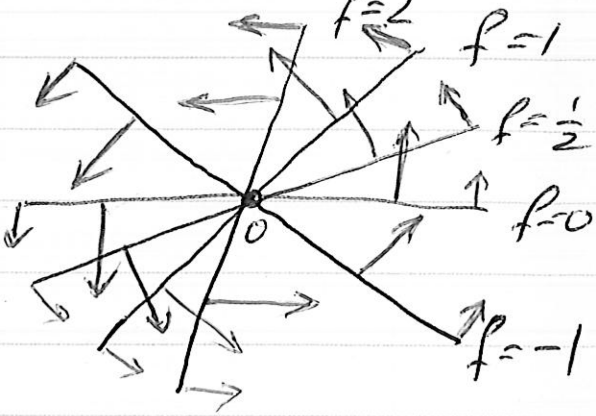
HW Solutions 9.6, 9.7 p. 402

#3) $f = \dfrac{y}{x}$, $\text{grad} f = \left\langle -\dfrac{y}{x^2}, \dfrac{1}{x} \right\rangle = \dfrac{1}{x^2}\langle -y, x\rangle$

Level curves $f = c \Rightarrow y = cx$ are lines through the origin



#8) From the product rule,

$$\begin{cases} \dfrac{\partial}{\partial x}(fg) = f\dfrac{\partial g}{\partial x} + g\dfrac{\partial f}{\partial x} \\ \dfrac{\partial}{\partial y}(fg) = f\dfrac{\partial g}{\partial y} + g\dfrac{\partial f}{\partial y} \end{cases}$$

Thus $\nabla(fg) = \left\langle \dfrac{\partial}{\partial x}(fg), \dfrac{\partial}{\partial y}(fg)\right\rangle =$

$$\left\langle f\frac{\partial g}{\partial x} + g\frac{\partial f}{\partial x}, f\frac{\partial g}{\partial y} + g\frac{\partial f}{\partial y}\right\rangle = f\nabla g + g\nabla f$$

#12) $f = \dfrac{x}{x^2+y^2}$, $P = (1,1)$

$$\nabla f = \left\langle \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2}\right\rangle$$

$$\nabla f(P) = \left\langle \frac{1}{2} - \frac{2}{4}, -\frac{2}{4}\right\rangle = \left\langle 0, -\frac{1}{2}\right\rangle$$

#15) $f = 4x^2 + 9y^2 + z^2$, $P = (5, -1, -11)$

$\nabla f = \langle 8x, 18y, 2z\rangle$

$\nabla f(P) = \langle 8\cdot 5 - 18\cdot 1 - 2\cdot 11\rangle = \langle 40, -18, -22\rangle$

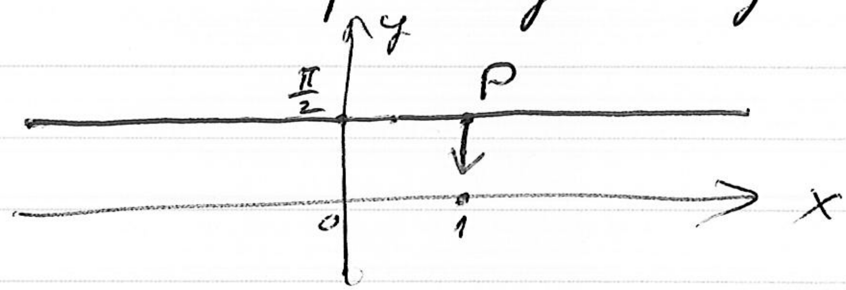
#21) $f = e^x \cos y$, $P = (1, \frac{\pi}{2})$

$\vec{v} = \text{grad} f = \langle e^x \cos y, -e^x \sin y \rangle$

$\vec{v}(P) = e \langle \cos\frac{\pi}{2}, -\sin\frac{\pi}{2} \rangle = \langle 0, -e \rangle$

$f(P) = e \cos\frac{\pi}{2} = 0 \Rightarrow \{f = 0\} = \{\cos y = 0\}$

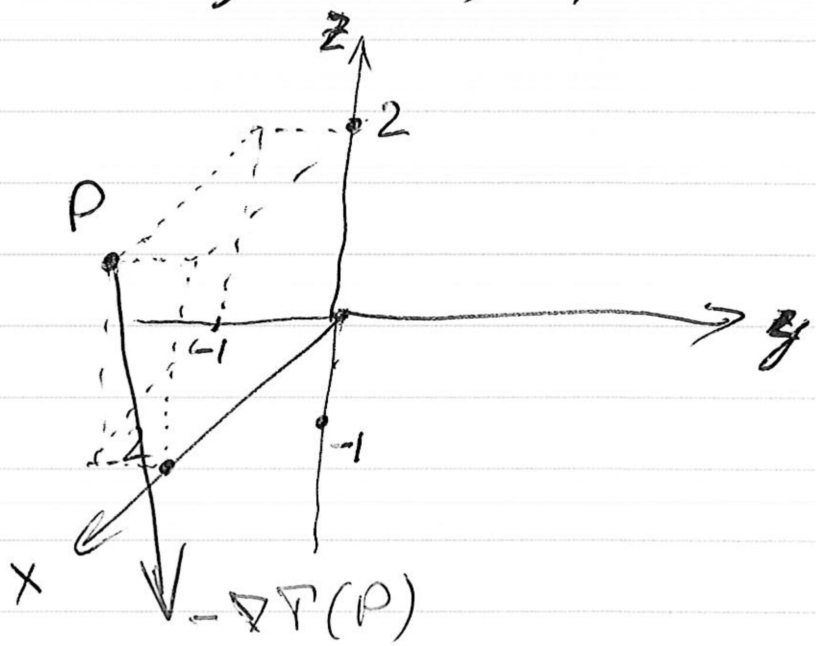
The curve passing through P is $\{y = \frac{\pi}{2}\}$



#26) $T = x^2 + y^2 + 4z^2$, $P = (2, -1, 2)$

<u>Max decrease</u> is $-\nabla T = \langle -2x, -2y, -8z \rangle$

$-\nabla T(P) = \langle -4, 2, -16 \rangle$



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#3) $\vec{v} = \left\langle \dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2} \right\rangle$

$$\operatorname{div} \vec{v} = \frac{\partial}{\partial x} \frac{x}{x^2+y^2} + \frac{\partial}{\partial y} \frac{y}{x^2+y^2} =$$

$$\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = 0$$

#5) $\vec{v} = \langle x^3y^2z^2, x^2y^3z^2, x^2y^2z^3 \rangle$,
$P = (3, -1, 4)$

$$\operatorname{div} \vec{v} = 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2 = 9x^2y^2z^2$$

$$\operatorname{div} \vec{v}(P) = 9 \cdot 9 \cdot 16 = 1296$$

#9) a) $\operatorname{div}(k\vec{v}) = \frac{\partial}{\partial x} k v_1 + \frac{\partial}{\partial y} k v_2 + \frac{\partial}{\partial z} k v_3$

$$= k\left(\frac{\partial}{\partial x} v_1 + \frac{\partial}{\partial y} v_2 + \frac{\partial}{\partial z} v_3\right) = k \operatorname{div} \vec{v}$$

b) $\operatorname{div}(f\vec{v}) = \frac{\partial}{\partial x}(f v_1) + \frac{\partial}{\partial y}(f v_2) + \frac{\partial}{\partial z}(f v_3)$

$$= f\frac{\partial v_1}{\partial x} + v_1 \frac{\partial f}{\partial x} + f \frac{\partial v_2}{\partial y} + v_2 \frac{\partial f}{\partial y} + f \frac{\partial v_3}{\partial z} + v_3 \frac{\partial f}{\partial z}$$

$$= f \operatorname{div} \vec{v} + \vec{v} \cdot \nabla f$$

c) $\operatorname{div}(f \nabla g) = f \operatorname{div}(\nabla g) + \nabla g \cdot \nabla f = f \nabla^2 g + \nabla f \cdot \nabla g$

from (b)

d) $\operatorname{div}(f\nabla g) - \operatorname{div}(g \nabla f) = f\nabla^2 g - g \nabla^2 f$ (from (c))

#17) $f = \ln(x^2+y^2)$

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f) = \operatorname{div}\left(\left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right\rangle\right)$$

$$= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} = 0$$

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#4) $\operatorname{curl}(\langle 2y^2, 5x, 0\rangle) =$

$$\left\langle 0, 0, \frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y^2)\right\rangle$$

$$= \langle 0, 0, 5-4y\rangle$$

#9) $\vec{v} = \langle 0, 3z^2, 0\rangle$

$\operatorname{div}\vec{v} = \frac{\partial}{\partial y}(3z^2) = 0 \Rightarrow \vec{v}$ is <u>incompressible</u>

$\operatorname{curl}\vec{v} = \left\langle -\frac{\partial}{\partial z}(3z^2), 0, \frac{\partial}{\partial x}(3z^2)\right\rangle \neq 0$

$\Rightarrow \vec{v}$ is <u>not irrotational</u>

Streamlines have tangent vectors parallel to $\langle 0,1,0\rangle$ when $z \neq 0$, thus they are straight lines parallel to the y-axis.
When $z=0$, each point $(x,y,0)$ is a stationary point of the flow.

#14) a) $\operatorname{curl}(\vec{u}+\vec{v}) =$

$$\left\langle \frac{\partial}{\partial y}(u_3+v_3) - \frac{\partial}{\partial z}(u_2+v_2), \frac{\partial}{\partial z}(u_1+v_1) - \frac{\partial}{\partial x}(u_3+v_3), \frac{\partial}{\partial x}(u_2+v_2) - \frac{\partial}{\partial y}(u_1+v_1) \right\rangle$$

$$= \left\langle \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right\rangle + \left\langle \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right\rangle$$

$$= \operatorname{curl}\vec{u} + \operatorname{curl}\vec{v}$$

b) $\operatorname{div}(\operatorname{curl}\vec{v}) = \frac{\partial}{\partial x}\left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z} + \frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x} + \frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}$$

$= 0$ because $\frac{\partial^2 v_3}{\partial y \partial x} = \frac{\partial^2 v_3}{\partial x \partial y}, \frac{\partial^2 v_2}{\partial z \partial x} = \frac{\partial^2 v_2}{\partial x \partial z}, \frac{\partial^2 v_1}{\partial z \partial y} = \frac{\partial^2 v_1}{\partial y \partial z}$

c) $\operatorname{curl}(f\vec{v}) =$

$$\left\langle \frac{\partial}{\partial y}(fv_3) - \frac{\partial}{\partial z}(fv_2), \frac{\partial}{\partial z}(fv_1) - \frac{\partial}{\partial x}(fv_3), \frac{\partial}{\partial x}(fv_2) - \frac{\partial}{\partial y}(fv_1) \right\rangle$$

$$= f\left\langle \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right\rangle +$$

$$\left\langle v_3\frac{\partial f}{\partial y} - v_2\frac{\partial f}{\partial z}, v_1\frac{\partial f}{\partial z} - v_3\frac{\partial f}{\partial x}, v_2\frac{\partial f}{\partial x} - v_1\frac{\partial f}{\partial y} \right\rangle =$$

$f \operatorname{curl} v + \nabla f \times \vec{v}$

d) curl (grad f) =

$$\left\langle \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right)-\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right), \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)-\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right\rangle$$

$$=\left\langle \frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}, \frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}, \frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right\rangle$$

$= \langle 0, 0, 0\rangle$ because

$$\frac{\partial^2 f}{\partial y\partial z}=\frac{\partial^2 f}{\partial z\partial y}, \quad \frac{\partial^2 z}{\partial z\partial x}=\frac{\partial^2 f}{\partial x\partial z}, \quad \frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

e) $\mathrm{div}(\vec{u}\times\vec{v}) = \frac{\partial}{\partial x}(u_2v_3-u_3v_2)+\frac{\partial}{\partial y}(u_3v_1-u_1v_3)+\frac{\partial}{\partial z}(u_1v_2-u_2v_1)$

$$= v_3\frac{\partial u_2}{\partial x}-v_2\frac{\partial u_3}{\partial x}+v_1\frac{\partial u_3}{\partial y}-v_3\frac{\partial u_1}{\partial y}+v_2\frac{\partial u_1}{\partial z}-v_1\frac{\partial u_2}{\partial z}$$

$$+u_2\frac{\partial v_3}{\partial x}-u_3\frac{\partial v_2}{\partial x}+u_3\frac{\partial v_1}{\partial y}-u_1\frac{\partial v_3}{\partial y}+u_1\frac{\partial v_2}{\partial z}-u_2\frac{\partial v_1}{\partial z}$$

$$=v_1\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z}\right)+v_2\left(\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x}\right)+v_3\left(\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)$$

$$+u_1\left(\frac{\partial v_2}{\partial z}-\frac{\partial v_3}{\partial y}\right)+u_2\left(\frac{\partial v_3}{\partial x}-\frac{\partial v_1}{\partial z}\right)+u_3\left(\frac{\partial v_1}{\partial y}-\frac{\partial v_2}{\partial x}\right)$$

$$=\vec{v}\cdot\mathrm{curl}\,\vec{u}-\vec{u}\cdot\mathrm{curl}\,\vec{v}$$

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#2) $\int_C \langle y^2, -x^2\rangle \cdot d\vec{r}$ where $C=\{y=4x^2,\ 0\le x\le 1\}$

equals (taking x as a parameter)

$$\int_0^1 \langle (4x^2)^2, -x^2\rangle \cdot \underbrace{\langle 1, 8x\rangle}_{\text{"}\vec{r}\,'\text{"}}\, dx =$$

$$\int_0^1 (16x^4 - 8x^3)\,dx = \frac{16}{5}x^5 - 2x^4\Big|_0^1 = \frac{16}{5} - 2 = \frac{6}{5}$$

#3) Same for $C=\{y=4x,\ 0\le x\le 1\}$

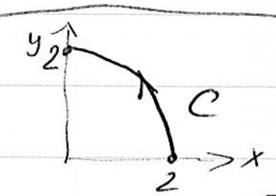
equals (taking x as a parameter)

$$\int_0^1 \langle (4x)^2, -x^2\rangle \cdot \langle 1, 4\rangle\, dx =$$

$$\int_0^1 (16x^2 - 4x^2)\,dx = \int_0^1 12x^2\,dx = 4x^3\Big|_0^1 = 4$$

<u>Integral depends on the path C.</u>

#5) $\int_C \langle xy, x^2y^2\rangle \cdot d\vec{r}$ where



$C=\{\langle 2\cos t, 2\sin t\rangle,\ 0\le t\le \frac{\pi}{2}\}$

equals $\int_0^{\pi/2} \langle 4\cos t\sin t, 16\cos^2 t\sin^2 t\rangle \cdot \underbrace{\langle -2\sin t, 2\cos t\rangle}_{\text{"}r'\text{"}}\,dt$

$$= \int_0^{\pi/2} (-8\cos t\sin^2 t + 32\cos^3 t\sin^2 t)\,dt = -\frac{8}{3}\sin^3 t + \frac{32}{3}\sin^3 t - \frac{32}{5}\sin^5 t\Big|_0^{\pi/2} = 8 - \frac{32}{5} = \frac{8}{5}$$

#6) $\displaystyle\int_C \langle x-y,\ y-z,\ z-x\rangle \cdot d\vec{r}$

where $C = \{ \langle 2\cos t,\ t,\ 2\sin t\rangle,\ 0 \le 2 \le 2\pi \}$

$\vec{r}\,' = \langle -2\sin t,\ 1,\ 2\cos t\rangle$

$$= \int_0^{2\pi} \langle 2\cos t - t,\ t - 2\sin t,\ 2\sin t - 2\cos t\rangle \cdot \langle -2\sin t, 1, 2\cos t\rangle\, dt$$

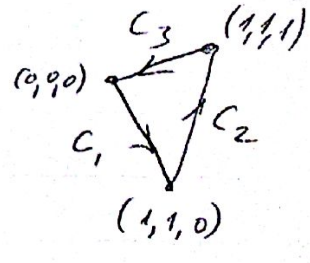
$$= \int_0^{2\pi} \left(-4\cos t\sin t + 2t\sin t + t - 2\sin t + 4\sin t\cos t - 4\cos^2 t\right) dt$$

$$= \int_0^{2\pi} \left(2t\sin t + t - 2\sin t - 4\cos^2 t\right) dt$$

$$= -2t\cos t + 2\sin t + \frac{t^2}{2} + 2\cos t - 2t - \sin 2t \Big|_0^{2\pi}$$

$$= \left(-4\pi + 2\pi^2 + 2 - 4\pi\right) - (2) = 2\pi^2 - 8\pi$$

#10) $\displaystyle\int_C \langle x,\ -z,\ 2y\rangle \cdot d\vec{r}$ where



$C = C_1 + C_2 + C_3$

$$= \int_{C_1} \langle x, -z, 2y\rangle \cdot d\vec{r} + \int_{C_2} \langle x, -z, 2y\rangle \cdot d\vec{r} + \int_{C_3} \langle x, -z, 2y\rangle \cdot d\vec{r}$$

On C_1, $\vec{r} = \langle t, t, 0\rangle$, $0 \le t \le 1$, $d\vec{r} = \langle 1, 1, 0\rangle\, dt$

On C_2, $\vec{r} = \langle 1, 1, t\rangle$, $0 \le t \le 1$, $d\vec{r} = \langle 0, 0, 1\rangle\, dt$

On C_3, $\vec{r} = \langle 1-t, 1-t, 1-t\rangle$, $0 \le t \le 1$, $d\vec{r} = \langle -1, -1, -1\rangle\, dt$

$$\int_0^1 \langle t, 0, 2t\rangle \cdot \langle 1, 1, 0\rangle\, dt + \int^1 \langle 1, -t, 2\rangle \cdot \langle 0, 0, 1\rangle\, dt + \int_0^1 \langle 1-t, t-1, 2-2t\rangle \langle -1, -1, -1\rangle\, dt$$

$$= \frac{t^2}{2}\Big|_0^1 + 2 + (t-1)^2 = \frac{1}{2} + 2 - 1 = \frac{3}{2}$$