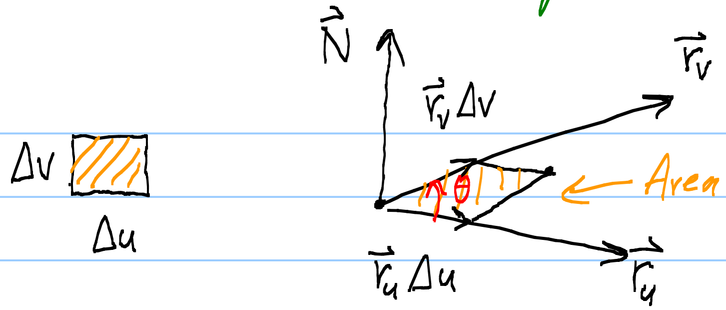
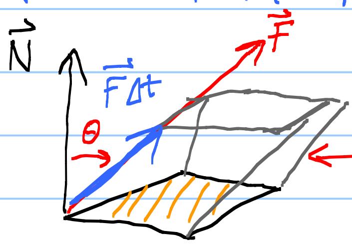
Lesson 8 on 10.6 Surface integrals

HWK 3: Lessons 6, 7, 8 due Wed



$\leftarrow$ Area $\|\vec{r}_u \times \vec{r}_v\| = \|\vec{r}_u\| \|\vec{r}_v\| |\sin\theta| \cdot \Delta u \Delta v$

Flux of a vector field $\vec{F}$ $\leftarrow$ velocity field



$\leftarrow$ height $(\vec{F} \cdot \hat{n})\Delta t$

$$\underset{\Delta u, \Delta v}{\text{Flux}} = \frac{\text{Volume of outflow}}{\Delta t} = \frac{\left(\vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \Delta t\right) \|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v}{\Delta t}$$

$$= \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \Delta u \Delta v$$

$$\text{Total flux} = \iint_R \vec{F}(\vec{r}) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= \frac{\underline{\text{net}} \text{ outflow}}{\Delta t}$$

<u>Notation</u>: $\displaystyle \iint_S \vec{F} \cdot \underbrace{\hat{n}\, dA}_{d\vec{A}} = \iint_S \vec{F} \cdot d\vec{A}$

<u>EX</u>: S top half of sphere. $\vec{F} = \hat{k}$

$$\vec{r}(u,v) = R\cos v \cos u\, \hat{\imath} + R\cos v \sin u\, \hat{\jmath} + R\sin v\, \hat{k}$$

$$0 \le v \le \frac{\pi}{2}, \quad 0 \le u \le 2\pi$$

$$\vec{r}_u \times \vec{r}_v = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -R\cos v \sin u & R\cos v \cos u & 0 \\ -R \sin v \cos u & R\sin v \sin u & R\cos v \end{bmatrix}$$

$$= \underline{R^2\cos^2 v \cos u \,\hat{i} + R^2\cos^2 v \sin u \,\hat{j} + R^2 \cos v \sin v \,\hat{k}}$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{(R^2\cos^2 v)^2 \underbrace{(\sin^2 u + \cos^2 u)}_{1} + (R^2\cos^2 v) R^2 \sin^2 v}$$

$$= R^2 \sqrt{\cos^2 v} = R^2 |\cos v|$$

Surface area: $\iint_R \|\vec{r}_u \times \vec{r}_v\|\, du =$

$$= \int_0^{\pi/2}\int_0^{2\pi} R^2 \cos v \;\underline{\underline{du}}\, dv$$

$\cos v > 0$, so no $|\;|$

$$= R^2\int_0^{\pi/2} 2\pi \cos v \, dv = 2\pi R^2 \checkmark$$

Whole sphere: $4\pi R^2$

Flux: $\iint_R \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, du\, dv$

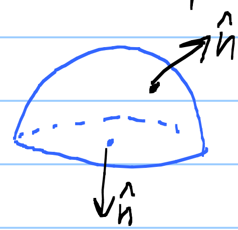
$$= \int_0^{\pi/2}\int_0^{2\pi} R^2 \cos v \sin v \;\underline{\underline{du}}\; dv$$

$$= R^2 \int_0^{\pi/2} 2\pi \overbrace{\cos v \underbrace{\sin v}_{w}\, dv}^{dw}$$

$$= 2\pi R^2 \left[\frac{1}{2}\sin^2 v\right]_0^{\pi/2} = \pi R^2$$

Hmmm. Same as flux past equator circle.
Of course! Think of water in a pipe.

Divergence Theorem:



outward pointing normal

$$\iint_S \vec{F}\cdot\hat{n}\, dA = \iiint_\Omega \mathrm{Div}\,\vec{F}\, dV$$

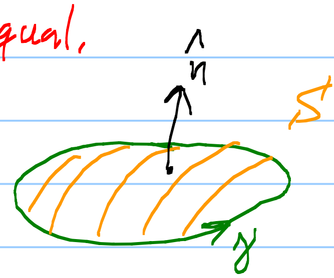
EX: Div $\hat{k} = 0$. ← incompressible

Flux $= 0$. (Flux out of top) + (Flux out bottom)

— Flux into bottom.

So Fluxes are equal.

Another biggy:



$\hat{n}$ chosen by Right

Stokes' Theorem: $\displaystyle\iint_S (\mathrm{Curl}\,\vec{F})\cdot\hat{n}\, dA = \int_\gamma \vec{F}\cdot d\vec{r}$

"Flux of vorticity" = "Circulation"

Another Stokes' Theorem $\displaystyle\int_{\text{Boundary}} \omega = \int_{\text{Inside}} d\omega$

Case of S as a graph: $z = f(x,y)$

$$\vec{r}(x,y) = x\hat{\imath} + y\hat{\jmath} + f(x,y)\hat{k}$$

$$\underbrace{\vec{r}_x \times \vec{r}_y}_{\vec{N}} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{bmatrix}$$

$$= -\frac{\partial f}{\partial x}\hat{\imath} - \frac{\partial f}{\partial y}\hat{\jmath} + \hat{k}$$

upward pointing normal.

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1^2}$$

$$\text{Area}(S') = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy = \iint_S dA$$

$$\text{Mass}(S') = \iint_R \rho(x, y, f(x,y)) \sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

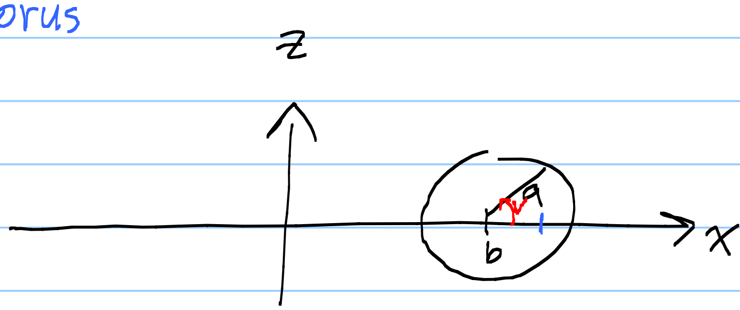
density fcn

$$= \iint_{S'} \rho(\vec{r})\, dA$$

$$\text{Flux}_{S'} \vec{F} = \iint_R \Big[-F_1(x,y,f(x,y))\frac{\partial f}{\partial x} - F_2(x,y,f(x,y))\frac{\partial f}{\partial y} + F_3(x,y,f(x,y)) \Big]\, dx\, dy$$

$$= \iint_S \vec{F} \cdot \hat{n}\, dA$$

EX: Torus



$$\begin{cases} x = (b + a\cos v)\cos u \\ y = (b + a\cos v)\sin u \quad \begin{pmatrix} 0 \le u \le 2\pi \\ 0 \le v \le 2\pi \end{pmatrix} \\ z = a\sin v \end{cases}$$

$$\text{Area} = (2\pi a)(2\pi b) = 4\pi^2 ab$$