Risk Analytics

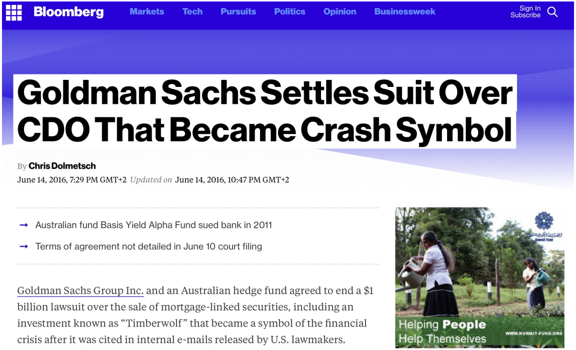
Correlated default

Dr. Paul Larsen

October 29, 2017

Table of contents

Introduction to correlated default


Bloomberg Markets Tech Pursuits Politics Opinion Businessweek Sign In Subscribe

Goldman Sachs Settles Suit Over CDO That Became Crash Symbol

By Chris Dolmetsch
June 14, 2016, 7:29 PM GMT+2 *Updated on* June 14, 2016, 10:47 PM GMT+2

- Australian fund Basis Yield Alpha Fund sued bank in 2011
- Terms of agreement not detailed in June 10 court filing

Goldman Sachs Group Inc. and an Australian hedge fund agreed to end a $1 billion lawsuit over the sale of mortgage-linked securities, including an investment known as "Timberwolf" that became a symbol of the financial crisis after it was cited in internal e-mails released by U.S. lawmakers.

Article: [Dolmetsch,]

Firm-value default models

Intuition: Default occurs if firm *i*s assets are below critical threshold at maturity T

- Counterparty i assets at time T: $A_T^{(i)}$
- Default threshold: C_i

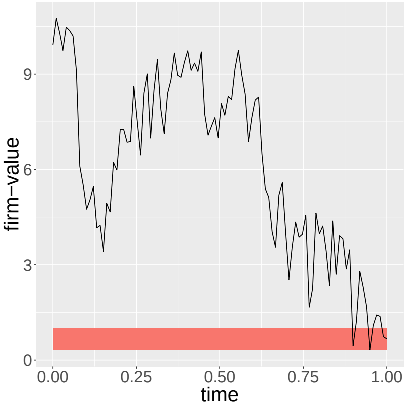
Definition

The *default event* in a firm-value model is $D_i = I_{\{A_T^{(i)} < C_i\}}$.

Firm-value default models

Definition

The *default event* in a firm-value model is $D_i = I_{\{A_T^{(i)} < C_i\}}$.



Firm value default models–key assumptions

Classic firm value model: [Merton, 1974] Assumptions:

- Under real-world measure, the firm's value (V_t) and debt B satisfies
$$dV_t = \mu_V V_t \, dt + \sigma_V V_t \, dW_t.$$
- Frictionless market with continuous trading.
- Risk-free interest rate deterministic and $>= 0$.
- The process (V_t) is independent of how the firm is financed. In particular, it is independent on B.

Less classical asset value model: Frank Hunziker, UBS, 2015 Banking Credit Risk Management Summit (Vienna)

Bernoulli default models

Set $p_i = \mathbb{E}_{\mathscr{P}}[D_i]$, probability of default at T. Then

$$D_i = \begin{cases} 1 \text{ (default)} & \text{with probability } p_i \\ 0 \text{ (no default)} & \text{with probability } 1 - p_i \end{cases}$$

i.e. $D_i \sim B(1, p_i)$.[1]

Definition

A *Bernoulli credit portfolio* is a vector of random variables $\mathbf{D} = (D_1, \ldots, D_m)$ such that $D_i \sim B(1, p_i)$ for all i. The *default count statistic* of D is

$$D = \sum_{i=1}^{m} D_i.$$

[1] $B(m,p)$ is the binomial distribution, with $\mathscr{P}(X = n) = \binom{m}{n} p^n (1-p)^{m-n}$

Bernoulli default models, II

Case: D_i independent, $D_i \sim B(1,p) \implies D \sim B(m,p)$
Example: $m = 2$, $p = 0.1$

Table: Default count distribution

0	1	2
0.81	0.18	0.01

Quick task: Derive the default count distribution for $m = 2$, arbitrary p.

Bernoulli default models, II

Case: D_i independent, $D_i \sim B(1,p) \implies D \sim B(m,p)$
Example: $m = 2$, $p = 0.1$

Table: Default count distribution

0	1	2
0.81	0.18	0.01

Quick task: Derive the default count distribution for $m = 2$, arbitrary p.
Solution:

$$\mathscr{P}(D=0) = (1-p)^2$$
$$\mathscr{P}(D=1) = 2p(1-p)$$
$$\mathscr{P}(D=2) = p^2$$

Bernoulli mixture models

Bernoulli credit portfolio as before, but now default probabilities random variables

- $\mathbf{P} = (P_1, \ldots, P_m) \sim F$, where
- F a distribution function with support in $[0,1]^m$

Definition

A *Bernoulli mixture default model* is a portfolio $\mathbf{D} = (D_1, \ldots, D_m)$ such that

- $D_i|_{P_i = p_i} \sim B(1, p_i)$
- $(D_i|_{\mathbf{P}=\mathbf{p}})$ independent

For $d_i \in \{0,1\}$, the probability distribution of $\mathbf{D}$ is

$$\mathscr{P}(D_1 = d_1, \ldots, D_m = d_m) = \int_{[0,1]^m} \prod_{i=1}^{m} p_i^{d_i} (1 - p_i)^{1-d_i} \, \mathrm{d}F(p_1, \ldots, p_m)$$

Bernoulli mixture models, II

Example (Single factor CreditMetrics™, KMV models)

- $D_i = I_{\{A_T^{(i)} < C_i\}} \sim B\left(1, \mathscr{P}\left(A_T^{(i)} < C_i\right)\right)$
- Asset value dependent on underlying factors, e.g. GDP, unemployment, region
- Normalizing log-returns $\log\left(A_T^{(i)}/A_0^{(i)}\right)$, asset-value is

$$r_i = \sqrt{\rho}\Phi + \sqrt{1-\rho}\epsilon_i$$

  - Φ: *composite systemic factor*, $\Phi \sim N(0,1)$[2]
  - ϵ_i: *idiosyncratic factor*, $\epsilon_i \sim N(0,1)$
  - All ϵ_i assumed independent of one another and of Φ
  - $\rho = \mathrm{Covar}(A_i, \Phi)$: *systemic risk* of counterparty i

$\Longrightarrow$ Default event is $D_i = I_{\{r_i < c_i\}} \sim B(1, \mathscr{P}(r_i < c_i))$

$\Longrightarrow$ (Normalized) default threshold satisfies $c_i = N^{-1}(p_i)$

[2] In general there are more systemic factors

Bernoulli mixture models, III

Recap

- Default $\iff r_i = \sqrt{\rho}\Phi + \sqrt{1-\rho}\epsilon_i < c_i$, where $r_i, \Phi, \epsilon_i \sim N(0,1)$, and $c_i = N^{-1}(p_i)$
- Default probalities P_i depend on ρ, Φ:
$$\mathscr{P}(P_i = p_i | \Phi = \phi) = N\left(c_i - \sqrt{\rho}\phi\right)$$
- The conditional default portfolio $\mathbf{D}|_{\Phi=\phi} = (D_1, \ldots, D_m)|_{\Phi=\phi}$ is independent $\implies$ **D** is a Bernoulli default mixture model

Bernoulli mixture example

Set $m = 2$, $\rho = 0.5$, $c = (c_1, c_2) = (-1.28, -2.88)$
$$\implies (p_1, p_2) = (0.1, 0.002)$$

Table: Default count distribution

0	1	2
0.8992	0.0995	0.0013

Alternate default dependence structures

Basics of t-distribution: Let $X_i \sim N(0,1)$ be independent.

Definition

The random variable $X = X_1^2 + \ldots + X_n^2$ is χ^2-distributed with n *degrees of freedom*

Let $\mathbf{Y} = (Y_1, \ldots, Y_m) \sim N(0, \Gamma)$, and let $\Theta = \sqrt{(n/X)}$, where $X \sim \chi^2(n)$, and $\mathbf{Y}, \Theta$ independent

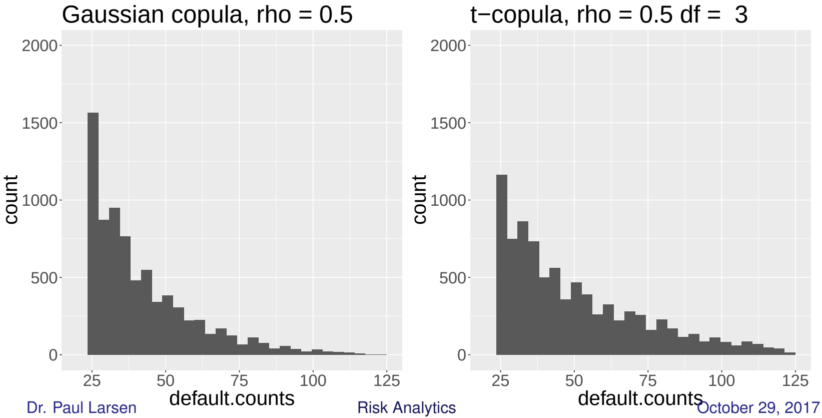
Definition

The random variable $Z = \Theta Y$ is a multivariate t-distribution with n degrees of freedom and correlation matrix Γ.

Bernoulli mixture model with t-copula, $m = 2, 125$

Table: Default count distribution, m=2

	0	1	2
Gaussian copula	0.8992	0.0995	0.0013
t-copula	0.8994	0.0991	0.0015



Stress testing Bernoulli mixture models

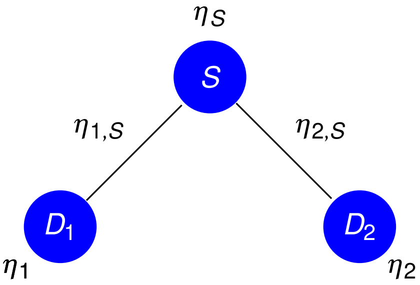
Goal: Quantify change in credit loss distribution in adverse economic situation

- Increase PDs
- Increase λ (loss-given-default)
- Truncate systemic factors Φ

Graphical default models

Single-factor, $m = 2$:

- $D_1, D_2 \in \{0, 1\}$ as before
- Systemic factor Φ replaced with $S \in \{0, 1\}$, where $S = 0 \implies \text{healthy}$, $S = 1 \implies \text{distress}$
- Model parameters η_i, η_F (nodes), $\eta_{i,S}$ (edges)



Graphical default models, II

Definition

A graphical default model is called *homogeneous single-factor* if $\eta_{i,S} = \eta_{D,S}$, and $\eta_i = \eta_D$ for all $i \in \{1, \dots, m\}$

If homogeneous, single-factor
$\implies$ total distribution function, $(\mathbf{d}, s) = (d_1, \dots, d_m, s) \in \{0,1\}^{m+1}$:

$$\mathscr{P}(\mathbf{D} = \mathbf{d}, S = s) = \frac{1}{Z} \exp\left(\eta_D \sum_{i=1}^{m} d_i + \eta_S s + \eta_{D,S}\, s \sum_{i=1}^{m} d_i\right),$$

where Z is *partition function* to ensure total probability = 1:

$$Z = \left(1 + \mathrm{e}^{\eta_D}\right)^m + \mathrm{e}^{\eta_S}\left(1 + \mathrm{e}^{\eta_D + \eta_{D,S}}\right)^m$$

Graphical default models, III

Assume homogenous, single-factor. Then
Counterparty marginal distribution:

$$\mathscr{P}(\mathbf{D} = \mathbf{d}) = \sum_{s \in \{0,1\}} \mathscr{P}(\mathbf{D} = \mathbf{d}, S = s)$$

$$= 1/Z \left(\exp\left(\eta_D \sum_{i=1}^{m} d_i \right) + \exp\left(\eta_S + (\eta_D + \eta_{D,S}) \sum_{i=1}^{m} d_i \right) \right)$$

And the default count distribution is

$$\mathscr{P}(D = n) = \mathscr{P}\left(\sum_{i=1}^{m} D_i = n \right)$$

$$= \frac{1}{Z} \binom{m}{n} \left(\exp(\eta_D n) + \exp\left(\eta_S + (\eta_D + \eta_{D,S}) n \right) \right)$$

Graphical default models, example

Set $m = 15$, $\eta_D = -0.7$, $\eta_D S = -2.1$, $\eta_S = 5.5$

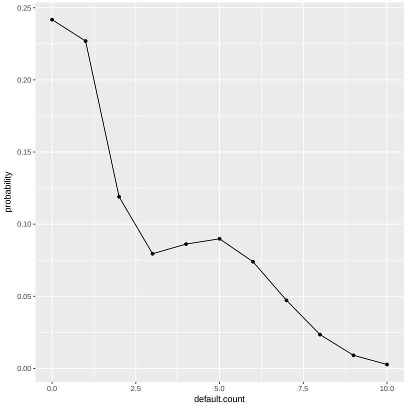
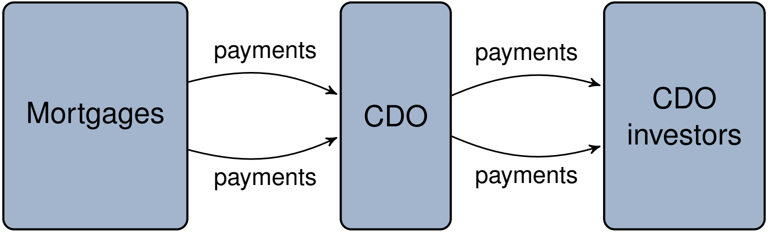


Figure: Graphical Model Default Count Distribution

Collateralized debt obligations and default models



- CDO investor payments separated into *tranches*: highest tranch (*senior*) gets payments first, lowest tranch (*equity*) gets remainder payments after defaults
- Spread per tranch $\iff$ default risk per tranch $\iff$ correlated default risk
- Gaussian copulas used to price CDOs leading up to 2007-8 crisis [**?**]

Main sources

- [Bluhm et al., 2016], Chapter 2
- [McNeil et al., 2015], Chapter 8
- [Rutkowski and Tarca, 2015]
- [Kalkbrener and Packham, 2015]
- [Filiz et al., 2012]

Bibliography I

Bluhm, C., Overbeck, L., and Wagner, C. (2016). *Introduction to credit risk modeling*. CRC Press.

Dolmetsch, C.

Filiz, I. O., Guo, X., Morton, J., and Sturmfels, B. (2012). Graphical models for correlated defaults. *Mathematical Finance*, 22(4):621–644.

Kalkbrener, M. and Packham, N. (2015). Correlation under stress in normal variance mixture models. *Mathematical Finance*, 25(2):426–456.

McNeil, A. J., Frey, R., and Embrechts, P. (2015). *Quantitative risk management: Concepts, techniques and tools*. Princeton university press.

Merton, R. C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *The Journal of finance*, 29(2):449–470.

Rutkowski, M. and Tarca, S. (2015). Regulatory capital modeling for credit risk. *International Journal of Theoretical and Applied Finance*, 18(05):1550034.