HW2 Solutions

Section 10.2, p. 425

p. 425 #3.

$$\int_{(\frac{\pi}{2},\pi)}^{(\pi,0)} P\,dx + Q\,dy \quad \text{where}$$

$$P = \frac{1}{2}\cos\frac{x}{2}\cos 2y, \quad Q = -2\sin\frac{x}{2}\sin 2y$$

$$\frac{\partial Q}{\partial x} = -\cos\frac{x}{2}\sin 2y = \frac{\partial P}{\partial y},$$

thus the integral is path independent ($P\,dx + Q\,dy = df$ where $f = \sin\frac{x}{2}\cos 2y$).

$$\int_{(\frac{\pi}{2},\pi)}^{(\pi,0)} P\,dx + Q\,dy = f(\pi,0) - f\left(\frac{\pi}{2},\pi\right) = 1 - \frac{\sqrt{2}}{2}$$

#5. $$\int_{(0,0,\pi)}^{(2,\frac{1}{2},\frac{\pi}{2})} P\,dx + Q\,dy + R\,dz \quad \text{where}$$

$$P = y\sin z\, e^{xy}, \quad Q = x\sin z\, e^{xy}, \quad R = \cos z\, e^{xy}$$

$$\text{curl}\langle P,Q,R\rangle = \langle x\cos z - x\cos z,\ y\cos z - y\cos z,\ (1+xy)\sin z - (1+xy)\sin z\rangle e^{xy} = \vec{0}.$$

Thus $P\,dx + Q\,dy + R\,dz = df$ where $f = \sin z\, e^{xy}$.

$$\int_{(0,0,\pi)}^{(2,\frac{1}{2},\pi/2)} P\,dx + Q\,dy + R\,dz = f\left(2,\frac{1}{2},\frac{\pi}{2}\right) - f(0,0,\pi) = e - 0 = e$$

#13. $P\,dx + Q\,dy$ where $P = 2xe^{x^2}\cos 2y$, $Q = -2e^{x^2}\sin 2y$ is exact, since

$$\frac{\partial Q}{\partial x} = -4xe^{x^2}\sin 2y = \frac{\partial P}{\partial y}$$

$P\,dx + Q\,dy = df$ where $f = e^{x^2}\cos 2y$

$$\int_{(0,0,0)}^{(a,b,c)} P\,dx + Q\,dy = f(a,b,c) - f(0,0,0) = e^{a^2}\cos 2b - 1$$

Note: The answer in the book is not correct.

#16. $P\,dx + Q\,dy + R\,dz$ where $P = e^y$, $Q = xe^y - e^z$, $R = -ye^z$ is exact, since

$$\text{curl}\,\langle P, Q, R\rangle = \langle -e^z - (-e^z), 0 - 0, e^y - e^y\rangle = \vec{0}.$$

$P\,dx + Q\,dy + R\,dz = df$ where $f = xe^y - ye^z$

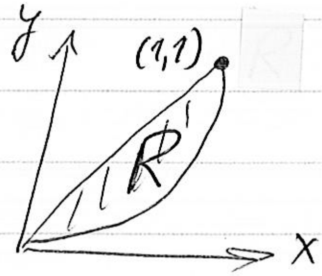
$$\int_{(0,0,0)}^{(a,b,c)} P\,dx + Q\,dy + R\,dz = f(a,b,c) - f(0,0,0) = ae^b - be^c.$$

Section 10.3, p. 432

#5. $$\int_0^1\int_{x^2}^{x}(1-2xy)\,dy\,dx=\int_0^1\left(y-xy^2\right)\Big|_{y=x^2}^{y=x}dx$$

$$=\int_0^1 x-x^3-x^2+x^5\,dx=\frac{x^2}{2}-\frac{x^4}{4}-\frac{x^3}{3}+\frac{x^6}{6}\Big|_0^1=$$

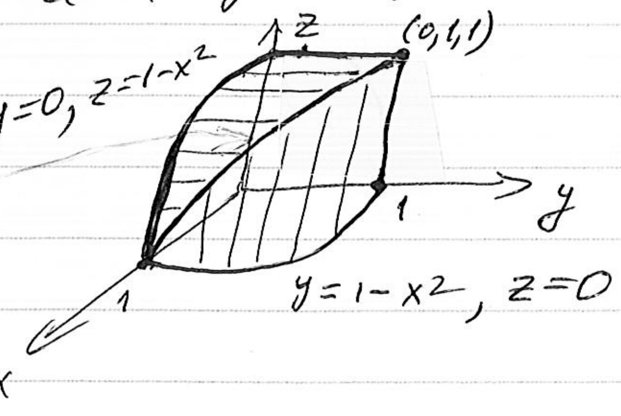
$$\frac{1}{2}-\frac{1}{4}-\frac{1}{3}+\frac{1}{6}=\frac{1}{12}$$



R is the region above $y=x^2$ and below $y=x$

#10. Find the volume of the first octant (e.g., $x\ge 0, y\ge 0, z\ge 0$) region E bounded by coordinate planes and $y=1-x^2$, $z=1-x^2$

The region E has an edge $y=z=1-x^2$

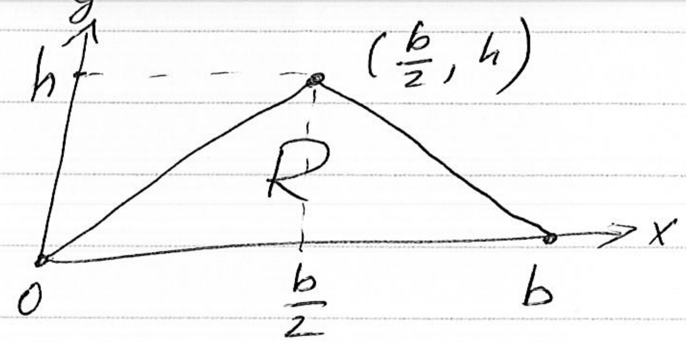


$$V(E)=\iint_R 1-x^2\,dA \quad \text{where } R=\{0\le x\le 1,\ 0\le y\le 1-x^2\}$$

$$=\int_0^1\int_0^{1-x^2}1-x^2\,dy\,dx=\int_0^1(1-x^2)^2\,dx=\int_0^1 1-2x^2+x^4\,dx$$

$$=x-\frac{2x^3}{3}+\frac{x^5}{5}\Big|_0^1=1-\frac{2}{3}+\frac{1}{5}=\frac{8}{15}$$

#12. Find the center of gravity $(\bar{x}, \bar{y})$ of a mass of density $f(x,y)=1$ in R:



Because of the symmetry, $\bar{x} = \frac{b}{2}$

Again because of the symmetry, $\bar{y}$ is the same as for $\underbrace{\{0 \le x \le \frac{b}{2}, 0 \le y \le \frac{2h}{b}x\}}_{R/2}$

mass $= \text{Area}\left(\frac{R}{2}\right) = \frac{1}{2}\frac{b}{2}h = \frac{bh}{4}$

moment $= \iint_{\frac{R}{2}} y \, dA = \int_0^{b/2} \int_0^{\frac{2h}{b}x} y \, dy \, dx$

$= \int_0^{b/2} \frac{y^2}{2}\Big|_0^{\frac{2h}{b}x} dx = \int_0^{b/2} \frac{2h^2}{b^2} x^2 dx =$

$\frac{2h^2}{3b^2} x^3 \Big|_0^{b/2} = \frac{h^2 b}{12}$

$\bar{y} = \frac{\text{moment}}{\text{mass}} = \frac{h}{3}$

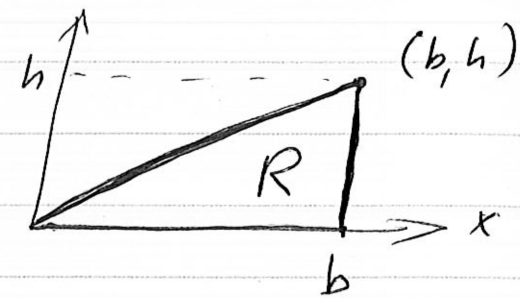
Answer: $(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{3}\right)$

#17. Find the moments of inertia

$$I_x = \iint_R y^2 f(x,y)\,dA,\quad I_y = \iint_R x^2 f(x,y)\,dA,\quad I_0 = I_x + I_y$$

of a mass of density $f(x,y) = 1$

in the region R



$$I_x = \iint_R y^2\,dA = \int_0^b \int_0^{\frac{h}{b}x} y^2\,dy\,dx$$

$$= \int_0^b \frac{y^3}{3}\Big|_0^{\frac{h}{b}x} dx = \int_0^b \frac{h^3}{3b^3} x^3\,dx$$

$$= \frac{h^3}{12b^3} x^4 \Big|_0^b = \frac{h^3 b}{12}$$

$$I_y = \iint_R x^2\,dA = \int_0^b \int_0^{\frac{h}{b}x} x^2\,dy\,dx = \int_0^b \frac{h}{b} x^3\,dx$$

$$= \frac{h}{4b} x^4 \Big|_0^b = \frac{hb^3}{4}$$

$$I_0 = \frac{h^3 b}{12} + \frac{hb^3}{4}$$