Notes on differential equation

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Abstract

Begin abstract

1 Introduction

2 Partial Differential Equation

2.1 Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{1}$$

According to the orthogonality relations for sine and cosine.

$$\begin{gathered} \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm} \\ \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm} \\ \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0 \end{gathered} \tag{2}$$

***Note: detailed notes in Integral Notes$-$ >Step 1

We multiply f(x) by $\sin \frac{n\pi x}{L}$ and take integral with respect to x.

$$\begin{aligned} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx &= \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx \\ &= \sum_{m=1}^{\infty} b_m \left(L\delta_{nm} \right) \\ &= Lb_n \end{aligned} \tag{3}$$

Then multiply f(x) by $\cos \frac{n\pi x}{L}$ and take integral with respect to x.

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx \tag{4}$$

2.2 Fourier sine and cosine series

We now know that the Fourier series of a periodic function with period $2L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

with

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

In this section, we provide a simplified version of fourier series if $f(x)$ is odd or even.

If $f(x)$ is an even function,

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = 0 \tag{5}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad f(x) \text{ even} \tag{6}$$

If $f(x)$ is an odd function,

$$a_n = 0, \quad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx \tag{7}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad f(x) \text{ odd} \tag{8}$$

2.3 Example of Fourier series

$$f(x) = 1 - \frac{2x}{\pi}, \quad 0 < x < \pi \tag{9}$$

Fourier expression,

$$f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots \right) \tag{10}$$

Since

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \text{ with } a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx \tag{11}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx dx = \frac{4}{n^2\pi^2}(1-\cos n\pi) = \begin{cases} 8/\left(n^2\pi^2\right), & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases} \tag{12}$$

2.4 Diffusion equation (separation of variables)

$$
\begin{gathered}
u_t = Du_{xx} \\
u_{0,t} = 0 \\
u_{L,t} = 0 \\
u(x,0) = f(x)
\end{gathered}
\tag{13}
$$

where,

$$D \text{ is a constant }, \quad D > 0. \tag{14}$$

We assume,

$$u(x,t) = X(x)T(t) \tag{15}$$

and substitute the ansatz into the diffusion equation

$$
\begin{gathered}
XT' = DX''T \\
\frac{X''}{X} = \frac{1}{D}\frac{T'}{T}
\end{gathered}
\tag{16}
$$

The left-hand side is independent to t and the right-hand side is independent to x. So, they need to be equal to a constant. Let's say λ.

$$X'' + \lambda X = 0, \quad T' + \lambda DT = 0 \tag{17}$$

Then it is easy to solve these two ODEs with the boundary conditions

$$u(0,t) = X(0)T(t) = 0, \quad u(L,t) = X(L)T(t) = 0 \tag{18}$$

Solution. 1

$$
\begin{gathered}
X'' + \lambda X = 0, \quad X(0) = X(L) = 0 \\
c^2 + \lambda = 0, \quad \lambda = \mu^2 \\
c = \pm \mu i \\
X(x) = A\cos\mu x + B\sin\mu x \\
A = 0 \quad (X(0) = 0), \quad B\sin\mu L = 0 \quad (X(L) = 0) \\
\sin\mu L = 0 \Rightarrow \mu_n = \frac{n\pi}{L} \\
X_n = \sin(n\pi x/L), \quad n = 1, 2, 3, ...
\end{gathered}
\tag{19}
$$

Solution. 2

$$
\begin{gathered}
T' + \left(n^2\pi^2 D/L^2\right) T = 0 \\
T_n = e^{-n^2\pi^2 Dt/L^2}
\end{gathered}
\tag{20}
$$

In the end,

$$u_n(x,t) = \sin(n\pi x/L)e^{-n^2\pi^2 Dt/L^2} \tag{21}$$

By superposition for homogeneous linear differential equations the states that the general solution with the spatial boundary conditions is given by

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)e^{-n^2\pi^2 Dt/L^2} \tag{22}$$

Then we can apply the last condition and if

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \tag{23}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{24}$$

2.5 Summary: Diffusion equation

Given a diffusion equation,

$$\begin{gathered} u_t = Du_{xx} \\ u_{0,t} = 0 \\ u_{L,t} = 0 \\ u(x,0) = f(x) \end{gathered} \tag{25}$$

where,

$$D \text{ is a constant }, \quad D > 0. \tag{26}$$

Solution.

$$\begin{gathered} f(x) = \textstyle\sum_{n=1}^{\infty} b_n \sin(n\pi x/L) \\ b_n = \tfrac{2}{L}\int_0^L f(x)\sin\tfrac{n\pi x}{L}dx \end{gathered} \tag{27}$$