Assignment 2

Sunday, January 24, 2016 2:02 PM

1. (a) i) scalar part

$x = (0, \vec{x})$, $Q = (q_0, \vec{q})$ $\therefore Q^* = (q_0, -\vec{q})$

$\therefore \vec{x} = 3\times 1$, $\vec{q} = 3\times 1$

According to textbook P33, $Q\cdot P = (q_0 p_0 - \vec{q}\cdot\vec{p},\ q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p})$

$\therefore QXQ^* = (q_0, \vec{q})\cdot(0, \vec{x})\cdot(q_0, -\vec{q})$

$= (0 - \vec{q}\cdot\vec{x},\ q_0\vec{x} + \vec{q}\times\vec{x})\cdot(q_0, -\vec{q})$

$= (-\vec{x}^T\vec{q},\ q_0\vec{x} + \vec{q}\times\vec{x})\cdot(q_0, -\vec{q})$

$$= \begin{bmatrix} -q_0\vec{x}^T\vec{q} + q_0\vec{x}^T\vec{q} + \vec{q}\times\vec{x}\cdot\vec{q} \\ x^T\vec{q}\,\vec{q} + q_0^2\vec{x} + q_0\vec{q}\times\vec{x} - q_0\vec{x}\times\vec{q} - (\vec{q}\times\vec{x})\times\vec{q} \end{bmatrix}$$

$\because$ $\vec{q}\times\vec{x}$ must be perpendicular to the plane of $\vec{q}, \vec{x}$

$\therefore \vec{q}\times\vec{x} \perp \vec{q}$ $\therefore \vec{q}\times\vec{x}\cdot q = 0$ $\therefore QXQ^* = \begin{bmatrix} 0 \\ \cdots \end{bmatrix}$

$\therefore$ It's a pure quaternion. QED

(a) ii)

vector part of QXQ^*

$= x^T\vec{q}\,\vec{q} + q_0^2\vec{x} + q_0\vec{q}\times\vec{x} - q_0\vec{x}\times\vec{q} - (\vec{q}\times\vec{x})\times\vec{q}$

$= x^T\vec{q}\,\vec{q} + q_0^2\vec{x} + 2q_0\vec{q}\times\vec{x} - (\vec{q}\times\vec{x})\times\vec{q}$

$\because (\vec{q}\times\vec{x})\times\vec{q} = -\vec{q}(\vec{x}\cdot\vec{q}) + \vec{x}(\vec{q}\cdot\vec{q})$

$\therefore = q_0^2\vec{x} - (\vec{q}\cdot\vec{q})\vec{x} + 2q_0\vec{q}\times\vec{x} + 2\vec{q}\cdot\vec{x}\cdot\vec{q}$

$\because \vec{q}\cdot\vec{x}\cdot\vec{q} = (\vec{q}\cdot\vec{x})\cdot\vec{q} = (\vec{x}\cdot\vec{q})\cdot\vec{q}$

$\therefore = (q_0^2 - \vec{q}\cdot\vec{q})\vec{x} + 2(q_0(\vec{q}\times\vec{x}) + (\vec{x}\cdot\vec{q})\vec{q})$

$\therefore$ QED

(a) iii)

Let $R = e^{\hat{w}\theta}$ $\therefore$ According to the textbook P34, the quaternion for rotation $R = e^{\hat{w}\theta}$ is $Q = (\cos\frac{\theta}{2}, w\sin\frac{\theta}{2})$ (θ – angle, w – axis 3×1)

$\therefore$ We need to prove that $Rx = QXQ^*$.

$$\therefore \text{Right hand side} = \begin{bmatrix} 0 \\ (q_0^2 - \vec{q}\cdot\vec{q})\vec{x} + 2[q_0(\vec{q}\times\vec{x}) + (\vec{x}\cdot\vec{q})\vec{q}] \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ (\cos^2\frac{\theta}{2} - w\cdot w\sin^2\frac{\theta}{2})\vec{x} + 2\cos\frac{\theta}{2}\cdot\vec{w}\sin\frac{\theta}{2}\times\vec{x} \\ + 2(\vec{x}\cdot\vec{w}\sin\frac{\theta}{2})\vec{w}\sin\frac{\theta}{2} \end{bmatrix}$$

$\because \|\vec{w}\| = 1 \therefore \vec{w}\cdot\vec{w} = \vec{w}^T\vec{w} = 1$, $\vec{x}\cdot\vec{w} = \vec{w}^T\vec{x}$

$\therefore 2\sin^2\frac{\theta}{2}(\vec{x}\,\vec{w})\cdot\vec{w} = 2\sin^2\frac{\theta}{2}(\vec{w}^T\vec{x})\cdot\vec{w} = (1-\cos\theta)(\vec{w}^T\vec{x})\cdot\vec{w}$

$$\therefore = \begin{bmatrix} 0 \\ \cos\theta\,\vec{x} + \sin\theta\cdot\vec{w}\times\vec{x} + (1-\cos\theta)(\vec{w}^T\vec{x})\vec{w} \end{bmatrix}$$

let $\vec{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $\vec{x} = \begin{bmatrix} i \\ j \\ k \end{bmatrix}$ $\therefore (\vec{w}^T\vec{x})\vec{w} = \begin{bmatrix} (ai+bj+ck)a \\ (ai+bj+ck)b \\ (ai+bj+ck)c \end{bmatrix}$

$$\vec{w}\vec{w}^T\vec{x} = \begin{bmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{bmatrix}\begin{bmatrix} i \\ j \\ k \end{bmatrix} = \begin{bmatrix} a^2i + abj + ack \\ bai + b^2j + bck \\ cai + cbj + c^2k \end{bmatrix}$$

$\therefore (\vec{w}^T\vec{x})\vec{w} = \vec{w}\vec{w}^T\vec{x}$

According to the textbook P28 (2.12), we have

$\hat{a}^2 = aa^T - \|a\|^2 I$ $\therefore \hat{w}^2\vec{x} = \vec{w}\vec{w}^T\vec{x} - \|w\|^2 I\vec{x}$

$= \vec{w}^T\vec{x}\vec{w} - I\vec{x}$

$$\therefore = \begin{bmatrix} 0 \\ \cos\theta I \vec{x} + \sin\theta \, \vec{w} \times \vec{x} + (1-\cos\theta) \left(\hat{w}^2 x + I\vec{x} \right) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ I\vec{x} + \sin\theta \, \hat{w} \vec{x} + (1-\cos\theta) \, \hat{w}^2 x \end{bmatrix}$$ textbook P_{28} | (2.14)

$\therefore$ vector part $= \left(I + \sin\theta \hat{w} + (1-\cos\theta) \hat{w}^2 \right) \vec{x} = e^{\hat{w}\theta} x$

$\therefore (q_0^2 - \vec{q}\cdot\vec{q}) \vec{x} + 2\left[q_0 (\vec{q} \times \vec{x}) + (\vec{x}\cdot\vec{q})\vec{q} \right] = e^{\hat{w}\theta} x$ $\therefore$ It's the point after rotation Q $\therefore$ QED

(b)

$\because$ According to (a), QXQ^* (Q-unit quaternion) describes to point of rotation under Q

$\therefore$ ① let $Q_1 = (q_0, \vec{q})$

$$\therefore Q_1 X Q_1^* = \begin{bmatrix} 0 \\ (q_0^2 - \vec{q}\cdot\vec{q})\vec{x} + 2\underbrace{\left[q_0 (\vec{q}\times\vec{x}) + (\vec{x}\cdot\vec{q})\vec{q} \right]}_{\text{}} \end{bmatrix}, \quad R_1 x = \ldots$$

② let $Q_2 = (-q_0, -\vec{q})$

$\therefore Q_2 X Q_2^*$

$$= \begin{bmatrix} 0 \\ \left[(-q_0)^2 - (-\vec{q})\cdot(-\vec{q})\right]\vec{x} + 2\left\{ (-q_0)(-\vec{q}\times\vec{x}) + \left[\vec{x}\cdot(-\vec{q})\right](-\vec{q}) \right\} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ (q_0^2 - \vec{q}\cdot\vec{q})\vec{x} + 2\left(q_0(\vec{q}\times\vec{x}) + (\vec{x}\cdot\vec{q})\vec{q} \right) \end{bmatrix} = Q_1 X Q_1^*, \quad R_2 x = \ldots$$

$\therefore R_1 = R_2$. for each R, there are 2 Quaternions $\therefore$ QED

(c) i)
$$\begin{bmatrix} a_1 & d_1 & g_1 \\ b_1 & e_1 & h_1 \\ c_1 & f_1 & i_1 \end{bmatrix} \cdot \begin{bmatrix} \ldots \end{bmatrix} = \begin{bmatrix} 1^{st}\text{row}\cdot 1^{st}\text{col} & 1^{st}\text{row}\cdot 2^{nd}\text{col} & \ldots \\ & \ldots & \\ & \ldots & \end{bmatrix}$$

$$\cancel{\begin{bmatrix} b_1 & e_1 & h_1 \\ c_1 & f_1 & i_1 \end{bmatrix} \cdot \begin{bmatrix} \cdots \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix}}$$

$$= \begin{bmatrix} 3\otimes 2\oplus & \cdots \\ \cdots \\ \cdots \end{bmatrix} \quad \therefore \begin{array}{l} 2\times 9 = 18 \text{ additions} \\ 3\times 9 = 27 \text{ multiplications} \end{array}$$

ii) $Q\cdot P = (q_0p_0 - \vec{q}\cdot\vec{p},\ q_0\vec{p} + p_0\vec{q} + \vec{q}\times\vec{p})$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf = 3\otimes\ 2\oplus$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \times \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ae\vec{k} - af\vec{j} - bd\vec{k} + bf\vec{i} + cd\vec{j} - ce\vec{i} = \begin{bmatrix} bf-ce \\ cd-af \\ ae-bk \end{bmatrix}$$

$$= 6\otimes\ 3\oplus$$

$\therefore Q\cdot P:\ \times: 1+3+3+3+6 = 16 \quad \therefore 16\times$

$\quad\quad +: 1+2+3+3+3 = 12 \quad 12+$

iii) $$\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot \begin{bmatrix} j \\ l \\ m \end{bmatrix} = \begin{pmatrix} aj+dl+gm \\ bj+el+hm \\ cj+fl+im \end{pmatrix} \quad \therefore \begin{array}{l} 9\times \\ 6+ \end{array}$$

iv) $$X' = QXQ^* = \begin{bmatrix} 0 \\ (q_0^2 - \vec{q}\cdot\vec{q})\vec{x} + 2[q_0(\vec{q}\times\vec{x}) + (\vec{x}\cdot\vec{q})\vec{q}] \end{bmatrix}$$

$\therefore \times: 1+3+3+3+6+3+3+3 = 25\times$

$+: 1+2+3+3+3+2 = 14+$

(d) $\|w\|=1$, ∵ rotation ∴ let the rotation be represented by

$\cancel{R = e^{\hat{w}t}, \therefore Q = (\cos\frac{\theta}{2}, \vec{w}\sin\frac{\theta}{2})}$ ~~∵ unit velocity~~ $\cancel{\therefore \vec{w} = 0}$

$\cancel{\therefore Q^* = (\cos\frac{\theta}{2}, \vec{w}\sin\frac{\theta}{2})}$

$R = e^{\hat{w}t}$, $\therefore Q = (\cos\frac{\theta}{2}, \vec{w}\sin\frac{\theta}{2})$

$\because$ unit velocity $\therefore \dot{\vec{w}} = 0$

$\therefore Q^* = (\cos\frac{\theta}{2}, -\vec{w}\sin\frac{\theta}{2})$

$\dot{Q} = (-\frac{1}{2}\sin\frac{\theta}{2}, \frac{\vec{w}}{2}\cos\frac{\theta}{2})$

$\therefore \dot{Q}Q^* = (-\frac{1}{2}\sin\frac{\theta}{2}, \frac{\vec{w}}{2}\cos\frac{\theta}{2}) \cdot (\cos\frac{\theta}{2}, -\vec{w}\sin\frac{\theta}{2})$

$$= \left(-\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{\vec{w}\cdot\vec{w}}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \ \frac{1}{2}\sin^2\frac{\theta}{2}\vec{w} + \frac{1}{2}\cos^2\frac{\theta}{2}\vec{w} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{\vec{w}\times\vec{w}}{2}\right)$$

$\because \vec{w}\cdot\vec{w} = \vec{w}^T\vec{w} = 1$, $\vec{w}\times\vec{w} = \vec{0}$

$\therefore = (0, \frac{1}{2}\vec{w})$ $\therefore$ QED

2. (a)

We want $e^{\hat{\xi}\theta} \in SE(3)$

① $\vec{w} = \vec{0}$

$\therefore \hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$ $\therefore (\hat{\xi})^2 = (\hat{\xi})^3 = \cdots = \vec{0}$

$\therefore e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{(\hat{\xi}t)^2}{2!} + \frac{(\hat{\xi}t)^3}{3!} + \cdots$ according to the textbook P40

$= I + \hat{\xi}t$

$\because \hat{\xi}\theta = \begin{bmatrix} 0 & v\theta \\ 0 & 0 \end{bmatrix}$

$$\therefore \underset{3\times3}{e^{\hat{\xi}\theta}} = \underset{3\times3}{I} + \underset{3\times3}{\hat{\xi}}\underset{\text{scalar}}{\theta} = \begin{bmatrix} I & \underset{2\times1}{v\theta} \\ 0 & 1 \end{bmatrix} = \underset{3\times3}{\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}}$$ — homogeneous transformation

$\therefore \hat{\xi} \in se(2) \longrightarrow e^{\hat{\xi}\theta} \in SE(2)$

~~② $\vec{w} \neq \vec{0}$ $\xi = (v, w)$ $\hat{\xi} = [\hat{w} \ v]$ $\hat{\xi}^2 = [\hat{w}^2 \ \hat{w}v]$~~

2) $\vec{w} \neq \vec{0}$

$\xi = (v, w)$. $\underset{3\times 3}{\hat{\xi}} = \begin{bmatrix} \overset{2\times 2}{\hat{w}} & \overset{2\times 1}{v} \\ 0 & 0 \end{bmatrix}$ $\quad \hat{\xi}^2 = \begin{bmatrix} \hat{w}^2 & \hat{w}v \\ 0 & 0 \end{bmatrix}$

But it's hard to prove here. ∴ we want to transform ξ to ξ' (textbook P59)

∴ we use $g = \begin{bmatrix} I & w\times v \\ 0 & 1 \end{bmatrix}$

$$\therefore \hat{\xi}' = g^{-1}\hat{\xi} g = \begin{bmatrix} I & -w\times v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & w\times v \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & w\times v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{w} & \hat{w}(w\times v) + v \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{w} & w\times(w\times v) + v \\ 0 & 0 \end{bmatrix}$$

$\because w\times(w\times v) = \underbrace{w(w\cdot v)}_{= w(w^T v) = w^T v w} - v\underbrace{(w\cdot w)}_{=1}$

$$\therefore = \begin{bmatrix} \hat{w} & w^T v w - v + v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{w} & w^T v w \\ 0 & 0 \end{bmatrix}.$$

$$\therefore (\hat{\xi}')^2 = \begin{bmatrix} \hat{w} & w^T v w \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{w} & w^T v w \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{w}^2 & \hat{w}(w^T v w) \\ 0 & 0 \end{bmatrix}$$

$\because$ $w^T v w$ is a vector on $\vec{w}$ direction $\therefore \hat{w}(w^T v w) = \vec{0}$

$\therefore (\hat{\xi}')^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}$ $\therefore (\hat{\xi}')^3 = \begin{bmatrix} \hat{w}^3 & 0 \\ 0 & 0 \end{bmatrix}$

$$\therefore e^{\hat{\xi}'\theta} = I + \hat{\xi}'\theta + \frac{(\hat{\xi}\theta)^2}{2!} + \cdots = I + \begin{bmatrix} \hat{w} & w^T v w \\ 0 & 0 \end{bmatrix}\theta$$

$$+ \begin{bmatrix} \frac{\hat{w}^2\theta^2}{2} & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} I + \hat{w}\theta + \frac{(\hat{w}\theta)^2}{2} + \cdots & w^T v w\theta \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\hat{w}\theta} & w^T v w\theta \\ 0 & 1 \end{bmatrix}$$

~~According to (2.35), we have~~ $\cancel{e^{\hat{\xi}\theta} = e^{g\hat{\xi}'\theta g^{-1}} = g e^{\hat{\xi}'\theta} g^{-1}}$

According to (2.35), we have $e^{\hat{\xi}\theta} = e^{g\hat{\xi}'\theta g^{-1}} = g e^{\hat{\xi}'\theta} g^{-1}$

$$\therefore e^{\hat{\xi}\theta} = g e^{\hat{\xi}'\theta} g^{-1} = \begin{bmatrix} \overset{2\times 2}{e^{\hat{w}\theta}} & (\overset{2\times 2}{I - e^{\hat{w}\theta}})(\overset{2\times 1}{w \times v}) + \overset{2\times 1}{ww^T v\theta} \\ 0 & 1 \end{bmatrix} \quad w \neq 0.$$

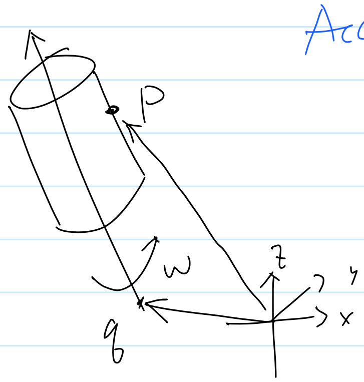
$\because \hat{w} = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}$, $\hat{w}^T = -\hat{w}$ $\therefore \hat{w}$ is skew symm matrix

$\therefore$ According to Textbook P28, $\hat{w} \in so(2)$ $\therefore e^{\hat{w}\theta} \in SO(2)$

$\because (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \in \mathbb{R}^2$ $\therefore$ According to Textbook P35

$\therefore e^{\hat{\xi}\theta} \in SE(2)$ $\therefore$ QED

(b) ① pure rotation:



According to textbook P39

$$\dot{p}(t) = w \times (p(t) - q)$$

$$\therefore \begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{w} & -w \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \hat{\xi} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

$$\therefore \xi = \begin{bmatrix} -w \times q \\ w \end{bmatrix} = \begin{bmatrix} -\hat{w} q \\ w \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & w \\ -w & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} \\ w \end{bmatrix}$$

$\because$ Assume $\|w\| = 1$ $\therefore \xi = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$ $\therefore$ QED

② pure translation $\therefore \hat{w} = 0$

$$\therefore \hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \therefore \xi = \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} \therefore \text{QED}$$

(c) In a moment, any RB motion can be described as SE(2). i.e. $\vec{g} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

$\because$ Now let's assume that there is a point in the rigid body system that is not moving at all ~~in~~ at this moment while the RB is moving like g

$\therefore$ let that point be $A = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{r_a}$

$\therefore \begin{bmatrix} \vec{r_a} \\ 1 \end{bmatrix} = \begin{bmatrix} R & V \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \vec{r_a} \\ 1 \end{bmatrix}$ $\therefore R\vec{r_a} + V = \vec{r_a}$

$\therefore (I - R)\vec{r_a} = V$

$\therefore$ ① $I = R$, i.e. there is no rotation (pure translation)

$\therefore \vec{r_a} = \begin{cases} \infty & \text{if } \vec{V} \neq \vec{0} \rightarrow \text{Entire RB is doing pure translation} \\ \forall \text{ pt in BodyFrame} & \text{if } \vec{V} = \vec{0} \rightarrow \text{Entire RB is Stationary} \end{cases}$

② $I \neq R$ i.e. there is a rotation and translation

$\therefore$ Now we need to verify if $(I-R)$ is singular or not.

$\because R$ in $\mathbb{R}^2$ is 2×2, and it's orthornormal matrix

$\therefore R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore I - R = \begin{bmatrix} 1-\cos\theta & \mp\sin\theta \\ \pm\sin\theta & 1-\cos\theta \end{bmatrix}$ $\therefore \det(I-R)$

$= (1-\cos\theta)^2 + \sin^2\theta$ $\therefore$ let $\det(I-R) = 0$,

i.e. $(\cos\theta - 1)^2 = -\sin^2\theta$ $\because$ LHS and RHS both ≥ 0

$\therefore \begin{cases} \cos\theta = 1 \\ \sin\theta = 0 \end{cases}$ $\therefore \theta = 0$, $\theta \in [0, 2\pi)$ $\therefore$ when $\theta = 0$, $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\therefore$ there is no rotation, $\therefore$ this should be in case 1

$\therefore$ In case 2 when $I \neq R$, $I-R$ is not singular

$\therefore$ $I-R$ is invertible $\therefore$ there must be a solution to

$\vec{r_a} = (I-R)^{-1} V$ $\therefore$ There must be a stationary point although the other points in RB are moving

$\therefore$ We can see the motion as a rotation around point A

The pt A is the pole of motion or Instant centre of rotation $\therefore$ QED

(d) According to the textbook P54

$$g = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$$

$$\hat{V}^S = \dot{g} g^{-1} = \begin{bmatrix} \dot{R} & \dot{P} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T P + \dot{P} \\ 0 & 0 \end{bmatrix}$$

According to Rodrigues' formula in textbook P28

$$e^{\hat{w}\theta} = I + \hat{w}\sin\theta + \hat{w}^2(1-\cos\theta)$$

$$\therefore R = e^{\hat{w}\theta} = I + \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \sin\theta + \begin{bmatrix} -w^2 & 0 \\ 0 & -w^2 \end{bmatrix} (1-\cos\theta)$$

$$= \begin{bmatrix} 1+0-w^2(1-\cos\theta) & -w\sin\theta \\ w\sin\theta & 1-w^2(1-\cos\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1-w^2+w^2\cos\theta & -w\sin\theta \\ w\sin\theta & 1-w^2+w^2\cos\theta \end{bmatrix}$$

$$\therefore \dot{R} = \begin{bmatrix} -w^2\sin\theta & -w\cos\theta \\ w\cos\theta & -w^2\sin\theta \end{bmatrix} \dot{\theta}, \quad R^T = \begin{bmatrix} 1-w^2+w^2\cos\theta & w\sin\theta \\ -w\sin\theta & 1-w^2+w^2\cos\theta \end{bmatrix}$$

$\because \|w\| = 1$ and $w \in \mathbb{R}$ $\therefore w = 1$

$$\therefore \dot{R} = \begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix} \dot{\theta} = \begin{bmatrix} -s\theta & -c\theta \\ c\theta & -s\theta \end{bmatrix} \dot{\theta}$$

$$R^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}$$

$\cancel{\begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} s\theta & c\theta \end{bmatrix}}$

$$\therefore \dot{R}R^T = \begin{bmatrix} -s\theta c\theta + s\theta c\theta & -s^2\theta - c^2\theta \\ c^2\theta + s^2\theta & c\theta s\theta - s\theta c\theta \end{bmatrix}\dot{\theta} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}$$

$$\therefore -\dot{R}R^T p + \dot{p} = \begin{bmatrix} 0 & \dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \end{bmatrix} + \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \begin{bmatrix} p_y\dot{\theta} + \dot{p}_x \\ -p_x\dot{\theta} + \dot{p}_y \end{bmatrix}$$

$$\therefore \vec{V}_s = \begin{bmatrix} p_y\dot{\theta} + \dot{p}_x \\ -p_x\dot{\theta} + \dot{p}_y \\ \dot{\theta} \end{bmatrix}$$

$$\hat{V}_b = g^{-1}\dot{g}$$

$$\therefore R^T\dot{R} = \begin{bmatrix} -s\theta c\theta + s\theta c\theta & -c^2\theta - s^2\theta \\ s^2\theta + c^2\theta & s\theta c\theta - s\theta c\theta \end{bmatrix}\dot{\theta} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix}$$

$$\therefore \vec{V}_b = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix} = \begin{bmatrix} \cos\theta\,\dot{p}_x + \sin\theta\,\dot{p}_y \\ -\sin\theta\,\dot{p}_x + \cos\theta\,\dot{p}_y \\ \dot{\theta} \end{bmatrix}$$

According to textbook P41. a twist is an element in $se(2)$

$$\hat{\xi} = \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

$$\therefore \hat{V}_s = \begin{bmatrix} 0 & -\dot{\theta} & p_y\dot{\theta} + \dot{p}_x \\ \dot{\theta} & 0 & -p_x\dot{\theta} + \dot{p}_y \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ They are both twists

skew symm

$$\hat{V}_b = \begin{bmatrix} 0 & -\dot{\theta} & \cos\theta\,\dot{p}_x + \sin\theta\,\dot{p}_y \\ \dot{\theta} & 0 & -\sin\theta\,\dot{p}_x + \cos\theta\,\dot{p}_y \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \vec{V}_s$ is the velocity of a pt on RB as it travels through the origin of spatial frame, viewing in spatial frame ~~$\begin{bmatrix} p_x\dot{\theta} \\ -p_x\dot{\theta} \end{bmatrix}$ is the velocity coming from rotation~~

~~$\begin{bmatrix} \dot{p}_x \end{bmatrix}$ is the translation~~

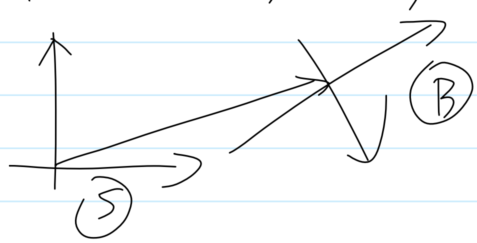
~~t~~ frame, $\begin{bmatrix} p_y\dot\theta \\ -p_x\dot\theta \end{bmatrix}$ is the velocity coming from rotation (similar to "centripetal velocity"), $\begin{bmatrix} \dot p_x \\ \dot p_y \end{bmatrix}$ is the translation velocity

From (b), we get pure rot $\xi = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix}$ and pure translation $\xi = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix}$

$\therefore$ We can view the motion = a pure rotation + a pure translation in spatial frame

On the other hand, $\vec{V_b}$ is the velocity of origin of body frame relative to spatial frame, viewed in body frame

$\therefore$ the $\hat\omega$ part of $\vec{V_b}$ is same as $\vec{V_s}$, but the $\vec{v}$ part only includes rotation, no translation ($\because$ viewed in body frame), and it's $= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot p_x \\ \dot p_y \end{bmatrix}$



(e) $\vec{V^s} = \begin{bmatrix} p_y\dot\theta + \dot p_x \\ -p_x\dot\theta + \dot p_y \\ \dot\theta \end{bmatrix}$ $\quad \vec{V^b} = \begin{bmatrix} \cos\theta\, \dot p_x + \sin\theta\, \dot p_y \\ -\sin\theta\, \dot p_x + \cos\theta\, \dot p_y \\ \dot\theta \end{bmatrix}$

$\because R = \begin{bmatrix} 1-w^2+w^2\cos\theta & -w\sin\theta \\ w\sin\theta & 1-w^2+w^2\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Assume $Ad_g = \begin{bmatrix} R & \begin{bmatrix} p_y \\ -p_x \end{bmatrix} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_y \\ \sin\theta & \cos\theta & -p_x \\ 0 & 0 & 1 \end{bmatrix}$

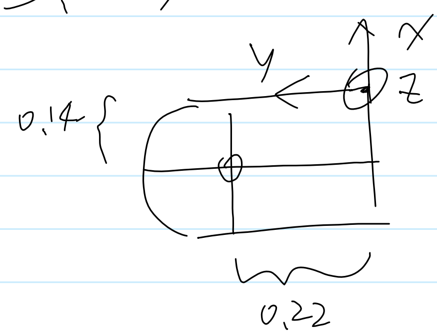
~~$\therefore Ad_g V_b = [\, c^2\theta \dot p_x + \cos\theta \dot p_y + s^2\theta \dot p_x - s\theta c\theta \dot p_y + p_y\dot\theta$~~

$$\therefore Ad_g \vec{V_b} = \begin{bmatrix} c^2\theta \dot{p}_x + \cos\theta \dot{p}_y + s^2\theta \dot{p}_x - s\theta c\theta \dot{p}_y + p_y\dot{\theta} \\ s\theta c\theta \dot{p}_x + s^2\theta \dot{p}_y - \cos\theta \dot{p}_x + c^2\theta \dot{p}_y - p_x\dot{\theta} \\ \dot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{p}_x + p_y\dot{\theta} \\ \dot{p}_y - p_x\dot{\theta} \\ \dot{\theta} \end{bmatrix} = \vec{V_s} \quad \therefore \text{QED}$$

---

3. (a) base is located at (0.75, 0.5, 1)



∴ marker tip is at

$$\begin{bmatrix} 0.75-0.14 \\ 0.5+0.22 \\ 1.0+0.91+0.346+0.12 \end{bmatrix}$$

$$= (0.61,\ 0.72,\ 2.376)$$

(b) code is pasted in this file (problem3.m)
The actual problem3.m file is submitted.
Also the result.txt is submitted

(C) From result.txt we can gather 3 points on the white board:

$A = (0.2743,\ 0.5291\ \ -0.1072)$

$B = (0.3035,\ 0.5210,\ -0.0923)$

$C = (0.3990,\ 0.4618,\ -0.0666)$

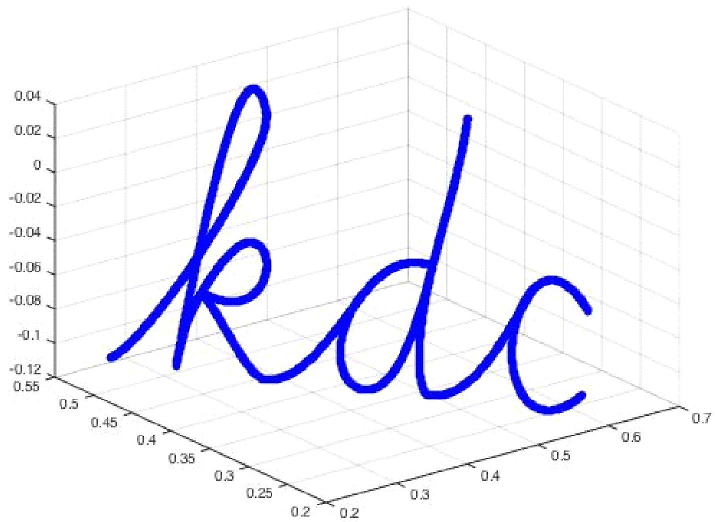
$\therefore \vec{a} = \vec{AB} = (0.0292, -0.0081, 0.0149)$

$\vec{b} = \vec{BC} = (0.0955, -0.0592, 0.0257)$

$\therefore \vec{a} \times \vec{b} = (0.6767, 0.6767, -0.9569) \times 10^{-3}$

$\Rightarrow (0.5, 0.5, -0.7071)$ after being normalized

(d) "KDC":



My program 1: problem3.m

```
clear all; close all; clc;
P = dlmread('JointData.txt',' ');
size = length(P(:,1));
disp (size);

tips = zeros(size, 3);
for i = 1:size
    joints_val = P(i,:);
    % we can use the joint 1 as our reference frame
    % the joint in reference to another joint
    j1_to_B = [cos(joints_val(1)) -sin(joints_val(1)) 0. 0.; ...
        sin(joints_val(1)) cos(joints_val(1)) 0. 0.; 0. 0. 1. 0.; 0. ...
        0. 0. 1.];
    j2_to_j1 = [cos(joints_val(2)) 0. sin(joints_val(2)) 0.; 0. 1. 0. ...
        0.; -sin(joints_val(2)) 0. cos(joints_val(2)) 0.; 0. 0. 0. 1.];
    j3_to_j2 = [cos(joints_val(3)) -sin(joints_val(3)) 0. 0.; ...
        sin(joints_val(3)) cos(joints_val(3)) 0. 0.; 0. 0. 1. 0.; 0. ...
        0. 0. 1.];
    j4_to_j3 = [cos(joints_val(4)) 0. sin(joints_val(4)) 0.045; 0. 1. ...
        0. 0.; -sin(joints_val(4)) 0. cos(joints_val(4)) 0.55; 0. 0. ...
        0. 1.];
    j5_to_j4 = [cos(joints_val(5)) -sin(joints_val(5)) 0. -0.045; ...
        sin(joints_val(5)) cos(joints_val(5)) 0. 0.; 0. 0. 1. 0.3; 0. ...
        0. 0. 1.];
    j6_to_j5 = [cos(joints_val(6)) 0. sin(joints_val(6)) -0.; 0. 1. 0. ...
        0.; -sin(joints_val(6)) 0. cos(joints_val(6)) 0.; 0. 0. 0. 1.];
    j7_to_j6 = [cos(joints_val(7)) -sin(joints_val(7)) 0. 0.; ...
        sin(joints_val(7)) cos(joints_val(7)) 0. 0.; 0. 0. 1. 0.; 0. ...
        0. 0. 1.];
    T_to_j7 = [1. 0. 0. 0.; 0. 1. 0. 0.; 0. 0. 1. 0.12+0.06; 0. 0. 0. 1.];
    tip = ...
        j1_to_B*j2_to_j1*j3_to_j2*j4_to_j3*j5_to_j4*j6_to_j5*j7_to_j6*T_to_j7;
    tip_pos = tip(1:3,4);
    tips(i,:) = tip_pos;
end


fid = fopen('result.txt', 'wt');
for i=1:size
   fprintf(fid, '%d %d %d\n', tips(i,:));
end
fclose(fid);

figure;
scatter3(tips(:,1),tips(:,2),tips(:,3),'blue','filled');
```