THE BERGMAN KERNEL AND QUADRATURE DOMAINS IN THE PLANE

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Abstract. A streamlined proof that the Bergman kernel associated to a quadrature domain in the plane must be algebraic will be given. A byproduct of the proof will be that the Bergman kernel is a rational function of z and one other explicit function known as the Schwarz function. Simplified proofs of several other well known facts about quadrature domains will fall out along the way. Finally, Bergman representative coordinates will be defined that make subtle alterations to a domain to convert it to a quadrature domain. In such coordinates, biholomorphic mappings become algebraic.

1. Introduction. In this paper, we will recombine a string of results by Aharonov and Shapiro [1], Gustafsson [14], Davis [12], Shapiro [17], and Avci [2] in light of recent results by the author in [7] and [9] to obtain elementary proofs of a number of results about quadrature domains and the classical functions associated to them. In particular, we present an efficient proof of the fact proved in [9] that the Bergman kernel associated to a quadrature domain in the plane is an algebraic function. In fact, we shall show that it is a rational function of z and the Schwarz function, and consequently it is also a rational combination of z and $Q(z)$ where $Q(z)$ is an explicit algebraic function given by

$$Q(z) = \int_{b\Omega} \frac{\bar{w}}{w-z}\ dw.$$

These results are all a natural outgrowth of the work of Aharonov and Shapiro [1] and Gustafsson [14] and many of the results obtained in those works will come as corollaries in the approach we take here. For example, Aharonov and Shapiro proved that Ahlfors maps associated to quadrature domains are algebraic, and we shall deduce this via the connection between Ahlfors maps, the double, and the Bergman kernel. The new approach we use shall also allow us to view Bergman representative coordinates in a new and very interesting light.

We shall call an n-connected domain Ω in the plane such that no boundary component is a point a *quadrature domain* if there exist finitely many points $\{w_j\}_{j=1}^N$ in the domain and non-negative integers n_j such that complex numbers c_{jk} exist satisfying

$$\int_\Omega f\ dA = \sum_{j=1}^{N}\sum_{k=0}^{n_j} c_{jk} f^{(k)}(w_j) \tag{1.1}$$

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for every function f in the Bergman space of square integrable holomorphic functions on Ω. Here, dA denotes Lebesgue area measure. Our results require the function $h(z) \equiv 1$ to be in the Bergman space, and so we shall also assume that the domain under study has *finite area.*

If Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point, then it is well known that the Bergman kernel function associated to Ω satisfies an identity of the form

$$1 \equiv \sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm} K^{(m)}(z, w_j) \tag{1.2}$$

where $K^{(m)}(z,w)$ denotes $(\partial^m/\partial\bar{w}^m)K(z,w)$ (and of course $K^{(0)}(z,w) = K(z,w)$) and where the points w_j are the points that appear in the characterizing formula (1.1) of quadrature domains. It can be seen by noting that the inner product of an analytic function against the function $h(z) \equiv 1$ and against the sum on the right hand side of (1.2) agree for all functions in the Bergman space. Hence the two functions must be equal. Note that we must assume that Ω has finite area here just so that $h(z) \equiv 1$ is in the Bergman space.

We first state a theorem about the Bergman kernel of a multiply connected domain in the plane with smooth boundary. Note that, although formula (1.2) is clearly in the background, we do not assume that the domain is a quadrature domain.

Theorem 1.1. *Suppose that Ω is an n-connected bounded domain in the plane whose boundary is given by n non-intersecting simple closed C^∞ smooth real analytic curves. Let $A(z)$ be a function of the form*

$$\sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm} K^{(m)}(z, w_j)$$

where the w_j are points in Ω. Let G_1 and G_2 be any two meromorphic functions on Ω that extend meromorphically to the double of Ω and form a primitive pair for the field of meromorphic functions on the double. There is a rational function $R(z_1, z_2, w_1, w_2)$ of four complex variables such that $K(z,w)$ is given by

$$K(z,w) = A(z)\overline{A(w)}R\left(G_1(z), G_2(z), \overline{G_1(w)}, \overline{G_2(w)}\right).$$

Theorem 1.1 is a corollary of Theorem 2.3 of [7]. We shall give a straightforward and direct proof of Theorem 1.1 in §3.

We remark here that two meromorphic functions on a compact Riemann surface are said to form a primitive pair if they generate the field of meromorphic functions, i.e., if every meromorphic function on the Riemann surface is a rational combination of the two. For basic facts about primitive pairs, see Farkas and Kra [13].

If Ω is an n-connected domain in the plane such that no boundary component is a point, then it is a standard construction in the subject to produce a biholomorphic mapping φ which maps Ω one-to-one onto a bounded domain Ω_s bounded by n smooth real analytic curves. The subscript s stands for "smooth" and we shall write

$K(z,w)$ for the Bergman kernel of Ω and $K_s(z,w)$ for the Bergman kernel associated to Ω_s. We shall say that a meromorphic function h on Ω *extends meromorphically to the double* of Ω if $h \circ \varphi^{-1}$ is a meromorphic function on Ω_s which extends meromorphically to the double of Ω_s. (This terminology might be considered to be rather non-standard, but it greatly simplifies the statements of many of our results below.) It is easy to verify that this definition does not depend on the choice of φ and Ω_s. We shall also say that G_1 and G_2 form a primitive pair for Ω if $G_1 \circ \varphi^{-1}$ and $G_2 \circ \varphi^{-1}$ extend meromorphically to the double of Ω_s and form a primitive pair for the double of Ω_s. Using this terminology, we may apply Theorem 1.1 to obtain the following result.

Theorem 1.2. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. Then the Bergman kernel function $K(z,w)$ associated to Ω is a rational combination of any two functions G_1 and G_2 that form a primitive pair for Ω in the sense that $K(z,w)$ is a rational combination of $G_1(z)$, $G_2(z)$, $\overline{G_1(w)}$, and $\overline{G_2(w)}$.*

We remark that the same conclusion in Theorem 1.2 can be made about the square $S(z,w)^2$ of the Szegő kernel. Furthermore, the classical functions F_j' (see §2 for definitions) are rational functions of G_1 and G_2 and so is any proper holomorphic mapping of Ω onto the unit disc. The reader may see [7] for the details of the more general statement.

The next theorem of Gustafsson [14] reveals that quadrature domains have nice boundaries. It also allows us to smoothly connect a quadrature domain to another domain with a double in the classical sense.

Theorem 1.3. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. Suppose that φ is a holomorphic mapping which maps Ω one-to-one onto a bounded domain Ω_s bounded by n smooth real analytic curves. Then φ^{-1} extends holomorphically past the boundary of Ω_s. Furthermore, φ^{-1} extends meromorphically to the double of Ω_s. It follows that the boundary of Ω is piecewise real analytic and the possibly finitely many non-smooth boundary points are easily described as cusps which point toward the inside of Ω. Furthermore, it follows that φ extends continuously to the boundary of Ω.*

Avci did not state Theorem 1.3 in [2], however, all the elements of a proof are there. If Avci had done things in a different order, he might have needed this theorem and he might very well have spelled out the proof as well. In fact, much of Avci's work in [2] is headed in a direction that could have easily have led to many of the results of this paper.

We shall give a short Bergman kernel proof of Theorem 1.3 in the §3.

We remark that at a cusp boundary point b of Ω mentioned in Theorem 1.3, the map φ behaves like a principal branch of the square root of $z-b$ mapping the plane minus a horizontal slit from b to the left to the right half plane. This map sends b to zero, and the inverse map φ^{-1} behaves like $b+z^2$ on the right half plane mapping zero to b. The power two in $b+z^2$ is the only power larger than one that makes such a map one-to-one on the right half plane near zero, and this kind of reasoning can be used to show that the derivative of φ^{-1} can have at most a simple zero at a boundary point of Ω_s.

Aharonov and Shapiro [1] showed that the Schwarz function associated to a quadrature domain extends meromorphically to the domain, i.e., that the function

$\bar{z}$ agrees on the boundary with a function $S(z)$, known as the Schwarz function, which is meromorphic on the domain and which extends continuously up to the boundary. We will follow Gustafsson [14] and modify somewhat his observation that z and $S(z)$ form a primitive pair (in the special sense we use here) to obtain a quick proof of the following result.

Theorem 1.4. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. The function z extends to the double as a meromorphic function. Consequently, there is a meromorphic function $S(z)$ on Ω (known as the Schwarz function) which extends continuously up to the boundary such that $S(z) = \bar{z}$ on $b\Omega$. The functions z and $S(z)$ form a primitive pair for Ω. It also follows that z and*

$$Q(z) = \int_{b\Omega} \frac{\bar{w}}{w - z}\, dw$$

form a primitive pair for Ω. Consequently, $S(z)$ and $Q(z)$ are algebraic functions. It also follows that the boundary of Ω is a real algebraic curve.

Aharonov and Shapiro [1] first showed that the boundary of a quadrature domain is an algebraic curve and Gustafsson [14] later gave a precise description of what these curves must be. Theorems 1.4 and 1.2 can now be combined to yield the following result.

Theorem 1.5. *Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. Then the Bergman kernel function $K(z, w)$ associated to Ω is a rational combination of the two functions z and the Schwarz function. It is also a rational function of z and $Q(z)$. Consequently, $K(z, w)$ is algebraic. Furthermore, the Szegő kernel is algebraic, the classical functions F_j' are algebraic, and every proper holomorphic mapping from Ω onto the unit disc is algebraic.*

Similar statements to the theorems above can be made for the Poisson kernel and first derivative of the Green's function. These results follow from formulas appearing in [6] and we do not spell them out here.

It is interesting to note here that, not only is the Bergman kernel of a quadrature domain Ω a rational combination of z and the Schwarz function, but the Schwarz function associated to Ω is a rational combination of $K(z, a)$ and $K(z, b)$ for two fixed points a and b in the domain by virtue of the fact that $S(z)$ extends to the double and because it is possible to find two such functions $K(z, a)$ and $K(z, b)$ of z that form a primitive pair for Ω.

Since $S(z) = \bar{z}$ on the boundary, the next theorem is an easy consequence of Theorem 1.5.

Theorem 1.6. *Suppose that Ω is an n-connected quadrature domain in the plane of finite area such that no boundary component is a point. The Bergman kernel $K(z, w)$ and the square $S(z, w)^2$ of the Szegő kernel are rational functions of z, $\bar{z}$, w, and $\bar{w}$ on $b\Omega \times b\Omega$ minus the boundary diagonal. The functions $F_j'(z)$ are rational functions of z and $\bar{z}$ when restricted to the boundary. Furthermore, the unit tangent vector function $T(z)$ is such that $T(z)^2$ is a rational function of z and $\bar{z}$ for $z \in b\Omega$.*

The Riemann mapping theorem can be viewed as saying that any simply connected domain in the plane that is not the whole plane is biholomorphic to the grandaddy of all quadrature domains, the unit disc. Gustafsson generalized this theorem to multiply connected domains. He proved that any finitely connected domain in the plane such that no boundary component is a point is biholomorphic to a quadrature domain. The circle of ideas we develop in this paper can be used to reformulate these theorems via a tool I would venture to call "Bergman representative coordinates." We shall show that any domain with C^∞ smooth boundary can be mapped by a biholomorphic mapping which is as C^∞ close to the identity map as desired to a quadrature domain. The biholomorphic map will be given in the form of a quotient of linear combinations of the Bergman kernel. This process begins with the following lemma, which was proved in [3]. Let $A^\infty(\Omega)$ denote the subspace of $C^\infty(\overline{\Omega})$ consisting of functions that are holomorphic on Ω.

Lemma 1.7. *Suppose that Ω is a bounded finitely connected domain bounded by simple closed C^∞ smooth curves. The complex linear span of the set of functions of z of the form $K(z,b)$ where b are points in Ω is dense in $A^\infty(\Omega)$.*

If Ω is a domain as in Lemma 1.7, let $\mathcal{K}_2(z)$ denote a finite linear combination $\sum c_j K(z,b_j)$ which is C^∞ close to the function $h(z)\equiv 1$ and let $\mathcal{K}_1(z)$ denote a finite linear combination $\sum a_j K(z,b_j)$ which is C^∞ close to the function $h(z)\equiv z$. I call the quotient $\mathcal{K}_1(z)/\mathcal{K}_2(z)$ a Bergman representative mapping function if it is one-to-one. By taking the linear combinations to be C^∞ close enough to their target functions, any such mapping can be made one-to-one and as C^∞ close to the indentity as desired. The mappings extend meromorphically to the double of Ω because $T(z)K(z,b) = -\overline{T(z)\Lambda(z,b)}$ for z in $b\Omega$. (See (3.1) and §3-4 for the details of this argument.) Hence, according to Gustafsson [14], the mapping sends Ω to a quadrature domain. Hence, the following theorem holds

Theorem 1.8. *Suppose that Ω is a bounded finitely connected domain bounded by simple closed C^∞ smooth curves. There is a Bergman representative mapping function which is as C^∞ close to the identity map as desired which maps Ω to a quadrature domain.*

Bergman representative mappings as defined here yield a rather fascinating change of coordinates. Indeed, they were used in several complex variables in [10] to locally linearize biholomorphic mappings. In one variable, if $\Phi:\Omega_1\to\Omega_2$ is a biholomorphic (or merely proper holomorphic mapping) between C^∞-smooth finitely connected domains in the plane, then Theorem 1.8 allows us to make changes of coordinates that are C^∞ close to the identity on each domain in such a way that the mapping Φ in the new coordinates is an *algebraic function.* Indeed, if $\Phi:\Omega_1\to\Omega_2$ is a proper holomorphic mapping between quadrature domains in the new coordinates, let f_a denote an Ahlfors mapping of Ω_2 onto the unit disc. Then f_a is a proper holomorphic mapping of the smooth quadrature domain Ω_2 onto the unit disc, and is therefore algebraic by Theorem 1.5. Now $f_a\circ\Phi$ is a proper holomorphic mapping of the smooth quadrature domain Ω_1 onto the unit disc, and is therefore algebraic. It follows that Φ itself must be algebraic. It is rather striking that subtle changes in the boundary can make conformal mappings become defined on the whole complex plane.

Since Bergman representative mappings extend to the double and are one-to-one, they are Gustafsson mappings (as defined in [9]), and they can be used to compress the classical kernel functions into a small data set as in [9]. Björn Gustafsson read a preliminary version of this paper and realized that every Gustafsson map can be expressed as a Bergman representative map. He has granted me permission to include his argument in §4 of the present paper.

The points in the quadrature identity associated to the Bergman representative domain in Theorem 1.8 can be arranged to fall in any small disc in the domain. This can be done using similar constructions to those used in [9] and by Gustafsson in [14]. We do not treat this problem here.

Another interesting way to view Theorem 1.8 is as follows. Shrink a smooth domain Ω by moving in along an inward pointing unit vector a fixed short distance. Now use a Bergman representative mapping which is sufficiently C^∞ close to the identity map so that the shrunken domain gets mapped to a domain inside of Ω. This shows that we may lightly "sand" the edges of our original domain to turn it into a quadrature domain. Similarly, by expanding the domain by first moving along an outward pointing normal and repeating this process, we can see that we can "paint" the edges of our domain with an arbitrarily thin coat of paint of variable thickness to turn it into a quadrature domain.

It is reasonable to allow more general functions of the form

$$\sum_{j=1}^{N}\sum_{m=0}^{n_j} c_{jm}K^{(m)}(z,w_j)$$

to appear as the numerator and denominator of a Bergman representative mapping because the individual elements of these functions satisfy the same kind of basic identity as the Bergman kernel itself (see formula (3.1)) and, consequently, such quotients extend to the double and have the same mapping properties as the Bergman representative mappings as we constructed them above. Under this stipulation, we can also prove a converse to Theorem 1.8.

Theorem 1.9. *If Ω is a finitely connected quadrature domain of finite area, then there is a Bergman representative mapping which is equal to the identity map.*

Another approach to representative coordinates is given by Jeong and Taniguchi in [15]. It might be interesting to see what comes of combining the two approaches.

Finally, we remark that, when the results of this paper are combined with the results in [8], it can be seen that the infinitesimal Carathéodory metric associated to a finitely connected quadrature domain of finite area such that no boundary component is a point is given by $\rho(z)|dz|$ where ρ is a rational combination of z, the Schwarz function, and the complex conjugates of these two functions.

Complete proofs of the theorems will be given in §3.

2. Preliminaries. It is a standard construction in the theory of conformal mapping to show that an n-connected domain Ω in the plane such that no boundary component is a point is conformally equivalent via a map φ to a bounded domain $\widetilde{\Omega}$ whose boundary consists of n simple closed C^∞ smooth real analytic curves. Since such a domain $\widetilde{\Omega}$ is a bordered Riemann surface, the double of $\widetilde{\Omega}$ is an easily realized compact Riemann surface. We shall say that an analytic or meromorphic function h on Ω *extends meromorphically to the double of* Ω if $h\circ\varphi^{-1}$ extends meromorphically to the double of $\widetilde{\Omega}$. Notice that whenever Ω is itself a bordered

Riemann surface, this notion is the same as the notion that h extends meromorphically to the double of Ω. We shall say that two functions G_1 and G_2 extend to the double and generate the meromorphic functions on the double of Ω, and that they therefore form a primitive pair for the double of Ω, if $G_1 \circ \varphi^{-1}$ and $G_2 \circ \varphi^{-1}$ extend to the double of $\widetilde{\Omega}$ and form a primitive pair for the double of $\widetilde{\Omega}$ (see Farkas and Kra [13] for the definition and basic facts about primitive pairs).

It is proved in [6] that if Ω is an n-connected domain in the plane such that no boundary component is a point, then almost any two distinct Ahlfors maps f_a and f_b generate the meromorphic functions on the double of Ω. It is also proved that any proper holomorphic mapping from Ω to the unit disc extends to the double of Ω.

Suppose that Ω is a bounded n-connected domain whose boundary consists of n non-intersecting C^∞ smooth simple closed curves. The Bergman kernel $K(z, w)$ associated to Ω is related to the Szegő kernel via the identity

$$K(z, w) = 4\pi S(z, w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z) \overline{F_j'(w)}, \tag{2.1}$$

where the functions $F_i'(z)$ are classical functions of potential theory described as follows. The harmonic function ω_j which solves the Dirichlet problem on Ω with boundary data equal to one on the boundary curve γ_j and zero on γ_k if $k \neq j$ has a multivalued harmonic conjugate. Let γ_n denote the outer boundary curve. The function $F_j'(z)$ is a single valued holomorphic function on Ω which is locally defined as the derivative of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial \omega_j / \partial z)$.

The Bergman and Szegő kernels are holomorphic in the first variable and antiholomorphic in the second on $\Omega \times \Omega$ and they are hermitian, i.e., $K(w, z) = \overline{K(z, w)}$. Furthermore, the Bergman and Szegő kernels are in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in b\Omega\})$ as functions of (z, w) (see [4, p. 100]).

We shall also need to use the Garabedian kernel $L(z, w)$, which is related to the Szegő kernel via the identity

$$\frac{1}{i} L(z, a) T(z) = S(a, z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega \tag{2.2}$$

where $T(z)$ represents the complex unit tangent vector at z pointing in the direction of the standard orientation of $b\Omega$. For fixed $a \in \Omega$, the kernel $L(z, a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z, a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z, w)$ is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in \overline{\Omega}\})$ as a function of (z, w) (see [4, p. 102]). Also, $L(z, a)$ is non-zero for all (z, a) in $\overline{\Omega} \times \Omega$ with $z \neq a$ and $L(a, z) = -L(z, a)$ (see [4, p. 49]).

For each point $a \in \Omega$, the function of z given by $S(z, a)$ has exactly $(n-1)$ zeroes in Ω (counting multiplicities) and does not vanish at any points z in the boundary of Ω (see [4, p. 49]).

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $C^\infty(\overline{\Omega})$, and it maps each boundary curve γ_j one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via (see [4, p. 49])

$$f_a(z) = \frac{S(z,a)}{L(z,a)}. \tag{2.3}$$

Note that $f_a'(a) = 2\pi S(a,a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z,a)$ at a accounts for the simple zero of f_a at a. The other $n-1$ zeroes of f_a are given by the $(n-1)$ zeroes of $S(z,a)$ in $\Omega - \{a\}$.

When Ω does not have smooth boundary, we define the kernels and domain functions above as in [5] via a conformal mapping to a domain with real analytic boundary curves.

3. Proofs of the theorems. In this section, we give complete proofs of Theorems 1.1–1.9.

Proof of Theorem 1.1. The Bergman kernel is related to the classical Green's function via ([11, p. 62], see also [4, p. 131])

$$K(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial \bar{w}}.$$

Another kernel function on $\Omega \times \Omega$ that we shall need is given by

$$\Lambda(z,w) = -\frac{2}{\pi}\frac{\partial^2 G(z,w)}{\partial z \partial w}.$$

(In the literature, this function is sometimes written as $L(z,w)$ with anywhere between zero and three tildes and/or hats over the top. We have chosen the symbol Λ here to avoid confusion with our notation for the Garabedian kernel above.)

The Bergman kernel and the kernel $\Lambda(z,w)$ satisfy an identity analogous to (2.2):

$$\Lambda(w,z)T(z) = -K(w,z)\overline{T(z)} \qquad \text{for } w \in \Omega \text{ and } z \in b\Omega \tag{3.1}$$

(see [4, p. 135]). We remark that it follows from well known properties of the Green's function that $\Lambda(z,w)$ is holomorphic in z and w and is in $C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z,z) : z \in \overline{\Omega}\})$. If $a \in \Omega$, then $\Lambda(z,a)$ has a double pole at $z = a$ as a function of z and $\Lambda(z,a) = \Lambda(a,z)$ (see [4, p. 134]). Since Ω has real analytic boundary, the kernels $K(z,w)$, $\Lambda(z,w)$, $S(z,w)$, and $L(z,w)$, extend meromorphically to $\overline{\Omega} \times \overline{\Omega}$ (see [4, p. 103, 132–136]). Let $A(z) = \sum_{j=1}^{N}\sum_{m=0}^{n_j} c_{jm}K^{(m)}(z,w_j)$ where the w_j are points in Ω. Notice that A cannot be the zero function because, if it were, it would be orthogonal to all functions in the Bergman space, and consequently every function in the Bergman space would have to satisfy the identity $0 \equiv \sum_{j=1}^{N}\sum_{m=0}^{n_j} \bar{c}_{jm} g^{(m)}(w_j)$, which is absurd. Notice that (3.1) shows that there is a meromorphic function $M(z)$ on Ω which extends meromorphically to a neighborhood of $\overline{\Omega}$ and which has no poles on $b\Omega$ such that

$$A(z)T(z) = \overline{M(z)T(z)} \qquad \text{for } z \in b\Omega. \tag{3.2}$$

Let $\mathcal{B}$ denote the class of holomorphic functions $B(z)$ on Ω that have the property that they extend holomorphically past the boundary and such that there exists a meromorphic function $m(z)$ on Ω which extends meromorphically to a neighborhood of $\overline{\Omega}$ and which has no poles on $b\Omega$ such that

$$B(z)T(z) = \overline{m(z)T(z)} \qquad \text{for } z \in b\Omega.$$

We have shown that $A(z)$ belongs to $\mathcal{B}$. Notice that, if $B(z)$ is in $\mathcal{B}$, then $B(z)/A(z)$ is equal to the complex conjugate of a meromorphic function $m(z)/M(z)$ for $z \in b\Omega$. This shows that $B(z)/A(z)$ extends meromorphically to the double of Ω. Now the Bergman kernel is given by

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} F_i'(z)\overline{F_j'(w)}$$

where

$$S(z,w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(c_0 S(z,a)\overline{S(w,a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z,a_i)\,\overline{S(w,a_j)} \right) \tag{3.3}$$

and where $f_a(z)$ denotes the Ahlfors map associated to a (see [5]). The functions F_j' belong to the class $\mathcal{B}$. Indeed

$$F_j'(z)T(z) = -\overline{F_j'(z)T(z)} \qquad \text{for } z \in b\Omega \tag{3.4}$$

(see [4, p. 80]). Furthermore, functions of z of the form $S(z,a_i)S(z,a_j)$ also belong to the class $\mathcal{B}$ by virtue of identity (2.2). Hence, $K(z,w)$ is given by a sum of terms of the form

$$\frac{B_1(z)\overline{B_2(w)}}{(1 - f_a(z)\overline{f_a(w)})^2}$$

plus a sum of functions of the form $B_1(z)\overline{B_2(w)}$ where B_1 and B_2 belong to $\mathcal{B}$. Therefore, if we divide $K(z,w)$ by $A(z)\overline{A(w)}$, we obtain a function which is a sum of terms of the form

$$\frac{g_1(z)\overline{g_2(w)}}{1 - f_a(z)\overline{f_a(w)})^2}$$

plus a sum of functions of the form $g_1(z)\overline{g_2(w)}$ where g_1 and g_2 extend meromorphically to the double. But f_a also extends meromorphically to the double. Hence, $K(z,w)$ is equal to $A(z)\overline{A(w)}$ times a rational function of $G_1(z)$, $G_2(z)$, $\overline{G_1(w)}$, and $\overline{G_2(w)}$ where G_1 and G_2 are any two functions that form a primitive pair for the double of Ω. This completes the proof of Theorem 1.1.

Proof of Theorems 1.2 and 1.3. Suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point. Suppose that φ is a holomorphic mapping which maps Ω one-to-one onto a bounded domain Ω_s bounded by n smooth real analytic curves. Let Φ denote φ^{-1}. The transformation formula for the Bergman kernels under φ can be written in the form

$$\Phi'(z)K(\Phi(z),w) = K_s(z,\varphi(w))\overline{\varphi'(w)}.$$

As mentioned in §1, since Ω is a quadrature domain of finite area, an identity of the form

$$1 \equiv \sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm} K^{(m)}(z, w_j)$$

holds. Let $A(z)$ denote the linear combination on the right hand side of this equation. It now follows that

$$\begin{aligned}
\Phi'(z) = \Phi'(z) \cdot (A \circ \Phi)(z) &= \Phi'(z) \sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm} K^{(m)}(\Phi(z), w_j) \\
&= \sum_{j=1}^{N} \sum_{m=0}^{n_j} c_{jm} \frac{\partial^m}{\partial \bar{w}^m} \left(K_s(z, \varphi(w)) \overline{\varphi'(w)} \right) \Big|_{w=w_j}.
\end{aligned}$$

Notice that the function on the last line of this string of equations is a finite linear combination of functions of the form $K_s^{(m)}(z, W)$ where the W are points in Ω_s. Let $A_s(z)$ denote this linear combination. We have shown two things. We have shown that

$$\Phi' = A_s$$

where A_s is a linear combination of functions of z of the form $K_s^{(m)}(z, W)$. We shall use this first fact momentarily to prove Theorem 1.3. Secondly, we have shown that

$$\Phi'(z) \cdot (A \circ \Phi)(z) = A_s(z),$$

and consequently, that

$$A(z) = \varphi'(z) \cdot (A_s \circ \varphi)(z).$$

This second fact will yield Theorem 1.2. Indeed, we may now combine this fact with Theorem 1.1 and the transformation formula for the Bergman kernels to obtain that

$$\begin{aligned}
K(z, w) = \frac{K(z, w)}{A(z)\overline{A(w)}} &= \frac{\varphi'(z) K_s(\varphi(z), \varphi(w)) \overline{\varphi'(w)}}{\varphi'(z) \cdot (A_s \circ \varphi)(z)\ \overline{\varphi'(w) \cdot (A_s \circ \varphi)(w)}} \\
&= \frac{K_s(\varphi(z), \varphi(w))}{A_s(\varphi(z))\ \overline{A_s(\varphi(w))}}.
\end{aligned}$$

If G_1 and G_2 form a primitive pair for the double of Ω_s, then Theorem 1.1 yields that the last function in the string of equations is a rational combination of $G_1 \circ \varphi$ and $G_2 \circ \varphi$, and this shows that the Bergman kernel $K(z, w)$ is generated by a primitive pair in the generalized sense that we defined in §1. This completes the proof of Theorem 1.2.

We now turn to the proof of Theorem 1.3. We have shown that Φ' is a linear combination of functions of the form $K_s^{(m)}(z, W_j)$ where $m \geq 0$ and $W_j = \varphi(w_j)$. All functions of this form extend holomorphically past the boundary of Ω_s (see [4, p. 41,133]). Hence Φ' extends holomorphically past the boundary of Ω_s and, consequently, so does Φ. It follows that Ω is a bounded domain and that the boundary of Ω is piecewise real analytic. The singular points, if any, in the boundary of Ω are given as the images of boundary points under Φ where Φ' has a simple zero.

(Notice that for Φ to be one-to-one on Ω_s, Φ' cannot have any zeroes of multiplicity greater than one on the boundary.) It is clear that φ extends continuously to the boundary of Ω. We will complete the proof of Theorem 1.3 during the course of the next proof when we show that Φ extends meromorphically to the double of Ω_s.

Proof of Theorem 1.4. As in the preceding proof, we suppose that Ω is an n-connected quadrature domain of finite area in the plane such that no boundary component is a point and we suppose that φ is a holomorphic mapping which maps Ω one-to-one onto a bounded domain Ω_s bounded by n smooth real analytic curves. Again, let Φ denote φ^{-1}. We now claim that Φ extends to the double of Ω_s as a meromorphic function. To see this, we use the inhomogeneous Cauchy integral formula,

$$u(z) = \frac{1}{2\pi i}\int_{w\in b\Omega_s} \frac{u(w)}{w-z}\, dw + \frac{1}{2\pi i}\iint_{w\in\Omega_s} \frac{\partial u/\partial\bar{w}}{w-z}\, dw\wedge d\bar{w}$$

with $\overline{\Phi}$ in place of u to obtain

$$\overline{\Phi(z)} = \frac{1}{2\pi i}\int_{w\in b\Omega_s} \frac{\overline{\Phi(w)}}{w-z}\, dw + \frac{1}{2\pi i}\iint_{w\in\Omega_s} \frac{\overline{\Phi'(w)}}{w-z}\, dw\wedge d\bar{w}.$$

We now let z approach the boundary of Ω_s from the inside and use a method developed in [4] to determine the boundary values of the two integrals. Since the boundary of Ω_s is real analytic and since Φ extends holomorphically past the boundary of Ω_s, the Cauchy-Kovalevskaya theorem (see [4, p. 39]) yields that there is a function v which is real analytic in a neighborhood of the boundary of Ω_s, which vanishes on the boundary, and such that $\frac{\partial v}{\partial\bar{w}} \equiv \overline{\Phi'(w)}$ on a neighborhood of the boundary. We may extend v to be C^∞ smooth in Ω_s. Now apply the inhomogeneous Cauchy Integral Formula to $\overline{\Phi} - v$ to obtain

$$\overline{\Phi(z)} - v(z) = \frac{1}{2\pi i}\int_{w\in b\Omega_s} \frac{\overline{\Phi(w)}}{w-z}\, dw + \frac{1}{2\pi i}\iint_{w\in\Omega_s} \frac{\overline{\Phi'(w)} - \frac{\partial v}{\partial\bar{w}}}{w-z}\, dw\wedge d\bar{w}.$$

The function given as the integral over the boundary of Ω_s is a holomorphic function $H(z)$ on Ω_s which extends C^∞ smoothly to the boundary (see [4, p. 7]). The function given as the integral over Ω_s is also C^∞ up to the boundary of Ω_s because the integrand has compact support. Furthermore, the boundary values of this function agree with the boundary values of

$$\iint_{w\in\Omega_s} \frac{\overline{\Phi'(w)} - \frac{\partial v}{\partial\bar{w}}}{w-z}\, dw\wedge d\bar{w}.$$

as z approaches the boundary of Ω_s from the *outside* of Ω_s. It was shown in the course of proving Theorem 1.3 that Φ' is equal to a linear combination of functions of z of the form $K_s^{(m)}(z,w_j)$. Note that this integral is a constant times the L^2 inner product of $\Phi' - \frac{\partial\bar{v}}{\partial w}$, and the holomorphic function $1/(w-z)$ of w when z is outside of Ω_s. Functions of the form $\partial\bar{v}/\partial w$ are orthogonal to smooth holomorphic functions when v vanishes on the boundary via integration by parts. Hence, the boundary values of the integral over Ω_s are given as a linear combination of terms of the form $1/(w_j - z)^m$, i.e., a rational function of z. Hence, for z in the boundary of Ω_s, we have shown that

$$\overline{\Phi(z)} = H(z) + \mathcal{R}(z)$$

where H is a holomorphic function on Ω_s which is C^∞ smooth up to the boundary and $\mathcal{R}(z)$ is a rational function of z. (In fact, the poles of $R(z)$ only fall at the points $\varphi(w_j)$ where w_j are the points appearing in the quadrature identity for Ω.) This last identity reveals that Φ extends to the double of Ω_s as a meromorphic function, and this completes the proof of Theorem 1.3, as promised.

Since $z = \Phi(\varphi(z))$ and Φ extends meromorphically to the double of Ω_s, it follows that z extends meromorphically to the double of Ω in the generalized sense that we use in this paper.

Let the symbol Φ also denote the function defined on the double of Ω_s which is the meromorphic extension of Φ. Let R denote the antiholomorphic reflection function on the double of Ω_s which maps Ω_s to reflected copy in the double of Ω_s and let $G = \overline{\Phi \circ R}$. It is now an easy matter to see that Φ and G form a primitive pair for the double of Ω_s. Indeed, G has poles only at finitely many points in the double which fall in the Ω_s side. Choose a complex number w_0 sufficiently close to the point at infinity so that the set $\mathcal{S} = G^{-1}(w_0)$ consists of finitely many points in Ω_s such that G takes the value w_0 with multiplicity one at each point in $\mathcal{S}$. Since Φ is one-to-one on Ω_s, it follows that Φ separates the points of $G^{-1}(w_0)$, and this implies that Φ and G form a primitive pair (see Farkas and Kra [13]). Consequently, there is an irreducible polynomial $P(z, w)$ such that $P(\Phi(z), G(z)) \equiv 0$ on the double of Ω_s.

The Schwarz function for Ω is now given as $S(z) = G(\varphi(z))$ and z and $S(z)$ form a primitive pair for Ω. It follows that $S(z)$ is meromorphic on Ω and continuous up to the boundary with boundary values equal to $\bar{z}$. By composing the polynomial identity $P(\Phi(z), G(z)) \equiv 0$ with φ, we see that $P(z, S(z)) \equiv 0$ on Ω, and this shows that $S(z)$ is an algebraic function. When this identity is restricted to the boundary, we see that $P(z, \bar{z}) = 0$ when z is in the boundary, i.e., that the boundary of Ω is contained in an algebraic curve. (Gustafsson refined this argument to show that the boundary is in fact equal to the algebraic curve minus perhaps finitely many points.)

We now turn to examine the function $Q(z)$. Let $p(z)$ denote the sum of the principal parts of $S(z)$ in Ω. Apply the Cauchy integral formula to $S(z) - p(z)$ and use the fact that $S(z) = \bar{z}$ on the boundary to see that $S(z) - p(z) = Q(z)$ plus a linear combination of integrals of the form $\int_{b\Omega} \frac{1}{(w-w_j)^k(w-z)}\,dw$ where the w_j are points in Ω where S has poles. But all such integrals are zero. Hence $S(z) = p(z) + Q(z)$. Since S is algebraic, so is Q. Since z and S generate the meromorphic functions on the double of Ω, and since $p(z)$ is rational, it follows that z and $Q(z)$ also generate the meromorphic functions on the double of Ω. This completes the proof of Theorem 1.4

Proof of Theorem 1.5. Theorem 1.4 together with Theorem 1.2 yield that the Bergman kernel $K(z, w)$ for Ω is a rational combination of z, $S(z)$, $\bar{w}$, and $\overline{S(w)}$. Thus, the Bergman kernel is algebraic. Similarly $K(z, w)$ is a rational combination of z and $Q(z)$. Since any proper holomorphic mapping of Ω onto the unit disc extends meromorphically to the double, it follows that all such maps are rational combinations of z and $S(z)$, and consequently, they are algebraic.

It is proved in [5] that if the Bergman kernel is algebraic, then so is the Szegő kernel, and so are the classical functions F_j'.

Proof of Theorem 1.6. We have seen that the kernels $K(z,w)$ and $S(z,w)^2$ and the proper holomorphic maps to the unit disc and the functions F_j' are all generated by z and $S(z)$, and since these functions are equal to z and $\bar{z}$, respectively on the boundary, we may deduce most of the rest of the claims made in Theorem 1.6. To finish the proof, note that identity (2.2) yields that

$$T(z)^2 = -\frac{S(a,z)^2}{L(z,a)^2}$$

where a is an arbitrary point chosen and fixed in Ω. The function $S(z,a)^2$ is a rational function of z and $\bar{z}$ on the boundary. Identity (2.3) yields that $L(z,a)^2 = S(z,a)^2/f_a(z)^2$, and so $L(z,a)^2$ is also a rational function of z and $\bar{z}$ on the boundary. Finally, it follows that $T(z)^2$ is a rational function of z and $\bar{z}$ on the boundary.

Proof of Theorem 1.9. Suppose Ω is a finitely connected quadrature domain of finite area. We know that $\bar{z}$ is equal to the Schwarz function $S(z)$ on the boundary and that $S(z)$ is meromorphic on Ω with finitely many poles. Let h be a holomorphic function on Ω that extends smoothly to the boundary. Notice that

$$\begin{aligned}\iint_\Omega z\,\overline{h(z)}\,dA &= \frac{i}{2}\iint_\Omega z\,\overline{h(z)}\,dz\wedge d\bar{z}\\ &= \frac{i}{4}\iint_\Omega \frac{\partial}{\partial z}\left(z^2\,\overline{h(z)}\right)\,dz\wedge d\bar{z}\\ &= \frac{i}{4}\int_{b\Omega} z^2\,\overline{h(z)}\,d\bar{z} = \frac{i}{4}\int_{b\Omega}\overline{S(z)^2}\,\overline{h(z)}\,d\bar{z},\end{aligned}$$

and the Residue Theorem yields that this last integral is equal to a fixed linear combination of values of h and finitely many of its derivatives at the points in Ω where $S(z)$ has poles. Since such functions h are dense in the Bergman space, this shows that the function z is a linear combination of functions of the form $K^{(m)}(z,w_k)$ where w_k are points where $S(z)$ has poles. Since the constant function 1 is also given by a linear combination as in formula (1.2), we may form a quotient to get a Bergman representative mapping which is equal to the function z.

4. Gustafsson's Theorem. Björn Gustafsson has granted me permission to include here his proof of his discovery that every Gustafsson map on a smoothly bounded domain is a Bergman representative map. After I give Gustafsson's elegant proof, I will offer an alternative, more pedestrian proof that sheds additional light on this phenomenon.

The main tool in Gustafsson's argument is the following lemma.

Lemma 4.1. *Suppose that Ω is a bounded finitely connected domain bounded by simple closed C^∞ smooth curves. If G is a holomorphic function on Ω which extends meromorphically to the double of Ω and has no poles on $\overline{\Omega}$, then G' must be equal to a complex linear combination of functions of z of the form $K^{(m)}(z,w_k)$ where w_k are points in Ω.*

Proof of Lemma 4.1. Let G denote both the function on Ω and its extension to the double $\widehat{\Omega}$ of Ω. Note that dG is a meromorphic differential on $\widehat{\Omega}$, and that since it is exact, it is free of residues. Identity (3.1) can be used in a standard way to establish the well know fact that differentials of the form $K^{(m)}(z, w_k)\,dz$ extend meromorphically to $\widehat{\Omega}$. Since the function $\Lambda(z, w)$ has a double pole at $z = w$ with no residue term, it is possible to choose a linear combination of the extensions of $K^{(m)}(z, w_k)\,dz$ so that the poles of dG on the back side of $\widehat{\Omega}$ are exactly cancelled by the sum. (We shall say more about this argument below.) Let $\mathcal{K}(z)\,dz$ denote such a linear combination. Now $dG - \mathcal{K}\,dz$ is a holomorphic differential on $\widehat{\Omega}$. All the periods of dG are zero, and it is well known that the β-periods of differentials of the form $\mathcal{K}\,dz$ vanish, i.e., the periods that go across Ω from one boundary curve to another and then return along the backside along the reflected curve. (This fact follows from the relationships between the Bergman kernel and the Λ kernel and the Green's function. It is explained in Schiffer and Spencer [16, p. 101-105]. Also, the arguments can be found in Gustafsson's paper [14] in the proofs of Theorems 1 and 2.)

Now a holomorphic differential vanishes if all its β-periods, or if all its α-periods, vanish. Hence $dG - \mathcal{K}\,dz$ is zero and it follows that $G' = \mathcal{K}$ on Ω. This completes the proof.

I call a function on a bounded finitely connected domain bounded by smooth curves a Gustafsson mapping if it is holomorphic and one-to-one on the domain and extends meromorphically to the double and has no poles on the boundary of the domain. Gustafsson mappings effect a conformal change of variables from the given domain to a quadrature domain.

Theorem 4.2. *Suppose that Ω is a bounded finitely connected domain bounded by simple closed C^∞ smooth curves. If g is a Gustafsson function on Ω, then g is equal to a Bergman representative mapping.*

Proof of Theorem 4.2. If g is a Gustafsson map, apply Lemma 4.1 to g and to $\frac{1}{2}g^2$ to get $g' = \mathcal{K}_2$ and $gg' = \mathcal{K}_1$. Since g is one-to-one, the equality $g' = \mathcal{K}_2$ shows that the linear combination $\mathcal{K}_2$ is non-vanishing on $\overline{\Omega}$. Now g is given by the quotient $\mathcal{K}_1/\mathcal{K}_2$, which is a Bergman representative mapping.

I shall now give an alternative proof of Lemma 4.1 that sheds further light on the property proved in [9] that, on a smooth quadrature domain, if G is a function on the domain that extends meromorphically to the double, then G' is also a function on the domain that extends meromorphically to the double. Suppose that Ω is a bounded finitely connected domain bounded by simple closed C^∞ smooth curves, and suppose that G is a holomorphic function on Ω that extends meromorphically to the double of Ω and has no poles on $\overline{\Omega}$. In this case, there is a meromorphic function H on Ω which extends smoothly up to the boundary such that $G = \overline{H}$ on $b\Omega$. Let $z(t)$ denote a parameterization of a boundary curve of Ω. Since $G(z(t)) = \overline{H(z(t))}$, we may differentiate with respect to t and divide the result by $|z'(t)|$ to obtain that $G'(z)T(z) = \overline{H'(z)T(z)}$ for $z \in b\Omega$. This last identity is very similar to identity (3.1). Indeed, we may differentiate (3.1) with respect to w and rewrite it to obtain

$$K^{(m)}(z, w)T(z) = -\overline{\Lambda^{(m)}(z, w)T(z)}$$

for $z \in b\Omega$, where $\Lambda^{(m)}(z, w)$ denotes the m-th derivative of $\Lambda(z, w)$ with respect to w (and of course $\Lambda^{(0)}(z, w) = \Lambda(z, w)$). The singular part of $\Lambda(z, w)$ is a constant times $(z - w)^{-2}$. Since H' is the derivative of a meromorphic function, the poles of H' are double or more. Hence, there is a unique linear combination $\mathcal{L}$ of the functions $\Lambda^{(m)}(z,w)$ so that the principal parts at the poles of $\mathcal{L}$ agree with the principal parts of H' at each pole in Ω. If $\mathcal{L}(z) = \sum_{j=1}^{N}\sum_{m=1}^{n_j} c_{jm}\Lambda^{(m)}(z,w_j)$, let $\mathcal{K}(z) = -\sum_{j=1}^{N}\sum_{m=1}^{n_j} \bar{c}_{jm}K^{(m)}(z,w_j)$. Notice that $\mathcal{K}(z)T(z) = \overline{\mathcal{L}(z)T(z)}$ on $b\Omega$. Now $(G'-\mathcal{K})T = \overline{(H'-\mathcal{L})T}$ on $b\Omega$ where both $G'-\mathcal{K}$ and $H'-\mathcal{L}$ are holomorphic on Ω and extend smoothly to the boundary. This implies $(G'-\mathcal{K})T$ is both orthogonal to the Hardy space and conjugates of functions in the Hardy space. Consequently, a theorem of Schiffer yields that $G'-\mathcal{K} = \sum_{j=1}^{n-1} c_j F_j'$ for some constants c_j. (See [4, p. 80] for a proof of this result that proves, rather than assumes, that the zeroes of the Szegö kernel are simple zeroes.) Now $\mathcal{K}(z)$ is $\partial/\partial z$ of a linear combination $\mathcal{G}$ of functions of the form $(\partial^m/\partial\bar{w}^m)G(z,w)$ where $G(z,w)$ is the classical Green's function. All linear combinations of this form vanish on the boundary in the z variable. Also, $F_j' = 2(\partial/\partial z)\omega_j$. Hence $G'-\mathcal{K} = \sum_{j=1}^{n-1} c_j F_j'$ yields that $G - \mathcal{G} - 2\sum_{j=1}^{n-1} c_j\omega_j$ is antimeromorphic on Ω. But $G = \overline{H}$ on the boundary. Hence, the boundary values of $\mathcal{G} + 2\sum_{j=1}^{n-1} c_j\omega_j$ agree with the boundary values of a function which is antimeromorphic on Ω and which extends smoothly to the boundary. Since this meromorphic function vanishes on one of the boundary curves, it must be identically zero. This forces us to conclude that all the c_j's are zero, and we have proved Gustafsson's theorem that $G' = \mathcal{K}$.

It should be remarked that the argument just given can be run in reverse to yield a converse to Gustafsson's lemma. Indeed, if $\mathcal{K}$ is a linear combination of the form used above that has vanishing periods along all the boundary curves of Ω, then an analytic antiderivative of $\mathcal{K}$ must extend meromorphically to the double.

Gustafsson's lemma can also be routinely generalized to yield the following result. Suppose that Ω is a bounded finitely connected domain bounded by simple closed C^∞ smooth curves. If G is a *meromorphic* function on Ω which extends meromorphically to the double of Ω, then $G'(z)$ must be equal to a function of the form

$$\sum_{k=1}^{N}\sum_{m=1}^{p_k} a_{km}K^{(m)}(z,x_k) + \sum_{k=1}^{M}\sum_{m=1}^{q_k} b_{km}\Lambda^{(m)}(z,y_k),$$

where the x_k are points in $\overline{\Omega}$ and the y_k are points in Ω. The points x_k are used to cancel the poles of the extension of G' on the back side of the double and the y_k are used to cancel the poles of G' on Ω. These results allow the field of meromorphic functions on the double to be handled like a linear space in many instances.

In case the domain under study is a quadrature domain of finite area with smooth boundary, then we know that the Schwarz function $S(z)$ agrees with $\bar{z}$ on the boundary. We can differentiate the identity $z = \overline{S(z)}$ along the boundary as we did above to see that $T(z) = \overline{S'(z)T(z)}$ for z in $b\Omega$. Notice that this identity reveals that $|S'(z)| = 1$ on the boundary. Now, if G is a meromorphic function on Ω which extends meromorphically to the double, we may write $G'T = \overline{H'T}$ on the boundary as we did above and divide by the identity for the Schwarz function to obtain that $G'(z)$ is equal to the conjugate of $H'(z)/S'(z)$ for z in the boundary. This yields another way to see that, on a smooth quadrature domain, any meromorphic function G on the domain that extends meromorphically to the double has the property that its derivative G' also extends meromorphically to the double. This proof also has the virtue that it gives an explicit formula for the extension of G' to the double.

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