HW 4 Solutions

10.7 p. 457

#11) $\vec{F} = \langle e^x, e^y, e^z \rangle$, $\operatorname{div} \vec{F} = e^x + e^y + e^z$

S is the surface of $T = \{|x| \le 1, |y| \le 1, |z| \le 1\}$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (e^x + e^y + e^z)\, dx\, dy\, dz =$$

$$4\int_{-1}^{1} e^x\, dx + 4\int_{-1}^{1} e^y\, dy + 4\int_{-1}^{1} e^z\, dz = 12\left(e - \frac{1}{e}\right).$$

#13) $\vec{F} = \langle \sin y, \cos x, \cos z \rangle$, $\operatorname{div} \vec{F} = -\sin z$

S is the surface of $T = \{x^2 + y^2 \le 4, |z| \le 2\}$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = \iint_R \int_{-2}^{2} -\sin z\, dz\, dA$$

$$= \text{Area}(R) \int_{-2}^{2} -\sin z\, dz = \text{Area}(R) \cos z \Big|_{-2}^{2} = 0$$

#17) $\vec{F} = \langle x^2, y^2, z^2 \rangle$, $\operatorname{div} \vec{F} = 2x + 2y + 2z$

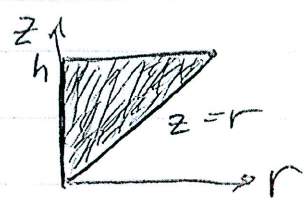
S is the surface of $T = \{x^2 + y^2 \le z^2, 0 \le z \le h\}$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T \operatorname{div} \vec{F}\, dV = 2\int_0^{2\pi}\int_0^h\int_r^h (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^h \left[(h-r)(r^2\cos\theta + r^2\sin\theta) + \frac{r(h^2 - r^2)}{2}\right] dr\, d\theta$$

$$= 2 \underbrace{\int_0^{2\pi} (\cos\theta + \sin\theta)\, d\theta}_{=0} \int_0^h r^2(h-r)\, dr + 2\pi \int_0^h r(h^2 - r^2)\, dr$$

$$= 2\pi\left(h^2\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_0^h = \frac{h^4 \pi}{2}$$

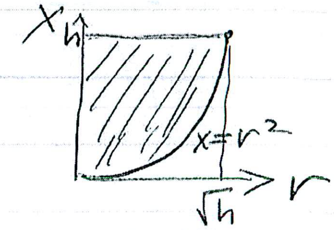


#22) Find moment of inertia $I_x = \iiint_T (y^2+z^2)\,dx\,dy\,dz$ about the x-axis of $T = \{y^2+z^2 \le x,\ 0 \le x \le h\}$

Polar coordinates in the yz-plane; $y = r\cos\theta$, $z = r\sin\theta$

$y^2+z^2 = r^2$

$$\iiint_T (y^2+z^2)\,dV = \int_0^h \int_0^{2\pi} \int_0^{\sqrt{x}} r^2\, r\,dr\,d\theta\,dx =$$



$$2\pi \int_0^h \frac{r^4}{4}\Big|_0^{\sqrt{x}} dx = 2\pi \int_0^h \frac{x^2}{4}\,dx = \frac{h^3}{6}\pi$$

10.8 p. 462

#1) Verify $\iint_S \frac{\partial f}{\partial \vec{n}}\,dA = 0$ for a harmonic function $f = 2z^2 - x^2 - y^2$ and S the surface of $\{0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c\}$

$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ where

$x=0$ on S_1, $x=a$ on S_2, $y=0$ on S_3, $y=b$ on S_4,

$z=0$ on S_5, $z=c$ on S_6.

$\frac{\partial f}{\partial \vec{n}} = 0$ on S_1, S_3, S_5, thus $\iint_S \frac{\partial f}{\partial \vec{n}}\,dA = \iint_{S_2} \frac{\partial f}{\partial \vec{n}}\,dA + \iint_{S_4} \cdots + \iint_{S_6} \cdots$

<u>On S_2</u>, $\vec{n} = \langle 1, 0, 0\rangle$, $\frac{\partial f}{\partial \vec{n}} = \frac{\partial f}{\partial x} = -2x = -2a$

<u>On S_4</u>, $\vec{n} = \langle 0, 1, 0\rangle$, $\frac{\partial f}{\partial \vec{n}} = \frac{\partial f}{\partial y} = -2y = -2b$

<u>On S_6</u>, $\vec{n} = \langle 0, 0, 1\rangle$, $\frac{\partial f}{\partial \vec{n}} = \frac{\partial f}{\partial z} = 4z = 4c$

$\iint_{S_2} \frac{\partial f}{\partial \vec{n}}\,dA = -2a\,\text{Area}(S_2) = -2abc$. Similarly,

$\iint_{S_4} \frac{\partial f}{\partial \vec{n}}\,dA = -2abc$, $\iint_{S_6} \frac{\partial f}{\partial \vec{n}}\,dA = 4abc \Rightarrow \iint_S \frac{\partial f}{\partial \vec{n}}\,dA = 0.$

#3) Verify $\iiint_T (f\nabla^2 g + \nabla f \cdot \nabla g)\,dV = \iint_S f \frac{\partial g}{\partial \vec{n}}\,dA$

for $f = 4y^2$, $g = x^2$, S the surface of

$\{0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$.

$$\iiint_T (f\nabla^2 g + \underbrace{\nabla f \cdot \nabla g}_{=0})\,dV = \int_0^1\int_0^1\int_0^1 8y^2\,dx\,dy\,dz = \frac{8}{3}$$

$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ where

$x = 0$ on S_1; $x = 1$ on S_2, $y = 0$ on S_3, $y = 1$ on S_4, $z = 0$ on S_5, $z = 1$ on S_6. $f = 0$ on S_3, $\frac{\partial g}{\partial n} = 0$ on S_1, S_3, S_4, S_5, S_6

Thus $\iint_S f\frac{\partial g}{\partial \vec{n}}\,dA = \iint_{S_2} f\frac{\partial g}{\partial \vec{n}}\,dA = \int_0^1\int_0^1 4y^2 \cdot 2x\,dy\,dz$

$= \int_0^1\int_0^1 8y^2\,dy\,dz = \frac{8}{3}$. Neither f nor g is harmonic.

#5) Verify $\iiint_T (f\nabla^2 g - g\nabla^2 f)\,dV = \iint_S \left(f\frac{\partial g}{\partial \vec{n}} - g\frac{\partial f}{\partial \vec{n}}\right)dA$

for $f = 6y^2$, $g = 2x^2$, S the surface as in #3.

$$\iiint_T (f\nabla^2 g - g\nabla^2 f)\,dV = \int_0^1\int_0^1\int_0^1 (24y^2 - 24x^2)\,dx\,dy\,dz = 0$$

$S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$ as in #3.

$f\frac{\partial g}{\partial \vec{n}} = 0$ on S_1, S_3, S_4, S_5, S_6; $g\frac{\partial f}{\partial \vec{n}} = 0$ on S_1, S_2, S_3, S_5, S_6

$$\iint_{S_2} f\frac{\partial g}{\partial \vec{n}}\,dA = \int_0^1\int_0^1 6y^2 \cdot 4x\,dy\,dz = \int_0^1\int_0^1 24y^2\,dy\,dz = 8$$

$$\iint_{S_4} g\frac{\partial f}{\partial \vec{n}}\,dA = \int_0^1\int_0^1 2x^2 \cdot 12y\,dx\,dz = \int_0^1\int_0^1 24x^2\,dy\,dz = 8$$

$$\iint_S \left(f\frac{\partial g}{\partial \vec{n}} - g\frac{\partial f}{\partial \vec{n}}\right)dA = \iint_{S_2} f\frac{\partial g}{\partial \vec{n}}\,dA - \iint_{S_4} g\frac{\partial f}{\partial \vec{n}}\,dA = 0$$

#11) Apply $\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy)$ (see (5) on p. 449).

10.9 p. 468

#3) $\vec{F} = \langle e^{-z}, e^{-z}\cos y, e^{-z}\sin y \rangle$

curl $\vec{F} = \langle 2e^{-z}\cos y, -e^{-z}, 0 \rangle$

$S = \{ z = y^2/2,\ -1 \le x \le 1,\ 0 \le y \le 1 \}$

Parameterize S by $(x,y) \Rightarrow \vec{N} = \langle 0, -y, 1 \rangle$

(assuming S oriented by <u>upward</u> normal).

$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, dA = \int_0^1 \int_{-1}^1 y e^{-y^2/2}\, dx\, dy = -2e^{-y^2/2}\Big|_0^1 = 2\left(1 - \frac{1}{\sqrt{e}}\right).$

#5) $\vec{F} = \langle z^2, \frac{3}{2}x, 0 \rangle$, curl $\vec{F} = \langle 0, 2z, \frac{3}{2} \rangle$

$S = \{ 0 \le x \le a, 0 \le y \le a, z = 1 \}$, $\hat{n} = \langle 0, 0, 1 \rangle$

(assuming S oriented by <u>upward</u> normal).

$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, dA = \iint_S \frac{3}{2}\, dA = \frac{3}{2}\,\text{Area}(S) = \frac{3}{2}a^2.$

#13) $\vec{F} = \langle -5y, 4x, z \rangle$, curl $\vec{F} = \langle 0, 0, 9 \rangle$

C is the boundary of $S = \{ x^2 + y^2 \le 16, z = 4 \}$.

$\vec{n} = \langle 0, 0, 1 \rangle$. $\oint_C \vec{F} \cdot d\vec{r} = \iint_S 9\, dA = 9\,\text{Area}(S) = 144\pi$

#15) $\vec{F} = \langle y^2, x^2, z + x \rangle$, curl $\vec{F} = \langle 0, -1, 2x - 2y \rangle$.

C is the boundary of a triangle S with vertices $(0,0,0), (1,0,0), (1,1,0)$. $\hat{n} = \langle 0, 0, 1 \rangle$.

$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (2x - 2y)\, dx\, dy = \int_0^1 \int_0^x (2x - 2y)\, dy\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$

#19) $\vec{F} = \langle z, e^z, 0 \rangle$, curl $\vec{F} = \langle -e^z, 1, 0 \rangle$.

C is the boundary of $S' = \{ z = \sqrt{x^2+y^2}, x \geq 0, y \geq 0, 0 \leq z \leq 1 \}$.

Parameterize S' by $(x,y) \Rightarrow \vec{N} = \langle -\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1 \rangle$.

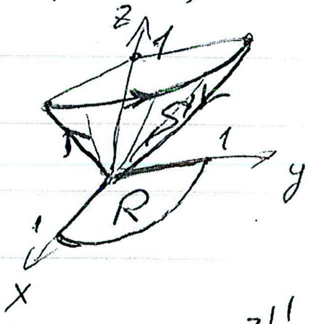
$$\iint_{S'} \text{curl } \vec{F} \cdot n \, dA = \iint_R \left(\frac{x}{\sqrt{x^2+y^2}} e^{\sqrt{x^2+y^2}} - \frac{y}{\sqrt{x^2+y^2}} \right) dx\,dy$$

$$= \int_0^{\pi/2} \int_0^1 (\cos\theta \, e^r - \sin\theta) \, r \, dr \, d\theta =$$

$$\int_0^{\pi/2} \cos\theta \, d\theta \int_0^1 r e^r dr - \int_0^{\pi/2} \sin\theta \, d\theta \int_0^1 r \, dr =$$

$$\sin\theta \Big|_0^{\pi/2} (r-1)e^r \Big|_0^1 + \cos\theta \Big|_0^{\pi/2} \frac{r^2}{2} \Big|_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

Note that C (with the same orientation) is also the boundary of $S_1' + S_2' + S_3'$ where $S_1' = \{x^2+y^2 \leq 1, x \geq 0, y \geq 0, z = 1\}$ oriented by $\vec{n} = \langle 0, 0, 1 \rangle$, $S_2' = \{ x = 0, 0 \leq z \leq 1, 0 \leq y \leq z \}$ oriented by $\vec{n} = \langle -1, 0, 0 \rangle$, $S_3' = \{ y = 0, 0 \leq z \leq 1, 0 \leq x \leq z \}$ oriented by $\vec{n} = \langle 0, -1, 0 \rangle$.



Thus $\iint_{S'} \text{curl } \vec{F} \cdot \vec{n} \, dA =$

$$\iint_{S_1'} \text{curl } \vec{F} \cdot \vec{n} \, dA + \iint_{S_2'} \text{curl } \vec{F} \cdot \vec{n} \, dA + \iint_{S_3'} \text{curl } \vec{F} \cdot \vec{n} \, dA = 0 + \int_0^1 \int_0^z e^z \, dx \, dz + \int_0^1 \int_0^z (-1) \, dy \, dz = (z-1)e^z \Big|_0^1 - \frac{1}{2} = \frac{1}{2}$$