HW 7 Solutions

14.2 p. 659

#9) Integrate counterclockwise around the unit circle C.

$f(z) = \exp(-z^2)$ is analytic in $\mathbb{C}$

$\Rightarrow \int_C \exp(-z^2)\, dz = 0$ by Cauchy's Theorem

#11) $f(z) = \frac{1}{2z-1} = \frac{1}{2} \frac{1}{z-\frac{1}{2}}$ is not analytic at $\frac{1}{2}$.

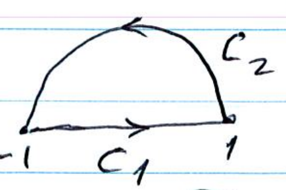
$$\int_C \frac{dz}{2z-1} = \frac{1}{2}\int_{|z-\frac{1}{2}|=\varepsilon} \frac{dz}{z-\frac{1}{2}} = \frac{1}{2}\cdot 2\pi i = \pi i$$

#13) $f(z) = \frac{1}{z^4 - 1.1}$ is analytic inside C, as all four points $z = (1.1)^{1/4}$ are outside unit disk. Thus $\int_C \frac{dz}{z^4-1.1} = 0$ by Cauchy's Theorem.

#18) $f(z) = \frac{1}{4z-3} = \frac{1}{4}\frac{1}{z-\frac{3}{4}}$ is not analytic at $\frac{3}{4}$.

$$\int_C \frac{dz}{4z-3} = \frac{1}{4}\int_{|z-\frac{3}{4}|=\varepsilon} \frac{dz}{z-3/4} = \frac{1}{4}\cdot 2\pi i = \frac{\pi i}{2}$$

#22)

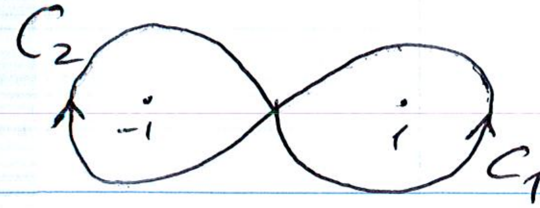


$$\int_C \mathrm{Re}\, z\, dz = \int_{C_1} \mathrm{Re}\, z\, dz + \int_{C_2} \mathrm{Re}\, z\, dz$$

Cauchy's Theorem does not apply since $\mathrm{Re}\, z$ is nowhere analytic. On C_1, $dz = dx$. $\int_{C_1} x\, dx = 0$. Parameterize C_2 as $z = e^{it}$, $0 \le t \le \pi$.

$$\int_{C_2} x\, dz = \int_0^\pi \cos t\ ie^{it} dt = i\int_0^\pi (\cos^2 t + i\cos t \sin t)\, dt = i\frac{\pi}{2}$$

#24) $\displaystyle\int_C \frac{dz}{z^2-1} =$



$$\int_{C_1} \frac{dz}{z^2-1} + \int_{C_2} \frac{dz}{z^2-1} =$$

Partial fractions: $\displaystyle\frac{1}{z^2-1} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z+1}$

$$= \frac{1}{2}\int_{C_1} \frac{dz}{z-1} - \frac{1}{2}\int_{C_2} \frac{dz}{z+1}$$

Since $\displaystyle\int_{C_1} \frac{dz}{z+1} = 0$ and $\displaystyle\int_{C_2} \frac{dz}{z-1} = 0$

$$= \frac{1}{2}\cdot 2\pi i - \frac{1}{2}(-2\pi i) = 2\pi i$$

Since C_2 is oriented clockwise,

$$\int_{C_2} \frac{dz}{z+1} = -2\pi i$$

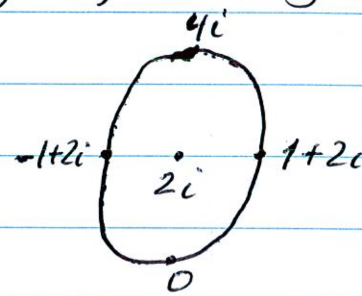
14.3 p. 663

#3) $\displaystyle\int_C \frac{z^2}{z^2-1}\,dz$ around $C = \{|z+i| = 1.4\}$

The function is not analytic at ± 1. Since $|-i \pm 1| = \sqrt{2} > 1.4$, both points are outside the contour. Thus

$$\int_C \frac{z^2}{z^2-1}\,dz = 0$$ by Cauchy's Theorem

#11) $\displaystyle\int_C \frac{dz}{z^2+4}$, $\quad C = \{4x^2 + (y-2)^2 = 4\}$



The function is not analytic at $\pm 2i$, but only $z = 2i$ is inside the contour. Thus

$$\int_C \frac{dz}{z^2+4} = \int_C \frac{\frac{1}{z+2i}}{z-2i}\,dz = 2\pi i \left.\frac{1}{z+2i}\right|_{z=2i} = \frac{\pi}{2}$$

by Cauchy's Formula.

#12) $\displaystyle\int_C \frac{z}{z^2+4z+3}\,dz$, $\quad C = \{|z+1| = 2\}$

The function is not analytic at -1 and -3. As -3 is on C, the integral <u>diverges.</u>

#13) $\displaystyle\int_C \frac{z+2}{z-2}\,dz, \qquad C = \{ |z-1| = 2 \}$

The function is not analytic at 2, which is inside the contour. Thus

$$\int_C \frac{z+2}{z-2}\,dz = 2\pi i\,(z+2)\Big|_{z=2} = 8\pi i$$

by Cauchy's Formula.

#15) $\displaystyle\int_C \frac{\cosh(z^2-\pi i)}{z-\pi i}\,dz$

C is the square with vertices $\pm 4, \pm 4i$

The function is not analytic at πi, which is inside the contour. Thus

$$\int_C \frac{\cosh(z^2-\pi i)}{z-\pi i}\,dz = 2\pi i\cosh(z^2-\pi i)\Big|_{z=\pi i} =$$

$$2\pi i\cosh(-\pi^2-\pi i) = 2\pi i\cosh(\pi^2+\pi i) =$$

$$\pi i\left(e^{\pi^2+\pi i} + e^{-\pi^2-\pi i}\right) = -\pi i\left(e^{\pi^2}+e^{-\pi^2}\right) =$$

$-2\pi i\cosh(\pi^2)$, since $e^{\pm\pi i} = -1$.

14.4. p. 667

#1) Integrate counterclockwise around the unit circle C.

$$\int_C \frac{\sin z}{z^4}\,dz = \frac{2\pi i}{3!}\,(\sin z)'''\Big|_{z=0} =$$

$$\frac{\pi i}{3}\left(-\cos 0\right) = -\frac{\pi i}{3}$$

#3) $$\int_C \frac{e^z}{z^n}\,dz = \frac{2\pi i}{(n-1)!}\left(e^z\right)^{(n)}\Big|_{z=0} = \frac{2\pi i}{(n-1)!}$$

#6) $$\int_C \frac{dz}{(z-2i)^2\left(z-\frac{i}{2}\right)^2} = 2\pi i\left(\frac{1}{(z-2i)^2}\right)'\Big|_{z=\frac{i}{2}}$$

$$= +\frac{2\pi i(-2)}{(z-2i)^3}\Big|_{z=\frac{i}{2}} = (-2)\frac{2\pi i}{\left(-\frac{3}{2}i\right)^3} = \frac{-32\pi}{27}$$

Since $2i$ is outside the contour.

#11) $$\int_C \frac{(1+z)\sin z}{(2z-1)^2}\,dz = \frac{1}{4}\int_C \frac{(1+z)\sin z}{\left(z-\frac{1}{2}\right)^2}\,dz$$

where $C = \{\,|z-i| = 2\,\}$ counterclockwise, so $\frac{1}{2}$ is inside the contour, as $\left|\frac{1}{2}-i\right| = \frac{\sqrt{5}}{2} < 2$.

$$= \frac{1}{4}\,2\pi i\left((1+z)\sin z\right)'\Big|_{z=\frac{1}{2}} =$$

$$\frac{\pi i}{2}\left(\sin z + (1+z)\cos z\right)\Big|_{z=\frac{1}{2}} = \frac{\pi i}{2}\left(\sin\frac{1}{2} + \frac{3}{2}\cos\frac{1}{2}\right).$$