EF4822 Review Chapter 3

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Econometric Modelling Strategy for Volatility

1. **General Volatility Model**
- $X_t = \mu_t + \epsilon_t$
- $\epsilon_t = h_t^{\frac{1}{2}} z_t$
- $\{z_t\} \sim \text{MDS}(0,1)$
- $E(z_t | I_{t-1}) = 0$ a.s and $E(z_t^2 | I_{t-1}) = 1$ a.s.
- $\mu_t = \mu(I_{t-1})$, conditional mean
- $h_t = h(I_{t-1})$, conditional variance
- How to interpret μ_t and h_t?
  $E(X_t | I_{t-1}) = \mu_t$
  $\text{var}(X_t | I_{T-1}) = h_t \text{var}(z_t) = h_t$
- The random variable

$$z_t = \frac{X_t - \mu_t}{h_t^{\frac{1}{2}}}$$

is called standarized innovation, which represents **unobsverable shocks** and **news**.

2. **Difference Sequance (MDS):** Let $\{y_t, I_t\}$ be a martingale sequence and $\{\epsilon_t, I_t\}$ is a MDS iff
   - **a.** $I_t < I_{t-1}$
   - **b.** $E(\epsilon_t | I_{t-1}) = 0$

3.2.7 *Martingales and Martingale Difference Sequences*

Let $\{y_t\}$ denote a sequence of random variables and let $I_t = \{y_t, y_{t-1}, \ldots\}$ denote a set of conditioning information or *information set* based on the past history of y_t. The sequence $\{y_t, I_t\}$ is called a *martingale* if

- $I_{t-1} \subset I_t$ (I_t is a filtration)
- $E[|y_t|] < \infty$
- $E[y_t|I_{t-1}] = y_{t-1}$ (martingale property)

The most common example of a martingale is the random walk model

$$y_t = y_{t-1} + \varepsilon_t, \ \varepsilon_t \sim WN(0, \sigma^2)$$

where y_0 is a fixed initial value. Letting $I_t = \{y_t, \ldots, y_0\}$ implies $E[y_t|I_{t-1}] = y_{t-1}$ since $E[\varepsilon_t|I_{t-1}] = 0$.

Let $\{\varepsilon_t\}$ be a sequence of random variables with an associated information set I_t. The sequence $\{\varepsilon_t, I_t\}$ is called a *martingale difference sequence* (MDS) if

- $I_{t-1} \subset I_t$
- $E[\varepsilon_t|I_{t-1}] = 0$ (MDS property)

If $\{y_t, I_t\}$ is a martingale, a MDS $\{\varepsilon_t, I_t\}$ may be constructed by defining

$$\varepsilon_t = y_t - E[y_t|I_{t-1}]$$

By construction, a MDS is an uncorrelated process. This follows from the *law of iterated expectations*. To see this, for any $k > 0$

$$\begin{aligned} E[\varepsilon_t \varepsilon_{t-k}] &= E[E[\varepsilon_t \varepsilon_{t-k}|I_{t-1}]] \\ &= E[\varepsilon_{t-k} E[\varepsilon_t|I_{t-1}]] \\ &= 0 \end{aligned}$$

In fact, if z_n is any function of the past history of ε_t so that $z_n \in I_{t-1}$ then

$$E[\varepsilon_t z_n] = 0$$

ARCH(Assuming μ_t is given)

Linaer ARCH

1. **Basic idea**

   The shock of times series$\{\epsilon_t\}$ is uncorrelated but depens on X_t

2. **Model**

- $X_t = \epsilon_t$
- $\epsilon_t = z_t h_t^{\frac{1}{2}}$
- $h_t = \beta_0 + \Sigma_{j=1}^{q} \beta_j X_{t-j}^2$
- z_t~i. i. d. (0,1)

3. **Property**

   $E(X_t)=E(\mu_t + \epsilon_t)=E\left(z_t h_t^{\frac{1}{2}}\right) = E\left(E\left(z_t h_t^{\frac{1}{2}} \middle| I_{t-1}\right)\right) = E\left(h_t^{\frac{1}{2}} E(z_t|I_{t-1})\right) = 0$

   $var(X_t|I_{t-1}) = var(\epsilon_t|I_{t-1}) = var\left(z_t h_t^{\frac{1}{2}} | I_{t-1}\right) = var\left(z_t\left(\beta_0 + \Sigma_{j=1}^{q} \beta_j X_{t-j}^2\right) \middle| I_{t-1}\right) = h_t var(z_t|I_{t-1}) = h_t = \beta_0 + \Sigma_{j=1}^{q} \beta_j X_{t-j}^2$

$\text{var}(X_t) = E\left(\text{var}(X_t|I_{t-1})\right) = \beta_0 + \Sigma_{j=1}^{q}\beta_j E\left(X_{t-j}^2\right) = \beta_0 + \Sigma_{j=1}^{q}\beta_j E\left(\epsilon_{t-j}^2\right)$

If $\{X_t\}$ is weakly stationary, we have

$$\sigma^2 = \beta_0 + \Sigma_{j=1}^{q}\beta_j\sigma^2 \Rightarrow \sigma^2 = \frac{\beta_0}{1 - \Sigma_{j=1}^{q}\beta_j} > 0$$

To ensure σ^2 is well − defined, we reuqire $\beta_0 > 0$, $\beta_j \geq 0$ for $j = 1, \dots, q$ and $\Sigma_{j=1}^{q}\beta_j < 1$

4. AR(q) representation in X_t^2

Let $v_t = X_t^2 - h_t$. Then $E(v_t|I_{t-1}) = h_t - h_t = 0$,

$$X_t^2 = E(X_t^2|I_{t-1}) + [X_t^2 - E(X_t^2|I_{t-1})]$$
$$= h_t + v_t$$
$$= \beta_0 + \Sigma_{j=1}^{q}\beta_j X_{t-j}^2 + v_t$$

where $E(X_{t-j}^2 v_t) = E\left(E\left(X_{t-j}^2 v_t|I_{t-1}\right)\right) = E\left(X_{t-j}^2 E(v_t|I_{t-1})\right) = 0.$

If $\mu_t \neq 0, X_t = \mu_t + \epsilon_t \Rightarrow z_t = X_t - \mu_t = \epsilon_t \Rightarrow AR(q)$ of z_t^2

Remark

- For an ARCH(q) process
  $\{X_t\}, \{X_t^2\}$ can be expressed as $AR(q)$ model. It can be used to determine the order of q by Consider {AIC/BIC, PACF} of $\{X_t^2\}$
- (2) OLS will deliver consistent estimators of $\{\beta_j\}_{j=0}^{q}$. This follows because the orthogonality condition that $E(Z_t v_t) = 0$, where $Z_t = (1, X_{t-1}^2, \cdots, X_{t-q}^2)'$.

  Trade on Xt

  (Eagle Model)

Example ARCH(1) holds stationary

$$\begin{cases} X_t = \epsilon_t \\ \epsilon_t = z_t h_t^{\frac{1}{2}}, \\ h_t = \beta_0 + \beta_1 X_{t-1}^2, \\ z_t \sim i.i.d.(0,1). \end{cases}$$

$\rho_2(j) = \text{corr}\left(X_t^2, X_{t-j}^2\right) = \frac{\text{cov}(X_t^2, X_{t-j}^2)}{\gamma(0)} = \beta_1^{|j|}$, which decays to zero at a gemoetric speed

$E(X_t^4) = E(z_t^4 h_t^2) = E\left(E(z_t^4 h_t^2|I_{t-1})\right) = E\left(h_t^2 E(z_t^4|I_{t-1})\right) = E\left(h_t^2\right)E(E(z_t^4|I_{t-1}))$

$= E((\beta_0 + \beta_1 X_{t-1}^2)^2)E(z_t^4)$

$= 3(\beta_0^2 + 2\beta_0\beta_1 E(X_{t-1}^2) + \beta_1^2 E(X_{t-1}^4))$

$E(X_t^4) = \frac{3\beta_0^2(1+\beta_1)}{(1-\beta_1)(1-3\beta_1^2)}$, to ensure fourth moment exists, $\beta_1 < \frac{1}{\text{sqrt}(3)} \approx 0.577$

Remark

$E(z_t^4)$ and $\{z_t\} \sim N(0,1)$ recall $MGF_z(t) = \exp\left(\mu_t + \frac{\sigma^2 t^2}{2}\right) \Rightarrow \exp\left(\frac{t^2}{2}\right)$ for $N(0,1)$

$MGF^4(t) = \left(\exp\left(\frac{t^2}{2}\right) + t\exp\left(\frac{t^2}{2}\right) + 2\exp\left(\frac{t^2}{2}\right) + 2t^2\exp\left(\frac{t^2}{2}\right) + t^4\exp\left(\frac{t^2}{2}\right)\right) \Rightarrow MGF^4(0) = 3$

So, $E(z_t^4) = 3$.

$$K = \frac{E(X_t^4)}{[E(X_t^2)]^2} = \frac{3E(h_t^2)}{\left(E(z_t^2 h_t)\right)^2} = \frac{3E(h_t^2)}{\left(E(h_t)\right)^2} = \frac{E(X_t^4)}{\left(var(X_t)\right)^2} = \frac{3\beta_0^2(1+\beta_1)}{(1-\beta_1)(1-3\beta_1^2)} \Big/ \left(\frac{\beta_0}{1-\beta_1}\right)^2$$

$=3\frac{1-\beta_1^2}{1-3\beta_1^2} > 3$ which is long tail distribution.

Build the ARCH Model

Model

- $X_t = \epsilon_t$
- $\epsilon_t = z_t h_t^{\frac{1}{2}}$
- $h_t = \beta_0 + \Sigma_{j=1}^{q} \beta_j X_{t-j}^2$
- $z_t \sim i.i.d.(0,1)$

$X_t^2 = h_t z_t^2 = \left(\beta_0 + \Sigma_{j=1}^{q}\beta_j X_{t-j}^2\right) z_t^2$

Step 1 Determine the order

Let $\eta_t = \sigma_t^2 - \delta_t^2$

$\widehat{X_{t-1}}(1) = E(X_t^2 | I_{t-1}) = E\left(\left(\beta_0 + \Sigma_{j=1}^{q}\beta_j X_{t-j}^2\right) z_t^2 \middle| I_{t-1}\right) = \left(\beta_0 + \Sigma_{j=1}^{q}\beta_j X_{t-j}^2\right) E(z_t^2 | I_{t-1}) = \left(\beta_0 + \Sigma_{j=1}^{q}\beta_j X_{t-j}^2\right)$

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + \eta_t$$
$$\Rightarrow \sigma_t^2 = \alpha_t^2 - \alpha_0 - \alpha_1 a_{t-1}^2 - \cdots - \alpha_m a_{t-m}^2 + \delta_t^2$$

? $$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2.$$

Similar to AR model(???)

? **Volatility clustering**

Limitation on ARCH Model

1. ARCH **assuming possitive and negative shock** has the same effect on volatility because it depends on the square of previous shocks$\{\epsilon_t^2\}$
2. The ARCH restricted in a sense only very small parameters is permitted to ensure the existence of certain fourth moments. It limits the ability of ARCH to simulate the **kurtosis**
3. ARCH doesn't provide new insight of the sourse of variations of financial time series.
4. ? It's likely to overpredict the volatility. (overfit/complex than truth)

GARCH Model

- $X_t = h_t^{\frac{1}{2}} z_t$
- $H_t = \alpha_0\ \Sigma^{p}{}_{j=1}\beta_j h_{t-j} + \Sigma_{j=1}^{q}\alpha_j X_{t-j}^2$
- $\{z_t\} \sim N(0,1)$
- Where $\alpha_0 > 0, \alpha_i \geq 0, \beta_i \geq 0$ and $\Sigma_{i=1}^{max(p,q)}(\alpha_i + \beta_i) < 1$

- $z_{1t} = h_t^{\frac{1}{2}} y_t$
- $h_t = \omega + \beta h_{t-1} + \alpha z_{1,t-1}^{2}$
- $\{z_t\} \sim N(0,1)$
- Where $\alpha_0 > 0, \alpha \geq 0, \beta \geq 0$ and $\alpha + \beta < 1$

Example GARCH(1,1) Model

$$
\begin{cases}
X_t = h_t^{1/2} z_t, \\
h_t = \omega + \beta h_{t-1} + \gamma X_{t-1}^2, \\
\{z_t\} \sim i.i.d.(0,1).
\end{cases} \tag{37}
$$

Question: What are the interpretations of β and γ?

Let $v_t = X_t^2 - h_t$.

$$
\begin{aligned}
h_t &= \omega + \beta h_{t-1} + \gamma X_{t-1}^2 \\
&= \omega + (\gamma + \beta) h_{t-1} + \gamma (X_{t-1}^2 - h_{t-1}) \\
&= \omega + (\gamma + \beta) h_{t-1} + \gamma v_{t-1}
\end{aligned}
$$

$\Rightarrow X_t^2 = h_t + v_t = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 h_{t-1} + v_t$

$= \alpha_0 + (\alpha_1 + \beta_1) X_{t-1}^2 - \beta_{1 v_{t-1}} + v_t \Rightarrow$ ARMA(1,1)

EXAM Question:

Given ARMA(1,1) on $X_T^2 = \cdots \Rightarrow$ GARCH

Exam Q:

given ARMA(1,1) on $X_t^2 = \cdots \Rightarrow$ GARCH ?

$\hat{X_t^2} = \hat{r}_0 + \hat{r}_1 X_{t-1}^2 + \hat{r}_2 v_{t-1}$. $\hat{r}_0 = \hat{\alpha}_0$, $\hat{r}_1 = \hat{\alpha}_1 + \hat{\beta}_1$, $\hat{r}_2 = -\hat{\beta}_1 \Rightarrow \begin{cases} X_t = \sqrt{h_t} z_t \\ h_t = \hat{\alpha}_0 + \hat{\alpha}_1 X_{t-1}^2 + \hat{\beta}_1 h_{t-1} \\ z_t \sim i.i.d.(0,1) \end{cases}$

Remarks:

(1) For GARCH(1,1), to ensure $h_t > 0$, we need $\omega > 0, \gamma \geq 0$, which are necessary and sufficient for the stationary of the GARCH(1,1) model.

$E(h_t) = \omega + \beta E(h_{t-1}) + \gamma E(X_{t-1}^2) \Rightarrow \sigma^2 = \omega + \beta\sigma^2 + \gamma\sigma^2$

$\Rightarrow \sigma^2 = \dfrac{\omega}{1 - (\beta + \gamma)}$

Remarks:

Therefore, to ensure weak stationarity, we need the following conditions for an GARCH(1,1) model,

$$
\begin{cases}
\omega > 0, \\
\beta \geq 0, \\
\gamma \geq 0, \\
\beta + \gamma < 1.
\end{cases} \tag{43}
$$

Remark: An alternative interpretation of an GARCH(1,1) model- an ARMA(1,1) representation in $\{X_t^2\}$:

Put $\nu_t = X_t^2 - h_t$, which is a MDS given $E(\nu_t | I_{t-1}) = h_t - h_t = 0$. Then

$$
\begin{aligned}
X_t^2 &= h_t + \nu_t \\
&= \omega + \beta h_{t-1} + \gamma X_{t-1}^2 + \nu_t \\
&= \omega + (\beta + \gamma) X_{t-1}^2 + \beta (h_{t-1} - X_{t-1}^2) + \nu_t \\
&= \omega + (\beta + \gamma) X_{t-1}^2 - \beta \nu_{t-1} + \nu_t.
\end{aligned} \tag{44}
$$