC]

Q3. Average & instantaneous rate. $s(t) = 3t^2$.

$$\text{Avg on } [2, 2.1] = \frac{s(2.1)-s(2)}{0.1} = \frac{13.23-12}{0.1} = \boxed{12.3 \text{ m s}^{-1}},$$

$$\text{Avg on } [2, 2.01] = \frac{12.1203-12}{0.01} = \boxed{12.03 \text{ m s}^{-1}}.$$

Instantaneous velocity $s'(t) = 6t$, hence $s'(2) = \boxed{12 \text{ m s}^{-1}}$, consistent with the averages.

Solution to Q323. [Back to Question p. 184] [Back to TOC]

Q4. Secant slopes → tangent slope. For $y = x^2$ at $(1,1)$:

$$m_{1.5} = \frac{2.25-1}{0.5} = 2.5, \quad m_{1.2} = \frac{1.44-1}{0.2} = 2.2, \quad m_{1.1} = \frac{1.21-1}{0.1} = 2.1, \quad m_{1.01} = \frac{1.0201-1}{0.01} = 2.01.$$

These approach $\boxed{2}$, the tangent slope (indeed $y' = 2x$, so at $x=1$ it is 2).

Solution to Q324. [Back to Question p. 184] [Back to TOC]

Q5. Interpreting derivative notation.

- $\frac{dy}{dx}$: y depends on x; rate of change of y with respect to x. If y is metres and x seconds, units are $m\ s^{-1}$.
- $f'(3)$: the derivative of f evaluated at $x=3$; instantaneous rate at $x=3$ with units (dependent)/(independent) e.g. m s^{-1}.
- $\frac{dV}{dr}$: volume wrt radius; units $m^3\ per\ m = m^2$. Context: how sphere volume changes with radius.
- $\frac{ds}{dt}$: distance wrt time; instantaneous speed; units $m\ s^{-1}$.

Solution to Q325. [Back to Question p. 185] [Back to TOC]

Q6. Estimating a trigonometric limit. Compute numerically (radians):

h	0.1	0.01	0.001
$\dfrac{\sin(\frac{\pi}{4}+h)-\sin(\frac{\pi}{4})}{h}$	0.6706	0.7036	0.7068

Values approach $\boxed{\cos(\pi/4) = \frac{\sqrt{2}}{2} \approx 0.7071}$.

Solution to Q326. [Back to Question p. 185] [Back to TOC]

Q7. Instantaneous rate from a table.

$$\text{Avg on } [20,30] = \frac{24.7-23.8}{10} = \boxed{0.09\ ^\circ\text{C min}^{-1}}, \quad \text{Avg on } [30,40] = \frac{25.1-24.7}{10} = \boxed{0.04\ ^\circ\text{C min}^{-1}}.$$

Symmetric difference at $t = 30$:

$$\frac{T(40)-T(20)}{40-20} = \frac{25.1-23.8}{20} = \boxed{0.065\ ^\circ\text{C min}^{-1}}.$$

Heating rate is smaller around $t = 30$ than earlier (warming is slowing).

Solution to Q327. [Back to Question p. 186] [Back to TOC]

Q8. Sign of slope from the sinusoidal curve. From the sketch the curve rises on approximately $[0, 0.75)$ and $(2.25, 3.0)$, so $f'(x) > 0$ there; it falls on $(0.75, 2.25)$, so $f'(x) < 0$ there. Horizontal tangents (where $f'(x) = 0$) occur near $x \approx \boxed{0.75}$ (peak) and $x \approx \boxed{2.25}$ (trough).

Solution to Q328. [Back to Question p. 186] [Back to TOC]

Q9. Limit vs. function value. For $x \neq 2$, $g(x) = \dfrac{(x-2)(x+1)}{x-2} = x+1$, hence

$$\lim_{x\to 2} g(x) = 2 + 1 = \boxed{3}.$$

But $g(2) = \boxed{5} \neq 3$, so there is a *removable* (hole) discontinuity at $x = 2$.

Solution to Q329. [Back to Question p. 186] [Back to TOC]

Q10. Tangent slope by estimation. Drawing a tangent at $x = 1$ and reading two points on it gives a slope about $\boxed{-0.4}$ (the curve is decreasing slightly there). Interpretation: near $x = 1$, y decreases by about 0.4 units for each 1 unit increase in x (instantaneous rate of change).

SL5.2 Increasing and decreasing functions

Solution to Q330. [Back to Question p. 188] [Back to TOC]

Analytic: polynomial.

$$f(x) = x^4 - 4x^2 + 1, \qquad f'(x) = 4x^3 - 8x = 4x(x^2 - 2).$$

Critical points: $x = 0,\ \pm\sqrt{2}$. Sign of f':

x	$(-\infty, -\sqrt{2})$	$(-\sqrt{2}, 0)$	$(0, \sqrt{2})$	$(\sqrt{2}, \infty)$
$4x(x^2-2)$	$-$	$+$	$-$	$+$

Therefore f decreases on $(-\infty, -\sqrt{2})$ and $(0, \sqrt{2})$, increases on $(-\sqrt{2}, 0)$ and $(\sqrt{2}, \infty)$. Stationary points: local minima at $x = \pm\sqrt{2}$ (since $- \to +$) with

$$f(\pm\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 + 1 = 4 - 8 + 1 = -3;$$

local maximum at $x = 0$ with $f(0) = 1$.

Solution to Q331. [Back to Question p. 188] [Back to TOC]

Read from a graph of f. From the plotted curve (a shifted/scaled cubic) the function is:

- *increasing* for approximately $x < -0.2$ and for $x > 2.1$;
- *decreasing* on roughly $(-0.2, 2.1)$.

Local *maximum* near $x \approx -0.15$ and local *minimum* near $x \approx 2.15$. At the turning points the slope $f'(x)$ is 0; on rising segments $f'(x) > 0$ and on falling segments $f'(x) < 0$.

Solution to Q332. [Back to Question p. 188] [Back to TOC]

Given the graph of f'. Where $f'(x) > 0$ (above the x–axis) the function f is increasing; where $f'(x) < 0$ it is decreasing. From the picture:

$$\begin{aligned}&\text{Increasing on } (-3, -2) \cup (-1, 0.5) \cup (2.8, 3],\\ &\text{Decreasing on } (-2, -1) \cup (0.5, 2.8).\end{aligned}$$

Zeros of f' (sign changes) give extrema for f: f has a local *max* where f' changes $+ \to -$ (near $x \approx 0.5$) and a local *min* where f' changes $- \to +$ (near $x \approx -2$ and $x \approx 2.8$). A rough sketch of f would rise to a peak near $x \approx 0.5$, dip either side as described.

Solution to Q333. [Back to Question p. 189] [Back to TOC]

Rational function (state the domain!).

$$f(x) = \frac{x+1}{x-2}, \qquad f'(x) = \frac{(x-2)-(x+1)}{(x-2)^2} = \boxed{-\frac{3}{(x-2)^2}}.$$

Since $(x-2)^2 > 0$ for $x \neq 2$, $f'(x) < 0$ on each domain interval. Hence f is strictly *decreasing* on $(-\infty, 2)$ and on $(2, \infty)$. The vertical asymptote $x = 2$ splits the domain and is not included.

Solution to Q334. [Back to Question p. 189] [Back to TOC]

Trigonometric on a closed interval.

$$f(x) = \sin x + \tfrac{1}{2}\cos(2x), \qquad f'(x) = \cos x - \sin(2x) = \cos x\,(1 - 2\sin x).$$

Critical points on $[0, 2\pi]$ from $\cos x = 0$ or $\sin x = \frac{1}{2}$:

$$x = \boxed{\tfrac{\pi}{6},\ \tfrac{\pi}{2},\ \tfrac{5\pi}{6},\ \tfrac{3\pi}{2}}.$$

Sign of f' by factors gives:

interval	$[0, \frac{\pi}{6})$	$(\frac{\pi}{6}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{5\pi}{6})$	$(\frac{5\pi}{6}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi]$
sign of f'	$+$	$-$	$+$	$-$	$+$

Thus f increases on $[0, \frac{\pi}{6}) \cup (\frac{\pi}{2}, \frac{5\pi}{6}) \cup (\frac{3\pi}{2}, 2\pi]$ and decreases on $(\frac{\pi}{6}, \frac{\pi}{2}) \cup (\frac{5\pi}{6}, \frac{3\pi}{2})$. Local maxima at $x = \frac{\pi}{6}, \frac{5\pi}{6}$ (change $+ \to -$); local minima at $x = \frac{\pi}{2}, \frac{3\pi}{2}$ $(- \to +)$.

Solution to Q335. [Back to Question p. 190] [Back to TOC]

Piecewise linear graph of f. From the polyline:

$$\begin{aligned} &\text{Increasing on } [-3, -1] \text{ and } [2, 4], \\ &\text{Constant on } [-1, 1], \\ &\text{Decreasing on } [1, 2]. \end{aligned}$$

Corners at $x = -1, 1, 2 \Rightarrow f'(x)$ is *undefined* at these x (non-differentiable sharp points).

Solution to Q336. [Back to Question p. 190] [Back to TOC]

Sign chart from a factored derivative.

$$f'(x) = (x-1)^2(x+2)(3-x).$$

Zeros at $x = -2,\ 1,\ 3$ with multiplicities $1, 2, 1$ respectively. Since $(x-1)^2 \geq 0$ never changes sign, the sign of f' is governed by $(x+2)(3-x)$:

x	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
sign of f'	$-$	$+$	$+$	$-$

Therefore f decreases on $(-\infty, -2)$ and $(3, \infty)$ and increases on $(-2, 1)$ and $(1, 3)$. Classification: at $x = -2$ $(- \to +)$ a *local minimum*; at $x = 1$ the sign does not change (double root) $\Rightarrow$ *stationary inflection/flat point*; at $x = 3$ $(+ \to -)$ a *local maximum*.

Solution to Q337. [Back to Question p. 190] [Back to TOC]

Table of derivative values. From the table $f'(x) > 0$ on $(-1, 2)$ and $f'(x) < 0$ on $(-3, -1)$ and $(2, 4)$. Thus f increases on $(-1, 2)$ and decreases on $(-3, -1)$ and $(2, 4)$. Sign changes suggest a local *minimum* near $x \approx -1$ (from $-$ to $+$) and a local *maximum* near $x \approx 2$ (from $+$ to $-$).

Solution to Q338. [Back to Question p. 191] [Back to TOC]

From monotonicity of f to f'. Because f is increasing on $(-\infty, -1)$ and $(2, \infty)$, one must have $f'(x) > 0$ on those intervals; since f is decreasing on $(-1, 2)$, $f'(x) < 0$ there. A consistent sketch of f' therefore:

f' positive for $x < -1$, crosses 0 at $x = -1$, negative on $(-1, 2)$, crosses 0 at $x = 2$, then positive for $x > 2$.

(One example is the cubic $f'(x) = (x+1)(x-2)$, but any curve with the same sign pattern is acceptable.)

SL5.3 Basic differentiation

Solution to Q339. [Back to Question p. 192] [Back to TOC]

Q1. Differentiate basic powers.

1. $f(x) = 7x^6 \;\Rightarrow\; f'(x) = \boxed{42x^5}$.
2. $f(x) = -4x^{-3} \;\Rightarrow\; f'(x) = \boxed{12x^{-4}} = \dfrac{12}{x^4}$.
3. $f(x) = 5x^{-1} \;\Rightarrow\; f'(x) = \boxed{-5x^{-2}} = -\dfrac{5}{x^2}$.
4. $f(x) = 12 \;\Rightarrow\; f'(x) = \boxed{0}$.

Solution to Q340. [Back to Question p. 192] [Back to TOC]

Q2. Polynomials with integer exponents.

$$g(x) = 3x^7 - 5x^4 + 2x^3 - 9x + 6 - 8x^{-2}.$$

$$g'(x) = \boxed{21x^6 - 20x^3 + 6x^2 - 9 + 16x^{-3}} = 21x^6 - 20x^3 + 6x^2 - 9 + \frac{16}{x^3}.$$

Solution to Q341. [Back to Question p. 192] [Back to TOC]

Q3. Slope at a point and tangent.

$$h(x) = 2x^5 - x^2 + 3x - 4, \qquad h'(x) = \boxed{10x^4 - 2x + 3}.$$

At $x = -1$ the slope is $h'(-1) = 10 + 2 + 3 = \boxed{15}$. Point: $h(-1) = 2(-1)^5 - (-1)^2 + 3(-1) - 4 = -10$. Tangent: $y + 10 = 15(x + 1) \;\Rightarrow\; \boxed{y = 15x + 5}$.

Solution to Q342. [Back to Question p. 193] [Back to TOC]

Q4. Tangent and normal at $x = 2$. For $y = x^4 - 2x^2 + 1$,

$$y'(x) = 4x^3 - 4x.$$

At $x = 2$: slope $m = 4(8) - 8 = 24$, point $(2, 9)$.

$$\text{Tangent: } y - 9 = 24(x - 2) \;\Rightarrow\; \boxed{y = 24x - 39}.$$

Normal slope $= -\dfrac{1}{24}$, hence

$$\text{Normal: } y - 9 = -\frac{1}{24}(x - 2) \;\Rightarrow\; \boxed{y = -\frac{1}{24}x + \frac{109}{12}}.$$

Solution to Q343. [Back to Question p. 193] [Back to TOC]

Q5. Stationary points of a cubic.

$$f(x) = x^3 - 6x^2 + 9x + 1, \quad f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3).$$

Critical $x = \boxed{1, 3}$. Using $f''(x) = 6x - 12$:

$$f''(1) = -6 < 0 \Rightarrow \text{local max at } (1, f(1)) = (1, 5),$$

$$f''(3) = 6 > 0 \Rightarrow \text{local min at } (3, f(3)) = (3, 1).$$

Solution to Q344. [Back to Question p. 193] [Back to TOC]

Q6. Increasing/decreasing via $f'(x) = x(x-3)^2(x+1)$. Zeros at $x = -1, 0, 3$ (with multiplicity 2 at $x = 3$). Sign of f':

x	$(-\infty, -1)$	$(-1, 0)$	$(0, 3)$	$(3, \infty)$
sign f'	$+$	$-$	$+$	$+$

Thus f increases on $(-\infty, -1) \cup (0, \infty)$ and decreases on $(-1, 0)$. Classification: local *max* at $x = -1$ ($+ \to -$), local *min* at $x = 0$ ($- \to +$), and a *flat/stationary inflection* at $x = 3$ (no sign change).

Solution to Q345. [Back to Question p. 194] [Back to TOC]

Q7. Find unknown coefficients. $p(x) = ax^3 + bx^2 + cx + 4$, so $p'(x) = 3ax^2 + 2bx + c$.

$$p'(1) = 0: \; 3a + 2b + c = 0, \qquad p'(2) = 6: \; 12a + 4b + c = 6, \qquad \text{horizontal at } x = 0: \; c = 0.$$

Hence $3a + 2b = 0, \; 12a + 4b = 6 \Rightarrow a = 1, \; b = -\frac{3}{2}, \; c = 0.$

$$\boxed{p(x) = x^3 - \tfrac{3}{2}x^2 + 4}.$$

Solution to Q346. [Back to Question p. 194] [Back to TOC]

Q8. Parallel/perpendicular tangents for $y = 2x^3 - x$. $y'(x) = 6x^2 - 1$.

1. Parallel to $y = 5x - 1$ (slope 5): $6x^2 - 1 = 5 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$. Points $(1, 1)$ and $(-1, -1)$, tangents:

$$\boxed{y = 5x - 4} \text{ at } (1, 1), \qquad \boxed{y = 5x + 4} \text{ at } (-1, -1).$$

2. Perpendicular to $y = \frac{1}{2}x + 3$ requires slope -2. But $6x^2 - 1 \geq -1$ for all x, so $6x^2 - 1 = -2$ has no real solution. *No perpendicular tangent exists.*

Solution to Q347. [Back to Question p. 194] [Back to TOC]

Q9. Applied rate of change. $s(t) = 4t^3 - 3t^2 + 2$ (m). Velocity $v(t) = s'(t) = \boxed{12t^2 - 6t}$ (m s^{-1}). Acceleration $a(t) = v'(t) = \boxed{24t - 6}$ (m s^{-2}). At $t = 2$: $v(2) = 48 - 12 = \boxed{36 \text{ m s}^{-1}}$, $a(2) = 48 - 6 = \boxed{42 \text{ m s}^{-2}}$.

Solution to Q348. [Back to Question p. 195] [Back to TOC]

Q10. Rational with integer powers.

$$r(x) = \frac{3x^2 - 1}{x^3} = 3x^{-1} - x^{-3}.$$

$$r'(x) = \boxed{-3x^{-2} + 3x^{-4}} = -\frac{3}{x^2} + \frac{3}{x^4}.$$

Solution to Q349. [Back to Question p. 195] [Back to TOC]

Q11. Optimisation of $V(x) = x(20 - 2x)^2$, $0 < x < 10$.

$$V(x) = 400x - 80x^2 + 4x^3, \qquad V'(x) = 400 - 160x + 12x^2.$$

Critical values from $12x^2 - 160x + 400 = 0 \;\Rightarrow\; 3x^2 - 40x + 100 = 0$,

$$x = \frac{40 \pm 20}{6} \Rightarrow x = 10, \quad \frac{10}{3}.$$

At $x = 10$ the volume is 0; the maximum occurs at

$$\boxed{x = \frac{10}{3}\text{ cm}}, \quad \text{since } V''(x) = -160 + 24x,\; V''(10/3) = -80 < 0.$$

Solution to Q350. [Back to Question p. 195] [Back to TOC]

Q12. Graph–based derivative check.

$$f(x) = x^4 - 4x^2 + 1, \quad f'(x) = 4x^3 - 8x = 4x(x^2 - 2).$$

$$f'(1) = 4(1)(-1) = \boxed{-4}.$$

Numerical secant with $x = 0.9$ and 1.1:

$$\frac{f(1.1) - f(0.9)}{1.1 - 0.9} = \frac{(1.4641 - 4.84 + 1) - (0.6561 - 3.24 + 1)}{0.2} = \frac{-2.3759 - (-1.5839)}{0.2} \approx \boxed{-3.96},$$

which is close to -4, confirming the derivative value.

SL5,4 Tangents and normals

Solution to Q351. [Back to Question p. 198] [Back to TOC]

Tangent and normal at a given x-value.

For $f(x) = x^3 - 2x^2 + 5x - 7$,

$$f'(x) = 3x^2 - 4x + 5.$$

At $x = 2$,

$$m_{\text{tan}} = f'(2) = 3(4) - 8 + 5 = 9, \qquad f(2) = 8 - 8 + 10 - 7 = 3.$$

Tangent: $y - 3 = 9(x - 2) \;\Rightarrow\; \boxed{y = 9x - 15}$.

Normal slope $m_{\text{nor}} = -\dfrac{1}{9}$, so

$$\boxed{y - 3 = -\tfrac{1}{9}(x - 2) \;\Rightarrow\; y = -\tfrac{1}{9}x + \tfrac{29}{9}}.$$

Solution to Q352. [Back to Question p. 198] [Back to TOC]

Tangent through a given point on the curve.

$y = \ln(3x)$ with $y'(x) = \dfrac{1}{x}$. At $P(1, \ln 3)$ the slope is 1, so

$$\boxed{y - \ln 3 = 1(x - 1) \;\Rightarrow\; y = x - 1 + \ln 3}.$$

Solution to Q353. [Back to Question p. 198] [Back to TOC]

Normal line.

For $y = e^{2x}$, $y'(x) = 2e^{2x}$. At $x_0 = \ln 2$, $y_0 = e^{2\ln 2} = 4$ and $m_{\text{tan}} = 2 \cdot 4 = 8$; hence $m_{\text{nor}} = -\dfrac{1}{8}$.

$$y - 4 = -\tfrac{1}{8}\,(x - \ln 2) \;\Longrightarrow\; \boxed{x + 8y - (32 + \ln 2) = 0}.$$

Solution to Q354. [Back to Question p. 199] [Back to TOC]

Tangent parallel to a given line.

For $y = x^3 - 3x$, $y' = 3x^2 - 3 = 3(x^2 - 1)$. Parallel to $y = 6x - 4$ means slope 6:

$$3(x^2 - 1) = 6 \;\Rightarrow\; x^2 = 3 \;\Rightarrow\; x = \pm\sqrt{3}.$$

Points $(\pm\sqrt{3}, 0)$. Tangents:

$$\boxed{y = 6x - 6\sqrt{3}} \quad \text{and} \quad \boxed{y = 6x + 6\sqrt{3}}.$$

Solution to Q355. [Back to Question p. 199] [Back to TOC]

Tangent perpendicular to a given line.

Line $3x + y = 0$ has slope -3. For $y = \sqrt{x}$, $y'(x) = \dfrac{1}{2\sqrt{x}}$. Perpendicular condition: $\dfrac{1}{2\sqrt{a}} \cdot (-3) = -1 \Rightarrow \dfrac{1}{2\sqrt{a}} = \dfrac{1}{3} \Rightarrow \sqrt{a} = \tfrac{3}{2} \Rightarrow a = \dfrac{9}{4}$. Point $\left(\dfrac{9}{4}, \dfrac{3}{2}\right)$. Normal slope $m_{\text{nor}} = -\dfrac{1}{y'(a)} = -\dfrac{1}{1/3} = -3$.

$$\boxed{y - \tfrac{3}{2} = -3\Big(x - \tfrac{9}{4}\Big)}.$$

Solution to Q356. [Back to Question p. 199] [Back to TOC]

Horizontal and vertical tangents.

$y = x^{2/3}(x-3)$. Using product rule,

$$y' = \frac{2}{3}x^{-1/3}(x-3) + x^{2/3} = x^{-1/3}\left(\frac{2}{3}(x-3) + x\right) = x^{-1/3}\left(\frac{5}{3}x - 2\right).$$

Horizontal tangents when $y' = 0 \Rightarrow \frac{5}{3}x - 2 = 0 \Rightarrow \boxed{x = \frac{6}{5}}$.

As $x \to 0^+$, $x^{-1/3} \to +\infty$ and $y' \to -\infty$; as $x \to 0^-$, $x^{-1/3} \to -\infty$ and $y' \to +\infty$. Slopes blow up with opposite signs $\Rightarrow$ *cusp* at $x = 0$ (not a vertical tangent).

Solution to Q357. [Back to Question p. 199] [Back to TOC]

Normal passing through a fixed point.

Curve $y = x^2 + 1$. At $x = x_0$, $y_0 = x_0^2 + 1$, $y'(x_0) = 2x_0$ and normal slope $m_{\text{nor}} = -\dfrac{1}{2x_0}$ (for $x_0 \neq 0$). Requiring the normal through $(0, 2)$:

$$2 - y_0 = m_{\text{nor}}(0 - x_0) \;\Rightarrow\; 2 - (x_0^2 + 1) = \frac{1}{2} \;\Rightarrow\; x_0^2 = \frac{1}{2}.$$

Thus the points are

$$\boxed{\left(\tfrac{1}{\sqrt{2}},\ \tfrac{3}{2}\right)} \quad \text{and} \quad \boxed{\left(-\tfrac{1}{\sqrt{2}},\ \tfrac{3}{2}\right)}.$$

(Optionally, normals: $y - \frac{3}{2} = -\frac{1}{\sqrt{2}}\left(x - \frac{1}{\sqrt{2}}\right)$ and $y - \frac{3}{2} = \frac{1}{\sqrt{2}}\left(x + \frac{1}{\sqrt{2}}\right)$.)

Solution to Q358. [Back to Question p. 200] [Back to TOC]

Tangent to a circle (analytic).

Circle $x^2 + y^2 = 25$, line $\ell : y = mx + 1$. Distance from the origin to ℓ is $\dfrac{|1|}{\sqrt{1+m^2}} < 1$ for all real m, but the circle has radius 5. Tangency requires the distance to equal 5, which is impossible.

$$\boxed{\text{No real slope } m \text{ makes } y = mx + 1 \text{ tangent to } x^2 + y^2 = 25.}$$

Solution to Q359. [Back to Question p. 200] [Back to TOC]

Exponential model; technology may help.

$f(x) = 5e^{-0.4x} + 1$, $f'(x) = -2e^{-0.4x}$.

(a) At $x = 2$, $f(2) = 5e^{-0.8} + 1 \approx 3.2467$, slope $m_{\tan} = -2e^{-0.8} \approx -0.8987$.

$$\boxed{y - f(2) = m_{\tan}(x-2)} \quad \text{or} \quad \boxed{y \approx -0.8987(x-2) + 3.2467}.$$

(b) A normal at $(x_0, f(x_0))$ has slope $m_{\text{nor}} = -1/f'(x_0) = \frac{1}{2}e^{0.4x_0}$. To pass through the origin we need $f(x_0) = m_{\text{nor}}\, x_0$, i.e.

$$5e^{-0.4x_0} + 1 = \frac{x_0}{2}e^{0.4x_0}.$$

Solving numerically gives $\boxed{x_0 \approx 2.33}$ (more precisely 2.334). Then $m_{\text{nor}} \approx \frac{1}{2}e^{0.4(2.334)} \approx 1.272$, so the normal is approximately

$$\boxed{y \approx 1.272\,x}.$$

Solution to Q360. [Back to Question p. 200] [Back to TOC]

Where is the tangent of given slope?

$y = \sin x + \dfrac{x}{2} \Rightarrow y' = \cos x + \dfrac{1}{2}$. Set $y' = 1 \Rightarrow \cos x = \dfrac{1}{2}$. On $[0, 2\pi]$:

$$\boxed{x = \frac{\pi}{3},\ \frac{5\pi}{3}}.$$

At $x = \pi/3$, $y = \dfrac{\sqrt{3}}{2} + \dfrac{\pi}{6}$ and the tangent ($m = 1$) is

$$\boxed{y - \left(\tfrac{\sqrt{3}}{2} + \tfrac{\pi}{6}\right) = 1\left(x - \tfrac{\pi}{3}\right) \ \Rightarrow\ y = x + \left(\tfrac{\sqrt{3}}{2} - \tfrac{\pi}{6}\right)}.$$

(At $x = 5\pi/3$, the tangent is $y = x - \dfrac{5\pi}{6} - \dfrac{\sqrt{3}}{2}$.)

Solution to Q361. [Back to Question p. 201] [Back to TOC]

Normal of minimal distance to a point.

Curve $y = x^2 - 4x + 7$, so $y'(x) = 2x - 4$. At a general point (x_0, y_0) with $y_0 = x_0^2 - 4x_0 + 7$, the normal slope is

$$m_{\text{nor}} = -\frac{1}{2x_0 - 4}.$$

For the normal to pass through $(0, 0)$ its slope must equal y_0/x_0, hence

$$\frac{y_0}{x_0} = -\frac{1}{2x_0 - 4}.$$

This gives

$$(2x_0 - 4)(x_0^2 - 4x_0 + 7) + x_0 = 0 \quad \Rightarrow \quad \boxed{2x_0^3 - 12x_0^2 + 31x_0 - 28 = 0}.$$

Solving the cubic numerically yields

$$\boxed{x_0 \approx 1.72137}, \qquad y_0 = x_0^2 - 4x_0 + 7 \approx \boxed{3.07810}.$$

Tangent slope: $m_{\tan} = 2x_0 - 4 \approx -0.55726$, so the tangent line is

$$\boxed{y - 3.07810 \approx -0.55726\,(x - 1.72137)}.$$

Normal slope: $m_{\text{nor}} = -1/m_{\tan} \approx 1.7943$, and since the normal was constrained to pass through the origin,

$$\boxed{y \approx 1.7943\,x}.$$

Solution to Q362. [Back to Question p. 201] [Back to TOC]

Graph-and-verify (technology).

$f(x) = \ln(x+2) - \dfrac{x}{3}$ for $x > -2$. $f'(x) = \dfrac{1}{x+2} - \dfrac{1}{3}$. At $x = 1$, $f'(1) = \dfrac{1}{3} - \dfrac{1}{3} = 0$ and $f(1) = \ln 3 - \dfrac{1}{3}$, so the tangent is the horizontal line

$$\boxed{y = \ln 3 - \frac{1}{3}}.$$

(Technology will show the line touches the curve only at $x = 1$.)

SL5.5 Integration

Solution to Q363. [Back to Question p. 204] [Back to TOC]

Q1. Indefinite integrals (power rule). Always add $+C$.

1. $\displaystyle\int \left(7x^5 - 3x^2 + 4\right) dx = \frac{7}{6}x^6 - x^3 + 4x + C.$
2. $\displaystyle\int \left(2x^{-3} - 5x^{-1} + 9x\right) dx = -x^{-2} - 5\ln|x| + \frac{9}{2}x^2 + C.$
3. $\displaystyle\int \left(-6x^7 + x - 8\right) dx = -\frac{3}{4}x^8 + \frac{1}{2}x^2 - 8x + C.$

Solution to Q364. [Back to Question p. 204] [Back to TOC]

Q2. Constant from a boundary condition.

$$\frac{dy}{dx} = 3x^2 + x \;\Rightarrow\; y = x^3 + \frac{1}{2}x^2 + C.$$

Use $y(1) = 10$: $1 + \frac{1}{2} + C = 10 \Rightarrow C = 8.5$.

$$\boxed{y = x^3 + \tfrac{1}{2}x^2 + 8.5}.$$

Solution to Q365. [Back to Question p. 205] [Back to TOC]

Q3. Initial value problem (velocity $\to$ displacement).

$$s'(t) = v(t) = 4t - 3 \Rightarrow s(t) = 2t^2 - 3t + C, \quad s(0) = 2 \Rightarrow C = 2.$$

Thus $s(t) = \boxed{2t^2 - 3t + 2}$. At $t = 5$: $s(5) = 50 - 15 + 2 = \boxed{37\text{ m}}$ (distance from origin).

Solution to Q366. [Back to Question p. 205] [Back to TOC]

Q4. Definite integral.

$$\int_2^6 (3x^2 + 4)\, dx = \left[x^3 + 4x\right]_2^6 = (216 + 24) - (8 + 8) = \boxed{224}.$$

Solution to Q367. [Back to Question p. 205] [Back to TOC]

Q5. Area under $f(x) = 4 - x^2$ on $[-1, 1]$.

$$A = \int_{-1}^{1} (4 - x^2)\, dx = \left[4x - \frac{x^3}{3}\right]_{-1}^{1} = \left(4 - \frac{1}{3}\right) - \left(-4 + \frac{1}{3}\right) = \boxed{\frac{22}{3}} \approx \boxed{7.33} \text{ square units.}$$

Solution to Q368. [Back to Question p. 206] [Back to TOC]

Q6. Area where the function changes sign, $g(x) = x^2 - 4x$. Zeros at $x = 0$ and $x = 4$. On $(0, 4)$, $g(x) < 0$, so the total area is

$$A = \int_0^4 |g(x)|\, dx = -\int_0^4 (x^2 - 4x)\, dx = \int_0^4 (4x - x^2)\, dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \boxed{\frac{32}{3}} \approx 10.67.$$

(The sketch should show the curve below the x–axis between 0 and 4.)

Solution to Q369. [Back to Question p. 206] [Back to TOC]

Q7. Recover f from f'.

$$f'(x) = 5x^4 - 2x \Rightarrow f(x) = x^5 - x^2 + C.$$

Use $f(2) = 7$: $32 - 4 + C = 7 \Rightarrow C = -21$. Hence $f(x) = \boxed{x^5 - x^2 - 21}$ and $f(0) = \boxed{-21}$.

Solution to Q370. [Back to Question p. 206] [Back to TOC]

Q8. Area for $y = 2x + 1$ above the axis on $[-0.5, 2]$.

$$A = \int_{-0.5}^{2} (2x+1)\,dx = \left[x^2 + x\right]_{-0.5}^{2} = (4+2) - \big(0.25 - 0.5\big) = 6 - (-0.25) = \boxed{\frac{25}{4}} = 6.25.$$

Solution to Q371. [Back to Question p. 207] [Back to TOC]

Q9. Average value on $[1, 4]$ for $f(x) = 3x^2 - x$.

$$\bar{f} = \frac{1}{4-1}\int_1^4 (3x^2 - x)\,dx = \frac{1}{3}\left[x^3 - \frac{x^2}{2}\right]_1^4 = \frac{1}{3}\Big(64 - 8 - \big(1 - \tfrac{1}{2}\big)\Big) = \frac{1}{3}\cdot\frac{111}{2} = \boxed{\frac{111}{6}} = \boxed{18.5}.$$

Solution to Q372. [Back to Question p. 207] [Back to TOC]

Q10. From a to v to s; distance travelled.

$$a(t) = 6t \Rightarrow v(t) = \int 6t\,dt = 3t^2 + C_1, \quad v(0) = 2 \Rightarrow C_1 = 2,$$

so $v(t) = \boxed{3t^2 + 2}$.

$$s(t) = \int (3t^2 + 2)\,dt = t^3 + 2t + C_2, \quad s(0) = 5 \Rightarrow C_2 = 5,$$

hence $s(t) = \boxed{t^3 + 2t + 5}$. Distance travelled on $[0, 3]$ (velocity is positive, so equals displacement):

$$\int_0^3 v(t)\,dt = \Big[t^3 + 2t\Big]_0^3 = 27 + 6 = \boxed{33 \text{ m}}.$$

SL 5.6 Local minimums and maximums

Solution to Q373. [Back to Question p. 209] [Back to TOC]

Solve $f'(x) = 0$ and classify for $f(x) = x^3 - 6x^2 + 9x + 2$.

$$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3).$$

Stationary x–values: $\boxed{x = 1,\ 3}$. Second derivative $f''(x) = 6x - 12$:

$$f''(1) = -6 < 0 \Rightarrow \text{local maximum at } x = 1, \quad f''(3) = +6 > 0 \Rightarrow \text{local minimum at } x = 3.$$

Coordinates:

$$f(1) = 1 - 6 + 9 + 2 = \boxed{6}, \qquad f(3) = 27 - 54 + 27 + 2 = \boxed{2}.$$

So $(1, 6)$ is a local max, $(3, 2)$ a local min.

Solution to Q374. [Back to Question p. 209] [Back to TOC]

$g(x) = x^4 - 4x^2$ on $[-3, 3]$.

$$g'(x) = 4x^3 - 8x = 4x(x^2 - 2) = 0 \ \Rightarrow \ x = \boxed{0,\ \pm\sqrt{2}}.$$

$$g''(x) = 12x^2 - 8.$$

At $x = 0$: $g''(0) = -8 < 0$ (local max); at $x = \pm\sqrt{2}$: $g''(\pm\sqrt{2}) = 16 > 0$ (local minima). Function values:

$$g(\pm 3) = 81 - 36 = 45, \quad g(0) = 0, \quad g(\pm\sqrt{2}) = 4 - 8 = -4.$$

Hence on $[-3, 3]$ the greatest value is $\boxed{45}$ at $x = \pm 3$ (endpoints), and the least value is $\boxed{-4}$ at $x = \pm\sqrt{2}$.

Solution to Q375. [Back to Question p. 209] [Back to TOC]

From a factored derivative $h'(x) = (x-2)^2(x+1)$.

Stationary x–values: $\boxed{x = -1,\ 2}$. Sign of h':

x	$(-\infty, -1)$	$(-1, 2)$	$(2, \infty)$
$(x-2)^2$	$+$	$+$	$+$
$(x+1)$	$-$	$+$	$+$
$h'(x)$	$-$	$+$	$+$

Thus at $x = -1$ the sign changes $- \to + \Rightarrow$ **local minimum**. At $x = 2$ the sign does not change (even multiplicity) $\Rightarrow$ **stationary point of inflection** (flat).

Solution to Q376. [Back to Question p. 210] [Back to TOC]

Technology turning point for $p(x) = x\,e^{-0.3x}$ on $[0, 10]$.

$$p'(x) = e^{-0.3x}(1 - 0.3x) = 0 \Rightarrow 1 - 0.3x = 0 \Rightarrow x = \boxed{\tfrac{10}{3} \approx 3.333}.$$

Since $p''(x) = e^{-0.3x}(-0.6 + 0.09x)$ gives $p''(10/3) = -0.2\,e^{-10/3} < 0$, this is a local *maximum*.

$$p\left(\tfrac{10}{3}\right) = \tfrac{10}{3}\,e^{-1} \approx \boxed{1.226}.$$

Solution to Q377. [Back to Question p. 210]   [Back to TOC]

Revenue $R(p) = -200p^2 + 5200p$**.**

$$R'(p) = -400p + 5200 = 0 \Rightarrow p = \boxed{13 \text{ dollars}}.$$

$R''(p) = -400 < 0$ so this is a maximum.

$$R(13) = -200(169) + 5200(13) = -33800 + 67600 = \boxed{\$33{,}800}.$$

Solution to Q378. [Back to Question p. 210]   [Back to TOC]

Sketch-based estimation.

From the curve (two turning points), the horizontal tangents occur roughly midway between successive x–intercepts. Visual read–off gives

$$\boxed{x \approx -0.9 \text{ (local max)}, \quad x \approx 1.2 \text{ (local min)}}$$

(answers within a small tolerance are acceptable; justification: tangent looks horizontal and the graph changes from increasing to decreasing at the first point, and vice versa at the second).

Solution to Q379. [Back to Question p. 211]   [Back to TOC]

Rational function $q(x) = \dfrac{x^3 - 3x}{x^2 + 1}$**.**

Quotient rule (denominator positive):

$$q'(x) = \frac{(3x^2 - 3)(x^2 + 1) - (x^3 - 3x)(2x)}{(x^2 + 1)^2} = \frac{x^4 + 6x^2 - 3}{(x^2 + 1)^2}.$$

Solve $x^4 + 6x^2 - 3 = 0$. Put $u = x^2$:

$$u^2 + 6u - 3 = 0 \Rightarrow u = -3 \pm 2\sqrt{3}.$$

Only $u = -3 + 2\sqrt{3} > 0$ is admissible, so

$$x = \boxed{\pm\sqrt{-3 + 2\sqrt{3}}} \approx \boxed{\pm 0.681}.$$

Sign of numerator: positive for $|x| > 0.681$, negative for $|x| < 0.681$. Therefore at $x \approx -0.681$: $+ \to -$ **local maximum**; at $x \approx 0.681$: $- \to +$ **local minimum**. Coordinates (approx.): $q(-0.681) \approx \boxed{1.18}$, $q(0.681) \approx \boxed{-1.18}$.

Solution to Q380. [Back to Question p. 211]   [Back to TOC]

$r'(x) = x(x^2 - 4) = x(x - 2)(x + 2)$**.**

Roots: $\boxed{x = -2, 0, 2}$. Sign chart $\Rightarrow$ increasing on $(-2, 0)$ and $(2, \infty)$; decreasing on $(-\infty, -2)$ and $(0, 2)$. Thus $x = -2$ and $x = 2$ are **local minima** $(- \to +)$, and $x = 0$ is a **local maximum** $(+ \to -)$.

Solution to Q381. [Back to Question p. 212]   [Back to TOC]

$s(x) = \sin x + 0.2x$ **on** $[-3\pi, 3\pi]$**.**

$$s'(x) = \cos x + 0.2 = 0 \Rightarrow \cos x = -0.2.$$

General solutions: $x = \pm \arccos(-0.2) + 2k\pi$. Within $[-3\pi, 3\pi]$ this gives

$$\boxed{x \approx -8.055,\ -4.511,\ -1.772,\ 1.772,\ 4.511,\ 8.055}.$$

Use $s''(x) = -\sin x$ at each root (where $|\sin x| \approx 0.980$): $s'' > 0$ (min) when $\sin x < 0$; $s'' < 0$ (max) when $\sin x > 0$. Hence *local minima* at $\boxed{x \approx -8.055,\ -1.772,\ 4.511}$ and *local maxima* at $\boxed{x \approx -4.511,\ 1.772,\ 8.055}$.

Solution to Q382. [Back to Question p. 212] [Back to TOC]

Local $\neq$ global on a closed interval.

(a) On a closed interval, a continuous function attains global extrema either at stationary points ($f'(x) = 0$) *or* at the endpoints. Thus a local max/min inside the interval need not be the greatest/least overall value.

(b) Candidates for the global maximum/minimum on $[-5, 4]$ are the stationary points and endpoints:

$$\boxed{x = -5, \quad -2, \quad 1, \quad 4}.$$

(One would evaluate $t(x)$ at these four x–values to decide the global extrema.)

SL 5.7 Optimisation

Solution to Q383. [Back to Question p. 213] [Back to TOC]

Price to maximise profit (linear demand).

Demand: $q = 120 - 3p$.
Revenue: $R = pq = p(120 - 3p) = 120p - 3p^2$.
Cost as a function of p:

$$C(q) = 420 + 8q = 420 + 8(120 - 3p) = 1380 - 24p.$$

Profit as a function of price:

$$P(p) = R - C = (-3)p^2 + 144p - 1380.$$

Differentiate and set to zero:

$$P'(p) = -6p + 144 = 0 \quad \Rightarrow \quad p = 24.$$

Since $P''(p) = -6 < 0$, this gives a maximum. The maximum profit is

$$P(24) = -3(24)^2 + 144(24) - 1380 = 348 \text{ dollars}.$$

Breakeven price(s): solve $P(p) = 0$:

$$-3p^2 + 144p - 1380 = 0 \iff p^2 - 48p + 460 = 0$$

$$p = \frac{48 \pm \sqrt{48^2 - 4 \cdot 460}}{2} = 24 \pm \sqrt{116} \approx 24 \pm 10.770,$$

so $p \approx \$13.23$ or $p \approx \$34.77$.

Solution to Q384. [Back to Question p. 213] [Back to TOC]

Rectangular paddock beside a river.

Let x be the side perpendicular to the river and w the length along the river. Only three sides need fencing, so $2x + w = L \Rightarrow w = L - 2x$. Area

$$A(x) = xw = x(L - 2x) = Lx - 2x^2.$$

$$A'(x) = L - 4x = 0 \Rightarrow x = \frac{L}{4}, \qquad A''(x) = -4 < 0 \text{ (maximum)}.$$

Hence $w = L - 2(L/4) = L/2$. The maximum area is

$$A_{\max} = \frac{L}{4} \cdot \frac{L}{2} = \frac{L^2}{8}.$$

Solution to Q385. [Back to Question p. 214] [Back to TOC]

Cylindrical can: minimum surface for fixed volume.

Volume constraint $V = \pi r^2 h = 500 \Rightarrow h = \dfrac{500}{\pi r^2}$.

Surface area

$$S(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \left(\frac{500}{\pi r^2}\right) = 2\pi r^2 + \frac{1000}{r}, \qquad r > 0.$$

$$S'(r) = 4\pi r - \frac{1000}{r^2}, \qquad S''(r) = 4\pi + \frac{2000}{r^3} > 0.$$

Set $S'(r) = 0$: $4\pi r^3 = 1000 \Rightarrow r^3 = 250/\pi$. Thus

$$\boxed{r = \left(\frac{250}{\pi}\right)^{1/3} \text{ cm}, \qquad h = \frac{500}{\pi r^2} = 2r}$$

(numerically $r \approx 4.31$ cm, $h \approx 8.62$ cm). Minimum surface area

$$S_{\min} = 2\pi r^2 + \frac{1000}{r} \approx 349\,\text{cm}^2 \text{ (to nearest cm}^2).$$

Solution to Q386. [Back to Question p. 214] [Back to TOC]

Packaging with different material costs.

Base area x^2 at \$0.06 $\Rightarrow$ cost $0.06x^2$. Lid area x^2 at \$0.03 $\Rightarrow$ cost $0.03x^2$. Four sides area $4xh$ at \$0.04 $\Rightarrow$ cost $0.16xh$. With volume $x^2h = 2000 \Rightarrow h = \dfrac{2000}{x^2}$,

$$C(x) = 0.06x^2 + 0.03x^2 + 0.16x\left(\frac{2000}{x^2}\right) = 0.09x^2 + \frac{320}{x}, \qquad x > 0.$$

Minimise: $C'(x) = 0.18x - \dfrac{320}{x^2} = 0 \Rightarrow 0.18x^3 = 320 \Rightarrow x = \left(\frac{320}{0.18}\right)^{1/3} \approx 12.11$ cm.

Then $h = \dfrac{2000}{x^2} \approx 13.64$ cm. At the optimum, $320/x = 0.18x^2$, hence

$$C_{\min} = 0.09x^2 + \frac{320}{x} = 0.27x^2 \approx 0.27(12.11)^2 \approx \$39.6.$$

So $x \approx 12.1$ cm, $h \approx 13.6$ cm, minimum cost about \$39.6.

Solution to Q387. [Back to Question p. 215] [Back to TOC]

Maximise the volume of a cylinder inside a sphere.

With sphere radius R and cylinder radius r, the half-height is $\sqrt{R^2 - r^2}$, so $h = 2\sqrt{R^2 - r^2}$. Thus

$$V(r) = \pi r^2 h = 2\pi r^2\sqrt{R^2 - r^2}, \qquad 0 < r < R.$$

Differentiate:

$$\frac{dV}{dr} = 2\pi\left(2r\sqrt{R^2 - r^2} - \frac{r^3}{\sqrt{R^2 - r^2}}\right) = \frac{2\pi r}{\sqrt{R^2 - r^2}}\left(2(R^2 - r^2) - r^2\right).$$

Set $\dfrac{dV}{dr} = 0$ (and $r \neq 0$): $2(R^2 - r^2) - r^2 = 0 \Rightarrow r^2 = \dfrac{2R^2}{3}$. Hence

$$\boxed{r = R\sqrt{\frac{2}{3}}, \qquad h = 2\sqrt{R^2 - r^2} = \frac{2R}{\sqrt{3}}}.$$

For $R = 5$ cm: $r = 5\sqrt{2/3} \approx 4.082$ cm, $h = 10/\sqrt{3} \approx 5.774$ cm, and

$$V_{\max} = \pi r^2 h = \pi\left(\frac{50}{3}\right)\left(\frac{10}{\sqrt{3}}\right) = \frac{500\pi}{3\sqrt{3}} \approx 3.02 \times 10^2 \text{ cm}^3.$$

Solution to Q388. [Back to Question p. 216] [Back to TOC]

Maximise profit with a saturation model.

Given $q(p) = \dfrac{900}{1 + e^{0.4(p-18)}}$ and $C(q) = 2000 + 6q$,

$$P(p) = pq(p) - C(q(p)) = \big(p - 6\big)\,\frac{900}{1 + e^{0.4(p-18)}} - 2000.$$

Differentiate. Let $K = 900$ and $k = 0.4$. Writing $g(p) = \dfrac{K}{1 + e^{k(p-18)}}$,

$$g'(p) = -k\,g(p)\left(1 - \frac{g(p)}{K}\right).$$

Hence

$$P'(p) = g(p) + (p - 6)g'(p) = g(p)\Big(1 - k(p - 6)\big(1 - g(p)/K\big)\Big).$$

Set $P'(p) = 0$ (note $g(p) > 0$):

$$1 - k(p - 6)\left(1 - \frac{g}{K}\right) = 0 \quad\Longleftrightarrow\quad (p - 6)\frac{e^{k(p-18)}}{1 + e^{k(p-18)}} = \frac{1}{k} = 2.5.$$

Let $E = e^{k(p-18)}$. Using $g/K = \dfrac{1}{1 + E}$, the equation becomes

$$(12 + 2.5 \ln E)\frac{E}{1 + E} = 2.5 \iff E(\ln E + 3.8) = 1.$$

Solving numerically gives $E \approx 0.360$, whence

$$p = 18 + \frac{1}{k}\ln E = 18 + 2.5 \ln(0.360) \approx 15.46 \text{ dollars}.$$

Then $q = \dfrac{900}{1 + E} \approx \dfrac{900}{1.360} \approx 662$ units and

$$P_{\max} \approx (p - 6)\,q - 2000 \approx 9.46 \times 662 - 2000 \approx \$4251 \text{ (nearest dollar)}.$$

Why very low or very high prices reduce profit: for very low p, the margin $p - 6$ is small (even negative if $p < 6$), so despite high demand, profit is low or negative. For very high p, demand $q(p)$ becomes very small by the saturation model, so revenue collapses while the fixed cost \$2000 remains, reducing profit.

SL 5.8 Numerical methods - Trapzium rule

Solution to Q389. [Back to Question p. 218] [Back to TOC]

Q1. From a function (equal subintervals).

Step size: $h = \dfrac{4-0}{8} = 0.5.$

Ordinates (values of f):

x	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	1	1.05	1.20	1.45	1.80	2.25	2.80	3.45	4.20

Composite trapezium rule:

$$T_8 = h\left[\tfrac{1}{2}f(0) + \sum_{i=1}^{7} f(x_i) + \tfrac{1}{2}f(4)\right] = 0.5\Big[\tfrac{1}{2}(1) + \underbrace{1.05 + 1.2 + 1.45 + 1.8 + 2.25 + 2.8 + 3.45}_{=14.00} + \tfrac{1}{2}(4.2)\Big] = \boxed{8.300}.$$

(For comparison, the exact area is $\int_0^4 (0.2x^2+1)\,dx = \dfrac{64}{15} + 4 = 8.266\overline{6}$.)

Solution to Q390. [Back to Question p. 218] [Back to TOC]

Q2. Velocity table to distance.

Step size $h = 5$ s. With velocities $v_0, \ldots, v_6 = \{0, 12, 21, 27, 30, 29, 26\}$,

$$\text{distance} \approx T = h\left[\tfrac{1}{2}v_0 + \sum_{i=1}^{5} v_i + \tfrac{1}{2}v_6\right] = 5\Big[0 + \ (12 + 21 + 27 + 30 + 29) + 13\Big] = \boxed{660\text{ m}}.$$

Average velocity over $0 \le t \le 30$ s:

$$\bar{v} \approx \frac{660}{30} = \boxed{22\text{ m s}^{-1}}.$$

Solution to Q391. [Back to Question p. 218] [Back to TOC]

Q3. Cross-sectional area from equally spaced measurements.

Spacing $h = 2$ m. Using $y_0 = \cdots, y_6 = \{0, 1.8, 2.5, 3.1, 2.7, 2.0, 0\}$,

$$A \approx T = h\left[\tfrac{1}{2}y_0 + \sum_{i=1}^{5} y_i + \tfrac{1}{2}y_6\right] = 2\big[0 + (1.8 + 2.5 + 3.1 + 2.7 + 2.0) + 0\big] = \boxed{24.2\text{ m}^2}.$$

This is reasonable because the end depths are 0 (the banks), the bed profile changes smoothly, and the trapezia closely follow the channel shape.

Solution to Q392. [Back to Question p. 219] [Back to TOC]

Q4. Overestimate or underestimate?

Here $f(x) = e^{-0.3x}$ on $[0,3]$ with $n = 6$, so $h = 0.5$. Ordinates (rounded): $f(0) = 1$, $f(0.5) = 0.860708$, $f(1) = 0.740818$, $f(1.5) = 0.637628$, $f(2) = 0.548812$, $f(2.5) = 0.472367$, $f(3) = 0.406570$.

$$T_6 = 0.5\Big[\tfrac{1}{2}(1) + \big(0.860708 + 0.740818 + 0.637628 + 0.548812 + 0.472367\big) + \tfrac{1}{2}(0.406570)\Big] = \boxed{1.9818}\text{ (approx).}$$

Since $f''(x) = 0.09e^{-0.3x} > 0$ on $[0,3]$, the curve is *concave up*, and the trapezoidal rule *overestimates*. (Exact value: $\int_0^3 e^{-0.3x}\,dx = \frac{1}{0.3}(1 - e^{-0.9}) \approx 1.9781$.)

Solution to Q393. [Back to Question p. 219] [Back to TOC]

Q5. Sine curve and comparison.

$h = \dfrac{\pi}{6}$, $x_k = k\dfrac{\pi}{6}$ for $k = 0, 1, \ldots, 6$. Ordinates: $\sin 0 = 0$, $\sin\dfrac{\pi}{6} = \frac{1}{2}$, $\sin\dfrac{\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin\dfrac{\pi}{2} = 1$, $\sin\dfrac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\sin\dfrac{5\pi}{6} = \frac{1}{2}$, $\sin\pi = 0$.

$$T_6 = \frac{\pi}{6}\left[\tfrac{1}{2}(0) + \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2} + 1 + \tfrac{\sqrt{3}}{2} + \tfrac{1}{2}\right) + \tfrac{1}{2}(0)\right] = \frac{\pi}{6}\left(2 + \sqrt{3}\right) \approx \boxed{1.9548}.$$

(The exact value is 2; since $\sin x$ is concave down on $[0, \pi]$, the trapezoidal estimate is an *underestimate.*)

AHL 5.9 Differentiation of further functions and related rates

Solution to Q394. [Back to Question p. 221] [Back to TOC]

Basic derivatives.

$$\frac{d}{dx}(\sin x) = \cos x, \qquad \frac{d}{dx}(\cos x) = -\sin x, \qquad \frac{d}{dx}(\tan x) = \sec^2 x,$$

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \frac{d}{dx}\left(x^{5/3}\right) = \frac{5}{3}x^{2/3}.$$

Solution to Q395. [Back to Question p. 221] [Back to TOC]

Chain rule (composites).

$$\frac{d}{dx}\big(\sin(3x^2)\big) = \cos(3x^2)\cdot 6x = 6x\cos(3x^2),$$

$$\frac{d}{dx}\big(e^{2x-1}\big) = 2e^{2x-1}, \qquad \frac{d}{dx}\left(\ln\sqrt{x^2+1}\right) = \frac{1}{2}\cdot\frac{2x}{x^2+1} = \frac{x}{x^2+1},$$

$$\frac{d}{dx}\big((5-2x)^7\big) = 7(5-2x)^6\cdot(-2) = -14(5-2x)^6,$$

$$\frac{d}{dx}\left((x^2+x+1)^{3/2}\right) = \frac{3}{2}(x^2+x+1)^{1/2}(2x+1).$$

Solution to Q396. [Back to Question p. 221] [Back to TOC]

Product rule.

$$\frac{d}{dx}\big(x^2e^{3x}\big) = 2xe^{3x} + x^2\cdot 3e^{3x} = e^{3x}(2x+3x^2),$$

$$\frac{d}{dx}\big((x+1)\ln x\big) = \ln x + (x+1)\frac{1}{x} = \ln x + 1 + \frac{1}{x},$$

$$\frac{d}{dx}\big(x\sin(2x)\big) = \sin(2x) + 2x\cos(2x).$$

Solution to Q397. [Back to Question p. 222] [Back to TOC]

Quotient rule.

$$\frac{d}{dx}\left(\frac{x^2+1}{x-1}\right) = \frac{(2x)(x-1)-(x^2+1)}{(x-1)^2} = \frac{x^2-2x-1}{(x-1)^2},$$

$$\frac{d}{dx}\left(\frac{\tan x}{x}\right) = \frac{x\sec^2 x - \tan x}{x^2}, \qquad \frac{d}{dx}\left(\frac{e^x}{x^2}\right) = \frac{e^x x^2 - e^x\cdot 2x}{x^4} = \frac{e^x(x-2)}{x^3}.$$

Solution to Q398. [Back to Question p. 222] [Back to TOC]

Mixed rules (and values at $x=0$).

$$\frac{d}{dx}\big(e^x\cos x\big) = e^x(\cos x - \sin x) \;\Rightarrow\; y'(0) = 1.$$

$$\frac{d}{dx}\big(\ln(x^2+1)\sin(3x)\big) = \frac{2x}{x^2+1}\sin(3x) + \ln(x^2+1)\cdot 3\cos(3x) \;\Rightarrow\; y'(0) = 0.$$

$$\frac{d}{dx}\big((x^2+1)e^{-x^2}\big) = -2x^3e^{-x^2} \;\Rightarrow\; y'(0) = 0.$$

Solution to Q399. [Back to Question p. 222] [Back to TOC]

Tangent and normal for $y = xe^{-x^2}$.

$$y' = \frac{d}{dx}\big(xe^{-x^2}\big) = e^{-x^2} + x \cdot e^{-x^2}(-2x) = e^{-x^2}(1 - 2x^2).$$

At $x = 1$: slope of the tangent $m_{\text{tan}} = -e^{-1}$ and point $(1, e^{-1})$.

$$\text{Tangent: } y - e^{-1} = -e^{-1}(x - 1).$$

Normal slope $m_{\text{nor}} = e$ (since $m_{\text{tan}} m_{\text{nor}} = -1$).

$$\text{Normal: } y - e^{-1} = e(x - 1).$$

Solution to Q400. [Back to Question p. 223] [Back to TOC]

Related rates: expanding circle. With $A = \pi r^2$ and $C = 2\pi r$,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(20)(0.30) = 12\pi \text{ m}^2 \text{ min}^{-1},$$

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi(0.30) = 0.60\pi \text{ m min}^{-1}.$$

Solution to Q401. [Back to Question p. 223] [Back to TOC]

Related rates: water in a cone. By similarity $r = \frac{R}{H}h = \frac{1}{3}h$. Volume

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h = \frac{\pi}{27}h^3.$$

Differentiate:

$$\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}.$$

At $h = 12$ cm and $\frac{dV}{dt} = 15 \text{ cm}^3 \text{ s}^{-1}$,

$$15 = \frac{\pi}{9} \cdot 144 \cdot \frac{dh}{dt} = 16\pi\frac{dh}{dt} \;\Rightarrow\; \frac{dh}{dt} = \frac{15}{16\pi} \text{ cm s}^{-1} \approx 0.298 \text{ cm s}^{-1}.$$

Solution to Q402. [Back to Question p. 224] [Back to TOC]

Related rates: sliding ladder. With $x^2 + y^2 = 25$,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \;\Rightarrow\; \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 3$ m, $y = 4$ m and $\frac{dx}{dt} = 0.8 \text{ m s}^{-1}$,

$$\frac{dy}{dt} = -\frac{3}{4}(0.8) = -0.6 \text{ m s}^{-1}.$$

So the top slides downward at 0.6 m s^{-1}.

Solution to Q403. [Back to Question p. 225] [Back to TOC]

Log and trig composite.

$$\frac{d}{dx}\big(\ln(\cos x)\big) = \frac{1}{\cos x}(-\sin x) = -\tan x, \qquad \text{so at } x = \frac{\pi}{4}: \; y' = -\tan\left(\frac{\pi}{4}\right) = -1.$$

AHL 5.10 Second derivative

Solution to Q404. [Back to Question p. 226] [Back to TOC]

Compute first and second derivatives.

1. $f(x) = 3x^4 - 5x^2 + 7$
   $f'(x) = 12x^3 - 10x, \qquad f''(x) = 36x^2 - 10.$
2. $f(x) = \dfrac{x^2+1}{x-2}$
   Using the quotient rule,
   $$f'(x) = \frac{2x(x-2) - (x^2+1)}{(x-2)^2} = \frac{x^2-4x-1}{(x-2)^2}.$$
   Differentiate again (let $u = x^2 - 4x - 1$, $v = (x-2)^2$):
   $$f''(x) = \frac{u'v - uv'}{v^2} = \frac{(2x-4)(x-2)^2 - 2(x-2)u}{(x-2)^4} = \frac{10(x-2)}{(x-2)^4} = \boxed{\frac{10}{(x-2)^3}}.$$
3. $f(x) = e^{2x}\sin x$
   $f'(x) = e^{2x}(2\sin x + \cos x)$, hence
   $$f''(x) = e^{2x}\big(2\sin x + \cos x\big)\cdot 2 + e^{2x}(2\cos x - \sin x) = \boxed{e^{2x}\big(3\sin x + 4\cos x\big)}.$$
4. $f(x) = \ln(x^2+1)$
   $f'(x) = \dfrac{2x}{x^2+1}$, and
   $$f''(x) = \frac{2(x^2+1) - 4x^2}{(x^2+1)^2} = \boxed{\frac{2(1-x^2)}{(x^2+1)^2}}.$$

Solution to Q405. [Back to Question p. 226] [Back to TOC]

Second derivative test (polynomial). For $f(x) = x^3 - 3x$:

$$f'(x) = 3x^2 - 3 = 3(x^2-1) = 0 \;\Rightarrow\; x = \pm 1.$$

$$f''(x) = 6x.$$

At $x = -1$, $f''(-1) = -6 < 0$ so $x = -1$ is a local maximum with $f(-1) = 2$. At $x = 1$, $f''(1) = 6 > 0$ so $x = 1$ is a local minimum with $f(1) = -2$.

Increasing where $f'(x) > 0$, i.e. $(-\infty, -1)$ and $(1, \infty)$; decreasing on $(-1, 1)$.

Solution to Q406. [Back to Question p. 227] [Back to TOC]

Point of inflection (sign-change test). For $g(x) = x^3 - 6x^2 + 9x$,

$$g''(x) = 6x - 12 = 0 \implies x = 2.$$

Since $g''(1) = -6 < 0$ and $g''(3) = 6 > 0$, the concavity changes at $x = 2$, so there is a point of inflection at $(2, g(2)) = (2, 2)$.

Solution to Q407. [Back to Question p. 227] [Back to TOC]

Concavity intervals from h''. For $h(x) = \ln x$ on $(0, \infty)$:

$$h'(x) = \frac{1}{x}, \qquad h''(x) = -\frac{1}{x^2} < 0 \text{ for } x > 0.$$

Hence the graph is concave-down for all $x > 0$ and there is no point of inflection (the sign of h'' does not change).

Solution to Q408. [Back to Question p. 227] [Back to TOC]

Inflection in a bell-shaped curve. For $y = e^{-x^2}$,

$$y' = -2xe^{-x^2}, \qquad y'' = (-2 + 4x^2)e^{-x^2} = (4x^2 - 2)e^{-x^2}.$$

Set $y'' = 0$: $4x^2 - 2 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$. Because $e^{-x^2} > 0$, the sign of y'' is that of $(4x^2 - 2)$: concave-down on $\big(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\big)$ and concave-up for $|x| > \frac{1}{\sqrt{2}}$. Inflection points:

$$\left(\pm\frac{1}{\sqrt{2}}, e^{-1/2}\right).$$

Solution to Q409. [Back to Question p. 228] [Back to TOC]

Second derivative test may be inconclusive. For $p(x) = x^4$,

$$p'(x) = 4x^3 = 0 \Rightarrow x = 0, \qquad p''(x) = 12x^2.$$

At $x = 0$, $p''(0) = 0$ so the second derivative test is inconclusive. Inspect $p'(x)$: $p'(x) < 0$ for $x < 0$ and $p'(x) > 0$ for $x > 0$, so the function decreases then increases; therefore $x = 0$ is a local (indeed global) minimum with $p(0) = 0$.

Solution to Q410. [Back to Question p. 228] [Back to TOC]

Concavity and sketch from derivative information. Given $f'(x) > 0$ on $(-3, -1)$, $f'(x) < 0$ on $(-1, 1)$, and $f'(x) > 0$ on $(1, 3)$: there is a local maximum at $x = -1$ and a local minimum at $x = 1$. Since $f''(x) < 0$ on $(-3, 0)$ and $f''(x) > 0$ on $(0, 3)$, the graph is concave-down to the left of 0 and concave-up to the right, with a point of inflection at $x = 0$ (the y-value is whatever $f(0)$ is). A consistent sketch shows rising to a peak at $x = -1$, falling to a trough at $x = 1$, and an inflection at $x = 0$ where the curvature changes.

Solution to Q411. [Back to Question p. 229] [Back to TOC]

Applied context (kinematics).

$$s(t) = t^3 - 6t^2 + 9t, \qquad v(t) = s'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3),$$

$$a(t) = v'(t) = 6t - 12 = 6(t-2).$$

The velocity is *increasing* when $a(t) > 0$ (i.e. $t > 2$) and *decreasing* when $a(t) < 0$ (i.e. $t < 2$). An inflection of s occurs when $s''(t) = a(t) = 0$, at $t = 2$; since a changes sign there, $t = 2$ is indeed a point of inflection. The position at that time is

$$s(2) = 8 - 24 + 18 = 2 \text{ m}.$$

Interpretation: at $t = 2$ s, the curvature of the position–time graph changes (acceleration switches from negative to positive), so after $t = 2$ the velocity starts to increase.

AHL 5.11 - Integration by substitution

Solution to Q412. [Back to Question p. 230] [Back to TOC]

Indefinite integral: power rule.

$$\int\left(3x^{5/2}-4x^{-3}+7\right)dx=3\cdot\frac{x^{7/2}}{7/2}-4\cdot\frac{x^{-2}}{-2}+7x+C=\frac{6}{7}x^{7/2}+2x^{-2}+7x+C.$$

Power rule used: $\displaystyle\int x^n\,dx=\frac{x^{n+1}}{n+1}+C$ for $n\neq -1$.

Solution to Q413. [Back to Question p. 230] [Back to TOC]

Basic trig and exponential.

$$\int\sin x\,dx=-\cos x+C,\qquad\int\cos(3x)\,dx=\frac{1}{3}\sin(3x)+C,\qquad\int e^{2x-5}\,dx=\frac{1}{2}e^{2x-5}+C.$$

Solution to Q414. [Back to Question p. 230] [Back to TOC]

Secant squared. Let $u=4x-\frac{\pi}{6}$, $du=4\,dx$. Then

$$\int\sec^2(4x-\tfrac{\pi}{6})\,dx=\frac{1}{4}\tan(4x-\tfrac{\pi}{6})+C.$$

Solution to Q415. [Back to Question p. 230] [Back to TOC]

Definite integral: powers.

$$\int_1^4\left(3x^{1/2}+\frac{2}{x^2}\right)dx=\left[2x^{3/2}-\frac{2}{x}\right]_1^4=\left(2\cdot4^{3/2}-\frac{2}{4}\right)-\left(2\cdot1^{3/2}-2\right)=\frac{31}{2}.$$

Solution to Q416. [Back to Question p. 231] [Back to TOC]

Definite integral: sine and cosine.

$$\int_0^{\pi/3}\cos x\,dx=\sin x\Big|_0^{\pi/3}=\frac{\sqrt{3}}{2},\qquad\int_0^{\pi}\sin x\,dx=-\cos x\Big|_0^{\pi}=2.$$

Solution to Q417. [Back to Question p. 231] [Back to TOC]

Substitution (inspection). Let $u=2x+5$, $du=2\,dx$:

$$\int\sin(2x+5)\,dx=-\tfrac{1}{2}\cos(2x+5)+C.$$

Solution to Q418. [Back to Question p. 231] [Back to TOC]

Substitution (linear). Let $u=3x+2$, $du=3\,dx$:

$$\int\frac{1}{3x+2}\,dx=\frac{1}{3}\ln|3x+2|+C.$$

Solution to Q419. [Back to Question p. 232] [Back to TOC]

Substitution with chain rule reverse. With $u = x^2$, $du = 2x\,dx$,

$$\int 4x\,\sin(x^2)\,dx = 2\int \sin u\,du = -2\cos(x^2) + C.$$

Solution to Q420. [Back to Question p. 232] [Back to TOC]

Quotient in derivative form. Let $u = 1 + \sin(5x)$, $du = 5\cos(5x)\,dx$:

$$\int \frac{\cos(5x)}{1 + \sin(5x)}\,dx = \frac{1}{5}\ln|1 + \sin(5x)| + C.$$

Solution to Q421. [Back to Question p. 232] [Back to TOC]

Definite integral via substitution. With $u = x^2$, $du = 2x\,dx$, and $u : 0 \to 1$,

$$\int_0^1 2x\,e^{x^2}\,dx = \int_0^1 e^u\,du = e - 1.$$

Solution to Q422. [Back to Question p. 232] [Back to TOC]

Mixed practice (indefinite).

$$\int \left(\frac{2x}{1 + x^2} + e^x - 5\cos x\right)dx = \ln(1 + x^2) + e^x - 5\sin x + C.$$

Solution to Q423. [Back to Question p. 233] [Back to TOC]

Initial value problem. Integrate:

$$F(x) = \int\left(\frac{2x}{1 + x^2} + e^x\right)dx = \ln(1 + x^2) + e^x + C.$$

Use $F(0) = 1$: $1 = \ln 1 + e^0 + C = 1 + C \Rightarrow C = 0$. Hence

$$\boxed{F(x) = \ln(1 + x^2) + e^x}.$$

AHL 5.12 - Area and volumes of revolution

Solution to Q424. [Back to Question p. 234] [Back to TOC]

(a) Signed area.

$$\int_0^5 (x^2 - 4x + 3)\,dx = \left[\tfrac{1}{3}x^3 - 2x^2 + 3x\right]_0^5 = \left(\tfrac{125}{3} - 50 + 15\right) - 0 = \frac{20}{3}.$$

(b) Total geometric area. Since $x^2 - 4x + 3 = (x-1)(x-3)$, the curve crosses the x-axis at $x = 1, 3$, is above the axis on $[0,1]$ and $[3,5]$, and below on $(1,3)$. With $F(x) = \frac{1}{3}x^3 - 2x^2 + 3x$,

$$\text{Area} = \int_0^1 f\,dx - \int_1^3 f\,dx + \int_3^5 f\,dx = \big(F(1)-F(0)\big) - \big(F(3)-F(1)\big) + \big(F(5)-F(3)\big) = \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \boxed{\frac{28}{3}}.$$

Solution to Q425. [Back to Question p. 234] [Back to TOC]

(a) Signed area.

$$\int_0^{3\pi} \sin x\,dx = \big[-\cos x\big]_0^{3\pi} = -\cos(3\pi) + \cos(0) = 1 + 1 = \boxed{2}.$$

(b) Total geometric area. On $[0, 3\pi]$, $y = \sin x$ is above the axis on $[0,\pi]$ and $[2\pi, 3\pi]$, and below on $[\pi, 2\pi]$. Thus

$$\text{Area} = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx + \int_{2\pi}^{3\pi} \sin x\,dx = 2 - (-2) + 2 = \boxed{6}.$$

Solution to Q426. [Back to Question p. 235] [Back to TOC]

The region is between $x = y^2$ (left) and $x = 4$ (right), from $y = 0$ (the x-axis) to $y = 2$. Integrating with respect to y,

$$\text{Area} = \int_0^2 \left(4 - y^2\right) dy = \left[4y - \tfrac{y^3}{3}\right]_0^2 = 8 - \frac{8}{3} = \boxed{\frac{16}{3}}.$$

Solution to Q427. [Back to Question p. 235] [Back to TOC]

(a) Intersections. Solve $e^{-x/2} = 0.2x + 0.2$. The function $h(x) = e^{-x/2} - (0.2x + 0.2)$ has $h'(x) = -\frac{1}{2}e^{-x/2} - 0.2 < 0$, so there is a unique solution:

$$\boxed{x \approx 1.43719, \quad y \approx 0.48744}.$$

(b) Enclosed area. *Note:* These two curves intersect only once, so no finite region is enclosed by the curves alone. If the intended region is the finite lens bounded by the two curves and the y-axis (from $x = 0$ to the intersection x^*), then

$$\text{Area} = \int_0^{x^*} \left(e^{-x/2} - (0.2x + 0.2)\right) dx = \left[-2e^{-x/2} - \left(0.1x^2 + 0.2x\right)\right]_0^{x^*}.$$

With $x^* \approx 1.43719$, this gives

$$\text{Area} \approx \boxed{0.53114} \text{ (square units)}.$$

Solution to Q428. [Back to Question p. 236] [Back to TOC]

Using discs about the x-axis:

$$V=\int_0^4 \pi\big(\sqrt{x}\big)^2\,dx=\int_0^4 \pi x\,dx=\pi\left[\tfrac{x^2}{2}\right]_0^4=\pi\cdot\frac{16}{2}=\boxed{8\pi}.$$

Solution to Q429. [Back to Question p. 236] [Back to TOC]

About the y-axis with $x=\sqrt{y}$ (discs in y):

$$V=\int_0^4 \pi\big(\sqrt{y}\big)^2\,dy=\int_0^4 \pi y\,dy=\pi\left[\tfrac{y^2}{2}\right]_0^4=\pi\cdot\frac{16}{2}=\boxed{8\pi}.$$

Solution to Q430. [Back to Question p. 237] [Back to TOC]

Washers about the x-axis on $0\le x\le 2$: outer radius $R=2$, inner radius $r=x$.

$$V=\int_0^2 \pi\big(R^2-r^2\big)\,dx=\int_0^2 \pi(4-x^2)\,dx=\pi\left[4x-\tfrac{x^3}{3}\right]_0^2=\pi\left(8-\frac{8}{3}\right)=\boxed{\frac{16\pi}{3}}.$$

Solution to Q431. [Back to Question p. 237] [Back to TOC]

T is bounded by $y=\frac{1}{2}(4-x)$, $x=0$, and $y=0$.

1. About the x-axis (washers; do not evaluate):

$$\boxed{V=\int_0^4 \pi\Big(\tfrac{1}{2}(4-x)\Big)^2\,dx}.$$

2. About the y-axis: write x as a function of y from $y=\frac{1}{2}(4-x)\Rightarrow x=4-2y$, with $0\le y\le 2$. Using discs in y,

$$\boxed{V=\int_0^2 \pi\big(4-2y\big)^2\,dy}.$$

AHL 5.13 - Kinematics

Solution to Q432. [Back to Question p. 238] [Back to TOC]

Given $s(t) = t^3 - 6t^2 + 9t - 2$.

1. $v(t) = \dfrac{ds}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3), \qquad a(t) = \dfrac{dv}{dt} = 6t - 12.$
2. At rest when $v(t) = 0 \Rightarrow t = 1, 3$ (both ≥ 0).
3. Displacement on $[0, 5]$: $s(5) - s(0) = (125 - 150 + 45 - 2) - (-2) = 18 + 2 = \boxed{20 \text{ m}}$.
4. Since $v > 0$ on $[0, 1] \cup (3, 5]$ and $v < 0$ on $(1, 3)$,
$$\text{distance} = \left|s(1) - s(0)\right| + \left|s(3) - s(1)\right| + \left|s(5) - s(3)\right| = 4 + 4 + 20 = \boxed{28 \text{ m}}.$$

Solution to Q433. [Back to Question p. 238] [Back to TOC]

$v(t) = 3t - 6$ on $0 \leq t \leq 5$.

1. Signed displacement:
$$\int_0^5 (3t - 6)\,dt = \left[\tfrac{3}{2}t^2 - 6t\right]_0^5 = \tfrac{75}{2} - 30 = \boxed{\tfrac{15}{2} \text{ m}}.$$
2. Break at $t = 2$ where $v = 0$:
$$\int_0^5 |3t - 6|\,dt = -\int_0^2 (3t - 6)\,dt + \int_2^5 (3t - 6)\,dt = 6 + 13.5 = \boxed{\tfrac{39}{2} \text{ m}}.$$
3. The car moves in the positive direction when $v > 0$, i.e. on $\boxed{(2, 5]}$.

Solution to Q434. [Back to Question p. 238] [Back to TOC]

Areas are read from the piecewise linear v-t graph.

Segment $0 \to 2$: trapezium with heights -2 and 2 has signed area 0; total distance there is two triangles of area 1 each $\Rightarrow 2$.

Segment $2 \to 5$: trapezium (heights $2, 3$, base 3): signed area $= \frac{2+3}{2} \cdot 3 = \frac{15}{2}$; distance the same.

Segment $5 \to 8$: crosses the axis at $t = 5 + \dfrac{3}{4} \cdot 3 = \boxed{\frac{29}{4} = 7.25}$. Signed area $= 3$ (trapezium); split for distance into a positive triangle area $\frac{1}{2} \cdot 2.25 \cdot 3 = \frac{27}{8}$ and a negative triangle area $\frac{1}{2} \cdot 0.75 \cdot 1 = \frac{3}{8}$.

Hence
$$\text{signed disp.} = 0 + \tfrac{15}{2} + 3 = \boxed{\tfrac{21}{2} \text{ m}}, \qquad \text{total distance} = 2 + \tfrac{15}{2} + \tfrac{27}{8} + \tfrac{3}{8} = \boxed{\tfrac{53}{4} \text{ m}}.$$

Direction changes at the zeros of v: $\boxed{t = 1 \text{ and } t = \frac{29}{4}}$.

Solution to Q435. [Back to Question p. 239] [Back to TOC]

Given $a(t) = 6t - 4$, $v(0) = 2$, $s(0) = -3$.

1. $v(t) = \int (6t - 4)\,dt = 3t^2 - 4t + C$. Using $v(0) = 2 \Rightarrow C = 2$, so $v(t) = 3t^2 - 4t + 2$.
Then $s(t) = \int v(t)\,dt = t^3 - 2t^2 + 2t + C_2$. Using $s(0) = -3 \Rightarrow C_2 = -3$, hence
$$\boxed{v(t) = 3t^2 - 4t + 2}, \qquad \boxed{s(t) = t^3 - 2t^2 + 2t - 3}.$$

2. $v(t) = 0 \Rightarrow 3t^2 - 4t + 2 = 0$ has discriminant $-8 < 0$. $\Rightarrow$ No real solution: $\boxed{\texttt{never at rest}}$.
3. Since $v(t) > 0$ for all t, distance $= s(5) - s(0)$:

$$(125 - 50 + 10 - 3) - (-3) = \boxed{85\text{ m}}.$$

Solution to Q436. [Back to Question p. 239] [Back to TOC]

Note (correction). With $a = -kv$ and $a = v\,\dfrac{dv}{ds}$,

$$v\frac{dv}{ds} = -kv \;\Rightarrow\; \frac{dv}{ds} = -k \;\Rightarrow\; \boxed{v(s) = v_0 - ks} \quad \text{(linear in } s\text{)}.$$

(Exponential decay holds in time: $v(t) = v_0 e^{-kt}$.)

1. As above, $v(s) = v_0 - ks$.
2. Set $v = \frac{1}{2}v_0$: $v_0 - ks = \frac{1}{2}v_0 \Rightarrow \boxed{s = \dfrac{v_0}{2k}}$.
3. Using $v(t) = v_0 e^{-kt}$, solve $v_0 e^{-kt} = \frac{1}{2}v_0$: $e^{-kt} = \frac{1}{2} \Rightarrow \boxed{t = \dfrac{\ln 2}{k}}$.

Solution to Q437. [Back to Question p. 240] [Back to TOC]

Given $a = -cv^2$ and $a = v\,\dfrac{dv}{ds}$:

1. $v\dfrac{dv}{ds} = -cv^2 \Rightarrow \dfrac{1}{v}\,dv = -c\,ds$. Integrate and use $v(0) = u$:

$$\ln v = -cs + \ln u \;\Rightarrow\; \boxed{v(s) = u\,e^{-cs}}.$$

2. Since $v(s) > 0$ for every finite s, the sled never reaches $v = 0$ in finite distance. For $v = \frac{u}{3}$:

$$ue^{-cs} = \tfrac{u}{3} \Rightarrow s = \boxed{\frac{\ln 3}{c}}.$$

3. From $\dfrac{dv}{dt} = -cv^2 \Rightarrow \dfrac{dv}{v^2} = -c\,dt$. Integrate $v : u \to u/3$:

$$\left[-\frac{1}{v}\right]_u^{u/3} = -c\,t \;\Rightarrow\; -\frac{3}{u} + \frac{1}{u} = -ct \;\Rightarrow\; \boxed{t = \frac{2}{cu}}.$$

Solution to Q438. [Back to Question p. 240] [Back to TOC]

$$\dot{x}(t) = 4e^{-t} - 2\sin t, \qquad x(0) = 1.$$

1. $\ddot{x}(t) = -4e^{-t} - 2\cos t$.
2. Integrate $\dot{x}$:
$$x(t) = -4e^{-t} + 2\cos t + C, \quad x(0) = -4 + 2 + C = 1 \Rightarrow C = 3.$$
Hence $\boxed{x(t) = -4e^{-t} + 2\cos t + 3}$.

3. Total distance $= \int_0^{2\pi} |\dot{x}(t)|\, dt$. Zeros of $\dot{x}$ in $[0, 2\pi]$ solve $4e^{-t} = 2\sin t \Rightarrow 2e^{-t} = \sin t$:

$$t_1 \approx 0.9210, \qquad t_2 \approx 3.0464.$$

On $[0, t_1]$ and $[t_2, 2\pi]$, $\dot{x} > 0$; on $[t_1, t_2]$, $\dot{x} < 0$. With antiderivative $F(t) = -4e^{-t} + 2\cos t$,

$$\text{distance} = \big[F(t_1) - F(0)\big] - \big[F(t_2) - F(t_1)\big] + \big[F(2\pi) - F(t_2)\big] \approx \boxed{7.590 \text{ (units)}}.$$

(The signed displacement is $F(2\pi) - F(0) = 4\,(1 - e^{-2\pi}) \approx 3.993$.)

Solution to Q439. [Back to Question p. 241] [Back to TOC]

$$|v(t)| = \begin{cases} 2t, & 0 \le t < 3, \\ 6 - t, & 3 \le t \le 6, \end{cases}$$

1. Distance (always forward):

$$\int_0^3 2t\, dt + \int_3^6 (6 - t)\, dt = 9 + \Big[6t - \tfrac{t^2}{2}\Big]_3^6 = 9 + 4.5 = \boxed{\tfrac{27}{2} \text{ m}}.$$

2. If the cyclist reverses direction at $t = 4$, then velocity is negative on $[4, 6]$ with the same speed.

$$\text{signed disp.} = \int_0^3 2t\, dt + \int_3^4 (6 - t)\, dt - \int_4^6 (6 - t)\, dt = 9 + 2.5 - 2 = \boxed{\tfrac{19}{2} \text{ m}},$$

$$\text{total distance} = 9 + 2.5 + 2 = \boxed{\tfrac{27}{2} \text{ m}}.$$

AHL 5.14 - Modelling with differential equations and solving by separation of variables

Solution to Q440. [Back to Question p. 242] [Back to TOC]

(a) "Rate proportional to $\sqrt{G}$" $\Rightarrow \dfrac{dG}{dt} = k\sqrt{G}$ with constant $k > 0$.

(b) Separate: $\dfrac{dG}{\sqrt{G}} = k\,dt \;\Rightarrow\; 2\sqrt{G} = kt + C$. Hence $\sqrt{G} = \dfrac{k}{2}t + C_1$ and

$$\boxed{G(t) = \left(\tfrac{k}{2}t + C_1\right)^2}.$$

(c) $G(0) = 9 \Rightarrow C_1 = 3$. Also $G(4) = 25 \Rightarrow \frac{k}{2} \cdot 4 + 3 = 5 \Rightarrow k = 1$. Thus

$$\boxed{G(t) = \left(3 + \tfrac{t}{2}\right)^2}.$$

Solution to Q441. [Back to Question p. 242] [Back to TOC]

(a) $\dfrac{dP}{dt} = kP \;\Rightarrow\; P(t) = Ce^{kt}$ is the general solution.

(b) $P(0) = 1200 \Rightarrow C = 1200$. Doubling time 8 h gives $1200e^{8k} = 2400 \Rightarrow e^{8k} = 2 \Rightarrow k = \dfrac{\ln 2}{8}$. Hence

$$\boxed{P(t) = 1200\,e^{(\ln 2/8)\,t} = 1200 \cdot 2^{t/8}}, \qquad \boxed{P(20) = 1200 \cdot 2^{20/8} = 4800\sqrt{2} \approx 6788}.$$

Solution to Q442. [Back to Question p. 242] [Back to TOC]

"Proportional decay": $\dfrac{dm}{dt} = -k\,m \Rightarrow m(t) = Ce^{-kt}$.

Half-life 3 years: $m(3) = \frac{1}{2}m(0) \Rightarrow e^{-3k} = \frac{1}{2} \Rightarrow k = \dfrac{\ln 2}{3}$. With $m(0) = 40$,

$$\boxed{m(t) = 40\,e^{-(\ln 2/3)\,t} = 40 \cdot 2^{-t/3}}.$$

To reach $m = 5$: $5 = 40 \cdot 2^{-t/3} \Rightarrow 2^{-t/3} = 1/8 \Rightarrow t = 9$ years.

$$\boxed{t = 9 \text{ years}}.$$

Solution to Q443. [Back to Question p. 243] [Back to TOC]

Newton cooling: $\dfrac{dT}{dt} = -k\,(T - T_a)$ with $T_a = 22$.

$$\begin{aligned} T(t) &= T_a + (T(0) - T_a)e^{-kt} = 22 + 60e^{-kt}, \\ 22 + 60e^{-10k} = 52 \;&\Rightarrow\; e^{-10k} = \tfrac{1}{2} \;\Rightarrow\; k = \frac{\ln 2}{10}, \\ T(t) &= 22 + 60\,2^{-t/10}, \\ 60e^{-kt} = 8 \;&\Rightarrow\; e^{-kt} = \tfrac{2}{15} \;\Rightarrow\; t = \frac{1}{k}\ln\!\left(\tfrac{15}{2}\right) = \frac{10}{\ln 2}\ln\!\left(\tfrac{15}{2}\right) \approx 29.07\,\text{min}. \end{aligned}$$

Solution to Q444. [Back to Question p. 243] [Back to TOC]

$$\begin{aligned}
\frac{dN}{dt} &= rN\Big(1-\frac{N}{K}\Big),\\
\frac{dN}{N(1-N/K)} &= r\,dt = \frac{K}{N(K-N)}\,dN,\\
\frac{K}{N(K-N)} &= \frac{1}{N}+\frac{1}{K-N} \;\Rightarrow\; \ln|N|-\ln|K-N| = rt+C,\\
\frac{N}{K-N} &= Ce^{rt} \;\Rightarrow\; N(t)=\frac{K}{1+C'e^{-rt}} \quad (C'=\tfrac{1}{C}>0).
\end{aligned}$$

With $K=500,\ r=0.6,\ N(0)=50$:

$$50=\frac{500}{1+C'} \Rightarrow C'=9 \;\Rightarrow\; N(t)=\frac{500}{1+9e^{-0.6t}}.$$

For $N=250$:

$$1+9e^{-0.6t}=2 \Rightarrow e^{-0.6t}=\frac{1}{9} \Rightarrow t=\frac{1}{0.6}\ln 9 \approx 3.662 \text{ yr}.$$

Solution to Q445. [Back to Question p. 243] [Back to TOC]

Volume $=100$ L. Let $y(t)$ be salt (kg). Inflow $=0.3\times 2=0.6$ kg/min; outflow $=2(y/100)=0.02y$.

$$\frac{dy}{dt}=0.6-0.02\,y.$$

Linear solution (or steady state + decay):

$$\begin{aligned}
&y(t)=y(\infty)+\big(y(0)-y(\infty)\big)e^{-0.02t}, \qquad y(\infty)=\frac{0.6}{0.02}=30,\\
&y(0)=0 \Rightarrow y(t)=30\big(1-e^{-0.02t}\big).
\end{aligned}$$

Concentration $=0.2$ kg/L $\Rightarrow y=0.2\cdot 100=20$:

$$30\big(1-e^{-0.02t}\big)=20 \Rightarrow e^{-0.02t}=\frac{1}{3} \Rightarrow t=50\ln 3\approx 54.93 \text{ min}.$$

Solution to Q446. [Back to Question p. 244] [Back to TOC]

For downward $v(t)$, with linear drag bv (upwards), Newton's 2nd law:

$$m\frac{dv}{dt}=mg-bv \quad\Rightarrow\quad \frac{dv}{dt}=g-\frac{b}{m}v.$$

Solve with $v(0)=0$:

$$v(t)=\frac{mg}{b}\Big(1-e^{-(b/m)t}\Big).$$

Terminal speed (as $t\to\infty$):

$$\boxed{v_T=\frac{mg}{b}}.$$

Solution to Q447. [Back to Question p. 244] [Back to TOC]

$\dfrac{dh}{dt} = -k\sqrt{h} \Rightarrow \dfrac{dh}{\sqrt{h}} = -k\,dt$. Integrate: $2\sqrt{h} = -kt + C \Rightarrow \sqrt{h} = C_1 - \frac{k}{2}t$; hence

$$\boxed{h(t) = \left(C_1 - \tfrac{k}{2}t\right)^2}.$$

With $h(0) = 1.6$, $C_1 = \sqrt{1.6}$. The tank empties when $h = 0$:

$$0 = \left(\sqrt{1.6} - \tfrac{k}{2}t\right)^2 \Rightarrow \boxed{t_{\text{empty}} = \frac{2\sqrt{1.6}}{k} = \frac{2\sqrt{1.6}}{0.25} \approx 10.12 \text{ s}}.$$

Solution to Q448. [Back to Question p. 244] [Back to TOC]

$\dfrac{dH}{dt} = -\dfrac{k}{H^2} \Rightarrow H^2\dfrac{dH}{dt} = -k$. Integrate: $\dfrac{1}{3}H^3 = -kt + C \Rightarrow H^3 = C' - 3kt$ and

$$\boxed{H(t) = (C' - 3kt)^{1/3}}.$$

With $H(0) = 6 \Rightarrow C' = 216$, so $H(t) = (216 - 3kt)^{1/3}$. For $k = 3$: $H(t) = (216 - 9t)^{1/3}$. When $H = 3$: $27 = 216 - 9t \Rightarrow \boxed{t = 21}$ (time units).

Solution to Q449. [Back to Question p. 245] [Back to TOC]

$\dfrac{dY}{dt} = aY^{2/3} \Rightarrow Y^{-2/3}dY = a\,dt$. Integrate: $3Y^{1/3} = at + C \Rightarrow Y^{1/3} = \dfrac{a}{3}t + C_1$ and

$$\boxed{Y(t) = \left(\frac{a}{3}t + C_1\right)^3}.$$

With $Y(0) = 8 \Rightarrow C_1 = 2$. Also $Y(9) = 27 \Rightarrow 3 = \dfrac{a}{3} \cdot 9 + 2 \Rightarrow a = \dfrac{1}{3}$. Thus

$$\boxed{Y(t) = \left(2 + \tfrac{t}{9}\right)^3}, \qquad \boxed{Y(16) = \left(\tfrac{34}{9}\right)^3 = \frac{39304}{729} \approx 53.9}.$$

AHL 5.15- Slope fields and their diagrams

Solution to Q450. [Back to Question p. 246] [Back to TOC]

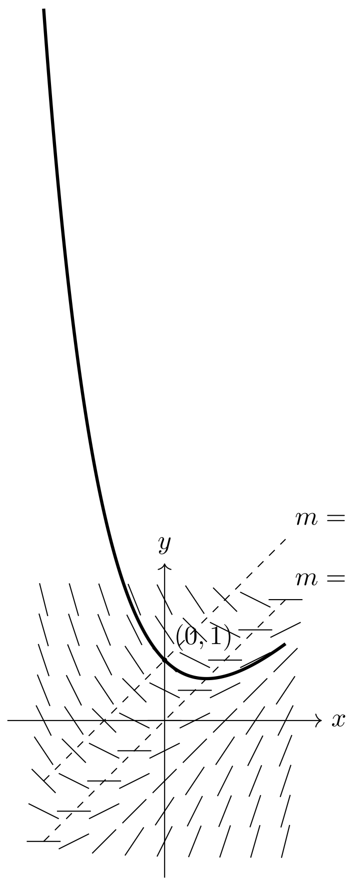
(a)–(b) Field and solution through $(0,1)$**.** Solve $y' = x - y \Rightarrow y' + y = x$. With integrating factor e^x,

$$\frac{d}{dx}(ye^x) = xe^x \Rightarrow ye^x = e^x(x-1) + C \Rightarrow y(x) = x - 1 + Ce^{-x}.$$

Through $(0,1)$ gives $1 = -1 + C \Rightarrow C = 2$, hence

$$\boxed{y(x) = x - 1 + 2e^{-x}}.$$

(c) Isoclines. For slope m, $x - y = m \iff y = x - m$. Thus $m = 0$ gives $y = x$ and $m = 1$ gives $y = x - 1$. These straight lines help place where solution curves are flat/steep.



Solution to Q451. [Back to Question p. 246] [Back to TOC]

Equilibria and stability. For $y' = y(1 - y/3)$, equilibria are where $y' = 0$:

$$y = 0, \quad y = 3.$$

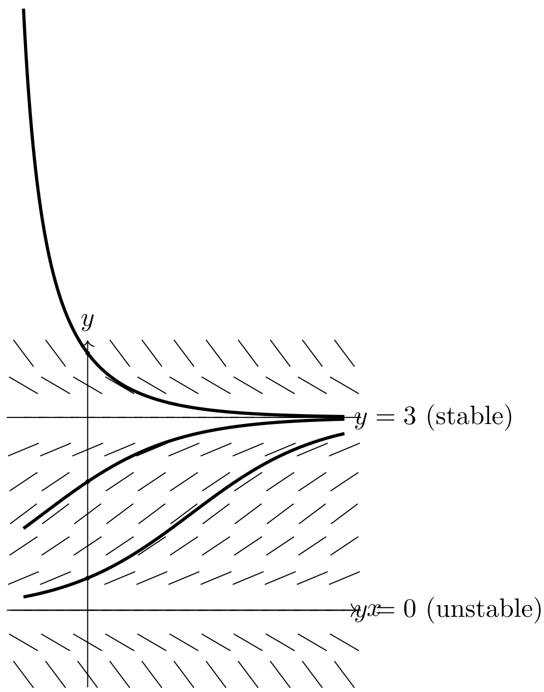
Since $f(y) = y(1 - y/3)$ has $f'(y) = 1 - \frac{2}{3}y$, we get $f'(0) = 1 > 0$ (unstable) and $f'(3) = -1 < 0$ (stable).

General solution and particular curves. Separate:

$$\frac{dy}{y(1-y/3)} = dx \;\Rightarrow\; \ln|y| - \ln|3-y| = x + C \;\Rightarrow\; \boxed{y(x) = \frac{3}{1+C'e^{-x}}}.$$

With $y(0) = y_0$, $C' = \dfrac{3}{y_0} - 1$. Thus

$$y(0) = 0.5 : C' = 5, \quad y = \frac{3}{1+5e^{-x}},$$
$$y(0) = 2 : C' = 0.5, \quad y = \frac{3}{1+0.5e^{-x}},$$
$$y(0) = 4 : C' = -0.25, \quad y = \frac{3}{1-0.25e^{-x}} \text{ (decreases to 3).}$$



Solution to Q452. [Back to Question p. 247] [Back to TOC]

Matching.

- **A** has slope depending only on x (same along vertical lines) $\Rightarrow$ $\boxed{dy/dx = x}$.
- **B** has slope depending only on y (same along horizontals) $\Rightarrow$ $\boxed{dy/dx = y}$.
- **C** has straight isoclines $x + y = \text{const}$ (slope constant on diagonals) $\Rightarrow$ $\boxed{dy/dx = x + y}$.

Reasoning via isoclines: set $dy/dx = m$ and observe the loci where m is constant.

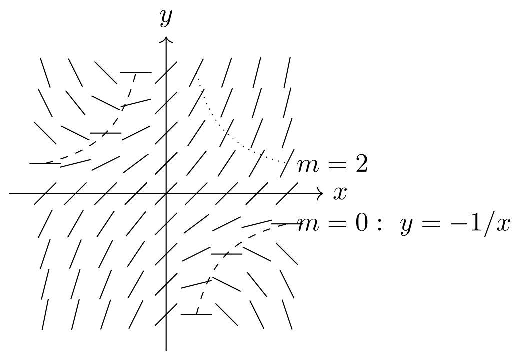
Solution to Q453. [Back to Question p. 247] [Back to TOC]

Isoclines. Given $y' = 1 + xy$, the isocline for slope m satisfies $1 + xy = m$, i.e.

$$\boxed{y = \frac{m-1}{x}} \quad (x \neq 0).$$

Zero-slope curve: $m = 0 \Rightarrow y = -\dfrac{1}{x}$ (a rectangular hyperbola).

Signs. $y' > 0$ where $1 + xy > 0$ (inside the hyperbola branches); $y' < 0$ where $1 + xy < 0$ (outside).



Solution to Q454. [Back to Question p. 248] [Back to TOC]

Equilibria are the real roots of $f(y) = y(y-2)(3-y)$:

$$\boxed{y = 0, \quad y = 2, \quad y = 3.}$$

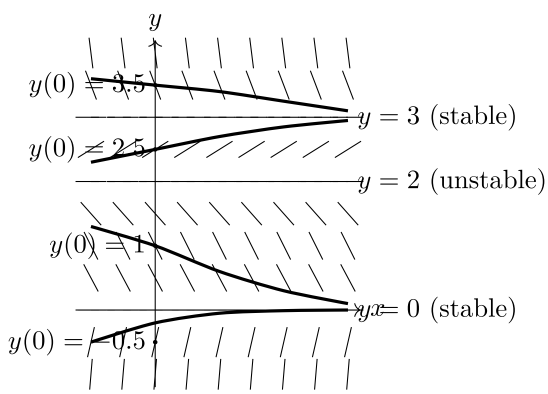
Stability via $f'(y) = -3y^2 + 10y - 6$:

$$f'(0) = -6 < 0 \text{ (stable)}, \qquad f'(2) = 2 > 0 \text{ (unstable)}, \qquad f'(3) = -3 < 0 \text{ (stable)}.$$

Qualitative behaviour:

y	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, 3)$	3 to $+\infty$
$f(y)$	$+$	0	$-$	0	$+$	0 (then $-$)

Thus $y(0) = -0.5$ and $y(0) = 1$ flow to 0; $y(0) = 2.5$ increases to 3; $y(0) = 3.5$ decreases to 3.



Solution to Q455. [Back to Question p. 248] [Back to TOC]

We can solve exactly. The linear ODE $y' - \frac{1}{2}y = \sin x$ has integrating factor $e^{-x/2}$ inverse, i.e. $e^{x/2}$:

$$\frac{d}{dx}\left(ye^{x/2}\right) = e^{x/2} \sin x.$$

Using $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}}{a^2+b^2}\,(a\sin bx - b\cos bx)$ with $a=\frac{1}{2}, b=1$,

$$\int e^{x/2}\sin x\,dx = e^{x/2}\left(\frac{2}{5}\sin x - \frac{4}{5}\cos x\right) + C.$$

Hence

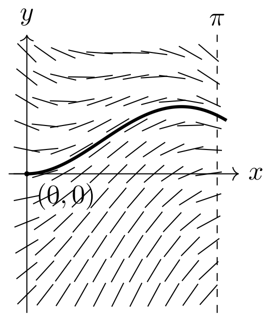
$$y(x) = \left(\frac{2}{5}\sin x - \frac{4}{5}\cos x\right) + Ce^{-x/2}.$$

Impose $y(0)=0$: $0 = -\frac{4}{5} + C \Rightarrow C = \frac{4}{5}$, so

$$\boxed{y(x) = \frac{2}{5}\sin x - \frac{4}{5}\cos x + \frac{4}{5}e^{-x/2}}.$$

Therefore

$$\boxed{y(\pi) = \tfrac{4}{5}\big(1 + e^{-\pi/2}\big) \approx 0.9663}.$$



AHL 5.16- Euler's method

Solution to Q456. [Back to Question p. 251] [Back to TOC]

IVP: $y' = x + y,\ y(0) = 1$, step $h = 0.2$, $x_n = 0, 0.2, \ldots, 1.0$. Euler update:

$$\boxed{y_{n+1} = y_n + h\,(x_n + y_n)}.$$

Compute:

n	x_n	y_n	$f(x_n, y_n) = x_n + y_n$
0	0.0	1.00000	1.00000
1	0.2	1.20000	1.40000
2	0.4	1.48000	1.88000
3	0.6	1.85600	2.45600
4	0.8	2.34720	3.14720
5	1.0	**2.97664**	–

Thus $y(1) \approx \boxed{2.97664}$. The exact solution is $y(x) = 2e^x - x - 1$, so

$$y(1) = 2e - 2 \approx 3.43656, \qquad \text{abs. error} \approx \boxed{0.45992}.$$

Solution to Q457. [Back to Question p. 252] [Back to TOC]

IVP: $y' = y(1 - y/3)$, $y(0) = 0.6$.

(a) $h = 0.5$ (four steps to $x = 2$):

n	x_n	y_n
0	0.0	0.600000
1	0.5	0.840000
2	1.0	1.142400
3	1.5	1.496087
4	2.0	**1.871084**

(b) $h = 0.25$ (eight steps):

n	x_n	y_n
0	0.00	0.600000
1	0.25	0.720000
2	0.50	0.856800
3	0.75	1.009824
4	1.00	1.177302
5	1.25	1.356123
6	1.50	1.541898
7	1.75	1.729251
8	2.00	**1.912371**

(Values rounded to 10^{-6}.)

Exact value: $y(x) = \dfrac{3}{1 + C'e^{-x}}$ with $C' = \dfrac{3}{0.6} - 1 = 4$, so $y(2) = \dfrac{3}{1 + 4e^{-2}} \approx 1.9461$. Hence halving h moves the estimate $1.8711 \to 1.9124$ toward the exact value.

Solution to Q458. [Back to Question p. 252] [Back to TOC]

ODE: $y' = -0.7y + 0.3$, $y(0) = 0$, $h = 0.2$. Euler update

$$y_{n+1} = y_n + h(-0.7y_n + 0.3) = 0.86\,y_n + 0.06.$$

First steps:

n	x_n	y_n
0	0.0	0.000000
1	0.2	0.060000
2	0.4	0.111600
3	0.6	0.155976
4	0.8	0.194139
5	1.0	0.226960
6	1.2	**0.255185**

Smallest t with $y_n \geq 0.25$ is $\boxed{t = 1.2\text{ s}}$. Linear interpolation between $(1.0, 0.226960)$ and $(1.2, 0.255185)$ gives

$$t_* = 1.0 + 0.2\,\frac{0.25 - 0.22695985}{0.25518547 - 0.22695985} \approx \boxed{1.163\text{ s}}.$$

Solution to Q459. [Back to Question p. 252] [Back to TOC]

IVP: $y' = \sin x - \frac{1}{2}y$, $y(0) = 1$, step $h = 0.1$.

(a) Spreadsheet formulas. If x_0 is in `A2`, y_0 in `B2`, and h in `D1`:

$$\texttt{A3:} \quad = A2 + \$D\$1, \qquad \texttt{B3:} \quad = B2 + \$D\$1 * (\sin(A2) - 0.5 * B2).$$

Copy down to $x = 1$.

(b) Euler approximation. Iterating gives $y(1) \approx \boxed{0.96317}$ (ten steps). (Exact solution with $y(0) = 1$ is $y(x) = \frac{2}{5}\sin x - \frac{4}{5}\cos x + \frac{9}{5}e^{-x/2}$, so $y(1) \approx 0.99610$.)

Solution to Q460. [Back to Question p. 253] [Back to TOC]

IVP: $y' = x - y$, $y(0) = 1$, $h = 0.5$.

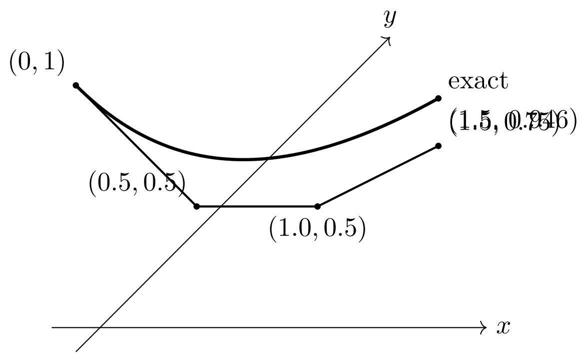
(a) Euler steps:

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$
0	0.0	1.00	-1.00
1	0.5	0.50	0.00
2	1.0	0.50	0.50
3	1.5	**0.75**	–

So $y(1.5) \approx \boxed{0.75}$.

(b) Exact $y(x) = x - 1 + 2e^{-x}$ gives $y(1.5) \approx 0.94626$, percentage error $\approx \dfrac{|0.94626 - 0.75|}{0.94626} \times 100\% \approx \boxed{20.7\%}$.

(c) Euler polygon vs exact.



Solution to Q461. [Back to Question p. 253] [Back to TOC]

IVP: $y' = y\cos x$, $y(0) = 1$; target $x = \pi/2$. Euler update:

$$y_{n+1} = y_n\big(1 + h\cos x_n\big), \qquad x_{n+1} = x_n + h.$$

(a) $h = \dfrac{\pi}{8}$ (4 steps):

$$y(\tfrac{\pi}{2}) \approx \boxed{2.78944}.$$

(b) $h = \dfrac{\pi}{16}$ (8 steps):

$$y(\tfrac{\pi}{2}) \approx \boxed{2.76463}.$$

(c) Richardson refinement ($O(h)$ global error):

$$y^* \approx y_{h/2} + (y_{h/2} - y_h) = 2.76463 + (2.76463 - 2.78944) \approx \boxed{2.73982}.$$

(Exact solution: $y = e^{\sin x}$, so $y(\frac{\pi}{2}) = e \approx 2.71828$; the refined value reduces the error.)

Solution to Q462. [Back to Question p. 254] [Back to TOC]

For $y' = \lambda y$ ($\lambda = -5$), Euler gives $y_{n+1} = (1 + h\lambda)y_n$.

h	$1 + h\lambda$
0.05	0.75 (monotone decay)
0.20	0 (one step to 0)
0.50	−1.5 (unstable, alternating growth)

Stability requires $|1 + h\lambda| < 1$ (here $h < 0.4$). Large h can flip signs or magnify errors, giving qualitatively wrong behaviour for stiff decay.

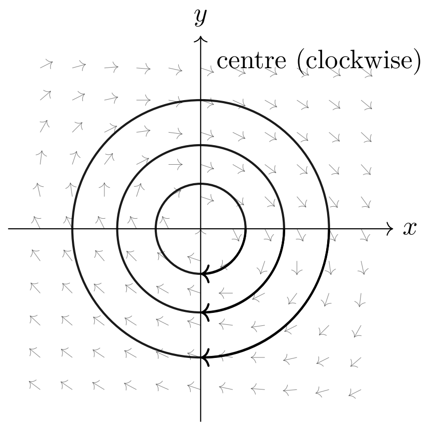
AHL 5.17- Phase portraits

Solution to Q463. [Back to Question p. 256] [Back to TOC]

Centre (purely imaginary).

$$A = \begin{pmatrix} 1 & 3 \\ -2 & -1 \end{pmatrix}, \quad \operatorname{tr} A = 0, \quad \det A = 1 \cdot (-1) - 3 \cdot (-2) = 5.$$

$\Delta = (\operatorname{tr} A)^2 - 4 \det A = -20 < 0$ with $\det A > 0 \Rightarrow$ **centre**. At $(1, 0)$: $(x', y') = (1, -2)$ points downwards, so motion is **clockwise**.

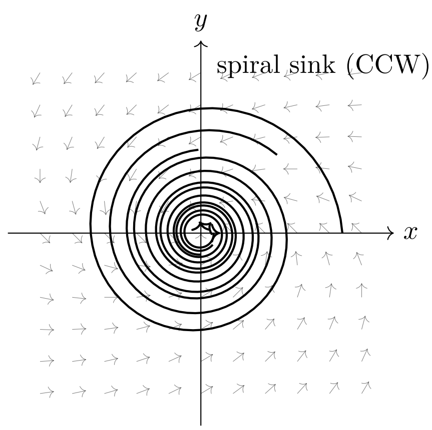


Solution to Q464. [Back to Question p. 256] [Back to TOC]

Spiral sink.

$$A = \begin{pmatrix} -2 & -5 \\ 2 & -3 \end{pmatrix}, \quad \operatorname{tr} A = -5, \quad \det A = 16 > 0, \quad \Delta = \operatorname{tr}^2 - 4 \det = -39 < 0.$$

Hence **stable spiral (sink)**. At $(1, 0)$: $(x', y') = (-2, 2)$ gives **counterclockwise** rotation.

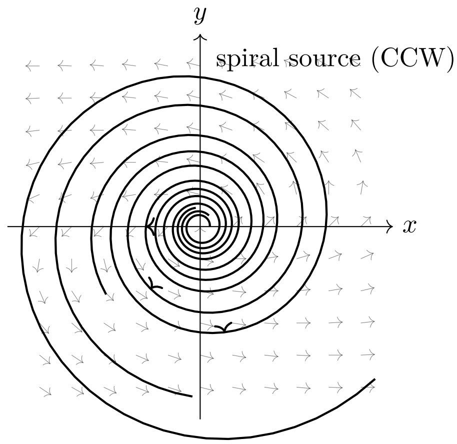


Solution to Q465. [Back to Question p. 256] [Back to TOC]

Spiral source.

$$A = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}, \quad \operatorname{tr} A = 2, \quad \det A = 5 > 0, \quad \Delta = 4 - 20 = -16 < 0.$$

Hence **unstable spiral (source)**. At $(1,0)$: $(x', y') = (1,1) \Rightarrow$ **counterclockwise** rotation.

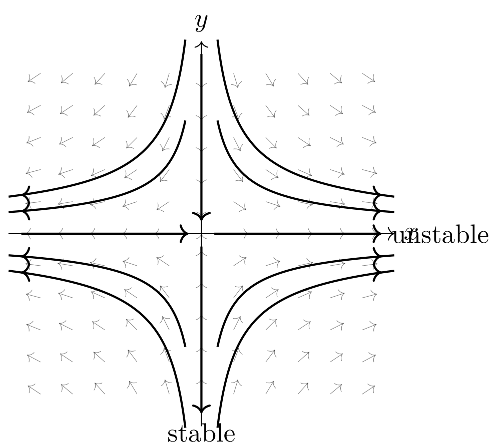


Solution to Q466. [Back to Question p. 257] [Back to TOC]

Saddle; eigenlines and sketch.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \lambda_1 = 3,\ v_1 = (1,0); \quad \lambda_2 = -2,\ v_2 = (0,1).$$

Unstable eigenline: x-axis; stable eigenline: y-axis.

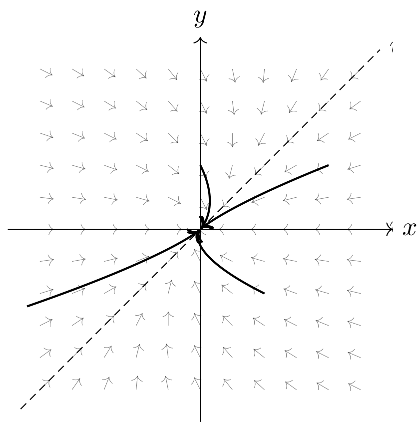


Solution to Q467. [Back to Question p. 257] [Back to TOC]

Stable node.

$$A = \begin{pmatrix} -3 & 1 \\ 0 & -2 \end{pmatrix} \Rightarrow \text{eigenvalues } -3, -2 \text{ (distinct, both } < 0).$$

For $\lambda = -3$: $v_1 = (1,0)$. For $\lambda = -2$: $(A + 2I)v = 0 \Rightarrow v_2 = (1,1)$. Thus **stable node**. As $t \to \infty$, trajectories align with the slower direction $v_2 = (1,1)$.



Solution to Q468. [Back to Question p. 258] [Back to TOC]

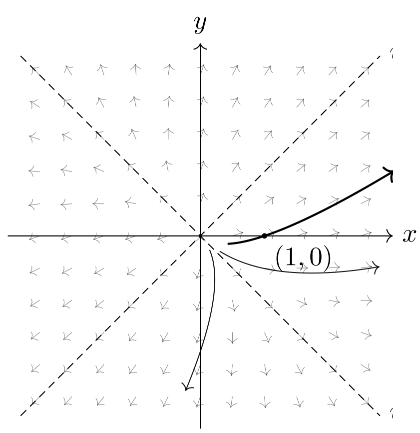
Exact solution (real, distinct).

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Eigenpairs: $\lambda_1 = 4,\ v_1 = (1,1);\ \lambda_2 = 2,\ v_2 = (1,-1)$. Decompose $(1,0) = \frac{1}{2}v_1 + \frac{1}{2}v_2$. Hence

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \tfrac{1}{2}e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2}e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$x(t) = \tfrac{1}{2}(e^{4t} + e^{2t}), \qquad y(t) = \tfrac{1}{2}(e^{4t} - e^{2t}).$$

Both eigenvalues $> 0 \Rightarrow$ **unstable node (source)**.

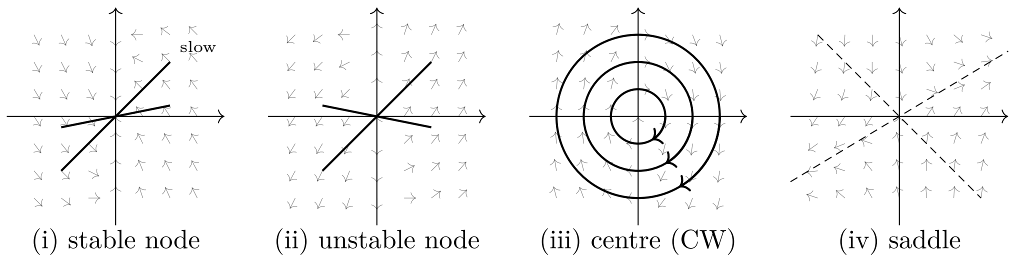


Solution to Q469. [Back to Question p. 258] [Back to TOC]

Trace–determinant classification (sketches).

(i) $\mathrm{tr} = -3,\ \det = 2 > 0,\ \Delta = 1 > 0 \Rightarrow$ **stable node**; (ii) $\mathrm{tr} = 3,\ \det = 2 > 0,\ \Delta = 1 > 0 \Rightarrow$ **unstable node**;

(iii) $\mathrm{tr} = 0,\ \det = 4 > 0,\ \Delta = -16 < 0 \Rightarrow$ **centre (clockwise)**; (iv) $\mathrm{tr} = -2,\ \det = -11 < 0 \Rightarrow$ **saddle**.

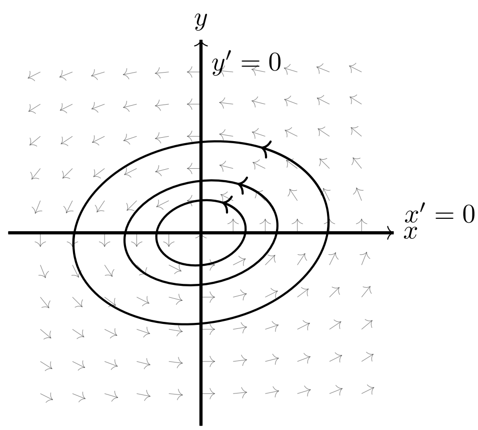


Solution to Q470. [Back to Question p. 258] [Back to TOC]

Nullclines and rotation.

$$x' = -2y, \quad y' = x.$$

Nullclines: $x' = 0 \Rightarrow y = 0$ (the x-axis), $y' = 0 \Rightarrow x = 0$ (the y-axis). At $(1,0)$: $(x', y') = (0,1)$ points upward $\Rightarrow$ **counterclockwise** rotation. Closed orbits (centre).



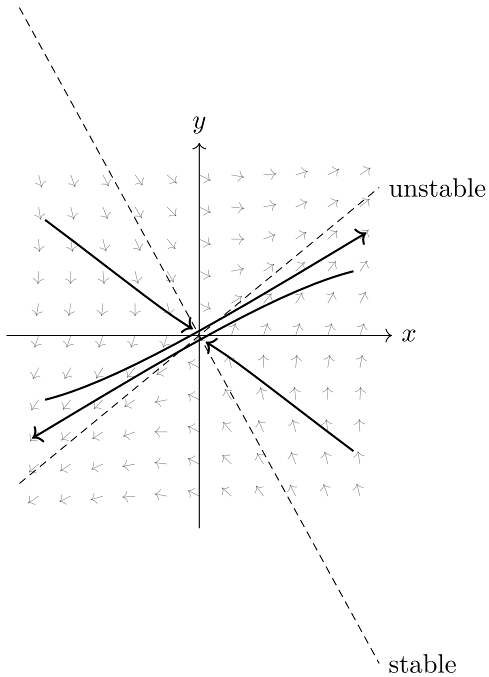
Solution to Q471. [Back to Question p. 259] [Back to TOC]

Saddle; eigenvectors and long–time behaviour.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}, \quad \operatorname{tr} A = 0, \ \det A = 1 \cdot (-1) - 2 \cdot 3 = -7 < 0.$$

$\det < 0 \Rightarrow$ **saddle**. Eigenvalues solve $\lambda^2 - 7 = 0 \Rightarrow \lambda_{\pm} = \pm\sqrt{7}$. For $\lambda = \sqrt{7}$: $(1-\sqrt{7})x + 2y = 0 \Rightarrow y = \frac{\sqrt{7}-1}{2}\,x$ (unstable line). For $\lambda = -\sqrt{7}$: $(1+\sqrt{7})x + 2y = 0 \Rightarrow y = -\frac{1+\sqrt{7}}{2}\,x$ (stable line).

As $t \to \infty$: points on the stable line approach the origin; off that line they move away, becoming tangent to the unstable line.



Solution to Q472. [Back to Question p. 259] [Back to TOC]

Exact solution and interpretation.

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Eigenpairs: $\lambda_1 = 1,\ v_1 = (1,1)$ (unstable); $\lambda_2 = -3,\ v_2 = (1,-1)$ (stable). With $(x(0), y(0)) = (0,1)$, write

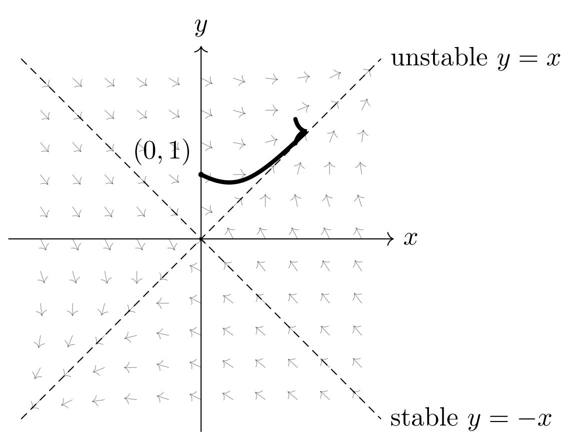
$$(0,1)^\top = \tfrac{1}{2}(1,1)^\top - \tfrac{1}{2}(1,-1)^\top.$$

Hence

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \tfrac{1}{2}e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$x(t) = \tfrac{1}{2}\big(e^t - e^{-3t}\big), \qquad y(t) = \tfrac{1}{2}\big(e^t + e^{-3t}\big).$$

As $t \to \infty$, e^t dominates $\Rightarrow$ trajectory moves **away** from the origin, asymptotic to the unstable direction $y = x$.



AHL 5.18- Second order differential equations

Solution to Q473. [Back to Question p. 262] [Back to TOC]

Rewrite as a first–order system.

1. Set $y = \dfrac{dx}{dt}$. Then the equivalent system is

$$\boxed{x' = y, \qquad y' = f(x,y,t)}, \qquad (x(0), y(0)) = (x_0, v_0).$$

2. For $f(x,\dot{x},t) = -\sin x - 0.3\,\dot{x} + 2\cos t$,

$$\boxed{x' = y, \qquad y' = -\sin x - 0.3\,y + 2\cos t}.$$

3. Phase–plane axes: horizontal x, vertical $y = \dot{x}$. An equilibrium in the (x,y)-plane requires $y = 0$ and $f(x,0,t) = 0$ *for all* t (otherwise the point is time–dependent and not an equilibrium). For the example in (b) there is *no* equilibrium because of the forcing $2\cos t \not\equiv 0$.

Solution to Q474. [Back to Question p. 262] [Back to TOC]

Euler scheme for second–order ODEs. With $x' = y,\ y' = f(x,y,t)$ and $t_{n+1} = t_n + h$, forward Euler gives

$$\boxed{x_{n+1} = x_n + h\,y_n, \qquad y_{n+1} = y_n + h\,f(x_n, y_n, t_n)}.$$

The *local* truncation error is $O(h^2)$; hence the global error after $O(1/h)$ steps is $O(h)$.

Solution to Q475. [Back to Question p. 262] [Back to TOC]

Euler steps on a nonlinear oscillator.

$$x'' = -\sin x - 0.2\,x', \qquad x(0) = 1.0, \quad x'(0) = 0.$$

(a) $x' = y,\ y' = -\sin x - 0.2y$.

(b) With $h = 0.1$ and $(x_0, y_0) = (1, 0)$:

$$f_0 = -\sin(1) - 0.2(0) = -0.8414709848 \approx -0.841471.$$

$$x_1 = x_0 + hy_0 = 1, \qquad y_1 = y_0 + hf_0 \approx 0 - 0.0841471 = -0.0841471.$$

Next

$$f_1 = -\sin(x_1) - 0.2y_1 = -\sin(1) + 0.0168294 \approx -0.824642,$$

$$x_2 = x_1 + hy_1 \approx 1 - 0.00841471 = 0.99158529, \qquad y_2 = y_1 + hf_1 \approx -0.0841471 - 0.0824642 = -0.166611.$$

So $\boxed{(x_1, y_1) \approx (1.000000, -0.084147)}$, $\boxed{(x_2, y_2) \approx (0.991585, -0.166611)}$.

(c) Plot these points in the (x,y)-plane and join with arrows from $(x_0, y_0) \to (x_1, y_1) \to (x_2, y_2)$.

Solution to Q476. [Back to Question p. 263] [Back to TOC]

Linear constant–coefficients (real distinct).

$$x'' - 5x' + 6x = 0, \qquad x(0) = 1,\ x'(0) = 0.$$

(a) $x' = y,\ y' = 5y - 6x$, so $A = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix}$.

(b) $\det(A-\lambda I) = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$. Hence $\lambda_1 = 2$ with $v_1 = (1,2)$, and $\lambda_2 = 3$ with $v_2 = (1,3)$.

(c) The general solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

From $x(0) = 1,\ y(0) = 0$:

$$c_1 + c_2 = 1, \qquad 2c_1 + 3c_2 = 0 \ \Rightarrow\ c_1 = 3,\ c_2 = -2.$$

Thus

$$\boxed{x(t) = 3e^{2t} - 2e^{3t}}, \qquad \boxed{y(t) = 6e^{2t} - 6e^{3t}}.$$

(d) Since both eigenvalues are > 0, the origin is an **unstable node (source)**; trajectories emerge and become tangent to the faster direction $v_2 = (1,3)$ backward in time and align with the slower $v_1 = (1,2)$ forward in time.

Solution to Q477. [Back to Question p. 263] [Back to TOC]

Critically damped case (sketch).

$$x'' + 4x' + 4x = 0.$$

(a) $x' = y,\ y' = -4y - 4x$, so $A = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix}$.

(b) $\operatorname{tr} A = -4,\ \det A = 4$, discriminant $\Delta = \operatorname{tr}^2 - 4\det = 16 - 16 = 0$. Hence a **stable (critically damped) node** with repeated eigenvalue $\lambda = -2$ and a single eigenvector $v = (1, -2)$.

(c) Sketch trajectories approaching the origin and tangent to the line $y = -2x$ (the slow/eigendirection). No oscillations.

Solution to Q478. [Back to Question p. 264] [Back to TOC]

Underdamped oscillator (portrait).

$$x'' + 2x' + 5x = 0 \ \Rightarrow\ x' = y,\ y' = -5x - 2y, \quad A = \begin{pmatrix} 0 & 1 \\ -5 & -2 \end{pmatrix}.$$

$\operatorname{tr} A = -2,\ \det A = 5,\ \Delta = \operatorname{tr}^2 - 4\det = 4 - 20 = -16 < 0 \Rightarrow$ **spiral sink**. At $(1,0)$: $(x', y') = (0, -5)$ points downward, giving **clockwise** rotation. Sketch a clockwise spiral into the origin.

Solution to Q479. [Back to Question p. 264] [Back to TOC]

Driven system; one Euler step.

$$x'' = -x - 0.4x' + 3\cos t, \qquad (x_0, y_0) = (0, 1),\ h = 0.1.$$

Coupled system: $x' = y,\ y' = -x - 0.4y + 3\cos t$. At $t_0 = 0$, $\cos t_0 = 1$:

$$f_0 = y_0' = -0 - 0.4(1) + 3 = 2.6.$$

Euler step:

$$\boxed{x_1 = x_0 + hy_0 = 0.1}, \qquad \boxed{y_1 = y_0 + hf_0 = 1 + 0.26 = 1.26}.$$

(c) Euler is first order and introduces numerical *damping/dispersion*; for oscillatory forcing it can mis-estimate both amplitude and phase unless h is very small (global error $O(h)$).

Solution to Q480. [Back to Question p. 265] [Back to TOC]

Mass–spring–damper.

$$mx'' + cx' + kx = 0, \qquad m, k > 0,\ c \geq 0.$$

(a) Divide by m: $x'' + \frac{c}{m}x' + \frac{k}{m}x = 0$. Define $\omega_n = \sqrt{k/m}$ and $\zeta = \dfrac{c}{2m\omega_n}$. Then

$$\boxed{x'' + 2\zeta\omega_n x' + \omega_n^2 x = 0}.$$

(b) System and matrix:

$$x' = y, \qquad y' = -\omega_n^2 x - 2\zeta\omega_n y, \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}.$$

Hence $\operatorname{tr} A = -2\zeta\omega_n$ and $\det A = \omega_n^2$.

(c) $\zeta < 1$: **underdamped spiral sink** (decays with oscillation). $\zeta = 1$: **critically damped** stable node (fastest non-oscillatory decay). $\zeta > 1$: **overdamped** stable node (monotone decay). All cases with $c > 0$ decay to 0.

Solution to Q481. [Back to Question p. 265] [Back to TOC]

Conservative oscillator.

$$x'' + \omega^2 x = 0, \qquad y = \dot{x}.$$

(a) Let $E = \frac{1}{2}y^2 + \frac{1}{2}\omega^2 x^2$. Then

$$\frac{dE}{dt} = y\,y' + \omega^2 x\,x' = y(-\omega^2 x) + \omega^2 xy = 0,$$

so E is constant along trajectories.

(b) Level sets $E = \text{const}$ give $y^2 + \omega^2 x^2 = C$: **ellipses** centred at the origin. Direction: since $x' = y$ and $y' = -\omega^2 x$, at $(1, 0)$ the vector points downward, hence motion is **clockwise**. Sketch three nested ellipses with arrows.

(c) The period $T = 2\pi/\omega$ is independent of amplitude because the linear system has constant angular speed ω on all energy levels (all ellipses correspond to the same frequency).

Solution to Q482. [Back to Question p. 265] [Back to TOC]

Compare Euler with exact.

$$x'' - x = 0 \iff x'' = x, \qquad (x_0, y_0) = (1, 0).$$

(a) $x' = y,\ y' = x,\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Eigenvalues $\lambda = \pm 1$ with eigenvectors $(1, 1)$, $(1, -1)$.

(b) Exact: $x(t) = c_1 e^t + c_2 e^{-t}$. From $x(0) = 1,\ y(0) = x'(0) = c_1 - c_2 = 0$ we get $c_1 = c_2 = \frac{1}{2}$, hence

$$\boxed{x(t) = \cosh t}, \qquad y(t) = \sinh t.$$

(c) Euler with $h = 0.1$:

$$\begin{aligned}
(x_1, y_1) &= (1 + 0.1 \cdot 0,\ 0 + 0.1 \cdot 1) = (1.000, 0.100), \\
(x_2, y_2) &= (1.000 + 0.1 \cdot 0.100,\ 0.100 + 0.1 \cdot 1.000) = (1.010, 0.200), \\
(x_3, y_3) &= (1.010 + 0.1 \cdot 0.200,\ 0.200 + 0.1 \cdot 1.010) = (1.030, 0.301).
\end{aligned}$$

(d) Exact $x(0.3) = \cosh(0.3) = \frac{e^{0.3}+e^{-0.3}}{2} \approx \mathbf{1.0453385}$. Euler gives $x_3 = 1.030$. The error is $x_3 - x(0.3) \approx -0.01534 < 0$: here Euler **underestimates** the true value (global $O(h)$ error).

Solution to Q483. [Back to Question p. 266] [Back to TOC]

Pendulum with damping; two Euler steps.

$$x'' + 0.1x' + \sin x = 0, \qquad x(0) = \tfrac{\pi}{2},\ y(0) = 0.$$

(a) $x' = y, \quad y' = -0.1\,y - \sin x$.

(b) With $h = 0.05$:

$$f_0 = -0.1(0) - \sin\left(\frac{\pi}{2}\right) = -1, \quad x_1 = x_0 + hy_0 = \frac{\pi}{2}, \quad y_1 = y_0 + hf_0 = -0.05.$$

$$f_1 = -0.1(-0.05) - \sin\left(\frac{\pi}{2}\right) = 0.005 - 1 = -0.995,$$

$$\boxed{x_2 = x_1 + hy_1 = \tfrac{\pi}{2} - 0.0025 \approx 1.56830}, \qquad \boxed{y_2 = y_1 + hf_1 = -0.05 - 0.04975 = -0.09975}.$$

(c) Near the origin the linearization $x'' + 0.1x' + x \approx 0$ yields a spiral sink; solutions decay to $(0,0)$ (pendulum comes to rest).

Solution to Q484. [Back to Question p. 266] [Back to TOC]

Matrix-to-second-order translation.

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad a, b > 0.$$

(a) From $x' = -ax - by$ and $y' = x$,

$$x'' = -ax' - by' = -ax' - bx \implies \boxed{x'' + ax' + bx = 0}.$$

(b) For $A = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}$,

$$\operatorname{tr} A = -a, \quad \det A = b, \quad \Delta = \operatorname{tr}^2 - 4\det = a^2 - 4b.$$

With $a > 0,\ b > 0$:

- Overdamped node: $\Delta > 0 \Leftrightarrow b < \frac{a^2}{4}$ (two distinct negative real eigenvalues).
- Critical damping: $\Delta = 0 \Leftrightarrow b = \frac{a^2}{4}$ (repeated negative eigenvalue).
- Underdamped spiral: $\Delta < 0 \Leftrightarrow b > \frac{a^2}{4}$ (complex pair with negative real part).

(c) Overdamped case ($b < \frac{a^2}{4}$). Let $r_{1,2} = \frac{-a \pm \sqrt{a^2-4b}}{2}$ (both < 0). With $x(0) = x_0$, $\dot{x}(0) = v_0$,

$$\boxed{x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \qquad C_1 = \frac{v_0 - r_2 x_0}{r_1 - r_2}, \quad C_2 = \frac{r_1 x_0 - v_0}{r_1 - r_2}}.$$