FINITELY GENERATED FUNCTION FIELDS AND COMPLEXITY IN POTENTIAL THEORY IN THE PLANE

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ABSTRACT. We prove that the Bergman kernel function associated to a finitely connected domain Ω in the plane is given as a rational combination of only *three* basic functions of one complex variable: an Alhfors map, its derivative, and one other function whose existence is deduced by means of the field of meromorphic functions on the double of Ω. Because many other functions of conformal mapping and potential theory can be expressed in terms of the Bergman kernel, our results shed light on the complexity of these objects. We also prove that the Bergman kernel is an *algebraic* function of a single Ahlfors map and its derivative. It follows that many objects of potential theory associated to a multiply connected domain are algebraic if and only if the domain is a finite branched cover of the unit disc via an algebraic holomorphic mapping.

1. Introduction. On a simply connected domain in the plane, the Riemann mapping function can be used to pull back the classical kernel functions of potential theory from the unit disc and it follows that the kernel functions are given as simple rational combinations of two functions of one complex variable, the Riemann map and its derivative. Indeed, if a is a point in a simply connected domain $\Omega \neq \mathbb{C}$ and $f_a(z)$ is the Riemann mapping function mapping Ω one-to-one onto the unit disc with $f_a(a) = 0$ and $f_a'(a) > 0$, then the Bergman kernel $K(z, w)$ associated to Ω is given by

$$K(z, w) = \frac{f_a'(z)\overline{f_a'(w)}}{\pi(1 - f_a(z)\overline{f_a(w)})^2}.$$

Another way to write the same formula is

$$K(z, w) = \frac{cK(z, a)\overline{K(w, a)}}{(1 - f_a(z)\overline{f_a(w)})^2}$$

where $c = \pi/f_a'(a)^2$. In this paper, I shall prove that, similarly, on an n-connected domain Ω such that no boundary component is a point, the Bergman kernel is a rational combination of only *three* basic functions of *one* complex variable. One of the basic functions is an Ahlfors mapping associated to the domain. We shall define Ahlfors maps carefully in the next section. Suffice it say for now that the Ahlfors mapping f_a associated to a point $a \in \Omega$ is a branched n-to-one covering map of Ω onto the unit disc with $f_a(a) = 0$ and with $f_a'(a) > 0$ and maximal. In many ways, the Ahlfors map can be thought of as a "Riemann mapping function" for a multiply connected domain. The following theorem shows that the Ahlfors map takes over the role of the Riemann map in the kernel identities mentioned above.

Theorem 1.1. *Suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. There exist points a and b in Ω with the property that the Bergman kernel $K(z, w)$ associated to Ω is a rational combination of the Ahlfors map f_a, its derivative f_a', and the function of one variable given by $K(\cdot, b)$. To be precise, there exists a formula of the form*

$$(1.1) \qquad K(z, w) = R(f_a(z), f_a'(z), K(z, b), \overline{f_a(w)}, \overline{f_a'(w)}, \overline{K(w, b)})$$

where R is a rational function on $\mathbb{C}^6$.

There is nothing particularly special about the points a and b in Theorem 1.1. In fact, the proof shall show that there is a dense open subset of points (a, b) in $\Omega \times \Omega$ satisfying the same properties. We shall prove Theorem 1.1 in §§2-4.

I shall also prove that, under the hypotheses of Theorem 1.1, there is an irreducible polynomial of two complex variables $P(z, w)$ such that the two functions $f_a(z)$ and $K(z, b)/f_a'(z)$ satisfy $P(f_a(z), K(z, b)/f_a'(z)) \equiv 0$, and so it follows that $K(z, b)$ is an algebraic function of f_a and f_a'. Consequently, formula (1.1) yields that $K(z, w)$ is an algebraic function of $f_a(z)$, $f_a'(z)$, and the conjugates of $f_a(w)$ and $f_a'(w)$. The polynomial $P(z, w)$ is a rather interesting algebraic geometric object attached to Ω. We show that it is conformally invariant, but we do not explore its possibly deeper significance here.

The Bergman kernel is at the heart of the Bergman metric and Ahlfors maps determine the Carathéodory metric. The results of this paper show that the two metrics are connected in an algebraic, but perhaps very complicated, manner on multiply connected domains.

Because most other objects of potential theory can be expressed in terms of the Bergman kernel, the results of this paper shall yield that many of the objects of potential theory on a finitely connected domain are given as rational combinations of an Ahlfors map, its derivative, and *one* other basic function of one variable. For example the classical "functions of the first kind," F_1', F_2', ... ,F_{n-1}', associated to an n-connected domain will be seen to be rational combinations of the three basic functions mentioned above, and it will follow that the square of the Szegő kernel is a rational combination of the three basic functions. Furthermore, the relationships proved in [4] between the Szegő kernel, the Poisson kernel, and the Green's function will shed light on the complexity of all these classical objects of potential theory.

An interesting example that is easily analyzed by the methods of this paper is the 2-connected domain $A(r)$ given by $\{z : |z+1/z| < r\}$ where r is a real constant bigger than 2. We shall show that the Bergman and Szegő kernels associated to $A(r)$ are algebraic and that every 2-connected domain in the plane such that no boundary component is a point is biholomorphic to exactly one domain $A(r)$ with $r > 2$. We shall show that biholomorphic maps between domains with algebraic Bergman kernel functions must be algebraic, and it shall follow that we may think of $A(r)$ as being the defining member of a conformal class. We shall also show that there is a polynomial $P(z, w)$ of two complex variables such that the Bergman kernel associated to $A(r)$ satisfies

$$P(K(z, w), f(z), \overline{f(w)}) = 0$$

on $A(r) \times A(r)$ where $f(z) = (1/r)(z + 1/z)$. Let $\hat{A}(r)$ denote the double of $A(r)$. The polynomial equation will reveal that it is possible to analytically continue

$K(z, w)$ to $\hat{A}(r) \times \hat{A}(r)$ as a finitely valued multivalued function with algebraic singularities.

Before we proceed to give all the details, I shall sketch the proof of the Theorem 1.1. For the purposes of this introduction, suppose that Ω is a bounded n-connected domain in the plane bounded by n non-intersecting real analytic curves. The first step in the proof is to show that the Bergman kernel generates itself in the following sense. Let $K(z, w)$ denote the Bergman kernel associated to Ω. Let $K_0(z, w)$ also denote the Bergman kernel and let $K_m(z, w)$ denote the function $(\partial^m / \partial \bar{w}^m) K(z, w)$. We prove that there exists a finite subset $\mathcal{A}$ of Ω and a positive integer N such that the Bergman kernel $K(z, w)$ is given as a rational combination of functions from the finite set of functions of z,

$$\{K_m(z, b) \,:\, b \in \mathcal{A}, \; 0 \le m \le N\},$$

and the finite set of functions of w,

$$\{\overline{K_m(w, b)} \,:\, b \in \mathcal{A}, \; 0 \le m \le N\}.$$

Let $f_a(z)$ denote an Ahlfors mapping associated to Ω which maps $a \in \Omega$ to the origin. We next show that $f_a(z)$ and functions of z of the form

$$\frac{K_m(z, b)}{f_a'(z)}$$

extend to the double of Ω as meromorphic functions. Next, we show that there are points a and b in Ω such that $f_a(z)$ and $K(z, b)/f_a'(z)$ form a primitive pair for the field of meromorphic functions on the double of Ω. This means that any meromorphic function on the double of Ω can be written as a rational combination of these two functions (see Farkas and Kra [10, page 249]). Finally, it follows that the functions of z of the form

$$\frac{K_m(z, w)}{f_a'(z)}$$

can be expressed as rational combinations of $f_a(z)$ and $K(z, b)/f_a'(z)$. Since there are finitely many functions of z of the form $K_m(z, w)$ that generate the Bergman kernel, the existence of formula (1.1) will be proved.

Theorem 1.1 can be interpreted to mean that the Ahlfors maps play the same role in the multiply connected setting that the Riemann maps play for simply connected domains. It is surprising, however, that the same reasoning as above will show that *any* proper holomorphic mapping of the domain onto the unit disc can be used in place of an Ahlfors map — even one of very high order.

I proved in [4] that the Bergman and Szegő kernels associated to an n-connected domain are generated by $n + 1$ basic functions of one complex variable. It is ironic that in order to prove that the Bergman kernel is generated by only the *three* basic functions of one variable mentioned above, I shall need to prove as an intermediate step that it is generated by $3(n^2 - 2n + 2)$ functions of one variable.

2. **Background information and the statement of a main theorem.** Before we can begin to carefully state and prove our main results, we need to review some known facts about the classical kernel functions. Many of these facts and formulas can be found in Stefan Bergman's book [8]. I have also written up most of these

results in [2] in the same spirit as this paper and I include cross references here to give the interested reader access to a uniform approach to the whole subject.

To begin with, we shall assume that Ω is a bounded n-connected domain in the plane with C^∞ smooth boundary. (Later, we shall consider general n-connected domains such that no boundary component is a point.)

Let γ_j, $j = 1, \ldots, n$, denote the n non-intersecting C^∞ simple closed curves which define the boundary $b\Omega$ of Ω, and suppose that γ_j is parameterized in the standard sense by $z_j(t)$, $0 \le t \le 1$. We shall use the convention that γ_n denotes the *outer boundary curve* of Ω. Let $T(z)$ be the C^∞ function defined on $b\Omega$ such that $T(z)$ is the complex number representing the unit tangent vector at $z \in b\Omega$ pointing in the direction of the standard orientation (meaning that $iT(z)$ represents the *inward pointing normal vector* at $z \in b\Omega$). This complex unit tangent vector function is characterized by the equation $T(z_j(t)) = z_j'(t)/|z_j'(t)|$.

The symbol $A^\infty(\Omega)$ will denote the space of holomorphic functions on Ω that are in $C^\infty(\overline{\Omega})$. The space of complex valued functions on Ω that are square integrable with respect to Lebesgue area measure dA will be denoted by $L^2(\Omega)$, and the space of complex valued functions on $b\Omega$ that are square integrable with respect to arc length measure ds by $L^2(b\Omega)$. The Bergman space of holomorphic functions on Ω that are in $L^2(\Omega)$ will be denoted by $H^2(\Omega)$ and the Hardy space of functions in $L^2(b\Omega)$ that are the L^2 boundary values of holomorphic functions on Ω by $H^2(b\Omega)$. The Bergman projection P_B is the orthogonal projection of $L^2(\Omega)$ onto $H^2(\Omega)$, and the Szegő projection P_S is the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$. The Bergman kernel $K(z, w)$ and the Szegő kernel $S(z, w)$ are the kernels for the respective projections in the sense that

$$(P_B\varphi)(z) = \iint_{w \in \Omega} K(z, w)\, \varphi(w)\, dA,$$

$$(P_S\psi)(z) = \int_{w \in b\Omega} S(z, w)\, \psi(w)\, ds.$$

The Bergman kernel $K(z, w)$ is related to the Szegő kernel via the identity

(2.1) $$K(z, w) = 4\pi S(z, w)^2 + \sum_{i,j=1}^{n-1} A_{ij} F_i'(z)\overline{F_j'(w)},$$

where the functions $F_i'(z)$ are classical functions of potential theory described as follows ([8, page 119], or see also [2, pages 94–96]). The harmonic function ω_j which solves the Dirichlet problem on Ω with boundary data equal to one on the boundary curve γ_j and zero on γ_k if $k \ne j$ has a multivalued harmonic conjugate. The function $F_j'(z)$ is a globally defined single valued holomorphic function on Ω which is locally defined as the derivative of $\omega_j + iv$ where v is a local harmonic conjugate for ω_j. The Cauchy-Riemann equations reveal that $F_j'(z) = 2(\partial\omega_j/\partial z)$. A very important fact that we shall need is that the matrix $[A_{ij}]$ appearing in formula (2.1) is non-singular. That this is so was proved by Hejhal in [11] (see Lemma 1 on page 74 and Theorems 30 and 31 on pages 81–83).

The Bergman and Szegő kernels are holomorphic in the first variable and anti-holomorphic in the second on $\Omega \times \Omega$ and they are hermitian, i.e., $K(w, z) = \overline{K(z, w)}$. Furthermore, the Bergman and Szegő kernels are in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in b\Omega\})$ as functions of (z, w) (see [2, page 100]).

We shall also need to mention the Garabedian kernel $L(z, w)$, which is related to the Szegő kernel via the identity

(2.2) $$\frac{1}{i} L(z, a) T(z) = S(a, z) \qquad \text{for } z \in b\Omega \text{ and } a \in \Omega.$$

For fixed $a \in \Omega$, the kernel $L(z, a)$ is a holomorphic function of z on $\Omega - \{a\}$ with a simple pole at a with residue $1/(2\pi)$. Furthermore, as a function of z, $L(z, a)$ extends to the boundary and is in the space $C^\infty(\overline{\Omega} - \{a\})$. In fact, $L(z, w)$ is in $C^\infty((\overline{\Omega} \times \overline{\Omega}) - \{(z, z) : z \in \overline{\Omega}\})$ as a function of (z, w) (see [2, page 102]). Also, $L(z, a)$ is non-zero for all (z, a) in $\overline{\Omega} \times \Omega$ with $z \neq a$ and $L(a, z) = -L(z, a)$ (see [2, page 49]).

For each point $a \in \Omega$, the function of z given by $S(z, a)$ has exactly $(n-1)$ zeroes in Ω (counting multiplicities) and does not vanish at any points z in the boundary of Ω (see [2, page 49]).

Given a point $a \in \Omega$, the Ahlfors map f_a associated to the pair (Ω, a) is a proper holomorphic mapping of Ω onto the unit disc. It is an n-to-one mapping (counting multiplicities), it extends to be in $A^\infty(\Omega)$, and it maps each boundary curve γ_j one-to-one onto the unit circle. Furthermore, $f_a(a) = 0$, and f_a is the unique function mapping Ω into the unit disc maximizing the quantity $|f_a'(a)|$ with $f_a'(a) > 0$. The Ahlfors map is related to the Szegő kernel and Garabedian kernel via (see [2, page 49])

(2.3) $$f_a(z) = \frac{S(z, a)}{L(z, a)}.$$

Note that $f_a'(a) = 2\pi S(a, a) \neq 0$. Because f_a is n-to-one, f_a has n zeroes. The simple pole of $L(z, a)$ at a accounts for the simple zero of f_a at a. The other $n - 1$ zeroes of f_a are given by the $(n-1)$ zeroes of $S(z, a)$ in $\Omega - \{a\}$. Let $a_1, a_2, \ldots, a_{n-1}$ denote these $n - 1$ zeroes (counted with multiplicity) and let $\mathcal{Z}(a)$ denote the set $\{a, a_1, \ldots, a_{n-1}\}$ which is equal to the set of zeroes of f_a on Ω. Next, we define a set $\mathcal{A}(a) = \mathcal{Z}(a) \cup_{k=1}^{n-1} \mathcal{Z}(a_k)$. I proved in [3] (see also [2, page 105]) that, as a tends to a boundary curve γ_j, the $n - 1$ zeroes $a_1, \ldots, a_{n-1}$ become distinct simple zeroes which separate and tend toward the $n - 1$ distinct boundary components of Ω which are different from γ_j. To be precise, as z approaches a point in γ_j, there is exactly one zero of $S(z, a)$ that approaches a point in γ_k, $k \neq j$. It follows from this result that there is a finite subset G of Ω such that $S(z, a)$ has $n - 1$ distinct simple zeroes in Ω as a function of z for every point a in $\Omega - G$. Hence, for points a in $\Omega - G$ that are sufficiently close to the boundary, the set $\mathcal{A}(a)$ is the set consisting of a together with the simple zeroes $a_1, \ldots, a_{n-1}$ together with the $n - 1$ simple zeroes of $S(z, a_k)$ for each k, $1 \leq k \leq n - 1$. Note that the set $\mathcal{A}(a)$ has at most $n + (n - 1)(n - 2) = n^2 - 2n + 2$ elements (because a is one of the zeroes of $S(z, a_k)$ for each k). One of the key ingredients in the work that follows is that the Bergman kernel finitely generates itself. To make precise what we mean by this, we must define some function spaces. Let $K_0(z, w)$ and $K(z, w)$ both denote the Bergman kernel associated to Ω and let $K_m(z, w)$ denote the function $(\partial^m / \partial \bar{w}^m) K(z, w)$. Let $\mathcal{R}_z(\mathcal{A}(a), N)$ denote the field of holomorphic functions of z that are finite rational combinations of

$$\{K_m(z, \alpha) : \alpha \in \mathcal{A}(a) \text{ and } 0 \leq m \leq N\}$$

where N is a positive integer. Let $\mathcal{R}_{\bar{w}}(\mathcal{A}(a), N)$ denote the field of antiholomorphic functions of w that are finite rational combinations of

$$\{\overline{K_m(w, \alpha)} \; : \; \alpha \in \mathcal{A}(a) \text{ and } 0 \leq m \leq N\},$$

and let $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a), N)$ denote the field of functions of z and w that are finite rational combinations of functions in $\mathcal{R}_z(\mathcal{A}(a), N)$ and $\mathcal{R}_{\bar{w}}(\mathcal{A}(a), N)$.

Theorem 2.1. *Suppose that Ω is an n-connected domain in the plane such that no boundary component of Ω is a point. For points $a \in \Omega$ that are sufficiently close to the boundary, the Bergman kernel function $K(z, w)$ associated to Ω is in $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a), 2)$. Furthermore, $S(z, w)^2$, is in $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a), 2)$ and the functions $F_j'(z)$, $j = 1, \ldots, n-1$ are each in $\mathcal{R}_z(\mathcal{A}(a), 2)$.*

We remark that similar reasoning can be used to prove that the kernels $\Lambda(z, w)$ and $L(z, w)^2$ are in $\mathcal{R}_{z,w}(\mathcal{A}(a), 2)$ (where the missing bar over the w means that the generating functions in w are taken to be holomorphic instead of conjugate-holomorphic).

The proof of Theorem 2.1 will be given in §3.

Fix a point a in Ω so that the zeroes $a_1, \ldots, a_{n-1}$ of $S(z, a)$ are distinct simple zeroes. I proved in [4, Theorem 3.1] that the Szegő kernel can be expressed in terms of the $n + 1$ functions of one variable, $S(z, a)$, $f_a(z)$, and $S(z, a_i)$, $i = 1, \ldots, n-1$ via the formula

$$(2.4) \quad S(z, w) = \frac{1}{1 - f_a(z)\overline{f_a(w)}} \left(c_0 S(z, a)\overline{S(w, a)} + \sum_{i,j=1}^{n-1} c_{ij} S(z, a_i) \overline{S(w, a_j)} \right)$$

where $f_a(z)$ denotes the Ahlfors map associated to (Ω, a), $c_0 = 1/S(a, a)$, and the coefficients c_{ij} are given as the coefficients of the inverse matrix to the matrix $[S(a_j, a_k)]$.

A similar identity exists for the Garabedian kernel (see [4]).

$$(2.5) \quad L(z, w) = \frac{f_a(w)}{f_a(z) - f_a(w)} \left(c_0 S(z, a)L(w, a) + \sum_{i,j=1}^{n-1} c_{ij} S(z, a_i)L(w, a_j) \right)$$

where the constants c_0 and c_{ij} are the same as the constants in (2.4).

Let $\mathcal{F}'$ denote the vector space of functions given by the complex linear span of the set of functions $\{F_j'(z) : j = 1, \ldots, n-1\}$ mentioned above. It is a classical fact that $\mathcal{F}'$ is $n-1$ dimensional. It shall be important for us to relate the functions in $\mathcal{F}'$ to the Szegő and Bergman kernel functions. Notice that $S(z, a_i)L(z, a)$ is in $A^\infty(\Omega)$ because the pole of $L(z, a)$ at $z = a$ is cancelled by the zero of $S(z, a_i)$ at $z = a$. Similarly, $S(z, a)L(z, a_i)$ is in $A^\infty(\Omega)$ because the pole of $L(z, a_i)$ at $z = a_i$ is cancelled by the zero of $S(z, a)$ at $z = a_i$. A theorem due to Schiffer ([14], see also [2, page 80]) states that the set of $n-1$ functions $\{S(z, a_i)L(z, a) : i = 1, \ldots, n-1\}$ form a basis for $\mathcal{F}'$. It is also shown in [2, page 80] that the linear span of $\{S(z, a_i)L(z, a) : i = 1, \ldots, n-1\}$ is the same as the linear span of $\{L(z, a_i)S(z, a) : i = 1, \ldots, n-1\}$. Hence, formula (2.1) can be rewritten in the form

$$(2.6) \qquad K(z, w) = 4\pi S(z, w)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} \mathcal{L}_i(z) \overline{\mathcal{L}_j(w)}$$

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where $\mathcal{L}_i(z) = L(z, a_i)S(z, a)$ and, because the matrix $[A_{ij}]$ in formula (2.1) is non-singular, the change of basis we have used yields a matrix $[\lambda_{ij}]$ that is also non-singular.

The Bergman kernel is related to the classical Green's function via ([8, page 62], see also [2, page 131])

$$K(z, w) = -\frac{2}{\pi} \frac{\partial^2 G(z, w)}{\partial z \partial \bar{w}}.$$

Another kernel function on $\Omega \times \Omega$ that we shall need is given by

$$\Lambda(z, w) = -\frac{2}{\pi} \frac{\partial^2 G(z, w)}{\partial z \partial w}.$$

In the literature, this function is sometimes written as $L(z, w)$ with anywhere between zero and three tildes and/or hats over the top. We have chosen the symbol Λ here to avoid confusion with our notation for the Garabedian kernel above. It follows from well known properties of the Green's function that $\Lambda(z, w)$ is holomorphic in z and w and is in $C^\infty(\overline{\Omega} \times \overline{\Omega} - \{(z, z) : z \in \overline{\Omega}\})$. If $a \in \Omega$, then $\Lambda(z, a)$ has a double pole at $z = a$ as a function of z and $\Lambda(z, a) = \Lambda(a, z)$ (see [2, page 134]).

The Bergman kernel is related to Λ via the identity

(2.7) $\Lambda(w, z)T(z) = -K(w, z)\overline{T(z)}$ for $w \in \Omega$ and $z \in b\Omega$

(see [2, page 135]).

The kernel $\Lambda(z, w)$ can also be expressed in terms of kernel functions associated to the boundary (see [4]).

(2.8) $$\Lambda(w, z) = 4\pi L(w, z)^2 + \sum_{i,j=1}^{n-1} \lambda_{ij} L(w, a_i)S(w, a)S(z, a_j)L(z, a),$$

holds for $z, w \in \Omega$, $z \neq w$. The coefficients λ_{ij} are the same as those appearing in (2.6). We may express the functions $S(z, a_j)L(z, a)$ in terms of the other basis $\{\mathcal{L}_j\}_{j=1}^{n-1}$ for $\mathcal{F}'$ in order to be able to rewrite formula (2.8) in the form

(2.9) $$\Lambda(w, z) = 4\pi L(w, z)^2 + \sum_{i,j=1}^{n-1} \mu_{ij} \mathcal{L}_i(z) \mathcal{L}_j(w)$$

where $\mathcal{L}_i(z) = L(z, a_i)S(z, a)$ and, because the matrix $[\lambda_{ij}]$ is non-singular, so is $[\mu_{ij}]$.

We now suppose that Ω is merely an n-connected domain in the plane such that no boundary component of Ω is a point. It is well known that there is a biholomorphic mapping Φ mapping Ω one-to-one onto a bounded domain Ω^a in the plane with real analytic boundary. The standard construction yields a domain Ω^a that is a bounded n-connected domain with C^∞ smooth boundary whose boundary consists of n non-intersecting simple closed real analytic curves. Let superscript a's indicate that a kernel function is associated to Ω^a. Kernels without superscripts are associated to Ω. The transformation formula for the Bergman kernels under biholomorphic mappings gives

(2.10) $$K(z, w) = \Phi'(z)K^a(\Phi(z), \Phi(w))\overline{\Phi'(w)}.$$

Similarly,

$$(2.11) \qquad \Lambda(z, w) = \Phi'(z)\Lambda^a(\Phi(z), \Phi(w))\Phi'(w).$$

It is well known that the function Φ' has a single valued holomorphic square root on Ω (see [2, page 43]). To avoid a discussion of the meaning of the Cauchy transform and the Szegő projection in non-smooth domains, we shall opt to *define* the Szegő and Garabedian kernels associated to Ω via the natural transformation formulas,

$$(2.12) \qquad S(z, w) = \sqrt{\Phi'(z)}\, S^a(\Phi(z), \Phi(w))\overline{\sqrt{\Phi'(w)}},$$

and

$$(2.13) \qquad L(z, w) = \sqrt{\Phi'(z)}\, L^a(\Phi(z), \Phi(w))\sqrt{\Phi'(w)}.$$

The Green's functions satisfy

$$(2.14) \qquad G(z, w) = G^a(\Phi(z), \Phi(w))$$

and the functions associated to harmonic measure satisfy

$$\omega_j(z) = \omega_j^a(\Phi(z)) \quad \text{and} \quad F_j'(z) = \Phi'(z)F_j^{a\,\prime}(\Phi(z)).$$

Finally, the Ahlfors map associated to a point $b \in \Omega$ is defined to be the solution to the extremal problem, $f_b : \Omega \to D_1(0)$ with $f_b'(b) > 0$ and maximal. It is easy to see that the Ahlfors map satisfies

$$f_b(z) = \lambda f_{\Phi(b)}^a(\Phi(z))$$

for some unimodular constant λ and it follows that $f_b(z)$ is a proper holomorphic mapping of Ω onto $D_1(0)$. Furthermore, the transformation formula (2.12) yields that $f_b(z)$ is given by $S(z, b)/L(z, b)$.

It is a routine matter to check that the transformation formulas for the functions above respect all the formulas given in this section where the variables range *inside* the domain. Therefore, statements made above about the zeroes of the Szegő kernel function and the Ahlfors mappings, etc., remain true in this more general setting.

3. The Bergman kernel finitely generates itself. Suppose that Ω is an n-connected domain in the plane such that no boundary component of Ω is a point. Because there is a biholomorphic mapping of Ω onto a bounded domain with real analytic boundary, there are plenty of L^2 holomorphic functions on Ω and so the Bergman projection can be defined in the standard way as the orthogonal projection of $L^2(\Omega)$ onto its subspace of holomorphic functions and the Bergman kernel is the kernel function associated to this projection.

It is a direct consequence of Theorem 2 in [6] (see also [7]) that any proper holomorphic mapping of Ω onto the unit disc is in a set like $\mathcal{R}_z(\mathcal{A}, N)$ mentioned in §2. Because I shall need some corollaries of the proof given in [6], I shall give a quick statement and proof of it here in the simpler case that the proper map is an Ahlfors map with simple zeroes.

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Lemma 3.1. *Suppose Ω is an n-connected domain domain in the plane such that no boundary component is a point. Suppose that f_a is an Ahlfors map associated to Ω such that f_a has exactly n distinct simple zeroes. Let $\mathcal{Z}(a) = f_a^{-1}(0)$. For a positive integer N, let $\mathcal{R}_z(\mathcal{Z}(a), N)$ denote the set of rational combinations of functions from the set*

$$\mathcal{K}(\mathcal{Z}(a), N) := \{K_m(z, \alpha) \,:\, \alpha \in \mathcal{Z}(a) \text{ and } 0 \le m \le N\}$$

where $K_0(z, w)$ is the Bergman kernel associated to Ω and $K_m(z, w)$ denotes the function $(\partial^m / \partial \bar{w}^m) K(z, w)$. The function f_a' is a linear combination of functions from $\mathcal{K}(\mathcal{Z}(a), 0)$ and $f_a' f_a$ is a linear combination of functions from $\mathcal{K}(\mathcal{Z}(a), 1)$. Hence f_a is in $\mathcal{R}_z(\mathcal{Z}(a), 1)$. Furthermore, partial derivatives of $f_a(z)$ or $f_a'(z)$ with respect to a or $\bar{a}$ belong to $\mathcal{R}_z(\mathcal{Z}(a), 2)$ as functions of z.

The following theorem is the more general result than Lemma 3.1 that is proved in [6]. We state it here for future use.

Theorem 3.2. *Suppose that a domain Ω is biholomorphic to a bounded domain in the plane and that f is a proper holomorphic mapping of Ω onto the unit disc. Let $\mathcal{Z} = f^{-1}(0)$. There exists a positive integer N such that f belongs to $\mathcal{R}_z(\mathcal{Z}, N)$.*

A holomorphic function $A(z, w)$ of two complex variables on an open set in $\mathbb{C} \times \mathbb{C}$ is called *algebraic* if there is a holomorphic polynomial $P(a, z, w)$ of three complex variables such that A satisfies $P(A(z, w), z, w) = 0$. It is a well established fact that a function $H(z, w)$ which is holomorphic in z and w on a product domain $\Omega_1 \times \Omega_2$ is algebraic if and only if, for each fixed $b \in \Omega_2$, the function $H(z, b)$ is algebraic in z, and for each fixed $a \in \Omega_1$, the function $H(a, w)$ is algebraic in w (see Bochner and Martin [9, pages 199-202]). We shall say that the Bergman kernel function $K(z, w)$ associated to a domain Ω is *algebraic* if it can be written as $R(z, \bar{w})$ where R is a holomorphic algebraic function of two variables on $\{(z, \bar{w}) \,:\, (z, w) \in \Omega \times \Omega\}$. Because the Bergman kernel is hermitian, the fact above about separate algebraicity implies that $K(z, w)$ is algebraic if and only if, for each point $b \in \Omega$, the function $K(z, b)$ is an algebraic function of z. In fact, $K(z, w)$ is algebraic if and only if there exists a small disc $D_\epsilon(w_0) \subset \Omega$ such that $K(z, b)$ is an algebraic function of z for each $b \in D_\epsilon(w_0)$. We mention here for future use that if $K(z, w)$ is algebraic, then differentiating a polynomial identity of the form $P(K(z, w), z, \bar{w}) = 0$ reveals that all the derivatives $(\partial^m / \partial \bar{w}^m) K(z, w)$ are also algebraic.

One important consequence of Theorem 3.2 is the following result.

Corollary 3.3. *If Φ is a biholomorphic mapping between two finitely connected domains in the plane such that no boundary component is a point that each have algebraic Bergman kernel functions, then Φ must be an algebraic function.*

Corollary 3.3 follows from Theorem 3.2 because if $\Phi : \Omega_1 \to \Omega_2$ is biholomorphic and if $f_a : \Omega_2 \to D_1(0)$ is an Ahlfors map, then $f_a \circ \Phi$ is a proper holomorphic map of Ω_1 onto the unit disc. Theorem 3.2 yields that f_a and $f_a \circ \Phi$ are algebraic, and hence there are germs such that $\Phi = f_a^{-1} \circ (f_a \circ \Phi)$ and it follows that Φ is algebraic.

Proof of Lemma 3.1. Suppose Ω and f_a are as in the statement of the lemma. It is known that f_a is an n-to-one branched covering map of Ω onto $D_1(0)$ (see [2, page 62–70]). The branch locus $\mathcal{B} = \{z \in \Omega : f_a'(z) = 0\}$ is a finite set, and for each

point w_0 in $D_1(0) - f_a(\mathcal{B})$, there are exactly n distinct points in $f_a^{-1}(w_0)$. Near such a point w_0, there is an $\epsilon > 0$ such that it is possible to define n distinct holomorphic maps $\Phi_1(w), \ldots, \Phi_n(w)$ on $D_\epsilon(w_0)$ which map into $\Omega - \mathcal{B}$ such that $f_a(\Phi_k(w)) = w$. These local inverses appear in the following transformation formula for the Bergman kernels under a proper holomorphic mapping. Let $K_D(z, w) = \pi^{-1}(1 - z\bar{w})^{-2}$ denote the Bergman kernel of the unit disc (and recall that $K(z, w)$ denotes the Bergman kernel for Ω). It is proved in [2, page 68] that the kernels transform via

$$f_a'(z) K_D(f_a(z), w) = \sum_{k=1}^{n} K(z, \Phi_k(w)) \overline{\Phi_k'(w)}.$$

Since the origin is in $D_1(0) - f(\mathcal{B})$, we may set $w = 0$ in the transformation formula for the Bergman kernels to obtain

$$(3.1) \qquad f_a'(z) = \pi \sum_{k=1}^{n} K(z, \Phi_k(0)) \overline{\Phi_k'(0)}.$$

This shows that $f_a'(z)$ is a linear combination of functions in $\mathcal{K}(\mathcal{Z}(a), 0)$. Now differentiate the transformation formula with respect to $\bar{w}$ and then set $w = 0$ again to obtain

$$(3.2) \qquad 2f_a'(z) f_a(z) = \pi \sum_{k=1}^{n} \frac{\partial}{\partial \bar{w}} K(z, \Phi_k(0)) \overline{\Phi_k'(0)^2} + \pi \sum_{k=1}^{n} K(z, \Phi_k(0)) \overline{\Phi_k''(0)}.$$

This shows that $f_a'(z) f_a(z)$ is a linear combination of functions in $\mathcal{K}(\mathcal{Z}(a), 1)$, and so it follows that $f_a(z)$ is in $\mathcal{R}_z(\mathcal{Z}(a), 1)$.

To finish the proof of the lemma, we must analyze the way the functions Φ_k depend on a and we shall begin here to write $\Phi_k(w, a)$ instead of $\Phi_k(w)$. Let $a_1, a_2, \ldots, a_{n-1}$ denote the simple zeroes of $S(z, a)$ as in §2 and renumber the Φ_k so that $\Phi_k(0, a) = a_k$ for $k = 1, \ldots, n-1$ and $\Phi_n(0, a) = a$. If we define a_n to be equal to a, then we may write $\Phi_k(0, a) = a_k$ for all k. Recall that $f_a(z) = S(z, a)/L(z, a)$. Hence derivatives of $f_a(z)$, $f_a'(z)$, or $f_a''(z)$ with respect to a or $\bar{a}$ are well defined and easy to compute. (We shall use the convention that primes denote differentiation with respect to the z variable.) Since $\Phi_k(f_a(z), a) = z$ near $z = a_k$, we may differentiate this formula with respect to z and set $z = a_k$ to see that

$$(3.3) \qquad \Phi_k'(0, a) = 1/f_a'(a_k).$$

Next, differentiate the formula $\Phi_k(f_a(z), a) = z$ twice with respect to z and set $z = a_k$ to obtain $\Phi_k''(0, a) f_a'(a_k)^2 + \Phi_k'(0, a) f_a''(a_k) = 0$. This shows that

$$(3.4) \qquad \Phi_k''(0, a) = -f_a''(a_k)/f_a'(a_k)^3.$$

To finish the proof, we must consider the way in which the zeroes a_k depend on a. We now write $a_k(a)$ in order to regard a_k as a function of a. (Of course, $a_n(a) = a$ and it is only for $k = 1, \ldots, n-1$ that we need to exert ourselves.) Let A_0 be a fixed point in Ω such that the zeroes of $S(z, A_0)$ are simple. Since the points $a_k(A_0)$ are distinct, we may choose an $\epsilon > 0$ such that the closed discs of radius ϵ about the points $a_k(A_0)$ are mutually disjoint and each contained in Ω. Thus, $a_k(A_0)$ is the

only zero of $S(z, A_0)$ in the closure of $D_\epsilon(a_k(A_0))$. The dependence of the zeroes of $S(z, a)$ on a can be read off from the formula,

$$a_k(a) = \frac{1}{2\pi i} \int_{|z - a_k(A_0)| = \epsilon} z \, \frac{\frac{\partial}{\partial z} S(z, a)}{S(z, a)} \, dz,$$

which is valid when a is close to A_0. Because $S(z, a)$ is antiholomorphic in a, this formula shows that $a_k(a)$ *is an antiholomorphic function of* a near A_0.

We may now differentiate (3.1) and (3.2) with respect to a or $\bar{a}$ and use the complex chain rule together with (3.3) and (3.4) to complete the proof of the lemma.

We now turn to the proof of Theorem 2.1. Assume that Ω is an n-connected domain in the plane such that no boundary component of Ω. Lemma 3.1 yields that the Ahlfors maps associated to Ω are rational combinations of the type $\mathcal{R}_z(\mathcal{Z}(a), 1)$.

The zeroes of $S(z, a)$ in the z variable become simple zeroes as a approaches the boundary ([3]) and so for $a \in \Omega$ that are sufficiently close to the boundary we know that $S(z, a)$ has exactly $n - 1$ simple zeroes in the z variable. As in §2, we shall denote these zeroes by a_i, $i = 1, \ldots, n-1$ and we shall let $\mathcal{Z}(a)$ denote the set of n points, $\{a, a_1, \ldots, a_{n-1}\}$. Because the Ahlfors map f_a is a proper holomorphic map onto the unit disc, and a together with $a_1, \ldots, a_{n-1}$ are the zeroes of f_a, it follows that the classical Green's function $G(z, w)$ associated to Ω satisfies

$$(3.5) \qquad \frac{1}{2} \ln |f_a(z)|^2 = G(z, a) + \sum_{i=1}^{n-1} G(z, a_i).$$

We now differentiate (3.5) with respect to z to obtain

$$\frac{f_a'(z)}{2 f_a(z)} = \frac{\partial}{\partial z} G(z, a) + \sum_{i=1}^{n-1} \frac{\partial}{\partial z} G(z, a_i).$$

During the course of the proof of Lemma 3.1, we showed that the zeroes $a_1, \ldots, a_{n-1}$ are antiholomorphic functions of a when a is near the boundary. Next, we differentiate with respect to a and use the complex chain rule to obtain

$$(3.6) \qquad \frac{\partial}{\partial a} \left(\frac{f_a'(z)}{2 f_a(z)} \right) = \frac{\partial^2 G(z, a)}{\partial z \partial a} + \sum_{i=1}^{n-1} \frac{\partial^2 G(z, a_i)}{\partial z \partial \bar{a}_i} \frac{\partial \bar{a}_i}{\partial a}.$$

Let $R(z, a)$ denote the left hand side of (3.6). Lemma 3.1 yields that, as a function of z, $R(z, a)$ is in $\mathcal{R}_z(\mathcal{Z}(a), 2)$. The function on the right hand side of (3.6) can be rewritten to yield

$$R(z, a) = -\frac{\pi}{2} \Lambda(z, a) - \frac{\pi}{2} \sum_{i=1}^{n-1} K(z, a_i) \frac{\partial \bar{a}_i}{\partial a}.$$

This last formula shows that, for each fixed a sufficiently close to the boundary of Ω, the function $\Lambda(z, a)$ is in $\mathcal{R}_z(\mathcal{Z}(a), 2)$. Note that we may also state that $\Lambda(z, a_k)$ is in $\mathcal{R}_z(\mathcal{Z}(a_k), 2)$ for $k = 1, \ldots, n - 1$ when a is sufficiently close to the boundary.

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Solve formula (2.6) for $S(z, w)^2$ and formula (2.9) for $L(z, w)^2$ and divide the two, noting that the Ahlfors map $f_w(z)$ is equal to $S(z, w)/L(z, w)$, to obtain

$$(3.7) \qquad f_w(z)^2 = \frac{K(z, w) - \sum_{i,j=1}^{n-1} \lambda_{ij} \mathcal{L}_i(z) \overline{\mathcal{L}_j(w)}}{\Lambda(w, z) - \sum_{i,j=1}^{n-1} \mu_{ij} \mathcal{L}_i(z) \mathcal{L}_j(w)}.$$

Recall that $\mathcal{L}_i(z) := L(z, a_i)S(z, a)$. We shall use formula (3.7) to show that the functions $\{\mathcal{L}_j(z)\}_{j=1}^{n-1}$ are in $\mathcal{R}_z(\mathcal{A}(a), 2)$ where

$$\mathcal{A}(a) = \mathcal{Z}(a) \cup_{k=1}^{n-1} \mathcal{Z}(a_k)$$

is the set described before the statement of Theorem 2.1. We may manipulate (3.7) to obtain

$$(3.8) \qquad \sum_{i,j=1}^{n-1} \mathcal{L}_i(z) \left(\lambda_{ij} \overline{\mathcal{L}_j(w)} - \mu_{ij} f_w(z)^2 \mathcal{L}_j(w) \right) = K(z, w) - f_w(z)^2 \Lambda(w, z).$$

Our plan now is to plug into this formula $w = a_1, a_2, \ldots, a_{n-1}$ where the a_k's are the zeroes of the Szegő kernel $S(z, a)$ that appear in the definition of the functions $\mathcal{L}_i(z)$. We shall obtain a system of $n - 1$ equations that we can use to solve for the functions $\mathcal{L}_i(z)$ and thereby see that the $\mathcal{L}_i(z)$ are in $\mathcal{R}_z(\mathcal{A}(a), 2)$. Define

$$A_{ik}(z) = \sum_{j=1}^{n-1} \left(\lambda_{ij} \overline{\mathcal{L}_j(a_k)} - \mu_{ij} f_{a_k}(z)^2 \mathcal{L}_j(a_k) \right).$$

Notice that, because $\mathcal{L}_j(z) = L(z, a_j)S(z, a)$ and because $S(a_k, a) = 0$, we see that

$$\mathcal{L}_j(a_k) = \begin{cases} 0 & \text{if } j \neq k \\ q_k & \text{if } j = k \end{cases}$$

where, because the zeroes of $S(z, a)$ are *simple* zeroes, q_k is a non-zero number given by $V'(a_k)$ where $V(z) = S(z, a)/(2\pi)$. This shows that the $(n - 1) \times (n - 1)$ matrix $[\mathcal{L}_j(a_k)]$ is non-singular. Notice that

$$A_{ik}(a) = \sum_{j=1}^{n-1} \lambda_{ij} \overline{\mathcal{L}_j(a_k)}$$

because $f_{a_k}(a)$ is zero via (2.3) and the fact that $S(a, a_k) = 0$. It therefore follows from the matrix identity

$$[A_{ik}(a)] = [\lambda_{ij}][\, \overline{\mathcal{L}_j(a_k)} \,]$$

and Hejhal's theorem that $[A_{ik}(a)]$ is non-singular. Hence, for z in a neighborhood of a, the matrix $[A_{ik}(z)]$ will be non-singular and we can use Cramer's rule to solve the $n - 1$ equations obtained from (3.8) by plugging in $w = a_1, a_2, \ldots, a_{n-1}$ to see that each of the functions $\mathcal{L}_j(z)$ is a rational combination of the functions $f_{a_k}(z)$ and $K(z, a_k) - f_{a_k}(z)^2 \Lambda(a_k, z)$, $k = 1, \ldots, n - 1$. Since all of these functions are in $\mathcal{R}_z(\mathcal{A}(a), 2)$, we conclude that $\mathcal{L}_j(z)$ is in $\mathcal{R}_z(\mathcal{A}(a), 2)$.

It now follows from (2.9) that $L(z,a)^2$ is also in $\mathcal{R}_z(\mathcal{A}(a),2)$. Next, we multiply (2.4) by $L(z,a)\overline{L(w,a)}$ to see that

$$S(z,w)L(z,a)\overline{L(w,a)}$$

is a rational combination of the functions $f_a(z)$ and $\overline{f_a(w)}$ times a linear combination of the functions $S(z,a)L(z,a)$, $\overline{S(w,a)L(w,a)}$, and $S(z,a_i)L(z,a)$, $\overline{S(w,a_j)L(w,a)}$ for $i,j = 1,\ldots,n-1$. This shows that $S(z,w)L(z,a)\overline{L(w,a)}$ is in $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a),2)$ because f_a is in $\mathcal{R}_z(\mathcal{A}(a),2)$, and $S(z,a)L(z,a) = f_a(z)L(z,a)^2$ is in $\mathcal{R}_z(\mathcal{A}(a),2)$ since f_a and $L(z,a)^2$ are, and each of the functions $S(z,a_i)L(z,a)$ is in $\mathcal{R}_z(\mathcal{A}(a),2)$ because, as mentioned in §2, the linear span of $\{\mathcal{L}_i : i = 1,\ldots,n-1\}$ is the same as the linear span of $\{S(z,a_i)L(z,a) : i = 1,\ldots,n-1\}$.

Finally, we multiply (2.6) by $L(z,a)^2\overline{L(w,a)^2}$ to obtain

$$K(z,w)L(z,a)^2\overline{L(w,a)^2} =$$

$$4\pi[S(z,w)L(z,a)\overline{L(w,a)}]^2 + L(z,a)^2\overline{L(w,a)^2}\sum_{i,j=1}^{n-1}\lambda_{ij}\mathcal{L}_i(z)\,\overline{\mathcal{L}_j(w)}.$$

We have shown that the right hand side of this equation is composed of functions in $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a),2)$. Since we know that $L(z,a)^2$ is in $\mathcal{R}_z(\mathcal{A}(a),2)$, we may divide the equation by $L(z,a)^2\overline{L(w,a)^2}$ to see that $K(z,w)$ is in $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a),2)$. Now formula (2.6) shows that $S(z,w)^2$ is in $\mathcal{R}_{z,\bar{w}}(\mathcal{A}(a),2)$. Similar reasoning using (2.5) and (2.9) reveals that $\Lambda(z,w)$ and $L(z,w)^2$ are in $\mathcal{R}_{z,w}(\mathcal{A}(a),2)$. The proof of Theorem 2.1 is now complete.

4. The double of a domain, proper maps, and the Bergman kernel. It is well known that the Bergman kernel can be extended to the double of a smooth finitely connected domain as a meromorphic *differential*. In this section, we show that certain simple combinations of the Bergman kernel and a proper holomorphic map onto the unit disc extend to the double as meromorphic *functions*.

Theorem 4.1. *Suppose that Ω is a bounded n-connected domain in the plane bounded by n non-intersecting C^∞ smooth real analytic curves and suppose that $f : \Omega \to D_1(0)$ is a proper holomorphic map. Then $f(z)$ extends meromorphically to the double of Ω. Also, $K(z,w)/f'(z)$ extends meromorphically to the double of Ω as a function of z for each $w \in \overline{\Omega}$ and so do the functions $K_m(z,w)/f'(z)$ for each non-negative integer m (where $K_0(z,w)$ denotes the Bergman kernel of Ω and $K_m(z,w) = (\partial^m/\partial\bar{w}^m)K(z,w)$). Furthermore, if the zeroes of $f(z)$ in Ω are simple zeroes, then for all but possibly finitely many b in $\overline{\Omega}$, the functions $f(z)$ and $K(z,b)/f'(z)$, as functions of z, generate the field of meromorphic functions on the double of Ω, i.e., they form a primitive pair for the double of Ω as in [10, page 249]. Hence, there is an irreducible polynomial $P(z,w)$ such that $P(f(z),K(z,b)/f'(z)) \equiv 0$ on Ω, and for fixed b as above, we may state that $K(z,b)/f'(z)$ is an algebraic function of $f(z)$.*

We shall be able to combine Theorems 2.1 and 4.1 to obtain the following theorem.

Theorem 4.2. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point and suppose that $f : \Omega \to D_1(0)$ is a proper holomorphic map. There is a point $b \in \Omega$ such that the Bergman kernel $K(z, w)$ associated to Ω is a rational combination of the six functions $f(z)$, $f'(z)$, and $K(z, b)$ and the conjugates of $f(w)$, $f'(w)$, and $K(w, b)$. Furthermore, there is an irreducible polynomial $P(z, w)$ such that $P(f(z), K(z, b)/f'(z)) \equiv 0$ on Ω, and therefore $K(z, b)/f'(z)$ is an algebraic function of $f(z)$. Hence, $K(z, w)$ is an algebraic function of $f(z)$ and $f'(z)$ and the conjugates of $f(w)$ and $f'(w)$.*

Theorem 4.2 gives a rather clean answer to a question posed in [5], "When is the Bergman kernel associated to a domain algebraic?"

Corollary 4.3. *Suppose that Ω is an n-connected domain in the plane such that no boundary component is a point. The Bergman kernel associated to Ω is algebraic if and only if there exists a proper holomorphic mapping of Ω onto the unit disc which is algebraic, i.e., if and only if there is a non-constant algebraic function f which has a single valued holomorphic sheet over Ω such that $|f(z)|$ tends to one as z tends to the boundary of Ω. Furthermore, if the Bergman kernel is algebraic, then every proper holomorphic mapping of Ω onto the unit disc is algebraic, the functions $F'_1, \ldots, F_{n-1}$ are algebraic, and so are the Szegő kernel, the Garabedian kernel, and the kernel $\Lambda(z, w)$.*

This corollary implies, for example, that all the classical domain functions associated to the 2-connected domain given by $\{z : |z + 1/z| < r\}$ are algebraic when r is a real constant bigger than 2. An interesting application of our results is that every 2-connected domain is biholomorphic to a domain with algebraic Bergman kernel. Indeed, standard arguments (see [12, pages 10-11]) show that the modulus of $\{z : |z + 1/z| < r\}$ is a continuous increasing function of r which goes to zero as r approaches 2 from above (since the domains have a narrow "pinch" near ± 1 as $r \to 2$) and which goes to infinity as r tends to infinity (since "fat" annuli can be put inside the domains). Hence, each 2-connected domain is biholomorphic to exactly one domain of the form $\{z : |z + 1/z| < r\}$ with $r > 2$. Since Corollary 3.3 states that biholomorphic maps between domains with algebraic Bergman kernel functions are algebraic, we may think of $\{z : |z + 1/z| < r\}$ as being the defining member of a class. This domain also has the virtue that the mapping $f(z) = (1/r)(z + 1/z)$ is a 2-to-one branched cover of $A(r)$ onto the unit disc that extends to be a one-to-one biholomorphic mapping from each connected component of the complement of $A(r)$ in the Riemann sphere onto the complement of the unit disc in the Riemann sphere.

An interesting problem remains to extend these ideas to n-connected domains. Can every n-connected domain be mapped to a domain of the form

$$\{|z + \sum_{k=1}^{n-1} a_k/(z - b_k)| < r\}?$$

When Corollary 4.3 is combined with results of §3, we obtain the following result.

Theorem 4.4. *Suppose Ω is a finitely connected domain in the plane such that no boundary component is a point. The following conditions are equivalent.*

(1) *The Bergman kernel associated to Ω is algebraic.*

(2) *The Szegő kernel associated to Ω is algebraic.*

(3) *There exists a single proper holomorphic mapping of Ω onto the unit disc which is algebraic.*

(4) *Every proper holomorphic mapping of Ω onto the unit disc is algebraic.*

We remark that results in [4] giving explicit formulas for the Poisson kernel in terms of the Szegő kernel also allow us to deduce that if the Szegő kernel associated to a finitely connected domain in the plane such that no boundary component is a point is algebraic, then the Poisson kernel of the domain is a real algebraic function of the variable that runs over the boundary. (It is possible to formulate a converse of this statement because the Poisson kernel is related to a derivative $(\partial/\partial\bar{w})G(z,w)$ of the Green's function and the Bergman kernel is a constant times $(\partial^2/\partial z\partial\bar{w})G(z,w)$. However, because the statement is rather awkward, and because the interesting direction of the implication is the one we stated here, we omit it.)

Whenever a proper map f of a domain onto the unit disc is algebraic, then so are the Schwarz Reflection Functions associated to the boundary curves of the domain because the antiholomorphic reflection functions can be written as $R(z) = f^{-1}(1/\overline{f(z)})$ near the boundary. Hence, the techniques used in this paper might have applications to the sorts of questions studied in Shapiro [13].

Proof of Theorem 4.1. Suppose that Ω and f are as in the statement of Theorem 4.1. It is well known that f extends holomorphically past the boundary of Ω, that f' is non-vanishing on $b\Omega$, and that there is a positive integer m such that f is an m-to-one branched covering map (see [2, page 62-66]). Furthermore, $|f(z)| = 1$ for $z \in b\Omega$.

Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic involution on $\widehat{\Omega}$ which fixes the boundary of Ω. Let $\widetilde{\Omega} = R(\Omega)$ denote the reflection of Ω across the boundary. It is easy to see that f extends to be a meromorphic function on $\widehat{\Omega}$ because $f(z) = 1/\overline{f(z)}$ for $z \in b\Omega$ and, since $R(z) = z$ on $b\Omega$, it follows that

$$f(z) = 1/\overline{f(R(z))} \quad \text{for } z \in b\Omega.$$

The function on the left hand side of this formula is holomorphic on Ω and the function on the right hand side is meromorphic on $\widetilde{\Omega}$ and the two functions extend continuously to $b\Omega$ from opposite sides and agree on $b\Omega$. Hence, the function given by $f(z)$ on $\overline{\Omega}$ and $1/\overline{f(R(z))}$ on $\widetilde{\Omega}$ is meromorphic on $\widehat{\Omega}$. We next note that, because $\ln|f(z)|^2 = 0$ for $z \in b\Omega$, we may differentiate $\ln|f(z(t))|^2$ with respect to t when $z(t)$ parameterizes the boundary to obtain

$$\frac{f'(z(t))}{f(z(t))}z'(t) + \overline{f'(z(t))z'(t)/f(z(t))} = 0.$$

Dividing this equation by $|z'(t)|$ reveals that

(4.1) $$\frac{f'(z)}{f(z)}T(z) = -\overline{f'(z)T(z)/f(z)}, \qquad \text{for } z \in b\Omega.$$

Notice the similarity of this formula to (2.7). Let w be a fixed point in $\overline{\Omega}$. The conjugate of (2.7) is

(4.2) $$K(z,w)T(z) = -\overline{\Lambda(w,z)}\,\overline{T(z)} \qquad \text{for } z \in b\Omega.$$

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We may divide (4.2) by (4.1) to obtain

$$\frac{f(z)K(z,w)}{f'(z)} = \overline{f(z)\Lambda(w,z)/f'(z)} \qquad \text{for } z \in b\Omega.$$

Finally, we may replace z by $R(z)$ (because $R(z) = z$ on $b\Omega$) in the right hand side of this last equation to obtain

$$(4.3) \qquad \frac{f(z)K(z,w)}{f'(z)} = \overline{f(R(z))\Lambda(w,R(z))/f'(R(z))} \qquad \text{for } z \in b\Omega.$$

The function on the left hand side of (4.3) is meromorphic on Ω and extends holomorphically up to the boundary. The function on the right hand side is meromorphic on $\widetilde{\Omega}$ and extends to the boundary of Ω from the outside of Ω. Since these two agree on the boundary, the right hand side defines the meromorphic extension of $f(z)K(z,w)/f'(z)$ to $\widehat{\Omega}$. Since $f(z)$ also extends to be meromorphic on $\widehat{\Omega}$, we may divide by $f(z)$ to see that $K(z,w)/f'(z)$ extends meromorphically to $\widehat{\Omega}$ for each $w \in \overline{\Omega}$.

This last argument can be repeated with $K_m(z,w)$ and $(\partial^m/\partial w^m)\Lambda(z,w) := \Lambda_m(z,w)$ in place of $K(z,w)$ and $\Lambda(z,w)$, respectively, because (2.7) can be differentiated with respect to w and then conjugated to yield

$$K_m(z,w)T(z) = -\overline{\Lambda_m(w,z)}\,\overline{T(z)} \qquad \text{for } z \in b\Omega.$$

Hence, $K_m(z,w)/f'(z)$ extends meromorphically to $\widehat{\Omega}$ for each w in $\overline{\Omega}$.

To finish the proof of Theorem 4.1, we must study conditions under which $f(z)$ and $K(z,w)/f'(z)$ form a *primitive pair*. We now assume that the zeroes of $f(z)$ are simple zeroes. Let $a_1, \ldots, a_m$ denote the m distinct zeroes of f on Ω. It is easy to see that the order of f as a meromorphic function on $\widehat{\Omega}$ is m (because the m zeroes of f on Ω get reflected by R to m poles on $\widetilde{\Omega}$). Hence, to prove that $f(z)$ and $K(z,w)/f'(z)$ form a primitive pair for most w, we must show that $K(z,w)/f'(z)$ separates the m points $a_1, \ldots, a_m$ as a function of z for most w (see [1, page 321-324]). Let S_{ij} denote the set of points w in $\overline{\Omega}$ such that $K(a_i,w) = c_{ij}K(a_j,w)$ where c_{ij} is a non-zero constant given by $f'(a_i)/f'(a_j)$. It is easy to see that the linear span of the set of functions of z given by $\mathcal{K} := \{K(z,w) \ : \ w \in \Omega\}$ is dense in $H^2(\Omega)$. (Indeed, if $h(z)$ is a function in $H^2(\Omega)$ that is orthogonal to all the functions in the spanning set, then $h(w) = 0$ for all $w \in \Omega$ by the reproducing property of the Bergman kernel, and this proves the density.) Let P be a polynomial of one complex variable such that $P(a_i) \neq c_{ij}P(a_j)$. Since it is possible to choose a linear combination of functions in the spanning set $\mathcal{K}$ which converge uniformly on compact subsets of Ω to P, it follows that the function of w given by $K(a_i,w) - c_{ij}K(a_j,w)$ cannot be identically zero on Ω. Since this function extends to be holomorphic on a neighborhood of $\overline{\Omega}$ as a function of w, it follows that S_{ij} is a finite set. Hence, the set of points $\cup_{i<j}S_{ij}$ where $K(z,w)/f'(z)$ might fail to separate the m points $a_1, \ldots, a_m$ as a function of z is at most a finite subset of $\overline{\Omega}$.

Finally, it is a classical fact that, whenever g and h form a primitive pair on a compact Riemann surface, there exists an irreducible polynomial $P(z,w)$ such that $P(g(z),h(z)) \equiv 0$ on the surface (see Farkas and Kra [10, page 248-250]). This finishes the proof of Theorem 4.1.

Proof of Theorem 4.2. Suppose that Ω is a domain as described in the hypotheses of Theorem 4.2. Suppose for the moment that the zeroes of f on Ω are simple zeroes. Theorem 2.1 yields that the Bergman kernel $K(z,w)$ generates itself. Let Φ denote a biholomorphic mapping which maps Ω one-to-one onto a bounded domain Ω^a in the plane with real analytic boundary curves and let $\varphi = \Phi^{-1}$ and $F = f \circ \Phi^{-1}$. Let $K^a(z,w)$ denote the Bergman kernel associated to Ω^a. Since F is a proper holomorphic mapping of Ω^a onto the unit disc with simple zeroes, Theorem 4.1 yields that there exists a point $b \in \Omega^a$ such that $F(z)$ and $K^a(z,b)/F'(z)$ form a primitive pair for the field of meromorphic functions on the double of Ω^a. Let $B = \varphi(b)$.

The transformation formula (2.10) for the Bergman kernels under biholomorphic mappings can also be written in the form

$$\varphi'(z)K(\varphi(z), w) = K^a(z, \Phi(w))\overline{\Phi'(w)}.$$

Hence, we may transform an expression of the form

$$K(z,w)/f'(z)$$

by replacing z by $\varphi(z)$ and by multiplying the whole thing by $\varphi'(z)/\varphi'(z)$ to obtain

$$(4.4) \qquad K(\varphi(z), w)/f'(\varphi(z)) = \overline{\Phi'(w)}K^a(z, \Phi(w))/F'(z).$$

Similar reasoning shows that

$$(4.5) \qquad K(z,w)/f'(z) = \overline{\Phi'(w)}K^a(\Phi(z), \Phi(w))/F'(\Phi(z)).$$

Differentiating (4.4) repeatedly with respect to $\bar{w}$ yields that the linear span of the set

$$\{K_m(\varphi(z), w)/f'(\varphi(z)) : m \leq N\}$$

is equal to the linear span of

$$\{K_m^a(z, \Phi(w))/F'(z) : m \leq N\}.$$

Theorem 4.1 states that functions of the form $K_m^a(z, \Phi(w))/F'(z)$ are rational combinations of $F(z)$ and $K^a(z,b)/F'(z)$. It follows that the field of functions generated by functions of the form $K_m(\varphi(z), w)/f'(\varphi(z))$ is equal to the field of functions generated by $F(z)$ and $K^a(z,b)/F'(z)$. Finally, by replacing z by $\Phi(z)$ and by using (4.5), we see that the field of functions generated by functions of the form $K_m(z,w)/f'(z)$ is equal to the field generated by $f(z) = F(\Phi(z))$ and $K(z,B)/f'(z)$. It now follows from Theorem 2.1 that the Bergman kernel $K(z,w)$ is in the field generated by by $f(z)$, $f'(z)$, and $K(z,B)$, and conjugates of $f(w)$, $f'(w)$, and $K(w,B)$.

Similar reasoning using (4.5) shows that, by replacing z by $\Phi(z)$, a polynomial identity of the form $P^a(F(z), K^a(z,b)/F'(z)) = 0$ can be transformed into one of the form $P(f(z), K(z,B)/f'(z)) = 0$ and the proof is complete in case f has simple zeroes.

To finish the proof, we must treat the case of a proper holomorphic mapping $f : \Omega \to D_1(0)$ that does not have simple zeroes. This is easy, however, because we

may compose f with a Möbius transformation ψ so that $\psi \circ f$ has simple zeroes. Since ψ is rational, it is easy to see that $\psi \circ f$ and $(\psi \circ f)'$ are contained in the function field generated by f and f' and the first part of the theorem follows. An explicit calculation shows that a polynomial identity of the form $P((\psi \circ f)(z), K(z, B)/(\psi \circ f)'(z)) = 0$ can be converted to one of the form $\widetilde{P}(f(z), K(z, B)/f'(z)) = 0$ after clearing the denominator terms by multiplying by a suitable polynomial in $f(z)$. The proof of Theorem 4.2 is complete.

We remark that if we define $I(z, w) = K(z, w)/[f'(z)\overline{f'(w)}]$ and $I^a(z, w) = K^a(z, w)/[F'(z)\overline{F'(w)}]$ in the proof of Theorem 4.2 above, we obtain a rather interesting invariant. The transformation formula (2.10) for the Bergman kernels together with the fact that $f'(z) = F'(\Phi(z))\Phi'(z)$ yields that

$$I(z, w) = I^a(\Phi(z), \Phi(w)),$$

and polynomials satisfying $P(f(z), I(z, b)) = 0$ can be viewed as genuine algebraic geometric invariants.

5. The case of algebraic kernel functions. When the results of this paper are combined with results proved in [5], we can show that if the Bergman kernel associated to a finitely connected domain in the plane is algebraic, then the Bergman kernel is a rational combination of just *two* holomorphic functions of one variable and it is an algebraic function of a single proper holomorphic map. To prove this, we shall need to use the following result from [5].

Theorem 5.1. *Suppose that Ω is a finitely connected domain in the plane such that no boundary component is a point. If the Bergman or the Szegő kernel associated to Ω is algebraic, then Ω can be realized as a subdomain of a compact Riemann surface $\widetilde{\mathcal{R}}$ such that all the kernel functions $S(z, w)$, $L(z, w)$, $K(z, w)$, $\Lambda(z, w)$ extend to $\widetilde{\mathcal{R}} \times \widetilde{\mathcal{R}}$ as single valued meromorphic functions. Furthermore, the Ahlfors maps $f_a(z)$ and every proper holomorphic mapping from Ω to the unit disc extend to be single valued meromorphic functions on $\widetilde{\mathcal{R}}$. Also, the functions $F'_k(z)$, $k = 1, \ldots, n-1$, extend to be single valued meromorphic functions on $\widetilde{\mathcal{R}}$. Furthermore, the complement of Ω in $\widetilde{\mathcal{R}}$ is connected.*

When this theorem is combined with results in §4, we derive the following result.

Theorem 5.2. *Suppose Ω is a finitely connected domain in the plane such that no boundary component is a point and suppose that the Bergman kernel $K(z, w)$ associated to Ω is algebraic. If $f : \Omega \to D_1(0)$ is a proper holomorphic mapping, then $K(z, w)$ is an algebraic function of $f(z)$ and $\overline{f(w)}$ and there exists an irreducible polynomial $P(u, v, w)$ of three complex variables such that*

$$P(K(z, w), f(z), \overline{f(w)}) \equiv 0.$$

Furthermore, there exists a holomorphic function $g(z)$ on Ω such that $K(z, w)$ is a rational combination of $f(z)$, $g(z)$, and conjugates of $f(w)$ and $g(w)$.

Let $\widehat{\Omega}$ denote the double of Ω and let $R(z)$ denote the antiholomorphic reflection function associated to the construction of $\widehat{\Omega}$. The polynomial equation in Theorem 5.2 reveals that it is possible to analytically continue $K(z, w)$ to $\widehat{\Omega} \times \widehat{\Omega}$ as

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a finitely valued multivalued function with algebraic singularities. Indeed, since $f(z) = \overline{f(R(z))^{-1}}$ for $z \in b\Omega$, the expression $P(K(z,w), \overline{f(R(z))^{-1}}, \overline{f(w)})$ is equal to $P(K(z,w), f(z), \overline{f(w)})$ for $z \in b\Omega$ and it defines the holomorphic continuation of $K(z,w)$ in the z variable to the reflection $R(\Omega)$ for each fixed $w \in \Omega$. Now the same thing can be done in the w variable for each fixed $z \in \widehat{\Omega}$ to complete the extension to $\widehat{\Omega} \times \widehat{\Omega}$.

Proof of Theorem 5.2. Theorem 5.1 states that the proper holomorphic map f extends as a meromorphic function to the compact Riemann surface $\widetilde{\mathcal{R}}$. Suppose that the order of f on $\widetilde{\mathcal{R}}$ is m. Choose a point $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ so that $f^{-1}(\lambda)$ consists of m distinct points. We may construct a meromorphic function g on $\widetilde{\mathcal{R}}$ as in Farkas and Kra [10, page 248-249] which is holomorphic on $\Omega \subset \widetilde{\mathcal{R}}$ such that f and g form a primitive pair for the field of meromorphic functions on $\widetilde{\mathcal{R}}$. The construction of $\widetilde{\mathcal{R}}$ in [5] reveals that functions of the form $K(z,b)$ and $K_m(z,b)$ extend to $\widetilde{\mathcal{R}}$ as meromorphic functions of z. Hence, by the properties of a primitive pair, these functions are generated by f and g. Now Theorem 4.2 yields that $K(z,w)$ is a rational combination of $f(z)$, $g(z)$, and conjugates of $f(w)$ and $g(w)$. Finally, there is an irreducible polynomial $G(z,w)$ such that $G(f(z), g(z)) \equiv 0$ on $\widetilde{\mathcal{R}}$. It follows that g is an algebraic function of f, and hence that $K(z,w)$ is an algebraic function of $f(z)$ and the conjugate of $f(w)$. Hence, a polynomial identity of the form $P(K(z,w)f(z), \overline{f(w)}) \equiv 0$ as in the statement of the theorem holds. This completes the proof.

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