Comparison of Some Finite Difference Schemes for Boussinesq Paradigm Equation

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Abstract. The aim of the paper is to propose and study families of finite difference schemes for solving the Boussinesq Paradigm Equation. The nonlinear term of the equation is approximated in three different ways. We obtain a pair of implicit (with respect to the nonlinearity) families of schemes and an explicit one. All schemes have second rate of convergence in space and time. Numerical tests performed confirm our theoretical results regarding accuracy and convergence of all three schemes.

Keywords: Boussinesq Paradigm Equation, finite difference method, conservative schemes, solitons

1 Introduction

In this paper we consider the Cauchy problem for the Boussinesq Paradigm Equation (BPE)([7])

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u),\ x \in \mathbb{R}^n,\ 0 < t \le T < \infty, \tag{1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \tag{2}$$

$$u(x,t) \to 0, \quad \Delta u(x,t) \to 0 \ \text{ as } \ |x| \to \infty. \tag{3}$$

Here u is surface elevation, $\beta_1, \beta_2 \ge 0$, $\beta_1 + \beta_2 \ne 0$ are two dispersion coefficients, and α is an amplitude parameter. The nonlinear term $f(u)$ has a form $f(u) = u^p$, $p = 2, 3, \ldots$. BPE first appears in the modeling of surface waves in shallow waters. Subsequently, it has been applied to many other areas of mathematical physics dealing with wave phenomena, such as acoustic waves, ion-sound waves, plasma and nonlinear lattice waves, etc. (see [4, 7] and references therein). A great effort of research work has been invested in recent years for the theoretical and numerical study of the BPE. Most of the results are obtained for the one dimensional case (1D) and when $\beta_1 = 0$ or $\beta_2 = 0$. The numerical solutions are based on finite element methods, finite difference methods, spectral, pseudo-spectral methods, and Adomian decomposition methods – [1, 3, 9, 12] and references therein. Conservative finite difference schemes for (1), $p = 2$ are first proposed and applied in [4, 6]. In contrast, the study of 2D BPE is in its initial stage (see e.g. [4, 8]).

2 Finite Difference Schemes (FDS)

We propose the following families of FDS for the equation (1):

$$B\left(\frac{v_{ij}^{n+1} - 2v_{ij}^{n} + v_{ij}^{n-1}}{\tau^2}\right) - \Lambda v_{ij}^{n} + \beta_2 \Lambda^2 v_{ij}^{n} = \alpha \Lambda g(v_{ij}^{n+1}, v_{ij}^{n}, v_{ij}^{n-1}), \tag{4}$$

$$B = I - (\beta_1 + \theta\tau^2)\Lambda + \theta\tau^2\beta_2\Lambda^2. \tag{5}$$

Here v_{ij}^{n} is a discrete approximation to u at (x_i, y_j, t_n), τ is a time-step, I is the identity operator, $\Lambda = \Lambda^{xx} + \Lambda^{yy}$ is the standard five-point discrete Laplacian, $\Lambda^2 = (\Lambda^{xxxx} + 2\Lambda^{xxyy} + \Lambda^{yyyy})$ is the discrete biLaplacian. In approximations to Λv and $\Lambda^2 v$ we use the symmetric θ-weighted approximation to v_{ij}^{n}: $v_{ij}^{\theta,n} = \theta v_{ij}^{n+1} + (1-2\theta)v_{ij}^{n} + \theta v_{ij}^{n-1}$, $\theta \in \mathbb{R}$. We omit the notion $_{(ij)}$ whenever possible.

The function $g(v^{n+1}, v^n, v^{n-1})$ in (4) is an approximation to the nonlinear term $f(u)$. There are different possibilities to treat the nonlinear term. According to that we develop three families of schemes:

$$\text{Family 1:}\quad g(v^{n+1}, v^n, v^{n-1}) = f(v^n), \tag{6}$$

$$\text{Family 2:}\quad g(v^{n+1}, v^n, v^{n-1}) = \frac{F(v^{n+1}) - F(v^{n-1})}{v^{n+1} - v^{n-1}}, \tag{7}$$

$$\text{Family 3:}\quad g(v^{n+1}, v^n, v^{n-1}) = 2\frac{F(0.5(v^{n+1} + v^n)) - F(0.5(v^n + v^{n-1}))}{v^{n+1} - v^{n-1}}, \tag{8}$$

where $F(u) = \displaystyle\int_0^u f(s)\,ds$, and $f(u) = u^p$. In such a way we obtain an explicit with respect to the nonlinearity family of FDS (4), (6) (Family 1) and a pair of implicit with respect to the nonlinearity families of FDS (4), (7) (Family 2), and (4), (8) (Family 3).

The properties of some particular cases of these three families are studied theoretically in our papers [10, 11]. We have proved that all schemes considered above have second order of convergence in space and time. The schemes are unconditionally stable for $\theta \geq 1/4$. For $\theta < 1/4$ the schemes are conditionally stable. Thus, for $\theta = 0$ the schemes are stable provided $\tau^2 < \frac{4}{9}\frac{\beta_1}{\beta_2}h^2$.

An important feature of the proposed Families 2 and 3 is their conservativeness. We introduce the discrete energy functional $E_h(v^n)$:

$$\begin{aligned} E_h(v^n) = &-\left\langle \Lambda^{-1} v_t^n, v_t^n \right\rangle + \beta_1 \left\langle v_t^n, v_t^n \right\rangle + \tau^2(\theta - 1/4)\left\langle (I - \beta_2\Lambda)v_t^n, v_t^n \right\rangle \\ &+ 1/4\left\langle v^n + v^{n+1} - \beta_2\Lambda(v^n + v^{n+1}), v^n + v^{n+1} \right\rangle + \tilde{E}_h(v^n), \end{aligned}$$

where $\langle , \rangle$ is a scalar product at time t^n, $v_t^n = (v^{n+1} - v^n)/\tau$, $\tilde{E}_h(v^n)$ is a term corresponding to the approximation of the nonlinearity $g(v^{n+1}, v^n, v^{n-1})$:

$$\tilde{E_h}(v^n) = \begin{cases} \alpha \left\langle F(v^{n+1}) + F(v^n), 1 \right\rangle, & \text{for Family 2,} \\ 2\alpha \left\langle F(0.5(v^{n+1} + v^n)), 1 \right\rangle, & \text{for Family 3.} \end{cases}$$

It is proven in [10, 11] that $E_h(v^{(n)}) = E_h(v^{(0)}), \quad n = 1, 2, \ldots$, i.e. the discrete energy is conserved in time.

Using the regularization method [13] we replace the operator B in (5) by the factorized operator $\tilde{B}$

$$\tilde{B} = (I - \theta\tau^2 \Lambda^{xx} + \theta\tau^2 \beta_2 \Lambda^{xxxx})(I - \theta\tau^2 \Lambda^{yy} + \theta\tau^2 \beta_2 \Lambda^{yyyy})(I - \beta_1 \Lambda).$$

In such a way we get stable factorized scheme with the same properties as the initial one. The main advantage of the factorized scheme is that it can be reduced to a sequence of three simpler schemes provided appropriate boundary conditions. Moreover, the factorized scheme leads to an economic algorithm, i.e. an algorithm with a linear complexity with respect to the number of nodes. Let us emphasize that the first two discrete operators of $\tilde{B}$ depend only on one spatial variable. This justifies the need to study the properties of the proposed schemes in the one dimensional case. Our aim is to analyze the proposed FDS for 1D BPE in terms of their rate of convergence, accuracy as well as the energy preservation of the conservative schemes.

3 Numerical method for 1D BPE

For the discretization of 1D BPE a regular mesh is used in the interval $[-L_1, L_2]$, $x_i = -L_1 + ih$, $h = (L_1 + L_2)/N$, $i = 0, \ldots, N$. An $O(h^2 + \tau^2)$ approximation to the initial conditions (2) is given by

$$\begin{aligned} v_i^0 &= u_0(x_i), \\ v_i^1 &= u_0(x_i) + \tau u_1(x_i) + 0.5\tau^2 B^{-1} \left(\Lambda^{xx}(u_0) - \beta_2 \Lambda^{xxxx}(u_0) + \alpha (f(u_0))_{xx} \right)(x_i). \end{aligned}$$

We consider (4) subject to the following boundary conditions

$$v_0^{n+1} = v_N^{n+1} = 0, \qquad v_{xx,0}^{n+1} = v_{xx,N}^{n+1} = 0. \tag{9}$$

In order to approximate the second boundary conditions in (9) the mesh is extended outside the interval $[-L_1, L_2]$ by one point and the symmetric second-order finite difference is used.

The nonlinear schemes Families 2 and 3 are linearized using successive iterations. In our calculations we use v^n as an initial approximation to the sought function v^{n+1}. The iterations stop when the relative error between two successive iterations is less than a given tolerance ε. The resulting systems of linear algebraic equations are five-diagonal with constant matrix coefficients. To solve them we apply a special kind of nonmonotonic Gaussian elimination with pivoting proposed in [5].

4 Numerical tests and discussion

In this section the accuracy and the convergence of the proposed FDS are studied. We also demonstrate the stability of FDS for different nonlinearities $f(u) = u^p$. Following [2] it can be shown that the 1D BPE admits a one parameter family of soliton solutions given by

$$u^s(x,t;c) = \left[\frac{(c^2-1)(p+1)}{2\alpha}\mathrm{sech}^2\left(\frac{1-p}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right)\right]^{\frac{1}{p-1}}, \quad p \neq 1,$$

where c is a phase velocity of the localized wave.

The proposed families of schemes are tested on two problems typical for the quadratic nonlinearity:
(i) Propagation of a Solitary Wave: In this case we consider the following initial data: $u(x,0) = u^s(x,0;c)$, $u_t(x,0) = u^s_t(x,0;c)$. It represents a single soliton at the initial time moment located at $x = 0$ that is then allowed to evolve according to the BPE.
(ii) Interaction of Two Solitary Waves: The following initial conditions:

$$\begin{aligned} u(x,0) &= u^s(x+x_0^1,0;c_1) + u^s(x-x_0^2,0;c_2),\\ u_t(x,0) &= u^s_t(x+x_0^1,0;c_1) + u^s_t(x-x_0^2,0;c_2) \end{aligned}$$

are used. At the initial time we take the superposition of two solitons initially located at points $x = -x_0^1$ and $x = x_0^2$ and traveling one against another with velocities c_1 and c_2.

The convergence and accuracy of the proposed schemes are numerically analyzed using embedded grids. Let us denote by $v_{[h]}$, $v_{[h/2]}$, and $v_{[h/4]}$ the solutions obtained on three embedded grids. Then we define the rate of convergence κ and the error R by

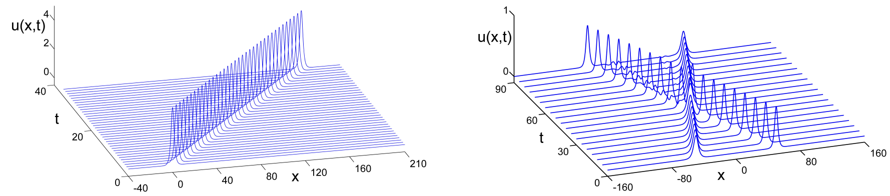
$$(i): \kappa = \log_2\left(\frac{||u^s - v_{[h]}||}{||u^s - v_{[h/2]}||}\right), \qquad \mathrm{R} = ||u^s - v_{[h]}||,$$

$$(ii): \kappa = \log_2\left(\frac{||u_{[h]} - u_{[h/2]}||}{||u_{[h/2]} - u_{[h/4]}||}\right), \mathrm{R} = \frac{(||u_{[h]} - u_{[h/2]}||)^2}{||u_{[h]} - u_{[h/2]}|| - ||u_{[h/2]} - u_{[h/4]}||},$$

where $||\cdot||$ is the maximum norm.

To evaluate the conservativeness of Family 2 and Family 3 we introduce a relative discrete energy E_r at time t_n: $E_r = \max_{0\le k\le n} \frac{|E_h(v^k) - E_h(v^0)|}{E_h(v^0)}$.

Numerical experiments are conducted for $p = 2, 3, 4$, and 5 ($f(u) = u^p$) and for variety values of the parameters β_1, β_2, α, and c. First, we investigate the propagation of the single soliton with different velocities c. Our numerical tests confirm the fact that the schemes capture the evolution of solitons correctly even for higher velocity. The snapshots of the solution with cubic nonlinearity and $c = 5$ computed at different evolution times are plotted on Fig. 1(a). The rest of parameters are $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$.



(a) (b)

Fig. 1. $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$: (a)– One solitory wave, $c = 5$, $0 \le t \le 40$; (b)– Interaction of two solitory waves, $c_1 = 1.1$, $c_2 = -1.3$, $0 \le t \le 80$.

Table 1 shows the rate of convergence κ and the errors R for one solitary wave solution with cubic nonlinearity $f(u) = u^3$, and $c = 2$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $x \in [-40, 120]$ for evolution time $T = 40$. One can see that the second rate of convergence is reached for all three families of schemes. The solutions obtained by Family 1 are almost 9 times more precise than ones obtained by Family 2, and 4.5 times more precise than ones obtained by Family 3. If one compares the conservative schemes, then Family 3 is about two times more precise than Family 2. The above observation fully confirms the theoretical result obtained in our paper [11]. For the example shown in Table 1 when $h = \tau = 0.025$ we compute the relative energy E_r at the evolution time $T = 40$. For both conservative Family 2 and Family 3 we obtain $E_r \approx 1 \times 10^{-9}$. The rate of convergence and the errors for the interaction between two solitary waves with cubic nonlinearity $f(u) = u^3$ are presented in Table 2. The results are obtained for $c_1 = 1.1$, $c_2 = -1.3$, $x_0^1 = x_0^2 = 50$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $x \in [-160, 160]$ for evolution time $T = 80$ (Fig. 1(b)). The observations concerning the rate of convergence and accuracy in this case are similar to the one solitary wave case discussed above.

Table 1. One solitary wave solution, $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $x \in [-40, 120]$, $\theta = 0.5$, $\varepsilon = 10^{-13}$, $T = 40$.

$h = \tau$	Error R			Rate κ		
	Family 1	Family 2	Family 3	Family 1	Family 2	Family 3
0.2	0.0944117	0.6079979	0.3483821	–	–	–
0.1	0.0227430	0.1841944	0.0949054	2.05	1.72	1.88
0.05	0.0056362	0.0480465	0.0242024	2.01	1.94	1.97
0.025	0.0014057	0.0121329	0.0060802	2.00	1.99	1.99
0.0125	0.0030431	0.0030431	0.0015243	2.01	1.99	2.00
0.00625	0.0000472	0.0008006	0.0004205	2.89	1.93	1.86

Table 2. Interaction of two solitary waves, $f(u) = u^3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = 1.1$, $c_2 = -1.3$, $x \in [-160, 160]$, $\theta = 0.5$, $\varepsilon = 10^{-13}$, $T = 80$.

$h = \tau$	Error R			Rate κ		
	Family 1	Family 2	Family 3	Family 1	Family 2	Family 3
0.1	0.0197330	0.1038638	0.0663246	1.97	1.90	1.94
0.05	0.0050042	0.0273100	0.0170770	1.99	1.98	1.99
0.025	0.0012437	0.0069027	0.0042895	2.04	2.00	2.01

The numerical tests confirm that all families of schemes are of order $O(h^2 + \tau^2)$. With respect to the error magnitude Family 1 performs much better than Family 2 and Family 3. One may expect that this is due to the smoothness of the initial data. If the initial data are not smooth enough, then the conservative Families 2 and 3 should be chosen. In this case Family 3 is preferable since it has higher accuracy than Family 2. The numerical experiments demonstrate a high reliability of the proposed schemes for the most studied case of quadratic nonlinearity $f(u) = u^2$. Our experience dealing with various nonlinearities confirms expectations of different phenomenology in the cases $f(u) = u^{2k+1}$ and $f(u) = u^{2k}$, $k \geq 1$ (global existence and blow-up of the solution). The studied families of FDS could be efficiently applied for the 2D BPE.

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