KDC Assignment 2

Clint Liddick

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1 Unit Quaternions

1.1

$$
\begin{aligned}
X &= (0, \bar{x}) \\
Q &= (q_0, \bar{q}) \\
Q^* &= (q_0, -\bar{q}) \\
QP &= (q_0 p_0 - \bar{q} \cdot \bar{p}, q_0 \bar{p} + p_0 \bar{q} + \bar{q} \times \bar{p})
\end{aligned}
$$

$$
\begin{aligned}
QXQ^* &= (q_0, \bar{q}) \cdot (0, \bar{x}) \cdot (q_0, -\bar{q}) \\
&= (0 - \bar{q} \cdot \bar{x}, q_0 \bar{x} + \bar{q} \times \bar{x}) \cdot (q_0, -\bar{q}) \\
&= (-\bar{x}^T \bar{q}, q_0 \bar{x} + \bar{q} \times \bar{x}) \cdot (q_0, -\bar{q}) \\
&= (-q_0 \bar{x}^T \bar{q} + q_0 \bar{x}^T \bar{q} + (\bar{q} \times \bar{x}) \cdot \bar{q}, \bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + q_0 \bar{q} \times \bar{x} - q_0 \bar{x} \times \bar{q} - (\bar{q} \times \bar{x}) \times \bar{q}) \\
&= ((\bar{q} \times \bar{x}) \cdot \bar{q}, \bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + q_0 \bar{q} \times \bar{x} - q_0 \bar{x} \times \bar{q} - (\bar{q} \times \bar{x}) \times \bar{q}) \\
\bar{q} \times \bar{x} &\perp \bar{q} \\
\bar{q} \times \bar{x} \cdot \bar{q} &= 0 \\
\therefore QXQ^* &= (0, \ldots) \quad \square \\
&\bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + q_0 \bar{q} \times \bar{x} - q_0 \bar{x} \times \bar{q} - (\bar{q} \times \bar{x}) \times \bar{q} \\
&= \bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + q_0 \bar{q} \times \bar{x} + q_0 \bar{q} \times \bar{x} - (\bar{q} \times \bar{x}) \times \bar{q} \\
&= \bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + 2 q_0 \bar{q} \times \bar{x} - (\bar{q} \times \bar{x}) \times \bar{q} \\
&= \bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + 2 q_0 \bar{q} \times \bar{x} + \bar{q} \times (\bar{q} \times \bar{x}) \\
&= \bar{x}^T \bar{q} \bar{q} + q_0^2 \bar{x} + 2 q_0 \bar{q} \times \bar{x} + -\bar{q}(\bar{x} \cdot \bar{q}) + \bar{x}(\bar{q} \cdot \bar{q}) \\
&= (q_0^2 - \bar{q} \cdot \bar{q}) \bar{x} + 2(q_0 (\bar{q} \times \bar{x}) + (\bar{x} \cdot \bar{q}) \bar{q}) \quad \square
\end{aligned}
$$

1.2

1.3

1.3.1

27 multiplications, 18 additions: matrix multiplication of two 3x3 matrices

1.3.2

16 multiplications, 12 additions: by quaternion product definition

$$pq = (p_0 q_0 + \bar{p} \cdot \bar{q}, p_0 \bar{q} + q_0 \bar{p} + \bar{p} \times \bar{q}) \tag{1}$$

and cross product definition

$$\bar{a} \times \bar{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix} \tag{2}$$

1.3.3

9 multiplications, 6 additions: 3x3 matrix times 3x1 matrix

1.3.4

25 multiplications, 14 additions

$$QXQ^* = (0, (q_0^2 - \bar{q} \cdot \bar{q})\bar{x} + 2(q_0(\bar{q} \times \bar{x}) + (\bar{x} \cdot \bar{q})\bar{q}))$$

1.4

$$\begin{aligned}
R &= e^{\hat{\omega} t} \\
Q &= (\cos\theta/2, \bar{\omega} \sin\theta/2) \\
Q* &= (\cos\frac{\theta}{2}, -\bar{\omega} \sin\frac{\theta}{2}) \\
\dot{Q} &= (-\frac{1}{2}\sin\frac{\theta}{2}, \frac{\bar{\omega}}{2}\cos\frac{\theta}{2}) \\
\dot{Q}Q^* &= (-\frac{1}{2}\sin\frac{\theta}{2}, \frac{\bar{\omega}}{2}\cos\frac{\theta}{2})(\cos\frac{\theta}{2}, -\bar{\omega}\sin\frac{\theta}{2}) \\
&= (-\frac{1}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{\bar{\omega}\cdot\bar{\omega}}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}, -\frac{1}{2}\sin^2\frac{\theta}{2}\bar{\omega} + \frac{1}{2}\cos^2\frac{\theta}{2}\bar{\omega} - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\frac{\bar{\omega}\times\bar{\omega}}{2}) \\
\bar{\omega}\cdot\bar{\omega} &= \bar{\omega}^T\bar{\omega} = 1 \\
\bar{\omega}\times\bar{\omega} &= \bar{0} \\
\therefore \dot{Q}Q^* &= (0, \frac{1}{2}\bar{\omega}) \quad \square
\end{aligned}$$

2 Planar Rigid Body Transformations

2.1

For pure translation, $\omega = 0$:

$$\begin{aligned}
e^{\hat{\xi}\theta} &= \mathbb{I} + \theta\hat{\xi} + \frac{\theta^2\hat{\xi}^2}{2!} + \frac{\theta^3\hat{\xi}^3}{3!} + \frac{\theta^4\hat{\xi}^4}{4!} + \cdots \\
\hat{\xi} &= \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & v_x \\ 0 & 0 & v_y \\ 0 & 0 & 1 \end{bmatrix} \\
\hat{\xi}^2 &= \hat{\xi}^3 = \hat{\xi}^4 = \cdots = 0 \\
e^{\hat{\xi}\theta} &= \mathbb{I} + \theta\hat{\xi} = \begin{bmatrix} \mathbb{I} & v\theta \\ 0 & 1 \end{bmatrix} \\
&\therefore e^{\hat{\xi}\theta} \in SE(2)
\end{aligned}$$

For generalized transformations $\omega \neq 0$, assume $||\omega|| = 1$, by scaling θ if necessary, and define transformation g by

$$g = \begin{bmatrix} \mathbb{I} & \omega \times v \\ 0 & 1 \end{bmatrix}$$

Using equation 2.34:

$$\begin{aligned}
\hat{\xi}' &= g^{-1}\hat{\xi}g \\
&= \begin{bmatrix} \hat{\omega} & h\omega \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

where $h := \omega^T v$.
So given

$$e^{\hat{\xi}\theta} = \mathbb{I} + \theta\hat{\xi} + \frac{\theta^2\hat{\xi}^2}{2!} + \frac{\theta^3\hat{\xi}^3}{3!} + \frac{\theta^4\hat{\xi}^4}{4!} + \cdots$$

the identity from (2.35)

$$e^{\hat{\xi}\theta} = e^{g(\hat{\xi}\theta)g^{-1}} = ge^{\hat{\xi}\theta}g^-1$$

and the fact that

$$\hat{\omega}\omega = \omega \times \omega = 0$$

we can show

$$(\hat{\xi}')^2 = \begin{bmatrix} \hat{\omega}^2 & 0 \\ 0 & 0 \end{bmatrix}, (\hat{\xi}')^3 = \begin{bmatrix} \hat{\omega}^3 & 0 \\ 0 & 0 \end{bmatrix}, \ldots$$

Hence,

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta \\ 0 & 1 \end{bmatrix}$$

and multiplying by g and g^{-1} as defined above and in equation 2.35 yields

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & (\mathbb{I} - e^{\hat{\omega}\theta})(\omega \times v) + \omega\omega^T v\theta \\ 0 & 1 \end{bmatrix}$$
$$\therefore e^{\hat{\xi}\theta} \in SE(2)$$

2.2

2.2.1 Pure Rotation

Given the (constant) velocity of point p as,

$$\dot{p}(t) = \omega \times (p(t) - q)$$
$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \hat{\xi} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

then

$$\xi = \begin{bmatrix} v \\ \omega \end{bmatrix}$$
$$v = -\omega \times q$$
$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} = \begin{bmatrix} -\hat{\omega} q \\ \omega \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} \\ \omega \end{bmatrix}$$
$$||\omega|| = 1$$
$$\xi = \begin{bmatrix} q_y \\ -q_x \\ 1 \end{bmatrix} \quad \square$$

2.2.2 Pure Translation

$$\hat{\omega} = 0$$
$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$$
$$\therefore \xi = \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ 0 \end{bmatrix} \quad \square$$

$=$

2.3

2.4

2.5

3 Forward Kinematics

Per-frame translations:

$$\begin{aligned}
base^S &= [0.75, 0.5, 1.0] \\
link_0^b &= [0.220, 0.140, 0.346] \\
link_1^0 &= [0, 0, 0] \\
link_2^1 &= [0, 0, 0] \\
link_3^2 &= [0.045?, 0, 0.550] \\
link_4^3 &= [0, 0, 0] \\
link_5^4 &= [-0.045?, 0, 0.300] \\
link_6^5 &= [0, 0, 0] \\
link_7^6 &= [0, 0, 0.060] \\
tool^7 &= [0, 0, 0.12]
\end{aligned}$$

3.1

$$\begin{aligned}
tool^S = &\begin{bmatrix} 0.75 + 0.22 + 0.045 - 0.045 \\ 0.5 + .14 \\ 1.0 + 0.346 + 0.55 + 0.3 + 0.06 + 0.12 \end{bmatrix} \\
&= [0.97, 0.64, 2.376]
\end{aligned}$$

3.2

See source code

3.3

For time-consecutive points

$$p_1 = [1.1795, 0.7993, 0.4310] p_2 = [1.1797, 0.7994, 0.4310] p_3 = [1.1799, 0.7995, 0.4310]$$

and the p_2-zeroed vectors

$$\begin{aligned}
\bar{v_1} &= p_1 - p_2 \\
&= 1^{-3} * [-0.1821, -0.0914, -0.0215] \\
\bar{v_2} &= p_3 - p_2 \\
&= 1^{-3} * [0.1832, 0.0919, 0.0215]
\end{aligned}$$

then the plane can be described by $\bar{n}$ through p_2

$$\begin{aligned}\bar{n} &= v_1 \times v_2 \\ &= 1^{10} * [0.0748, -0.1284, -0.0877]\end{aligned}$$

3.4

"kdc"