A RIEMANN MAPPING THEOREM FOR TWO-CONNECTED DOMAINS IN THE PLANE

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Abstract. We show how to express a conformal map Φ of a general two connected domain in the plane, such that neither boundary component is a point, to a representative domain of the form $\mathcal{A}_r = \{z : |z + 1/z| < 2r\}$, where $r > 1$ is a constant. The domain $\mathcal{A}_r$ has the virtue of having an explicit algebraic Bergman kernel function, and we shall explain why it is the best analogue of the unit disc in the two connected setting. The map Φ will be given as a simple and explicit algebraic function of an Ahlfors map of the domain associated to a specially chosen point. It will follow that the conformal map Φ can be found by solving the same extremal problem that determines a Riemann map in the simply connected case. In the last section, we show how these results can be used to give formulas for the Bergman kernel in two-connected domains.

1. Main results

The Riemann map f_a associated to a point a in a simply connected domain $\Omega \neq \mathbb{C}$ in the complex plane maps Ω one-to-one onto the unit disc $\{z : |z| < 1\}$ and is the solution to an extremal problem: among all holomorphic mappings of Ω into the unit disc, f_a is the map such that $f_a'(a)$ is real and as large as possible. The objects of potential theory associated to the unit disc are particularly simple. The Bergman and Szegő kernel functions are explicit and simple rational functions on the disc, for example. The Poisson kernel is also very simple. Consequently, the Bergman, Poisson, and Szegő kernels associated to Ω can be expressed very simply in terms of a Riemann map.

It was noted in [B4] that any two-connected domain Ω in the plane such that neither boundary component is a point can be mapped conformally via a mapping Φ to a representative domain of the form

$$\mathcal{A}_r := \{z : |z + 1/z| < 2r\},$$

where r is a constant greater than one. See [J-T] for a nice proof of this fact which also proves a much more general theorem. See Figure 1 for a picture of $\mathcal{A}_r$. The dotted circle in the figure is the unit circle. Notice that $1/z$ is an automorphism of $\mathcal{A}_r$ that fixes the unit circle.

In this paper, we show that the conformal map $\Phi : \Omega \to \mathcal{A}_r$ can be expressed simply in terms of an Ahlfors map of Ω. The Bergman kernel associated to

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a representative domain $\mathcal{A}_r$ is an algebraic function (see [B4, B5]). In fact, the Bergman kernel has recently been written down explicitly in [D]. The Szegő kernel is also algebraic and the Poisson kernel can be expressed in terms of rather simple functions (see [B6]). It is proved in [B3] that neither the Bergman nor Szegő kernel can be rational in a two-connected domain. Thus, the representative domain $\mathcal{A}_r$ can be thought of as perhaps the best analogue of the unit disc in the two-connected setting. We shall show that the conformal map Φ can be determined by solving *the same* extremal problem as the one that determines a Riemann map in the simply connected case, but we must be careful to choose the base point properly. In fact, we show how to find a point a in Ω so that Φ is given by $cf_a + \sqrt{c^2 f_a^2 - 1}$ where f_a is the Ahlfors mapping associated to a, and c is a complex constant that we will determine. We will explain along the way how $cf_a + \sqrt{c^2 f_a^2 - 1}$ can be understood to be a single valued holomorphic function on Ω. The Ahlfors map f_a is the solution to the extremal problem: among all holomorphic mappings of Ω into the unit disc, f_a is the map such that $f_a'(a)$ is real and as large as possible. See [B2, p. 49] for the basic properties of the Ahlfors map, including the property that $f_a(a) = 0$, which follows from the extremal condition.

In order to choose a, we will inadvertently discover a new way to compute the modulus of the domain Ω.

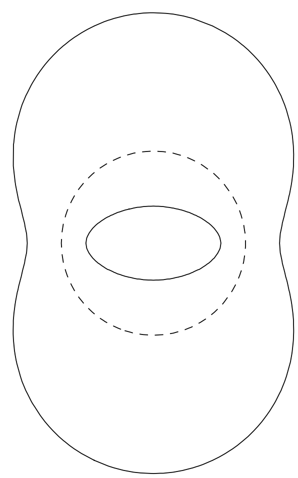

Figure 1. The domain $\mathcal{A}_r$ with $r = 1.05$.

In the last section of this paper, we explore how the mapping Φ can be used to study the Bergman kernel of two-connected domains.

2. Proof of the mapping theorem

In this section, we shall reduce the proof of the following mapping theorem to two key lemmas.

Theorem 2.1. *Suppose that Ω is a two-connected domain in the plane such that neither boundary component is a point. Then, there exists a point a in Ω and a complex constant c such that a one-to-one conformal mapping Φ of Ω onto the representative domain $\mathcal{A}_r$, where $r = |c|$, can be expressed in terms of the Ahlfors map f_a associated to the point a via $\Phi(z) = cf_a + \sqrt{c^2 f_a^2 - 1}$.*

Let a_ρ denote the annulus $\{z : \frac{1}{\rho} < |z| < \rho\}$ when $\rho > 1$, and let

$$J(z) := \frac{1}{2}\left(z + \frac{1}{z}\right)$$

denote the Joukowsky map. Note that $\frac{1}{r}J(z)$ is a proper holomorphic map of $\mathcal{A}_r$ onto the unit disc which is a 2-to-one branched covering map.

Several authors have found explicit biholomorphic maps between a_ρ and $\mathcal{A}_r$ and have determined the relationship between ρ and r (see [J-T, JOT, C, D]). Since we shall need such a map, and since we can construct one in a minimum of space here using the method from [D], we shall include the construction for completeness. The biholomorphic mapping $\Psi : a_\rho \to \mathcal{A}_r$ will be an explicit map involving the Jacobi *sn* function. There is a standard conformal map of the annulus $\{z : \frac{1}{\rho} < |z| < 1\}$ onto the unit disc minus a slit from $-L$ to $+L$ along the real axis. (See Nehari [N, p. 293] for a construction where the relationship between L and ρ is given explicitly.) The mapping is given by a constant times

$$sn\left(K + \frac{2iK}{\pi}\log \rho z\,;\, \rho^{-4}\right),$$

where K is a constant that can be determined explicitly. The map takes the unit circle to the unit circle and the circle of radius $1/\rho$ to the slit from $-L$ to L. By composing with an inversion of the annulus, we get a conformal map that maps the annulus $\{z : \frac{1}{\rho} < |z| < 1\}$ one-to-one onto the unit disc minus a slit from $-L$ to L and takes the unit circle to the slit from $-L$ to L and the circle of radius $1/\rho$ to the unit circle. The Joukowsky map $J(z)$ is a biholomorphic mapping of $\{z \in \mathcal{A}_r : |z| > 1\}$ onto the disc of radius r minus a slit from -1 to $+1$. It takes the outer boundary of $\mathcal{A}_r$ to the circle of radius r and the unit circle two-to-one onto the slit. Thus, by choosing r properly, scaling, and then composing, we can write down a biholomorphic mapping from the annulus $\{z : \frac{1}{\rho} < |z| < 1\}$ onto $\{z \in \mathcal{A}_r : |z| > 1\}$ which maps the unit circle to the unit circle. We may now extend this map by reflection to a biholomorphic map from a_ρ onto $\mathcal{A}_r$. Finally, we may compose with the automorphism $1/z$ and a rotation, if necessary, to obtain a conformal mapping $\Psi : a_\rho \to \mathcal{A}_r$ such that the outer boundary of a_ρ maps to the outer boundary of $\mathcal{A}_r$ and such that $\Psi(i) = i$. Note that Ψ is uniquely determined by these conditions. We shall need to know that $\Psi(-i) = -i$. Since $-1/\Psi(-1/z)$ satisfies the same properties, namely that it is a conformal map that takes the outer boundary to the outer boundary and fixes i, it must be that $\Psi(z) \equiv -1/\Psi(-1/z)$. From this, we deduce that $\Psi(-i) \equiv -1/\Psi(-i)$, and hence that $\Psi(-i)$ is a square root of -1. Because Ψ

is one-to-one, it cannot be that $\Psi(-i)$ is i, and so $\Psi(-i) = -i$. To summarize, there is a biholomorphic mapping $\Psi : a_\rho \to \mathcal{A}_r$ such that

$$\Psi(i) = i \quad \text{and} \quad \Psi(-i) = (-i), \tag{2.1}$$

and which takes the unit circle to the unit circle, the outer boundary to the outer boundary, and the inner boundary to the inner boundary.

It is well known that any two-connected domain Ω in the plane such that neither boundary component is a point can be mapped biholomorphically via a map F_ρ to a unique annulus of the form a_ρ. Let C_1 denote the unit circle. We shall call ρ^2 the modulus of Ω and we shall call the set $F_\rho^{-1}(C_1)$ the *median* of the two-connected domain. The median is the unique connected one-dimensional set that is left invariant by every automorphism of the two-connected domain. The biholomorphic map Ψ of a_ρ onto $\mathcal{A}_r$ takes the unit circle to itself. Hence, the median of $\mathcal{A}_r$ is also the unit circle.

Our main results will follow from two key lemmas, which we now list.

Lemma 2.2. *Suppose that Ω is a two-connected domain in the plane such that neither boundary component is a point. Then, the Ahlfors map f_a associated to any point a in Ω is such that it has two distinct and simple branch points on the median of Ω.*

Lemma 2.2 was proved by McCullough and Mair in [M-M] and independently and by other means in [Te].

Lemma 2.3. *The Ahlfors map associated to the point i in $\mathcal{A}_r$ is given by $\frac{1}{r}J(z)$.*

Lemma 2.3 was proved in [D]. For completeness, and because it will not take much space, we will give alternative proofs of the lemmas which are shorter than the originals in the next section.

Suppose that Ω is a two-connected domain in the plane such that neither boundary component is a point. We now describe the procedure to determine the mapping Φ, assuming the truth of the lemmas. Pick any point P in Ω. Lemma 2.2 guarantees the existence of a point a on the median of Ω such that $f_P'(a) = 0$. (Note that the Ahlfors map is eminently numerically computable if the boundary of Ω is sufficiently smooth (see [K-S, K-T, Tr, B1]), and the zeroes of a function with only two simple zeroes can easily be located numerically as well.) Now the Ahlfors map f_a associated to the point a is such that there are two distinct points p_1 and p_2 on the median of Ω such that $f_a'(p_i) = 0$ for $i = 1, 2$. The points p_i are also different from a because $f_a'(a) \neq 0$ (since $f_a'(a)$ is, in fact, the extremal value by definition of the Ahlfors map). We now claim that there exists an $r > 1$ and a biholomorphic mapping from Ω to $\mathcal{A}_r$ such that $\Phi(a) = i$. We may also arrange for the outer boundary of Ω to get mapped to the outer boundary of $\mathcal{A}_r$. To see this, we use the biholomorphic mapping F_ρ of Ω onto an annulus a_ρ. By composing with $1/z$, if necessary, we may map the outer boundary of Ω to the outer boundary of a_ρ. Since a is on the median, $F_\rho(a)$ is on the unit circle, and we may compose with a rotation so that $F_\rho(a) = i$. Now we may compose with the biholomorphic map Ψ (of a_ρ onto $\mathcal{A}_r$) constructed above, noting that i is fixed under Ψ and that outer boundaries are preserved, to obtain Φ. (At the moment, we can only say that the number r exists. We will explain how to find it momentarily.) Lemma 2.3 states that $\frac{1}{r}J(z)$ is the Ahlfors map of $\mathcal{A}_r$ associated to the point i. The extremal property of Ahlfors maps makes it easy to see that there is a unimodular complex number λ such that

$$\frac{1}{r}J(\Phi(z)) = \lambda f_a(z). \tag{2.2}$$

Since the branch points of $J(z)$ occur at ± 1, this last identity reveals that $\Phi(p_i) = \pm 1$ for $i = 1, 2$. We may assume that $\Phi(p_1) = -1$ and $\Phi(p_2) = 1$. Since $J(-1) = -1$ and $J(1) = 1$, equation (2.2) now gives $-\frac{1}{r} = \lambda f_a(p_1)$ and $\frac{1}{r} = \lambda f_a(p_2)$. Either one of these two equations determines both λ and r. We can now solve equation (2.2) locally for Φ to obtain

$$\Phi(z) = J^{-1}(cf_a(z)),$$

where $c = r\lambda$ and, of course, $J^{-1}(w) = w + \sqrt{w^2 - 1}$. This formula extends by analytic continuation from any point in Ω that is not a branch point of f_a. The apparent singularities at the branch points are removable because the construction aligned the two algebraic singularities of J^{-1} to occur at the two images of the branch points under cf_a. We shall explain how to compute Φ from its boundary values in §4. The modulus ρ^2 of Ω can now be gotten from r via any of the formulas relating ρ and r proved in [J-T, JOT, C-M, D].

3. Proofs of the Lemmas

To prove Lemmas 2.2 and 2.3, we will make use of the simple fact that if $G : \Omega_1 \to \Omega_2$ is a biholomorphism between bounded domains, the Ahlfors maps transform according to the following formula. Let $f_j(z; w)$ denote the Ahlfors map f_w associated to the point w in Ω_j. The extremal property of the Ahlfors maps can be exploited to show that

$$f_1(z; w) = \lambda f_2\left(G(z); G(w)\right), \tag{3.1}$$

where λ is a unimodular constant determined by the condition that the derivative of $f_1(z; w)$ in the z variable at $z = w$ must be real and positive.

Proof of Lemma 2.2. The biholomorphic map F_ρ from Ω to a_ρ together with equation (3.1) reveals that it is sufficient to prove Lemma 2.2 for an annulus a_ρ with base point $a > 0$. The function $h(z) = e^{iz}$ maps the horizontal strip $H = \{z : -\log\rho < \operatorname{Im} z < \log\rho\}$ onto a_ρ, takes the real line onto C_1, and is 2π-periodic. Consider $g(z) = f_a(h(z))$. Notice that g is 2π-periodic, maps H onto the unit disk, and maps the boundary of H onto the unit circle. It can be extended via a sequence of reflections to a function on $\mathbb{C}$. Because of these reflections, g has a second period of $(4\log\rho)i$, making g an elliptic function. Because f_a is two-to-one, g is a second order elliptic function.

Any second order elliptic function ϕ satisfies the equation $\phi(w - z) = \phi(z)$, where w is the sum of two noncongruent zeroes of ϕ (see [V, p. 474], [Nev, p. 74], or [H]). From [T-T], we know that if a is real, f_a has zeroes at a and $-1/a$. We will prove this fact below (Lemma 3.1) to make this paper self contained. It now follows that g will have zeroes at $-i\log a$ and $i\log a + \pi$. Thus, we can choose $w = \pi$. The mapping $z \mapsto \pi - z$ is symmetric with respect to $\pi/2$ and has $\pi/2$ as a fixed point, while the point $3\pi/2$ gets mapped to the congruent point $-\pi/2$. These two points will be branch points of g, and i and $-i$ will be the branch points of f_a, lying on the median of a_ρ.

Proof of Lemma 2.3. The biholomorphism Ψ from a_ρ onto $\mathcal{A}_r$ that we constructed in §2 takes i to i and $-i$ to $-i$ (see (2.1)). Let f_i denote the Ahlfors map of a_ρ with base point i. From [T-T], we know that f_i will have i and $-i$ as its zeroes (see Lemma 3.1 below). Equation (3.1) tells us that the Ahlfors map of $\mathcal{A}_r$ with base point i will have i and $-i$ as its zeroes. Because this Ahlfors map has the same zeroes as $\frac{1}{r}J(z)$, and both functions map the boundary of $\mathcal{A}_r$ to the unit circle, they are the same up to multiplication by a unimodular constant. Since $J'(i) = 1 > 0$, this constant must be 1. Lemma 2.3 is proved.

We shall now flesh out the proofs of the lemmas by giving a shorter and simpler proof than the one given in [T-T] of the following result.

Lemma 3.1. *The Ahlfors map f_a associated to a point a in a_ρ has simple zeroes at a and $-1/\bar{a}$. Thus f_a has simple zeroes at a and $-1/a$ if a is real, and f_i has simple zeroes at i and $-i$.*

The Ahlfors map f_a always has a simple zero at a. Since the Ahlfors map is two-to-one (counting multiplicities), f_a must have exactly one other simple zero distinct from a. The Ahlfors map is given via

$$f_a(z) = S(z,a)/L(z,a) \tag{3.2}$$

where $S(z,a)$ is the Szegő kernel for a_ρ and $L(z,a)$ is the Garabedian kernel. For the basic properties of the Szegő and Garabedian kernels, see [B2, p. 49]. We note here that $S(z,a) = \overline{S(a,z)}$ and that $L(z,a) \neq 0$ if $a \in \Omega$ and $z \in \overline{\Omega}$ with $z \neq a$. Also, $L(z,a)$ has a single simple pole at a in the z variable, and $L(z,a) = -L(a,z)$. It follows from equation (3.2) that the other zero of f_a is the one and only zero of $S(z,a)$ in the z variable. Let $Z(a)$ denote this zero. It is proved in [B4] that $Z(a)$ is a proper antiholomorphic self-correspondence of the domain. In the two-connected case that we are in now, it is rather easy to see that $Z(a)$ is an antiholomorphic function of a that maps a_ρ one-to-one onto itself. To see this, let $S'(z,a)$ denote the derivative of $S(z,a)$ in the z variable and notice that the residue theorem yields

$$Z(a) = \frac{1}{2\pi i}\int_{z\in b\Omega} \frac{zS'(z,a)}{S(z,a)}\, dz.$$

(See [B2] for proofs that $S(z,a)$ is smooth up to the boundary and non-vanishing on the boundary in the z variable when a is held fixed in a_ρ.) It can be read off from this last formula that $Z(a)$ is antiholomorphic in a. Now hold z fixed in a_ρ and let a tend to a boundary point b. Equation (3.2) shows that $f_a(z)$ tends to $S(z,b)/L(z,b)$. But $S(z,b) = -\frac{1}{i}L(z,b)T(b)$ where $T(b)$ is the complex unit tangent vector at b (see [B2, p. 107]), and so $f_a(z)$ tends to $iT(b)$, a unimodular constant. This shows that the zeroes of $f_a(z)$ must tend to the boundary as a tends to the boundary. Consequently, $Z(a)$ is a proper antiholomorphic self mapping of the annulus. The only such maps are one-to-one and onto (see [M-R]). We shall say that $Z(a)$ is an antiholomorphic automorphism of a_ρ. Note that $Z(a)$ must satisfy $Z(Z(a)) = a$. The only such antiholomorphic automorphisms are $Z(a) = \lambda\bar{a}$, where λ is a unimodular constant, and $Z(a) = 1/\bar{a}$, and $Z(a) = -1/\bar{a}$. Since f_a does not have a double zero at a, $Z(a)$ cannot have a fixed point. Only $Z(a) = -1/\bar{a}$ is without fixed points in a_ρ, so f_a has zeroes at a and $-1/\bar{a}$. The proof is complete.

Exactly the same argument applies to Ahlfors maps associated to $\mathcal{A}_r$. We state the following lemma here for use in §5

Lemma 3.2. *The Ahlfors map f_a associated to a point a in $\mathcal{A}_r$ has simple zeroes at a and $-1/\bar{a}$. In particular, the Ahlfors map associated to the point i has simple zeroes at i and $-i$, and the Ahlfors map associated to the point 1 has simple zeroes at 1 and -1.*

4. How to compute Φ and the median of Ω.

Suppose that Ω is a bounded two connected domain bounded by two non-intersecting C^∞ Jordan curves. The Kerzman-Stein-Trummer method [K-S, K-T] can be used to compute the boundary values of Ahlfors mappings associated to Ω. (See [B1] and [B2], Chapter 26, for a description of how to do this in the multiply connected setting.) Thus, we may pick any point P in Ω and compute f_P. We shall now explain how to compute the two branch points of f_P on the median of Ω. Let f denote the Ahlfors mapping associated to the point P and let P_1 and P_2 denote the two simple zeroes of f' on the median of Ω. The residue theorem yields that

$$P_1^n + P_2^n = \frac{1}{2\pi i}\int_{b\Omega} \frac{f''(z)z^n}{f'(z)}\, dz.$$

The two points can be determined from the values of $P_1^n + P_2^n$ for n equal to one and two. Indeed the coefficients A and B of the polynomial $z^2 - Az + B = (z - P_1)(z - P_2)$ are given as $A = P_1 + P_2$ and $B = \frac{1}{2}[(P_1 + P_2)^2 - (P_1^2 + P_2^2)]$, and so P_1 and P_2 can be found via the quadratic formula.

Next, let a denote one of the two points P_1 or P_2. We may now compute the branch points p_1 and p_2 of f_a as we did above for f_P. We shall now show how to compute the function Φ using only the boundary values of the Ahlfors map f_a.

We may calculate $c = r\lambda = -1/f_a(p_1)$ by the procedure described in §2. There are two holomorphic branches of $J^{-1}(w) = w + \sqrt{w^2 - 1}$ near the boundary of the disc of radius r corresponding to the two choices of the square root. One branch j_o takes the circle of radius r one-to-one onto the outer boundary of $\mathcal{A}_r$ and the other branch j_i takes the circle of radius r one-to-one onto the inner boundary of $\mathcal{A}_r$. The mapping Φ that maps Ω one-to-one onto $\mathcal{A}_r$ can be determined from its boundary values on Ω. The boundary values of Φ on the outer boundary of

Ω can be taken to be $j_o(cf_a(z))$ and the boundary values on the inner boundary are then $j_i(cf_a(z))$. The extension of Φ to Ω can be gotten from the boundary values of Φ via the Cauchy integral formula.

We now turn to a method to determine the median of Ω. Equation (2.2) shows that $J(\Phi(z)) = cf(z)$, where f is shorthand for f_a and $c = \lambda r$. This shows that f maps the median of Ω to the line segment $\mathcal{L}$ from $-1/c$ to $1/c$. Hence, the median of Ω is given by $f^{-1}(\mathcal{L})$. We already know the points p_1, p_2, and a on the median. To compute other points w in this set such that $f(w) = \tau$ for some τ in $\mathcal{L}$ where τ is not one of the endpoints of $\mathcal{L}$, we may again use the residue theorem to obtain

$$w_1^n + w_2^n = \frac{1}{2\pi i} \int_{z \in b\Omega} \frac{f'(z) z^n}{f(z) - \tau} \, dz,$$

where w_1 and w_2 are the two points in $f^{-1}(\tau)$. Finally, we may use the Newton identity for $n = 2$ and the quadratic formula as we did above to determine p_1 and p_2 to get w_1 and w_2. Note that p_1 and p_2 correspond to the inverse images of the two endpoints of $\mathcal{L}$, and so we may generate the rest of the median by letting τ move between the endpoints.

5. Other consequences.

We describe in this section how the Bergman kernel of a two-connected domain can be expressed rather concretely in terms of an Ahlfors map. It was proved in [D] that the Bergman kernel associated to $\mathcal{A}_r$ is given by

$$K_r(z, w) = C_1 \frac{2k^2 S(z, \bar{w}) + kC(z, \bar{w}) D(z, \bar{w}) + C_2}{z\bar{w}\sqrt{1 - k^2 J(z)^2}\sqrt{1 - k^2 J(\bar{w})^2}},$$

where $k = 1/r^2$, and C_1 and C_2 are constants that only depend on r, and

$$S(z, w) = -\left(\frac{J(z)\frac{w^2-1}{2w}\sqrt{1 - k^2 J(z)^2} + J(w)\frac{z^2-1}{2z}\sqrt{1 - k^2 J(w)^2}}{1 - k^2 J(z)^2 J(w)^2} \right)^2,$$

and

$$C(z, w) = \frac{-\frac{z^2-1}{2z}\frac{w^2-1}{2w} - J(z)J(w)\sqrt{1 - k^2 J(z)^2}\sqrt{1 - k^2 J(w)^2}}{1 - k^2 J(z)^2 J(w)^2},$$

and

$$D(z, w) = \frac{\sqrt{1 - k^2 J(z)^2}\sqrt{1 - k^2 J(w)^2} + k^2 J(z)J(w)\frac{z^2-1}{2z}\frac{w^2-1}{2w}}{1 - k^2 J(z)^2 J(w)^2}.$$

We showed that the biholomorphic map Φ that we constructed above from a two-connected domain to its representative domain $\mathcal{A}_r$ is such that $\frac{1}{r}J(\Phi(z)) = \lambda f_a(z)$, where f_a is an Ahlfors map associated to Ω for a point a on the median of Ω and λ is a unimodular constant (equation (2.2)). Differentiate this identity to see that

$$\Phi'(z) = 2cf_a'(z)/(1 - \Phi(z)^{-2}) = 2cf_a'(z)/(1 - J^{-1}(cf_a(z))^{-2}),$$

where $J^{-1}(w) = w + \sqrt{w^2-1}$. The Bergman kernel associated to Ω is given by

$$\Phi'(z)K_r(\Phi(z),\Phi(w))\overline{\Phi'(w)}.$$

We may read off from this and the formulas above that the Bergman kernel associated to Ω is $f_a'(z)\,\overline{f_a'(w)}$ times a function which is a rational combination of $f_a(z)$, $\overline{f_a(w)}$, simple algebraic functions of f_a of the form $\sqrt{Af_a(z)^2-1}$ and the conjugate of $\sqrt{Af_a(w)^2-1}$, where A is equal to c, λ, or one. The degree of the rational function does not depend on the modulus of the domain, and the coefficients of the rational function depend in a straightforward way on the modulus.

It was proved in [B5] that the Bergman kernel associated to a two connected domain such that neither boundary component is a point can be expressed in terms of two Ahlfors maps f_a and f_b as

$$f_a'(z)R(z,w)\overline{f_a'(w)}$$

where $R(z,w)$ is a rational function of $f_a(z)$, $f_b(z)$, $\overline{f_a(w)}$, and $\overline{f_b(w)}$. The proof in [B5] is an existence proof, and the rational function R has never been given explicitly. We are now in a position to write down this rational function for the domain $\mathcal{A}_r$ using the points $a=i$ and $b=1$. It was proved in [D] that the Ahlfors map associated to the point 1 of $\mathcal{A}_r$ is

$$g(z) = \frac{1}{2r}\frac{z-\frac{1}{z}}{\sqrt{1-k^2J(z)^2}}$$

where $k=1/r^2$. The square root in the formula can be taken to be the principal branch because the modulus of $kJ(z)$ is less than one in $\mathcal{A}_r$. We shall prove this result here because all the ingredients are on the table. It is a pleasant exercise to check that g is a two-to-one branched covering map of $\mathcal{A}_r$ onto the unit disc. Notice that g has zeroes at 1 and -1. Theorem 3.2 states that the Ahlfors map f_1 has zeroes at 1 and -1. Hence f_1/g is a unimodular constant. Checking derivatives at the point 1 yields that the unimodular constant is 1. So g is equal to the Ahlfors map f_1. Recall that the Ahlfors map f_i associated to i is just $\frac{1}{r}J(z)$.

To determine the rational function, we make note that the terms $\frac{z^2-1}{2z}$ can be multiplied by unity in the form of

$$\frac{\sqrt{1-k^2J(z)^2}}{\sqrt{1-k^2J(z)^2}}$$

to see that

$$\frac{z^2-1}{2z} = r\frac{1}{2r}\left(z-\frac{1}{z}\right) = rf_1(z)\sqrt{1-k^2J(z)^2} = rf_1(z)\sqrt{1-r^{-2}f_i(z)^2}.$$

The terms in the denominator in the expression for $K_r(z,w)$ can be manipulated by multiplying by unity in the form of f_i'/f_i', noting that $f_i'(z) = \frac{1}{r}(1-\frac{1}{z^2})$, to see that

$$z\sqrt{1-k^2J(z)^2} = \frac{2}{f_i'(z)} f_1(z)(1-r^{-2}f_i(z)^2).$$

When these procedures are carried out in each of the expressions that comprise the Bergman kernel, and when all the terms are combined, the square roots all become squared and we obtain

$$K_r(z,w) = f_i'(z)Q(f_i(z), f_1(z), \overline{f_i(w)}, \overline{f_1(w)})\overline{f_i'(w)},$$

where $Q(z_1,z_2,w_1,w_2)$ is a rational function of four variables which can be written as

$$C_1\left(\frac{\sigma(z_1,z_2,w_1,w_2)+\delta(z_1,z_2,w_1,w_2)+C_2}{q(z_1,z_2,w_1,w_2)}\right),$$

where

$$\sigma(z_1,z_2,w_1,w_2) = -2\left(\frac{z_1w_2(1-r^2w_1^2)+w_1z_2(1-r^2z_1^2)}{1-z_1^2w_1^2}\right)^2$$

and

$$\delta(z_1,z_2,w_1,w_2) = -\left(\frac{(z_2w_2+z_1w_1)(1-r^2z_1^2)(1-r^2w_1^2)(1+z_1w_1z_2w_2)+C_2}{(1-z_1^2w_1^2)^2}\right),$$

and

$$q(z_1,z_2,w_1,w_2) = z_2w_2(1-r^2z_1^2)(1-r^2w_1^2).$$

This is more than twice as complicated as any formula on the one-connected unit disc, but one might have expected it to be worse. A fascinating feature that can be read off is that the coefficients depend very simply on r and the degree of the rational function does not depend on r.

For the general two connected domain Ω, we use the mapping Φ and the transformation formula for the Bergman kernels to read off that the Bergman kernel $K(z,w)$ for Ω is

$$K(z,w) = f_a'(z)R(z,w)\overline{f_a'(w)},$$

where $R(z,w)$ is a rational function of $f_a(z)$, $f_b(z)$, $\overline{f_a(w)}$, and $\overline{f_b(w)}$. We take a to be the point we chose in the construction of Φ and b to be equal to p_2. We obtain a formula for the rational function and we note that the degree does not depend on r.

It will be interesting to investigate if similar properties hold for domains of higher connectivity using techniques of Crowday [C] and Crowdy and Marshall [C-M], and it is a safe bet that the Szegő kernel can be expressed in a similar manner on a two-connected domain.

References

[B1] Bell, Steven R., *Numerical computation of the Ahlfors map of a multiply connected planar domain*. J. Math. Anal. Appl. **120** (1986), 211–217.

[B2] ______, *The Cauchy transform, potential theory, and conformal mapping*, CRC Press, Boca Raton, 1992.

[B3] ______, *Complexity of the classical kernel functions of potential theory*. Indiana Univ. Math. J. **44** (1995), 1337–1369.

[B4] ______, *Finitely generated function fields and complexity in potential theory in the plane*, Duke Math. J. **98** (1999), 187–207.
[B5] ______, *Complexity in complex analysis*, Advances in Math. **172** (2002), 15–52.
[B6] ______, *The Greens function and the Ahlfors map*, Indiana Math. J., in press.
[C] D. Crowdy, *Conformal mappings from annuli to canonical doubly connected Bell representations,* J. Math. Anal. and Applications **340/1** (2007), 669–674.
[C-M] D. Crowdy and M. Marshall, *Conformal mappings between canonical multiply connected planar domains,* Computational Methods and Function Theory **6** (2006), 59–76.
[D] Ersin Deger, *A biholomorphism from the Bell representative domain onto an annulus and kernel functions*, `http://docs.lib.purdue.edu/dissertations/AAI3291018/`, Purdue Univ. PhD thesis, 2007.
[H] E. Hille, *Analytic Function Theory*, vol. 2, Blaisdell Publ. Co., Waltham, MA, 1962.
[J-T] M. Jeong and M. Taniguchi, *Bell representations of finitely connected planar domains*, PAMS **131** (2003), 2325–2328.
[JOT] M. Jeong, J.-W. Oh, and M. Taniguchi, *Equivalence problem for annuli and Bell representations in the plane,* J. Math. Anal. Appl. **325** (2007), 1295-1305.
[K-S] N. Kerzman and E. M. Stein, *The Cauchy kernel, the Szegő kernel, and the Riemann mapping function*, Math. Annalen **236** (1978), 85–93.
[K-T] N. Kerzman and M. Trummer, *Numerical conformal mapping via the Szegő kernel*, in Numerical conformal mapping, L. N. Trefethen, ed., North Holland, Amsterdam, 1986.
[M-M] Mair, B. A. and S. McCullough, *Invariance of extreme harmonic functions on an annulus; applications to theta functions*. Houston J. Math. **20** (1994), 453–473.
[M-R] Mueller, Carl and Walter Rudin, *Proper holomorphic self-maps of plane regions*. Complex Variables **17** (1991), 113-121.
[N] Nehari, Z., *Conformal Mapping*, Dover, New York, 1952.
[Nev] E. Neville, *Jacobian elliptic functions*, Oxford Univ. Press, Oxford, 1944.
[Te] Thomas Tegtmeyer, *The Ahlfors map and the Szegő kernel in multiply connected domains*, Purdue Univ. PhD thesis, 1998.
[T-T] Thomas Tegtmeyer and Anthony Thomas, *The Ahlfors map and the Szegő kernel for an annulus*, Rocky Mountain Math. J. **29** (1999), 709–723.
[Tr] M. Trummer, *An efficient implementation of a conformal mapping method based on the Szegő kernel*, SIAM J. of Numer. Anal. **23** (1986), 853–872.
[V] G. Valiron, *Théorie des fonctions*, Masson et Cis, Paris, 1942.

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