A New Conservative Finite Difference Scheme for Boussinesq Paradigm Equation

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Abstract. A family of nonlinear conservative finite difference schemes for the multidimensional Boussinesq Paradigm Equation is considered. A second order of convergence and a preservation of the discrete energy for this approach are proved. The schemes have been numerically tested on the models of the propagation of a soliton and the interaction of two solitons. The numerical experiments demonstrate that the proposed family of schemes is about two times more accurate than the family of schemes studied in [14].

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1. INTRODUCTION

Consider the Cauchy problem for the Boussinesq Paradigm Equation (BPE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \;\; x \in \mathbb{R}^n, \;\; 0 < t \le T, \;\; T < \infty \tag{1}$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$$u(x,t) \to 0, \quad \Delta u(x,t) \to 0, \quad |x| \to \infty, \tag{2}$$

and initial conditions

$$u(x,0) = u_0(x), \;\; \frac{\partial u}{\partial t}(x,0) = u_1(x). \tag{3}$$

Here $f(u) = u^p, p = 2,3,...$, Δ is the Laplace operator and the constants α, β_1 and β_2 are positive. Problem (1)–(3) may have either a bounded global solution or a blowing up solution. Sufficient conditions for the global existence and for the blow up of the weak solution can be found in [9, 19, 22].

The derivation of equation (1) from the original Boussinesq system can be found in [7]. The BPE or similar Boussinesq type equations arise in many models of real-life processes, such as surface waves in shallow waters, acoustic waves and ion-sound waves [3, 7].

A lot of papers are devoted to computational simulations and to appropriate physical interpretations of the one-dimensional problem (1D) – see e.g. [5, 6, 13, 16]. A variety of exact solutions for $f(u) = u^p, p = 1,2,3,...$ in 1D can be found in Yan and Bluman [23] and Wazwaz [21]. Adomian decomposition method for solving (1)–(3) is applied in [1]. An implicit finite difference scheme based on rational approximation to the matrix-exponential term in the time level recurrence formula for the numerical solution to the improved Boussinesq equation has been proposed by Bratsos in [2].

The nonlinear stability and the convergence of some finite difference methods for the "good" Boussinesq equation, i.e. $\beta_1 = 0$ in (1), are presented by Ortega and Sanz-Serna [17]. Pani and Saranga [18] have used H^2 finite element method for the numerical solution of the "good" Boussinesq equation. Pseudospectral numerical methods are examined by De Frutos, Ortega and Sanz-Serna in [11] for the "good" Boussinesq equation and by Choo [4] for the damped Boussinesq equation. These methods are proved to have second order of convergence.

Unlike the 1D case, in the two-dimensional case (2D) there are few studies on the numerical simulations - see e.g. [3] and [8]. Two finite difference schemes for multi-dimensional BPE have been proposed in [14, 15]. The first scheme is linearized with respect to the nonlinearity while the second scheme is nonlinear and conservative. The second order of convergence is proved for both schemes. The extensive numerical tests show clear advantage of the linearized scheme over the nonlinear scheme for smooth solutions. But the linearized scheme does not preserve the discrete energy.

The aim of this paper is to give a new family of conservative schemes for numerical solution to the BPE which, as the previous conservative scheme, preserves the discrete energy and has second order of convergence. The numerical experiments reported in the present paper and in [12] show that the approximation error of the new scheme is about a half of the approximation error of the conservative scheme from [14].

The outline of the paper is as follows. In Section 2 we introduce some preliminaries and construct a new family of finite difference schemes. A conservation law and error estimates are proved in Section 3. Results of the numerical simulations in the 1D case are reported and compared in Section 4.

2. FINITE DIFFERENCE SCHEME

We discretize BPE (1)–(3) on a sufficiently large space domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$. We assume that the solution and its first and second derivatives are negligible outside Ω. For integers N_1, N_2 set the space steps $h_i = L_i/N_i, i = 1,2$ and $h = (h_1, h_2)$. Let $\Omega_h = \{(x_i, y_j) : x_i = ih_1, i = -N_1, \dots, N_1, y_j = jh_2, j = -N_2, \dots, N_2\}$. Next, for integer N we denote the time step by $\tau = T/N$. For each of the time levels $t^k = k\tau,\ k = 0,1,2,\dots,N$ we consider a mesh function $v^k_{i,j}$ defined on $\Omega_h \times \{t^k\}$. Whenever possible the sub-indexes i, j of the mesh functions are omitted.

The discrete scalar product $\langle v, w\rangle = \sum_{i,j} h_1 h_2 v_{i,j} w_{i,j}$ is associated with the space of mesh functions v, w, which vanish on the boundary of Ω_h. Denote by Δ_h the standard 5-point discrete Laplacian.

The finite differences $v^k_t = \left(v^{k+1} - v^k\right)\tau^{-1}$ and $v^k_{\bar{t}t} = \left(v^{k+1} - 2v^k + v^{k-1}\right)\tau^{-2}$ are used for the approximation of the first and second time derivatives. For a real parameter θ the symmetric θ-weighted approximation

$$v^{\theta,k} = \theta v^{k+1} + (1-2\theta)v^k + \theta v^{k-1} = v^k + \theta\tau^2 v^k_{\bar{t}t} \tag{4}$$

to v^k will be applied in constructing approximations to the discrete Laplacians $\Delta_h v$ and $(\Delta_h)^2 v$. The nonlinear term f in (1) can be treated in different ways, see [17, 18]. In this paper we approximate $\alpha f(u)$ by

$$g(v^{k+1}, v^k, v^{k-1}) = 2\frac{F(0.5(v^{k+1}+v^k)) - F(0.5(v^k+v^{k-1}))}{v^{k+1} - v^{k-1}}, \ \text{where}\ F(u) = \alpha\int_0^u f(s)ds = \alpha\frac{u^{p+1}}{p+1}. \tag{5}$$

Our finite difference scheme defines the approximate solution $v^k_{i,j}$ to $u(x_i, y_j, t_k)$ as the solution of

$$v^k_{\bar{t}t} - \beta_1 \Delta_h v^k_{\bar{t}t} - \Delta_h v^{\theta,k} + \beta_2 (\Delta_h)^2 v^{\theta,k} = \Delta_h g(v^{k+1}, v^k, v^{k-1}) \tag{6}$$

at the internal mesh points of Ω_h, i.e. $|i| < N_1$ and $|j| < N_2$, with initial conditions

$$\begin{aligned} v^0_{i,j} &= u_0(x_i, y_j), \\ v^1_{i,j} &= u_0(x_i, y_j) + \tau u_1(x_i, y_j) + 0.5\tau^2 (I - \beta_1\Delta_h)^{-1}\left(\Delta_h u_0 - \beta_2(\Delta_h)^2 u_0 + \alpha\Delta_h f(u_0)\right)(x_i, y_j) \end{aligned} \tag{7}$$

and boundary conditions at the boundary mesh points, i.e. $|i| = N_1$ or $|j| = N_2$, of Ω_h

$$v^k_{i,j} = 0,\ \ \Delta_h v^k_{i,j} = 0,\ \ k = 1,2,\dots,N. \tag{8}$$

In order to implement the second boundary condition of (8) the grid is overlapping the domain Ω_h by one line at each boundary.

Equations (6)–(8) constitute a family (*Family 1*) of finite difference schemes depending on the parameter θ. In our previous paper [14] we considered the following approximation to the nonlinearity $\alpha f(u)$

$$\widetilde{g}(v^{k+1}, v^k, v^{k-1}) = \frac{F(v^{k+1}) - F(v^{k-1})}{v^{k+1} - v^{k-1}}. \tag{9}$$

The second family of schemes (*Family 2*) is generated by equation (6) with g in the right-hand side replaced by $\widetilde{g}$, (7) and (8). Some properties of this family of schemes are obtained in [14]. Efficient algorithms for evaluation of the solution v from (6)–(8) can be found in [12] and [14].

3. ANALYSIS OF THE FINITE DIFFERENCE SCHEME

In the space of functions, which vanish on the boundary of Ω_h we define operators $A=-\Delta_h$ and $B=I-\beta_1\Delta_h+\tau^2\theta(-\Delta_h+\beta_2(\Delta_h)^2)$. Here I stands for the identity operator. Note that these operators are self-adjoint and positive definite operators ($\theta\geq 0$ is required in the case of B). In the analysis of Family 1, we use the representation (4) and rewrite (6) in the operator form

$$Bv_{\bar{t}t}+Av+\beta_2A^2v=-Ag. \tag{10}$$

It is well known that the energy conservation law

$$E\left(u(\cdot,t)\right)=E\left(u(\cdot,0)\right),\ 0<t\leq T,$$

is valid for the solution of the BPE (1)–(3). Here the energy functional E is given by the equality

$$E\left(u(\cdot,t)\right)=\left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}(\cdot,t)\right\|^2+\beta_1\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|^2+\|u(\cdot,t)\|^2+\beta_2\|\Delta u(\cdot,t)\|^2+2\int_{R^n}F(u(\cdot,t))dx \tag{11}$$

and $||\cdot||$ stands for the standard norm in $L_2(\mathbb{R}^n)$.

Theorem 1 below states that, as in the continuous case, the conservation property holds for the solution to the finite difference scheme (6)–(8). We define the full discrete energy functional $E_h(v^k)$ and the linear functional $E_{h,L}(v^k)$ as

$$\begin{aligned}E_h(v^k)=&\left\langle A^{-1}v_t^k,v_t^k\right\rangle+\beta_1\left\langle v_t^k,v_t^k\right\rangle+1/4\left\langle v^k+v^{k+1}+\beta_2A(v^k+v^{k+1}),v^k+v^{k+1}\right\rangle\\&+\tau^2(\theta-1/4)\left\langle(I+\beta_2A)v_t^k,v_t^k\right\rangle+2\left\langle F(0.5(v^{k+1}+v^k)),1\right\rangle=E_{h,L}(v^k)+2\left\langle F(0.5(v^{k+1}+v^k)),1\right\rangle.\end{aligned} \tag{12}$$

Multiplying (10) by $A^{-1}\left(v^{k+1}-v^{k-1}\right)$ and using summation by parts together with the boundary conditions (8) we prove

Theorem 1 (Discrete conservation law). *The discrete energy (12) of the solution to the difference scheme (6)–(8) is conserved in time*

$$E_h(v^k)=E_h(v^0),\ \ k=1,2,\ldots,N.$$

Our calculations given in Section 4 confirm that the discrete energy functional $E_h(v^k)$ is preserved with high accuracy.

The key result of the paper is the following theorem for the convergence of the solution v of the finite difference scheme to the exact solution u to BPE.

Theorem 2 (Convergence of the method). *Assume that $f(u)=u^p, p=2,3,4,...$ and that*

(i) *the parameter θ satisfies*

$$\theta>\frac{1}{4}-\frac{\beta_1}{\tau^2||I+\beta_2A||}; \tag{13}$$

(ii) *the solution u to BPE possesses bounded continuous derivatives up to the fourth order with respect to x and t on $\mathbb{R}^2\times[0,T]$;*

(iii) *the solution v to the finite difference scheme (6), (7) and (8) is bounded in the maximal norm.*

Let M be a constant such that

$$M\geq\max_{i,j,k\leq N}\left(|u(x_i,y_j,t_k)|,|v_{i,j}^k|\right).$$

Then for $\tau<0.5\min\{0.25M^{-1},\beta_1\}$ the discrete solution v converges to the exact solution u as $|h|,\tau\to 0$ and there exists a constant C (independent of h, τ and u) such that

$$||v^k-u(\cdot,t^k)||+||A^{1/2}(v^k-u(\cdot,t^k))||\leq Ce^{Mt^k}\left(|h|^2+\tau^2\right),\quad k=1,2,...,N. \tag{14}$$

Proof. Denote by $z=v-u$ the error of the numerical solution. We substitute $v=z+u$ into (6) and obtain that z satisfies the following equation

$$Bz_{\bar{t}t}+Az+A^2z=-Ag(v^{k+1},v^k,v^{k-1})-Bu_{\bar{t}t}-Au-A^2u. \tag{15}$$

Using equation (1) and Taylor series expansion for u about the node (x_i, y_j, t_k) it is straightforward to show that the right-hand side of (15) can be written as

$$-Ag(v^{k+1}, v^k, v^{k-1}) - Bu_{\bar{t}t} - Au - A^2u = -A\psi_1 + \psi_2$$

with $\psi_1^k = g(v^{k+1}, v^k, v^{k-1}) - \alpha f(u(t_k))$ and $\psi_2 = O(|h|^2 + \tau^2)$. Using Taylor series for u we estimate ψ_1^k by

$$|\psi_1^k| < C\left(M_1\tau^2 + M\left(|z^{k+1} + z^k| + |z^{k-1} + z^k|\right)\right). \tag{16}$$

As in Theorem 1 we multiply (10) by $A^{-1}\left(v^{k+1} - v^{k-1}\right)$, use summation by parts and obtain

$$E_{h,L}(z^k) \le E_{h,L}(z^{k-1}) + \tau||\psi_1^k||^2 + \tau||A^{-1}\psi_2^k||^2 + \tau||z_t^k||^2 + \tau||z_t^{k-1}||^2.$$

We apply estimate (16) in the above inequality and get

$$E_{h,L}(z^k) \le E_{h,L}(z^{k-1}) + \tau||z_t^k||^2 + \tau||z_t^{k-1}||^2 + \tau M||z^k + z^{k+1}||^2 + \tau M||z^k + z^{k-1}||^2 + \tau||A^{-1}\psi_2^k||^2$$

Now we sum the above inequalities for $k = 1, 2, ..., K$ $(K \le N)$ and obtain

$$E_{h,L}(z^K) \le E_{h,L}(z^0) + 2\tau\sum_{k=0}^{K}||z_t^k||^2 + 2\tau M\sum_{k=0}^{K}||z^k + z^{k+1}||^2 + \tau\sum_{k=0}^{K}||A^{-1}\psi_2^k||^2. \tag{17}$$

The initial conditions (3) are approximated locally by (7) with error $O(|h|^2 + \tau^2)$ and $\psi_2 = O(|h|^2 + \tau^2)$. In view of (13) the linear functional $E_{h,L}$ is positive definite. The terms $||z_t^K||^2$ and $||z^K + z^{K+1}||^2$ from the right-hand side of (17) can be moved to the left-hand side of (17) because the time step τ is sufficiently small by hypothesis. Now the statement of the theorem follows by Gronwall's inequality. □

Theorem 2 gives second order of convergence of the FDS in discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem. Thus, the nonlinearity does not deteriorate the rate of convergence.

Note that the L_∞ norm of the exact solution u is included in the exponent in the right-hand sides of the error estimate (14) in Theorem 2 and in the restriction for the parameter τ. Thus, if u blows up at a moment T_0 and if one would like to evaluate the solution in a neighborhood of the blow up moment, then the parameter τ should be chosen very small and the convergence of the finite difference scheme will slow up.

Corollary 1. *For $\theta \ge 1/4$ the convergence of the numerical solution of (6) - (8) to the exact solution to BPE is of second order when $|h|$ and τ go independently to zero. For $\theta = 0$ the convergence of the numerical solution to the exact solution is of second order when $|h|, \tau$ go to zero subject to the condition $\tau^2 < \frac{4}{9}\frac{\beta_1}{\beta_2}h^2$.*

Combining Theorem 2 with the embedding theorems we get

Corollary 2. *Under the assumptions of the Theorem 2 the finite difference scheme (6)–(8) admits the following error estimate in the uniform norm:*

$$\max_i |v_i^k - u(x_i, t^k)| \le Ce^{Mt^k}\left(|h|^2 + \tau^2\right), \qquad d = 1;$$

$$\max_{i,j} |v_{i,j}^k - u(x_i, y_j, t^k)| \le Ce^{Mt^k}\sqrt{\ln(\max\{N_1, N_2\})}\left(|h|^2 + \tau^2\right), \qquad d = 2.$$

The above estimates are optimal for the 1D case and almost *optimal (up to a logarithmic factor) for the 2D case.*

4. NUMERICAL RESULTS

In this section we compare the conservative scheme (6)–(8), denoted as *Family 1*, with the conservative *Family 2* with the approximation to the nonlinear term $\tilde{g}$ given by (9), proposed in [14], for the typical quadratic nonlinearity $f(u) = u^2$ in the one-dimensional case. The comparison of both families for nonlinearity $f(u) = u^3$ is given in our paper [12]. Both families are implicit with respect to the value v^{k+1} on the $k+1$ time level (functions g and $\tilde{g}$ depend on the unknown value v^{k+1}). Thus a standard iterative procedure for evaluation of v^{k+1} from (6) is required. The inner iterations stopped when the relative error between two consecutive iterations is found to be less that a threshold $\varepsilon = 10^{-13}$. At each time level, four to five iterations are required to satisfy the stop condition, for moderate time step $\tau = 0.025$. We apply both conservative schemes with parameter $\theta = 0.5$. The parameters of BPE (1) in all cases are fixed to $\alpha = 3$, $\beta_1 = 1.5$ and $\beta_2 = 0.5$.

It is well known that BPE (1) possesses an analytical solution

$$\tilde{u}(x,t;x_0,c) = \frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-x_0-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}\right),$$

which is a solitary wave traveling with velocity c and possessing a peak located at the point x_0 at the initial moment $t = 0$.

Example 1. *Propagation of one solitary wave.*

Consider BPE (1)–(3) with initial conditions

$$u(x,0) = \tilde{u}(x,0;0,2), \quad \frac{du}{dt}(x,0) = \frac{d\tilde{u}}{dt}(x,0;0,2).$$

Table 1 shows the error δ_h defined as the maximal norm $||\cdot||_\infty$ of the difference between the exact solution u and the numerical solution v_h obtained by *Family 1* or *Family 2*, i.e. $\delta_h = ||\tilde{u} - v_h||_\infty$. The numerical rate of convergence is evaluated as $\log_2(\delta_h/\delta_{h/2})$.

Table 1. Numerical comparison at time $T = 40$ of *Family 1* and *Family 2* for one solitary wave with velocity $c = 2$ and with $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $x \in [-40, 120]$.

$h = \tau$	Error δ_h, *Family 1*	Rate *Family 1*	Error δ_h, *Family 2*	Rate *Family 2*	Error *Fam.2*/Error *Fam.1*
0.2	0.144106	–	0.265115	–	1.83
0.1	0.037527	1.9411	0.071849	1.8836	1.91
0.05	0.009478	1.9852	0.018315	1.9720	1.93
0.025	0.002376	1.9961	0.004601	1.9929	1.94
0.0125	0.000596	1.9961	0.001153	1.9966	1.93

Our calculations confirm that both families are of order $O(h^2 + \tau^2)$. The last column of Table 1 shows that for one and the same steps h and τ, *Family 1* is about two times more precise than *Family 2*.

Example 2. *Interaction of two solitary waves.*

The initial conditions in this case are:

$$u(x,0) = \tilde{u}(x,0;-40,2) + \tilde{u}(x,0;50,-1.5), \quad \frac{du}{dt}(x,0) = \frac{d\tilde{u}}{dt}(x,0;-40,2) + \frac{d\tilde{u}}{dt}(x,0;50,-1.5).$$

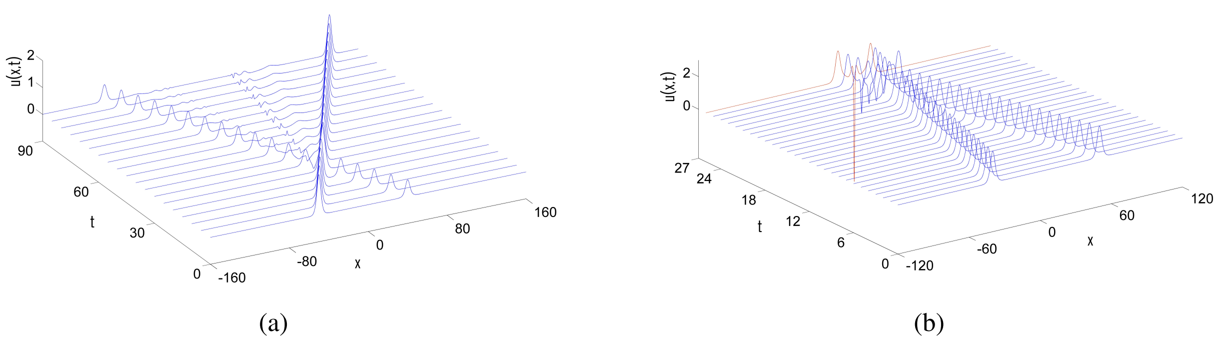

Figure 1. Interaction of two solitary waves, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$: (a) $c_1 = 2$, $c_2 = -1.5$, $0 \le t \le 90$; (b) $c_1 = -c_2 = 2.2$, $t^* \approx 27$, t^* - blow up time.

On Figure 1 two typical interactions of two solitary waves with different velocities are plotted at successive time steps. On Figure 1(a) the velocities of the waves are relatively small and both waves keep traveling preserving their shapes after the interaction. The interaction also generates additional smaller waves. In Figure 1(b) the velocities of the waves are relatively big and the solution blows up after the collision increasing the absolute value of its amplitude.

Table 2. Numerical comparison at time $T = 80$ of *Family 1* and *Family 2* for the interaction of two solitary waves traveling with velocities $c_1 = 2$, $c_2 = -1.5$ and with $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $x \in [-160, 170]$.

$h = \tau$	Error δ_h, *Family 1*	Rate *Family 1*	Error δ_h, *Family 2*	Rate *Family 2*	Error *Fam.2*/Error *Fam.1*
0.1	–	–	–	–	–
0.05	0.066214	1.9819	0.126497	1.9634	1.91
0.025	0.016692	2.0000	0.032210	1.9931	1.93
0.0125	0.004034	2.1789	0.007785	2.1730	1.93

The same quantities as on the Table 1 are shown on Table 2, for interaction with relatively small velocities. The error is calculated using Runge method, as $\delta_h = \rho_{2h}^2/(\rho_{2h} - \rho_h)$, $\rho_h = ||v_h - v_{h/2}||_\infty$. The calculations for the two solitary waves confirm that both families of schemes are of order $O(|h|^2 + \tau^2)$. The second observation in this case is that, as in the case of one soliton, the solution obtained by *Family 1* is about two times more precise than the solution obtained by *Family 2*. An explanation of this observation is the following one: both families differ by the approximation of the nonlinear term only. Using Taylor series for $u(\cdot, t^k) = u^k$ about the time level t^k (the spatial argument is fixed) we find

$$g(u^{k+1}, u^k, u^{k-1}) - f(u^k) = \tau^2 R_1 + O(\tau^3), \quad \tilde{g}(u^{k+1}, u^k, u^{k-1}) - f(u^k) = \tau^2 R_2 + O(\tau^3),$$
$$R_1 = \frac{1}{4}\alpha \frac{\partial f}{\partial u}(u^k) \frac{\partial^2 u}{\partial t^2}(t^k), \quad R_2 = \frac{1}{2}\alpha \frac{\partial f}{\partial u}(u^k) \frac{\partial^2 u}{\partial t^2}(t^k).$$

Thus, the leading terms of the error for *Family 1* and *Family 2* are $\tau^2 R_1$ and $\tau^2 R_2$, respectively, with $R_2 = 2R_1$. This fact explains the better accuracy of the numerical solution to *Family 1* compared with the numerical solution to *Family 2*.

Remark. The numerical experiments for one solitary wave and two solitary waves show that the discrete conservation law is maintained with high precision by both *Family 1* and *Family 2*. The discrete relative energy $E_{h,rel}$ is defined as $E_{h,rel} = \max_{0 \le k \le n} \frac{|E_h(v^k) - E_h(v^0)|}{E_h(v^0)}$. For one solitary wave, spatial step $h = 0,05$ and time $T = t^n = 40$, the discrete relative energy is $1.0 \cdot 10^{-10}$ and $1.01 \cdot 10^{-10}$ respectively for the *Family 1* and for the *Family 2*. Note that each family has its own discrete energy functional. The two discrete functionals do not coincide but they both approximate the energy functional (11) of the continuous problem.

5. CONCLUSION

In this paper a new family of conservative finite difference schemes is applied for numerical solving of multidimensional BPE. The second order of accuracy in the W_2^1 mesh norm and in the maximal norm of the solution are proved theoretically and are confirmed numerically. Comparison of the new family of schemes with the family introduced in [14] is given. Even though the new schemes have the same properties (the conservativeness and the second order of convergence) as the family [14], the new schemes have clear advantages in precision of the solution – about two times smaller approximation error for a single soliton and for two colliding solitary waves.

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