Risk Analytics

Correlated default

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November 1, 2017

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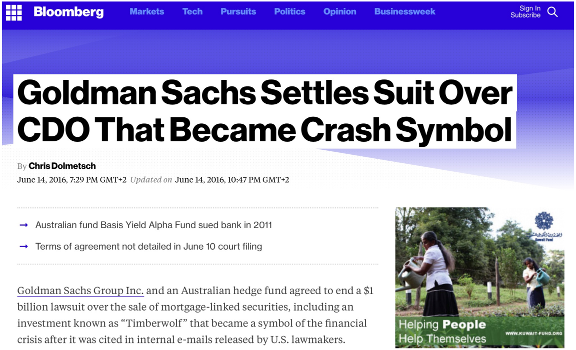
Introduction to correlated default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Goldman Sachs Settles Suit Over CDO That Became Crash Symbol

By **Chris Dolmetsch**
June 14, 2016, 7:29 PM GMT+2 *Updated on* June 14, 2016, 10:47 PM GMT+2

- Australian fund Basis Yield Alpha Fund sued bank in 2011
- Terms of agreement not detailed in June 10 court filing

Goldman Sachs Group Inc. and an Australian hedge fund agreed to end a $1 billion lawsuit over the sale of mortgage-linked securities, including an investment known as "Timberwolf" that became a symbol of the financial crisis after it was cited in internal e-mails released by U.S. lawmakers.

Article: [Dolmetsch,]

Firm-value default models

Intuition: Default occurs if firm *i*s assets are below critical threshold at maturity T

- Counterparty i assets at time T: $A_T^{(i)}$
- Default threshold: C_i

Definition

The *default event* in a firm-value model is $D_i = I_{\{A_T^{(i)} < C_i\}}$.

Firm-value default models

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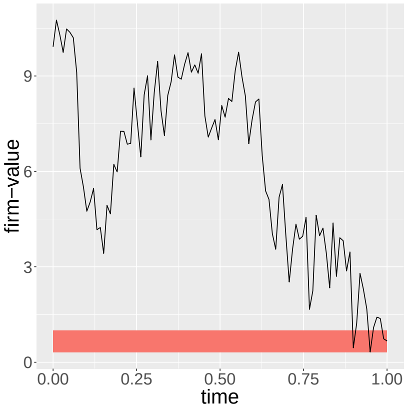


Firm value default models–key assumptions

Classic firm value model: [Merton, 1974] Assumptions:

- Under real-world measure, the firm's value (V_t) and debt B satisfies

$$dV_t = \mu_V V_t \, dt + \sigma_V V_t \, dW_t.$$

- Frictionless market with continuous trading.
- Risk-free interest rate deterministic and ≥ 0.
- The process (V_t) is independent of how the firm is financed. In particular, it is independent on B.

Less classical asset value model: Frank Hunziker, UBS, 2015 Banking Credit Risk Management Summit (Vienna)

Bernoulli default models

Set $p_i = \mathbb{E}_{\mathscr{P}}[D_i]$, probability of default at T. Then

$$D_i = \begin{cases} 1 \text{ (default)} & \text{with probability } p_i \\ 0 \text{ (no default)} & \text{with probability } 1 - p_i \end{cases}$$

i.e. $D_i \sim B(1, p_i)$.[1]

Definition

A *Bernoulli credit portfolio* is a vector of random variables $\mathbf{D} = (D_1, \ldots, D_m)$ such that $D_i \sim B(1, p_i)$ for all i. The *default count statistic* of D is

$$D = \sum_{i=1}^{m} D_i.$$

[1] $B(m, p)$ is the binomial distribution, with $\mathscr{P}(X = n) = \binom{m}{n} p^n (1-p)^{m-n}$

Bernoulli default models, II

Case: D_i independent, $D_i \sim B(1,p) \implies D \sim B(m,p)$
Example: $m = 2$, $p = 0.1$

Table: Default count distribution

0	1	2
0.81	0.18	0.01

Quick task: Derive the default count distribution for $m = 2$, arbitrary p.

Bernoulli default models, II

Case: D_i independent, $D_i \sim B(1,p) \implies D \sim B(m,p)$
Example: $m=2$, $p=0.1$

Table: Default count distribution

0	1	2
0.81	0.18	0.01

Quick task: Derive the default count distribution for $m=2$, arbitrary p.
Solution:

$$\mathscr{P}(D=0) = (1-p)^2$$
$$\mathscr{P}(D=1) = 2p(1-p)$$
$$\mathscr{P}(D=2) = p^2$$

Bernoulli mixture models

Bernoulli credit portfolio as before, but now default probabilities random variables

- $\mathbf{P} = (P_1, \ldots, P_m) \sim F$, where
- F a distribution function with support in $[0,1]^m$

Definition

A *Bernoulli mixture default model* is a portfolio $\mathbf{D} = (D_1, \ldots, D_m)$ such that

- $D_i|_{P_i = p_i} \sim B(1, p_i)$
- $(D_i|_{\mathbf{P}=\mathbf{p}})$ independent

For $d_i \in \{0,1\}$, the probability distribution of $\mathbf{D}$ is

$$\mathscr{P}(D_1 = d_1, \ldots, D_m = d_m) = \int_{[0,1]^m} \prod_{i=1}^{m} p_i^{d_i}(1-p_i)^{1-d_i}\, \mathrm{d}F(p_1, \ldots, p_m)$$

Bernoulli mixture models, II

Example (Single factor CreditMetrics™, KMV models)

- $D_i = I_{\{A_T^{(i)} < C_i\}} \sim B\left(1, \mathscr{P}\left(A_T^{(i)} < C_i\right)\right)$
- Asset value dependent on underlying factors, e.g. GDP, unemployment, region
- Normalizing log-returns $\log\left(A_T^{(i)}/A_0^{(i)}\right)$, asset-value is

$$r_i = \sqrt{\rho}\Phi + \sqrt{1-\rho}\epsilon_i$$

  - Φ: *composite systemic factor*, $\Phi \sim N(0,1)$[2]
  - ϵ_i: *idiosyncratic factor*, $\epsilon_i \sim N(0,1)$
  - All ϵ_i assumed independent of one another and of Φ
  - $\rho = \mathrm{Covar}(A_i, \Phi)$: *systemic risk* of counterparty i

$\Longrightarrow$ Default event is $D_i = I_{\{r_i < c_i\}} \sim B(1, \mathscr{P}(r_i < c_i))$

$\Longrightarrow$ (Normalized) default threshold satisfies $c_i = N^{-1}(p_i)$

[2] In general there are more systemic factors

Bernoulli mixture models, III

Recap

- Default $\iff r_i = \sqrt{\rho}\Phi + \sqrt{1-\rho}\epsilon_i < c_i$, where $r_i, \Phi, \epsilon_i \sim N(0,1)$, and $c_i = N^{-1}(p_i)$
- Default probalities P_i depend on ρ, Φ:
$$\mathscr{P}(P_i = p_i | \Phi = \phi) = N\left(c_i - \sqrt{\rho}\phi\right)$$
- The conditional default portfolio $\mathbf{D}|_{\Phi=\phi} = (D_1, \ldots, D_m)|_{\Phi=\phi}$ is independent $\implies$ $\mathbf{D}$ is a Bernoulli default mixture model

Bernoulli mixture example

Set $m = 2$, $\rho = 0.5$, $c = (c_1, c_2) = (-1.28, -2.88)$
$$\implies (p_1, p_2) = (0.1, 0.002)$$

Table: Default count distribution

0	1	2
0.8992	0.0995	0.0013

Alternate default dependence structures

Basics of t-distribution: Let $X_i \sim N(0,1)$ be independent.

Definition

The random variable $X = X_1^2 + \ldots + X_n^2$ is χ^2-distributed with n *degrees of freedom*

Let $\mathbf{Y} = (Y_1, \ldots, Y_m) \sim N(0, \Gamma)$, and let $\Theta = \sqrt{(n/X)}$, where $X \sim \chi^2(n)$, and $\mathbf{Y}, \Theta$ independent

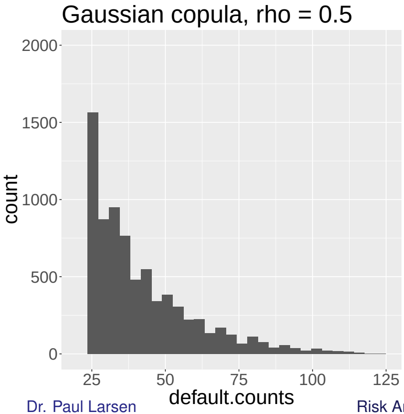
Definition

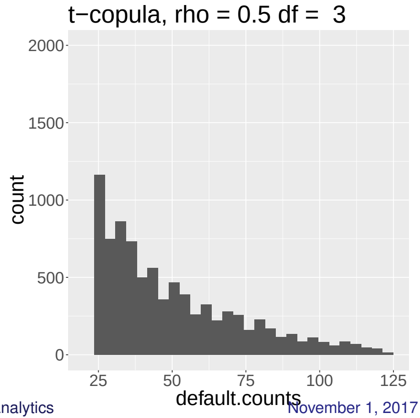
The random variable $Z = \Theta Y$ is a multivariate t-distribution with n degrees of freedom and correlation matrix Γ.

Bernoulli mixture model with t-copula, $m = 2, 125$

Table: Default count distribution, m=2

	0	1	2
Gaussian copula	0.8992	0.0995	0.0013
t-copula	0.8994	0.0991	0.0015





Stress testing Bernoulli mixture models

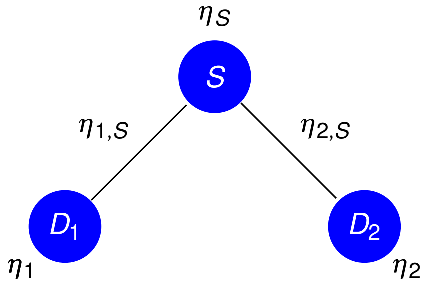
Goal: Quantify change in credit loss distribution in adverse economic situation

- Increase PDs
- Increase λ (loss-given-default)
- Truncate systemic factors Φ

Graphical default models

Single-factor, $m = 2$:

- $D_1, D_2 \in \{0, 1\}$ as before
- Systemic factor Φ replaced with $S \in \{0, 1\}$, where $S = 0 \implies \text{healthy}$, $S = 1 \implies \text{distress}$
- Model parameters η_i, η_F (nodes), $\eta_{i,S}$ (edges)



Graphical default models, II

Definition

A graphical default model is called *homogeneous single-factor* if $\eta_{i,S} = \eta_{D,S}$, and $\eta_i = \eta_D$ for all $i \in \{1, \ldots, m\}$

If homogeneous, single-factor
$\implies$ total distribution function, $(\mathbf{d}, s) = (d_1, \ldots, d_m, s) \in \{0,1\}^{m+1}$:

$$\mathscr{P}(\mathbf{D} = \mathbf{d}, S = s) = \frac{1}{Z} \exp\left(\eta_D \sum_{i=1}^{m} d_i + \eta_S s + \eta_{D,S}\, s \sum_{i=1}^{m} d_i\right),$$

where Z is *partition function* to ensure total probability = 1:

$$Z = \left(1 + \mathrm{e}^{\eta_D}\right)^m + \mathrm{e}^{\eta_S}\left(1 + \mathrm{e}^{\eta_D + \eta_{D,S}}\right)^m$$

Graphical default models, III

Assume homogenous, single-factor. Then
Counterparty marginal distribution:

$$\mathscr{P}(\mathbf{D}=\mathbf{d}) = \sum_{s\in\{0,1\}} \mathscr{P}(\mathbf{D}=\mathbf{d}, S=s)$$

$$= 1/Z\left(\exp\left(\eta_D \sum_{i=1}^{m} d_i\right) + \exp\left(\eta_S + (\eta_D + \eta_{D,S})\sum_{i=1}^{m} d_i\right)\right)$$

And the default count distribution is

$$\mathscr{P}(D=n) = \mathscr{P}\left(\sum_{i=1}^{m} D_i = n\right)$$

$$= \frac{1}{Z}\binom{m}{n}\left(\exp(\eta_D n) + \exp\left(\eta_S + (\eta_D + \eta_{D,S})n\right)\right)$$

Graphical default models, example

Set $m = 15$, $\eta_D = -0.7$, $\eta_D S = -2.1$, $\eta_S = 5.5$

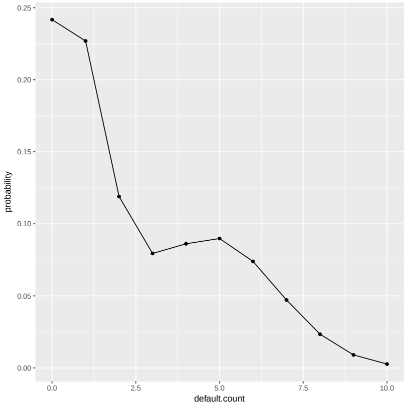
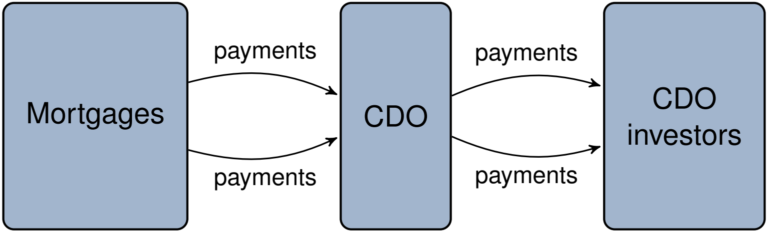


Figure: Graphical Model Default Count Distribution

Collateralized debt obligations and default models



- CDO investor payments separated into *tranches*: highest tranch (*senior*) gets payments first, lowest tranch (*equity*) gets remainder payments after defaults
- Spread per tranch $\iff$ default risk per tranch $\iff$ correlated default risk
- Gaussian copulas used to price CDOs leading up to 2007-8 crisis [Li, 2000]

Main sources

- [Bluhm et al., 2016], Chapter 2
- [McNeil et al., 2015], Chapter 8
- [Rutkowski and Tarca, 2015]
- [Kalkbrener and Packham, 2015]
- [Filiz et al., 2012]

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