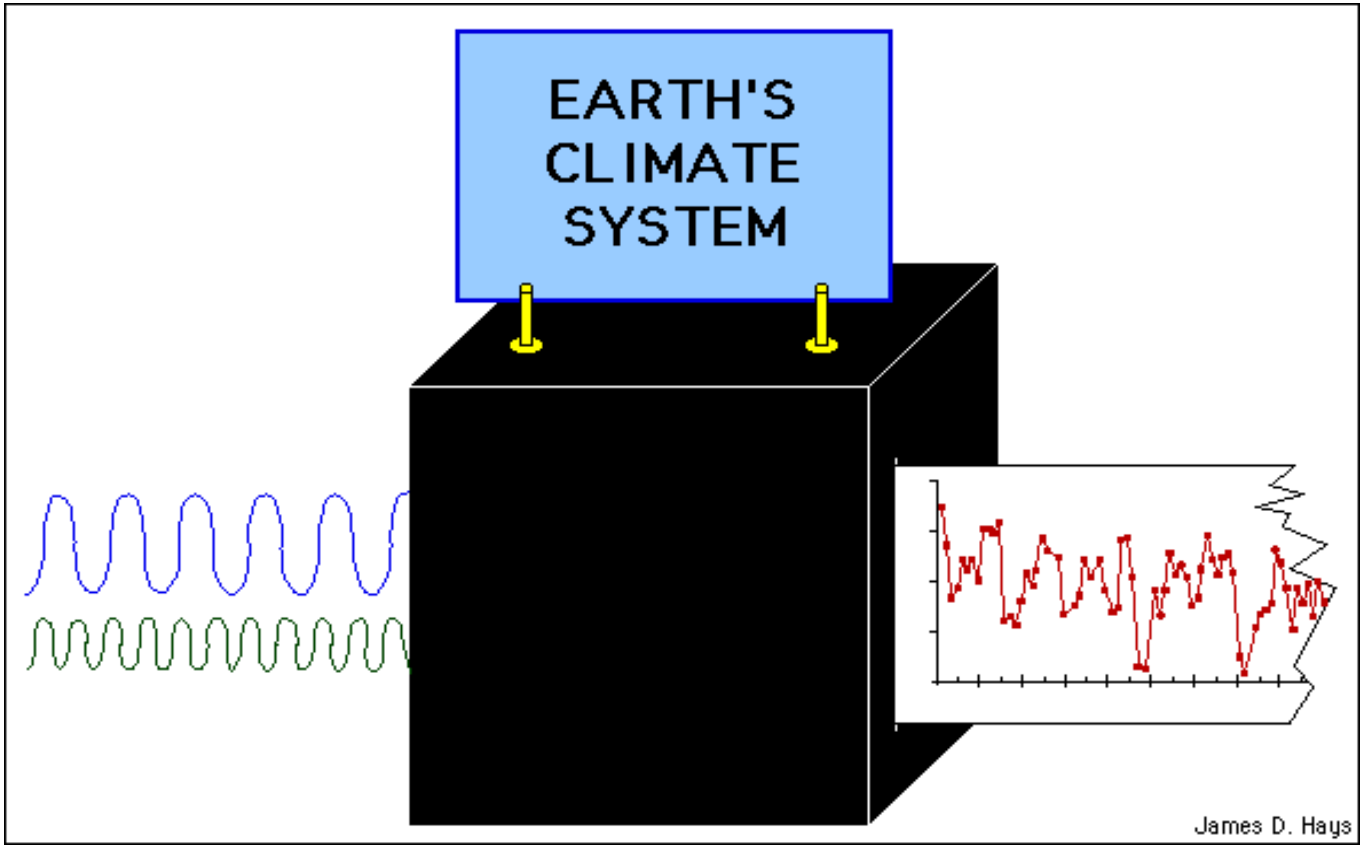
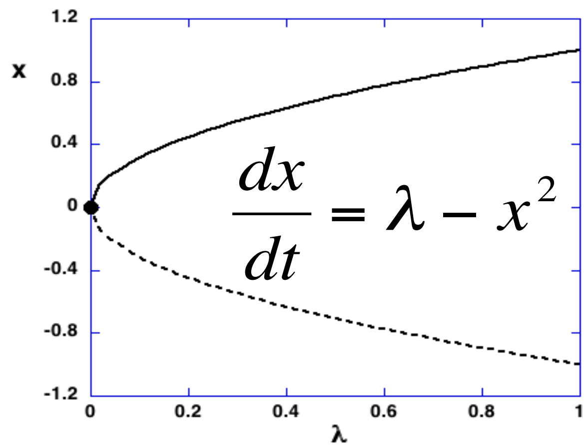
Basics of Dynamical Systems Theory



Navid & Henk, ANU, Jan 20-29, 2020

Elementary bifurcations

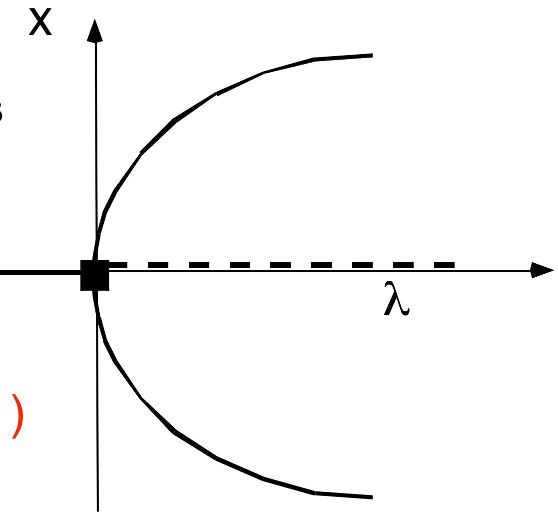
Elementary transitions (co-dim 1 bifurcations)



$$\frac{dx}{dt} = \lambda - x^2$$

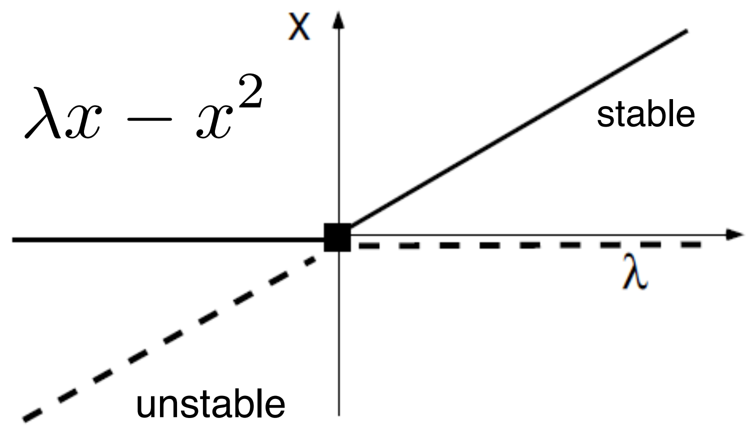
Saddle-node (non-existence)

$$\frac{dx}{dt} = \lambda x - x^3$$



Pitchfork
(symmetry-breaking)

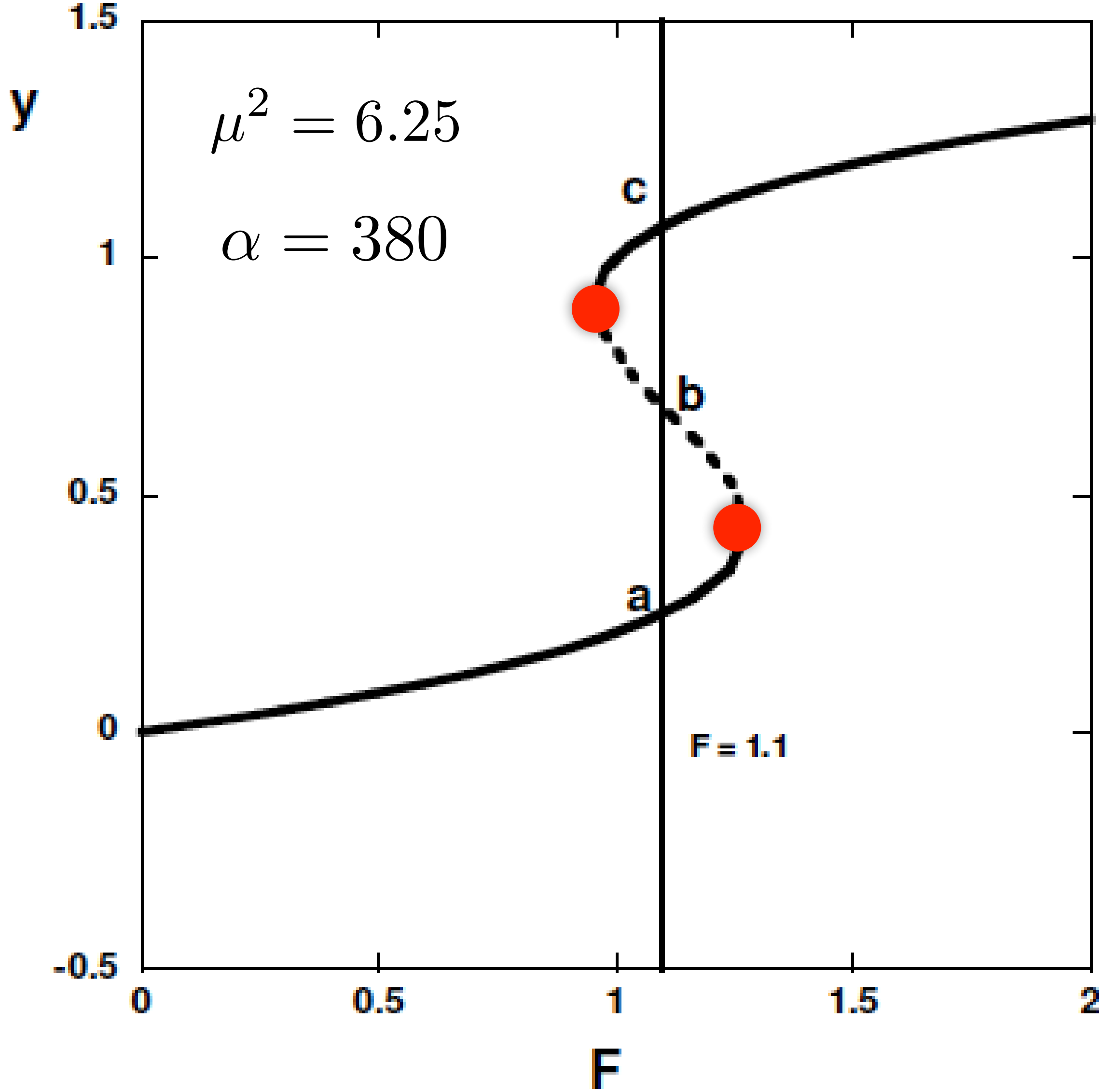
$$\frac{dx}{dt} = \lambda x - x^2$$



Transcritical (exchange of stability)

Equilibria dimensionless model

$$F = \bar{F}$$

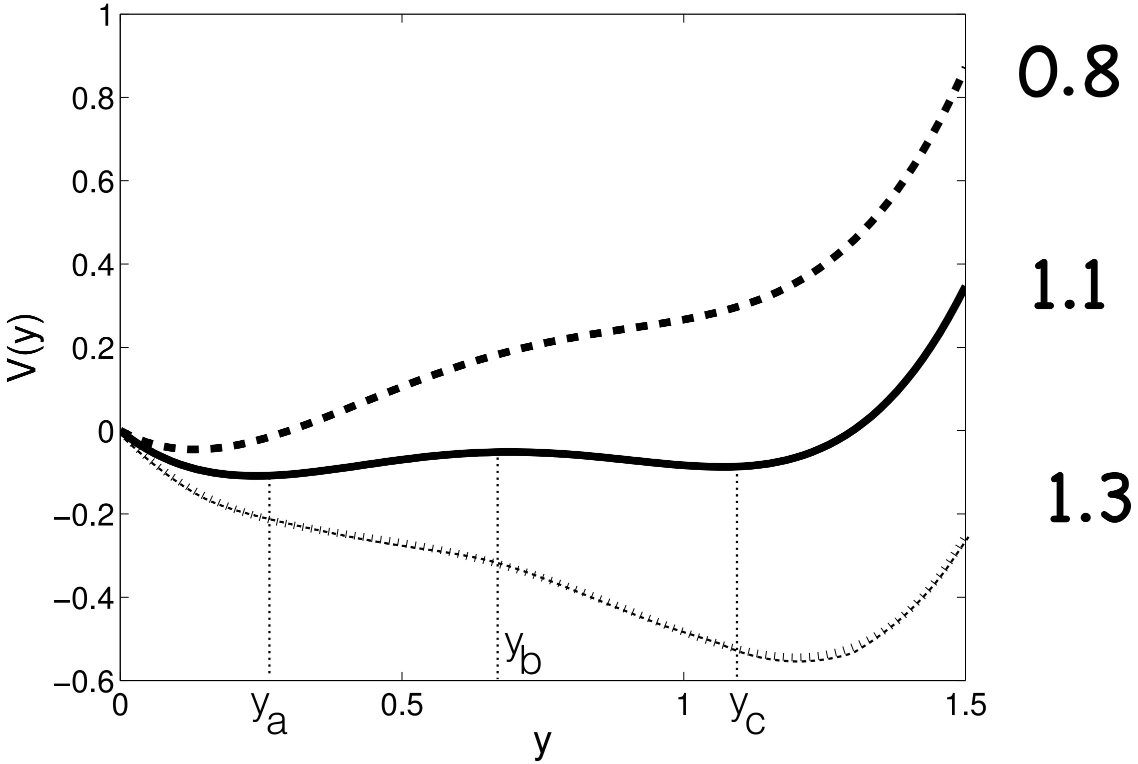


Equilibria dimensionless model

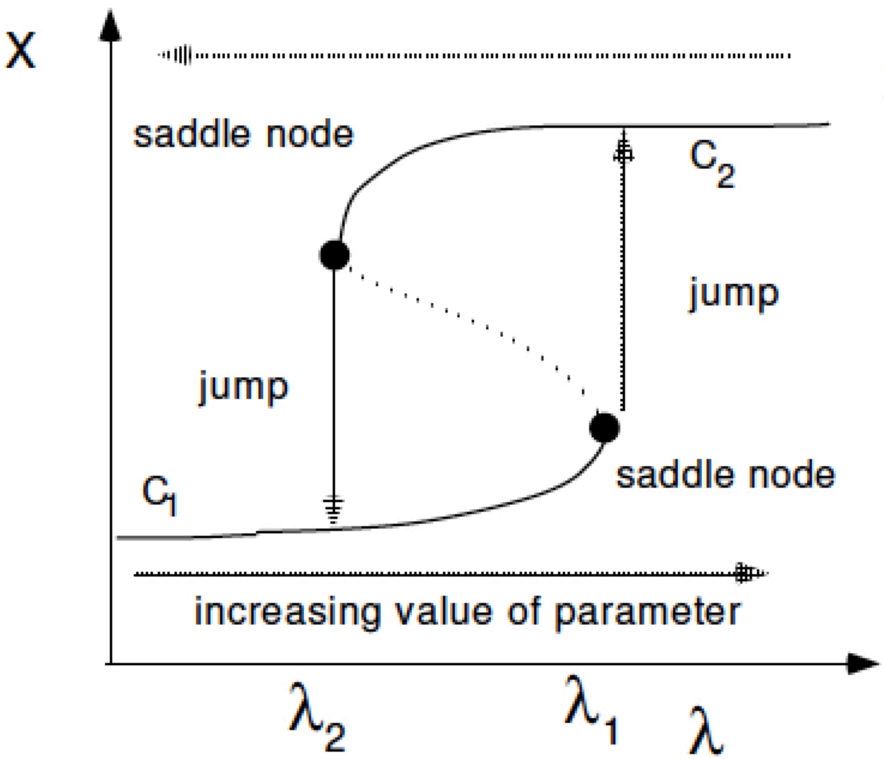
$$\alpha \rightarrow \infty$$

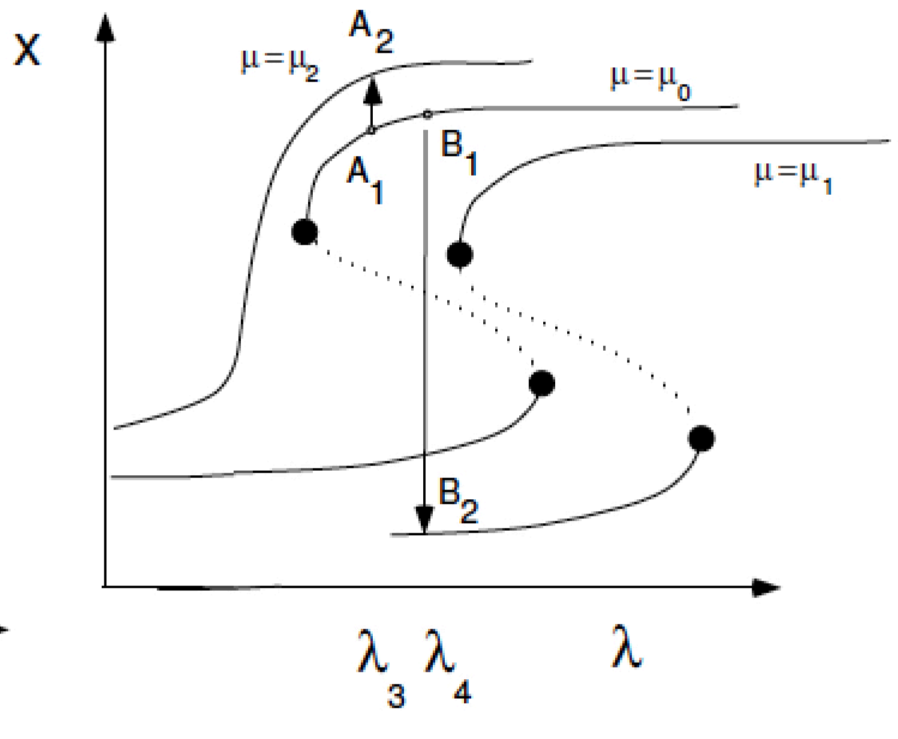
$$\dot{y} = -V'(y) \; ; \; V(y) = -(Fy - \frac{y^2}{2} - \mu^2(\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}))$$

$\bar{F}$



Why bifurcation diagrams?

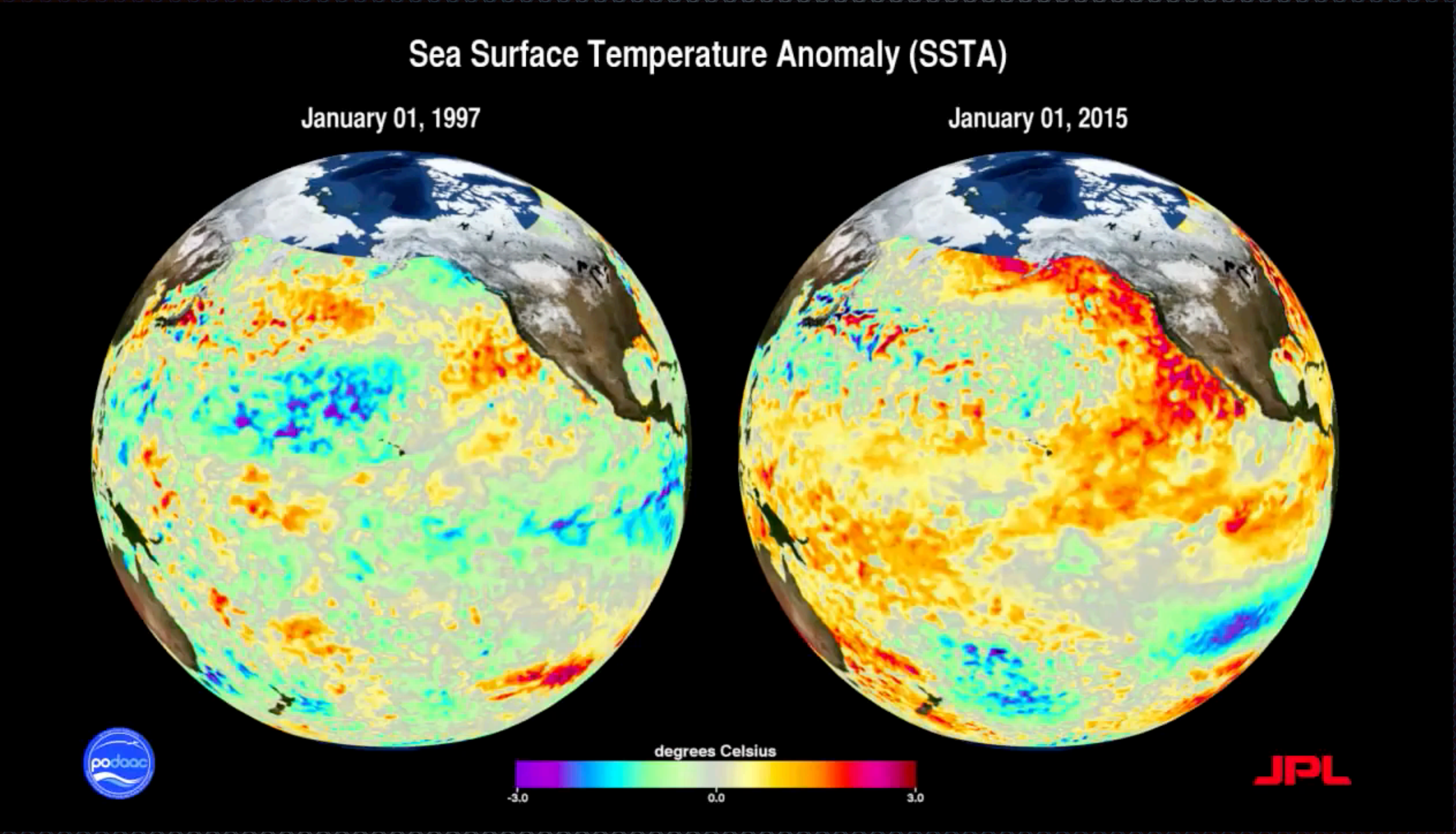




Interpretation sensitivity studies

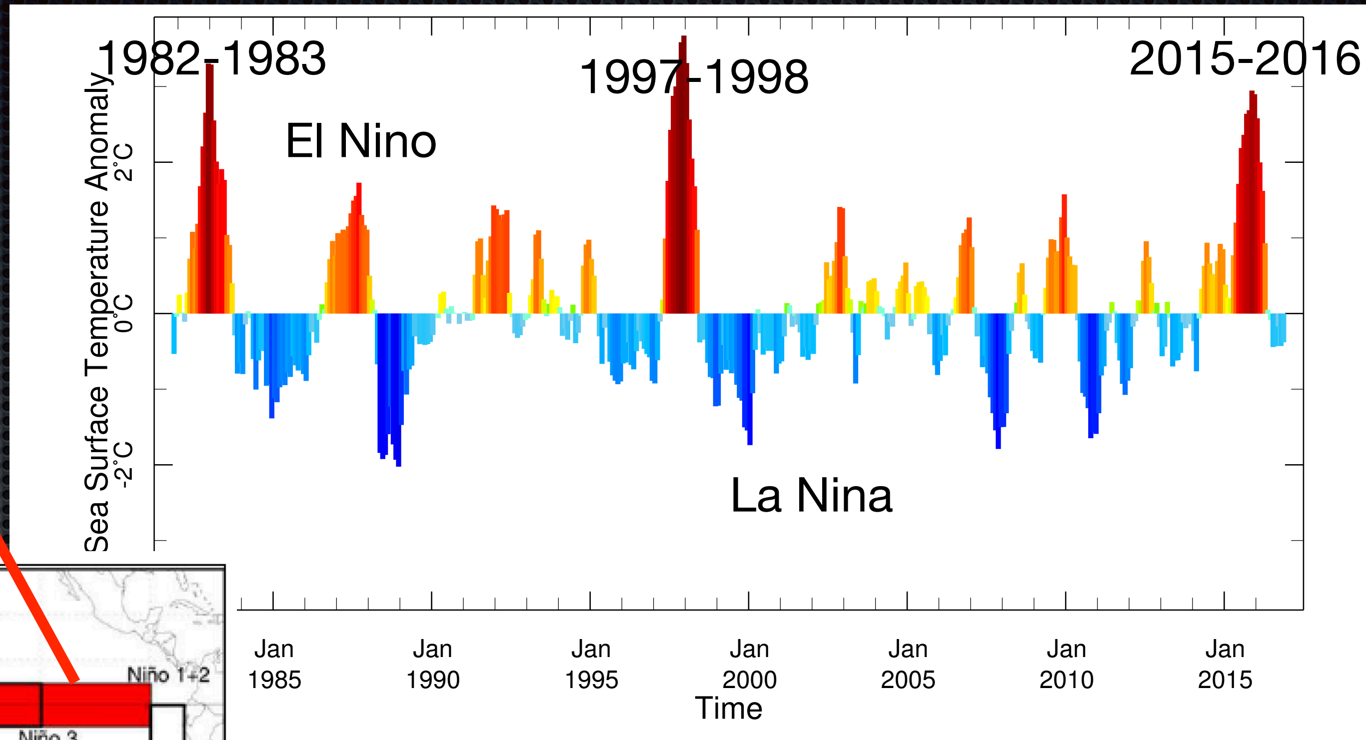
El Nino-Southern Oscillation

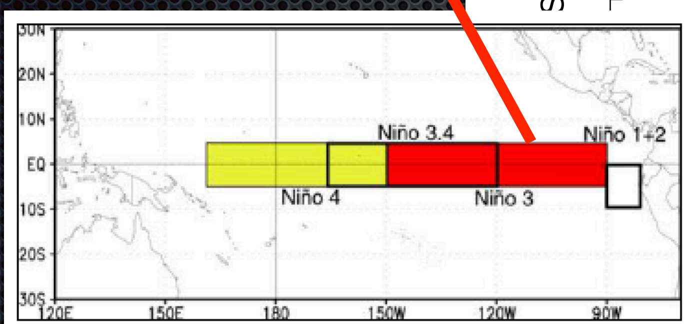
1997-1998 vs 2015-2016

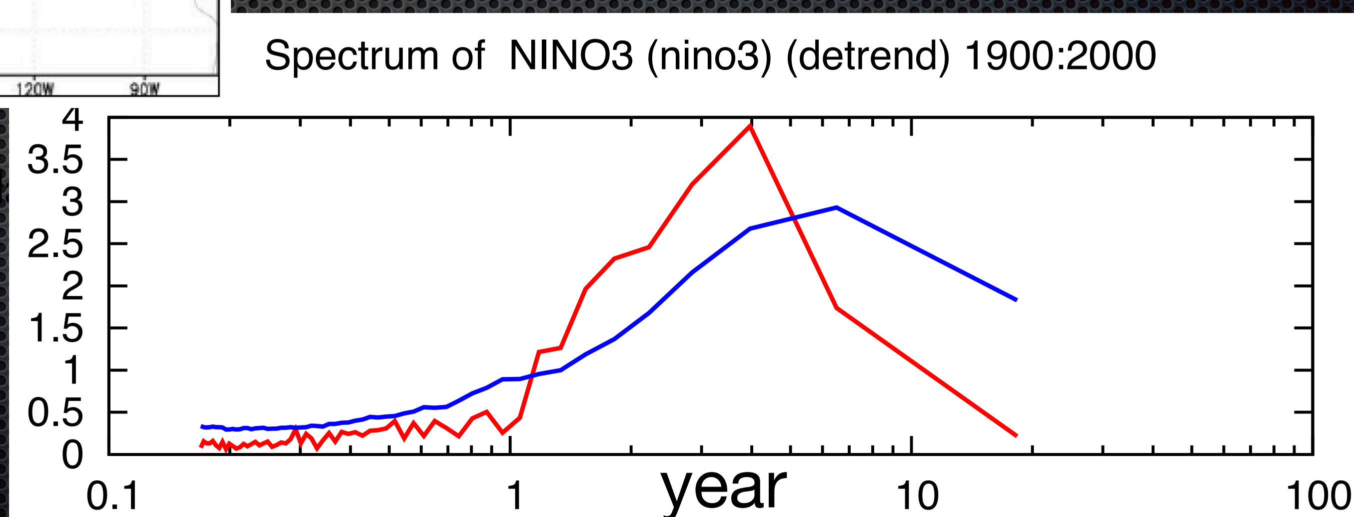


Temporal variability

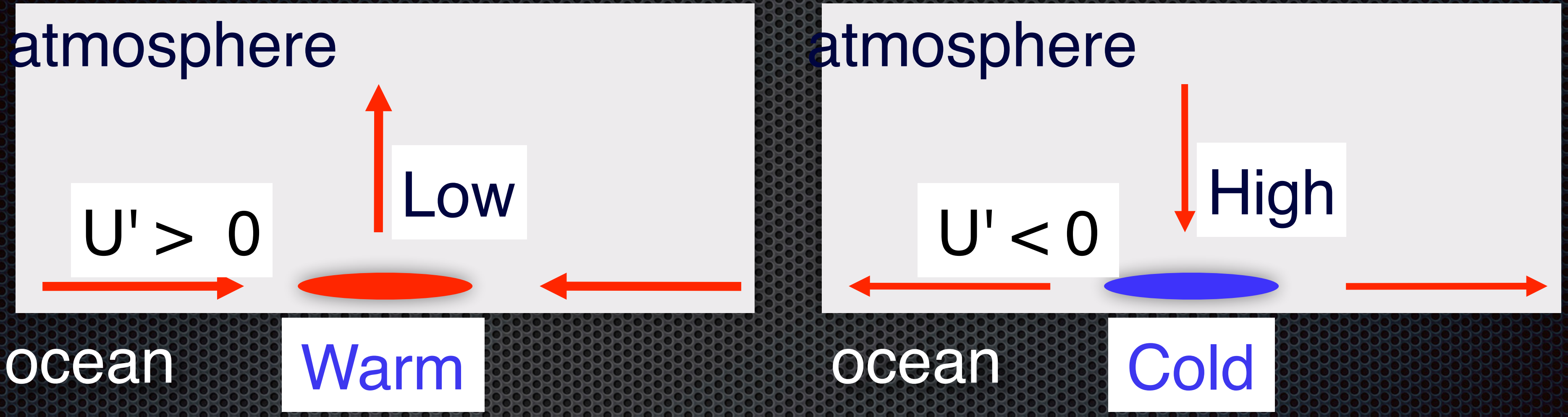
NINO3





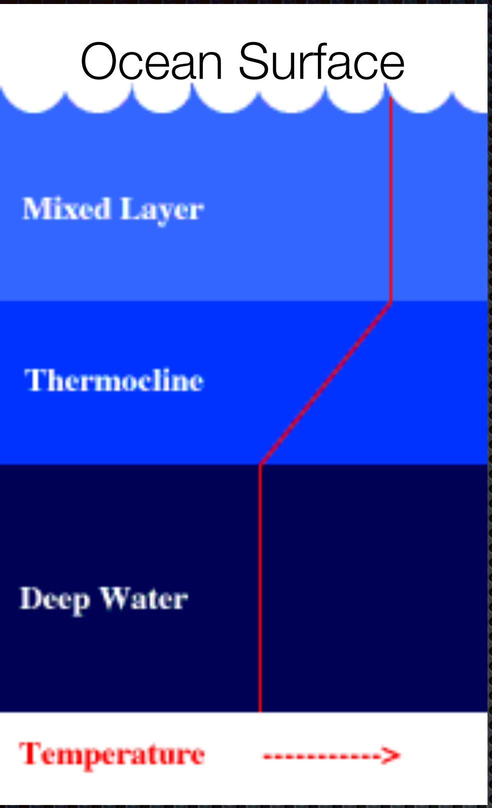


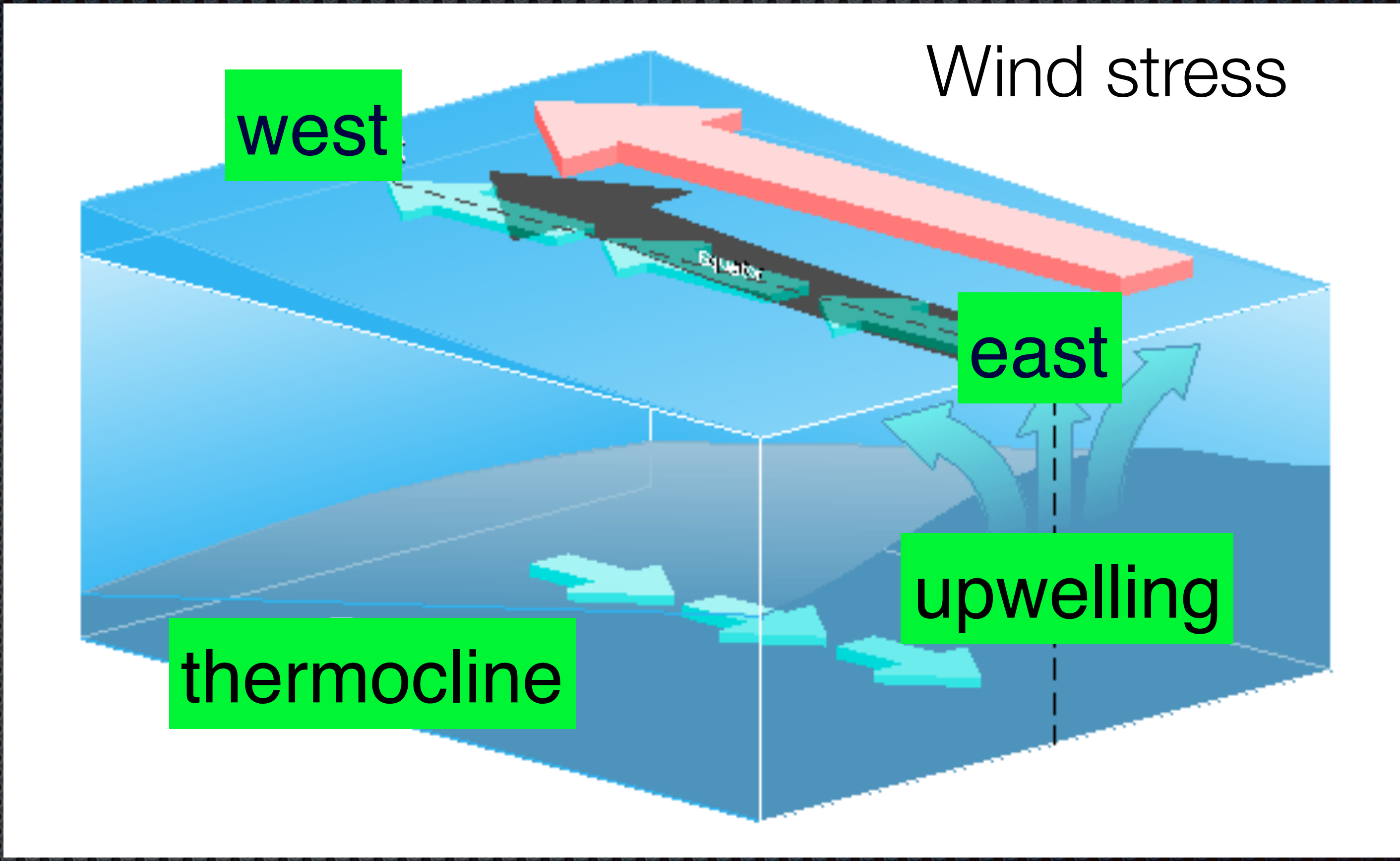
Ingredient 1:
Wind response to sea surface temperature anomalies



A positive sea surface temperature anomaly induces a westerly (towards the east) wind anomaly west of the sea surface temperature anomaly

Ingredient 2:
Effect of winds on ocean upwelling & thermocline slope



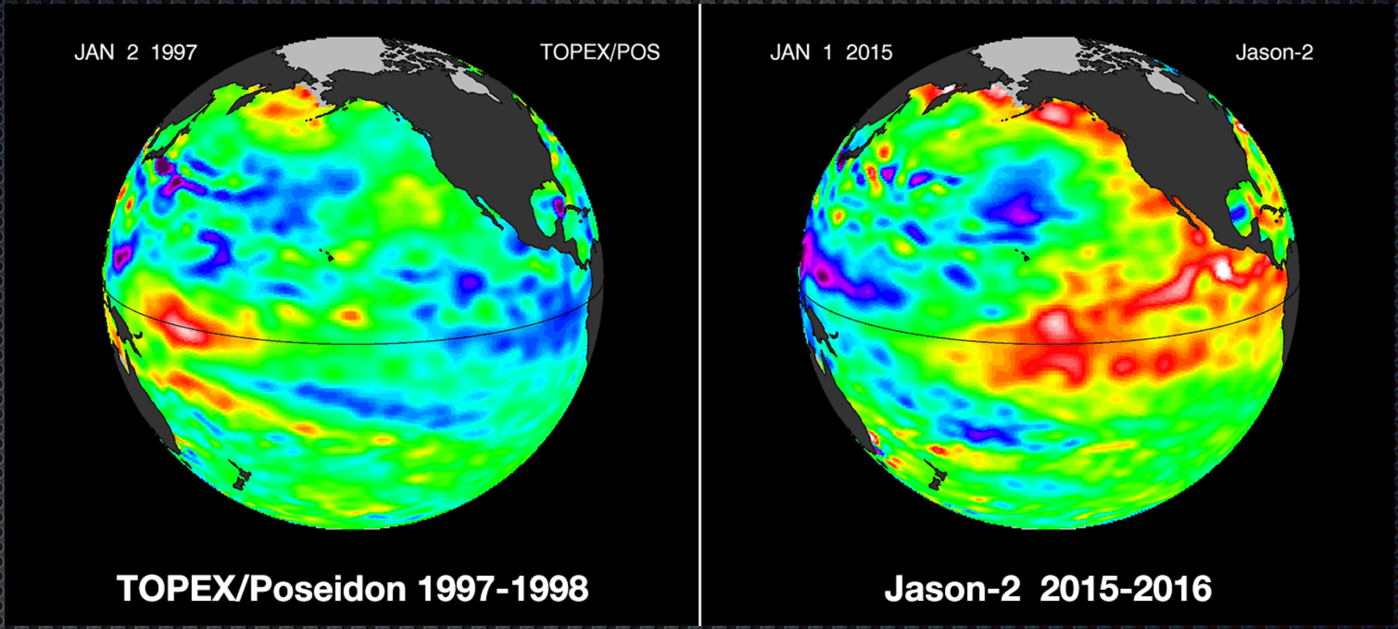


A westerly wind anomaly causes a:
- reduction in upwelling
- smaller thermocline slope

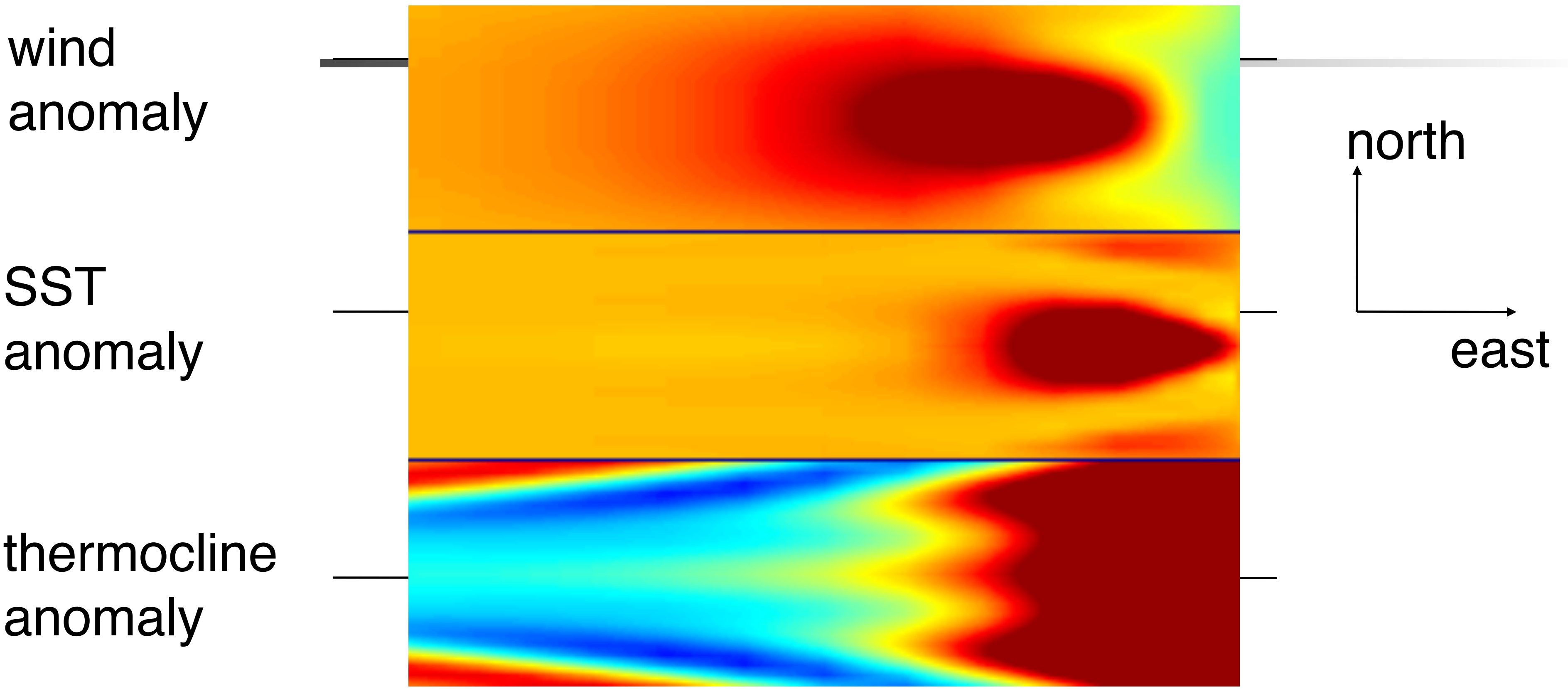
Ingredient 3:

Equatorial ocean wave dynamics

Sea level anomalies

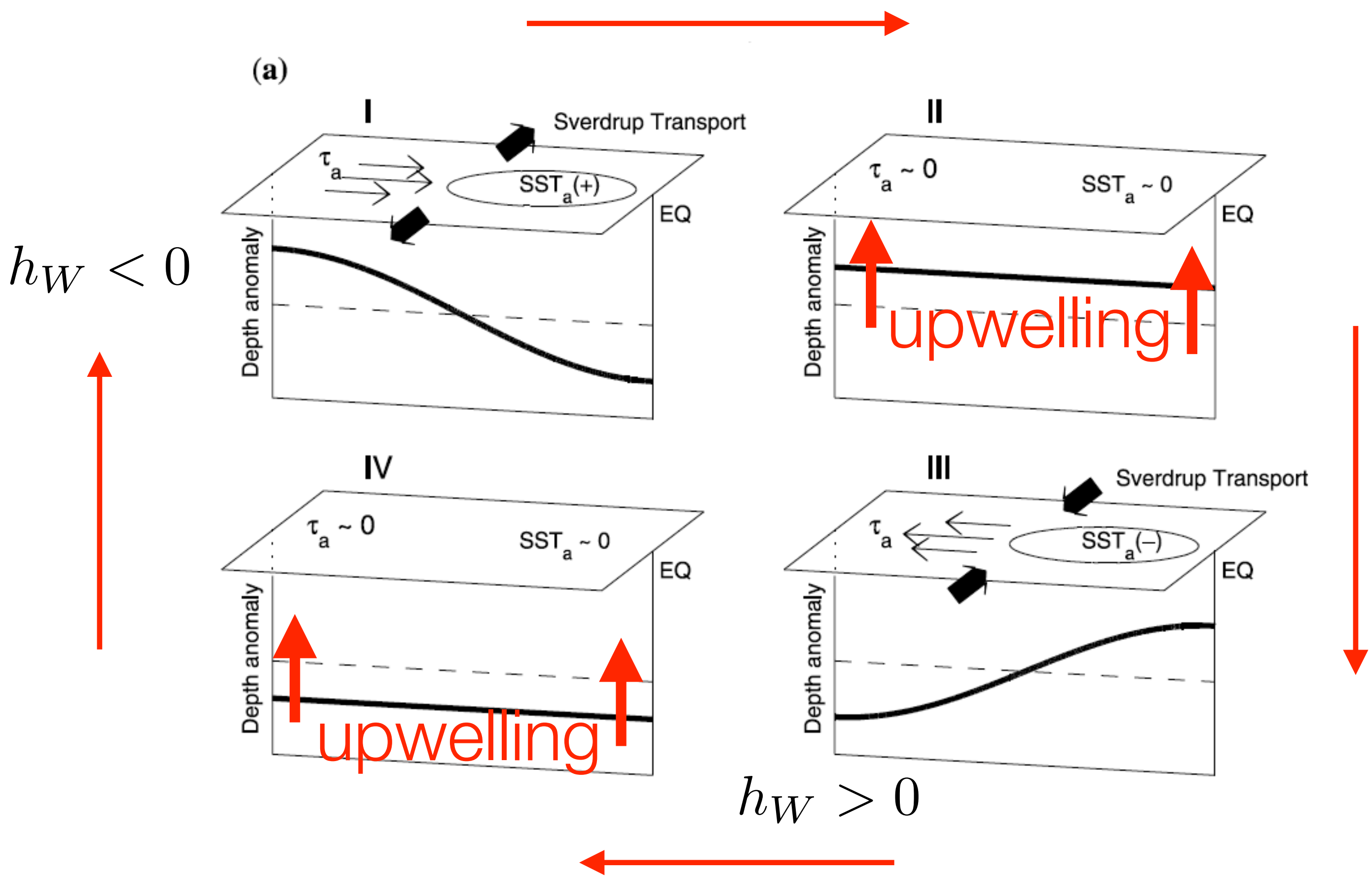


Basic ENSO variability

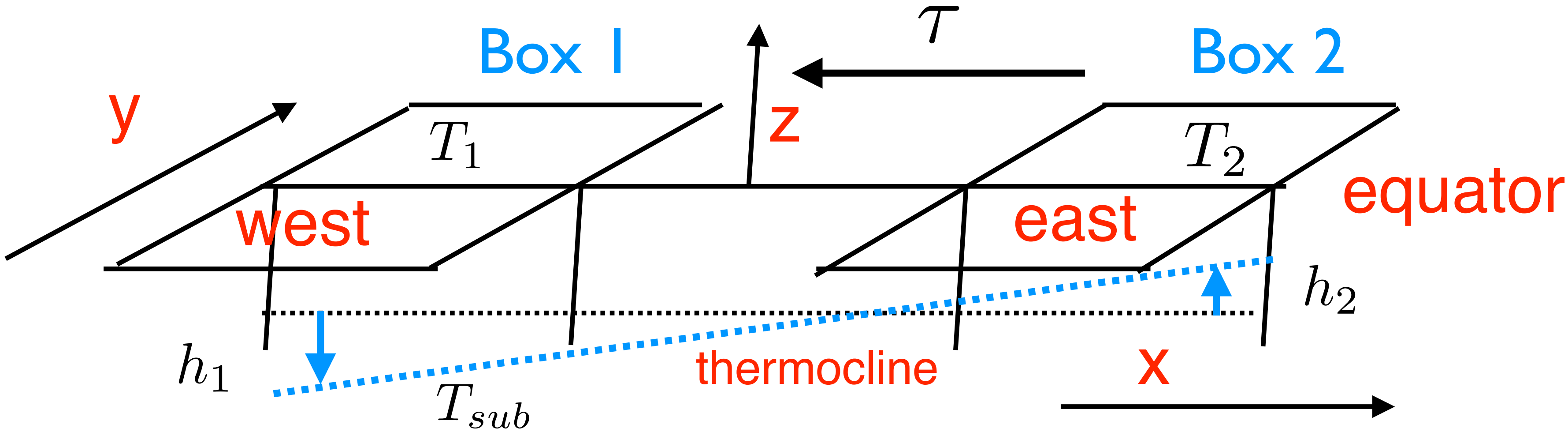


Spatial patterns: background state
Period: ocean wave dynamics + SST adjustment

ENSO mechanism: recharge oscillator



Jin-Timmermann model



$$
\begin{aligned}
h_2 &= h_1 + bL\,\tau, \\
\frac{dh_1}{dt} &= -r(h_1 - \frac{bL}{2}\tau),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dT_1}{dt} &= -\alpha(T_1 - T_r) - \frac{u(T_2 - T_1)}{L/2}, \\
\frac{dT_2}{dt} &= -\alpha(T_2 - T_r) - \frac{w(T_2 - T_{sub}(h_2))}{H_m},
\end{aligned}
$$

$$
\begin{aligned}
\tau &= \frac{\mu}{\beta}(T_2 - T_1), \\
\frac{u}{L/2} &= \mu\epsilon\beta\tau, \\
\frac{w}{H_m} &= -\mu\zeta\beta\tau.
\end{aligned}
$$

$$
T_{sub}(h) = T_r + \frac{T_r - T_{r0}}{2}(1 - \tanh\left[\frac{h + H - z_0}{h_*}\right]),
$$

Dimensionless model

$$\delta = \frac{rbL}{\zeta h^* \beta}, \quad \rho = \frac{\varepsilon h^* \beta}{rbL}, \quad a = \frac{\alpha bL}{\varepsilon \beta h^*}, \quad c = \frac{T_r - T_{r0}}{2S_0},$$

$$k = \frac{H - z_0}{h^*}, \quad S_0 = T_0 = \frac{h^* \beta}{bL\mu}, \quad h_0 = h^*, \quad t_0 = \frac{bL}{\beta \zeta h^*}.$$

Parameter	Value
$T_0 = S_0$	2.8182°C
h_0	62 m
t_0	104.9819 days
δ	0.2625
ρ	0.3224
a	6.8927
k	0.4032
c	2.3952

$$\begin{cases} x' = \rho\delta(x^2 - ax) + x[x + y + c - c\tanh(x + z)], \\ y' = -\rho\delta(ay + x^2), \\ z' = \delta\left(k - z - \dfrac{x}{2}\right), \end{cases}$$

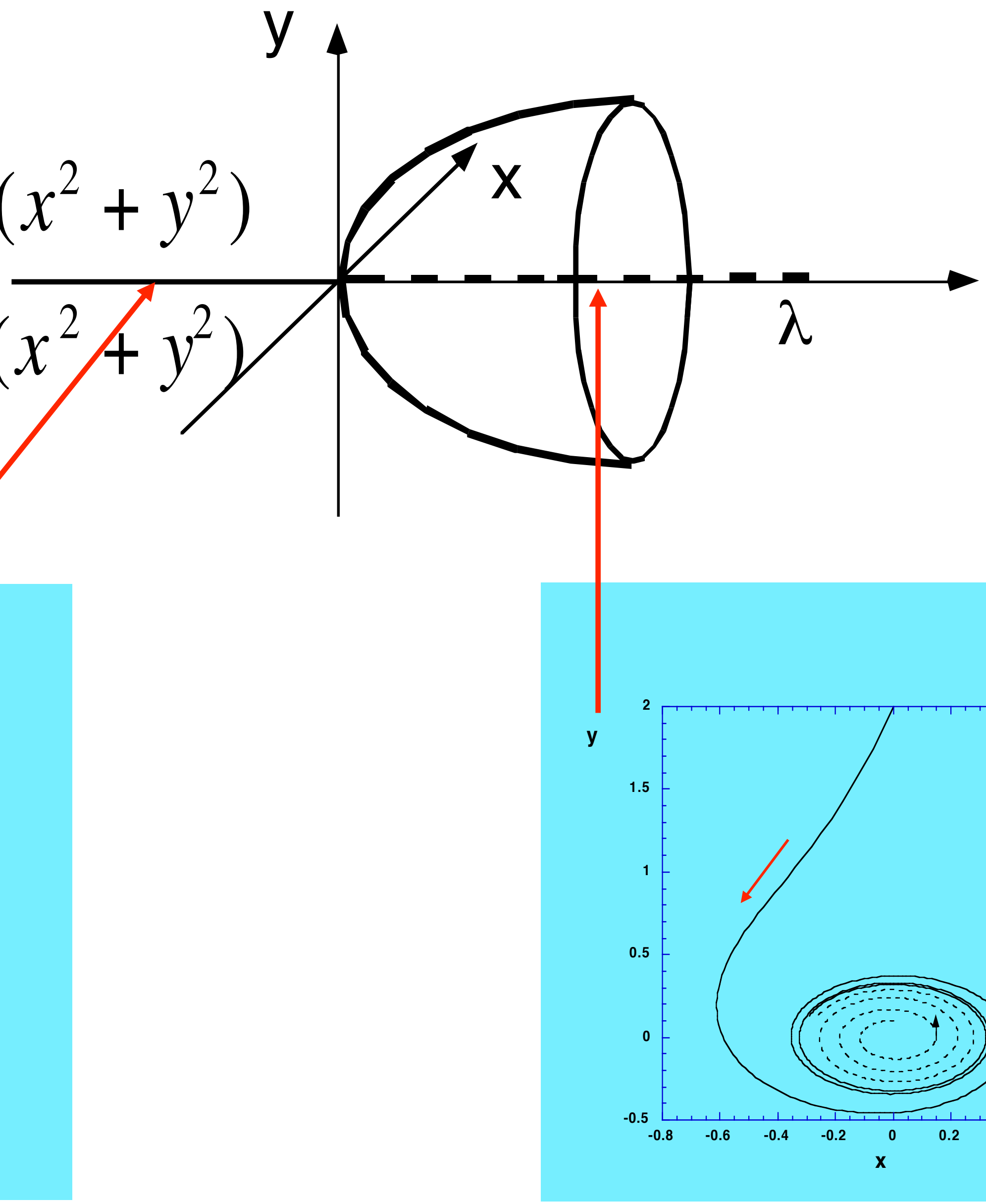
Roberts et al., JAS, (2016)

Hopf bifurcation

Hopf bifurcation

$$\dot{x} = \lambda x - \omega y - x(x^2 + y^2)$$

$$\dot{y} = \lambda y + \omega x - y(x^2 + y^2)$$



Ex: Hopf bifurcation