Notes

20/1/2020

3. After discretization, any model can be written as

$$\frac{d\underline{x}}{dt} = \underline{f}(\underline{x}, \lambda, t) \quad \text{(DS1)}$$

$\underline{x}$: state vector

$\underline{f}$: vector field

λ : parameter

t : time

$\underline{x} \in \mathbb{R}^n$, n: dimension

Two classes:

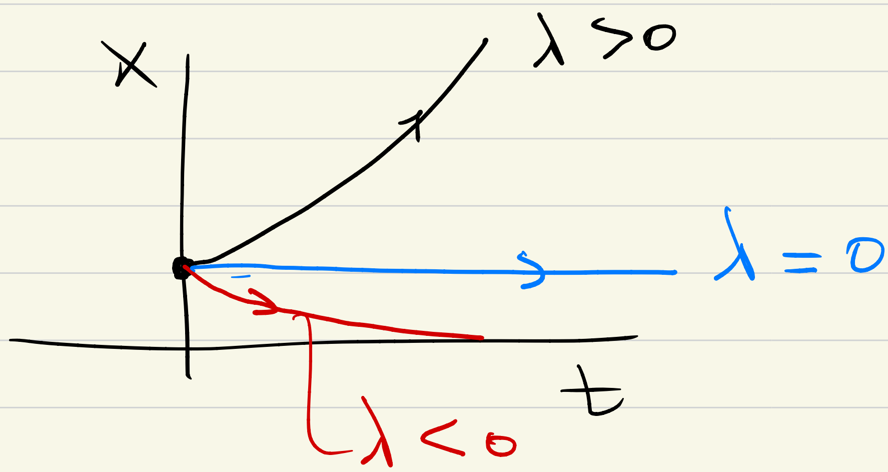
autonomous $\underline{f}(\underline{x}, \lambda)$

non-autonomous $\underline{f}(\underline{x}, \lambda, t)$

Example 1 : $\dot{x} = \lambda x$

$n = 1$

Solution: $x(t) = x_0 e^{\lambda t}$

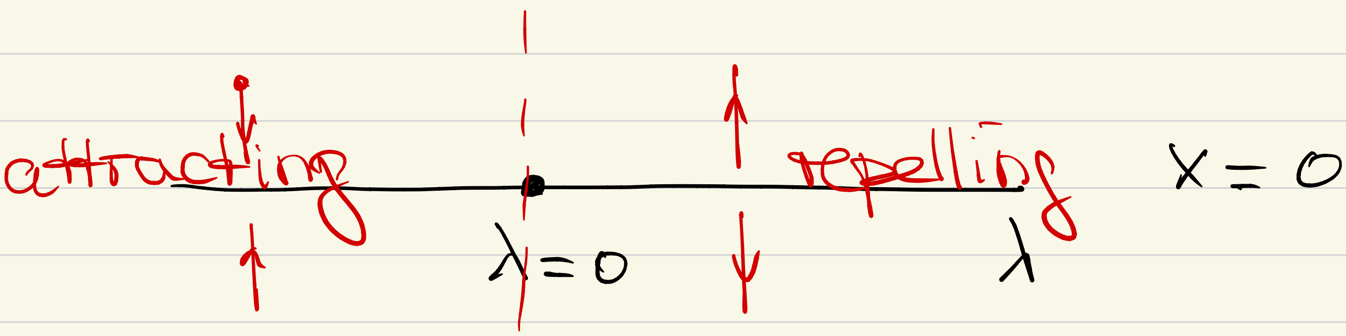


Concepts : $x(t)$ trajectory

x_0 initial condition

Fixed points : $\dot{x} = 0$

$\longrightarrow \bar{x} = 0 \quad \forall \lambda \in \mathbb{R}$



Example 1 : $\dot{x} = -x + t$

$$x(s) = x_0$$

Solution : $x_H = e^{-t}$

P.S. : $x_p = \alpha(t) e^{-t}$

$$\rightarrow \quad \dot{\alpha} e^{-t} - \cancel{\alpha e^{-t}} = -\cancel{\alpha e^{-t}} + t$$

$$\longrightarrow \quad \dot{\alpha} = t e^{+t}$$

$$\alpha(t) = \int t' e^{t'} dt'$$

$$= t e^{t} - \int e^{t'} dt'$$

$$= e^{t} (t-1)$$

Total

$$\begin{cases} x(t) = t - 1 + C e^{-t} \\ x(s) = x_0 \end{cases}$$

$$\rightarrow \quad s - 1 + C e^{-s} = x_0$$

$$C = e^{s} (x_0 + 1 - s)$$

Final solution

$$x(t) = t - 1 + e^{s-t}\left(x_0 + 1 - s\right)$$

$$= x_0 e^{s-t} + t - s e^{s-t} - \left(1 - e^{s-t}\right)$$

✓

Fix s and $t \to \infty$: $x(t)$ unbounded

Fix t and $s \to -\infty$:

$$x(t) \longrightarrow t - 1$$

pullback attractor.

show picture (corrected)

<u>Example 3</u> $(n = 2)$

$$\dot{x} = -2x - 3y$$
$$\dot{y} = 3x - 2y$$

<u>Vector plot</u> $\longrightarrow$ Phase portrait

- stable focus

<u>General</u>:

$$\dot{\underline{x}} = A\,\underline{x}$$

with

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det A \neq 0$$

$$\underline{\tilde{x}} = T\,\underline{x} \longrightarrow \dot{\underline{\tilde{x}}} = T\,\dot{\underline{x}}$$
$$= T A\,\underline{x}$$
$$= T A T^{-1}\,\underline{\tilde{x}}$$

$\boxed{7}$

$$\dot{\tilde{\underline{x}}} = B\,\tilde{\underline{x}} \qquad B = TAT^{-1}$$

where B has the same eigenvae σ as A. $\underline{x} = 0$ fixed point.

Case 1. $\sigma_1 \neq \sigma_2$ real

Case 2. $\sigma_{1,2} = \alpha \pm i\beta$

other special cases

1. $\sigma_1 < 0 < \sigma_2$

$$B = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

$$\begin{cases} \tilde{x}(t) = e^{\sigma_1 t} c_1 \\ \tilde{y}(t) = e^{\sigma_2 t} c_2 \end{cases}$$

Example 4.

$$\dot{x} = x + 2y$$
$$\dot{y} = 2x + y$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

eigenvalues: $0 = \begin{vmatrix} 1-\sigma & 2 \\ 2 & 1-\sigma \end{vmatrix}$

$\longrightarrow \quad (1-\sigma)^2 - 4 = 0$

$\sigma_1 = -1 , \quad \sigma_2 = 3$

$\longrightarrow$ saddle

Show all cases on a slide!

Python/Matlab phase plots

Summary

- trajectory
- fixed point
- attractor | repeller
- phase portrait

Still linear models $n = 1, 2$.

4. Conceptual climate models

A AMOC

- explain
- model results

Box model Stommel - type

(i) - equations slide

(ii) $\Delta T = T_e - T_p$

$$\rightarrow \quad \frac{d\Delta T}{dt} = -\frac{1}{t_r}\left(\Delta T - \theta\right) - Q(\Delta g)\,\Delta T$$

$$\Delta S = S_e - S_p$$

$$\rightarrow \quad \frac{d\Delta S}{dt} = \frac{F_s}{H} S_e - Q(\Delta g)\,\Delta S$$

$$\tilde{t} = t / t_d$$

(iii) $\Delta T = x\,\theta$, $\Delta S = y \frac{\alpha_T}{\alpha_S}\theta$

$$\rightarrow Q(\Delta\rho) = \frac{1}{t_d} + \frac{q}{\rho_0^2 V}(\Delta\rho)^2$$

$$\Delta\rho = \rho_e - \rho_p =$$

$$= -\alpha_T \Delta T + \alpha_S \Delta S$$

$$= -\alpha_T \theta x + \alpha_T \theta y$$

$$= \theta \alpha_T (y - x)$$

$$\Rightarrow \frac{\cancel{\theta}}{t_d}\dot{x} = -\frac{1}{t_r}(x\cancel{\theta} - \overset{1}{\cancel{\theta}}) +$$

$$+ \left(\frac{1}{t_d} + \frac{q}{V}\theta^2\alpha_T^2(y-x)^2\right) *$$

$$* \; x\,\cancel{\theta}$$

$$\longrightarrow \dot{x} = -\alpha(x-1) + x \, *$$

$$* \left(1 + \mu^2 (y-x)^2\right)$$

$$\frac{\frac{\alpha_T \theta}{\alpha_s}}{t_d} \dot{y} = \frac{F_s}{H} S_0 -$$

$$- \left(\frac{1}{t_d} + \frac{\mu^2}{t_d} (y - x)^2 \right) \frac{\alpha_T}{\alpha_s} \theta \, y$$

$$\rightarrow \quad \dot{y} = F - \left(1 + \mu^2 (y - x)^2 \right) y$$

where

$$\boxed{\alpha} = \frac{t_d}{t_r}$$

$$\boxed{\mu^2} = \frac{9 \theta^2 \alpha_T^2 t_d}{V}$$

$$\boxed{F} = \frac{F_s}{H} S_0 \cdot \frac{t_d \alpha_s}{\alpha_T \theta}$$

<u>Values</u> : slide

$$\alpha = 325$$

$$\mu^2 = 6.2 \qquad F = 1.1$$

Finally

$$\begin{cases} \dot{x} = -\alpha(x-1) - x\left(1+\mu^2(x-y)^2\right) \\ \dot{y} = F - y\left(1+\mu^2(x-y)^2\right) \end{cases}$$

Control parameter : F.

Show different phase portraits for $F = 0.5$, 1.1 & 2.0