A Multicomponent Alternating Direction Method for Numerical Solution of Boussinesq Paradigm Equation

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Abstract. We construct and analyze a multicomponent alternating direction method (a vector additive scheme) for the numerical solution of the multidimensional Boussinesq Paradigm Equation (BPE). In contrast to the standard splitting methods at every time level a system of many finite difference schemes is solved. Thus, a vector of the discrete solutions to these schemes is found. It is proved that these discrete solutions converge to the continuous solution in the uniform mesh norm with $O(|h|^2 + \tau)$ order. The method provides full approximation to BPE and is efficient in implementation. The numerical rate of convergence and the altitudes of the crests of the traveling waves are evaluated.

Keywords: Boussinesq Equation, multicomponent ADI method, vector additive scheme, Sobolev type problem.

1 Introduction

Consider the Cauchy problem for the Boussinesq Paradigm Equation (BPE)

$$\frac{\partial^2 u}{\partial t^2} - \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} = \Delta u - \beta_2 \Delta^2 u + \alpha \Delta f(u),\ (x,y) \in \mathbb{R}^2,\ 0 < t \le T,\ \ T < \infty \quad (1)$$

on the unbounded region $\mathbb{R}^2$ with asymptotic boundary conditions

$$u(x,y,t) \to 0, \quad \Delta u(x,y,t) \to 0, \quad |(x,y)| \to \infty, \quad (2)$$

and initial conditions

$$u(x,y,0) = u_0(x,y), \quad \frac{\partial u}{\partial t}(x,y,0) = u_1(x,y). \quad (3)$$

Here $f(u) = u^p$, $p \in \mathbb{N}$, $p \ge 2$, Δ is the Laplace operator and the constants α, β_1 and β_2 are positive. It is shown in [6] how equation (1) could be derived from the original Boussinesq system. Note that equation (1) is unsolved relative to the time derivative $\frac{\partial^2}{\partial t^2}$ (the Laplace operator acts on the second time derivative). Thus, problem (1)–(3) is of Sobolev type according to the terminology of [16].

I. Dimov, I. Faragó, and L. Vulkov (Eds.): NAA 2012, LNCS 8236, pp. 371–378, 2013.

A lot of papers are devoted to computational simulations of one dimensional BPE. In contrast the two dimensional problems are essentially less studied. The efficient algorithms for evaluation of the discrete approximation to the solution u of BPE presented in [5,7,10] are based on the representation of an implicit finite difference scheme as pair of an elliptic and a hyperbolic discrete equations. In [11,19] the regularization method is applied and the operator of the same finite difference scheme is factorized in order to reduce the evaluation of the numerical solution to a sequence of three simple schemes.

Numerous papers are dealing with the construction and investigation of splitting methods for numerical solution of second order evolutionary problems, see e.g. [8,9,13,14] and the references there. A multicomponent alternating direction method (ADI) for solving evolutionary problems is proposed and analyzed by Abrashin in [1]. In the method at each time step a system of finite difference equations is solved and a vector of discrete solutions to these schemes is found. This method is called a 'vector additive scheme' in [3,15,18]. Varying applications of the method can be found in [2,3,4,15,17]. The method has the following advantages. First, each finite difference scheme from the system approximates the initial continuous problem. Second, the method can be applied to equations with mixed derivatives and to problems posed on complicated domains. Third, the discrete solutions to the linear multicomponent ADI scheme satisfy a discrete identity which is an approximation to the conservation law valid for the solution of the linear initial problem. As a result the multicomponent method for linear problems is unconditionally stable. Thus, the method can be treated as a generalization of the classical ADI methods to cases of space dimensions $n > 2$. Fourth, the numerical implementation of the method is efficient. The main disadvantage of the vector additive schemes is that their implementation demands more computational resources (memory and time) compared to the standard schemes since at each time level two discrete equations have to be solved.

The aim of the paper is to construct and analyze a multicomponent ADI method for evaluation of the numerical solution to BPE (1)–(3). The algorithm for evaluation of the numerical solutions is proposed in Section 2. In Section 3 the numerical method is analyzed theoretically. First, it is proved for the particular case of linear BPE ((1)–(3) with $f(u) \equiv 0$) that the discrete solutions satisfy an identity, which is a proper discretization to the exact conservation law. Then the convergence of the method for the nonlinear BPE is established in the energy semi-norm. Important error estimates in the uniform and Sobolev mesh norms are derived and summarized. In Section 4 the evolution of 2D solitary waves with different velocities is computed with the multicomponent ADI scheme. The numerical rate of convergence and the altitudes of the crests of the traveling waves are also evaluated.

2 A Multicomponent ADI Finite Difference Scheme

We discretize BPE (1)–(3) on a sufficiently large space domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$. We assume that the solution and its derivatives are negligibly small outside Ω. For integers N_1, N_2 set the space steps $h_1 = L_1/N_1, h_2 = L_2/N_2$ and $h = (h_1, h_2)$. Let $\Omega_h = \{(x_i, y_j) : x_i = ih_1, i = -N_1, \ldots, N_1, y_j = jh_2, j = -N_2, \ldots, N_2\}$. Next, for integer K we denote the time step by $\tau = T/K$.

We consider mesh functions $v^{(k)}_{(i,j)}$ defined on $\Omega_h \times \{t^k\}$ on the time levels $t^k = k\tau,\ k = 0, 1, 2, \ldots, K$. Whenever possible the subscripts (i, j) of the mesh functions are omitted.

The discrete scalar product $\langle v, w\rangle = \sum_{i,j} h_1 h_2 v^{(k)}_{(i,j)} w^{(k)}_{(i,j)}$ and the corresponding $L_{2,h}$ discrete norm $|| \cdot || = \langle v, v\rangle^{\frac{1}{2}}$ are associated with the space of mesh functions which vanish on the boundary of Ω_h.

The operators A_1 and A_2 are defined as second finite differences of the mesh functions in the x the direction and in the y direction, i.e. $A_1 v^{(k)}_{(l,m)} = -(v^{(k)}_{(l-1,m)} - 2v^{(k)}_{(l,m)} + v^{(k)}_{(l+1,m)})h_1^{-2}$. Then $A_1^2 v$ will stand for the fourth finite difference in the first space direction times $(h_1)^4$ and A_1A_2 will be an approximation of the mixed fourth derivative $\frac{\partial^4}{\partial x^2 \partial y^2}$. The finite differences $v^{(k)}_t = \left(v^{(k+1)} - v^{(k)}\right)\tau^{-1}$ and $v^{(k)}_{\bar{t}t} = \left(v^{(k+1)} - 2v^{(k)} + v^{(k-1)}\right)\tau^{-2}$ are used for the approximation of the first and second time derivatives respectively.

We start with the construction of a multicomponent finite difference scheme for (1)–(3). At each time level k we consider two discrete approximations $v^{(1)(k)}_{(i,j)}$ and $v^{(2)(k)}_{(i,j)}$ to $u(ih_1, jh_2, k\tau)$. We deal with the following system of implicit finite difference schemes

$$\begin{aligned} v^{(1)(k)}_{\bar{t}t} &+ \beta_1 A_1 v^{(1)(k)}_{\bar{t}t} + \beta_1 A_2 v^{(1)(k-1)}_{\bar{t}t} + A_1 v^{(1)(k+1)} + A_2 v^{(2)(k)} \\ &+ \beta_2 A_1^2 v^{(1)(k+1)} + \beta_2 A_1 A_2 v^{(1)(k)} + \beta_2 A_2^2 v^{(2)(k)} + \beta_2 A_1 A_2 v^{(2)(k)} \\ &= -\alpha A_1 f(v^{(1)(k)}) - \alpha A_2 f(v^{(2)(k)}), \end{aligned} \tag{4}$$

$$\begin{aligned} v^{(2)(k)}_{\bar{t}t} &+ \beta_1 A_1 v^{(1)(k)}_{\bar{t}t} + \beta_1 A_2 v^{(2)(k)}_{\bar{t}t} + A_1 v^{(1)(k+1)} + A_2 v^{(2)(k+1)} \\ &+ \beta_2 A_1^2 v^{(1)(k+1)} + \beta_2 A_1 A_2 v^{(1)(k+1)} + \beta_2 A_2^2 v^{(2)(k+1)} + \beta_2 A_1 A_2 v^{(2)(k)} \\ &= -\alpha A_1 f(v^{(1)(k)}) - \alpha A_2 f(v^{(2)(k)}). \end{aligned} \tag{5}$$

Note that the nonlinear function $f(u)$ is evaluated on the time level k and the approximation $A_2 v^{(1)(k-1)}_{\bar{t}t}$ on the previous time level is used in (4). Thus, the scheme (4)–(5) is a four-level scheme and values of the numerical solution on the first three time levels $t = -\tau$, $t = 0$ and $t = \tau$ are required in order to start the method. We evaluate initial values for $v^{(1)}$ and $v^{(2)}$ on time levels $t = 0$ and $t = \tau$ using formulas

$$v^{(m)(0)}_{(i,j)} = u_0(x_i, y_j),\ m = 1, 2, \tag{6}$$

$$\begin{aligned} v^{(m)(1)}_{(i,j)} &= u_0(x_i, y_j) + \tau u_1(x_i, y_j) \\ &- 0.5\tau^2 \left(I + \beta_1 \mathcal{A}\right)^{-1} \left(\mathcal{A} u_0 + \beta_2 \mathcal{A}^2 u_0 + \alpha \mathcal{A} f(u_0)\right)(x_i, y_j),\ m = 1, 2, \end{aligned} \tag{7}$$

where $\mathcal{A} = A_1 + A_2$. The third initial value $v^{(m)(-1)}$ for $m = 1, 2$ at time level $t = -\tau$ is found from the equation

$$v^{(m)(0)}_{\bar{t}t(i,j)} = -\left(I + \beta_1 \mathcal{A}\right)^{-1} \left(\mathcal{A}u_0 + \beta_2 \mathcal{A}^2 u_0 + \alpha \mathcal{A} f(u_0)\right)(x_i, y_j),\ m = 1, 2. \quad (8)$$

For approximation of the second boundary condition in (2) the mesh is extended outside the domain Ω_h by one line at each spatial boundary and the symmetric second-order finite difference is used for the approximation of the second derivatives in (2).

Suppose in the following that the exact solution to (1)–(3) is smooth enough, e.g. $u \in C^{6,6}\left(\mathbb{R}^2 \times [0, T]\right)$. Straightforward calculations via Taylor series expansion at point (x_i, y_j, t_k) show that the local approximation error of the discrete equations (4)–(5) is $O(h^2 + \tau)$. Also (6)–(7) approximate the initial conditions locally with $O(|h|^2 + \tau^2)$ error.

The numerical algorithm for evaluation of $v^{(1)}$ and $v^{(2)}$ is as follows. Suppose the values of $v^{(1)}$ and $v^{(2)}$ on the three consecutive time levels $(k-2)$, $(k-1)$ and (k) are known. Then (4) is an implicit scheme along the x direction and is explicit along the y direction. Thus, for each $j = -N_2, \cdots, N_2$ the vector $\{v^{(1)(k+1)}_{(i,j)}, i = -N_1, \cdots, N_1\}$ can be found from equation (4) as a solution of linear five-diagonal system. Analogously, (5) is an explicit scheme in the x direction and is implicit in the y direction with respect to $v^{(2)(k+1)}$. Thus, for each $i = -N_1, \cdots, N_1$ the vectors $\{v^{(2)(k+1)}_{(i,j)}, j = -N_2, \cdots, N_2\}$ can be evaluated from equation (5) as a solution of linear five-diagonal system. As a result the implementation of the schemes (4)–(5) can be done by efficient numerical algorithms.

Remark 1. *In order to achieve efficient algorithm in equation (4) we use* $A_2 v^{(1)(k-1)}_{\bar{t}t}$ *for approximation of* $\frac{\partial^4 u}{\partial t^2 \partial y^2}$ *instead of the straightforward* $A_2 v^{(1)(k)}_{\bar{t}t}$. *But we pay for this efficiency by having low order of approximation – the error of discretization of equation (4) is* $O(\tau + |h|^2)$ *only. In addition the scheme becomes a four-level one. Note that the straightforward approximation on level k could easily lead to a* $O(\tau^2 + |h|^2)$ *scheme, but the efficiency would be lost in this case.*

In the case of "good" (or "proper") Boussinesq equation the combined time-space derivative is removed from (1), i.e. $\beta_1 = 0$ *is set in (1), and a three-level multicomponent ADI scheme with* $O(|h|^2 + \tau^2)$ *approximation error can be proposed and analyzed following the ideas of this paper.*

3 Theoretical Analysis

First we consider the linear problem, i.e. (1)–(3) with $f \equiv 0$. We define operators $\Lambda_1(u) = -\frac{\partial^2 u}{\partial x^2}$ and $\Lambda_2(u) = -\frac{\partial^2 u}{\partial y^2}$ in the space of functions which vanish at infinity together with their second derivatives.

Let $||\cdot||$ stand for the standard norm in $L_2(\mathbb{R}^2)$. Denote by E the energy functional

$$E(u)(t) = \left\|\Lambda_1^{\frac{1}{2}}\frac{\partial u}{\partial t}(\cdot,t)\right\|^2 + \left\|\Lambda_2^{\frac{1}{2}}\frac{\partial u}{\partial t}(\cdot,t)\right\|^2 + \beta_2\left\|(\Lambda_1+\Lambda_2)\frac{\partial u}{\partial t}(\cdot,t)\right\|^2$$
$$+\beta_1\left\|\Lambda_1^{\frac{1}{2}}\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2 + \beta_1\left\|\Lambda_2^{\frac{1}{2}}\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2 + \left\|\frac{\partial^2 u}{\partial t^2}(\cdot,t)\right\|^2. \tag{9}$$

It is straightforward to prove that the solution to problem (1)–(3) with $f=0$ satisfies the identity $E(u)(t) = E(u)(0)$ for every $t \geq 0$ or, equivalently the energy functional $E(u)$ is preserved in time.

We shall obtain a similar discrete identity for the solution to (4)–(8) with $f(u)=0$. First we define $\mathbf{v}^{(k)}$ as the couple of solutions $(v^{(1)(k)}, v^{(2)(k)})$ and the semi-norm (or the energy norm) $N(\mathbf{v}^{(k)})$ by

$$N(\mathbf{v}^{(k)}) = \|A_1^{\frac{1}{2}}v_t^{(1)(k)}\|^2 + \|A_2^{\frac{1}{2}}v_t^{(2)(k)}\|^2 + \beta_2\|A_1v_t^{(1)(k)} + A_2v_t^{(2)(k)}\|^2$$
$$+\beta_1\|A_1^{\frac{1}{2}}v_{\bar{t}t}^{(1)(k)}\|^2 + \beta_1\|A_2^{\frac{1}{2}}v_{\bar{t}t}^{(2)(k)}\|^2 + \|v_{\bar{t}t}^{(2)(k)}\|^2. \tag{10}$$

Following the proof of Theorem from page 318 of [1], it can be established

Theorem 1 (Discrete summation identity). *For every $Q = 1,2,3,\ldots,K$ the solution $\mathbf{v}^{(Q)}$ to problem (4)–(8) with $f(u)=0$ satisfies the equality*

$$N(\mathbf{v}^{(Q)}) + \tau\sum_{k=1}^{Q}\tau\left(\|A_1^{\frac{1}{2}}v_{\bar{t}t}^{(1)(k)}\|^2 + \beta_2\|A_1v_{\bar{t}t}^{(1)(k)}\|^2 + \beta_1\|A_1^{\frac{1}{2}}v_{\bar{t}t\bar{t}}^{(1)(k)}\|^2\right)$$
$$+\tau\sum_{k=1}^{Q}\tau\left(\|A_2^{\frac{1}{2}}v_{\bar{t}t}^{(2)(k)}\|^2 + \beta_2\|A_2v_{\bar{t}t}^{(2)(k)}\|^2 + \beta_1\|A_2^{\frac{1}{2}}v_{\bar{t}t\bar{t}}^{(2)(k)}\|^2\right)$$
$$+\tau\sum_{k=1}^{Q}\tau\|A_1v_t^{(1)(k)} + \beta_2A_1^2v_t^{(1)(k)} + \beta_2A_1A_2v_t^{(2)(k-1)} + \beta_1A_1v_{\bar{t}tt}^{(1)(k-1)}\|^2$$
$$+\tau\sum_{k=1}^{Q}\tau\|A_2v_t^{(2)(k)} + \beta_2A_2^2v_t^{(2)(k)} + \beta_2A_1A_2v_t^{(1)(k)} + \beta_1A_2v_{\bar{t}tt}^{(2)(k-1)}\|^2$$
$$= N(\mathbf{v}^{(0)}). \tag{11}$$

Consequently the energy norm of the numerical solution at each fixed time level deviates from the energy norm of the initial data by a small term of first order in time step, i.e. $N(\mathbf{v}^{(Q)}) - N(\mathbf{v}^{(0)}) = O(\tau)$ for $Q = 1,2,\ldots,K$.

We state now our main theorem

Theorem 2 (Convergence of the Multicomponent ADI Scheme). *Assume that the solution u to BPE obeys $u \in C^{6,6}\left(\mathbb{R}^2\times[0,T]\right)$ and the solutions $v^{(1)(k)}$, $v^{(2)(k)}$ to the multicomponent ADI scheme (4)–(8) are bounded in the* *maximum norm for every* $k = 1, 2, 3, \cdots, K$. *Then* $v^{(1)}$ *and* $v^{(2)}$ *converge to the exact solution* u *as* $|h|, \tau \to 0$ *and the energy norm estimate*

$$N(\mathbf{z}^{(k)}) \leq C \left(|h|^2 + \tau\right)^2, \quad k = 1, 2, \cdots, K$$

holds with a constant C *independent on* h *and* τ, *where* $z^{(1)(k)} = v^{(1)(k)} - u(\cdot, k\tau)$ *and* $z^{(2)(k)} = v^{(2)(k)} - u(\cdot, k\tau)$ *are the errors of the method.*

Sufficient conditions for global existence of bounded solution to (1)–(3) in $C^{6,6}\left(\mathbb{R}^2 \times [0, T]\right)$ are given in [21]. Stability and instability of solitary wave solutions to (1) are treated in many papers, see e.g. [20] and the references therein.

The problem for the boundedness of the discrete solutions to (4)–(8) imposed in Theorem 2 is still open. The boundedness (locally in time) of the discrete solution to a conservative finite difference scheme for BPE can be found in [12].

Corollary 1. *Under the assumptions of the Theorem 2 the multicomponent ADI scheme admits the following error estimates for every* $k = 1, 2, \cdots, K$, $m = 1, 2$

$$\begin{aligned} &\|z^{(1)(k)}\| + \|z^{(2)(k)}\| + \|A_1^{\frac{1}{2}} z^{(1)(k)}\| + \|A_2^{\frac{1}{2}} z^{(2)(k)}\| \leq C \left(|h|^2 + \tau\right), \\ &\|A_1 z^{(1)(k)} + A_2 z^{(2)(k)}\| \leq C \left(|h|^2 + \tau\right), \\ &\|z^{(m)(k)}\|_{L_\infty} \leq C \left(|h|^2 + \tau\right), \ \|z_t^{(m)(k)}\| + \|z_{\bar{t}t}^{(m)(k)}\| \leq C \left(|h|^2 + \tau\right). \end{aligned}$$

4 Numerical Results

In this section some numerical tests concerning the convergence of the multicomponent ADI method and the evolution of the numerical solution are presented in the 2D case. The computational domain is $[-30, 30] \times [-30, 30]$. The numerical solutions are evaluated for parameters $\alpha = 3$, $\beta_1 = 3$, $\beta_2 = 1$, $p = 2$ and initial conditions u_0, u_1 given in [5]. These initial conditions correspond to a solitary wave which moves along the y-axis with velocity c.

Table 1. Dependence of the convergence rate on time step and space steps

τ	$h_1 = h_2$	Rate $v^{(1)}$	Rate $v^{(2)}$	τ	$h_1 = h_2$	Rate $v^{(1)}$	Rate $v^{(2)}$
0.08	0.075	-	-	0.02	0.3	-	-
0.04	0.075	0.9384	0.9450	0.02	0.15	2.5502	2.6853
0.02	0.075	-	-	0.02	0.075	-	-

Table 1 contains the numerical rate of convergence at time $T = 8$. The accuracy of the proposed schemes in the uniform norm is calculated by Runge method using three nested meshes. We observe that the experimental rate of convergence with respect to time step approximates the theoretical rate of convergence $O(\tau)$. Regarding the convergence with respect to spatial steps, the numerical rate of convergence is better than the rate of convergence $O(|h|^2)$ proved in Corollary 1.

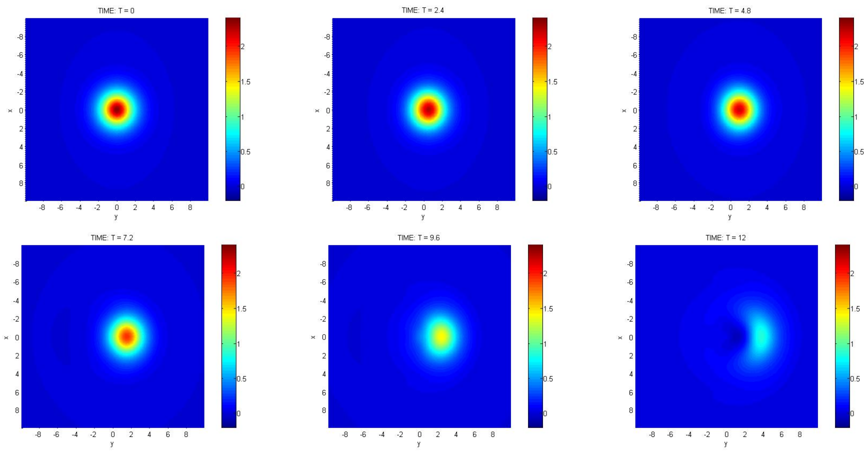

Fig. 1. (Color on-line) Evolution with velocity $c = 0.2$ of the numerical solution in time, $t = 0; 2.4; 4.8; 7.2; 9.6; 12$

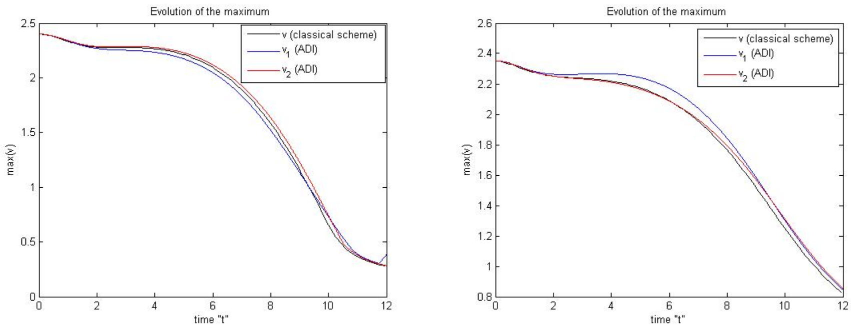

Fig. 2. (Color on-line) Evolution in time of the the altitudes of the crests of the solutions for velocity $c = 0$ (left) and velocity $c = 0.2$ (right)

On Figure 1 evolution of the numerical solution with velocity $c = 0.2$ is shown. For $t < 4.8$ the shape of the numerical solution is similar to the initial solution. For larger times the numerical solution changes its initial form and transforms into a diverging propagating wave. Evolutions of the altitudes of the crests of the solutions $v^{(1)}$ and $v^{(2)}$ in time are shown on Figure 2. For comparison the same quantity obtained by the conservative scheme from [12] is also plotted. It can be observed that the behavior of the altitudes of the crests of the numerical solutions obtained by the multicomponent ADI scheme is similar to the altitudes of the crests of the numerical solution given by the conservative scheme [7,12]. Thus, the proposed numerical method corresponds very well to the results evaluated by the well studied (theoretically and numerically) finite difference schemes from [7,12].

Acknowledgments. This work has been partially supported by the Bulgarian Science Found under grant DDVU 02/71.

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