Convergence of Finite Difference Schemes for a Multidimensional Boussinesq Equation

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Abstract. *Conservative finite difference schemes* for the numerical solution of multi-dimensional Boussinesq-type equations are constructed and studied theoretically. Depending on the way the nonlinear term $f(u)$ is approximated, two families of finite difference schemes are developed. Error estimates for these numerical methods in the uniform metric and the Sobolev space W_2^1 are obtained. The extensive numerical experiments given in [7] for the one-dimensional problem show good precision and full agreement between the theoretical results and practical evaluation for single soliton and the interaction between two solitons.

1 Introduction

1.1. Consider the Cauchy problem of the Boussinesq type equation (BE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^d, \; t > 0; \tag{1}$$

$$u(x,0) = u_0(x); \quad \frac{\partial u}{\partial t}(x,0) = u_1(x); \quad u(x,t) \to 0, \; \Delta u(x,t) \to 0, \; |x| \to \infty,$$

where f is a smooth non-linear function, say $f(u) = u^2$, the amplitude parameter α is a real number and the dispersion parameters β_1 and β_2 are positive constants. BE (1) occurs in a number of mathematical models of real processes, for example, in the modeling of surface waves in shallow water. The essentials of the derivation of (1) from the full Boussinesq model can be found, e.g. in [3].

BE (1) called in [3] “Boussinesq Paradigm Equation” and similar BE, called “good BE”, “damped BE”, “improved BE”, “generalized double dispersion equation”, have been studied by many authors in the case of one dimensional (1D) space variable x (i.e. $d = 1$). The existence (both local and global in time) and uniqueness of weak and strong solutions in Sobolev spaces for the 1D problem are treated in [8,13,14]. Sufficient conditions for blow-up of the solution are given in [6,13]. Numerical solutions based on finite difference methods, spectral and pseudo-spectral methods and finite element methods can be found in [3,5,8,10,11].

The multidimensional version of BE (i.e. $d > 1$) is less studied. The dependence of existence, smoothness and blow-up of the solution on the nonlinear function $f(u)$ is investigated in [14,15] for isotropic Sobolev spaces and in [12] for specially designed anisotropic Sobolev spaces. The numerical investigation of the 2D BE is also in its initial stage (see e.g. [1,2]).

In the present paper we study two families of finite difference schemes (FDS) for numerical computation of the multidimensional BE introduced in [7]. They differ on the way the approximation of the nonlinear term $\Delta f(u)$ is done. In Section 3 we show that one of the FDS retains an important property – the conservation law of the solution to the initial BE, while the other obeys a proper balance equation and demonstrates smaller approximation errors in experiments.

Section 4 contains error estimates for both FDS in the uniform metric and in the Sobolev space W_2^1 on the fixed time layer, as well as a number of corollaries and comments. The main results are contained in the convergence theorems 4 and 5. We establish second order of convergence for both FDS in the discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem. The convergence of both schemes (in the 1D case) is demonstrated in [7] on two basic examples of one solitary wave and interaction of two solitary waves traveling with different speeds towards each other. A variant of the proposed 2D FDS is implemented in [4].

Other FDS properties connected with the algorithms for their implementation can be found in [7]. Here we only mention that both FDS can be split as pairs of an elliptic and a hyperbolic 2D discrete equations, thus, their numerical solutions can be efficiently evaluated with stable algorithms.

1.2. By the linear change of variables $\frac{1}{\sqrt{\beta_1}}x = \xi$, $\frac{\sqrt{\beta_2}}{\beta_1}t = \theta$ equation (1) is rewritten in the form

$$\frac{\partial^2 U}{\partial \theta^2} = \Delta U + \Delta \frac{\partial^2 U}{\partial \theta^2} - \Delta^2 U + \Delta \frac{\beta_1}{\beta_2}\left(\alpha f(U) + \left(1 - \frac{\beta_2}{\beta_1}\right) U\right),$$

with $U(\xi,\theta) = u(x,t)$. Therefore, without loss of generality, we shall study the following problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \Delta \frac{\partial^2 u}{\partial t^2} - \Delta^2 u + \Delta g(u), \quad x \in \mathbb{R}^d, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \mathbb{R}^d, \tag{3}$$

$$u(x,t) \to 0, \quad \Delta u(x,t) \to 0, \quad |x| \to \infty, \quad t > 0, \tag{4}$$

where g is connected to f by

$$g(u) = \frac{\beta_1}{\beta_2}\left(\alpha f(u) + \left(1 - \frac{\beta_2}{\beta_1}\right) u\right).$$

We assume in this paper that the solution u to problem (2) – (4) belongs to $C^{6,4}\left(\mathbb{R}^d \times (0,T)\right)$. Here $C^{m,n}\left(\mathbb{R}^d \times (0,T)\right)$ denotes the space of continuous functions with continuous derivatives up to order m with respect to x and order n with respect to t. The existence of a classical (local or global) solution with the smoothness prescribed above is proved in the 1D case in [14], while for the multi-dimensional case similar results for *local* solutions are established in [15].

2 Numerical method

The numerical methods described here work for any space dimension. For simplicity we present them in the case $d = 2$.

Let L_1, L_2 be sufficiently large numbers. We consider the discrete problem in the computational domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, assuming that the solution with its derivatives is negligible outside this domain. We introduce a uniform grid with steps h_1, h_2 in Ω and let τ denote the uniform time step. The grid points are (x_i, y_j, t_k), where $x_i = ih_1, i = -N_1, \ldots, N_1$; $y_j = jh_2, j = -N_2, \ldots, N_2$; $t_k = k\tau, k = 0, 1, 2, ...$ with $N_1 = L_1/h_1$, $N_2 = L_2/h_2$. The discrete approximation to u at mesh point (x_i, y_j, t_k) is denoted by $v^{(k)}_{(i,j)}$. In the following, whenever possible, we omit the notation ${}^{(k)}_{(i,j)}$ for the arguments of the mesh function v.

By the symbol C with different indexes we shall denote positive constants, which does not depend on parameters h, τ, γ, σ and on the functions u_0, u_1, g, u, v. By the symbol M with different indexes we shall denote positive constants, which depend on the norms of the functions u, v.

The standard 5-point discrete Laplacian is denoted by Δ_h. The finite difference approximation to the second time derivative is

$$v^{(k)}_{\bar{t}t,(i,j)} = \left(v^{(k+1)}_{(i,j)} - 2v^{(k)}_{(i,j)} + v^{(k-1)}_{(i,j)} \right) \tau^{-2}.$$

For a real parameter σ denote by v^σ the symmetric σ-weighted approximation to $v^{(k)}_{(i,j)}$ given by $v^{\sigma(k)}_{(i,j)} = \sigma v^{(k+1)}_{(i,j)} + (1 - 2\sigma) v^{(k)}_{(i,j)} + \sigma v^{(k-1)}_{(i,j)}$. We apply approximations with parameter σ to the purely spatial operators Δ_h and $(\Delta_h)^2$ in (2). The simplest way to approximate $g(v)$ at (x_i, y_j, t_k) is to take $g(v^{(k)}_{(i,j)})$. Thus, at interior grid points we obtain a first family of finite difference methods depending on the parameter σ

$$v_{\bar{t}t} - \Delta_h v_{\bar{t}t} - \Delta_h v^\sigma + (\Delta_h)^2 v^\sigma = \Delta_h g(v). \tag{5}$$

Another well known approximation to the nonlinear term at (x_i, y_j, t_k) is

$$g_1(v^{(k)}_{(i,j)}) = \frac{G(v^{(k+1)}_{(i,j)}) - G(v^{(k-1)}_{(i,j)})}{v^{(k+1)}_{(i,j)} - v^{(k-1)}_{(i,j)}}, \quad \text{where } G(u) = \int_0^u g(s)\, ds. \tag{6}$$

Note that in the classical case $f(u) = u^2$ the function g is a second degree polynomial and the anti-derivative G used in g_1 is explicitly evaluated. In this way we get the second family of finite difference schemes

$$v_{\bar{t}t} - \Delta_h v_{\bar{t}t} - \Delta_h v^\sigma + (\Delta_h)^2 v^\sigma = \Delta_h g_1(v). \tag{7}$$

An $O(|h|^2 + \tau^2)$ approximation to the initial conditions (3) is given by

$$v^{(0)}_{(i,j)} = u_0(x_i, y_j), \tag{8}$$

$$\begin{aligned} v^{(1)}_{(i,j)} = {} & u_0(x_i, y_j) + \tau u_1(x_i, y_j) + \\ & 0.5\, \tau^2 (I - \Delta_h)^{-1} \left(\Delta_h u_0 - (\Delta_h)^2 u_0 + \Delta_h g(u_0) \right) (x_i, y_j). \end{aligned} \tag{9}$$

For the approximation of the second boundary condition the mesh is extended outside the domain Ω by one line at each space boundary and the symmetric second-order finite difference is used for the approximation of the second spatial derivative in (4).

Equations (5) or (7) with initial conditions (8), (9) and boundary conditions described above form two families of finite difference schemes indexed by σ. The efficient algorithms for evaluation of their solutions are given in [7].

3 Discrete identities

For given time moment t_k we consider the space of mesh functions $v^{(k)}$ which vanish at the points on the boundary of Ω and we define the operator $A = -\Delta_h$. In this space denote by $\langle v^{(k)}, w^{(k)} \rangle = \sum_{i,j} h_1 h_2 v^{(k)}_{(i,j)} w^{(k)}_{(i,j)}$ the discrete scalar product of mesh functions $v^{(k)}, w^{(k)}$ with respect to the spatial variables.

In the space of functions, which satisfy both asymptotic conditions on the computational boundary (2) we define the operator $B = (I + A)(I + \sigma\tau^2 A)$. Note that A and B are self-adjoint positive definite operators.

For the analysis of difference schemes, we use the representation $v^\sigma = v + \sigma\tau^2 v_{\bar{t}t}$ and rewrite the equations (5) and (7) in the operator form

$$Bv_{\bar{t}t} + Av + A^2 v = -Ag, \tag{10}$$

$$Bv_{\bar{t}t} + Av + A^2 v = -Ag_1. \tag{11}$$

Following [7], we first define the functional E_h^L given by

$$\begin{aligned}(E_h^L v)^{(k)} &= \left\langle A^{-1/2} v_t^{(k)}, A^{-1/2} v_t^{(k)} \right\rangle + \tau^2(\sigma - 1/4) \left\langle (I + A) v_t^{(k)}, v_t^{(k)} \right\rangle \\ &+ \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + 1/4 \left\langle v^{(k)} + v^{(k+1)} + A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle .\end{aligned}$$

and then, by incorporating the non-linear term g_1, the full discrete "energy" functional

$$(E_h v)^{(k)} = (E_h^L v)^{(k)} + \left\langle G(v^{(k+1)}), 1 \right\rangle + \left\langle G(v^{(k)}), 1 \right\rangle .$$

The following theorems are proved in [7]:

Theorem 1 (Discrete conservation law) *The discrete "energy" $(E_h v)^{(k)}$ of the solution v to the scheme (11) is preserved in time, i.e. it satisfies the equalities*

$$(E_h v)^{(k)} = (E_h v)^{(0)}, \qquad k = 1, 2, \ldots . \tag{12}$$

The discrete balance law (12) valid for the solution to the scheme (11) fully corresponds to the energy equation [14] valid for the solution to the initial problem (2)–(4). The scheme (10) does not have a strict conservation of the discretized energy functional $(E_h v)^{(k)}$, but it satisfies similar *balance identities* given below.

Theorem 2 *The solution to the scheme (10) satisfies the equalities*

$$(E_h^L v)^{(k)} - (E_h^L v)^{(k-1)} + \left\langle g(v^k), v^{(k+1)} - v^{(k-1)} \right\rangle = 0, \qquad k = 1, 2, \ldots . \tag{13}$$

4 Convergence of the FDS

4.1 Analysis of the linear problem

We begin with the analysis of the following discrete linear problem

$$Bv_{\bar{t}t} + Av + A^2 v = -A\psi_1 + \psi_2, \tag{14}$$

where ψ_1 and ψ_2 are given functions. The initial conditions to (14) are (8) and (9) with v_0, v_1 on the place of u_0, u_1 and $-A\psi_1 + \psi_2$ on the place of $-Ag(u_0)$.

Using the stability theory from [9], Chapter 6, we get the following theorem:

Theorem 3 *Let γ be a positive real number. Assume that for some steps h and τ the parameter σ satisfies the inequality*

$$\sigma > \frac{1+\gamma}{4} - \frac{1}{\tau^2 ||A||}. \tag{15}$$

Then the finite difference method (14), (8), (9) is stable with respect to the initial data and the right-hand side. Moreover, the following estimate holds:

$$\begin{aligned}\left(v^{(k)}, v^{(k)}\right) + \left(Av^{(k)}, v^{(k)}\right) \leq C\frac{1+\gamma}{\gamma} \Bigg[\left(Bv^{(0)}, v^{(0)}\right) + \left(A^{-1}Bv_t^{(0)}, A^{-1}Bv_t^{(0)}\right) \\ + \sum_{s=1}^{k-1} \tau \left(\psi_1^{(s)}, \psi_1^{(s)}\right) + \sum_{s=1}^{k-1} \tau \left(A^{-1}\psi_2^{(s)}, A^{-1}\psi_2^{(s)}\right) \Bigg]. \end{aligned} \tag{16}$$

4.2 Convergence of the FDS's for the non-linear problem

Now we are ready to study the convergence of FDS. We begin with FDS (10) assuming for the smoothness of the non-linear term $g \in W^1_\infty(\mathbb{R})$. Denote by $z = v - u$ the error of the solution. We substitute $v = z + u$ into the problem (10) and obtain the following problem for the error z:

$$Bz_{\bar{t}t} + Az + A^2 z = -Ag(v) - Bu_{\bar{t}t} - Au - A^2 u. \tag{17}$$

Now we use the equation (2) and Taylor series for the function u about the node (x_i, y_j, t_k). It is straightforward to show that

$$-Ag(v) - Bu_{\bar{t}t} - Au - A^2 u = -A\psi_1 + \psi_2$$

with $\psi_1 = g(v) - g(u)$, $\psi_2 = O(|h|^2 + \tau^2)$. Thus, we get that (17) has the form of (14) and we can apply Theorem 3. We estimate ψ_1 by $|g(v^{(k)}) - g(u(t_k))| \leq M^{(k)}|z^{(k)}|$ with a constant $M^{(k)}$ chosen so that $\max_{i,j}(|u(x_i, y_j, t_k)|, |v_{i,j}^{(k)}|) \leq M^{(k)}$. Also (8) and (9) approximate the initial conditions (3) locally with $O(|h|^2 + \tau^2)$ error. In this way we get

$$\begin{aligned}\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \\ \leq C\frac{1+\gamma}{\gamma} \left(C_1(|h|^2 + \tau^2)^2 + \sum_{s=1}^{k-1} \tau M^{(s)} \left(z^{(s)}, z^{(s)}\right) \right). \end{aligned} \tag{18}$$

Proceeding by induction on k if we assume the boundedness of $z^{(s)}$ for $s = 1, 2, \ldots, k-1$ we shall obtain from (18) that $|z_{i,j}^{(k)}|$ is bounded and hence $|v_{i,j}^{(k)}|$ is bounded whenever $\|u(\cdot, \cdot, t_k)\|$ is bounded.

Now we use the Gronwall's lemma and conclude

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq \frac{1+\gamma}{\gamma} C e^{M t_k} \left(|h|^2 + \tau^2\right)^2 \tag{19}$$

with $M = \max_k M^{(k)}$. In this way we proved the following theorem

Theorem 4 *Assume $g \in W^1_\infty(\mathbb{R})$, the parameter σ satisfies (15) for some $\gamma > 0$ and the solution u to the problem (2) – (4) obey $u \in C^{6,4}\left(\mathbb{R}^2 \times (0,T)\right)$. Then the solution v to the finite difference scheme (10), (8), (9) converges to u as $|h|, \tau \to 0$ and the estimate (19) holds for the error $z = y - u$ of the scheme.*

Now we turn to FDS (11) assuming for the smoothness of the non-linear term $g \in W^2_\infty(\mathbb{R})$. We may use the same arguments as in the previous scheme, but taking into account that ψ_1 is different. Here

$$\psi_1^{(s)} = \frac{G(v^{(s+1)}) - G(v^{(s-1)})}{v^{(s+1)} - v^{(s-1)}} - g(u(t_s)).$$

We first expand $G(v^{(s+1)})$ in Taylor series about the point $v^{(s-1)}$ and then we expand $g(v^{(s-1)}) = G'(v^{(s-1)})$ in Taylor series about the point $u(t_s)$. Thus, we get

$$|\psi_1^{(s)}| < C\left(M_1^{(s)}\tau^2 + M_2^{(s)}\left(|z^{(s-1)}| + |z^{(s)}| + |z^{(s+1)}|\right)\right),$$

where $M_2^{(s)}$ is a constant satisfying

$$M_2^{(s)} \geq \max_{i,j}\left(|u(x_i, y_j, t_s)|, \left|\frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_s)\right|, |v_{i,j}^{(s-1)}|, |v_{i,j}^{(s)}|, |v_{i,j}^{(s+1)}|\right).$$

Now Theorem 3 gives

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq C_2 \frac{1+\gamma}{\gamma}\left(C_1(|h|^2 + \tau^2)^2 + \sum_{s=1}^{k} \tau M_2^{(s)}\left(z^{(s)}, z^{(s)}\right)\right).$$

The above inequality differs from (18) by the term containing $(z^{(k)}, z^{(k)})$ in the right-hand side. If τ is sufficiently small, say $\tau \leq 0.5\gamma\left(C_2(1+\gamma)M_2^{(s)}\right)^{-1}$, then this term can be moved to the left-hand side and, thus, we see that $z^{(k)}$ satisfies (18) (with a bigger constant C). Using once more the Gronwall's lemma we obtain the following result:

Theorem 5 *Assume $g \in W^2_\infty(\mathbb{R})$ and the parameter σ satisfies (15) with some $\gamma > 0$. Assume that the solution u to (2) – (4) obeys $u \in C^{6,4}\left(\mathbb{R}^2 \times (0,T)\right)$ and* *the solution v to the finite difference scheme (11), (8), (9) is bounded in the maximal norm. Let M be a constant such that*

$$M \geq \max_{i,j,s} \left(|u(x_i, y_j, t_s)|, \left| \frac{\partial^2 u}{\partial t^2}(x_i, y_j, t_s) \right|, |v_{i,j}^{(s)}| \right)$$

and τ be sufficiently small, $\tau < \gamma \left(C_2(1+\gamma)M\right)^{-1}$. Then v converges to the exact solution u as $|h|, \tau \to 0$ and the following estimate holds for the error $z = y - u$:

$$\left(z^{(k)}, z^{(k)}\right) + \left(Az^{(k)}, z^{(k)}\right) \leq \frac{1+\gamma}{\gamma} C e^{Mt_k} \left(|h|^2 + \tau^2\right)^2. \tag{20}$$

The assumption in Theorem 5 for boundedness of the discrete solution could be dropped. It can be derived from the other assumptions by proving separately that the iterative process for obtaining $v^{(k+1)}$ from (11) is convergent. The proof uses that some mappings are contractive as in [14]. Here we skip the proof due to its length. We underline that the other difference between Theorems 4 and 5 – the hypothesis for the upper estimate on τ in Theorem 5 – is essential.

4.3 Corollaries

The main feature of Theorems 4 and 5 is the established second order of convergence in discrete W_2^1 norm, which is compatible with the rate of convergence of the similar linear problem.

Corollary 1 *(i) The convergence of the solution to FDS (10) or FDS (11) with $\sigma > 0.25$ to the exact solution is of second order when $|h|$ and τ go* independently *to zero.*

(ii) The convergence of the solution to the explicit *FDS (10) or FDS (11) with $\sigma = 0$ to the exact solution is of second order when $|h|$ and τ go to 0 provided: $\tau < \frac{|h|}{\sqrt{1+\gamma}}$ for the 1D problem or $\tau < \frac{|h|}{\sqrt{2(1+\gamma)}}$ for the 2D case.*

The error estimates obtained in Theorems 4 and 5 are in the discrete W_2^1 norm on the $t^{(k)}$ time layer. Using embedding theorems for the uniform norm we derive

Corollary 2 *Under the assumptions of Theorems 4 or 5 the FDS (10) or (11) admits the following error estimate in the uniform norm:*

$$\max_i |z_i^{(k)}| < Ce^{Mt_k} \sqrt{\frac{1+\gamma}{\gamma}} \left(|h|^2 + \tau^2\right), \qquad d = 1;$$
$$\max_{i,j} |z_{i,j}^{(k)}| < Ce^{Mt_k} \sqrt{\ln N} \sqrt{\frac{1+\gamma}{\gamma}} \left(|h|^2 + \tau^2\right), \qquad d = 2.$$

The above estimates are optimal for the 1D case and *almost* optimal (up to a logarithmic factor) for the 2D case.

One of the main assumptions in Theorems 4 and 5 is the boundedness of the exact solution u to the BE on the time interval $[0, T]$. Such assumption is natural because the BE may have both bounded on the time interval $[0, \infty)$ solutions and blowing up solutions. The L_∞ norm of the solution is included in the exponent in the right-hand sides of the error estimates in Theorems 4 and 5. Hence, if u blows up at a moment T_0 which is slightly bigger than T, then $\|u\|_{L_\infty[0,T]}$ will be big and, hence, the term e^{MT} will be big and the convergence will slow up. Additional, but not so important restriction on the time step τ, is the upper bound in Theorem 5 containing the reciprocal of $\|u\|_{L_\infty[0,T]}$. In any case the FDS should be applied with very small τ's if one would like to evaluate the solution in a neighborhood of the blow up moment.

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