EE 321 - Fall 2020
Project 3: Fourier Series Reconstruction
Due Date: October 23, 2020

1 Introduction

In this project, we will focus on Fourier series reconstruction: the process of combining the frequency components to give the signal representation in the time domain.

2 Fourier Series Reconstruction

The Fourier series can be used to represent periodic signals as a linear combination of a set of harmonic frequencies.

2.1 Sum of Sinusoids

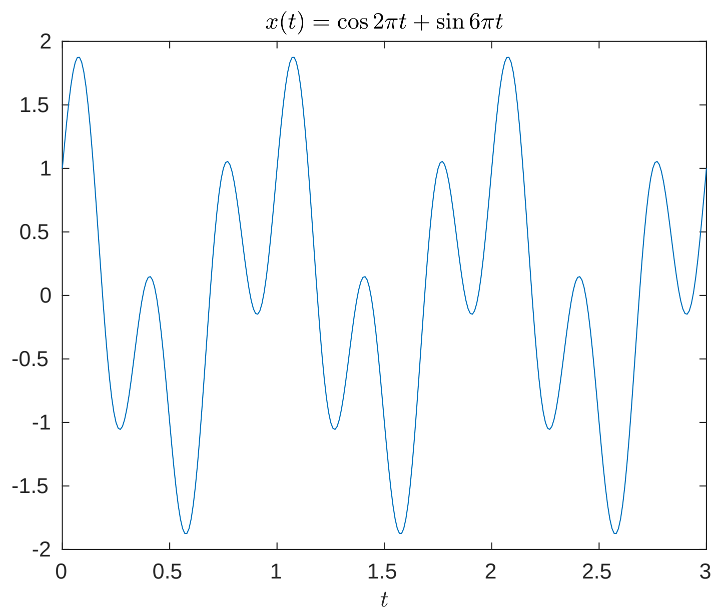

Figure 1: A sum of sinusoids, $x(t) = \cos 2\pi t + \sin 6\pi t$.

Take, for example, a sum of sinusoids, $x(t) = \cos 2\pi t + \sin 6\pi t$, and shown in Figure 1. Signals, such as $x(t)$, that are continuous and continuously differentiable, are called "smooth" signals. As we have seen in class, and in various problems, smooth signals have a finite number of harmonics in their representation; $x(t)$, for example, is represented using only four Fourier series coefficients: $X_k = 0.5\,[\delta(k-1) + \delta(k+1) + \delta(k-3) - \delta(k+3)]$ with $\omega_0 = 2\pi$. Since we have a finite number of coefficients, we achieve perfect reconstruction of the signal in time, by using:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}, \tag{1}$$

which in this specific case simplifies to:

$$\begin{aligned} x(t) &= \frac{1}{2}\left[X_{-1}e^{j(-1)\omega_0 t} + X_1 e^{j(1)\omega_0 t} + X_{-3}e^{j(-3)\omega_0 t} + X_3 e^{j(3)\omega_0 t}\right] \\ &= \frac{1}{2}\left[e^{j(-1)\omega_0 t} + e^{j(1)\omega_0 t} - e^{j(-3)\omega_0 t} + e^{j(3)\omega_0 t}\right], \end{aligned} \tag{2}$$

which can be easily verified to be $x(t)$, using Euler's expansion.

2.2 Periodic Pulses

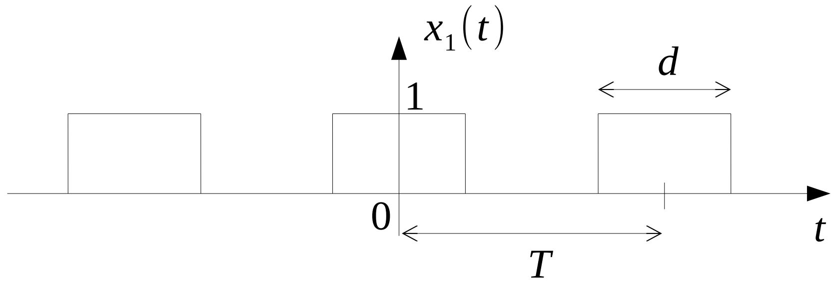

Figure 2: Periodic pulses.

Consider, now, the signal $x_1(t)$ shown in Figure 2. We know that the Fourier series representation of this signal is given by:

$$X_k = \frac{d}{T}\,\mathrm{sinc}\left(\frac{k\omega_0 d}{2}\right), \tag{3}$$

where $\omega_0 = 2\pi/T$, and $x_1(t)$ can be reconstructed as:

$$x_1(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}. \tag{4}$$

In this case, since $x_1(t)$ is non-smooth (it is not continuous), it requires infinitely many components to reconstruct. In general, any non-smooth signal requires infinitely many components to reconstruct.

3 Implementation of Fourier series reconstruction

Consider here, the ability to implement Fourier series reconstruction. In the case of $x(t)$, a finite number of terms are being combined, whereas for $x_1(t)$, we need to combine an infinite number of terms. In any computational scenario, combining an infinite number of terms is not possible. We can argue that we do not need an infinite number of terms, but a large number of terms. We need to verify if this is true; and if it is true, how large is "large"? In what follows, we will set out to answer these questions.

3.1 Error Calculation

We start by defining the signal reconstructed by a finite ($N < \infty$) number of terms:

$$\hat{x}_N(t) = \sum_{k=-N}^{N} X_k e^{jk\omega_0 t}. \tag{5}$$

The mean square error between the original signal and the reconstructed signal, as a function of the number of components is given by

$$MSE(N) = \frac{1}{T} \int_0^T |x(t) - \hat{x}_N(t)|^2 \,. \tag{6}$$

Since integrals cannot be computed exactly in MATLAB, we can make the following approximation:

$$MSE(N) = \frac{1}{M} \sum_{m=1}^{M} |x(m) - \hat{x}_N(m)|^2 \,, \tag{7}$$

where $\hat{x}_N(m)$ is the reconstructed signal in MATLAB, $x(m)$ is the "original" signal in MATLAB, M is the length of the sequence, x, and N is the number of FS coefficients used in reconstruction.

3.2 Coding Hints

Some hints to help with coding:

1. Do not use symbolic math in MATLAB. Avoid the use of commands such as `syms`. This starts up the Maple engine from within MATLAB, and slows down the execution of your code.

2. One period of $x_1(t)$, can be computed in MATLAB as $x(m)$ with $T = 4$ using the following commands:

```
>> t = 0.001:0.01:4;
>> x = 1*((t<=1)|((t>=3)&(t<=4)));
```

3. To compute the mean square error, exploit MATLAB's ability to perform matrix operations on arrays:

```
>> mse(N) = mean((x - xhat).^2)
```

4. Keep MSE in terms of N. That will help analyze your results as you change N, loop over N, etc.

4 Evaluating Fourier series reconstruction error

1. For the following signals represented in the frequency-domain:

   (i) Reconstruct the time-domain signals.

   (ii) Compute MSE vs N. Tabulate your results.

(a) $X_k = \delta(k) + \frac{1}{4}\delta(k-1) + \frac{1}{4}\delta(k+1) + \frac{1}{j2}\delta(k-2) - \frac{1}{j2}\delta(k+2)$

(b) $X_k = \begin{cases} jk & |k| < 3 \\ 0 & \text{otherwise} \end{cases}$

2. For the following signals in the time-domain:

   (i) Compute the FS coefficients by hand.

   (ii) Reconstruct the time-domain signals using MATLAB.

   (iii) Compute and plot $MSE(N)$ vs N. For better clarity at low values of $MSE(N)$, you can plot $MSE(N)$ in dB.

   (iv) Determine the minimum number of components, N, in order for $MSE(N)$ to be held below

      i. 0.01

      ii. 0.001

   (a) $x_1(t)$ as shown in Figure 2.

   (b) $x_2(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \end{cases}, \qquad T = 2.$

5 Reporting Requirements

Submit a project report that includes a brief description of what you did, and provide responses to all the prompts in Section 4. Make sure the report is clearly organized with all figures numbered and labeled. The text should refer to each figure and describe the takeaways from each figure. Remember that the project grade depends on both the deliverables and the quality of the project report. Finally, at the end of the report, as an appendix, include all your MATLAB code.

Submit your report as a single .zip file containing:

1. Your report in .PDF format.

2. Your code

Name the file "`Project3_TeamX.zip`", where `X` is your team number.