Contents

Stochastic Volatility Modelling

Cliff

March 5, 2021

1 Introduction

1.1 Characterizing a usable model-the Black-Sholes equation

In trading floor it's really important to get a correct pricing, but the question is that how to evaluate a valid model. There are several standards,

- Consistent with the terminal payoff: $f(T, S(T)) = P(T, S(T)), \forall S$.

- Able to do risk-neutral hedging.

Let's consider a scenario of shorting options through delta-hedging,

$$P\&L = \underbrace{-[P(t+\delta t, S+\delta S) - P(t, S)]}_{\text{PnL from shorting option}} + \underbrace{rP(t, S)\delta t}_{\text{interest accrued from money account}} - \underbrace{\Delta[(r-q)S\delta t - \delta S]}_{\text{PnL from hedging}} \tag{1}$$

and write as differential form and using $\Delta = \frac{\partial P}{\partial S}$

$$\begin{aligned} P\&L &= \frac{\partial P}{\partial t}dt - \frac{\partial P}{\partial S}dS - \frac{1}{2}S^2\frac{\partial^2 P}{\partial S^2}(\frac{dS}{S})^2 + rPdt - \frac{\partial P}{\partial S}[(r-q)Sdt - dS] \\ &= -\underbrace{[\frac{\partial P}{\partial t} + (r-q)S\frac{\partial P}{\partial S} - rP]dt}_{\text{theta portion}} - \underbrace{\frac{1}{2}S^2\frac{\partial^2 P}{\partial S^2}(\frac{dS}{S})^2}_{\text{gamma portion}} \end{aligned} \tag{2}$$

1.1.1 Single Asset Case

Our daily PnL reads,

$$P\&L = -A(t, S)\delta t - B(t, S)(\frac{\delta S}{S})^2 \tag{3}$$

In order to be risk-neutral, there should be some constraints on $A(t, S)$ and $B(t, S)$. First, A and B cannot be both positive or negative. Second, they should obey an equation: $\frac{\delta S}{S} = \pm\sqrt{-\frac{A(t,S)}{B(t,S)}}\sqrt{\delta t}$. If we assume underlying S satisfies a lognormal distribution, and we could have $\left\langle \left(\frac{\delta S}{S}\right)^2 \right\rangle = \hat{\sigma}^2\delta t$, where $\hat{\sigma}$ is the average of historical volatility.

Then we could get an approximation of $A(t, S)$ through $A(t, S) = -\hat{\sigma}^2 B(t, S)$, and substitute it into equation (3),

$$P\&L = -\frac{S^2}{2}\frac{\partial^2 P_{\hat{\sigma}}}{\partial S^2}\left(\frac{\delta S^2}{S^2} - \hat{\sigma}^2\delta t\right) \tag{4}$$

1.1.2 Multiple Assets Case

$$P\&L = -A(t, S)\delta t - \frac{1}{2}\sum_{ij}\phi_{ij}(t, S)\frac{\delta S_i}{S_i}\frac{\delta S_j}{S_j} \tag{5}$$

where $\phi_{ij}(t, S) = S_i S_j \dfrac{d^2 P}{dS_i dS_j}|_{t,S}$ and S denotes the vector of S_i. Using eigenvalue decomposition,

$$\phi = T\varphi T^T \tag{6}$$

furthermore,

$$\begin{gathered}\varphi = \{\frac{d^2 P}{dS_i dS_j}\}_{i,j\in\{0,..,n\}} \\ and \\ T = (S_i)_{i\in\{0,\ldots,n\}}\end{gathered} \tag{7}$$

The gamma portion of our $P\&L$ can be written as

$$\sum_{ij}\phi_{ij}\frac{\delta S_i}{S_i}\frac{dS_j}{S_j} = U^T\phi U = U^T T\varphi T^T U = (T^T U)^T\varphi(T^T U) \tag{8}$$

where $U = \{\frac{\delta S_i}{S_i}\}$ and $\delta z_k = T_k^T U$. our PnL reads:

$$P\&L = -A\delta t - \frac{1}{2}\sum_k \varphi_k \delta z_k^2 \tag{9}$$

As in mono-asset case, the condition for our model to be usable is

$$A = -\frac{1}{2}\sum_k \varphi_k w_k \tag{10}$$

so our PnL reads:

$$P\&L = -\frac{1}{2}\sum_k \varphi_k(\delta z_k^2 - w_k\delta t) \tag{11}$$

Let us express A differently, so as to give our P&L a more symmetric form.

$$A = -\frac{1}{2}\sum_k \varphi_k\omega_k = -\frac{1}{2}tr(\varphi\omega) = -\frac{1}{2}tr(T^T\phi T\omega) = -\frac{1}{2}tr(\phi T\omega T^T) = -\frac{1}{2}tr(\phi C) = -\frac{1}{2}\sum_{ij}\phi_{ij}C_{ij} \tag{12}$$

where $C = T\omega T^T$ is a positive matrix by construction,

$$A = -\frac{1}{2}\sum_k \; phi_k(\frac{\delta S_i}{S_i}\frac{dS_j}{S_j} - C_{ij}\delta t) \tag{13}$$

Because C is a positive matrix, it can be interpreted as an (implied) covariance matrix.

1.1.3 Conclusion

In the general case of multiple hedge instruments, the condition that our model is usable - no situation in which our carry P&L is systematically positive or negative- is that there exists a positive break-even covariance matrix $C(t,S), \forall S, \forall t$.

In addition, only suitable model can be used in the trading purposes, which called market model.

1.2 How (in)effective is delta hedging

The attitude of this section is to verify the average and standard deviation of P&L incurred over option's life.
We start from the Black-Sholes assumption:

- The underlying follows a lognormal process with constant volatility σ same as implied volatility in option pricing and risk management.

- The hedging interval is approaching to zero, $\delta \to 0$.

So, the sum of P&Ls vanishes with probability one.

However, in real-time life, the analysis should be more complex. Let's reduce the restrictions step by step.

Step 1

The dollar gamma weighted implied volatility equals future realized volatility over the option's life.

$$\left\langle \int_0^T e^{-rt}S^2\frac{d^2P_{\hat{\sigma}}}{dS^2}\sigma_t^2 dt \right\rangle = \left\langle \int_0^T e^{-rt}S^2\frac{d^2P_{\hat{\sigma}}}{dS^2}\hat{\sigma}^2 dt \right\rangle \tag{14}$$

with the above condition holds, our final P&L is not biased on average and let us concentrated on dispersion.

Step 2

Assume that the option is delta-hedging daily at times $t_i : \delta t = 1$ day. Recall (32):

$$P\&L = -\sum_i e^{-rt_i} \frac{S_i^2}{2} \frac{d^2 P_{\hat{\sigma}}}{dS^2}(t_i, S_i)(r_i^2 - \hat{\sigma}^2 \delta t) \tag{15}$$

where r_i are daily returns, given by $r_i = \dfrac{S_{i+1} - S_i}{S_i}$. It is dependent on the option's payoff since it related with gamma.

Step 3

In addition, we assume option's discounted dollar gamma is constant, and $e^{-rt_i} S^2 \dfrac{d^2 P_{\hat{\sigma}}}{dS^2} =$ initial value: $S_0^2 \dfrac{d^2 P_{\hat{\sigma}}}{dS^2}(t_0, S_0)$. Let's write the daily return r_i as:

$$r_i = \sigma_i \sqrt{\delta t} z_i, \quad \langle z_i \rangle = 0, \quad \langle z_i^2 \rangle = 1 \tag{16}$$

Let's assume z_i is iid and independent with σ_i, our P&L:

$$P\&L = -\sum_i e^{-rt_i} \frac{S_i^2}{2} \frac{d^2 P_{\hat{\sigma}}}{dS^2}(t_i, S_i)(\sigma_i^2 z_i^2 - \hat{\sigma}^2 \delta t) \tag{17}$$

Step 4

Let's assume σ_i is time-homogeneous, so that, σ_i is independent with i and has $\hat{\sigma}^2 = \langle \sigma_i^2 \rangle$. The variance of $\sum_i (\sigma_i^2 z_i^2 - \hat{\sigma}^2)\delta t$ is given by:

$$\begin{aligned}
&\left\langle \textstyle\sum_{ij} \left(\sigma_i^2 z_i^2 - \widehat{\sigma}^2\right) \delta t \left(\sigma_j^2 z_j^2 - \widehat{\sigma}^2\right) \delta t \right\rangle \\
&= \textstyle\sum_i \left(\left\langle \sigma_i^4 z_i^4 \right\rangle + \widehat{\sigma}^4 - 2\widehat{\sigma}^4\right) \delta t^2 + \sum_{i \neq j} \left\langle \sigma_i^2 \sigma_j^2 z_i^2 z_j^2 + \widehat{\sigma}^4 - 2\widehat{\sigma}^4 \right\rangle \delta t^2 \\
&= \textstyle\sum_i (2 + \kappa) \widehat{\sigma}^4 \delta t^2 + \sum_{i \neq j} \left(\left\langle \sigma_i^2 \sigma_j^2 \right\rangle - \widehat{\sigma}^4\right) \delta t^2 \\
&= \textstyle\sum_i (2 + \kappa) \widehat{\sigma}^4 \delta t^2 + \left(\left\langle \sigma^4 \right\rangle - \widehat{\sigma}^4\right) \sum_{i \neq j} f_{ij} \delta t^2 \\
&= \widehat{\sigma}^4 \left(\textstyle\sum_i (2 + \kappa) \delta t^2 + \Omega \sum_{i \neq j} f_{ij} \delta t^2\right)
\end{aligned} \tag{18}$$

where the kurtosis κ and variance correlation function f_{ij} is given by:

$$\kappa = \frac{\left\langle \sigma_i^4 z_i^4 \right\rangle}{\widehat{\sigma}^4} - 3, \quad f_{ij} = \frac{\left\langle \left(\sigma_i^2 - \widehat{\sigma}^2\right) \left(\sigma_j^2 - \widehat{\sigma}^2\right) \right\rangle}{\sqrt{\left\langle \sigma_i^4 \right\rangle - \widehat{\sigma}^4} \sqrt{\left\langle \sigma_j^4 \right\rangle - \widehat{\sigma}^4}} \tag{19}$$

and dimensionless factor,

$$\Omega = \frac{\left\langle \sigma^4 \right\rangle - \widehat{\sigma}^4}{\widehat{\sigma}^4} = \frac{\left\langle \sigma^4 \right\rangle - \left\langle \sigma^2 \right\rangle^2}{\left\langle \sigma^2 \right\rangle^2} \tag{20}$$

In the end, the standard deviation is given by

$$\mathrm{StDev}(P\&L) = \left|\frac{S_0^2}{2}\frac{d^2P_{\widehat{\sigma}}}{dS^2}(t_0, S_0)\right| \sqrt{\widehat{\sigma}^4 \left(\sum_i (2+\kappa)\delta t^2 + \Omega \sum_{i\neq j} f_{ij}\delta t^2\right)} \quad (21)$$

Step 5

We know the vega and gamma has the following relationship according to the Black-Sholes formula

$$\frac{dP_{\widehat{\sigma}}}{d\widehat{\sigma}} = S^2 \frac{d^2P_{\widehat{\sigma}}}{dS^2}\widehat{\sigma}T \quad (22)$$

Using Vega, the standard deviation of PnL is

$$\mathrm{StDev}(P\&L) = \left|\widehat{\sigma}\frac{dP_{\widehat{\sigma}}}{d\widehat{\sigma}}\right| \frac{1}{2T} \sqrt{\sum_i (2+\kappa)\delta t^2 + \Omega \sum_{i\neq j} f_{ij}\delta t^2} \quad (23)$$

1.3 The Black-Scholes case

Let's first assume that S follows the lognormal Black-Scholes dynamics σ_i, is constant and equal to $\hat{\sigma}$. So, $\Omega = 0, \kappa = 0, f = 0$.

$$StDev(P\&L) = \frac{1}{2N}\left|\hat{\sigma}\frac{dP_{\hat{\sigma}}}{d\hat{\sigma}}\right| \quad (24)$$

Note that $\dfrac{\hat{\sigma}}{\sqrt{2N}}$ is approximately the standard deviation of the historical volatility estimator. The historical volatility estimator is given by:

$$\bar{\sigma}^2 = \frac{1}{N\delta t}\sum_i \left(\frac{S_{i+1} - S_i}{S_i}\right)^2 \quad (25)$$

In the Black-Scholes assumption, daily return is assumed as Gaussian distribution.

$$\bar{\sigma}^2 \approxeq \frac{\hat{\sigma}^2}{N}\sum_i z_i^2 \quad (26)$$

And we know $Var(\hat{\sigma}^2) = \dfrac{2\hat{\sigma}^2}{N}$ since

$$\begin{aligned} Var(\hat{\sigma}^2) &= Var\left(\frac{\hat{\sigma}^2}{N}\sum_i z_i^2\right) \\ &= \frac{\hat{\sigma}^4}{N^2}\sum_i Var(z_i^2) \\ &= \frac{\hat{\sigma}^4}{N}\left[E(z_i^4) - E(z_i^2)^2\right] \end{aligned} \quad (27)$$

Then since the moment-generating-function of normal distribution is, $mgf(t) = \exp(\mu t + \frac{\sigma^2}{2})$ and let $\mu = 0$

$$\begin{aligned} mgf^{(1)}(t) &= \sigma^2 t \exp(\frac{\sigma^2 t^2}{2}) \\ mgf^{(2)}(t) &= (\sigma^2 + \sigma^4 t^2) \exp(\frac{\sigma^2 t^2}{2}) \\ mgf^{(3)}(t) &= (\sigma^6 t^3 + 3\sigma^4 t) \exp(\frac{\sigma^2 t^2}{2}) \\ mgf^{(4)}(t) &= (\sigma^8 t^4 + 6\sigma^6 t^2 + 3\sigma^4) \exp(\frac{\sigma^2 t^2}{2}) \end{aligned} \tag{28}$$

$$mgf^{(4)}(0) = 3\sigma^4 = 3$$

So, the variance of historical volatility estimator is

$$Var(\hat{\sigma}^2) = \frac{\hat{\sigma}^4}{N}[3-1] = \frac{2\hat{\sigma}^4}{N}$$

The relative standard deviation $StDev(\bar{\sigma}^2)/\langle\bar{\sigma}^2\rangle = \sqrt{\frac{2}{N}}$, if it is not too large, the relative standard deviation of historical volatility estimator is half of this, $\frac{1}{\sqrt{2N}}$.

Given the above conclusion, the StDev(P&L) is approximately vega multiplies the standard deviation of historical volatility estimator with the same hedging schedule.

example 1. *Let's assume a one-year ATM option with* $\hat{\sigma} = 20\%, S = 1, P = 7.97\% \quad N = 250$. *As a result,* $\frac{1}{\sqrt{(2N)}} \approx 0.045$. *And there are 2 Golden Thumb: (1)* $P_{\hat{\sigma}} \approx \frac{1}{\sqrt{2\pi}} S\hat{\sigma}\sqrt{T}$*; (2)*$\hat{\sigma}\frac{dP_{\hat{\sigma}}}{d\hat{\sigma}} \approx P$. *And it implies the StDev(P&L) = 0.045P, which is approximately* 5% *premium. And the bid/ask spread is approximately 10 %.*

1.3.1 The real case

Analysis In real life, the σ_i is not constantly equal to $\hat{\sigma}$ and f_{ij} will not vanish. We assume the dynamics is time-homogeneous. Then variance correlation function f_{ij} is the function of the difference $|i-j|$.

$$\sum_{ij} f_{ij}\delta t^2 \approx \int_0^T du \int_0^T dt f(t-u) = 2\int_0^T f(T-\tau)f(\tau) \tag{29}$$

We now have the from (23):

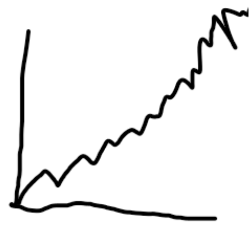

Figure 1: Final PnL

$$
\begin{aligned}
StDev(P\&L) &\approx |\hat{\sigma}\frac{dP_{\hat{\sigma}}}{d\hat{\sigma}}|\frac{1}{2T}\sqrt{(2+\kappa)\frac{T^2}{N}+2\Omega\int_0^T (T-\tau)f(\tau)d\tau} \\
&= |\hat{\sigma}\frac{dP_{\hat{\sigma}}}{d\hat{\sigma}}|\sqrt{\frac{2+\kappa}{4N}+\frac{\Omega}{2T^2}\int_0^T (T-\tau)f(\tau)d\tau}
\end{aligned}
\tag{30}
$$

Now we can investigate the two contributors to StDev(P&L).

- The first part: Let's assume the daily variances are constant, so Ω vanishes. It is similar in (24), but the κ is not zero.

- The second part: the prefactor Ω quantifies the dispersion of daily variances while $f(\tau)$ quantifies how a fluctuation in daily variance σ_i on day t_i impacts daily variances $\sigma^2_{i+\tau}$ on subsequent days.

$$
\Omega = \frac{\langle \sigma^4 \rangle - \langle \sigma^2 \rangle^2}{\langle \sigma^2 \rangle^2}, \text{ and } f_{ij} = \frac{\langle (\sigma_i^2 - \widehat{\sigma}^2)(\sigma_j^2 - \widehat{\sigma}^2) \rangle}{\sqrt{\langle \sigma_i^4 \rangle - \widehat{\sigma}^4}\sqrt{\langle \sigma_j^4 \rangle - \widehat{\sigma}^4}} \tag{31}
$$

  Thus, if Ω vanishes slowly then the daily variance would be strongly correlated. If we say that σ_i is higher than $\hat{\sigma}$, the daily variances σ_j will be likely higher than $\hat{\sigma}$. It will cause the daily P&L keeps the same sign which generating strong correlation among daily P&Ls:

$$
P\&L = -\frac{S^2}{2}\frac{\partial^2 P_{\hat{\sigma}}}{\partial S^2}\left(\frac{\delta S^2}{S^2} - \hat{\sigma}^2 \delta t\right) \tag{32}
$$

  and increasing the variance of our final P&L. (see Figure 1 and Eq 30)

  example 2. *Set $f(\tau) = 1$, then we have the second piece of (30) is $\frac{\Omega}{4}$. If Ω is small, the impact of this term is equivalent to the impact of a relative displacement of $\hat{\sigma}$ by $\hat{\sigma}\frac{\sqrt{\Omega}}{2}$, regardless of the number N of daily rehedges. (For this sentence consider the Taylor expansion on $\sqrt{\Delta x}$)*

Estimating $f(\tau), \Omega, \kappa$