Group Project for EF4821 : Exact Simulation for CIR Process

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1. Abstract

The Cox-Ingersoll-Ross Model (CIR) model becomes popular in the financial industry because (i) it is capable of providing good fits to various types of implied volatility curves observed in the marketplace (ii) it's square root volatility provides analytical solution to naive bond pricing problem (iii)it guarantees non-negative interest rate with suitable parameters setting. However, no analytical solution to the CIR model exists that can be simulated directly, which introduces difficulties when pricing more complicated derivative products. This paper explores the possibility of exact simulation for the CIR model.

Primary difficulties involved in are how to construct a non-central chi-square random variable with the same distribution with CIR process $\{X_t|X_0, t \geq 0\}$, and how to sample an non-central chi-square random variable.

We derived the non-central chi-square random variable by Fourier Transform and Kolmogorov Equation, and derived the sampling scheme of this random variable by manipulating its p.d.f.

We also presented the result by Euler Scheme, which is known to be biased. The empirical result also suggests that our own constructed scheme outperformed the other scheme in estimating US interest rates.

2. Introduction

We consider the following stochastic process in probability space $(\Omega, \mathcal{F}, P)$,

$$dX(t) = \alpha(b - X(t))dt + \sigma\sqrt{X(t)}dW(t), \quad X(0) = X_0 \geq 0,$$

which is known as CIR process. We assume that $2\alpha b \geq \sigma^2$ to guarantee $X_t(\omega)$ is non-negative for $t \geq 0$ and $\omega \in \Omega$.

This model gains its popularity due to its advantages mentioned in the previous section, but the absence of analytical solution involves difficulties when pricing more complicated derivative products than zero coupon bond, for which the CIR model provides analytical solution to Bond Pricing Formula. In light of the absence of closed-form derivatives pricing formulas and the drawbacks of PDE-based asymptotic expansions and conventional discretization simulation, the aim of this paper is to explore the possibility of exact simulation for the CIR model.

In the simulation literature, Discretization Simulation or Euler Scheme is unsatisfying as (i) it's a biased scheme and it's hard to correct its bias (ii) the issue arise at the boundary 0 for CEV-type diffusion, that is, under suitable parameters setting, the diffusion process is guaranteed non-negative, however, the discretization methods may generate negative values in the intermediate steps with a significant probability. Though we have absorbing and reflecting methods to deal with negative values in the literature, this brings a new distortion to the original price distribution. Thus, the exact simulation scheme is desired.

The remainder of this paper is organized as follows. In section 3, we introduce the conditional distribution of CIR model by Fourier Transformation of its Kolmogorov equation and sampling scheme of random variables with such distribution. Thus we derived an exact simulation scheme for the CIR model and we prove the consistency and correctness of distribution properties and required theorem of the simulation algorithm. In section 4, we test the difference between theortical mean and simulation mean of four simulation schemes among three devised CIR process. In section 5, we estimate the corresponding CIR parameters of several market datasets and conduct the our scheme and the rest schemes as well to test the empirical performance.In the end, we summarized the main results and analysis of the new scheme in section 6.

3. Main Result

In this section, we will derive and prove two theorems about the conditional distribution and sampling methods of $\{X_t|X_0, t \geq 0\}$ in CIR process as our main result of this paper.

3.1. **Conditional Distribution of X_t.** The following theorem shows that the distribution of the X_t is an non-central chi-square distribution.

Theorem 1. *Suppose that X_t follows the CIR stochastic differential equation*

$$dX(t) = \alpha(b - X(t))dt + \sigma\sqrt{X(t)}dW(t), \quad X(0) = X_0 \geq 0$$

then the conditional distribution of X_t given X_0 is a non-central chi-square distribution $c\chi_d'^2(\lambda)$. Where $d = \frac{4b\alpha}{\sigma^2}, c = \frac{\sigma^2\left(1-e^{-\alpha(t-u)}\right)}{4\alpha}$ and $\lambda = \frac{4\alpha e^{-\alpha(t-u)}}{\sigma^2\left(1-e^{-\alpha(t-u)}\right)}X_0$.

Proof. According to the generator of the CIR process, the p.d.f of X_t $p(x,t)$ derived from the Kolmogorov Equations is the solution to the following PDE,

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\alpha(b-x)p) + \frac{1}{2}\frac{\partial^2}{x^2}\left(\sigma^2 xp\right),$$

with initial condition $p(x,0) = \delta\left(x - x_0\right)$.

Then let f(t, ω) characteristic function, p.d.f. Fourier transformation,

$$f(t,\omega) = \int_{-\infty}^{\infty} e^{i\omega y}p(x,t)dx.$$

Function $f(t,\omega)$ is solution of first order linear partial differential equation:

$$\frac{\partial f}{\partial t} = ik\omega f + \left(-k\omega + \frac{1}{2}i\theta^2\omega^2\right)\frac{\partial f}{\partial \omega},$$

with initial condition $f(0,\omega) = e^{i\omega x_0}$, and this PDE has analytical solution. Then by plugging the solution of $f(t,\omega)$ to the following inverse equation

$$p(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x} f(t,\omega)d\omega,$$

we can easily calculate any moment of distribution using the following formula,

$$E\left(x^k\right) = \int_{-\infty}^{\infty} x^k p(x,t)dx = (-i)^k \frac{df^k(t,\omega)}{d\omega}\bigg|_{\omega=0}.$$

Then plug $k=1$ and $k=2$ into the above equation, we can get

$$EX_t = \frac{1}{2}AB\left(1-e^{-\alpha t}\right) + x_0 e^{-\alpha t},$$

and

$$\operatorname{var}(X_t) = 0.25AB^2\left(1-e^{-2\alpha t}\right) + x_0 B\left(1-0.5\frac{AB}{x_0}\right)e^{-\alpha t}\left(1-e^{-\alpha t}\right),$$

where $A = \frac{2\alpha b}{\sigma^2}$ and $B = \frac{\sigma^2}{\alpha}$.

Non-central Chi-Squared distribution os distribution of sum of squares d (degree of freedom) independent Gaussian random variables with mean μ and variance 1. The characteristic function of this distribution is

$$\begin{aligned}
&\varphi_0(\omega,d,\lambda) = (1-2i\omega)^{-\frac{n}{2}} e^{\frac{i\lambda}{\omega^{-1}-2i}} \\
&\lambda = n\mu^2
\end{aligned}$$

If variance equals σ, the characteristic function

$$\varphi(\omega,d,\lambda,\sigma) = \varphi_0\left(\sigma^2\omega, d, \frac{\lambda}{\sigma^2}\right).$$

If the CIR process X_t has non-central chi-squared distribution, we must choose $\sigma^2 = \mathrm{var}(X_t)$ and parameters λ, d such that the characteristic functions are identical. We can show that $d = \frac{4b\alpha}{\sigma^2}, c = \frac{\sigma^2\left(1-e^{-\alpha(t-u)}\right)}{4\alpha}$ and $\lambda = \frac{4\alpha e^{-\alpha(t-u)}}{\sigma^2\left(1-e^{-\alpha(t-u)}\right)}X_0$ are parameters satisfied by simple algebra. The theorem is proved.

3.2. **Sampling Non-Central Chi-Square Random Variable.**

Theorem 2. *A non-central chi-square random variable can be sampled by*

$$\chi_d^2(\lambda) \stackrel{d}{=} \chi_d^2(0) + \mathcal{Y}(\lambda, Z_1, Z_2, U)$$

where $\chi_d^2(0)$ has a gamma distribution $\mathcal{G}(d/2, 2)$, U has uniform distribution on [0,1], and Z_1, Z_2 has standard normal distribution, and

$$y(\lambda, z, \tilde{z}, u) = \begin{cases} 0, & : \text{ if } \lambda + 2\ln(U) \leq 0 \\ (Z + \sqrt{\lambda + 2\ln(U)})^2 + \tilde{Z}^2, & : \text{ if } \lambda + 2\ln(U) > 0 \end{cases}$$

Proof.

The p.d.f $g(x|\lambda, d)$ of $x \sim \chi_d^2(\lambda)$ is given by

$$g(x|\lambda, d) = e^{-\frac{\lambda}{2}} \sum_{j=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \frac{e^{-\frac{x}{2}}}{\Gamma\left(\frac{d}{2}+j\right)} \frac{x^{\frac{d}{2}+j-1}}{2^{\frac{d}{2}+j}}$$

then by Taylor's expansion,

$$e^{\frac{\lambda}{2}} g(\lambda, d, x) = g(0, d, x) + \sum_{j=1}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^j}{j!} \frac{e^{-\frac{x}{2}}}{\Gamma\left(\frac{d}{2}+j\right)} \frac{x^{\frac{d}{2}+j-1}}{2^{\frac{d}{2}+j}}$$

where $g(0, d, x) = \frac{e^{-\frac{x}{2}}}{\Gamma\left(\frac{d}{2}\right)} \frac{x^{\frac{d}{2}-1}}{2^{\frac{d}{2}}}$ is the density function of $\chi_d^2(0)$.

Then differentiating both sides w.r.t λ leads to the following relationship

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(e^{\frac{\lambda}{2}}g(\lambda,d,x)\right) &= \frac{1}{2}\sum_{j=1}^{\infty}\frac{(\lambda/2)^{j-1}}{(j-1)!}g(0,d+2j,x)\\ &= \frac{1}{2}\sum_{j=0}^{\infty}\frac{(\lambda/2)^{j}}{j!}g(0,d+2+2j,x)\\ &= \frac{1}{2}e^{\frac{\lambda}{2}}g(\lambda,d+2,x)\end{aligned}$$

then integrating both sides, we can get

$$e^{\frac{\lambda}{2}}g(\lambda,d,x) = g(0,d,x) + \frac{1}{2}\int_0^{\lambda} e^{\frac{t}{2}}g(t,d+2,x)dt$$

which is equivalent to

$$g(\lambda,d,x) = e^{-\frac{\lambda}{2}}g(0,d,x) + \int_{e^{-\frac{\lambda}{2}}}^{1} g(\lambda+2\ln s,d+2,x)ds.$$

Then we can claim that

$$\chi_d^2(\lambda) \stackrel{d}{=} \begin{cases} \chi_d^2(0), & : \text{ if } \lambda+2\ln(U)\le 0\\ \chi_{d+2}^2(\lambda+2\ln U), & : \text{ if } \lambda+2\ln(U)>0\end{cases}$$

where $\chi_{d+2}^2(\lambda+2\ln U) \stackrel{d}{=} \chi_d^2(0) + (Z_1+\sqrt{\lambda+2\ln U})^2 + Z_2^2$. The theorem is proved.

4. Implementation and Numerical Examples

In this section, we conduct different parameters from CIR process to simulate the finite sample results of our scheme and to compare them with several existing schemes in terms of bias through the mean value of simulated samples.

4.1. **CIR Parameter Setting.** We designed three cases with different standard deviations and mean-reverting coefficients, among which the degree of freedom of all cases is less than 1 to make the difference between traditional methods of non-central chi-square sampling and two modified exact simulation methods applied in this part.

Parameters	Case 1	Case II	Case III
σ	2	1.2	1
α	0.1	0.2	0.4
b	0.4	0.2	0.1
R(0)	0.3	0.1	0.05

The time interval was considered as 1 and we simulated Euler Schemes with time points($1/\delta_t$) 1,5,7,10,14,20,30,50, respectively to display its convergence process. In addition, we simulated all schemes for 10000 times, which ensures the reliability of results.

4.2. **Test results.** The theoretical mean of the three simulation cases equals to $0.310, 0.118, 0.0067$. According to the simulation results (See in Appendix), we compared the accuracy and efficiency of our method with the reest three methods. For accuracy,the bias is larger in Euler Schemes, particularly in those small time points, comparing to the New Scheme and Poisson Scheme. For time consumption, although the Poisson scheme is also unbiased, it would need more time to do sampling when time points become larger.

5. Collaboration and Comparison With Market Data

We examined four sets of interest rates, the Monthly U.S. interest rate, 1 year U.S.interest rate, 5 year U.S. interest rate and 10 year U.S. interest rate from 2015 to 2018. All data sets are downloaded from the Econometric dataset Ecdat. Then we use maximum likelihood estimation method to estimate the CIR parameters which are used in the following simulations. The CIR parameters are shown in the table below.

Data	sigma	alpha	b
Monthly US rate	0.8255179	0.1654958	5.555786
1 year US rate	0.6321328	0.106498	6.743644
5 year US rate	0.4395853	0.060666	8.292693
10 year US rate	0.3535297	0.045739	9.250609

For ease of comparsion, we compute the sample mean of interest rate data and the simulation mean for four simulation methods. The simulation parameters are the same for each method and all time points of Euler Schemes in empirical study equal to 15.

	Simulation				Sample mean
Data	Euler1*	Euler2 **	Possion	New	
Monthly US rate	4.824900353	4.827194344	4.754728	4.820158	4.820158
1 year US rate	5.624360024	5.622647043	5.637709	5.643843	5.525836
5 year US rate	6.578005277	6.542709617	6.392608	6.188323	5.973013
10 year US rate	7.307490655	7.327338032	7.285307	6.986284	6.157467

* Truncation Scheme with $(1/\delta_t = 15)$

** Reflection Scheme with $(1/\delta_t = 15)$

Comparing to the simulation results, the empirical mean is not as close as expected and we conjecture that it is because the parameters estimated from data have a little difference between true CIR process and the average number is also not exactly equivalent to the true mean. However, our new simulation still acquires better results than the Euler Schemes and faster than Poisson Schemes.

6. Conclusion

In this paper, we have constructed three simulation schemes based on different choices of parameters and compared the result of the simulation to the existing schemes, namely Euler Scheme and Poisson Scheme. The simulation result shows that Euler Scheme has relatively larger bias typically cases with smaller time points compared to the other cases. Although Poisson Scheme is unbiased, it required more time to perform sampling especially in the case of larger time points.

The empirical tests are then carried out on four interest rate data sets with different maturity, namely 1 month, 1 year, 5 years and 10 years. CIR model are then fitted to these data sets using maximum likelihood method. The sample means of interest rates and their corresponding simulation result using three different schemes are then calculated and compared. The empirical results shows that the new schemes that we suggested performed relatively better than both Euler Scheme and Poisson Scheme, thus improving the capability of the estimation of interest rate using CIR process.

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7. Appendix

Case 1				
$1/\delta_t$(Time points)	Euler1*	Euler2**	Possion	New
1.0	0.302817351	0.302817351	0.307954	0.311864
5.0	0.493452209	0.493452209		
7.0	0.527677855	0.527677855		
10.0	0.492632118	0.492632118		
14.0	0.486798479	0.486798479		
20.0	0.459647826	0.459647826		
30.0	0.433452458	0.433452458		
50.0	0.438306913	0.438306913		

* Truncation Scheme

** Reflection Scheme

Case 2

$1/\delta_t$(Time points)	Euler1*	Euler2**	Possion	New
1.0	0.118489728	0.31921895	0.11836	0.120729
5.0	0.191444173	0.333648696		
7.0	0.183742956	0.325290328		
10.0	0.188240916	0.3084493		
14.0	0.177398534	0.291376042		
20.0	0.175460637	0.258102744		
30.0	0.170019202	0.254387055		
50.0	0.163376047	0.225588142		

* Truncation Scheme

** Reflection Scheme

Case 3

$1/\delta_t$(Time points)	Euler1*	Euler2*	Possion	New
1.0	0.069880676	0.185371731	0.064048	0.066071
5.0	0.110738825	0.206499106		
7.0	0.109309116	0.200733785		
10.0	0.111518887	0.186203603		
14.0	0.105094282	0.175199517		
20.0	0.102259513	0.16716013		
30.0	0.100123085	0.154684751		
50.0	0.098855629	0.141650649		

* Truncation Scheme

** Reflection Scheme