Cent. Eur. J. Math.
DOI: 10.2478/s11533-012-0011-0

Central European Journal of **Mathematics**

A new conservative finite difference scheme for Boussinesq paradigm equation

Research Article

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Received 19 September 2011; accepted 21 November 2011

Abstract: A family of nonlinear conservative finite difference schemes for the multidimensional Boussinesq Paradigm Equation is considered. A second order of convergence and a preservation of the discrete energy for this approach are proved. Existence and boundedness of the discrete solution on an appropriate time interval are established. The schemes have been numerically tested on the models of the propagation of a soliton and the interaction of two solitons. The numerical experiments demonstrate that the proposed family of schemes is about two times more accurate than the family of schemes studied in [Kolkovska N., Two families of finite difference schemes for multidimensional Boussinesq paradigm equation, In: Application of Mathematics in Technical and Natural Sciences, Sozopol, June 21–26, 2010, AIP Conf. Proc., 1301, American Institute of Physics, Melville, 2010, 395–403].

MSC: 65M06, 65M15, 76B25

Keywords: Conservative finite difference scheme • Boussinesq Paradigm Equation • Solitary waves

1. Introduction

Consider the Cauchy problem for the Boussinesq Paradigm Equation (BPE):

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \qquad x \in \mathbb{R}^n, \quad 0 < t \le T < \infty, \tag{1}$$

on the unbounded region $\mathbb{R}^n$ with asymptotic boundary conditions

$$u(x,t) \to 0, \qquad \Delta u(x,t) \to 0, \qquad |x| \to \infty, \tag{2}$$

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and initial conditions

$$u(x,0) = u_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = u_1(x). \tag{3}$$

Here $f(u) = u^p, p \in \mathbb{N}, p \geq 2$, Δ is the Laplace operator and the constants α, β_1 and β_2 are positive. The derivation of equation (1) from the original Boussinesq system can be found in [11]. The BPE or similar Boussinesq type equations arise in many models of real-life processes, such as surface waves in shallow waters, acoustic waves and ion-sound waves, etc.

Problem (1)–(3) may have either a bounded global solution or a blowing up solution. Existence of a unique local, weak (or classical) solution on an appropriate time interval $[0, T^*)$ is proved in [25, 26]. A variety of conditions for global existence or for blow up of the weak solution can be found in [19, 25, 26, 28, 29]. If the full initial energy functional $E(u(\cdot, 0))$ is smaller than a critical constant, then sufficient conditions for global existence of the solution to (1)–(3) are given in [19, 28] in terms of smoothness of the initial data u_0, u_1, nonlinearity f and an additional, sign preserving functional. For BPE with full initial energy bigger than the critical constant or for some values of p the problem for global existence or blow up of the solution to (1)–(3) is still open, even in the one dimensional case.

There are many papers devoted to computational simulations and to appropriate physical interpretations of the one-dimensional problem – see e.g. [4, 5, 7, 10, 11, 13, 15, 20]. A variety of exact solutions for $f(u) = u^p, p \in \mathbb{N}, p \geq 2$ and $n = 1$ can be found in [27, 30]. The Adomian decomposition method for solving (1)–(3) was applied in [1]. Conservative finite difference schemes for quasilinear hyperbolic problems were proposed and studied in 1982 by Abrashin [2]. A conservative scheme for a similar but more complicated case of BPE was presented and applied in 1996 by Christov [10] in the case $p = 2$. Nonlinear stability and convergence of some finite difference methods for the "good" Boussinesq equation, i.e. $\beta_1 = 0$ in (1), were presented by Ortega and Sanz-Serna [23]. Pani and Saranga [24] used H^2 finite element method for the numerical solution of the "good" Boussinesq equation. Pseudospectral numerical methods were examined by De Frutos, Ortega and Sanz-Serna in [16] for the "good" Boussinesq equation, and by Choo [8] for the damped Boussinesq equation. These methods are proved to have second order of convergence.

Unlike the one-dimensional case, in the two-dimensional case there are few studies on the numerical simulations – see e.g. [6, 9, 12]. Two finite difference schemes for multi-dimensional BPE were proposed in [17]. The first scheme is linearized with respect to nonlinearity, while the second is nonlinear and conservative. The second order of convergence is proved for both schemes in [18]. Efficient algorithms for evaluation of the discrete solution were given in [14, 17]. The extensive numerical tests performed in [14, 17] for $n = 1$ show clear advantage of the linearized scheme over the nonlinear scheme for smooth solutions. But the linearized scheme does not preserve the discrete energy.

The aim of this paper is to study a new family of conservative schemes for numerical solution to BPE which preserves the discrete energy and has the second order of convergence. The new family of schemes is a modification of the conservative schemes from [17], with a different approximation to the nonlinear term in BPE. The discrete solution to these schemes is bounded in the energy norm locally in time. The numerical experiments reported in the present paper and in [14] show that the approximation error of the new scheme for smooth initial data is about a half of the approximation error of the conservative scheme from [17].

The outline of the paper is as follows. In Section 2 we introduce some preliminaries and construct a family of finite difference schemes. A conservation law, existence and boundedness of the solution on an appropriate time interval are proved in Section 3. Error estimates of the method are established in Section 4. Results of the numerical simulations in the one-dimensional case are reported and compared in Section 5.

2. Finite difference scheme

We discretize BPE (1)–(3) on a sufficiently large space domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$. We assume that the solution and its first and second derivatives are negligible outside Ω. For integers N_1 and N_2 set the space steps $h_i = L_i/N_i$, $i = 1, 2$, and $h = (h_1, h_2)$. Let $\Omega_h = \left\{(x_i, y_j) : x_i = ih_1, i = -N_1, \ldots, N_1;\ y_j = jh_2, j = -N_2, \ldots, N_2\right\}$. Next, for an integer K we denote the time step by $\tau = T/K$. For each of the time levels $t^k = k\tau$, $k = 0, 1, 2, \ldots, K$, we consider a mesh function $v_{i,j}^k$ defined on $\Omega_h \times \{t^k\}$. Whenever possible the sub-indexes i, j of the mesh functions are omitted. The discrete scalar product $\langle v, w \rangle = \sum_{i,j} h_1 h_2 v_{i,j} w_{i,j}$ and the corresponding $L_{2,h}$ discrete norm $\|\cdot\|$ are associated with the space of mesh functions v, w, which vanish on the boundary of Ω_h. Denote by Δ_h the standard 5-point discrete Laplacian.

The finite differences $v_t^k = \left(v^{k+1} - v^k\right)\tau^{-1}$ and $v_{\bar{t}t}^k = \left(v^{k+1} - 2v^k + v^{k-1}\right)\tau^{-2}$ are used for the approximation of the first and second time derivatives. For a real parameter θ the symmetric θ-weighted approximation

$$v^{\theta,k} = \theta v^{k+1} + (1-2\theta)v^k + \theta v^{k-1} = v^k + \theta\tau^2 v_{\bar{t}t}^k \tag{4}$$

to v^k will be applied in constructing approximations to the discrete Laplacians $\Delta_h v$ and $(\Delta_h)^2 v$. The nonlinear term f in (1) can be treated in different ways, see [2, 17, 23, 24]. In this paper we approximate $\alpha f(u)$ by

$$g\left(v^{k+1}, v^k, v^{k-1}\right) = 2\,\frac{F\left((v^{k+1}+v^k)/2\right) - F\left((v^k+v^{k-1})/2\right)}{v^{k+1} - v^{k-1}}, \qquad \text{where} \quad F(u) = \alpha\int_0^u f(s)\,ds = \alpha\,\frac{u^{p+1}}{p+1}\,.$$

Our finite difference scheme defines the approximate solution $v_{i,j}^k$ to $u(x_i, y_j, t^k)$ as the solution of

$$v_{\bar{t}t}^k - \beta_1\Delta_h v_{\bar{t}t}^k - \Delta_h v^{\theta,k} + \beta_2(\Delta_h)^2 v^{\theta,k} = \Delta_h g\left(v^{k+1}, v^k, v^{k-1}\right) \tag{5}$$

at the internal mesh points of Ω_h, i.e. $|i| < N_1$ and $|j| < N_2$. The initial conditions are

$$v_{i,j}^0 = u_0(x_i, y_j), \qquad v_{i,j}^1 = u_0(x_i, y_j) + \tau u_1(x_i, y_j) + \frac{\tau^2}{2(I - \beta_1\Delta_h)}\left(\Delta_h u_0 - \beta_2(\Delta_h)^2 u_0 + \alpha\Delta_h f(u_0)\right)(x_i, y_j) \tag{6}$$

and the boundary conditions at the boundary mesh points, i.e. $|i| = N_1$ or $|j| = N_2$, of Ω_h, are

$$v_{i,j}^k = 0, \qquad \Delta_h v_{i,j}^k = 0, \qquad k = 1, 2, \ldots, K. \tag{7}$$

In order to implement the second boundary condition of (7) the grid is overlapping the domain Ω_h by one line at each boundary.

Remark 2.1.

FDS (5) is a nonlinear system with respect to the unknowns v^{k+1} and, hence, it may have more than one real-valued solution. But v is a discrete approximation of the smooth solution u, which implies that v^{k+1} belongs to a neighborhood of v^k for small τ. We determine v^{k+1} from (5)–(7) with known v^{k-1} and v^k using the method of successive iterations: starting with $v^{k+1,\text{old}} = v^k$, we evaluate the nonlinear term in the right-hand side of (5), then solve the *linear* system of equations (5) with known right-hand side in order to get $v^{k+1,\text{new}}$ and continue the iteration process with $v^{k+1,\text{old}} = v^{k+1,\text{new}}$. Simple calculations of the norm of the operator of this process show that for sufficiently small time step τ (satisfying (21) below) the operator is a contraction mapping. Hence from the Banach fixed point theorem there exists a unique solution v^{k+1} to equations (5) in the neighborhood of v^k and FDS (5) is uniquely solvable in this case.

Equations (5)–(7) constitute a family (*Family 1*) of finite difference schemes depending on the parameter θ. In our previous paper [17] we considered the following approximation to the nonlinearity $\alpha f(u)$:

$$\tilde{g}\left(v^{k+1}, v^k, v^{k-1}\right) = \frac{F\left(v^{k+1}\right) - F\left(v^{k-1}\right)}{v^{k+1} - v^{k-1}}. \tag{8}$$

The second family of schemes (*Family 2*) is generated by equation (5) with g in the right-hand side replaced by $\tilde{g}$, (6) and (7). Some properties of this family of schemes are obtained in [17, 18]. Efficient algorithms (based on splitting procedures) for evaluation of the solution v^{k+1} from (5)–(7) can be found in [14, 17].

3. Analysis of the finite difference scheme

3.1. Discrete conservation law

In the space of functions which vanish on the boundary of Ω_h, we define operators $A = -\Delta_h$ and $B = I - \beta_1\Delta_h + \tau^2\theta\big(-\Delta_h + \beta_2(\Delta_h)^2\big)$. Here I stands for the identity operator. Note that these operators are self-adjoint and positive definite operators ($\theta \geq 0$ is required in the case of B). In the analysis of *Family 1*, we use the representation (4) and rewrite (5) in the operator form

$$Bv_{\bar{t}t} + Av + \beta_2 A^2 v = -Ag. \tag{9}$$

It is well known that the energy conservation law

$$E\left(u(\cdot,t)\right) = E\left(u(\cdot,0)\right), \qquad 0 < t \leq T,$$

is valid for the solution of BPE (1)–(3). Here the energy functional E is given by the equality

$$E\left(u(\cdot,t)\right) = \left\|(-\Delta)^{-1/2}\frac{\partial u}{\partial t}(\cdot,t)\right\|^2 + \beta_1\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|^2 + \|u(\cdot,t)\|^2 + \beta_2\|\nabla u(\cdot,t)\|^2 + 2\int_{\mathbb{R}^n} F(u(\cdot,t))\,dx, \tag{10}$$

where $\|\cdot\|$ stands for the standard norm in $L_2(\mathbb{R}^n)$. Theorem 3.1 below states that, as in the continuous case, the conservation property holds for the solution to the finite difference scheme (5)–(7). We define the linear functional $E_{h,L}\big(v^k, v^{k+1}\big)$ as

$$\begin{aligned} E_{h,L}\big(v^k, v^{k+1}\big) &= \big\langle A^{-1}v_t^k, v_t^k\big\rangle + \beta_1\big\langle v_t^k, v_t^k\big\rangle + \frac{1}{4}\Big\langle v^k + v^{k+1} + \beta_2 A\big(v^k + v^{k+1}\big), v^k + v^{k+1}\Big\rangle \\ &\quad + \tau^2\left(\theta - \frac{1}{4}\right)\big\langle (I+\beta_2 A)v_t^k, v_t^k\big\rangle \end{aligned} \tag{11}$$

and the full discrete energy functional $E_h\big(v^k, v^{k+1}\big)$ as

$$E_h\big(v^k, v^{k+1}\big) = E_{h,L}\big(v^k, v^{k+1}\big) + 2\Big\langle F\big((v^{k+1}+v^k)/2\big), 1\Big\rangle. \tag{12}$$

Multiplying (9) by $A^{-1}\big(v^{k+1} - v^{k-1}\big)$ and using summation by parts together with the boundary conditions (7) we prove

Theorem 3.1 (discrete conservation law).
The discrete energy (12) of the solution to the difference scheme (5)–(7) is conserved in time:

$$E_h\big(v^k, v^{k+1}\big) = E_h\big(v^0, v^1\big), \qquad k = 1, 2, \ldots, K-1.$$

Our calculations given in Section 5 confirm that the discrete energy functional $E_h\big(v^k, v^{k+1}\big)$ is preserved with a high accuracy.

We suppose in the following that the parameter θ of the scheme (5)–(7) satisfies

$$\theta \geq \frac{1+\epsilon}{4} - \frac{\beta_1}{\tau^2\|I + \beta_2 A\|} \tag{13}$$

for a positive number ϵ, independent of h and τ. Note that condition (13) ensures that the functional $E_{h,L}\big(v^k, v^{k+1}\big)$ is positive definite and can be viewed as an energy norm. Such combined norms depending on the values of function on several time layers are typical for the three-layers schemes.

3.2. Boundedness of the discrete solution

In this subsection we shall prove existence of a time interval $(0, T]$ such that the problem (5)–(7) admits for small time steps τ a bounded in the energy norm $E_{h,L}\left(v^k, v^{k+1}\right)$ discrete solution v^k on this time interval $(0, T]$. Here T is bounded by a number T^* depending only on values v^0, v^1 of the discrete solution at the first two time levels. We start with two technical results.

Lemma 3.2.

Suppose that the sequence w_k, $k = 1, 2, \ldots$, of positive numbers satisfies the recurrence formula

$$w_k - w_{k-1} \le A\tau w_{k-1} + B\tau w_{k-1}^q, \qquad k = 1, 2, \ldots, \quad w_0 > 0,$$

where A, B and τ are positive constants and $q > 1$. Then the following estimate holds:

$$w_k \le e^{Ak\tau}\left[\frac{1}{w_0^{q-1}} - (q-1)\frac{B\tau}{(1+A\tau)^{q-1}-1}\,\frac{e^{Ak\tau(q-1)}-1}{1+A\tau}\right]^{1/(1-q)} \tag{14}$$

for all k satisfying the inequality

$$k\tau < \frac{1}{A(q-1)}\ln\left(1 + \frac{(1+A\tau)((1+A\tau)^{q-1}-1)}{w_0^{q-1}(q-1)B\tau}\right). \tag{15}$$

The relation (14) is a discrete analog to the solution of a Bernoulli equation. Its proof can be found in [3]. The requirement for positiveness of the elements w_k leads to inequality (15) for the indexes k for which (14) holds. Discrete inequalities for the solutions to other discrete Bernoulli equations can be found in [21].

Lemma 3.3.

Let $b > 0$, $q > 1$, w satisfy

$$0 < w \le \frac{q-1}{q}\left(\frac{1}{qb}\right)^{1/(q-1)} \tag{16}$$

and x_0 be the smaller of the two positive roots of the equation

$$x - bx^q = w. \tag{17}$$

Then the following inequality holds:

$$x_0 \le w + b\mu_q w^q \qquad \textit{with} \quad \mu_q = \left(\frac{q}{q-1}\right)^q.$$

The proof of this lemma follows by analyzing differential properties of the function $x - bx^q$. The second inequality in (16) is a necessary and sufficient condition for existence of a positive solution to (17).

Now we shall prove an a priori estimate for the discrete solution to (5)–(7) for an appropriate time interval. Let D_p be the constant of the embedding of $W_2^1(\Omega_h)$ into $L_{2p}(\Omega_h)$ valid for the discrete functions and $p \ge 1$ (see (24) below). The constant D_p can be chosen independent on h, as shown in [22]. We set

$$a = \frac{\alpha}{2^{p+1}}, \qquad b = \frac{\alpha D_p^{2p}}{2^{p+1}\beta_1 \min\{1, \beta_2\}}. \tag{18}$$

Denote by $w_0 = E_{h,L}\left(v^0, v^1\right)$ the linear part of the discrete energy of equations (5)–(7) given by (11). We define the time

$$T^* = \frac{1}{a(p-1)} \ln\left(1 + w_0^{1-p}\right) \tag{19}$$

and fix a time T, $T = (K+1)\tau < T^*$. Then $e^{at(p-1)} - 1 < w_0^{1-p}$ follows for every $t \leq T$. Define also a constant

$$M = e^{aT}\left[w_0^{1-p} - \left(e^{aT(p-1)} - 1\right)\right]^{1/(1-p)}. \tag{20}$$

Let the time step τ be sufficiently small, e.g. satisfying the inequality

$$\tau < \min\left\{\frac{1}{6a(p-1)},\ \frac{M^{1-p}}{b(p-1)}\left(\frac{5}{12}\right)^p e^{-2/3},\ \sqrt{\beta_1} M^{(1-p)/2}\right\}. \tag{21}$$

Theorem 3.4.

Consider the finite difference scheme (5)–(7) (solved in accordance with Remark 2.1) with time step τ and determine the parameters a, b, T^*, T, K and M by (18)–(20). Let τ satisfy (21) and θ satisfy (13). Then the FDS admits a bounded solution v satisfying

$$E_{h,L}\left(v^k, v^{k+1}\right) \leq M, \qquad k = 1, 2, \ldots, K.$$

Proof. We establish boundedness of the discrete solution in three steps. For the first step we prove for $w_k = E_{h,L}\left(v^k, v^{k+1}\right)$ the inequality

$$w_k - \frac{b\tau}{1-a\tau} w_k^p \leq w_{k-1} \frac{1+a\tau}{1-a\tau} + \frac{b\tau}{1-a\tau} w_{k-1}^p, \qquad k = 1, 2, \ldots, K, \tag{22}$$

for the discrete solution v^k to (5)–(7) and constants a and b given by (18). We start with conservation of the full discrete energy on time levels k and $k-1$, valid from Theorem 3.1,

$$w_k + 2\left\langle F\left(\left(v^{k+1} + v^k\right)/2\right), 1\right\rangle = w_{k-1} + 2\left\langle F\left(\left(v^k + v^{k-1}\right)/2\right), 1\right\rangle \tag{23}$$

and estimate both nonlinear terms as follows:

$$\begin{aligned}\left\langle F\left(\left(v^{k+1} + v^k\right)/2\right), 1\right\rangle - \left\langle F\left(\left(v^k + v^{k-1}\right)/2\right), 1\right\rangle &= \frac{\alpha}{2^{p+1}(p+1)}\left\langle \left(v^{k+1} + v^k\right)^{p+1} - \left(v^k + v^{k-1}\right)^{p+1}, 1\right\rangle \\ &\leq \frac{\tau\alpha}{2^{p+1}}\left[\beta_1\left\|v_t^k\right\|^2 + \beta_1\left\|v_t^{k-1}\right\|^2 + \frac{1}{4\beta_1}\left\|\left(v^{k+1} + v^k\right)^p\right\|^2 + \frac{1}{4\beta_1}\left\|\left(v^k + v^{k-1}\right)^p\right\|^2\right].\end{aligned}$$

Further, we use the discrete embedding $W_2^1(\Omega_h) \to L_{2p}(\Omega_h)$ with a constant D_p:

$$\left\|\left(v^{k+1} + v^k\right)^p\right\|^2 \leq \left(D_p\left\|v^{k+1} + v^k\right\|_{W_2^1(\Omega_h)}\right)^{2p} \leq \frac{D_p^{2p}}{\min\{1, \beta_2\}} w_k^p. \tag{24}$$

From (23) and (24) and the inequalities derived above we get the inequality (with constants a, b given in (18))

$$w_k \leq w_{k-1} + a\tau w_k + a\tau w_{k-1} + b\tau w_k^p + b\tau w_{k-1}^p,$$

which is equivalent to (22).

In the second step of the proof we consider a new sequence of positive numbers W_k, $k = 1, 2, \dots, K$, given by

$$W_0 = w_0, \qquad W_k = W_{k-1}\frac{1+a\tau}{1-a\tau} + \frac{\tau b(1+\mu_p 3^p)}{1-a\tau}\, W_{k-1}^p, \qquad k = 1, 2, \dots, K,$$

with a constant μ_p given in Lemma 3.3. The condition (15) is fulfilled for $k \le K$ with $A = a$, $B = b(1+\mu_p 3^p)$ and $q = p$. Thus for any $k \le K$ we obtain the estimate

$$W_k \le e^{ak\tau}\left[\frac{1}{w_0^{p-1}} - (p-1)\frac{\tau b(1+\mu_p 3^p)}{\left(1+\tau b(1+\mu_p 3^p)\right)^{p-1} - 1}\,\frac{e^{ak\tau(p-1)}-1}{1+a\tau}\right]^{1(1-p)} \le M. \tag{25}$$

The first inequality in (25) follows from Lemma 3.2 and the second follows by straightforward calculations and (20).

Finally, in the third step of the proof, we get by induction that $w_k \le W_k$ for $k = 1, \dots, K$, which in view of (25) proves the theorem. Suppose that $w_k \le W_k$ for $k = 1, 2, \dots, k_0$. The next element of the sequence, w_{k_0+1}, satisfies (22) with the right-hand side

$$d_{k_0} = w_{k_0}\frac{1+a\tau}{1-a\tau} + \frac{b\tau}{1-a\tau}\, w_{k_0}^p. \tag{26}$$

From the restrictions on τ given in (21) and $w_{k_0} \le M$ it follows that

$$d_{k_0} \le \frac{p-1}{p}\left[\frac{1-\tau a}{p\tau b}\right]^{1/(p-1)},$$

which allows us to apply Lemma 3.3 and to obtain

$$w_{k_0+1} \le d_{k_0} + \mu_p\frac{\tau b}{1-\tau a}\, d_{k_0}^p. \tag{27}$$

We substitute d_{k_0} from (26) into (27). Then from the restrictions for τ given in (21) and the inequality $w_{k_0} \le W_{k_0}$ we conclude

$$w_{k_0+1} \le w_{k_0}\frac{1+a\tau}{1-a\tau} + \frac{\tau b(1+\mu_p 3^p)}{1-a\tau}\, w_{k_0}^p \le W_{k_0}\frac{1+a\tau}{1-a\tau} + \frac{\tau b(1+\mu_p 3^p)}{1-a\tau}\, W_{k_0}^p = W_{k_0+1}.$$

This completes the proof of the theorem. □

Theorem 3.4 establishes boundedness of the discrete solution to (5)–(7) in the energy norm generated by the linear functional $E_{h,L}\left(v^k, v^{k+1}\right)$ on the time interval $[0, T]$, where $T < T^*$ and T^* is given by (19). The critical time T^* depends on the input data (parameters of the nonlinearity p and α) and the linear part $w_0 = E_{h,L}\left(v^0, v^1\right)$ of the energy of initial data. The smaller the linear part of the initial energy, the longer the time interval for existence of the bounded discrete solution. Our calculations show that the maximal time of existence of the discrete solution could be much bigger than the time T^* given by (19).

As a consequence of boundedness of the discrete solution in the energy norm $E_{h,L}\left(v^k, v^{k+1}\right)$, the theorems for embedding of $W_2^1(\Omega_h)$ into $L_\infty(\Omega_h)$ for $n = 1, 2$ and condition (13) for θ, we get immediately boundedness of the discrete solution in discrete uniform norm (cf. hypothesis (ii) of Theorem 4.1).

Remark 3.5.

A similar analysis for existence and boundedness of the discrete solution on an appropriate time interval can be done for the conservative *Family 2* with the right-hand side (8).

4. Convergence of the numerical method

The key result of the paper is the following theorem for convergence of the solution v of the finite difference scheme to the exact solution u to BPE.

Theorem 4.1 (convergence of the method).
Assume that θ satisfies (13), $f(u) = u^p$, $p \in \mathbb{N}$, $p \geq 2$ and that

(i) *the solution u to BPE possesses bounded continuous derivatives up to the fourth order with respect to x and t on $\mathbb{R}^2 \times [0, T]$;*

(ii) *the solution v to the finite difference scheme (5)–(7) is bounded in the maximal norm on $\Omega_h \times [0, T]$, $T = K\tau$.*

Let M_0 be a constant such that

$$M_0 \geq \max_{\substack{|i|\leq N_1 \\ |j|\leq N_2,\, k\leq K}} \left(\left| u\left(x_i, y_j, t^k\right)\right|, \left|v_{i,j}^k\right| \right).$$

Then for $\tau < 0.5 \min \left\{0.25 M_0^{-2(p-1)}, \beta_1\right\}$ the discrete solution v converges to the exact solution u as $|h|, \tau \to 0$ and there exists a constant C (independent of h and τ) such that

$$\left\|v^k - u\left(\cdot, t^k\right)\right\| + \left\|A^{1/2}\left(v^k - u\left(\cdot, t^k\right)\right)\right\| \leq C e^{M_0^{p-1} k\tau} \left(|h|^2 + \tau^2\right), \qquad k = 1, 2, \ldots, K. \tag{28}$$

Proof. Denote by $z = v - u$ the error of the numerical solution. We substitute $v = z + u$ into (9) and obtain that z satisfies the following equation:

$$B z_{\bar{t}t} + Az + \beta_2 A^2 z = -Ag\left(v^{k+1}, v^k, v^{k-1}\right) - B u_{\bar{t}t} - Au - \beta_2 A^2 u. \tag{29}$$

Using equation (1) and the Taylor series expansion for u about the node $\left(x_i, y_j, t^k\right)$ it is straightforward to show that the right-hand side of (29) can be written as

$$-Ag\left(v^{k+1}, v^k, v^{k-1}\right) - B u_{\bar{t}t} - Au - \beta_2 A^2 u = -A\psi_1 + \psi_2$$

with $\psi_1^k = g\left(v^{k+1}, v^k, v^{k-1}\right) - \alpha f\left(u\left(t^k\right)\right)$ and $\psi_2 = O\left(|h|^2 + \tau^2\right)$. Using the Taylor series for u we estimate ψ_1^k by

$$\left|\psi_1^k\right| < C\left(M_1 \tau^2 + M_0^{p-1}\left(\left|z^{k+1} + z^k\right| + \left|z^{k-1} + z^k\right|\right)\right), \tag{30}$$

where M_1 depends only on the upper bound of the time derivatives of u. As in Theorem 3.1 we multiply (29) by $A^{-1}\left(z^{k+1} - z^{k-1}\right)$, use summation by parts and obtain

$$E_{h,L}\left(z^k, z^{k+1}\right) \leq E_{h,L}\left(z^{k-1}, z^k\right) + \tau\left\|\psi_1^k\right\|^2 + \tau\left\|A^{-1}\psi_2^k\right\|^2 + \tau\left\|z_t^k\right\|^2 + \tau\left\|z_t^{k-1}\right\|^2.$$

We substitute estimate (30) into the above inequality and get

$$E_{h,L}\left(z^k, z^{k+1}\right) \leq E_{h,L}\left(z^{k-1}, z^k\right) + \tau\left\|z_t^k\right\|^2 + \tau\left\|z_t^{k-1}\right\|^2 + \tau M_0^{2(p-1)}\left\|z^k + z^{k+1}\right\|^2 + \tau M_0^{2(p-1)}\left\|z^k + z^{k-1}\right\|^2 + \tau\left\|A^{-1}\psi_2^k\right\|^2.$$

Now we sum the above inequalities for $k = 1, 2, \ldots, K_0$, $K_0 \leq K$, and obtain

$$E_{h,L}\left(z^{K_0}, z^{K_0+1}\right) \leq E_{h,L}\left(z^0, z^1\right) + 2\tau \sum_{k=0}^{K_0} \left\|z_t^k\right\|^2 + 2\tau M_0^{2(p-1)} \sum_{k=0}^{K_0} \left\|z^k + z^{k+1}\right\|^2 + \tau \sum_{k=0}^{K_0} \left\|A^{-1}\psi_2^k\right\|^2. \tag{31}$$

The initial conditions (3) are approximated by (6) with error $O\left(|h|^2 + \tau^2\right)$ and $\psi_2 = O\left(|h|^2 + \tau^2\right)$. In view of (13) the linear functional $E_{h,L}$ is positive definite. The terms $\left\|z_t^{K_0}\right\|^2$ and $\left\|z^{K_0} + z^{K_0+1}\right\|^2$ from the right-hand side of (31) can be moved to the left-hand side of (31) because the time step τ is sufficiently small by hypothesis. Now the statement of the theorem, for $k = K_0$, follows by Gronwall's inequality. $\square$

Remark 4.2.

Sufficient conditions for hypothesis (i) of Theorem 4.1 are given in several papers, see e.g. [26, 28, 29], while Theorem 3.4 provides a sufficient condition for hypothesis (ii). Another way of verifying that the numerical solution remains bounded by a prescribed in advance constant M_0 is the direct verification of this condition at every time level during the computational process. Thus, the time interval $[0, T]$ in Theorem 4.1 could be different from the similar interval from Theorem 3.4.

Theorem 4.1 gives the second order of convergence of the FDS in discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem. Thus, the nonlinearity does not deteriorate the rate of convergence. Note that the uniform norms of the exact and discrete solutions are included in the exponent in the right-hand sides of the error estimate (28) in Theorem 4.1 and in the restriction for the parameter τ. Thus, if u blows up at a moment T_0 and if one would like to evaluate the solution in a neighborhood of the blow up moment, then the parameter τ should be chosen very small and in the fixed time interval the convergence of the finite difference scheme will slow up.

Corollary 4.3.

For $\theta > 1/4$ the convergence of the numerical solution to (5)–(7) to the exact solution to BPE is of the second order when $|h|$ and τ go independently to zero. For $\theta = 0$ the convergence of the numerical solution to the exact solution is of the second order when $|h|, \tau$ go to zero subject to the condition $9\beta_2\tau^2 < 4\beta_1 h^2$.

Combining estimate (28) with the theorems for embedding of $W_2^1(\Omega_h)$ into $L_\infty(\Omega_h)$ for $n = 1$ and $n = 2$, we get

Corollary 4.4.

Under the assumptions of Theorem 4.1 the finite difference scheme (5)–(7) admits the following error estimate in the uniform norm:

$$\max_{|i|\le N_1,\, k\le K} \left|v_i^k - u\left(x_i, t^k\right)\right| \le C e^{M_0^{p-1} k\tau}\left(|h|^2 + \tau^2\right), \qquad n = 1;$$

$$\max_{\substack{|i|\le N_1,\\ |j|\le N_2,\, k\le K}} \left|v_{i,j}^k - u\left(x_i, y_j, t^k\right)\right| \le C e^{M_0^{p-1} k\tau}\sqrt{\ln \max\left\{N_1, N_2\right\}}\left(|h|^2 + \tau^2\right), \qquad n = 2.$$

The above estimates are optimal in the one-dimensional case and almost *optimal (up to a logarithmic factor) in the two-dimensional case.*

Remark 4.5.

The above results remain true when the symmetric space domain Ω is replaced by a non-symmetric one.

5. Numerical results

In this section we compare the conservative scheme (5)–(7), denoted as *Family 1*, with the conservative *Family 2* with the approximation to the nonlinear term $\tilde{g}$ given by (8), proposed in [17], for the typical quadratic nonlinearity $f(u) = u^2$ in the one-dimensional case. The comparison of both families for nonlinearity $f(u) = u^3$ is given in our paper [14]. Both families are implicit with respect to the value v^{k+1} on the $k+1$ time level (functions g and $\tilde{g}$ depend on the unknown value v^{k+1}). Thus a standard iterative procedure for evaluation of v^{k+1} from (5) is required. The inner iterations stopped when the relative error between two consecutive iterations is found to be less that a threshold $\epsilon = 10^{-13}$. At each time level, four to five iterations are required to satisfy the stop condition, for moderate time step $\tau = 0.025$. We apply both conservative schemes with parameter $\theta = 0.5$. The parameters of BPE (1) in all cases are fixed to $\alpha = 3$, $\beta_1 = 1.5$ and $\beta_2 = 0.5$.

It is well known that for $n = 1$ BPE (1) possesses an analytical solution

$$\tilde{u}(x, t; x_0, c) = \frac{3}{2}\,\frac{c^2 - 1}{\alpha}\,\mathrm{sech}^2\left(\frac{x - x_0 - ct}{2}\sqrt{\frac{c^2 - 1}{\beta_1 c^2 - \beta_2}}\right),$$

which is a solitary wave traveling with velocity c and possessing a peak located at the point x_0 at the initial moment $t = 0$.

Example 1 (propagation of one solitary wave)

Consider BPE (1)–(3) with initial conditions

$$u(x,0) = \tilde{u}(x,0;0,2), \qquad \frac{du}{dt}(x,0) = \frac{d\tilde{u}}{dt}(x,0;0,2).$$

Table 1 shows the error δ_h defined as the maximal norm $\|\cdot\|_\infty$ of the difference between the exact solution u and the numerical solution v_h obtained by *Family 1* or *Family 2*, i.e. $\delta_h = \|\tilde{u} - v_h\|_\infty$. The numerical rate of convergence is evaluated as $\log_2(\delta_h/\delta_{h/2})$.

Table 1. Numerical comparison at time $T = 40$ of *Family 1* and *Family 2* for one solitary wave with velocity $c = 2$ and with $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $x \in [-40, 120]$.

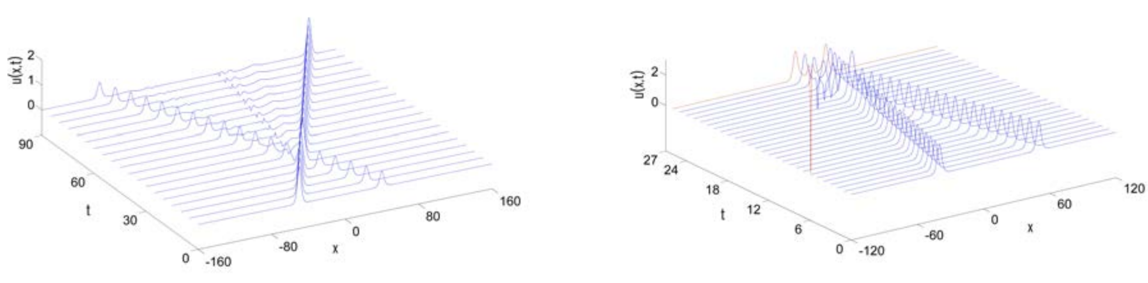
$h = \tau$	Error δ_h, *Family 1*	Rate, *Family 1*	Error δ_h, *Family 2*	Rate, *Family 2*	Error *Family 2*/Error *Family 1*
0.2	0.144106	–	0.265115	–	1.83
0.1	0.037527	1.9411	0.071849	1.8836	1.91
0.05	0.009478	1.9852	0.018315	1.9720	1.93
0.025	0.002376	1.9961	0.004601	1.9929	1.94
0.0125	0.000596	1.9961	0.001153	1.9966	1.93

Our calculations confirm that both families are of order $O(h^2 + \tau^2)$. The last column of Table 1 shows that for one and the same steps h and τ, *Family 1* is about two times more precise than *Family 2*.

Example 2 (interaction of two solitary waves)

The initial conditions in this case are:

$$u(x,0) = \tilde{u}\left(x,0;x_0^1,c_1\right) + \tilde{u}\left(x,0;x_0^2,c_2\right), \qquad \frac{du}{dt}(x,0) = \frac{d\tilde{u}}{dt}\left(x,0;x_0^1,c_1\right) + \frac{d\tilde{u}}{dt}\left(x,0;x_0^2,c_2\right).$$



(a) (b)

Figure 1. Interaction of two solitary waves, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $x_0^1 = -40$, $x_0^2 = 50$:
(a) $c_1 = 2$, $c_2 = -1.5$, $0 \le t \le 90$;
(b) $c_1 = -c_2 = 2.2$, $t^* \approx 27$, t^* – blow up time.

In Figure 1 two typical interactions of two solitary waves with different velocities are plotted at successive time steps. In Figure 1 (a) velocities of the waves are relatively small and both waves keep traveling preserving their shapes after the interaction. The interaction also generates additional smaller waves. In Figure 1 (b) the velocities are relatively big and the solution blows up after the collision increasing the absolute value of its amplitude.

Table 2. Numerical comparison at time $T = 80$ of *Family 1* and *Family 2* for the interaction of two solitary waves traveling with velocities $c_1 = 2$, $c_2 = -1.5$, and with $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $x_0^1 = -40$, $x_0^2 = 50$, $x \in [-160, 170]$.

$h = \tau$	Error δ_h, *Family 1*	Rate, *Family 1*	Error δ_h, *Family 2*	Rate, *Family 2*	Error *Family 2*/Error *Family 1*
0.1	–	–	–	–	–
0.05	0.066214	1.9819	0.126497	1.9634	1.91
0.025	0.016692	2.0000	0.032210	1.9931	1.93
0.0125	0.004034	2.1789	0.007785	2.1730	1.93

The same quantities as in Table 1 are shown in Table 2, for interaction with relatively small velocities. The error is calculated using the Runge method, as $\delta_h = \rho_{2h}^2/(\rho_{2h} - \rho_h)$, $\rho_h = \|v_h - v_{h/2}\|_\infty$. The calculations for the two solitary waves confirm that both families of schemes are of order $O\big(|h|^2 + \tau^2\big)$. The second observation in this case is that, as in the case of one soliton, the solution obtained by *Family 1* is about two times more precise than the solution obtained by *Family 2*. An explanation of this observation is the following one: both families differ by the approximation of the nonlinear term only. Using the Taylor series for $u(\cdot, t^k) = u^k$ about the time level t^k (the spatial argument is fixed) we find

$$g\big(u^{k+1}, u^k, u^{k-1}\big) - f\big(u^k\big) = \tau^2 R_1 + O(\tau^3), \qquad \tilde{g}\big(u^{k+1}, u^k, u^{k-1}\big) - f\big(u^k\big) = \tau^2 R_2 + O(\tau^3),$$
$$R_1 = \frac{1}{4}\,\alpha\,\frac{\partial f}{\partial u}\big(u^k\big)\,\frac{\partial^2 u}{\partial t^2}\big(t^k\big), \qquad R_2 = \frac{1}{2}\,\alpha\,\frac{\partial f}{\partial u}\big(u^k\big)\,\frac{\partial^2 u}{\partial t^2}\big(t^k\big).$$

Thus, the leading terms of the error for *Family 1* and *Family 2* are $\tau^2 R_1$ and $\tau^2 R_2$, respectively, with $R_2 = 2R_1$. This fact explains the better accuracy of the numerical solution to *Family 1* compared with the numerical solution to *Family 2*.

Remark 5.1.
The numerical experiments for one solitary wave and two solitary waves show that the discrete conservation law is maintained with high precision by both *Family 1* and *Family 2*. The discrete relative energy $E_{h,\mathrm{rel}}$ is defined as

$$E_{h,\mathrm{rel}} = \max_{0 \le k \le n-1} \frac{\big|E_h\big(v^k, v^{k+1}\big) - E_h\big(v^0, v^1\big)\big|}{E_h\big(v^0, v^1\big)}.$$

For one solitary wave, spatial step $h = 0.05$ and time $T = t^n = 40$, the discrete relative energy is $1.0 \cdot 10^{-10}$ and $1.01 \cdot 10^{-10}$ respectively for the *Family 1* and for the *Family 2*. Note that each family has its own discrete energy functional. The two discrete functionals do not coincide but they both approximate the energy functional (10) of the continuous problem.

6. Conclusion

In this paper a new family of conservative finite difference schemes is applied for numerical solving of multidimensional BPE. Existence and boundedness of the discrete solution on an appropriate time interval are established. The second order of accuracy in the W_2^1 mesh norm and in the uniform norm of the solution are proved theoretically and are confirmed numerically. Comparison of the new family of schemes with the family introduced in [17] is given. Even though the new schemes have the same properties (the conservativeness and the second order of convergence) as the family [17], the new schemes have clear advantages in precision of the solution – about two times smaller approximation error for a single soliton and for two colliding solitary waves.

Acknowledgements

This work is partially supported by the Bulgarian Science Fund under grant DDVU 02/71. The authors are very grateful to the referees for their valuable suggestions and remarks.

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