Lesson 11 on 10.9 Stokes' Theorem

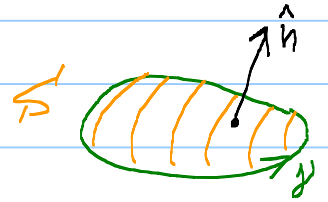
HWK 4: 9, 10, 11 due Wed.

Exam 1: Tues, Feb. 9 8-9pm EE-129

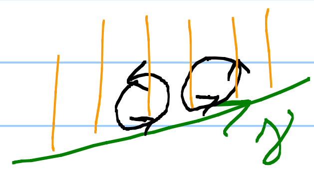
[Off. Hrs. M, T, W 2-3pm
WebEx Off. Hr. Monday 8-9pm

Monday: Review

Two practice exams on Home Page



$$\iint_S (\text{Curl}\,\vec{F})\cdot\hat{n}\,dA = \int_\gamma \vec{F}\cdot d\vec{r}$$



Right hand rule

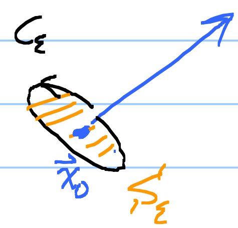
Möbius band ← Can't choose $\hat{n}$
Non-orientable surface.

Stokes' Thm makes it easy to understand implication:

$$\vec{F} \text{ conservative} \Longrightarrow \text{Curl}\,\vec{F} = 0$$

Why: $\vec{F}$ conservative $\Longrightarrow \int_\gamma \vec{F}\cdot d\vec{r} = 0$ any closed loop γ

Say $\text{Curl}\,\vec{F}\,\Big|_{\vec{x}_0} \neq 0.$

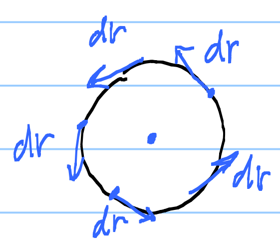


Stokes $\quad \pi\varepsilon^2 \|\text{Curl}\,\vec{F}\| \approx \int_{C_\varepsilon} \vec{F}\cdot d\vec{r} = 0$ ↯

So curl $\vec{F}$ must be zero.

Aha! Same idea yields

$$\text{Curl}\,\vec{F} = \lim_{\varepsilon\to 0} \frac{\hat{n}}{\pi\varepsilon^2} \int_{C_\varepsilon} \vec{F}\cdot d\vec{r}$$



Circulation measures spin

Curl $\vec{F} = 0$ means fluid is irrotational

Green's Thm = Stokes' Thm in plane.

$$\vec{F} = F\hat{\imath} + G\hat{\jmath} + 0\cdot\hat{k}$$

$$\text{Curl}\,\vec{F} = \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\hat{k}$$

$$\iint_{\Omega}\left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right)\underbrace{\hat{n}\cdot\hat{k}}_{1}\,dx\,dy = \int_{\gamma} F\,dx + G\,dy$$

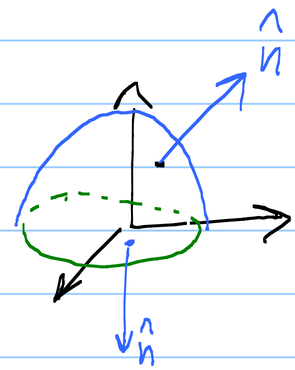
$$= \int_a^b \left(F(x(t),y(t))\frac{dx}{dt} + G(x(t),y(t))\frac{dy}{dt}\right)dt$$

$$= \int_a^b \left[F\hat{\imath} + G\hat{\jmath} + 0\hat{k}\right]\cdot\underbrace{\left[\frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} + 0\hat{k}\right]}_{\frac{d\vec{r}}{dt}}dt$$

✓ Stokes

$$= \int_{\gamma}\vec{F}\cdot d\vec{r}$$

Interesting viewpoint:



Div Thm: $$\iint_{S}(\text{Curl}\,\vec{F})\cdot\hat{n}\,dA = \iiint_{\Omega}\underbrace{\text{Div}\,\text{Curl}\,\vec{F}}_{0}\,dV = 0$$

$$\iint_{\substack{\text{Top Half}\\ S}}(\text{Curl}\,\vec{F})\cdot\hat{n}\,dA + \iint_{\substack{\text{Bottom half}\\ S}}(\text{Curl}\,\vec{F})\cdot\hat{n}\,dA = 0$$

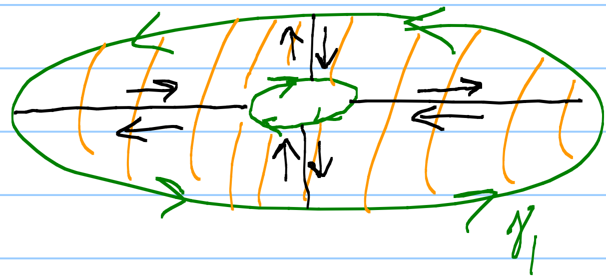
So $\iint_{\text{Top Half}} = \iint_{\text{Plane part}} = \int_\gamma \vec{F}\cdot d\vec{r}$

($-\hat{n}$ makes $\iint_{\text{Bottom}} = -$ plane part)

Proof of Stokes: Do on a little patch.

Choose nice coords to make it a graph of fcn in all 3 coord dirs. Fund. Thm. Calc.

Big S: Add up Stokes on patches.

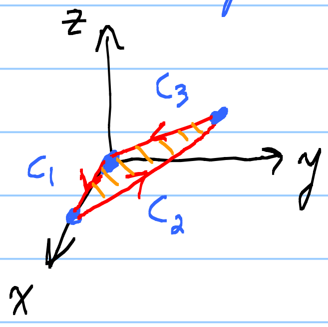


EX: Maxwell's Eqns

Curl (Magnetic Field) = c (current density)

EX: C bndry of Δ: $(0,0,0)$, $(2,0,0)$, $(0,2,1)$

$\vec{F} = -3y^2\,\hat{\imath} + 4z\,\hat{\jmath} + 6x\,\hat{k}$



Right Hand Rule: $\hat{n}$ upward pointing normal

$\vec{F} = -3y^2\hat{\imath} + 4z\hat{\jmath} + 6x\hat{k}$

C_1: $\vec{r}(t) = (2,0,0)t \quad 0 \le t \le 1$
$= (2t, 0, 0)$

$\vec{F} = (0, 0, 12t)$ on C_1

C_2: $\vec{r}(t) = (2,0,0) + \left[(0,2,1) - (2,0,0)\right]t \quad 0 \le t \le 1$
$= (2-2t, 2t, t) \quad 0 \le t \le 1.$

$\vec{F} = (-12t^2, 4t, 12-12t)$

$C_3: \quad \vec{r}(t) = (0,2,1)(-t) \qquad -1 \le t \le 0.$

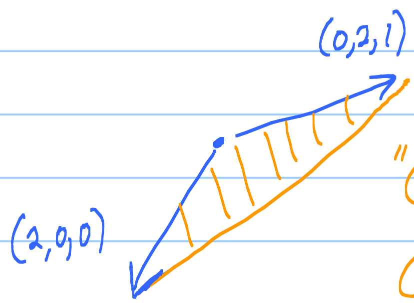
$\vec{F} = (-12t^2, -4t, 0)$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 0\,dt + \int_0^1 (24t^2+8t+12-12t)\,dt + \int_{-1}^0 8t\,dt = 14$$

By Stokes':

$$\text{Curl}\,\vec{F} = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3y^2 & 4z & 6x \end{bmatrix}$$

$$= -4\hat{i} - 6\hat{j} + 6y\,\hat{k}$$



"Convex hull" $= S$

Convex combinations $\lambda\vec{v}_1 + (1-\lambda)\vec{v}_2$

$u\vec{v}_1 + v\vec{v}_2 \quad \begin{cases} u+v \le 1 \\ u,v \ge 0 \end{cases}$

$\vec{r}(u,v) = u(2,0,0) + v(0,2,1) = (2u, 2v, v)$

$0 \le u \le 1$

$0 \le v \le 1-u$

$$\vec{r}_u \times \vec{r}_v = (2,0,0)\times(0,2,1) = \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= -2\hat{j} + 4\hat{k}$$

So $\displaystyle\iint_S (\text{Curl}\,\vec{F})\cdot\hat{n}\,dA =$

$$= \int_0^1 \int_0^{1-u} [-4,-6,12v]\cdot[0,-2,4]\;dv\,du$$

($\|\vec{r}_u \times \vec{r}_v\|$'s cancel)

$$= \int_0^1 \int_0^{1-u} 12 + 48v \, dv \, du$$

$$= \int_0^1 \left[12v + 24v^2 \right]_0^{1-u} du$$

$$= \int_0^1 12(1-u) + 24(1-u)^2 \, du = \underline{\underline{14}}$$