Overview

Background

- Volatility Clustering
- GARCH Model is widely applied in the financial industry
- High-dimension models have not well developed

Notations

Consider a financial time series $\{y_t\}_{t=1}^{T}$,

$$\begin{aligned} y_t &= \mu_t(\theta) + \varepsilon_t, \quad t = 1, \ldots, T \\ \varepsilon_t &= H_t^{1/2}(\theta) z_t, \quad t = 1, \ldots, T \end{aligned}$$

where θ is an information variable, y_t is an $p \times 1$ vector, $E_{t-1}(y_t) = \mu_t(\theta)$ and we assuming the mean and variance of the normal random variable z_t as following.

$$\mathrm{E}(z_t) = 0$$

$$\operatorname{Var}(z_t) = I_N$$

Therefore, the covariance matrix of y_t can be expressed in the following way,

$$\begin{aligned} \operatorname{Var}(y_t \mid I_{t-1}) = \operatorname{Var}_{t-1}(y_t) &= \operatorname{Var}_{t-1}(\varepsilon_t) \\ &= H_t^{1/2} \operatorname{Var}_{t-1}(z_t) \left(H_t^{1/2}\right)' \end{aligned}$$

,the matrix H_t is positive definite.

Bollerslev's VEC Model

The VEC (1,1) model is defined as,

$$h_t = c + A\eta_{t-1} + Gh_{t-1}$$

where,

$$\begin{aligned} h_t &= \text{vech}\left(H_t\right) \\ \eta_t &= \text{vech}\left(\varepsilon_t \varepsilon_t'\right) \\ \text{eigen}(A + G) &< 1 \end{aligned}$$

General Form

$$\text{vech}\left(H_t\right) = \text{vech}\left(C^*\right) + \sum_{i=1}^{q} A_i \,\text{vech}\left(\varepsilon_{t-i}\varepsilon'_{t-i}\right) + \sum_{j=1}^{p} B_j \,\text{vech}\left(H_{t-j}\right)$$

A_i and B_i are square parameter matrices of order $(N+1)N/2$ and C^* is a $(N+1)N/2 \times 1$ parameter vector.

$$\begin{aligned}
\text{num}(VEC) &= \text{num}(C^*) + \text{num}(A_i) + \text{num}(B_i) \\
&= \tilde{N}/2 + \tilde{N}^2/4 + \tilde{N}^2/4 \\
&= N(N+1)(N(N+1)+1)/2
\end{aligned}$$

Our Improvements

$$
\begin{aligned}
\mathrm{L}(\zeta, \theta; \Lambda) = {} & \frac{1}{T}\sum_{t=1}^{T} \log|H_t(\theta)| + \frac{1}{T}\sum_{t=1}^{T} \mathrm{tr}\left(H_t^{-1}(\theta)R_t\right)\lambda_1 + \sum_{Z=A,B,C} \|Z_L\|_* \\
& + \lambda_2 \sum_{Z=A,B,C} |Z_D|_1 - \sum_{Z=A,B,C} \langle \Lambda_{Z_L}, \zeta_{Z_L} - Z_L \rangle \\
& - \sum_{Z=A,B,C} \langle \Lambda_{Z_D}, \zeta_{Z_D} - Z_S \rangle \\
& + \sum_{Z=A,B,C} \frac{1}{2\mu_{Z_L}} \|\zeta_{Z_L} - Z_L\|_F^2 + \sum_{Z=A,B,C} \frac{1}{2\mu_{Z_D}} \|\zeta_{Z_D} - Z_S\|_F^2
\end{aligned}
$$

Application

$$\begin{aligned} \min \quad & w'\Sigma w \\ \text{w.r.t} \quad & w'1 = 1 \end{aligned}$$

Take the partial derivatives,

$$\frac{\partial}{\partial w} w'\Sigma w + \lambda \left(w'1 - 1 \right)$$
$$2\Sigma w + \lambda 1 = 0$$

Then the result is below and Σ can be estimated by $\{H_t\}$.

$$w = -\frac{\Sigma^{-1}1}{1'\Sigma^{-1}1}$$

Snapshot of Data Results

	LW	ISEE	UT	AT	MT	DCC	VEC
annual volatility	0.3280	0.3587	0.3286	0.336311	0.3588	0.3570	0.1845
sharpe ratio	0.2313	0.2018	0.2327	0.2100	0.2015	0.2065	0.4187

[1] In the above table LW is Ledoit and Wolf; DCC is Dynamic Conditional Covariance; DCP is Dynamic Conditional Precision; ISEE is Innovated Scalable Efficient Estimation; UT is Universal Thresholding; AT is Adaptive Thresholding; MT is Multiple Testing.

[2] Since the data is S&P 100 component stock price from 2008 to 2018, we rebalance the portfolio in every 2 years.