Rational solutions of the Boussinesq equation

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I dedicate this paper to the memory of Martin Kruskal in acknowledgement of his significant contribution to my research.

Abstract

Rational solutions of the Boussinesq equation are expressed in terms of special polynomials associated with rational solutions of the second and fourth Painlevé equations, which arise as symmetry reductions of the Boussinesq equation. Further generalized rational solutions of the Boussinesq equation, which involve an infinite number of arbitrary constants, are derived. The generalized rational solutions are analogs of such solutions for the Korteweg-de Vries and nonlinear Schrödinger equations.

1 Introduction

In this paper we are concerned with rational solutions of the Boussinesq equation

$$\sigma^2 u_{tt} + p u_{xx} + (u^2)_{xx} + \tfrac{1}{3} u_{xxxx} = 0, \tag{1.1}$$

where $\sigma^2 = \pm 1$ and p is an arbitrary constant, which also is a soliton equation solvable by inverse scattering [2, 22, 23, 35, 99]. The Boussinesq equation arises in several physical applications: (i), propagation of long waves in shallow water [19, 20, 94, 96]; (ii), one-dimensional nonlinear lattice-waves [90, 98]; (iii), vibrations in a nonlinear string [99]; and (iv), ion sound waves in a plasma [49, 86].

There has been considerable interest in partial differential equations solvable by inverse scattering, the *soliton equations*, since the discovery in 1967 by Gardner, Greene, Kruskal, and Miura [38] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2}$$

During the past thirty years or so there has been much interest in rational solutions of the soliton equations. For some soliton equations, solitons are given by rational solutions, e.g. for the Benjamin-Ono equation [62, 85] and the Kadomtsev-Petviashvili equation [61, 84]. Further applications of rational solutions to soliton equations include the description of explode-decay waves [68] and vortex solutions of the complex sine-Gordon equation [16, 76].

The idea of studying the motion of poles of solutions of the KdV equation (1.2) is attributed to Kruskal [57]; see also [89]. Airault, McKean, and Moser [10] studied the motion of the poles of rational solutions of the KdV equation (1.2) and the Boussinesq equation (1.1). Further they related the motion to an integrable many-body problem, the Calogero-Moser system with constraints; see also [3, 7, 25]. Ablowitz and Satsuma [3] have derived some rational solutions of the KdV equation (1.2) and the Boussinesq equation (1.1) by finding a long-wave limit of the known N-soliton solutions of these equations. Studies of rational solutions of other soliton equations include for the classical Boussinesq system [83] and the nonlinear Schrödinger (NLS) equation [32, 47, 68].

Ablowitz and Segur [4] demonstrated a close relationship between completely integrable partial differential equations solvable by inverse scattering and the Painlevé equations. For example, the second Painlevé equation ($\mathrm{P_{II}}$)

$$w'' = 2w^3 + zw + \alpha, \tag{1.3}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$ and α is an arbitrary constant, whose solutions are denoted $w(z;\alpha)$, arises as a scaling reduction of the KdV equation (1.2), see [4], and the fourth Painlevé equation ($\mathrm{P_{IV}}$)

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-\alpha)w + \frac{\beta}{w}, \tag{1.4}$$

where α and β are arbitrary constants, whose solutions are denoted $w(z;\alpha,\beta)$, arises as a scaling reduction of the Boussinesq equation (1.5). Consequently some special solutions can of the KdV equation (1.2) and the Boussinesq equation (1.5) can be expressed in terms of solutions of $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$, respectively.

The six Painlevé equations ($\mathrm{P_I}$–$\mathrm{P_{VI}}$) are nonlinear ordinary differential equations, whose solutions are called the *Painlevé transcendents*, were discovered about a hundred years ago by Painlevé, Gambier, and their colleagues whilst studying which second order ordinary differential equations have the property that the solutions have no movable branch points, i.e. the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property*. Painlevé, Gambier *et al.* showed that there were fifty canonical equations with this property, forty-four are either integrable in terms of previously known functions (such as elliptic functions or are equivalent to linear equations) or reducible to one of six new nonlinear ordinary differential equations, which define new transcendental functions (cf. [48]). The Painlevé equations can be thought of as nonlinear analogues of the classical special functions (cf. [31, 41, 50, 91]). Indeed Iwasaki, Kimura, Shimomura, and Yoshida [50] characterize the Painlevé equations as "the most important nonlinear ordinary differential equations" and state that "many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions". Further Umemura [91] states that "Kazuo Okamoto and his circle predict that in the 21st century a new chapter on Painlevé equations will be added to Whittaker and Watson's book". It is well known that $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ have rational solutions, algebraic solutions, and solutions expressed in terms of the classical special functions, though these solutions do not depend on two arbitrary constants and so are special solutions, sometimes known as "*classical solutions*" (see, e.g. [8, 15, 31, 36, 40, 46, 56, 63–67, 72–75] and the references therein).

Vorob'ev [95] and Yablonskii [97] expressed the rational solutions of $\mathrm{P_{II}}$ (1.3) in terms of certain special polynomials, which are now known as the *Yablonskii–Vorob'ev polynomials* (see §2 below). Okamoto [72] derived analogous special polynomials, which are now known as the *Okamoto polynomials*, related to some of the rational solutions of $\mathrm{P_{IV}}$ (1.4). Subsequently Okamoto's results were generalized by Noumi and Yamada [71] who showed that all rational solutions of $\mathrm{P_{IV}}$ can be expressed in terms of logarithmic derivatives of two sets of special polynomials, called the *generalized Hermite polynomials* and the *generalized Okamoto polynomials* (see §3 below). Clarkson and Mansfield [34] investigated the locations of the roots of the Yablonskii–Vorob'ev polynomials in the complex plane and showed that these roots have a very regular, approximately triangular structure. An earlier study of the distribution of the roots of the Yablonskii–Vorob'ev polynomials is given by Kametaka, Noda, Fukui, and Hirano [55]. The structure of the (complex) roots of the special polynomials associated with rational solutions of $\mathrm{P_{IV}}$ is described in [27], which either have an approximate rectangular structure and or are a combination of approximate rectangular and triangular structures. The term "approximate" is used since the patterns are not exact triangles and rectangles as the roots lie on arcs rather than straight lines.

In this paper our interest is in rational solutions and associated polynomials of the special case of Boussinesq equation (1.1) with $\sigma = 1$ and $p = 0$, i.e.

$$u_{tt} + (u^2)_{xx} + \tfrac{1}{3}u_{xxxx} = 0, \tag{1.5}$$

which has symmetry reductions to $\mathrm{P_{II}}$ (1.3) and $\mathrm{P_{IV}}$ (1.4). Consequently rational solutions of (1.5) can be obtained in terms the Yablonskii–Vorob'ev, generalized Hermite and generalized Okamoto polynomials. Further some of the rational solutions that are expressed in terms of the generalized Okamoto polynomials are generalized to give the rational solutions of the Boussinesq equation (1.5) obtained by Galkin, Pelinovsky, and Stepanyants [37] and Pelinovsky [80] which are analogs of the rational solutions of the KdV equation (1.2) [3, 7, 10, 25], the classical Boussinesq system [83] and the NLS equation [32, 47]; see also [30].

This paper is organized as follows. In §2 and §3 we review the special polynomials associated with rational solutions of $\mathrm{P_{II}}$ (1.3) and $\mathrm{P_{IV}}$ (1.4), respectively. In §4 we use the special polynomials discussed in §2 and §3 to derive special polynomials and associated rational solutions of the Boussinesq equation (1.5). We also derive generalized rational solutions which involve an infinite number of arbitrary constants. All these rational solutions are expressed in terms of Wronskians. Finally in §5 we discuss our results.

2 Special polynomials associated with rational solutions of $\mathrm{P_{II}}$

Rational solutions of $\mathrm{P_{II}}$ (1.4), with $\alpha = n \in \mathbb{Z}$, can be expressed in terms of the logarithmic derivative of special polynomials which are defined through a second order, bilinear differential-difference equation — see equation (2.2) below — that is equivalent to the Toda equation. These special polynomials were introduced by Vorob'ev [95] and Yablonskii [97] and are called the *Yablonskii–Vorob'ev polynomials*, which are summarized in the following theorem (see also [43, 87, 91, 93]).

Theorem 2.1. *Rational solutions of* $\mathrm{P_{II}}$ *exist if and only if* $\alpha = n \in \mathbb{Z}$, *which are unique, and have the form*

$$w_n = w(z; n) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \left\{ \frac{Q_{n-1}(z)}{Q_n(z)} \right\}, \tag{2.1}$$

for $n \geq 1$, where the polynomials Q_n satisfy the differential-difference equation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\left\{Q_nQ_n'' - (Q_n')^2\right\}, \tag{2.2}$$

with $Q_0(z) = 1$ and $Q_1(z) = z$. The other rational solutions of $\mathrm{P_{II}}$ *are given by $w_0 = 0$ and $w_{-n} = -w_n$.*

The Yablonskii–Vorob'ev polynomials are monic polynomials of degree $\frac{1}{2}n(n+1)$ with integer coefficients [43]. It is clear from the recurrence relation (2.2) that the Q_n are rational functions, though it is not obvious that in fact they are polynomials since one is dividing by Q_{n-1} at every iteration. Hence it is somewhat remarkable that the Q_n defined by (2.2) are indeed polynomials.

Kajiwara and Ohta [53] proved the following theorem on the representation of rational solutions of $\mathrm{P_{II}}$ using determinants (see also [34, 52, 92]).

Theorem 2.2. *Let $p_k(z)$ be the polynomial defined by*

$$\sum_{k=0}^{\infty} p_k(z)\lambda^k = \exp\left(z\lambda - \tfrac{4}{3}\lambda^3\right), \tag{2.3}$$

and $\tau_n(z)$ be the $n \times n$ determinant

$$\tau_n(z) = \mathcal{W}\left(p_1(z), p_3(z), \ldots, p_{2n-1}(z)\right), \qquad n \geq 1, \tag{2.4}$$

where $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the Wronskian defined by

$$\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}, \tag{2.5}$$

with $\varphi_j^{(m)} \equiv \mathrm{d}^m\varphi_j/\mathrm{d}z^m$, then

$$w(z;n) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\frac{\tau_{n-1}(z)}{\tau_n(z)}\right\}, \tag{2.6}$$

satisfies $\mathrm{P_{II}}$ *with $\alpha = n$, for $n \geq 1$.*

Remarks 2.3.

1. Flaschka and Newell [39], following the earlier work of Airault [8], expressed the rational solutions of $\mathrm{P_{II}}$ (1.3) in terms of determinants.

2. The "tau-functions" τ_n defined in Theorem 2.2, or equivalently the Yablonskii–Vorob'ev polynomials, can also be expressed in terms of Schur polynomials [52, 53].

3. The Yablonskii–Vorob'ev polynomials arise as partition functions in string theory [51].

4. The *Adler-Moser polynomials* [7, 10], which are generalizations of the Yablonskii–Vorob'ev polynomials and describe rational solutions of the Korteweg-de Vries equation (1.2), arise in the description of stationary vortex patterns [12–14, 59].

3 Special polynomials associated with rational solutions of $\mathrm{P_{IV}}$

3.1 Rational solutions of $\mathrm{P_{IV}}$

Simple rational solutions of $\mathrm{P_{IV}}$ (1.4) are

$$w_1(z;\pm 2,-2) = \pm 1/z, \qquad w_2(z;0,-2) = -2z, \qquad w_3(z;0,-\tfrac{2}{9}) = -\tfrac{2}{3}z. \tag{3.1}$$

It is known that there are three sets of rational solutions of $\mathrm{P_{IV}}$, which have the solutions (3.1) as the simplest members. These sets are known as the "$-1/z$ hierarchy", the "$-2z$ hierarchy" and the "$-\frac{2}{3}z$ hierarchy", respectively (cf. [15]). The "$-1/z$ hierarchy" and the "$-2z$ hierarchy" form the set of rational solutions of $\mathrm{P_{IV}}$ (1.4) with parameters given by (3.2) and the "$-\frac{2}{3}z$ hierarchy" forms the set with parameters given by (3.3). The rational solutions of $\mathrm{P_{IV}}$ (1.4) with parameters given by (3.2) lie at the vertices of the "Weyl chambers" and those with parameters given by (3.3) lie at the centres of the "Weyl chamber" [93].

Theorem 3.1. $\mathrm{P_{IV}}$ *(1.4) has rational solutions if and only if the parameters* α *and* β *are given by either*

$$\alpha = m, \qquad \beta = -2(2n - m + 1)^2, \tag{3.2}$$

or

$$\alpha = m, \qquad \beta = -2(2n - m + \tfrac{1}{3})^2, \tag{3.3}$$

with $m, n \in \mathbb{Z}$*. For each given* m *and* n*, there exists only one rational solution of* $\mathrm{P_{IV}}$ *with parameters given by either (3.2) or (3.3).*

Proof. See Lukashevich [60], Gromak [45] and Murata [66]; also Bassom, Clarkson, and Hicks [15], Gromak, Laine, and Shimomura [46, §26], Umemura and Watanabe [93]. □

In a comprehensive study of properties of solutions of $\mathrm{P_{IV}}$, Okamoto [72] introduced two sets of polynomials associated with rational solutions of $\mathrm{P_{IV}}$, analogous to the Yablonskii–Vorob'ev polynomials associated with rational solutions of $\mathrm{P_{II}}$, which are special polynomials associated with rational solutions of $\mathrm{P_{II}}$ [95, 97], see also [34]. Noumi and Yamada [71] generalized Okamoto's results and introduced the *generalized Hermite polynomials* $H_{m,n}(z)$, defined in Theorem 3.2, and the *generalized Okamoto polynomials* $Q_{m,n}(z)$, defined in Theorem 3.4; see also [27]. Noumi and Yamada [71] expressed both the generalized Hermite polynomials and the generalized Okamoto polynomials in terms of Schur functions related to the so-called modified Kadomtsev-Petviashvili hierarchy. Kajiwara and Ohta [54] also expressed rational solutions of $\mathrm{P_{IV}}$ (1.4) in terms of Schur functions by expressing the solutions in the form of determinants, see equations (3.9).

3.2 Generalized Hermite polynomials

Here we consider the generalized Hermite polynomials $H_{m,n}(z)$ which are defined in the following theorem.

Theorem 3.2. *Suppose* $H_{m,n}(z)$ *satisfies the recurrence relations*

$$2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 + 2mH^2_{m,n}, \tag{3.4a}$$

$$2nH_{m,n+1}H_{m,n-1} = -H_{m,n}H''_{m,n} + \left(H'_{m,n}\right)^2 + 2nH^2_{m,n}, \tag{3.4b}$$

with $H_{0,0} = H_{1,0} = H_{0,1} = 1$ *and* $H_{1,1} = 2z$*, then*

$$w^{[1]}_{m,n} = \frac{\mathrm{d}}{\mathrm{d}z} \ln \left\{ \frac{H_{m+1,n}(z)}{H_{m,n}(z)} \right\}, \tag{3.5a}$$

$$w^{[2]}_{m,n} = \frac{\mathrm{d}}{\mathrm{d}z} \ln \left\{ \frac{H_{m,n}(z)}{H_{m,n+1}(z)} \right\}, \tag{3.5b}$$

$$w^{[3]}_{m,n} = -2z + \frac{\mathrm{d}}{\mathrm{d}z} \ln \left\{ \frac{H_{m,n+1}(z)}{H_{m+1,n}(z)} \right\}, \tag{3.5c}$$

where $w^{[j]}_{m,n} = w(z; \alpha^{[j]}_{m,n}, \beta^{[j]}_{m,n})$ *for* $j = 1, 2, 3$*, are solutions of* $\mathrm{P_{IV}}$*, respectively for*

$$\alpha^{[1]}_{m,n} = 2m + n + 1, \qquad \beta^{[1]}_{m,n} = -2n^2, \tag{3.6a}$$

$$\alpha^{[2]}_{m,n} = -(m + 2n + 1), \qquad \beta^{[2]}_{m,n} = -2m^2, \tag{3.6b}$$

$$\alpha^{[3]}_{m,n} = n - m, \qquad \beta^{[3]}_{m,n} = -2(m + n + 1)^2. \tag{3.6c}$$

Proof. See Theorem 4.4 in Noumi and Yamada [71]; also Theorem 3.1 in [27]. □

The polynomials $H_{m,n}(z)$ defined by (3.4) are called the *generalized Hermite polynomials* since $H_{m,1}(z) = H_m(z)$ and $H_{1,m}(z) = \mathrm{i}^{-m}H_m(\mathrm{i}z)$, where $H_m(z)$ is the standard Hermite polynomial defined by

$$H_m(z) = (-1)^m \exp(z^2) \frac{\mathrm{d}^m}{\mathrm{d}z^m} \left\{ \exp(-z^2) \right\} \tag{3.7}$$

or alternatively through the generating function

$$\sum_{m=0}^{\infty} \frac{H_m(z)\, \xi^m}{m!} = \exp(2\xi z - \xi^2) \tag{3.8}$$

(cf. [6, 11, 88]). The rational solutions of $\mathrm{P_{IV}}$ defined by (3.5) include all solutions in the "$-1/z$" and "$-2z$" hierarchies, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (3.2), and can be expressed in terms of determinants whose entries are Hermite polynomials [54, 71], see equations (3.9). These rational solutions of $\mathrm{P_{IV}}$ (1.4) are special cases of the special function solutions which are expressible in terms of parabolic cylinder functions $D_\nu(\xi)$.

The polynomial $H_{m,n}(z)$ has degree mn with integer coefficients [71]; in fact $H_{m,n}(\frac{1}{2}\zeta)$ is a monic polynomial in ζ with integer coefficients. Further $H_{m,n}(z)$ possesses the symmetry $H_{n,m}(z) = \mathrm{i}^{-mn}H_{m,n}(\mathrm{i}z)$.

Examples of generalized Hermite polynomials and plots of the locations of their roots in the complex plane are given in [27]; see also [30, 32]. These plots, which are invariant under reflections in the real and imaginary z-axes, take the form of $m \times n$ "rectangles", though these are only approximate rectangles since the roots lie on arcs rather than straight lines as can be seen by looking at the actual values of the roots.

Remarks 3.3.

1. The generalized Hermite polynomials $H_{m,n}(z)$ can be expressed in determinantal form as follows

$$H_{m,n}(z) = c_{m,n}\mathcal{W}(H_m, H_{m+1}, \ldots, H_{m+n-1}), \qquad c_{m,n} = \prod_{j=1}^{n-1} \frac{(\frac{1}{2})^j}{j!}, \tag{3.9a}$$

where H_n is the Hermite polynomial and $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the usual Wronskian. An alternative representation is

$$H_{m,n}(z) = \widetilde{c}_{m,n}\begin{vmatrix} H_m & H_{m+1} & \ldots & H_{m+n-1} \\ H_{m+1} & H_{m+2} & \ldots & H_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n-1} & H_{m+n} & \ldots & H_{m+2n-2} \end{vmatrix}, \qquad \widetilde{c}_{m,n} = \prod_{j=1}^{n-1} \frac{(-\frac{1}{2})^j}{j!}, \tag{3.9b}$$

since H_n satisfies the recurrence relation $H_{n+1} = 2zH_n - H_n'$ (cf. [6, 11, 88]). The generalized Hermite polynomials $H_{m,n}(z)$ can also be expressed in terms of Schur polynomials; for further details see [54, 71].

2. Using the Hamiltonian formalism for $\mathrm{P_{IV}}$ [72], it can be shown that the generalized Hermite polynomials $H_{m,n}(z)$, which are defined by the differential-difference equations (3.4), also satisfy fourth order bilinear ordinary differential equation

$$\begin{aligned} H_{m,n}H''''_{m,n} - 4H'_{m,n}H'''_{m,n} + 3\left(H''_{m,n}\right)^2 - 4(z^2+2n-2m)\left[H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2\right] \\ + 4zH_{m,n}H'_{m,n} - 8mnH^2_{m,n} = 0, \end{aligned} \tag{3.10}$$

and homogeneous difference equations [29, 30].

3. The polynomial $H_{m,n}(z)$ can be expressed as the multiple integral

$$H_{m,n}(z) = \frac{\pi^{m/2}\prod_{k=1}^m k!}{2^{m(m+2n-1)/2}}\int_{-\infty}^{\infty} \overset{n}{\cdots} \int_{-\infty}^{\infty} \prod_{i=1}^{n}\prod_{j=i+1}^{n}(x_i-x_j)^2\prod_{k=1}^{n}(z-x_k)^m \exp\left(-x_k^2\right)\mathrm{d}x_1\,\mathrm{d}x_2\ldots\mathrm{d}x_n, \tag{3.11}$$

which arises in random matrix theory (for further details see Brézin and Hikami [21], Forrester and Witte [42]).

4. The orthogonal polynomials on the real line with respect to the weight $w(x; z, m) = (x-z)^m \exp(-x^2)$, satisfy the three-term recurrence relation

$$xp_n(x) = p_{n+1}(x) + a_n(z;m)p_n(x) + b_n(z;m)p_{n-1}(x), \tag{3.12a}$$

where

$$a_n(z;m) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{H_{n+1,m}}{H_{n,m}}, \qquad b_n(z;m) = \frac{nH_{n+1,m}H_{n-1,m}}{2H^2_{n,m}} \tag{3.12b}$$

(for further details see Chan and Feigin [24]).

3.3 Generalized Okamoto polynomials

Here we consider the generalized Okamoto polynomials $Q_{m,n}(z)$ which were introduced by Noumi and Yamada [71] and are defined in following Theorem.

Theorem 3.4. *Suppose $Q_{m,n}(z)$ satisfies the recurrence relations*

$$Q_{m+1,n}Q_{m-1,n} = \tfrac{9}{2}\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] + \left[2z^2 + 3(2m+n-1)\right]Q^2_{m,n}, \tag{3.13a}$$

$$Q_{m,n+1}Q_{m,n-1} = \tfrac{9}{2}\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] + \left[2z^2 + 3(1-m-2n)\right]Q^2_{m,n}, \tag{3.13b}$$

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}\,z$, then

$$\widetilde{w}^{[1]}_{m,n} = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\frac{Q_{m+1,n}(z)}{Q_{m,n}(z)}\right\}, \tag{3.14a}$$

$$\widetilde{w}^{[2]}_{m,n} = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\frac{Q_{m,n}(z)}{Q_{m,n+1}(z)}\right\}, \tag{3.14b}$$

$$\widetilde{w}^{[3]}_{m,n} = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\ln\left\{\frac{Q_{m,n+1}(z)}{Q_{m+1,n}(z)}\right\}, \tag{3.14c}$$

where $\widetilde{w}^{[j]}_{m,n} = w(z;\widetilde{\alpha}^{[j]}_{m,n},\widetilde{\beta}^{[j]}_{m,n})$ for $j = 1,2,3$, are solutions of $\mathrm{P_{IV}}$*, respectively for*

$$\widetilde{\alpha}^{[1]}_{m,n} = 2m+n, \qquad \widetilde{\beta}^{[1]}_{m,n} = -2(n-\tfrac{1}{3})^2, \tag{3.15a}$$

$$\widetilde{\alpha}^{[2]}_{m,n} = -(m+2n), \qquad \widetilde{\beta}^{[2]}_{m,n} = -2(m-\tfrac{1}{3})^2, \tag{3.15b}$$

$$\widetilde{\alpha}^{[3]}_{m,n} = n-m, \qquad \widetilde{\beta}^{[3]}_{m,n} = -2(m+n+\tfrac{1}{3})^2. \tag{3.15c}$$

Proof. See Theorem 4.3 in Noumi and Yamada [71]; also Theorem 4.1 in [27]. □

The polynomials $Q_{m,n}(z)$ defined by (3.13) are called the *generalized Okamoto polynomials* since Okamoto [72] defined the polynomials in the cases when $n = 0$ and $n = 1$. The rational solutions of $\mathrm{P_{IV}}$ defined by (3.14) include all solutions in the "$-\frac{2}{3}z$" hierarchy, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (3.3), which also can be expressed in the form of determinants [54, 71], see (3.17).

The polynomial $Q_{m,n}(z)$ has degree $d_{m,n} = m^2 + n^2 + mn - m - n$ with integer coefficients [71]; in fact $Q_{m,n}(\frac{1}{2}\sqrt{2}\,\zeta)$ is a monic polynomial in ζ with integer coefficients. Further $Q_{m,n}(z)$ possesses the symmetries

$$Q_{n,m}(z) = \exp(-\tfrac{1}{2}\pi\mathrm{i}d_{m,n})\,Q_{m,n}(\mathrm{i}z), \tag{3.16a}$$

$$Q_{1-m-n,n}(z) = \exp(-\tfrac{1}{2}\pi\mathrm{i}d_{m,n})\,Q_{m,n}(\mathrm{i}z). \tag{3.16b}$$

Note that $d_{m,n} = m^2 + n^2 + mn - m - n$ satisfies $d_{m,n} = d_{n,m} = d_{1-m-n,n}$.

Examples of generalized Okamoto polynomials and plots of the locations of their complex roots are given in [27]; see also [30, 32]. The roots of the polynomial $Q_{m,n}(z)$, with $m,n \geq 1$, take the form of $m \times n$ "rectangle" with an "equilateral triangle", which have either $m-1$ or $n-1$ roots, on each of its sides. The roots of the polynomial $Q_{-m,-n}$, with $m,n \geq 1$, take the form of $m \times n$ "rectangle" with an "equilateral triangle", which now have either m or n roots, on each of its sides. These are only approximate rectangles and equilateral triangles as can be seen by looking at the actual values of the roots. The plots are invariant under reflections in the real and imaginary z-axes.

Due to the symmetries (3.16), the roots of the polynomials $Q_{-m,n}$ and $Q_{m,-n}$, with $m,n \geq 1$ take similar forms as these polynomials they can be expressed in terms of $Q_{m,n}(z)$ and $Q_{-M,-N}$ for suitable $M,N \geq 1$. Specifically, the roots of the polynomial $Q_{-m,n}$, with $m \geq n \geq 1$, has the form of a $n \times (m-n+1)$ "rectangle" with an "equilateral triangle", which have either $n-1$ or $n-m-1$ roots, on each of its sides. Also the roots of the polynomial $Q_{-m,n}$ with $n > m \geq 1$, has the form of a $m \times (n-m-1)$ "rectangle" with an "equilateral triangle", which have either m or $n-m-1$ roots, on each of its sides. Further, we note that $Q_{-m,m} = Q_{m,1}$ and $Q_{1-m,m} = Q_{m,0}$, for all $m \in \mathbb{Z}$, where $Q_{m,0}$ and $Q_{m,0}$ are the original polynomials introduced by Okamoto [72]. Analogous results hold for $Q_{m,-n}$, with $m,n \geq 1$.

Remarks 3.5.

1. The generalized Okamoto polynomials $Q_{m,n}(z)$ can be expressed in determinantal form as follows

$$Q_{m,n}(z) = c_{m,n}\mathcal{W}\left(\psi_1, \psi_4, \ldots, \psi_{3m+3n-5}, \psi_2, \psi_5, \ldots, \psi_{3n-4}\right), \tag{3.17a}$$

$$Q_{-m,-n}(z) = \widetilde{c}_{m,n}\mathcal{W}\left(\psi_1, \psi_4, \ldots, \psi_{3n-2}, \psi_2, \psi_5, \ldots, \psi_{3m+3n-1}\right), \tag{3.17b}$$

for $m, n \geq 0$, with $c_{m,n}$ and $\widetilde{c}_{m,n}$ constants, $\mathcal{W}(\psi_1, \psi_2, \ldots, \psi_n)$ the Wronskian, and $\psi_n(z)$ given by

$$\sum_{n=0}^{\infty} \psi_n(z)\,\zeta^n = \exp\left(2z\zeta + 3\zeta^2\right), \qquad \psi_n(z) = \frac{3^{n/2}\mathrm{e}^{-n\pi\mathrm{i}/2}}{n!}\,H_n\left(\tfrac{1}{3}\sqrt{3}\,\mathrm{i}z\right). \tag{3.18}$$

This follows from equation (20) in Kajiwara and Ohta [54], though it is not given there explicitly (see also [32]). The generalized Okamoto polynomials $Q_{m,n}(z)$ can also be expressed in terms of Schur polynomials (cf. [32, 54, 71]).

2. As for the generalized Hermite polynomials, using the Hamiltonian formalism for $\mathrm{P_{IV}}$ [72], it can be shown that the generalized Okamoto polynomials $Q_{m,n}(z)$, which are defined by the differential-difference equations (3.13), also satisfy the fourth order bilinear ordinary differential equation

$$\begin{aligned} Q_{m,n}Q''''_{m,n} - 4Q'_{m,n}Q'''_{m,n} + 3\left(Q''_{m,n}\right)^2 + \tfrac{4}{3}z^2\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] \\ + 4zQ_{m,n}Q'_{m,n} - \tfrac{8}{3}(m^2+n^2+mn-m-n)Q^2_{m,n} = 0, \end{aligned} \tag{3.19}$$

and homogeneous difference equations [29, 30].

4 Rational Solutions of the Boussinesq Equation

4.1 Symmetry reductions of the Boussinesq equation

The classical method for finding symmetry reductions of partial differential equations is the Lie group method [18, 77]. There have been several generalizations of the classical Lie group method for symmetry reductions. For example, Bluman and Cole [17], in their study of symmetry reductions of the linear heat equation, proposed the so-called *nonclassical method of group-invariant solutions*. Motivated by the fact that symmetry reductions of the Boussinesq equation (1.5) are known that are not obtained using the classical Lie group method [69, 78, 79, 81, 82], Clarkson and Kruskal [33] developed a direct, algorithmic method for finding symmetry reductions, often referred to as the *direct method*, which they used to obtain previously unknown reductions of the Boussinesq equation, which are summarized in the theorem 4.1 below. Subsequently Levi and Winternitz [58] gave a group theoretical explanation of these results by showing that all the new reductions of the Boussinesq equation obtained by Clarkson and Kruskal could be obtained using the nonclassical method of Bluman and Cole [17]. The novel characteristic about the direct method, in comparison to the others mentioned above, is that it involves no use of group theory; for further details see [26].

Theorem 4.1. *The Boussinesq equation (1.5) has the following symmetry reductions*

$$u_1(x,t) = w_1(z), \qquad z = x + \mu_1 t, \tag{4.1}$$

$$u_2(x,t) = t^2 w_1(z) - x^2/(2t^2), \qquad z = xt, \tag{4.2}$$

$$u_3(x,t) = w_2(z) - 2\mu_2^2 t^2, \qquad z = x + \mu_2 t^2, \tag{4.3}$$

$$u_4(x,t) = t^2 w_2(z) - \left(x + \tfrac{2}{5}\mu_2 t^5\right)^2/(2t^2), \qquad z = xt + \tfrac{1}{15}\mu_2 t^6, \tag{4.4}$$

$$u_5(x,t) = \left[t w_4(z) - (x - 3\mu_4 t^2)^2\right]/(8t^2), \qquad z = \left(x + \mu_4 t^2\right)/(\tfrac{4}{3}t)^{1/2}, \tag{4.5}$$

where μ_1, μ_2, and μ_4 are arbitrary constants and $w_1(z)$, $w_2(z)$, and $w_4(z)$ satisfy

$$\begin{aligned} &w_1'''' + 6w_1w_1'' + 6(w_1')^2 = 0, \\ &w_2'''' + 6w_2w_2'' + 6(w_2')^2 + 6\mu_2 w_2' = 12\mu_2^2, \\ &w_4'''' + w_4w_4'' + (w_4')^2 + 4zw_4' + 8w_4 = 32z^2, \end{aligned}$$

which are solvable in terms of $\mathrm{P_I}$, $\mathrm{P_{II}}$, *and* $\mathrm{P_{IV}}$, *respectively.*

Proof. See Clarkson and Kruskal [33]. □

We remark that only the travelling wave reduction (4.1) and the scaling reduction (4.5) with $\mu_5 = 0$ can be obtained using the classical Lie method, all the other reductions are so-called nonclassical reductions which can be obtained using Bluman and Cole's nonclassical method or Clarkson and Kruskal's direct method.

Since there are no special solutions of $\mathrm{P_I}$

$$w'' = 6w^2 + z, \tag{4.7}$$

then it is not possible to derive special solutions of the Boussinesq equation (1.5) using either the symmetry reductions (4.1) and (4.2). In the next three subsections, we discuss rational solutions of the Boussinesq equation (1.5) arising from the symmetry reductions (4.3)–(4.5) which are expressed in terms of rational solutions of $\mathrm{P_{II}}$ (1.3) and $\mathrm{P_{IV}}$ (1.4).

4.2 Accelerating wave reduction

Consider the accelerating wave reduction (4.3), i.e.

$$u(x,t;\mu) = v(\zeta) - 2\mu^2 t^2, \qquad \zeta = x + \mu t^2, \tag{4.8}$$

where μ is an arbitrary constant and $v(\zeta)$ satisfies

$$\frac{\mathrm{d}^3 v}{\mathrm{d}\zeta^3} + 6v\frac{\mathrm{d}v}{\mathrm{d}\zeta} + 6\mu v = 12\mu^2\zeta, \tag{4.9}$$

with the constant of integration set to zero (without loss of generality). This is solvable in terms of solutions of $\mathrm{P_{II}}$ (1.3) as it is equivalent to

$$W''' + 6WW' = 2W + zW', \tag{4.10}$$

since setting

$$v(\zeta) = \mu\zeta + (6\mu)^{2/3}W(z), \qquad z = -(6\mu)^{1/3}\zeta, \tag{4.11}$$

in (4.9) yields (4.10). It is shown in [28] that equation (4.10), which also arises from scaling reductions of the KdV equation (1.2), has rational solutions

$$W(z) = 2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\{Q_n(z)\}, \tag{4.12}$$

where $Q_n(z)$ are the Yablonskii–Vorob'ev polynomials. Thus we obtain the rational solutions of the Boussinesq equation (1.5)

$$u_n(x,t;\mu) = \mu(x-\mu t^2) + 2(6\mu)^{2/3}\,\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q_n(z), \qquad z = -(6\mu)^{1/3}\left(x+\mu t^2\right). \tag{4.13}$$

We can also derive the rational solutions (4.13) directly. Consider the polynomials $\theta_n(x,t;\mu)$ defined by

$$\sum_{j=0}^{\infty}\theta_j(x,t;\mu)\lambda^j = \exp\left\{\left(x+\mu t^2\right)\lambda + \frac{2\lambda^3}{9\mu}\right\}, \tag{4.14}$$

and then let

$$\Theta_n(x,t;\mu) = \mathcal{W}_x(\theta_1,\theta_3,\ldots,\theta_{2n-1}), \tag{4.15}$$

where $\mathcal{W}_x(\theta_1,\theta_3,\ldots,\theta_{2n-1})$ is the Wronskian with respect to x, which is a polynomial in x of degree $\frac{1}{2}n(n+1)$ with coefficients that are polynomials in t. Then the Boussinesq equation (1.5) has rational solutions in the form

$$u_n(x,t;\mu) = \mu(x-\mu t^2) + 2\frac{\partial^2}{\partial x^2}\ln\{\Theta_n(x,t;\mu)\}, \tag{4.16}$$

which are equivalent to (4.13).

4.3 Nonclassical symmetry reduction

Consider the nonclassical symmetry reduction (4.4), i.e.

$$u(x,t;\mu) = t^2 v(\zeta) - \frac{(x+\frac{2}{5}\mu t^5)^2}{2t^2}, \qquad \zeta = xt + \tfrac{1}{15}\mu t^6, \tag{4.17}$$

where μ is an arbitrary constant and $v(\zeta)$ satisfies (4.9), which is solvable in terms of solutions of $\mathrm{P_{II}}$ (1.3). Thus we obtain the rational solutions of the Boussinesq equation (1.5)

$$u_n(x,t;\mu) = -\frac{x^2}{2t^2} + \frac{3\mu x t^3}{5} - \frac{\mu^2 t^8}{75} + 2(6\mu)^{2/3}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln Q_n(z), \qquad z = -(6\mu)^{1/3}(xt + \tfrac{1}{15}\mu t^6). \tag{4.18}$$

As above, we can also derive the rational solutions (4.18) directly. Consider the polynomials $\widetilde{\theta}_n(x,t;\mu)$ defined by

$$\sum_{j=0}^{\infty}\widetilde{\theta}_j(x,t;\mu)\lambda^j = \exp\left\{\left(xt + \frac{\mu t^6}{15}\right)\lambda + \frac{2\lambda^3}{9\mu}\right\}, \tag{4.19}$$

and then let

$$\widetilde{\Theta}_n(x,t;\mu) = \mathcal{W}_x(\widetilde{\theta}_1, \widetilde{\theta}_3, \ldots, \widetilde{\theta}_{2n-1}). \tag{4.20}$$

Then the Boussinesq equation (1.5) has rational solutions in the form

$$u_n(x,t;\mu) = -\frac{x^2}{2t^2} + \frac{3\mu x t^3}{5} - \frac{\mu^2 t^8}{75} + 2\frac{\partial^2}{\partial x^2}\ln\left\{\widetilde{\Theta}_n(x,t;\mu)\right\}, \tag{4.21}$$

which are equivalent to (4.18).

4.4 Generalized scaling reduction

Consider the generalized scaling reduction (4.5), i.e.

$$u(x,t) = \frac{V(z)}{8\,t} - \frac{\left(x - 3\mu t^2\right)^2}{8\,t^2}, \qquad z = \frac{x + \mu t^2}{(\frac{4}{3}t)^{1/2}}, \tag{4.22}$$

where μ is an arbitrary constant and $V(z)$ satisfies

$$V'''' + VV'' + (V')^2 + 4zV' + 8V = 32z^2. \tag{4.23}$$

This equation is solvable in terms of $\mathrm{P_{IV}}$ (1.4) since if

$$V = -6\left(w' + w^2 + 2zw\right) - 4z^2 + 4(\alpha - 1), \tag{4.24}$$

where $w = w(z;\alpha,\beta)$ satisfies $\mathrm{P_{IV}}$ then V satisfies (4.23); in fact there is a one-to-one relationship between solutions of (4.23) and those of $\mathrm{P_{IV}}$ (see [33] for further details). It is shown in [29] that equation (4.23) has rational solutions

$$V_{m,n}(z) = -4z^2 + 8(m-n) + 12\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\{H_{m,n}(z)\}, \qquad \widetilde{V}_{m,n}(z) = \tfrac{4}{3}z^2 + 12\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\{Q_{m,n}(z)\}, \tag{4.25}$$

with $H_{m,n}(z)$ and $Q_{m,n}(z)$ the generalized Hermite and generalized Okamoto polynomials, respectively. Hence we obtain rational solutions of the Boussinesq equation (1.5) in the form

$$u_{m,n}(x,t;\mu) = -\frac{x^2}{2t^2} + \frac{m-n}{t} - \frac{3\mu^2 t^2}{2} + \frac{3}{2t}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\{H_{m,n}(z)\}, \tag{4.26}$$

$$\widetilde{u}_{m,n}(x,t;\mu) = \mu(x - \mu t^2) + \frac{3}{2t}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\{Q_{m,n}(z)\}, \tag{4.27}$$

where $z = (x + \mu t^2)/(\frac{4}{3}t)^{1/2}$. We remark that the rational solution $\widetilde{u}_{m,n}$ (4.27) has the property that $u \to 0$ as $|x| \to \infty$ only in the special case when $\mu = 0$. Further the generalized scaling reduction (4.22) is a classical Lie point symmetry reduction only if $\mu = 0$, otherwise it is a nonclassical symmetry reduction.

We can also derive the rational solutions (4.26) and (4.27) directly. For the rational solutions (4.26), consider the polynomials $\varphi_n(x,t;\mu)$ defined by

$$\sum_{n=0}^{\infty}\varphi_n(x,t;\mu)\lambda^n = \exp\left\{\left(x + \mu t^2\right)\lambda - \tfrac{1}{3}t\lambda^2\right\}, \tag{4.28}$$

and then let

$$\Phi_{m,n}(x,t;\mu) = \mathcal{W}_x(\varphi_m, \varphi_{m+1}, \ldots, \varphi_{m+n-1}), \tag{4.29}$$

where $\mathcal{W}_x(\varphi_m, \varphi_{m+1}, \ldots, \varphi_{m+n-1})$ is the Wronskian with respect to x. Then the Boussinesq equation (1.5) has rational solutions in the form

$$u_{m,n}(x,t;\mu) = -\frac{x^2}{2t^2} + \frac{m-n}{t} - \frac{3\mu^2 t^2}{2} + 2\frac{\partial^2}{\partial x^2}\ln\{\Phi_{m,n}(x,t;\mu)\}, \tag{4.30}$$

which are equivalent to (4.26).

Analogously, for the rational solutions (4.27), consider the polynomials $\psi_n(x,t;\mu)$ defined by

$$\sum_{n=0}^{\infty} \psi_n(x,t;\mu)\lambda^n = \exp\left\{\left(x+\mu t^2\right)\lambda + t\lambda^2\right\}, \tag{4.31}$$

and then let

$$\Psi_{m,n}(x,t;\mu) = \mathcal{W}_x\left(\psi_1, \psi_4, \ldots, \psi_{3m+3n-5}, \psi_2, \psi_5, \ldots, \psi_{3n-4}\right), \tag{4.32a}$$

$$\Psi_{-m,-n}(x,t;\mu) = \mathcal{W}_x\left(\psi_1, \psi_4, \ldots, \psi_{3n-2}, \psi_2, \psi_5, \ldots, \psi_{3m+3n-1}\right), \tag{4.32b}$$

for $m, n \geq 0$, where $\mathcal{W}_x(\psi_1, \psi_2, \ldots, \psi_n)$ is the Wronskian with respect to x. Then the Boussinesq equation (1.5) has rational solutions in the form

$$\widetilde{u}_{m,n}(x,t;\mu) = \mu(x-\mu t^2) + 2\frac{\partial^2}{\partial x^2}\ln\{\Psi_{m,n}(x,t;\mu)\}, \tag{4.33}$$

which are equivalent to (4.27).

In [9], Airault constructs a family of rational solutions of the Boussinesq equation (1.5). The polynomials $\theta_d(x,\eta)$, which are of degree d in x, listed in §2 of [9] are the polynomials $\Psi_{m,n}(x,t;0)$, with $\eta = 2t$ and $d = m^2 + n^2 + mn - m - n$. This explains why Airault [9] found no polynomials $\theta_d(x,\eta)$ for certain values of d and two distinct polynomials for other values of d.

4.5 Generalized Rational Solutions of the Boussinesq Equation

It is known that there exist generalized rational solutions of the KdV equation (1.2) that are expressed in terms of the Adler-Moser polynomials, which are generalizations of the Yablonskii–Vorob'ev polynomials discussed in §2 and involve an infinite number of arbitrary parameters [7, 10]. Similarly generalized rational solutions depending on an infinite number of arbitrary parameters also exist for the classical Boussinesq system [83] and the NLS equation [32, 47].

We shall now discuss analogous generalized rational solutions for the Boussinesq equation (1.5). Consider the polynomials $\widehat{\psi}_n(x,t;\boldsymbol{\kappa})$ defined by

$$\sum_{n=0}^{\infty} \widehat{\psi}_n(x,t;\boldsymbol{\kappa})\lambda^n = \exp\left(x\lambda + t\lambda^2 + \sum_{j=3}^{\infty}\kappa_j\lambda^j\right), \tag{4.34}$$

where $\boldsymbol{\kappa} = (\kappa_3, \kappa_4, \ldots)$, with $\kappa_3, \kappa_4, \ldots$ arbitrary constants, and then let

$$\widehat{\Psi}_{m,n}(x,t;\boldsymbol{\kappa}) = \mathcal{W}_x\left(\widehat{\psi}_1, \widehat{\psi}_4, \ldots, \widehat{\psi}_{3m+3n-5}, \widehat{\psi}_2, \widehat{\psi}_5, \ldots, \widehat{\psi}_{3n-4}\right), \tag{4.35a}$$

$$\widehat{\Psi}_{-m,-n}(x,t;\boldsymbol{\kappa}) = \mathcal{W}_x\left(\widehat{\psi}_1, \widehat{\psi}_4, \ldots, \widehat{\psi}_{3n-2}, \widehat{\psi}_2, \widehat{\psi}_5, \ldots, \widehat{\psi}_{3m+3n-1}\right), \tag{4.35b}$$

for $m, n \geq 0$, where $\mathcal{W}_x(\widehat{\psi}_1, \widehat{\psi}_2, \ldots, \widehat{\psi}_n)$ is the Wronskian with respect to x. Then the Boussinesq equation (1.5) has generalized rational solutions in the form

$$\widehat{u}_{m,n}(x,t;\boldsymbol{\kappa}) = 2\frac{\partial^2}{\partial x^2}\ln\left\{\widehat{\Psi}_{m,n}(x,t;\boldsymbol{\kappa})\right\}, \tag{4.36}$$

which have the property that $\widehat{u}_{m,n} \to 0$ as $|x| \to \infty$.

Galkin, Pelinovsky, and Stepanyants [37] (see also Pelinovsky [80]) also derived the generalized rational solution (4.36). As above, consider the polynomials $\widehat{\psi}_n(x,t;\boldsymbol{\kappa})$ defined by (4.34) and then let

$$\tau_{m,n}(x,t;\boldsymbol{\kappa}) = \mathcal{W}_x(\widehat{\psi}_1, \widehat{\psi}_4, \ldots, \widehat{\psi}_{3m-2}, \widehat{\psi}_2, \widehat{\psi}_5, \ldots, \widehat{\psi}_{3n-1}), \tag{4.37}$$

for $m, n \geq 0$, where $\mathcal{W}_x(\widehat{\psi}_1, \widehat{\psi}_2, \ldots, \widehat{\psi}_n)$ is the Wronskian with respect to x, which is a polynomial in x of degree $m^2 + n(n+1) - mn$ with coefficients that are polynomials in t. Then the Boussinesq equation (1.5) has rational solutions in the form

$$u_{m,n}(x,t;\boldsymbol{\kappa}) = 2\frac{\partial^2}{\partial x^2} \ln\left\{\tau_{m,n}(x,t;\boldsymbol{\kappa})\right\}. \tag{4.38}$$

The polynomials $\widehat{\Psi}_{m,n}(x,t;\boldsymbol{\kappa})$ and $\tau_{m,n}(x,t;\boldsymbol{\kappa})$ defined by (4.35) and (4.37), respectively are related, up to a multiplicative constant, as follows

$$\tau_{m,n}(x,t;\boldsymbol{\kappa}) \sim \begin{cases} \widehat{\Psi}_{m-n,n+1}(x,t;\boldsymbol{\kappa}), & \text{if} \quad m \geq n \geq 0, \\ \widehat{\Psi}_{m-n,-m}(x,t;\boldsymbol{\kappa}), & \text{if} \quad n > m \geq 0, \end{cases}$$

$$\widehat{\Psi}_{m,n}(x,t;\boldsymbol{\kappa}) \sim \begin{cases} \tau_{m+n-1,n-1}(x,t;\boldsymbol{\kappa}), & \text{if} \quad m \geq 0, \ \ n \geq 1, \\ \tau_{-n,-m-n}(x,t;\boldsymbol{\kappa}), & \text{if} \quad m \leq 0, \ \ n \leq 0. \end{cases}$$

It is interesting to note that if we try a similar generalization procedure for the rational solutions $u_{m,n}$ (4.26), or equivalently (4.30), and $\widetilde{u}_{m,n}$ (4.27), or equivalently (4.33), by considering

$$\sum_{n=0}^{\infty} \varphi_n(x,t;\mu,\boldsymbol{\kappa})\lambda^n = \exp\left\{\left(x+\mu t^2\right)\lambda - \tfrac{1}{3}t\lambda^2 + \sum_{j=3}^{\infty}\kappa_j\lambda^j\right\}, \tag{4.39}$$

$$\sum_{n=0}^{\infty} \psi_n(x,t;\mu,\boldsymbol{\kappa})\lambda^n = \exp\left\{\left(x+\mu t^2\right)\lambda + t\lambda^2 + \sum_{j=3}^{\infty}\kappa_j\lambda^j\right\}, \tag{4.40}$$

where $\boldsymbol{\kappa} = (\kappa_3, \kappa_4, \ldots)$, with $\kappa_3, \kappa_4, \ldots$ arbitrary constants, rather than (4.28) and (4.31), respectively. Then for (4.30) to be a solution of the Boussinesq equation (1.5) we find that necessarily $\boldsymbol{\kappa} = 0$ in (4.39) and for (4.33) to be a solution then either $\mu = 0$ or $\boldsymbol{\kappa} = 0$ in (4.40). Similarly, the rational solutions expressed in terms of the Yablonskii–Vorob'ev polynomials given in §4.2 and §4.3 do not appear to have generalizations. Hence we conclude that generalized rational solutions, i.e. rational solutions which depend on an infinite number of arbitrary parameters, of the Boussinesq equation (1.5) exist only if the rational solution obtained through the reduction to a Painlevé equation decays as $|x| \to \infty$. This also true for generalized rational solutions of the KdV equation (1.2) [7, 10, 25], the classical Boussinesq system [83], and the NLS equation [32, 47].

5 Discussion

In this paper we have studied special polynomials associated with rational solutions of the Boussinesq equation (1.5) through special polynomials associated with rational solutions of $\mathrm{P_{II}}$ (1.3) and $\mathrm{P_{IV}}$ (1.4). Further we have derived some generalized rational solutions of (1.5) which are analogues of the generalized rational solutions of the KdV equation (1.2)[7, 10, 25], the classical Boussinesq system [83], and the NLS equation [32, 47]; see also [30].

However there are further rational solutions of the Boussinesq equation (1.5), for example the rational solution

$$u(x,t) = -\tfrac{1}{2} - \frac{4(x^2+t^2-1)}{(x^2-t^2+1)^2}, \tag{5.1}$$

is not a special case of the rational solutions derived above. In fact none of the rational solutions of the Boussinesq equation derived by Ablowitz and Satsuma [3] by taking a long-wave limit of the known N-soliton solutions are included in the rational solutions of the Boussinesq equation discussed in §4.

The classical orthogonal polynomials, such as Hermite, Laguerre, Legendre, and Tchebychev polynomials which are associated with rational solutions classical special functions, play an important role in a variety of applications (cf. [6, 11, 88]). Hence it seems likely that the special polynomials discussed here which are associated with rational solutions of nonlinear special functions, i.e. the soliton and Painlevé equations, will also arise in variety of applications, such as in numerical analysis.

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