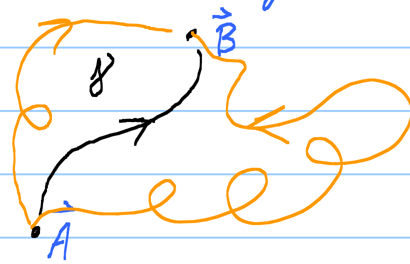
Lesson 4 on 10.2 Independence of Path (IoP)

HWK 1 due tonight 11:59 pm on Blackboard

Fundamental Theorem of Calculus for line integrals

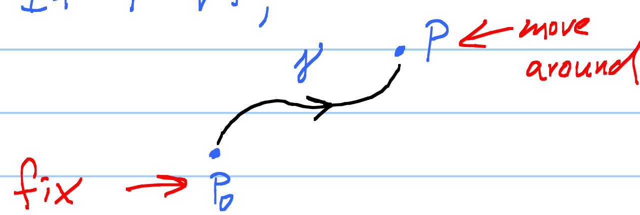
$$\int_\gamma \nabla f \cdot d\vec{r} = f(\vec{B}) - f(\vec{A})$$



Fact: If $\vec{F} = \nabla f$, ← f is a potential fcn
then $\int_\gamma \vec{F} \cdot d\vec{r}$ is IoP.

Reverse is true: If $\int_\gamma \vec{F} \cdot d\vec{r}$ is IoP, then $\vec{F} = \nabla f$ for a potential fcn f.

Why: Hmmm. If $\vec{F} = \nabla f$,

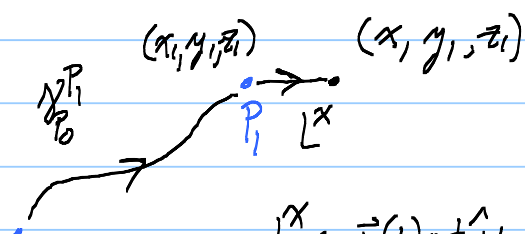


$$f(P) - f(P_0) = \int_{\gamma_{P_0}^{P}} \vec{F} \cdot d\vec{r}$$

So $$f(P) = \underset{\text{const.}}{f(P_0)} + \int_{\gamma_{P_0}^{P}} \vec{F} \cdot d\vec{r}$$

Big idea: Define $f(P) = \int_{\gamma_{P_0}^{P}} \vec{F} \cdot d\vec{r}$. Well defined.

How to see that $\frac{\partial f}{\partial x} = F_1$:



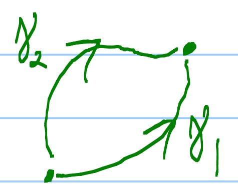
L^x: $\vec{r}(t) = t\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}$
$x_1 \le t \le x$

$$f(x,y_1,z_1) = \int_{\gamma_{P_0}^{P_1}} \vec{F}\cdot d\vec{r} + \underbrace{\int_{x_1}^{x} F_1(t,y_1,z_1)\cdot 1\cdot dt}_{\int_{\gamma_x} \vec{F}\cdot d\vec{r}}$$

$$\frac{\partial f}{\partial x}(x,y_1,z_1) = 0 + \underbrace{F_1(x,y_1,z_1)}_{\text{By Fund. Thm. Calc.}} \checkmark$$

Similarly for $\frac{\partial f}{\partial y_1}$, $\frac{\partial f}{\partial z}$.

Note:



$C = \gamma_1 \cup (-\gamma_2)$ is a closed curve.

IoP means that

$$\int_C = \int_{\gamma_1} + \underbrace{\int_{-\gamma_2}}_{-\int_{\gamma_2}} = 0$$

Fact: $\int_\gamma \vec{F}\cdot d\vec{r}$ is IoP $\iff \int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve.

Notation: $\int_\gamma \vec{F}\cdot d\vec{r} = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right)dt$

$$= \int_\gamma \underbrace{F_1\,dx + F_2\,dy + F_3\,dz}_{\text{differential form}}$$

Def^n: If $\vec{F} = \nabla f$, then

$$\underbrace{F_1\,dx + F_2\,dy + F_3\,dy}_{\text{exact differential}} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= df \leftarrow f = \text{potential}$$

Test for exactness (Same as test for $\vec{F} = \nabla f$.)

$\vec{F} = \nabla f \iff \operatorname{Curl} \vec{F} = 0$ [On simply connected regions]

Easy direction: Curl $\nabla f = 0$ ← Identity.

How to find f (after you check that Curl $\vec{F} = 0$)

EX: $\vec{F} = \frac{2xy}{1+x^2}\hat{\imath} + \left(\ln(1+x^2) + \cos y\right)\hat{\jmath} + z\hat{k}$

Step 1: Check that Curl $\vec{F} = 0$:

$$\text{Curl } \vec{F} = \det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{2xy}{1+x^2} & \ln(1+x^2)+\cos y & z \end{bmatrix}$$

$$= 0\cdot\hat{\imath} - 0\cdot\hat{\jmath} + \left(\frac{2x}{1+x^2} - \frac{2x}{1+x^2}\right)\hat{k}$$

$$= \vec{0} \checkmark$$

Step 2: Want f with

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{2xy}{1+x^2} & (A) \\ \frac{\partial f}{\partial y} = \ln(1+x^2) + \cos y & (B) \\ \frac{\partial f}{\partial z} = z & (C) \end{cases}$$

First use (A): $f = \int \frac{2xy}{1+x^2}\,dx = y\ln(1+x^2) + g(y,z)$

treat other vars like constants

arb. const. might depend on other vars.

Next use (B): $\frac{\partial}{\partial y}\left(y\ln(1+x^2) + g(y,z)\right) \overset{\text{want}}{=} \ln(1+x^2) + \cos y$

$$\ln(1+x^2) + \frac{\partial g}{\partial y} = \ln(1+x^2) + \cos y$$

$$\frac{\partial g}{\partial y} = \cos y$$

So $g(y,z) = \int \cos y \, dy = \sin y + h(z)$

arb. const. might depend on z.

Finally, use (c):

$$\frac{\partial}{\partial z}\left(y \ln(1+x^2) + \sin y + h(z) \right) \overset{\text{want}}{=} z$$

$$0 + 0 + h'(z) = z$$

$$h(z) = \int z \, dz = \tfrac{1}{2} z^2 + C$$

Done! $\boxed{f = y \ln(1+x^2) + \sin y + \tfrac{1}{2} z^2 + C}$

Note: Must work to get f, but once we have it, computing $\int_\gamma \vec{F} \cdot d\vec{r} = f(\text{END}) - f(\text{START})$ is a snap.

EX: $$\int_{(0,\frac{\pi}{2},1)}^{(1,\pi,-2)} \vec{F} \cdot d\vec{r} = f(1,\pi,-2) - f(0,\tfrac{\pi}{2},1)$$

$$= \left(\pi \ln 2 + 2 + C\right) - \left(1 + \tfrac{1}{2}\cdot 16 + C\right)$$

$$= \pi \ln 2 - 7$$ (C's cancel!)

If all you need is a potential, take $C=0$.

Meaning of $\operatorname{Curl} \vec{F} = 0$: $\vec{F} = F_1(x_1,x_2,x_3)\hat{e}_1 + \cdots + F_3(x_1,x_2,x_3)\hat{e}_3$

$\operatorname{Curl} \vec{F} = 0$ means that

$$\boxed{\frac{\partial F_1}{\partial x_2} = \frac{\partial F_2}{\partial x_1}, \quad \frac{\partial F_1}{\partial x_3} = \frac{\partial F_3}{\partial x_1}, \quad \frac{\partial F_2}{\partial x_3} = \frac{\partial F_3}{\partial x_2}} \quad *$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right), \ \ldots$$

Aha! From point of view of $\vec{F} = \nabla f$, (*) only means that the <u>mixed</u> partials of f are equal.

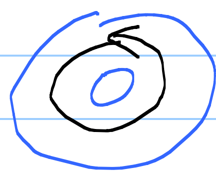
<u>Simply connected</u>: A region is s.c. if every curve in it can be contracted to a point.

<u>EX</u>: (Ball of rad 2) − (Ball of rad 1)
<u>is</u> s.c.

(Cylinder diam 2) − (Cyl. of diam 1)
is <u>not</u> s.c.

<u>Book</u>: EX 4 in 10.2 $\vec{F}$ blows up in inner cylinder.

Curl $\vec{F} = 0$



$$\int_C \vec{F}\cdot d\vec{r} \neq 0$$