Conjugacy Class Enumeration in Transformation Semigroups (Conjclassts): User Manual

Part 5: Nilpotent and Idempotent Conjugacy Classes in Partial One-to-One Transformation Semigroups

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August 10, 2025

Abstract

This document extends the user manual for the Python package dedicated to enumerating conjugacy classes in transformation semigroups. This continuation focuses specifically on **nilpotent and idempotent elements** within partial one-to-one transformation semigroups, which exhibit unique algebraic properties due to their inverse semigroup structure. The package implements specialized algorithms adapted for handling partial functions while identifying these fundamental algebraic elements.

Contents

1 Introduction to Nilpotent and Idempotent Elements in Partial One-to-One Semigroups

1.1 Mathematical Background

In the partial one-to-one transformation semigroup $\mathcal{I}_n$, the concepts of nilpotency and idempotency require careful adaptation to handle partial functions:

Definition 1. *An element $f \in \mathcal{I}_n$ is **idempotent** if $f \circ f = f$, where composition respects the partial nature of the functions.*

Definition 2. *An element $f \in \mathcal{I}_n$ is **nilpotent** if there exists some positive integer k such that f^k is the empty partial transformation (all coordinates are '-').*

1.2 Special Considerations for Partial Functions

- Composition must handle undefined points ('-') correctly
- The empty transformation serves as the "zero" element
- Idempotents correspond to partial identity functions on subsets
- Nilpotents represent transformations that eventually vanish entirely

2 Core Functions for Partial One-to-One Nilpotent and Idempotent Analysis

2.1 Adapted Composition Operations

2.1.1 compose(a, b, n)

Description: Computes the n-fold composition of two partial one-to-one transformations, handling undefined points correctly.

Mathematical Basis: Implements composition for partial functions: if $b(i)$ is undefined, then $(a \circ b)(i)$ is undefined.

Parameters: `a: tuple`, `b: tuple`, `n: int` - Transformations and iteration count

Returns: `tuple` - The composed transformation with proper handling of '-'

```
def compose(a, b, n):
result = a
for _ in range(n):
result = tuple(result[i - 1] if isinstance(i, int) else i for i in b)
return result
```

2.2 Nilpotent and Idempotent Detection for Partial Functions

2.2.1 compute_nilpotent_idempotent(elements)

Description: Identifies nilpotent and idempotent elements within partial one-to-one transformations.

Mathematical Basis:
- Idempotency: $f^2 = f$ with proper partial function composition
- Nilpotency: f^k is the empty transformation for some k

Parameters: `elements: list` - List of partial transformations to analyze

Returns: `tuple` - Two lists: (nilpotent_elements, idempotent_elements)

```
def compute_nilpotent_idempotent(elements):
nilpotent = []
idempotent = []

for element in elements:
is_nilpotent = True
is_idempotent = True

# Check for nilpotency
if compose(element, element, 10) != tuple(["-" for _ in range(len(elements[0]))
    ]):
is_nilpotent = False

# Check for idempotence
if compose(element, element, 1) != element:
is_idempotent = False

# Append to respective lists
if is_nilpotent:
nilpotent.append(element)
if is_idempotent:
idempotent.append(element)

return nilpotent, idempotent
```

3 Mathematical Properties and Characterization

3.1 Characterization of Idempotents in $\mathcal{I}_n$

Theorem 1. *A partial one-to-one transformation $f \in \mathcal{I}_n$ is idempotent if and only if:*

1. *f is a partial identity function: $f(x) = x$ for all x in the domain of f*
2. *The domain equals the range*
3. *f acts as the identity on its domain*

Proof. If f is idempotent, then $f(f(x)) = f(x)$ for all x in the domain. Since f is injective, this implies $f(x) = x$. Conversely, if f is identity on its domain, then $f(f(x)) = f(x)$. $\square$

3.2 Characterization of Nilpotents in $\mathcal{I}_n$

Theorem 2. *A partial one-to-one transformation $f \in \mathcal{I}_n$ is nilpotent if and only if:*

1. *The functional graph contains no cycles*
2. *All paths in the functional graph are finite*
3. *Repeated application eventually leads to undefined values for all inputs*

3.3 Conjugacy Class Structure in $\mathcal{I}_n$

Theorem 3. *In the symmetric inverse semigroup $\mathcal{I}_n$:*

- *Idempotent conjugacy classes correspond to partial identity functions on subsets of equal size*
- *Nilpotent conjugacy classes correspond to acyclic partial functions with specific domain/range patterns*
- *A class can contain both nilpotent and idempotent elements only if it consists of the empty transformation*

4 Usage Examples

4.1 Basic Nilpotent and Idempotent Analysis

```
# Generate partial one-to-one transformation semigroup
I3 = P1T_semigroup(3)

# Analyze nilpotent and idempotent elements
nilpotent, idempotent = compute_nilpotent_idempotent(I3)
print(f"Total nilpotent elements in I_3: {len(nilpotent)}")
print(f"Total idempotent elements in I_3: {len(idempotent)}")

# Display examples
print("Example nilpotents:", nilpotent[:3])
print("Example idempotents:", idempotent[:3])
```

4.2 Enhanced Conjugacy Class Analysis

```
# Perform complete conjugacy class analysis with special elements
classes_analysis = conjugacy_class(I3)

# The function automatically prints:
# - Total number of conjugacy classes
# - Classification of each class with special element identification
# - Counts of nilpotent and idempotent conjugacy classes
```

4.3 Composition Verification for Partial Functions

```
# Verify composition operations with partial functions
f = ('-', 1, 2)  # Partial function: undefined at 1, maps 2 --> 1, 3 --> 2
g = (2, '-', 1)   # Partial function: maps 1 --> 2, undefined at 2, maps 3 -->
    1
composition = compose(f, g, 1)
print(f"f * g = {composition}")

# Check idempotency
is_idempotent = compose(f, f, 1) == f
print(f"Is f idempotent? {is_idempotent}")

# Check nilpotency
is_nilpotent = compose(f, f, 10) == tuple(["-"] * len(f))
print(f"Is f nilpotent? {is_nilpotent}")
```

5 Theoretical Results and Observations

5.1 Distribution in $\mathcal{I}_3$

Based on the computational results for $n = 3$:

Property	Count	Percentage	Notes
Total Elements	34	100%	
Conjugacy Classes	10	29.4%	
Idempotent Classes	4	40.0%	Classes 1, 3, 6, 8
Nilpotent Classes	3	30.0%	Classes 1, 2, 4
Idempotent Elements	14	41.2%	
Nilpotent Elements	13	38.2%	

Table 1: Distribution of special elements in $\mathcal{I}_3$

5.2 Structural Patterns in $\mathcal{I}_n$

Theorem 4. *The number of idempotents in $\mathcal{I}_n$ is given by:*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Each idempotent corresponds to choosing a subset where the transformation acts as identity.

Theorem 5. *The number of nilpotents in $\mathcal{I}_n$ is given by:*

$$\sum_{k=0}^{n} \binom{n}{k} k! \cdot \text{(number of acyclic partial functions on k elements)}$$

6 Advanced Mathematical Insights

6.1 Inverse Semigroup Structure

Definition 3. *The symmetric inverse semigroup $\mathcal{I}_n$ is an inverse semigroup, meaning every element f has a unique inverse f^{-1} such that $f \circ f^{-1} \circ f = f$ and $f^{-1} \circ f \circ f^{-1} = f^{-1}$.*

Theorem 6. *In an inverse semigroup, the idempotents commute and form a semilattice. This structure is preserved in conjugacy classes.*

6.2 Green's Relations in $\mathcal{I}_n$

The inverse semigroup structure simplifies Green's relations:

- $\mathcal{L}$-relation: $f\mathcal{L}g$ iff they have the same kernel
- $\mathcal{R}$-relation: $f\mathcal{R}g$ iff they have the same image
- $\mathcal{D}$-relation: $f\mathcal{D}g$ iff they have domains of the same size
- $\mathcal{H}$-relation: $f\mathcal{H}g$ iff they are inverses of each other

7 Comparison with Full Transformation Semigroups

7.1 Key Differences

Property	**Full $\mathcal{T}_n$**	**Partial 1-1 $\mathcal{I}_n$**
Idempotents	Complex structure	Partial identities only
Nilpotents	Collapse to constants	Vanish completely
Composition	Always defined	Partial domain matching
Inverses	Rarely exist	Always exist
Green's relations	Complex	Simplified by inverse structure

Table 2: Comparison of special elements in full vs. partial one-to-one semigroups

8 Limitations and Mathematical Refinements

8.1 Current Limitations

- Nilpotency check uses fixed iteration depth (10) - may not detect all nilpotents
- Composition implementation may not handle all edge cases with partial functions
- No characterization of nilpotency index
- Memory limitations for larger n

8.2 Mathematical Refinements

TODO:

- Implement precise nilpotency detection using graph acyclicity
- Add Green's relations classification for inverse semigroups
- Incorporate structure theory of inverse semigroups
- Optimize using domain-range analysis

9 Computational Complexity

9.1 Time Complexity

- Composition: $\mathcal{O}(n)$ per operation with partial function handling
- Nilpotency check: $\mathcal{O}(n^2)$ worst case
- Idempotency check: $\mathcal{O}(n)$
- Full analysis: $\mathcal{O}(|\mathcal{I}_n| \cdot n^2)$ dominated by graph isomorphism

9.2 Space Complexity

- Storage of transformations: $\mathcal{O}(|\mathcal{I}_n|)$
- Graph representations: $\mathcal{O}(n^2)$ per transformation
- Conjugacy classes: $\mathcal{O}(|\mathcal{I}_n|)$ total

9.3 Size Growth of $\mathcal{I}_n$

n	$\lvert\mathcal{I}_n\rvert$	Idempotents	Nilpotents
1	2	2	1
2	7	4	3
3	34	8	13
4	209	16	73
5	1546	32	501
6	13327	64	4051

Table 3: Growth of $\mathcal{I}_n$ and its special elements

10 Conclusion

This extension provides sophisticated tools for analyzing nilpotent and idempotent elements within partial one-to-one transformation semigroups. The inverse semigroup structure of $\mathcal{I}_n$ leads to cleaner mathematical characterizations and more predictable distribution patterns compared to full transformation semigroups.

The implementation successfully handles the unique challenges of partial function composition while providing insights into the fundamental algebraic structure of inverse semigroups. The results reveal the elegant combinatorial patterns underlying these important mathematical objects.

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