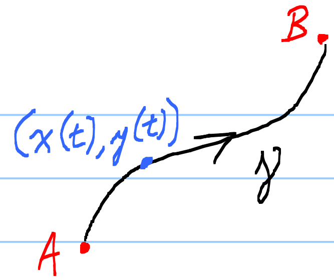
Lesson 20 on 14.1 $\underbrace{\text{path integration}}_{\substack{\text{complex line integrals} \\ \text{contour integration}}}$ HWK 6 : Lessons 18, 19, 20 Wed.



$\gamma: \ z(t) = x(t) + i\,y(t), \quad a \le t \le b$

Start : $A = z(a)$

End : $B = z(b)$

$z(t)$ is a complex valued fcn of a real var t.

Define $z'(t) = \lim\limits_{\Delta t \to 0} \dfrac{z(t+\Delta t) - z(t)}{\Delta t}$

$= \lim\limits_{\Delta t \to 0} \left[\left(\text{DQ for } x(t) \right) + i \left(\text{DQ for } y(t) \right) \right]$

$= x'(t) + i\,y'(t)$

Fact: $\displaystyle\int_a^b |z'(t)|\,dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= \text{Length}(\gamma)$

$$\int_\gamma \underbrace{f(z)}_{= u + iv} dz \overset{\text{Def}^n}{=} \int_a^b f(z(t)) \underbrace{z'(t)\,dt}_{dz = \frac{dz}{dt} dt}$$

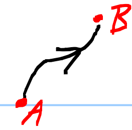
$$= \int_a^b \left[u(x(t), y(t)) + i v(x(t), y(t)) \right] \cdot \left[\frac{dx}{dt} + i \frac{dy}{dt} \right] dt$$

$$= \int_a^b \left(u \frac{dx}{dt} - v \frac{dy}{dt} \right) + i \left(v \frac{dx}{dt} + u \frac{dy}{dt} \right) dt$$

$$\boxed{\int_\gamma f\,dz = \left(\int_\gamma u\,dx - v\,dy \right) + i \left(\int_\gamma v\,dx + u\,dy \right)}$$

Brace yourself to use Green's and C-R Eqns!

Fund Thm Calculus for path integrals:



$$\int_\gamma F' \, dz = F(B) - F(A) \longleftarrow$$ F analytic on an open set containing γ

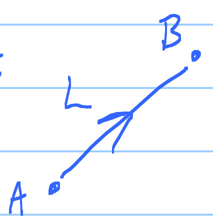
$F = u + iv \qquad F' = \begin{cases} u_x + i v_x \longleftarrow F' \\ v_y - i u_y \end{cases}$

$$\boxed{\int_\gamma F' dz = \left(\int_\gamma u_x \, dx - \overset{-u_y}{v_x} \, dy \right) + i \left(\int_\gamma v_x \, dx + \overset{v_y}{u_x} \, dy \right)}$$

$$= \int_a^b \Big(\underbrace{\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}}_{\frac{d}{dt} u(x(t), y(t))} \Big) dt + i \int_\gamma dv$$

$$= \big[u(B) - u(A) \big] + i \big[v(B) - v(A) \big] \checkmark$$

EX:



$L: z(t) = A + t(B-A), \; 0 \le t \le 1$

$z'(t) = B - A$

Calculate $\displaystyle\int_L e^z \, dz \overset{\text{Smart way}}{=} \int_L \frac{d}{dz}(e^z) \, dz$

$= e^B - e^A$

EX: $A = (1+i)$, $B = (3+5i)$

$$z(t) = \underbrace{(1+i)}_{A} + t \underbrace{(2+i4)}_{B-A} = (1+2t) + i(1+4t)$$

$0 \le t \le 1$

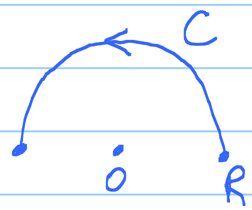
$z'(t) = 2 + i4$

$$\int_L e^z \, dz = \int_0^1 e^{(1+2t)+i(1+4t)} \cdot [2+4i] \, dt$$

$$= \int_0^1 e^{(1+2t)} \left(\cos(1+4t) + i \sin(1+4t) \right)(2+4i)\, dt$$

$$= \int_0^1 (\text{Real part})\, dt + i \int_0^1 (\text{Im part})\, dt$$

EX:



$$z(t) = Re^{it}, \quad 0 \le t \le \pi$$

$$z(t) = R\cos t + i R\sin t$$

$$z'(t) = -R\sin t + i R\cos t$$

$$= iRe^{it} \leftarrow \text{Aha!}$$

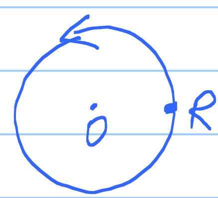
Special Chain Rule: $f(z(t))$, f analytic

$$\frac{d}{dt} f(z(t)) = f'(z(t))\, z'(t)$$

complex derivative

derivative $\mathbb{R} \to \mathbb{C}$

EX:



$$C: \; z(t) = Re^{it}, \quad 0 \le t \le 2\pi$$

$$z'(t) = iRe^{it}$$

$$\int_C \frac{1}{z}\, dz = \int_0^{2\pi} \frac{1}{(Re^{it})} \underbrace{\left[iRe^{it}\, dt \right]}_{dz = \frac{dz}{dt} dt}$$

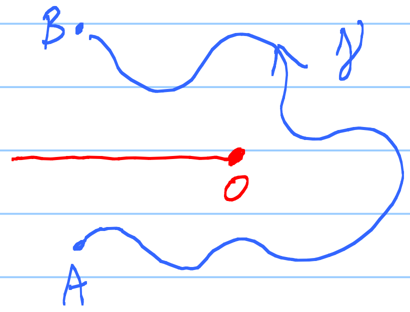
$$= \int_0^{2\pi} i\, dt = 0 + i \int_0^{2\pi} dt$$

$$= 2\pi i$$

Basis for the "Residue Theorem" later!

Using Fund Thm Calc: $\displaystyle\int_\gamma e^{3z}\, dz =$

$$\int_\gamma \frac{d}{dz}\left[\frac{1}{3}e^{3z}\right] dz = \frac{1}{3}e^{3z}\Big|_{START}^{END}$$



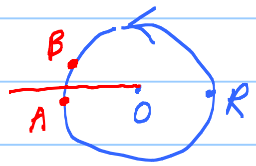
$$\int_\gamma \frac{1}{z}\, dz =$$

$$\int_\gamma \frac{d}{dz}\left[\operatorname{Log} z\right] dz$$

Princ. Branch

$$= \operatorname{Log}(B) - \operatorname{Log}(A)$$

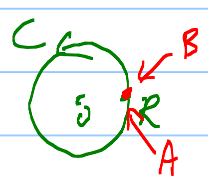
Hmmm:



Let A slide up, B slide down. Lim $= i\Delta \operatorname{Arg} z$
$= i2\pi$ ✓

<u>EX</u>: $\displaystyle\int_\gamma z^n\, dz = \left[\frac{1}{n+1} z^{n+1}\right]_{START}^{END}$

$n \neq 1$: $\displaystyle\int_\gamma z^{-n}\, dz = \left[\frac{1}{-n+1} z^{-n+1}\right]_{START}^{END}$



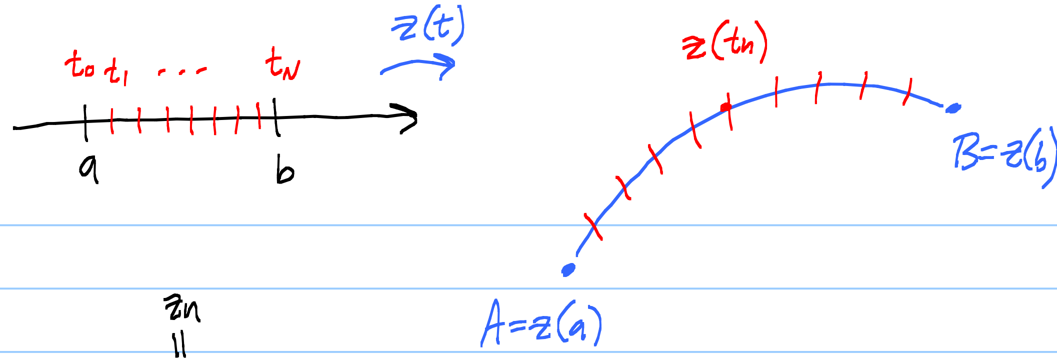
$$\int_C \frac{1}{z^5}\, dz = -\frac{1}{4}\cdot\frac{1}{z^4}\Big|_A^B = 0$$

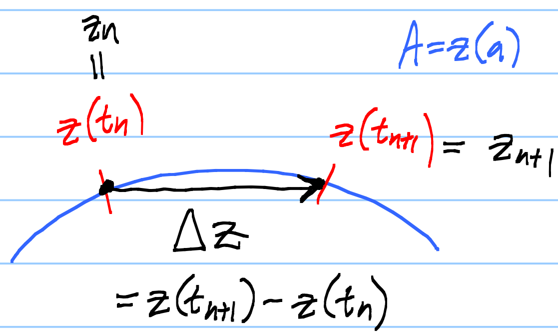
<u>EX</u>: f <u>not</u> analytic. No Fund Thm Calc.

e.g. $f(z) = \operatorname{Re} z$, $f(z) = \bar{z}$, $f(z) = |z|$

Must calculate from Def^n $\int_\gamma f\, dz$.

Physical meaning of $\int_\gamma f\, dz$





$$\int_\gamma f(z)\,dz = \lim_{\Delta z \to 0} \sum f(z_n)\,\Delta z$$

$$= \lim_{\Delta t \to 0} \sum f(z(t_n)) \underbrace{\left[z(t_{n+1}) - z(t_n) \right]}_{\approx \frac{dz}{dt}(t_n)\,\Delta t}$$

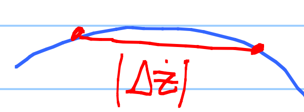
$$= \int_a^b f(z(t))\, z'(t)\, dt \quad \checkmark$$

<u>Basic Estimate</u>: $\left| \int_\gamma f\, dz \right| \le M \cdot \text{Length}(\gamma)$

where $M = \max_\gamma |f| = \max_{a \le t \le b} |f(z(t))|$.

<u>Why</u>: $\left| \sum f(z_n)\,\Delta z \right| \le \sum \underbrace{|f(z_n)\,\Delta z|}_{\underbrace{|f(z_n)|}_{\le M}\, |\Delta z|}$

$$\le \underbrace{\sum M\, |\Delta z|}_{M \sum |\Delta z|}$$



$\le$ length curved part

$$\le M\, \text{Length}(\gamma) \quad \checkmark$$

Take limit, $\Delta t \to 0$.

Important Ineq's:

Numerator estimate: $|z+w| \le |z| + |w|$

Denominator estimate: $|z+w| \ge \big||z| - |w|\big|$

$$|z+w| = |z-(-w)| \ge \big||z| - |-w|\big| \quad ✓$$