Contents

Stochastic Calculus Notes

Cliff

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1 Option Basics

The bound for a European Option.

$$S \geq c(S, K) \geq \max(S - KP(0,T), 0)$$

Proof The upper bound is simple, since$S > C_\infty(S,K) > c_\infty(S,K) > c(S,K)$. The lower bound is because $c(S,X) + KP(t,T) > S$ and option value is non-negative. $X = K$ and $B(\tau) = P(0,T)$

1.1 Convexity Properties

Write $X_2 = \lambda X_3 + (1-\lambda)X_1$, where $0 \leq \lambda \leq 1, X_1 \leq X_2 \leq X_3$, the convexity

$$c(S, \tau; X_2) \leq \lambda c(S, \tau; X_3) + (1-\lambda) c(S, \tau; X_1)$$

$c(S,\tau;X)$ is a decreasing function of X; furthermore, $\left|\frac{\partial c}{\partial X}\right| < B(\tau)$

2 Black-Sholes Model

Define the underlying $S(t)$ as

$$\frac{dS(t)}{S(t)} = \mu_s(t)dt + \sigma_s(t)dW(t)$$

Define the discount factor $D(t)$ as

$$\frac{dD(t)}{D(t)} = -R(t)dt$$

$$\begin{aligned}
\frac{d(D(t)S(t))}{D(t)S(t)} &= \frac{dD(t)}{dD(t)} + \frac{dS(t)}{dS(t)} + (\frac{dD(t)}{dD(t)})(\frac{dS(t)}{dS(t)}) \\
&= (\mu_s(t) - R(t))dt + \sigma_s(t)dW(t) \\
&= \sigma_s(t)[\sigma_s(t)^{-1}(\mu_s(t) - R(t))dt + dW(t)] \\
&= \sigma_s(t)[\Theta(t)dt + dW(t)] \\
&= \sigma_s(t)d\tilde{W}(t)
\end{aligned}$$

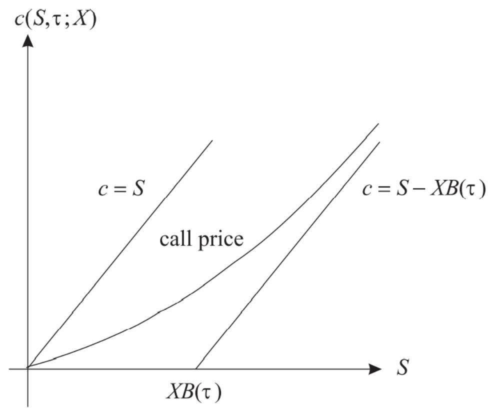

Figure 1: Option Price Bound

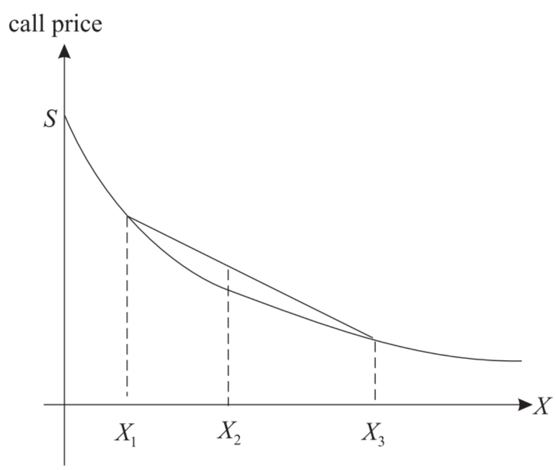

Figure 2: Convexity

	American call	American put
time value of strike, X	loss	gain
dividend, D_{total}	gain	loss
insurance value associated with holding of the option	loss	loss
Necessary condition for early exercise	$D_{\text{total}} > X[1-B(\tau)]$	$D_{\text{total}} < X[1-B(\tau)]$
Sufficiently deep-in-the-money	Sufficiently high asset price	Sufficiently low asset price

Figure 3: American Options

The last equality is held because Girsanov's Theorem.

Theorem 1 (Radon-Nikodym derivatives)**.** *Consider a random variable X under two different probability measures $\mathcal{P}$ and $\tilde{\mathcal{P}}$, we define symbolically*

$$\begin{aligned} \mathrm{d}P_X(x) = P\left[X \in \left(x - \tfrac{\mathrm{d}x}{2}, x + \tfrac{\mathrm{d}x}{2}\right)\right] = f_X^P(x)\mathrm{d}x \\ \mathrm{d}\widetilde{P}_X(x) = \widetilde{P}\left[X \in \left(x - \tfrac{\mathrm{d}x}{2}, x + \tfrac{\mathrm{d}x}{2}\right)\right] = f_X^{\widetilde{P}}(x)\mathrm{d}x \end{aligned}$$

The expectation calculations of X under $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are related by

$$\begin{aligned} E_{\widetilde{P}}[X] &= \int x \,\mathrm{d}\widetilde{P}_X(x) = \int x f_X^{\widetilde{P}}(x)\mathrm{d}x = \int x \left[\frac{f_X^{\widetilde{P}}(x)}{f_X^P(x)}\right] f_X^P(x)\mathrm{d}x \\ &= \int x \left[\frac{f_X^{\widetilde{P}}(x)}{f_X^P(x)}\right] \mathrm{d}P_X(x) = E_P\left[X \frac{\mathrm{d}\widetilde{P}}{\mathrm{d}P}\right] \end{aligned}$$

where

$$f_X^{\widetilde{P}}(x)/f_X^P(x) = \frac{\mathrm{d}\widetilde{P}_X(x)}{\mathrm{d}P_X(x)}$$

is the likelihood ratio of the density functions of X under $\mathcal{P}$ and $\tilde{\mathcal{P}}$. It is coined as the Radon-Nikodym derivative.

Theorem 2 (Girsanov, One dimension)**.** *In a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $W(t)$ be a Brownian Motion and $\Theta(t)$ be an adapted process with filtration $\mathcal{F}_t$. Define $\tilde{\mathbb{P}}_A = \int_A Z d\mathbb{P}$,*

$$Z(t) = \exp\{\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_t^\infty \Theta^2(u)du\}$$

with $\mathbb{E}(\int_t^\infty \Theta^2(u)du) < \infty$ Then

$$\int d\tilde{W}(t) = \int (\Theta(t)dt + dW(t))$$

is a Brownian Motion under measure $\tilde{\mathbb{P}}$.

Define the capital amount as $X(t)$

$$\begin{aligned}dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt \\ &= R(t)X(t)dt + \Delta(t)(\mu_s(t) - R(t))S(t)dt + \Delta(t)\sigma_s(t)S(t)dW(t) \\ &= R(t)X(t)dt + \Delta(t)\sigma_s(t)[\Theta(t)dt + dW(t)]S(t)\end{aligned}$$

and

$$d(D(t)X(t)) = \Delta(t)\sigma_s(t)S(t)[\Theta(t)dt + dW(t)]$$

Change to measure $\tilde{P}$ using Girsanov Theorem

$$d(D(t)X(t)) = \sigma_s(t)\Delta(t)S(t)d\tilde{W}(t)$$

Then using $V(t)$ denotes the payoff of derivatives, with the condition $X(0) = V(0)$ and $X(T) = V(T)$

$$D(t)X(t) = \tilde{\mathbb{E}}[D(T)X(T)|\mathcal{F}_t] = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t]$$

$$D(t)V(t) = \tilde{\mathbb{E}}[D(T)V(T)|\mathcal{F}_t]$$

In conclusion,

$$V(t) = \tilde{\mathbb{E}}[D(t)^{-1}D(t)V(T)|\mathcal{F}_t] = \tilde{\mathbb{E}}[e^{\int_t^T -R(t)dt}V(T)|\mathcal{F}_t]$$

and the last equation is also called *risk-neutral pricing formula* in continuous-time model.Note that the above formula is the induction of Fundamental Asset Pricing Formula using $B(t) = e^{\int_0^t R(t)dt}$ as the numeraire.

$$V(t) = B(t)\mathbb{E}^B\left[\frac{V(T)}{B(T)}\right]$$

And we compute the value of derivatives with payoff $(S(T) - K)^+$, $c(x, t)$ where $x = S(t)$.

$$c(x, t) = c(S(t), t) = \tilde{\mathbb{E}}[e^{\int_t^T -R(t)dt}(S(T) - K)^+|\mathcal{F}_t]$$

Let $R(t) = r$, $\sigma_s(t) = \sigma_s$

$$\begin{aligned}c(x, t) = c(S(t), t) &= \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T) - K)^+|\mathcal{F}_t] \\ &= e^{-r(T-t)}\tilde{\mathbb{E}}[(S(T) - K)^+|\mathcal{F}_t] \\ &= e^{-r(T-t)}\tilde{\mathbb{E}}[(S(t)\exp\{\sigma(\tilde{W}(T) - \tilde{W}(t)) + (r - \frac{1}{2}\sigma^2)(T - t)\} - K)^+|\mathcal{F}_t]\end{aligned}$$

Then we have $Y(\tau) = \tilde{W}(T) - \tilde{W}(t)$ as a normal random variable.

$$
\begin{aligned}
c(x,t) &= e^{-r(T-t)}\tilde{\mathbb{E}}[(S(t)\exp\{\sigma Y(\tau) + (r - \frac{1}{2}\sigma^2)\tau\} - K)^+|\mathcal{F}_t] \\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x,t)} (S(t)\exp\{-\sigma\sqrt{\tau}y + (r - \frac{1}{2}\sigma^2)\tau\} - K)e^{-\frac{y^2}{2}} \quad dy \\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x,t)} S(t)e^{\{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau\}}e^{-\frac{y^2}{2}} \quad dy \\
&- e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x,t)} Ke^{-\frac{y^2}{2}} \quad dy \\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(x,t)} S(t)e^{\{-\sigma\sqrt{\tau}y+(r-\frac{1}{2}\sigma^2)\tau\}}e^{-\frac{y^2}{2}} \quad dy - e^{-r\tau}KN(d_-(x,t)) \\
&= \int_{-\infty}^{d_-(x,t)} S(t)e^{-\frac{1}{2}(\sigma\sqrt{\tau}-y)^2} \quad dy - e^{-r\tau}KN(d_-(x,t)) \\
&= S(t)N(d_+(x,t)) - e^{-r\tau}KN(d_-(x,t))
\end{aligned}
$$

where $d_+(x,t) = \frac{1}{\sigma\sqrt{\tau}}[\log\frac{x}{K} + (r + \frac{1}{2}\sigma_s^2)\tau]$ and $d_-(x,t) = \frac{1}{\sigma\sqrt{\tau}}[\log\frac{x}{K} + (r - \frac{1}{2}\sigma_s^2)\tau]$.

In conclusion, $BSM(S(t), K, \tau, r, \sigma_s) = S(t)N(d_+(S(t),t)) - e^{-r\tau}KN(d_-(S(t),t))$

Note*: Risk Neutral Pricing

For any non-dividend-paying assets X(t),

$$\frac{dX(t)}{X(t)} = \alpha(t)dt + \sigma(t)dW(t)$$

In risk neutral measure, $d\tilde{W}(t) = \Theta(t)dt + dW(t)$, and $\frac{dX(t)}{X(t)} = (\alpha(t) - \sigma(t)\Theta(t))dt + \sigma(t)d\tilde{W}(t)$, where $\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}$. Substitute into the equation $\frac{dX(t)}{X(t)} = R(t)dt + \sigma(t)d\tilde{W}(t)$.

2.1 Currency Options

Its payoff is $(X(T) - K)^+$, using $BSM(x, K, \tau, r, q, \sigma_s)$, we have

$$BSM(X(t), K, \tau, r, r_f, \sigma_x)$$

2.2 Options on Foreign Assets Struck in Foreign Currency

Its payoff is $(X(T)S(T) - K)^+$, using $BSM(x, K, \tau, r, q, \sigma_s)$, we have

$$BSM(X(T)S(T), K, \tau, r, q, \sigma_{X(t)S(t)}) \text{ with } \sigma_{X(t)S(t)} = \sqrt{\sigma_x^2 + \sigma_s^2 + 2\rho\sigma_x\sigma_s}$$

Since

$$\begin{aligned}
\sigma^2_{X(t)S(t)}dt &= (\frac{d(X(t)S(t))}{X(t)S(t)})(\frac{d(X(t)S(t))}{X(t)S(t)}) \\
&= (\frac{dX(t)}{X(t)} + \frac{dS(t)}{S(t)} + \frac{dX(t)}{X(t)}\frac{dS(t)}{S(t)})(\frac{dX(t)}{X(t)} + \frac{dS(t)}{S(t)} + \frac{dX(t)}{X(t)}\frac{dS(t)}{S(t)}) \\
&= \{(\mu_x + \mu_s + \rho_{x,s}\sigma_x\sigma_s)dt + \sigma_x dW_x(t) + \sigma_s dW_s(t)\} \\
&\quad \{(\mu_x + \mu_s + \rho_{x,s}\sigma_x\sigma_s)dt + \sigma_x dW_x(t) + \sigma_s dW_s(t)\} \\
&= [\sigma_x dW_x(t) + \sigma_s dW_s(t)]\,[\sigma_x dW_x(t) + \sigma_s dW_s(t)] \\
&= (\sigma_x^2 + \sigma_s^2 + 2\rho_{x,s}\sigma_x\sigma_s)dt
\end{aligned}$$

2.3 Quantos

Before compute the price of the quantos, we introduce a risky numeriare $Y(t)$.

$$\text{Let } Q(t) = \frac{V(t)}{Y(t)}, \text{ where } V(t) \text{ is the underlying portfolio}$$

$$V(t) = e^{\int_{-\infty}^{t} q du} S(t)$$

$$\begin{aligned}
\frac{dQ(t)}{Q(t)} &= \frac{dV(t)}{V(t)} - \frac{dY(t)}{Y(t)} - (\frac{dV(t)}{V(t)})(\frac{dY(t)}{Y(t)}) + (\frac{dY(t)}{Y(t)})^2 \\
&= (q - \rho\sigma_s\sigma_y + \sigma_y^2)dt + \frac{dS}{S} - \frac{dY}{Y}
\end{aligned}$$

and consider dY/Y when Y as numeraire

$$\frac{d(R/Y)}{R/Y} = (r + \sigma_y^2)dt - \frac{dY}{Y}$$

that

$$\frac{dY}{Y} = (r + \sigma_y^2)dt + \sigma_y dW^*$$

substitute the above equation into dQ/Q, we have

$$\begin{aligned}
\frac{dQ(t)}{Q(t)} &= (q - \rho\sigma_s\sigma_y + \sigma_y^2)dt + \frac{dS}{S} - (r + \sigma_y^2)dt - \sigma_y dW^* \\
&= (q - r - \rho\sigma_s\sigma_y)dt + \frac{dS}{S} - \sigma_y dW^* \\
\frac{dS}{S} &= (r - q + \rho\sigma_s\sigma_y)dt + \sigma_s dW^*
\end{aligned}$$

Its payoff is $\bar{X}S(T)$ and we select $Z(t) = X(t)e^{qt}S(t)$ as numeraire.

Through Fundamental Pricing Formula, $\bar{X}S(0) = Z(0)\mathbb{E}^Z\left[\frac{\bar{X}S(T)}{Z(T)}\right]$ with $Z(t) = X(t)e^{qt}S(t)$.

$$\begin{aligned} Z(0)\mathbb{E}^Z\left[\frac{\bar{X}S(T)}{Z(T)}\right] &= X(0)S(0)e^{-qT}\mathbb{E}^Z\left[\frac{\bar{X}S(T)}{X(T)S(T)}\right] \\ &= \bar{X}S(0)e^{-qT}\mathbb{E}^Z\left[\frac{X(0)}{X(T)}\right] \end{aligned}$$

Now, we compute dX/X under numeraire Z through the above formula,

(i) Compute the correlation between X and Z.

$$\begin{aligned} \frac{dZ}{Z} &= qdt + \frac{d(XS)}{XS} \\ &= qdt + \frac{dX}{X} + \frac{dS}{S} + \frac{dX}{X}\frac{dS}{S} \\ &= (q + \mu_s + \mu_x + \rho_{x,s}\sigma_x\sigma_s)dt + \sigma_x dW_x + \sigma_s dW_s \\ &= (q + \mu_s + \mu_x + \rho_{x,s}\sigma_x\sigma_s)dt + \sigma(\frac{\sigma_x}{\sigma}dW_x + \frac{\sigma_s}{\sigma}dW_s) \end{aligned}$$

$$dW = (\frac{\sigma_x}{\sigma}dW_x + \frac{\sigma_s}{\sigma}dW_s)$$

The correlation is $\rho dt = (dW)(dW_x) = \frac{\sigma_x + \rho_{x,s}\sigma_s}{\sigma}dt$.

We substitute $r = r, q = r_f, \rho = \frac{\sigma_x + \rho_{x,s}\sigma_s}{\sigma}$ and $\sigma_s = \sigma_x$,

$$\frac{dX}{X} = (r - r_f + \sigma_x^2 + \rho_{x,s}\sigma_x\sigma_s)dt + \sigma_x dW_x^*$$

and

$$\frac{d(1/X)}{1/X} = -\frac{dX}{X} + (\frac{dX}{X})^2 = (r_f - r + \rho_{x,s}\sigma_x\sigma_s)dt - \sigma_x dW_x^*$$

is equivalent as

$$\frac{d(1/X)}{1/X} = -\frac{dX}{X} + (\frac{dX}{X})^2 = (r_f - r + \rho_{x,s}\sigma_x\sigma_s)dt + \sigma_x dW_x^*$$

under numeraire $Z(t)$.

$$\frac{1}{X(T)} = \exp\{\int_{-\infty}^{T}(r_f - r + \rho_{x,s}\sigma_x\sigma_s - \frac{1}{2}\sigma_x^2)dt + \int_{-\infty}^{T}\sigma_x dW_x^*\}$$

$$\begin{aligned} \mathbb{E}\left[\frac{X(0)}{X(T)}\right] &= \mathbb{E}\left[\exp\{\int_{-\infty}^{T}(r_f - r + \rho_{x,s}\sigma_x\sigma_s - \frac{1}{2}\sigma_x^2)dt + \int_{-\infty}^{T}\sigma_x dW_x^*\}\right] \\ &= \exp\{(r_f - r + \rho_{x,s}\sigma_x\sigma_s)T\} \end{aligned}$$

The quanto price at $t = 0$ is $\bar{X}S(0)\exp\{(r_f - r - q + \rho_{x,s}\sigma_x\sigma_s)T\}$.

2.4 Quanto Forwards

Let the forward price $F^*(t)$, and its payoff is $\bar{X}S(T) - F^*(t)$. The quanto forward price $F^*(t)$ is

$$F^*(t) = e^{r(T-t)}V(t) = \exp\{(r_f - q - \rho\sigma_x\sigma_s)(T-t)\}\bar{X}S(t)$$

2.5 Quanto Options

Let $V(T) = \bar{X}S(T)$ and $\sigma_{\bar{X}S(T)} = \sigma_s$. The value of a quanto call is

$$\begin{aligned}
&V(0)N(d_1) - e^{-rT}KN(d_2) \\
&\quad = \bar{X}S(0)\exp\{(r_f - r - q + \rho_{x,s}\sigma_x\sigma_s)T\}N(d_1) - e^{-rT}KN(d_2)
\end{aligned}$$

where

$$d_1 = \frac{\log\left(\frac{V(0)}{K}\right) + (r + \frac{1}{2}\sigma_s^2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma_s\sqrt{T}$$

Likewise, the value of a quanto put is given by the BS formula,

$$e^{-rT}KN(-d_2) - V(0)N(-d_1)$$

2.6 Return Swaps

The fair swap spread, which equates the value at date 0 of receiving the cash flow

$$CF(T) = \left(a + \frac{S_f(T)}{S_f(0)} - \frac{S_d(T)}{S_d(0)}\right)$$

at date T to zero, is

$$a = \exp\{(r - q_d)T\} - \exp\{(r_f - q_f - \rho\sigma_x\sigma_s)\}$$

Sketch of Proof: Since the value of receiving $\frac{S_d(0)}{S_d(0)}$ at date 0 is $\dfrac{e^{-qTS(0)}}{S(0)} = e^{-q_dT}$. And if we think $\frac{1}{S(0)}$ is a fixed exchange rate $\bar{X}$ in quanto, we can substitute the exchange rate volatility σ_x, foreign underlying volatility σ_s and their correlation ρ into quanto price formula at date 0.

$$\bar{X}S(0)\exp\{(r_f - r - q + \rho\sigma_x\sigma_s)T\}$$

In conclusion,

$$\mathbb{E}_0\left[CF(T)\right] = e^{-rT}a - e^{-q_dT} + \bar{X}S(0)\exp\{(r_f - r - q + \rho\sigma_x\sigma_s)T\} = 0$$

That

$$a = \exp\{(r - q_d)T\} - \exp\{(r_f - q_f - \rho\sigma_x\sigma_s)\}$$

2.7 Uncovered Interest Parity in the Risk-Neutral Probabilities

If we regard the foreign exchange rate, X, as an asset with dividends rate r_f, referring underlying dynamics under the risk-neutral measure, we have

$$\frac{dX}{X} = (r - r_f)dt + \sigma_x dB_x^*$$

It shows the no-risk return of different foreign currencies will converge. That we cannot earn profit by cross-currency arbitrage theoretically.

3 Black-Sholes Extension

3.1 Margrabe's Formula

Consider a payoff $\max(S_1(T) - S_2(T), 0)$, its price can be evaluated by BSM through thinking $\max(S_1(T) - S_2(T), 0) = \max(\frac{S_1(T)}{S_2(T)} - 1, 0)$ and $S_2(t)$ is a FX rate.

Assume $\frac{dS_i}{S_i} = \mu_i dt + \sigma_i dW(t)$, we have

$$e^{-q_1T}S_1(0)N(d_+) - e^{-q_2T}S_2(0)N(d_-)$$

where $d_+ = \frac{\ln(S_1/S_2(0)) + (q_2 - q_1 + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$

3.2 Black's Formula

Black's Model is used to price the option on forward contracts, and futures contracts with deterministic interest rates.

Consider a forward contract mature at T', $T' > T$, its call option has payoff $\max(F(T) - K, 0)$ at T.

Assume $\frac{dF}{F} = \mu dt + \sigma dW(t)$ and interest rate is random which can be written as $P(t,T) = e^{\int_t^T -R(u)du}$. Its payoff at T is $\max(P(T,T')F(T) - P(T,T')K, 0)$. Recall Margrabe's Formula, let $S_1(T) = P(T,T')F(T)$ and $S_2(T) = P(T,T')K$

$$P(0,T')F(0)N(d_+) - P(0,T')KN(d_-)$$

where $d_+ = \frac{\ln(F(0)/K) + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$ and σ is the volatility of $\frac{F}{K}$.

Put-Call Parity: $C + P(0.T')K = P + P(0,T')F(0)$.

3.3 Merton's Formula

The Merton's Formula is similar to the Black-Sholes Formula, consider a payoff $\max(F(T) - K, 0)$ and assume the interest rate is random,

$$\frac{dF}{F} = \mu dt + \sigma dW(t)$$

$$\text{Call Price} = e^{-qT}S(0)N(d_+) - P(0,T)KN(d_-)$$
$$\text{Put Price} = P(0,T)KN(-d_-) - e^{-qT}S(0)N(-d_+)$$

where $d_+ = \dfrac{\log\left(\dfrac{S(0)}{KP(0,T)}\right) - qT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.

Attention: The σ is forward volatility and we can estimate the forward volatility through underlying volatility and interest rate volatility.

$$\frac{dS}{S} = \mu_s dt + \sigma_s dB_s,$$
$$\frac{dP}{P} = \mu_p dt + \sigma_p dB_p, F(t) \quad = e^{-q(T-t)}S(t)/P(t,T)$$

The forward volatility is $\sigma = \sqrt{\sigma_s^2 + \sigma_p^2 - 2\rho\sigma_s\sigma_p}$.

3.4 Deferred Exchange Options

Consider an option maturity at T and will exchange two assets at T' > T. So , its payoff is $\max(P(0,T')F_1(0) - P(0,T')F_2(0), 0)$. Using Margrabe's formula, and let

$$S_1^*(t) = P(t,T')F_1(t)$$
$$S_2^*(t) = P(t,T')F_2(t)$$

The option price is $S_1(0)e^{-q_1T'}N(d_+) - S_2(0)e^{-q_2T'}N(d_-)$, where $d_+ = \dfrac{\log(\dfrac{S_1(0)}{S_2(0)}) + (q_2 - q_1)T' + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.

Note: the forward price is $F(t) = \dfrac{e^{-qT}S(t)}{P(t,T)}$.

3.5 Generic Option

$$PV_1N(d_+) - PV_2N(d_-), \quad d_+ = \frac{\log\dfrac{PV_1}{PV_2} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$$

Time	Profits from futures	Bond position	Net position
0	—	$G_{0,3}$	$G_{0,3}$
1	$\frac{1}{B_{0,1}}(G_{1,3}-G_{0,3})$	$\frac{G_{0,3}}{B_{0,1}}$	$\frac{G_{1,3}}{B_{0,1}}$
2	$\frac{1}{B_{0,1}B_{1,2}}(G_{2,3}-G_{1,3})$	$\frac{G_{1,3}}{B_{0,1}B_{1,2}}$	$\frac{G_{2,3}}{B_{0,1}B_{1,2}}$
3	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}(G_{3,3}-G_{2,3})$	$\frac{G_{2,3}}{B_{0,1}B_{1,2}B_{2,3}}$	$\frac{G_{3,3}}{B_{0,1}B_{1,2}B_{2,3}}=\frac{S_3}{B_{0,1}B_{1,2}B_{2,3}}$

Figure 4: Dynamic Hedging Strategy of Futures

3.6 Dynamic Hedging Strategy of Futures

Consider the futures matured at t_N has price $G(t_i, t_N)_{i=1,2,3...,n}$, if we want long 1 unit of futures from t_0, we will have the following strategy.

3.7 The relation of Futures Prices to Forward Prices

There are three facts:

(1) The futures price is a martingale under risk-neutral measure.
(2) The forward price is a martingale under Zero-Coupon Bonds as a numeriaire.
(3) When interest rate is non-random, the futures price is equal to the forward price.

Prof (1). Consider a portfolio, long a futures with reinvesting and withdrawing from a money account with risk-free rate r.

$$dV = dF^* + rV dt$$

Then

$$\begin{aligned}\frac{d(V/R)}{V/R} &= \frac{dV}{V} - \frac{dR}{R}\\ &= \frac{dF^*}{V}\end{aligned}$$

the drift of $\dfrac{dF^*}{V}$ is zero implies the drifts of dF^* is zero.

Prof (2). Recall the forward price $F(t) = \dfrac{e^{-qT}S(t)}{P(t,T)}$, $P(t,T)F(t)$ is a non-dividend paying assets, so $\dfrac{P(t,T)F(t)}{P(t,T)} = F(t)$ is a martingale.

Prof(3) Assume interest rate is non random,

$$\mathbb{P}_A^R = \mathbb{E}^P(1_A\phi(T)\frac{P(T,T)}{P(0,T)}) = \exp(\int_0^T r(u)du)\mathbb{E}^P(1_A\phi(T))$$

$$\mathbb{P}_A^R = \mathbb{E}^R(1_A\phi(T)\frac{R(T)}{R(0)}) = \exp(\int_0^T r(u)du)\mathbb{E}^R(1_A\phi(T))$$

So,

$$F^*(t) = \mathbb{E}^R(F^*(T)) = \mathbb{E}^P(F^*(T)) = \mathbb{E}^P(F(T)) = \mathbb{E}^R(F(T)) = F(t)$$

3.7.1 Extension. (1)

The difference between futures and forward is

$$\begin{aligned} F^*(t) - F(t) &= \mathbb{E}^R[S_T] - \frac{S_t}{P(t,T)} \\ &= \frac{\mathbb{E}^R[S_T]\mathbb{E}^R[P(t,T)] - \mathbb{E}^R[P(t,T)S_T]}{P(t,T)} \\ &= -\frac{cov[P(t,T),S_T]}{P(t,T)} \end{aligned}$$

3.8 Futures Option

The difference between forward option and futures option is that futures option is mark to market. So, the payoff of futures option in Margarabe's formula is

$$S_1 = P(0,T)F^*(0) \quad and \quad S_2 = P(0,T)K$$

Substitute into the Margrabe's formula, the **call option** price is

$$P(0,T)F^*(0)N(d_+) - P(0,T)KN(d_-)$$

$$\text{where } d_+ = \frac{\log(\frac{F^*(0)}{K}) + \frac{1}{2}\sigma^2}{\sigma\sqrt{T}}.$$

3.9 Hedging with Forward and Futures

Let $t < u$, consider the portfolio purchase $x(t)$ forward at $F(t)$ and sell $x(t)$ forward at u with $F(u)$. Since entering a forward contract is free that the PV at time u of portfolio is

$$\begin{aligned} x(t)P(u,T)(F(u) - F(t)) &= x(t)\left[P(t,T)\left[F(u) - F(t)\right] + \left[P(u,T) - P(t,T)\right]\left[F(u) - F(t)\right]\right] \\ &= x(t)\left[P(t,T)\Delta F + (\Delta P)(\Delta F)\right] \end{aligned}$$

which can be written as

$$x(t)\left[P(t,T)dF(t) + dP(t,T) \times dF(t)\right]$$

Change of futures price is simpler

$$x(t)dF^*(t)$$

The value doesn't need to discount since futures is mark to market.

Assume the risk free rate is constant,

Change of Forward

$$x(t)e^{r(T-t)}dF(t)$$

Change of Futures

$$x(t)dF^*(t) \quad , where \quad F^*(t) = F(t)$$

Finally, we conclude if x(t) is the number of forward contracts should be hedged, then $e^{-r(T-t)}x(t)$ is the number of futures should be hedges.

3.10 TBC

Sect. 7.6 & Sect. 7.10

4 Fixed Income

4.1 The yield curve

Define $y(t)$ as the yield at t,

$$P(t,T) = e^{-y(t)(T-t)}$$

Then let $\tau_1 < \tau_2 < ... < \tau_N$,

$$P = \sum_{j=1}^{N} e^{-y(\tau_j)\tau_j} C_j$$

The we can use cubic spline to fit the yield curve

$$y(t) = \begin{cases} a_0t^3 + b_0t^2 + c_0t + d_0 & , \quad 0 < t < t_1 \\ a_1t^3 + b_1t^2 + c_1t + d_1 & , \quad t_1 < t < t_2 \\ a_2t^3 + b_2t^2 + c_2t + d_2 & , \quad t_2 < t < t_3 \\ ... \end{cases}$$

And use the Equality of yields, first order derivatives and second order derivatives to iteratively compute the parameter set.

4.2 Spot Rate, Swap Rate and Forward Rate

Spot Rate

$$\frac{1}{P(t,u)} = 1 + \mathcal{R}(u-t)$$

Swap Rate

Consider the value of swap for fixed payer

$$P(t,t_0) - P(t,t_N) - \Delta t \bar{R} \sum_{i=1}^{N} P(t,t_i)$$

where $P(t,t_0)$ is the cost of receiving \$1 at t_0.

Forward Rate

$$\frac{P(t,u)}{P(t,u+\Delta t)} = 1 + R\Delta t$$

4.3 Duration and Convexity

$$\frac{dP}{P} = -Duration \times dy$$

$$Convexity = \frac{1}{P}\frac{d^2P}{dy^2}$$

4.4 Duration Hedge

Assume the portfolio price equals

$$P = f(t, y(t))$$

$$dP = P_t dt + P_y dy + \frac{1}{2}P_{yy}(dy)^2$$

which can be written as

$$dP = P_t dt - Duration \times P + \frac{1}{2}P_{yy}(dy)^2$$

4.5 Caps and Floors

For a Caps, the buyer's cash flow at each reset date t_i is

$$P(t_i, t_{i+1}) \max(R(t_i) - \bar{R}, 0)$$

For Floors,

$$P(t_i, t_{i+1}) \max(\bar{R} - R(t_i), 0)$$

4.5.1 The Market Model of Caps

We can regard the float leg is an asset, since $R(t_i)\Delta t$ can be replicated by $P(t,t_i) - P(t,t_{i+1})$ at $t < t_i$ and $P(t,t_{i+1})R_i\Delta t$ at $t > t_i$.

$$S_i(t) = \begin{cases} P(t,t_i) - P(t,t_{i+1}), & t < t_i \\ P(t,t_{i+1})R_i\Delta t, & t_i < t < t_{i+1} \end{cases}$$

$$F_i(t) = \begin{cases} \dfrac{P(t,t_i)}{P(t,t_{i+1})} - 1, & t < t_i \\ R_i\Delta t, & t_i < t < t_{i+1} \end{cases}$$

Recall the black's formula,

$$\text{Value of Caps} = P(0,t_{i+1})R_i(0)N(d_1) - P(0,t_{i+1})\bar{R}$$

where $d_1 = \dfrac{\log(R_i(0)/\bar{R}) + \frac{1}{2}\sigma^2 t_i}{\sigma\sqrt{t^i}}$.

4.6 Swaptions

Consider a swap with maturity at t_N,

Fixed leg:

$$Z(t) = \sum_{i=1}^{N} P(t,t_i)\bar{R}\Delta t$$

Floating leg:

$$S(t) = P(t,t_1) - P(t,t_N)$$

Its payoff

$$\max(S(t) - Z(t), 0)$$

Since we can write S(t) under Z(t) as a numeriaire,

$$\frac{S(t)}{Z(t)} = \frac{\mathcal{R}\Delta t\sum_{i=1}^{N} P(t,t_i)}{\sum_{i=1}^{N} P(t,t_i)\bar{R}\Delta t} = \frac{\mathcal{R}}{\bar{R}}$$

and assume swap rate $\mathcal{R}$ has constant volatility σ. The present value of swaptions for a payer is

$$(P(t,t_1 - P(t,t_N)))N(d_+) - \bar{R}\Delta t\sum_{i=1}^{N} P(t,t_i)N(d_-)$$

where $d_+ = \dfrac{\log(P(t,t_1 - P(t,t_N))) - \log(\bar{R}\Delta t\sum_{i=1}^{N} P(t,t_i)) + \frac{1}{2}\sigma^2}{\sigma\sqrt{t_N}}$

A Fundamental Assets Pricing Formula

For arbitrary non-dividends paying assets matured at T, we have its price

$$V(t) = \mathbb{E}[V(T)|\mathcal{F}_t]$$

and we know that if use risk-free asset as a numeraire, $R(t) = e^{-\int_t^T rds}$, the ratio of V and R is a martingale under risk neutral measure,

$$\mathbb{E}^R[\frac{V(T)}{R(T)}|\mathcal{F}_t] = \mathbb{E}^R[\frac{V(T)}{R(T)}]$$

So, the asset price is

$$V(t) = \mathbb{E}^R[V(T)|\mathcal{F}_t] = R(t)\mathbb{E}^R[\frac{V(T)}{R(T)}|\mathcal{F}_t] = R(t)\mathbb{E}^R[\frac{V(T)}{R(T)}]$$

B Feyman-Kac Formula

B.1 Simplified Version

Consider a PDE is written as follows

$$\frac{\partial F}{\partial t} + \mu(X,t)\frac{\partial F}{\partial X} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2 F}{\partial X^2} = 0$$

subject to the terminal condition

$$F(X(T),T) = h(X(T))$$

Suppose the Ito process $X(t)$ is governed by the differential equation

$$dX(s) = \mu(X(s),s)ds + \sigma(X(s),s)dZ(s), \quad t \le s \le T$$

with initial condition:$X(t) = x$

Consider a smooth function $F(X(t),t)$, by Ito Lemma,

$$dF = \left[\frac{\partial F}{\partial t} + \mu(X,t)\frac{\partial F}{\partial X} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2 F}{\partial X^2}\right]dt + \sigma\frac{\partial F}{\partial X}dZ$$

Recall the PDE in the beginning,

$$dF = \sigma\frac{\partial F}{\partial X}dZ$$

Combing the termination condition $F(X(T),T) = h(X(T))$,

$$F(x,t) = E_{x,t}[h(X(T))], \quad t < T$$

B.2 General Version

Consider the partial differential equation

$$\frac{\partial u}{\partial t}(x,t)+\mu(x,t)\frac{\partial u}{\partial x}(x,t)+\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)-V(x,t)u(x,t)+f(x,t)=0$$

defined for all $x\in\mathbb{R}$ and $t\in[0,T]$, subject to the terminal condition

$$u(x,T)=\psi(x)$$

The solution of the above PDE is

$$u(x,t)=E^Q\left[\int_t^T e^{-\int_t^T V(X_\tau,\tau)d\tau} f\left(X_r,r\right)dr+e^{-\int_t^T V(X_\tau,\tau)d\tau}\psi\left(X_T\right)\mid X_t=x\right]$$

Proof. Consider a partial differential equation

$$\frac{\partial u}{\partial t}(x,t)+\mu(x,t)\frac{\partial u}{\partial x}(x,t)+\frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2}(x,t)-V(x,t)u(x,t)+f(x,t)=0$$

Assume the solution of this PDE is $u(X_t,t)$, we can construct a process

$$Y(s)=e^{-\int_t^s V(X_\tau,\tau)d\tau}u\left(X_s,s\right)+\int_t^s e^{-\int_t^r V(X_\tau,\tau)d\tau}f\left(X_r,r\right)dr$$

and take the partial differential of $Y(t)$ by Ito Lemma, we have

$$\begin{aligned}dY=&d\left(e^{-\int_t^s V(X_\tau,\tau)d\tau}\right)u\left(X_s,s\right)+e^{-\int_t^s V(X_\tau,\tau)d\tau}du\left(X_s,s\right)\\&+d\left(e^{-\int_t^s V(X_\tau,\tau)d\tau}\right)du\left(X_s,s\right)+d\left(\int_t^s e^{-\int_t^r V(X_\tau,\tau)d\tau}f\left(X_r,r\right)dr\right)\end{aligned}$$

$$\begin{aligned}dY=&e^{-\int_t^s V(X_\tau,\tau)d\tau}\left(-V\left(X_s,s\right)u\left(X_s,s\right)+f\left(X_s,s\right)+\mu\left(X_s,s\right)\frac{\partial u}{\partial X}+\frac{\partial u}{\partial s}+\frac{1}{2}\sigma^2\left(X_s,s\right)\frac{\partial^2 u}{\partial X^2}\right)ds\\&+e^{-\int_t^s V(X_\tau,\tau)d\tau}\sigma(X,s)\frac{\partial u}{\partial X}dW\end{aligned}$$

Then we can substitute the PDE in the beginning to make the terms in parentheses is zero, and get

$$dY=e^{-\int_t^s V(X_\tau,\tau)d\tau}\sigma(X,s)\frac{\partial u}{\partial X}dW$$

□