B. ENSO

- explain
- model results

Box model

$$\dot{h}_1 = -r\left(h_1 + \frac{bL}{2}\frac{\mu}{\beta}\left(T_2 - T_1\right)\right)$$

$$\dot{T}_1 = -\alpha\left(T_1 - T_r\right) - \varepsilon\mu\left(T_2 - T_1\right)^2$$

$$\dot{T}_2 = -\alpha\left(T_2 - T_r\right) + \zeta\mu\left(T_2 - T_1\right) * \left(T_2 - T_{sub}(h_2)\right)$$

$$T_{sub}(h) = T_r - \frac{T_r - T_0}{2} * \left(1 - \tanh\left(\frac{h + H - z_0}{h_*}\right)\right)$$

$$h_2 = h_1 + bL\frac{\mu}{\beta}\left(T_2 - T_1\right)$$

<u>Define</u> $S = T_2 - T_1$

$T = T_1 - T_r$

$h = h_1 + H - z_e$

$$\rightarrow \quad \dot{S} = -\alpha S + \mu \zeta \$ \,*$$

$$* \left(S + T + \frac{T_r - T_0}{2} \left(1 - \right.\right.$$

$$\left.\left. + \tanh\left(\frac{h}{h_*} + \frac{bL\mu}{\beta h_*} S \right) \right) \right)$$

$$+ \varepsilon \mu S^2$$

$$\dot{T} = -\alpha T - \varepsilon \mu S^2$$

$$\dot{h} = -\gamma \left(h + z_0 - H + \right.$$

$$\left. + \frac{b\mu L}{2\beta} S \right)$$

$$S = x S_0 \, , \quad T = y T_0$$

$$h = z h_0 \, , \quad t = t_0 \tau$$

$$\longrightarrow \frac{S_0}{t_0} \dot{x} = \varepsilon \mu S_0^2 x^2 - \alpha S_0 x +$$

$$\mu \zeta S_0 x \left(S_0 x + T_0 y + \frac{T_I - T_0}{2} * \right.$$

$$\left. \left(1 - \tanh \left(z \frac{h_0}{h_*} + \frac{bL\mu}{\beta h_*} S_0 x \right) \right) \right) \Bigg)$$

$$\frac{T_0}{t_0} \dot{y} = -\alpha T_0 y - \varepsilon \mu S_0^2 x^2$$

$$\frac{h_0}{t_0} \dot{z} = -r \left(z h_0 + z_0 - H + \right.$$

$$\left. \frac{b\mu L}{2\beta} S_0 x \right)$$

$$S_0 = T_0 = \frac{h_* \beta}{bL\mu} \qquad h_0 = h_*$$

$$t_0 = \frac{bL}{\beta \zeta h_*}$$

$$\overset{\circ}{x} = \varepsilon \mu t_0 S_0 x^2 - \alpha t_0 x +$$

$$S_0 t_0 \mu \zeta x \Big(x + y + \frac{T_T - T_0}{2 S_0} *$$

$$* \left(1 - \tanh \left(z + x \right) \right) \Big)$$

$$S_0 t_0 \mu \zeta = \frac{h^* \beta}{b L \mu} \cdot \frac{b L}{\beta \zeta h^*} \cdot \mu \zeta = 1$$

$$\boxed{c} = \frac{T_T - T_0}{2 S_0}$$

$$\boxed{S} = \frac{\varepsilon h_* \beta}{r b L} \qquad \boxed{\delta} = \frac{r b L}{\zeta h_* \beta}$$

$$S^\delta = \varepsilon / \zeta \qquad \boxed{a} = \frac{\alpha}{\varepsilon \mu} \cdot \frac{b L \mu}{h_* \beta}$$

$$S_0 t_0 = \frac{1}{\mu \zeta}$$

$$\alpha t_0 = \varepsilon \mu t_0 S_0 \cdot \frac{\alpha}{\varepsilon \mu S_0}$$

$$= S^\delta \cdot a$$

Finally

$$\mathring{x} = \rho\delta\left(x^2 - ax\right) + x \, * \left(x + y + c - c\tanh\left(x+z\right)\right)$$

$$\mathring{y} = -\alpha t_0 \, y - \varepsilon\mu S_0 t_0 \, x^2$$

$$\rightarrow \quad = -\left(ay + x^2\right)\rho\delta$$

$$\dot{z} = -r t_0\left(z + \frac{z_0 - H}{h_*} + \right.$$

$$\left. + \, x/2\right) \qquad \delta = r t_0$$

$$= \delta\left(k - z - x/2\right)$$

$$\boxed{k} = \frac{H - z_0}{h_*}$$

5. Bifurcations of fixed points

Scalar equation, autonomous
(n=1)

$$\dot{x} = f(x,\lambda)$$

Fixed points $\bar{x}$

$$f(\bar{x},\lambda) = 0$$

Small perturbations $\tilde{x}$,
with $|\tilde{x}| \ll \bar{x}$ $\longrightarrow$

$$\dot{\tilde{x}} + \underset{\substack{\| \\ 0}}{\dot{\bar{x}}} = f(\bar{x}+\tilde{x},\lambda)$$

f smooth $\longrightarrow$

$$\dot{\tilde{x}} = f(\bar{x},\lambda) + \tilde{x}\, f_x(\bar{x},\lambda) + O(|\tilde{x}|^2)$$

Let $\sigma = f_x(\bar{x},\lambda)$

then $\qquad \mathring{\tilde{x}} = \sigma \tilde{x}$

$\rightarrow \quad \tilde{x}(t) = \tilde{x}_0 e^{\sigma t}$

$\sigma < 0$: $\bar{x}$ stable ; $\sigma > 0$: $\bar{x}$ unstable

Changes in perturbation behavior occur at $\sigma = 0$, i.e. at values of λ_c for which $f_x(\bar{x}, \lambda_c) = 0$;

λ_c are called bifurcation points.

Classification of bifurcation points based on Taylor series of f.

For a single parameter λ, there are three cases:

(i) saddle-node

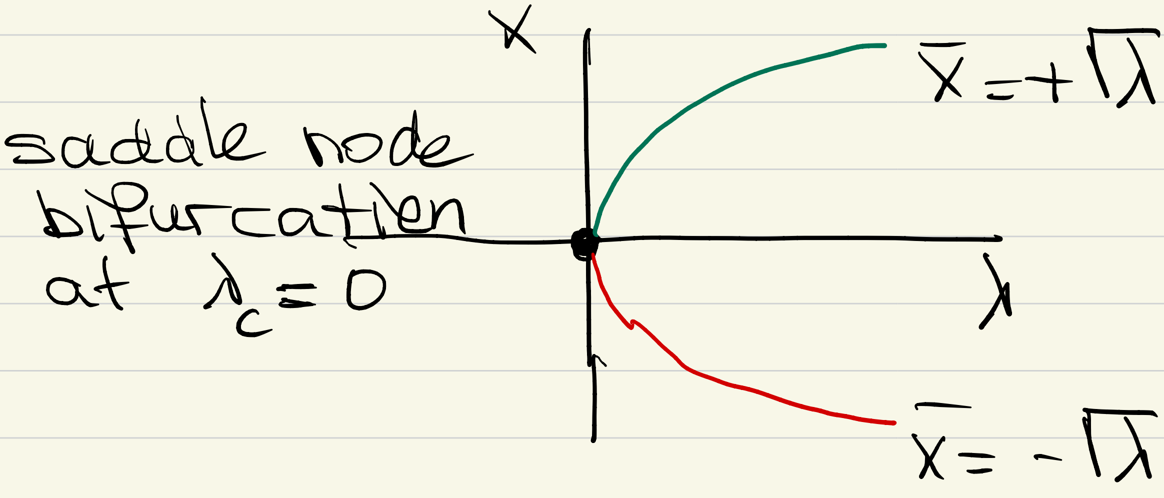
normal form: $\dot{x} = \lambda - x^2$

fixed points: $\bar{x} = \pm\sqrt{\lambda}$, $\lambda \geq 0$

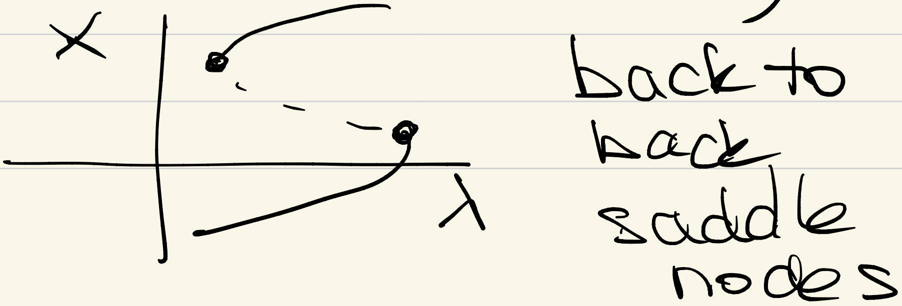
stability: $f_x(\bar{x}, \lambda) = -2\bar{x} = \sigma$

<u>hence</u>: $\bar{x} = +\sqrt{\lambda} \rightarrow \sigma < 0$

$\bar{x} = -\sqrt{\lambda} \rightarrow \sigma > 0$



- physical systems (bounded solutions)



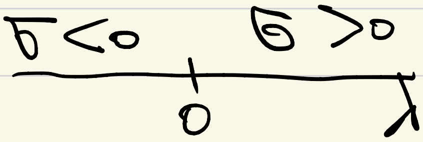
(ii) pitchfork bifurcation

normal form : $\dot{x} = x(\lambda - x^2)$

fixed points : $\bar{x} = 0$, $\bar{x} = \pm\sqrt{\lambda}, \lambda > 0$

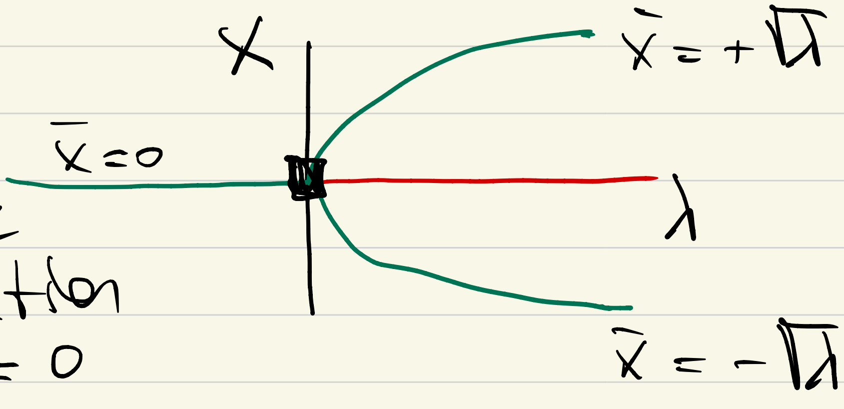
stability : $\sigma = f_x(\bar{x}, \lambda) = \lambda - 3\bar{x}^2$

$\bar{x} = 0 \to \sigma = \lambda$



$\bar{x} = +\sqrt{\lambda} \to \sigma = -2\lambda \quad < 0$

$\bar{x} = -\sqrt{\lambda} \to \sigma = -2\lambda \quad < 0$



pitchfork bifurcation at $\lambda_c = 0$

- physical systems

symmetry breaking

(iii) transcritical bifurcation

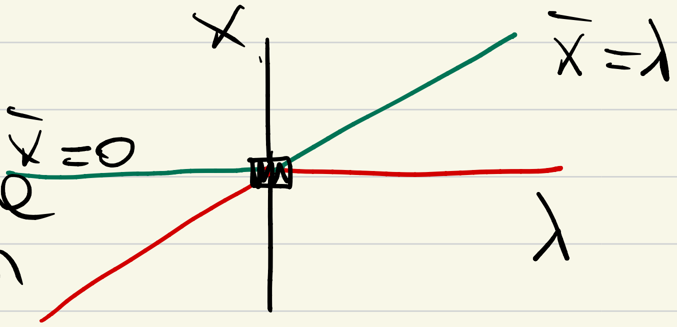
normal form : $\dot{x} = \lambda x - x^2$

fixed points : $\bar{x} = 0$, $\bar{x} = \lambda$

stability : $\sigma = f_x(\bar{x}, \lambda) = \lambda - 2\bar{x}$

$\bar{x} = 0 \longrightarrow \sigma = \lambda$ $\quad$ $\sigma < 0$, $\sigma > 0$ on the λ axis at 0

$\bar{x} = \lambda \longrightarrow \sigma = -\lambda$ $\quad$ $\sigma > 0$, $\sigma < 0$ on the λ axis at 0



transcritical bifurcation at $\lambda_c = 0$

- physics

  - no symmetry, (trivial) solution for all values of λ.

Two - dimensional case

$$\begin{cases} \dot{x} = f(x, y, \lambda) \\ \dot{y} = g(x, y, \lambda) \end{cases}$$

fixed points : $f(\bar{x}, \bar{y}, \lambda) = 0$

$g(\bar{x}, \bar{y}, \lambda) = 0$

stability : $\begin{cases} x = \bar{x} + \tilde{x} \\ y = \bar{y} + \tilde{y} \end{cases}$

$$\dot{\bar{x}} + \dot{\tilde{x}} = f(\bar{x} + \tilde{x}, \bar{y} + \tilde{y}, \lambda)$$

$$\dot{\bar{y}} + \dot{\tilde{y}} = g(\bar{x} + \tilde{x}, \bar{y} + \tilde{y}, \lambda)$$

$$\rightarrow \begin{cases} \dot{\tilde{x}} = f_x \tilde{x} + f_y \tilde{y} + \ldots \\ \dot{\tilde{y}} = g_x \tilde{x} + g_y \tilde{y} + \ldots \end{cases}$$

Jacobian matrix

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

$$\rightarrow \quad \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = J \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

Solution: $\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{y}} \end{pmatrix} = e^{tJ} \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \end{pmatrix}$

$$J = T \underbrace{\Sigma}_{\text{eigenvalues of } J} T^{-1}$$

In the 2-dimensional case, there is one additional bifurcation

(iv) Hopf bifurcation

normal form
$$\begin{cases} \dot{x} = \lambda x - \omega y - x(x^2+y^2) \\ \dot{y} = \omega x + \lambda y - y(x^2+y^2) \end{cases}$$

ω fixed ; λ varying

fixed point : $\bar{x} = \bar{y} = 0$

stability :

$$J(\bar{x}=\bar{y}=0) = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$$

eigenvalues : $\det(J - \lambda I) = 0$

$$\begin{vmatrix} \lambda-\sigma & -\omega \\ \omega & \lambda-\sigma \end{vmatrix} = 0$$

$$\rightarrow (\lambda-\sigma)^2 + \omega^2 = 0$$

$$\sigma_{1,2} = \lambda \pm i\omega$$

$$\lambda > 0 \quad : \mathrm{Re}(\sigma) > 0$$

$$\lambda < 0 \quad : \mathrm{Re}(\sigma) < 0$$

Solutions :

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} \tilde{x}_0 \cos\omega t + \tilde{y}_0 \sin\omega t \\ -\tilde{x}_0 \sin\omega t + \tilde{y}_0 \cos\omega t \end{pmatrix}$$

Polar coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} \dot{x} = \dot{r}\cos\theta - \dot{\theta} r \sin\theta \\ \dot{y} = \dot{r}\sin\theta + \dot{\theta} r\cos\theta \\ x^2 + y^2 = r \end{cases}$$

→

$$\dot r\cos\theta - \dot\theta r\sin\theta = \lambda r\cos\theta - \\ - \omega r\sin\theta - r^3\cos\theta \qquad (1)$$

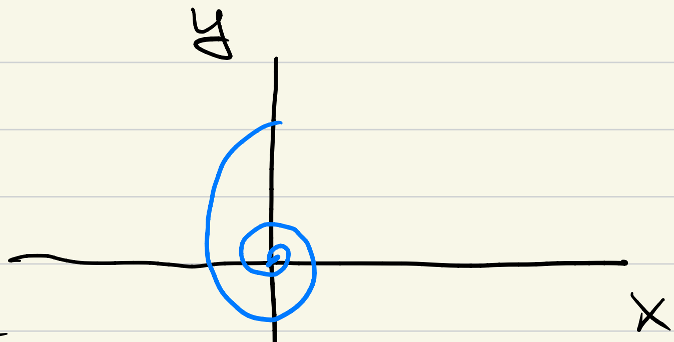
$$\dot r\sin\theta + \dot\theta r\cos\theta = \omega r\sin\theta + \\ + \lambda r\sin\theta - r^3\sin\theta \qquad (2)$$

$\cos\theta\ (1) + \sin\theta\ (2) \Rightarrow$

$$\dot r = \lambda r - r^3 \qquad (\text{'pitchfork'})$$

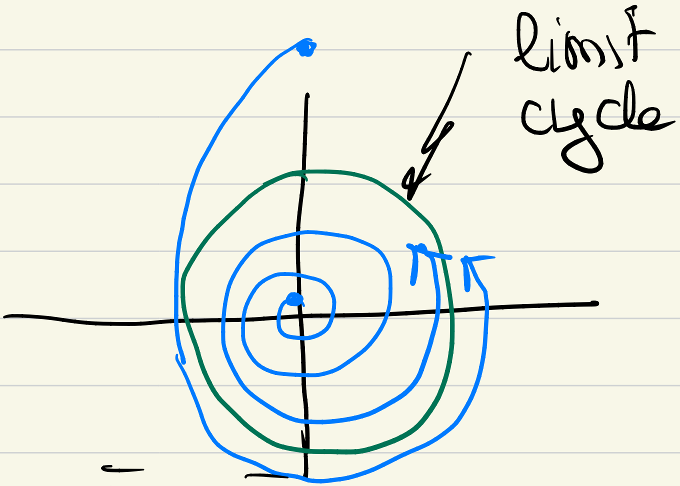
$-\sin\theta\ (1) + \cos\theta\ (2) \Rightarrow$

$$r^2\dot\theta = -\omega r^2 \longrightarrow \dot\theta = \omega$$



$\bar x = \bar y = 0$
stable focus

$\lambda < 0$



$\bar x = \bar y = 0$

unstable focus