New Boundary Condition to a Two Dimensional Stationary Boussinesq Equation

Abstract

In this paper we examine once again the stationary propagating wave solution to the two dimensional Boussinesq Equation. We focus on the boundary condition and examine and validate an explicit function for the asymptotic behavior of the wave solution. This gives more precise information on the boundaries of the domain with less computational effort and will help significantly any further numerical calculations on the topic.

1 Introduction

In this paper we consider the two dimensional Boussinesq Equation (BE)

$$
\begin{aligned}
&u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u + \Delta f(u) = 0, \quad for\ (x,y) \in \Re^2, t \in \Re^+ \\
&u(x,y,0) = u_0(x,y), u_t(x,y,0) = u_1(x,y) \\
&u(x,y) \to 0, \Delta u(x,y) \to 0, \quad for \sqrt{x^2+y^2} \to \infty,
\end{aligned}
$$

where $f(u) = \alpha u^2$, $\alpha, \beta_1, \beta_2 \succ 0$ are real constants, and Δ is the Laplace operator. The BE arises in several physical applications: propagation of long waves in shallow water, vibrations in a nonlinear string, etc. A derivation of the BE from the original Boussinesq system can be found in . Our goal is to show numerically that there exists a soliton solution to 1.1 and that the equation itself is a law that governs the overtaking of such wave solutions as shown in the 1D case]. In order to achieve that we first clear out a few problems that appear on the computational boundary of the domain. The article shows that for this type of problems one could use a certain analytical function which describes very well the behavior of the unknown function on the boundary. For now we proceed as in and try to find a traveling wave solution to in direction y with phase velocity c, i.e. solutions of type $u(x,y,t) = U(x, y-ct)$. The waves U satisfy the nonlinear fourth order elliptic equation:

$$c^2(E - \beta_1 \Delta) U_{yy} = \Delta U - \beta_2 \Delta^2 U - \Delta f(U),$$

The solution of both the hyperbolic and elliptic problems are examined in , , , , and , , respectively. We will use the same techniques defined in in order to test our new explicit formula used as boundary condition.

The asymptotic decay of the solution and the boundary condition used in are given by:

$$
\begin{gathered}
u \sim \frac{C_u}{r^2}, \quad \text{for } r \gg 1 \\
\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right)\Big|_{\partial\Omega} = -2u(x,y)\Big|_{\partial\Omega}.
\end{gathered}
$$

An important generalization in is the following rule

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \right)\Big|_{\partial\Omega} \approx -nu(x,y)\Big|_{\partial\Omega}, \quad r \gg 1$$

which applies if the expected behavior at infinity is algebraic of n-th order (i.e. $u \sim \frac{C_u}{r^n}$). The formula that we provide below satisfies all conditions , and .

2 Asymptotic behavior of the solution

Choosing a grid in order to prototype the numerical method and further select the right tools for a solution to both problems and is an essential step. The uniform and non-uniform grids define two different investigation approaches. The meshing that has been predominantly used in most cited papers for the numerical analysis is the non-uniform one. It has big time advantage of generating a fast solution for the elliptic equation but also creates a major problem when moving the wave/waves in the hyperbolic equation . The situation reverses when working with the uniform grid. The uniform grid is the right choice here because of the goals, formulated in the beginning of the article. Let's summarize this into the following chart:

	Uniform grid advantages	Non-uniform grid advantages
Hyperbolic equation	1. Better capabilities of moving the soliton in a larger time domain [0, T] 2. Possibility for colliding two or more solitons	
Elliptic equation		1. Less points in the computational box Ω increases the convergence speed of the iterative algorithm quadratically

This raises the problem of the solution asymptotic and the boundary condition. The behavior of the solution U as $r = \sqrt{x^2 + y^2} \to \infty$ is studied in details in . From the mathematical analysis and numerical results provided there it follows that u has $O(r^{-2})$ asymptotic decay at infinity.

We would like to go further by estimating which terms in the equation define the asymptotic behavior and then validate our proposition by numerical tests. At first, suppose that the second order derivatives of U ($\Delta U, c^2 U_{yy}$) are of order $O(r^{-4})$ whereas the fourth order derivatives and nonlinear term ($\Delta^2 U, c^2\Delta U_{yy}, \Delta f(U)$) in equation are of order $O(r^{-6})$. Now consider equation for sufficiently large r. We insert the asymptotic values of all terms in and neglect the higher order terms of the r-expansion (of order $O(r^{-6})$). In this way we obtain the following problem valid for large r

$$\begin{aligned}
&c^2 U_{yy} = \Delta U \\
&U(x,y), \Delta U(x,y) \to 0 \quad \text{as} \quad \sqrt{x^2+y^2} \to \infty \\
&U(x,y) \sim \frac{1}{x^2+y^2}
\end{aligned}$$

The following variable change is very suitable for further calculations:

$$\begin{aligned}
&\overline{x} = \sqrt{1-c^2}\,x \\
&\overline{y} = y \\
&v(\overline{x},\overline{y}) := U(x,y)
\end{aligned}$$

because

$$\begin{aligned}
&U_{yy} = v_{\overline{yy}} \\
&U_{xx} = (1-c^2) v_{\overline{xx}}
\end{aligned}$$

which transforms into the Laplace equation:

$$\Delta \overline{v} = 0$$

with the following restriction:

$$\overline{v}(\overline{x},\overline{y}) \sim \frac{1}{\overline{r}^2} \quad \text{for} \quad \overline{r} \to \infty.$$

in polar coordinates is:

$$\frac{\partial}{\partial \overline{r}^2}\overline{v}(\overline{r},\theta) + \frac{1}{r}\frac{\partial}{\partial \overline{r}}\overline{v}(\overline{r},\theta) + \frac{1}{r^2}\frac{\partial}{\partial \theta^2}\overline{v}(\overline{r},\theta) = 0.$$

After a separation of variables $\overline{v}(\overline{r},\theta) = H(\overline{r})G(\theta)$ is applied on the last equation , it transforms into:

$$-GH = \overline{r}^2 G_{\theta\theta}\left(H_{\overline{rr}} + \frac{1}{\overline{r}} H_{\overline{r}} \right)$$

The general solution of for G and H is found to be:

$$H(\overline{r}) = \mu_1 \frac{1}{\overline{r}^n} + \mu_2 \overline{r}^n,$$
$$G(\theta) = \mu_3 \sin(n\theta) + \mu_4 \cos(n\theta),$$

where the different μ symbols stand for the integration constants. Having in mind we get

$$\overline{v}(\overline{r},\theta) = \frac{\mu_3 \sin(n\theta) + \mu_4 \cos(n\theta)}{\overline{r}^n}.$$

Further if the profiles of the solution in are taken into account one can set $\mu_3 = 0$ and $n = 2$ which results into

$$\overline{v}(\overline{r},\theta,n=2) = \mu \frac{\cos(\theta)^2 - \sin(\theta)^2}{\overline{r}^2}$$
$$\overline{v}(\overline{x},\overline{y},n=2) = \mu \frac{\overline{x}^2 - \overline{y}^2}{(\overline{x}^2 + \overline{y}^2)^2},$$

where $\mu := \mu_4$ is the integration constant which needs to be defined. In the old (x,y) coordinate system reads:

$$v(x,y) = \mu \frac{(1-c^2)x^2 - y^2}{((1-c^2)x^2 + y^2)^2}.$$

Remark:
One could easily transform to Cartesian coordinates

$$\cos(n\theta) = \sum_{k=0}^{n} \binom{n}{k} \cos^k \theta \sin^{n-k} \theta \cos\left(\frac{1}{2}(n-k)\pi\right) =$$
$$= \frac{1}{r^n} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \cos\left(\frac{1}{2}(n-k)\pi\right)$$

by reconstructing the cosine/sinus functions. Then by using simple but tedious arithmetic which we exclude from the article, it is proven that (and in particular) holds true for (and in particular).

3 Numerical Method

The solution of the hyperbolic problem described in make use of the following variable change

$$\tilde{x} = \frac{x}{\sqrt{\beta_1}}, \tilde{y} = \frac{y}{\sqrt{\beta_1}}.$$

Later on, we will need the calculations presented here together with the energy functional from . Thus the same transformation is applied to which converts it into:

$$\beta c^2(E-\Delta)U_{yy} = \beta\Delta U - \Delta^2 U - \beta\Delta f(U).$$

Without loss of generality, it is clear that the same formula governs the asymptotic behavior because the β_1 constant could be extracted at the front of the fraction and could be masked into a new scale factor $\mu_{new} = \mu / \beta_1$:

$$\mu_{new}\frac{(1-c^2)\tilde{x}^2 - \tilde{y}^2}{((1-c^2)\tilde{x}^2 + \tilde{y}^2)^2}.$$

The solver developed for follows the same procedures described in (which is for equation). At first equation is rewritten as a system of two nonlinear second order elliptic equations. Then the simple iteration method is used to solve the system numerically. This is done along with the following important contributions that are described above:

- use a uniform grid
- apply formula on the boundary of the computational box
- use high order symmetric finite difference schemes

In order to resolve the boundary function completely one needs the μ parameter. It is obtained iteratively, at each time level of the iterative algorithm described in , by the minimization procedure showed below. For a given numerical solution $\hat{v}^k$ at the time level t_k we choose μ as minimizer of

$$\mu = \min_{\mu>0}\left\| v(x_i, y_i) - \hat{v}^k_{i,j} \right\|_{L_2,\hat{\Omega}}$$

where $(x_i, y_i) \in \Omega$ describes the computational box meshed with uniform grid, $\hat{v}_{i,j}$ is the numerical solution at (x_i, y_i) and $\hat{\Omega}$ includes not only the boundary nodes on $\partial\Omega$ but also inner nodes near boundary. The minimization problem produces a simple linear equation, which depends on the type of norm used in .

4 Validation of the Boundary Condition

Two tests are made to verify the new condition on the computational boundary. The first one review the behavior of the μ parameter defined in the previous section. In Table 2 the following quantities are presented at the end of the iteration procedure for several computational domains $L_x = L_y = 20; 40; 80; 160$ $\left(\Omega = [-L_x, L_x] \times [-L_y, L_y]\right)$: the values of the solution $\hat{v}_{i,j}$ at the point $(x_i = 0, y_i = L_y)$; the μ parameter, and the

$\left\| v(x_i, y_i) - \hat{v}^k_{i,j} \right\|_{L_2,\hat{\Omega}}$ norm included in the minimization procedure. The initial problem is solved with parameters $\alpha = 1$, $\beta = 3$, $c = 0.45$, $h = 0.5$ (domain discretization step size).

$L_x = L_y$	$\hat{v}_{i,j}$ at $x_i = 0, y_i = L_y$	μ	$\left\| v(x_i, y_i) - \hat{v}^k_{i,j} \right\|_{L_2,\hat{\Omega}}$
20	-2.23e-04	1.9355e-01	4.17e-05
40	-5.65e-05	1.9369e-01	4.42e-06
80	-1.41e-05	1.9378e-01	7.56e-07
160	-3.53e-06	1.9381e-01	7.44e-10

Table 2: Characteristic parameters of the minimization procedure for different computational domains

The results in Table 2 demonstrate that the μ parameter converge as the domain becomes larger. Further the values of $\hat{v}_{i,j}$ (shown in the second column in Table 2) decay with a rate of $1/r^2$.

The second test reveals the asymptotic of the numerical solution presented in log-log plots. Pictures in Figure 1 demonstrate important aspects of solution's cross sections on four different grids. The size of the computational domain [-50; 50]x[-50; 50] is kept constant and only the domain discretization step h changes, h = 0.1; 0.2; 0.4; 0.8. We set the following parameters: $\alpha = 1$, $\beta = 3$, $c = 0.45$.

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Figure 1

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The first two horizontal pictures in Figure 1 present logarithmic scaled plots of the absolute value of the numerical solution $\hat{v}$. One can see the decay $1/r^2$ at infinity guided by the black line. The next two horizontal pictures show the numerical solution scaled by a factor r^2. Thus these graphs display $\hat{v}r^2$ along the vertical z axis. One can observe that the scaled profile of the solution approximates a constant for large values of r. These plots are in agreement with the new boundary function v found in and with the asymptotic of the solution. Further using equation for $x = 0$ or for $y = 0$ one has for sufficiently large r

$$v(0, y) = -\frac{\mu}{y^2}, \quad v(x, 0) = \frac{\mu}{(1-c^2)x^2},$$

which explains the connection between the two constants displayed at legends on bottom pictures in Figure 1.

5 Results and Conclusory Remarks

It is obvious that the boundary function satisfies:

$$v(r,\theta)\Big|_{\partial\Omega}\begin{cases} >0, & \text{for } \theta\in(-\phi,\phi)\cup(\pi-\phi,\pi+\phi) \\ <0, & \text{for } \theta\in(\phi,\pi-\phi)\cup(\pi+\phi,-\phi) \end{cases}, \phi=\arctan\left(\sqrt{1-c^2}\right)$$

whereas the best-fit formula in obey the law:

$$v(r,\theta)\Big|_{\partial\Omega}\begin{cases} >0, & \text{for } \theta\in(-\psi,\psi)\cup(\pi-\psi,\pi+\psi) \\ <0, & \text{for } \theta\in(\psi,\pi-\psi)\cup(\pi+\psi,-\psi) \end{cases}, \psi=\frac{\pi}{4}$$

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Figure 2

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The straight black line in Figure 2 where the solution is zero further validates the asymptotic behavior given by . The iterative method (the simple iteration method) used here and in corrects the initial condition until it converges to a solution away from the boundary $\partial\Omega$. Near $\partial\Omega$ the function is fixed and cannot be corrected. If one chooses the best-fit formula from as initial condition for the simple iteration method then for slightly larger phase speeds *c* a difference occurs between the best-fit formulas and on $\partial\Omega$ as it is seen from and (see Remark I). Further the proposed boundary condition in cannot correct the sign of the iterative solution on the boundary $\partial\Omega$. This observation should be taken into account for the simulation of colliding two solitons when solving the hyperbolic equation .

I. Remark:

The variable change is trivial with respect to numerical calculations in and and in particular to the best-fit formula derived in since it only scales *x*,*y* axes by the same factor $1/\beta_1$. Furthermore the best-fit formula was converted for the case of equation .

II. Remark

It is clear that the scope of and focuses on small phase speeds *c* and non uniform grid. Nevertheless the hyperbolic problem with larger phase speed *c* and also the energy functionals ??(nqkoq ot tvoite posledni statii) shifts the priorities into a more detailed investigation of the elliptic Boussinesq equations. The obtained analytical formula gives a great advantage for the numerical computation of the stationary Boussinesq soliton for equations and respectively. Instead of choosing a bigger domain to represent the zero boundary conditions at infinity one could use now.

Other results obtained here which concern convergence of the iterative method, shape of the solution, comparison of the numerical solution from the iterative algorithm with the best-fit formula will be discussed in another article.

References

J. Choudhury, C. Christov, 2D Solitary waves of Boussinesq equation, CP75, (2005) 8590

N. Kolkovska, Two Families of Finite Difference Schemes for Multidimensional Boussinesq Paradigm Equation, AIP Conf. Proc. 1301, 395 (2010) http://dx.doi.org/10.1063/1.3526638

C. I. Christov, Numerical implementation of the asymptotic boundary conditions for steadily propagating 2D solitons of Boussinesq

M. Christou, C. I. Christov, Fourier Galerkin method for 2D solitons of Boussinesq equation, Mathematics and Computers in Simulation 74 (2007) 8292

M. Christou, C. I. Christov, Galerkin Spectral Method for the 2D Solitary Waves of Boussinesq Paradigm Equation, CP 1186 (2009) 217 { 225

C. I. Christov, J. Choudhury, Perturbation solution for the 2D Boussinesq equation, Mech. Res. Commun., 38 (2011) 274 { 281