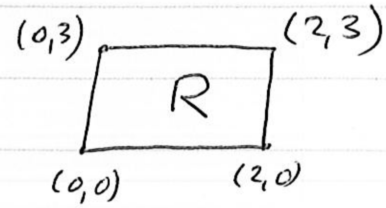
HW 3 solutions

10.4 p. 438

#3) $\vec{F} = \langle x^2 e^y, y^2 e^x \rangle$



$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(y^2 e^x - x^2 e^y \right) dx\, dy$$

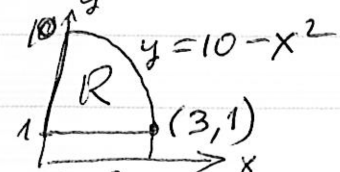
$$= \int_0^2 e^x dx \int_0^3 y^2 dy - \int_0^2 x^2 dx \int_0^3 e^y dy$$

$$= (e^2 - 1) \cdot 9 - \frac{8}{3}(e^3 - 1) = 9e^2 - \frac{8}{3} e^3 - \frac{19}{3}$$

#7) $\vec{F} = \nabla\left(x^3 \cos^2(xy)\right) \Rightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$

Integral is path independent, curl $\vec{F} = 0$

#15) $w = e^x \cos y + xy^3$,



$$\int_C \frac{\partial w}{\partial n} ds = \iint_R \nabla^2 w \, dA = \int_0^3 \int_1^{10-x^2} 6xy \, dy\, dx$$

$$= \int_0^3 3xy^2 \Big|_{y=1}^{y=10-x^2} dx = \int_0^3 3x(10-x^2)^2 - 3x \, dx$$

$$= -\frac{1}{2}(10-x^2)^3 - \frac{3}{2}x^2 \Big|_0^3 = -\frac{1}{2} - \frac{27}{2} + \frac{1000}{2} = 486$$

#16) $w = x^2+y^2$, $\quad C = \{x^2+y^2 = 4\}$

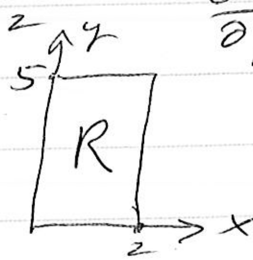
$\vec{n} = \frac{1}{2}\langle x, y\rangle$, $\quad \frac{\partial w}{\partial \vec{n}} = \langle 2x, 2y\rangle \cdot \langle \frac{x}{2}, \frac{y}{2}\rangle$

$= x^2+y^2 = 4$

$$\oint_C \frac{\partial w}{\partial \vec{n}}\, ds = 4 \text{ length}(C) = 16\pi$$

$$\nabla^2 w = 4 \Rightarrow \iint_R \nabla^2 w\, dA = 4 \text{ area}(R) = 16\pi$$

#19) $w = e^x \sin y$, $\quad \nabla^2 w = \underbrace{e^x \sin y}_{\frac{\partial^2 w}{\partial x^2}} \underbrace{- e^x \sin y}_{\frac{\partial^2 w}{\partial y^2}} = 0$



$$\int_C w \frac{\partial w}{\partial \vec{n}}\, ds = \iint_R |\nabla w|^2\, dA$$

$$= \int_0^2 \int_0^5 |\langle e^x \sin y, e^x \cos y\rangle|^2\, dy\, dx$$

$$= \int_0^2 \int_0^5 e^{2x}\, dy\, dx = \frac{5}{2}(e^4 - 1)$$

10.5 p.442

#2) xy-plane in polar coords $\vec{r} = \langle u\cos v, u \sin v\rangle$

$u \geq 0$, $0 \leq v \leq 2\pi$

u = const is a circle centered at (0,0), radius u

v = const is a ray with the end at (0,0), angle v with the positive x-axis

$\vec{r}_u = \langle \cos v, \sin v\rangle$, $\quad \vec{r}_v = \langle -u \sin v, u \cos v\rangle$,

$\vec{N} = \vec{r}_u \times \vec{r}_v = u \quad$ (or $\langle 0, 0, u\rangle$)

(cross product of vectors in the plane is a scalar)

#3) <u>cone</u> $\vec{r} = \langle u\cos v, u\sin v, cu\rangle$ $(c \neq 0)$

$u=$ const is a <u>circle</u> in the plane $z=cu$
except $u=0$ is a <u>point</u> $(0,0,0)$
$v=$ const is a line through the origin
with the direction vector $\langle \cos v, \sin v, c\rangle$

$\vec{r}_u = \langle \cos v, \sin v, c\rangle$, $\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle -cu\cos v, -cu\sin v, u\rangle$

<u>Equation</u> $\frac{z^2}{c^2} = x^2+y^2$ (<u>Not</u> $z = c\sqrt{x^2+y^2}$)

#5) <u>Paraboloid</u> $\vec{r} = \langle u\cos v, u\sin v, u^2\rangle$

$z = x^2+y^2$

$u=$ const is a <u>circle</u> in the plane $z=u^2$.
(Note that u and $-u$ gives the same circle.)
except $u=0$ is a <u>point</u> $(0,0,0)$.
$v=$ const is a <u>parabola</u> $z=x^2$ rotated by
the angle v about the z-axis.

$\vec{r}_u = \langle \cos v, \sin v, 2u\rangle$, $\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle -2u^2\cos v, -2u^2\sin v, u\rangle$.

#7) <u>Ellipsoid</u> $\vec{r} = \langle a\cos v\cos u, b\cos v\sin u, c\sin v\rangle$
$(a>0, b>0, c>0)$ <u>Equation</u> $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$u=$ const is an <u>ellipse</u> in the plane $\frac{y}{x} = \frac{b}{a}\tan u$
$v=$ const is an <u>ellipse</u> in the plane $z = c\sin v$
except $v = \frac{\pi}{2}$ is a <u>point</u> $(0,0,c)$.

$\vec{r}_u = \langle -a\cos v\sin u, b\cos v\cos u, 0\rangle$,

$\vec{r}_v = \langle -a\sin v\cos u, -b\sin v\sin u, c\cos v\rangle$

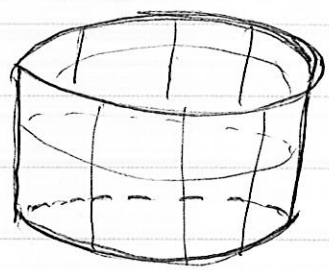
$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle bc\cos^2 v\cos u, ac\cos^2 v\sin u, ab\cos v\sin v\rangle$

#14) Plane $4x+3y+2z=12$ is a graph $z=6-2x-\frac{3}{2}y$, parameterized by x and y. (or $x=u, y=v, z=6-2u-\frac{3}{2}v$)

#15) Cylinder $(x-2)^2+(y+1)^2=25$

$x-2=5\cos u, \quad y+1=5\sin u, \quad z=v$

$x=2+5\cos u, \ y=-1+5\sin u, \quad z=v$

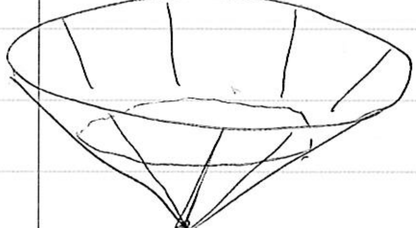


#18) Elliptic cone $z=\sqrt{x^2+4y^2}$

(this is a half-cone)

$z^2=x^2+4y^2, \quad z\geq 0$

$x=u\cos v, \quad y=\frac{u}{2}\sin v, \quad z=u\geq 0$



Alternatively, as a graph it can be parameterized by x and y

10.6 p. 450

#3) $\vec{F} = \langle 0, x, 0 \rangle$, $S = \{x^2+y^2+z^2=1,\ x\geq 0, y\geq 0, z\geq 0\}$

S is part of the unit sphere in first octant

$\vec{n} = \langle x, y, z \rangle$, $\vec{F}\cdot\vec{n} = xy$

Parameterization! $\vec{r} = \langle \cos v \cos u, \cos v \sin u, \sin v \rangle$

$0 \leq u \leq \frac{\pi}{2}$, $0 \leq v \leq \frac{\pi}{2}$ (because $x\geq 0, y \geq 0, z\geq 0$)

$\vec{r}_u = \langle -\cos v \sin u, \cos v \cos u, 0 \rangle$

$\vec{r}_v = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle \cos^2 v \cos u, \cos^2 v \sin u, \cos v \sin v \rangle$

$\vec{F}\cdot\vec{N} = \cos^3 v \cos u \sin u$

$$\int_S \vec{F}\cdot\vec{n}\, dA = \int_0^{\pi/2}\int_0^{\pi/2} \cos^3 v \cos u \sin u \; du\, dv$$

$$= \int_0^{\pi/2} \cos u \sin u \, du \int_0^{\pi/2} \cos^3 v \, dv$$

$$= \frac{\sin^2 u}{2}\Big|_0^{\pi/2} \cdot \left(\sin v - \frac{\sin^3 v}{3}\right)\Big|_0^{\pi/2} = \frac{1}{3}$$

(use $\cos^3 v = \cos v\,(1-\sin^2 v)$ to integrate)

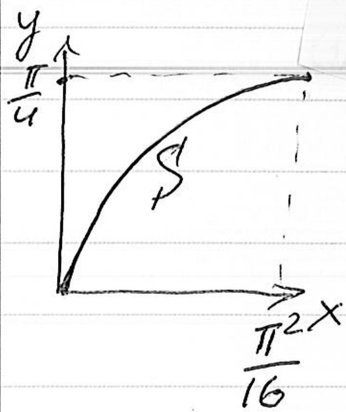
#5) $\vec{F} = \langle x, y, z \rangle$, $S = \{\vec{r} = \langle u\cos v, u \sin v, u^2 \rangle,\ 0 \leq u \leq 4,\ -\pi \leq v \leq \pi\}$

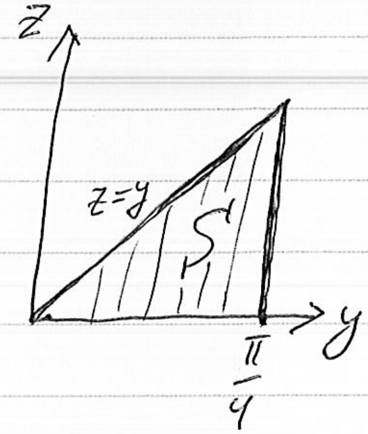
S is part of a paraboloid $z = x^2+y^2$ for $z \leq 16$

$\vec{N} = \langle -2u^2 \cos v, -2u^2 \sin v, u \rangle$ from 10.5 #5

$$\int_S \vec{F}\cdot\vec{n}\, dA = \int_{-\pi}^{\pi}\int_0^4 -u^3 \, du\, dv = -2\pi \frac{u^4}{4}\Big|_0^4 = -128\pi$$

#7) $\vec{F} = \langle 0, \sin y, \cos z \rangle$, $S = \{ x = y^2,\ 0 \le y \le \frac{\pi}{4},\ 0 \le z \le y \}$





S is part of a cylinder over $x = y^2$

Parameterize S by y and z:

$\vec{r} = \langle y^2, y, z \rangle$, $0 \le y \le \frac{\pi}{4}$, $0 \le z \le y$

$\vec{r}_y = \langle 2y, 1, 0 \rangle$, $\vec{r}_z = \langle 0, 0, 1 \rangle$

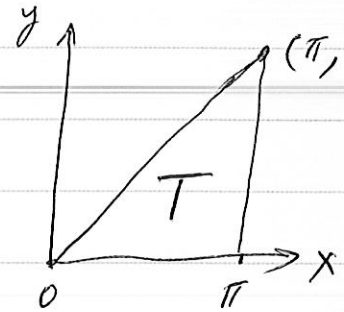
$\vec{N} = \vec{r}_y \times \vec{r}_z = \langle 1, -2y, 0 \rangle$, $\vec{F} \cdot \vec{N} = -2y \sin y$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \int_0^{\pi/4} \int_0^y -2y \sin y \, dz\, dy =$$

$$-2 \int_0^{\pi/4} y^2 \sin y \, dy = 2\left(y^2 \cos y - 2y \sin y - 2\cos y\right)\Big|_0^{\frac{\pi}{4}}$$

$$= 2\left(\frac{\pi^2}{16}\frac{\sqrt{2}}{2} - \frac{\pi}{2}\frac{\sqrt{2}}{2} - \sqrt{2} + 2\right) = \frac{\pi^2\sqrt{2}}{16} - \frac{\pi\sqrt{2}}{2} + 4 - 2\sqrt{2}$$

#13) $G = x + y + z$, $S = \{ z = x + 2y,\ 0 \le x \le \pi,\ 0 \le y \le x \}$



S is part of the plane $z = x + 2y$ over a triangle T with vertices $(0,0)$, $(\pi, 0)$, (π, π)

$\vec{r} = \langle x, y, x + 2y \rangle$

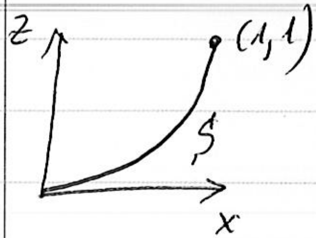
$\vec{r}_x = \langle 1, 0, 1 \rangle$, $\vec{r}_y = \langle 0, 1, 2 \rangle$

$\vec{N} = \vec{r}_x \times \vec{r}_y = \langle -1, -2, 1 \rangle$, $|\vec{N}| = \sqrt{6}$

$$\iint_S G\, dA = \int_0^{\pi} \int_0^{x} (2x + 3y)\sqrt{6}\, dy\, dx =$$

$$\sqrt{6} \int_0^{\pi} \left(2x^2 + \frac{3x^2}{2}\right) dx = \frac{7\sqrt{6}}{6} \pi^3$$

#15) $G = (1+9xz)^{3/2}, \quad S = \left\{ \vec{r} = \langle u, v, u^3 \rangle, \begin{matrix} 0 \le u \le 1 \\ -2 \le v \le 2 \end{matrix} \right\}$



S is part of a cylinder over $z = x^3$

$\vec{r}_u = \langle 1, 0, 3u^2 \rangle, \quad \vec{r}_v = \langle 0, 1, 0 \rangle$

$\vec{N} = \vec{r}_u \times \vec{r}_v = \langle -3u^2, 0, 1 \rangle$

$|\vec{N}| = \sqrt{1+9u^4}, \quad G|\vec{N}| = (1+9u^4)^2$

$$\iint_S G\,dA = \int_{-2}^{2} \int_0^1 (1+9u^4)^2\,du\,dv =$$

$$4 \int_0^1 (1 + 18u^4 + 81u^8)\,du =$$

$$4\left(u + \frac{18}{5}u^5 + 9u^9\right)\Big|_0^1 = \frac{272}{5}$$