$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u),$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

$$u(x,t) \to 0, \quad \Delta u(x,t) \to 0, \ |x| \to \infty$$

$$B\left(\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2}\right) - (\Lambda^{xx} + \Lambda^{yy})u^n + \beta_2(\Lambda^{xxxx} + \Lambda^{yyyy} + 2\Lambda^{xxyy})u^n =$$

$$\alpha(\Lambda^{xx} + \Lambda^{yy})F(u^{n+1}, u^n, u^{n-1})$$

$$B = I - (\beta_1 + \sigma\tau^2)(\Lambda^{xx}\Lambda^{yy} + \Lambda^{yy}) + s_1\tau^2\beta_2(\Lambda^{xxxx} + \Lambda^{yyyy}) + 2s_2\tau^2\beta_2\Lambda^{xxyy}$$

$\sigma, \quad s_1, \quad s_2$ - parameters

$$\tilde{B} = (I - \sigma\tau^2\Lambda^{xx} + s_1\tau^2\beta_2\Lambda^{xxxx})(I - \sigma\tau^2\Lambda^{yy} + s_1\tau^2\beta_2\Lambda^{yyyy})[I - \beta_1(\Lambda^{xx} + \Lambda^{yy})]$$

$$u_{tt} = u_{xx} + \beta_1 u_{ttxx} - \beta_2 u_{xxxx} + \alpha(u^2)_{xx}$$

$$x \in [-L_1, L_2], \quad Nh = (-L_1 + L_2), \quad x_i = ih, \; i = 1, \ldots, N-1$$

$$D\left(\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}\right) = Qu_i^n + \alpha\Lambda^{xx}(u_i^n)^2, \text{ or}$$

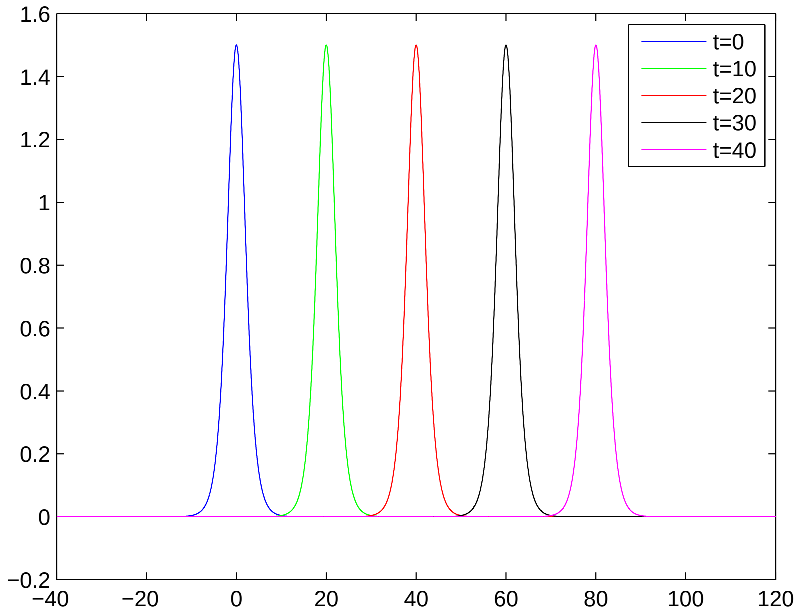
$$D\left(\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}\right) = Qu_i^n + \alpha\Lambda^{xx}[(u_i^{n+1})^2 + u_i^{n+1}u_i^{n-1} + (u_i^{n-1})^2]/3$$

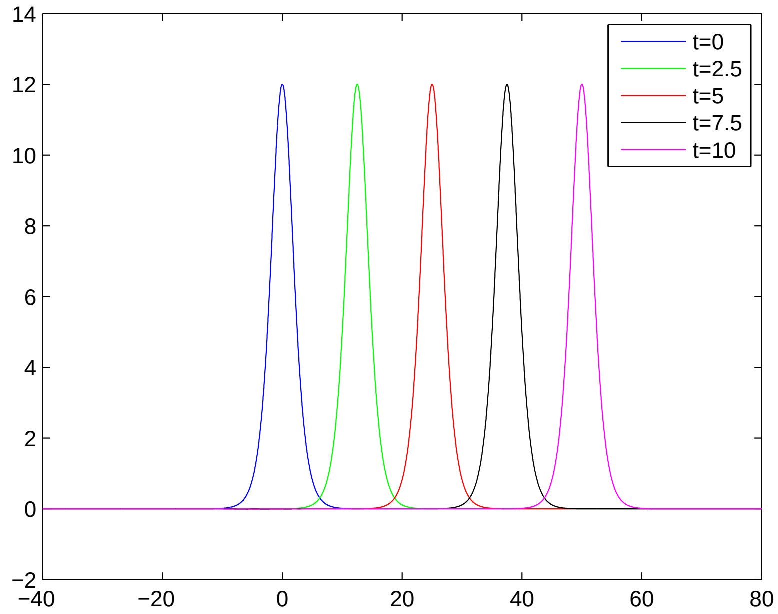
$$D = (I - \beta_1\Lambda^{xx} - \sigma\tau^2\Lambda^{xx} + \beta_2 s\tau^2\Lambda^{xxxx}), \qquad Q = \Lambda^{xx} - \beta_2\Lambda^{xxxx}$$

$$u_0^{n+1} = u_N^{n+1} = 0, \quad u_{xx,0}^{n+1} = u_{xx,N}^{n+1} = 0$$

$$\Lambda^{xx}u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \qquad \Lambda^{xxxx}u_i = \frac{u_{i+2} - 4u_{i+1} + 6u_i - 4u_{i-1} + u_{i-2}}{h^4}$$

$$\tilde{u}(x,t;c) = \frac{3}{2}\frac{(c^2-1)}{\alpha}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right)$$





c=2 c=5

Figure 1: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$; $\sigma = s = 0.5$

Table 1: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $t = 40$

$h = \tau$	Rate M1	Rate M2	Error M1	Error M2
0.2	-	-	0.0111339769	0.2651151649
0.1	2.0403	1.8836	0.0027068659	0.0718490891
0.05	2.0094	1.9720	0.0006723417	0.0183149097
0.025	2.0033	1.9929	0.0001677038	0.0046012352
0.0125	2.0578	1.9966	0.0000402788	0.0011529897

$$E_1 = ||\tilde{u} - u_{[h]}||, \quad E_2 = ||\tilde{u} - u_{[h/2]}|| \quad \text{Rate} = \log_2(E_1/E_2)$$

$$\text{Error} = \max_{0 \le i \le N} |\tilde{u}_i - u_{[h],i}|$$

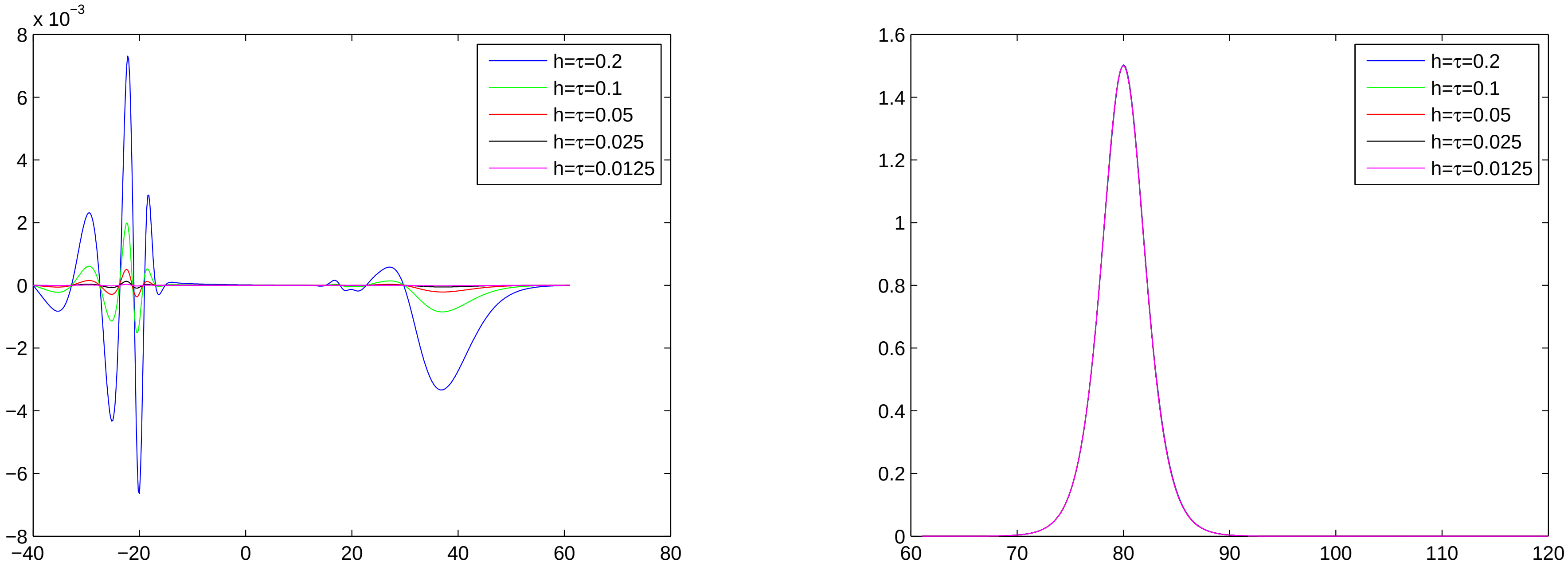


Figure 2: Method 1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $\sigma = s = 0.5$, $t = 40$

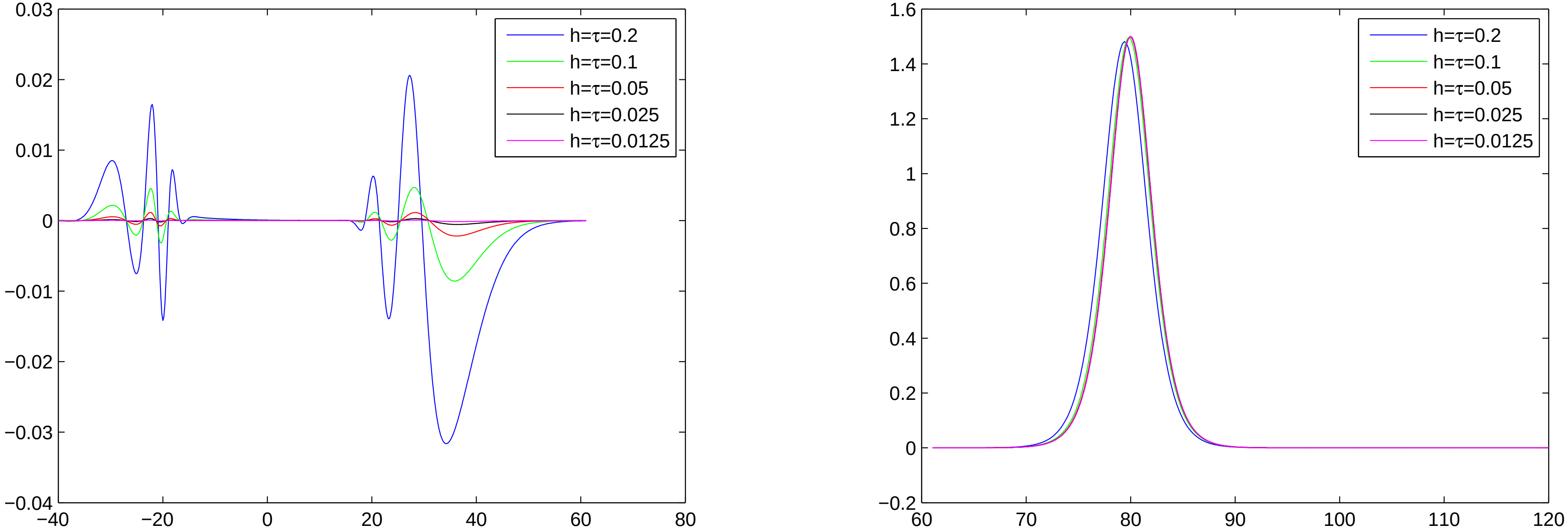


Figure 3: Method 2, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $\varepsilon = 10^{-13}$, $\sigma = s = 0.5$, $t = 40$

$E(t^j)$ - discrete "energy" for $t = t^j$

$E_{error} = \max\limits_{1\le j\le n} \dfrac{|E(t^j) - E(0)|}{E(0)}$ - relative error

$t^n = 40$

$$h = \tau = 0.2 \qquad E_{error} = 2.09546213767 \times 10^{-12}$$

$$h = \tau = 0.1 \qquad E_{error} = 3.92258760110 \times 10^{-11}$$

$$h = \tau = 0.05 \qquad E_{error} = 5.45994876224 \times 10^{-10}$$

$$h = \tau = 0.025 \qquad E_{error} = 4.60008425216 \times 10^{-9}$$

$$h = \tau = 0.0125 \qquad E_{error} = 8.15913691568 \times 10^{-8}$$

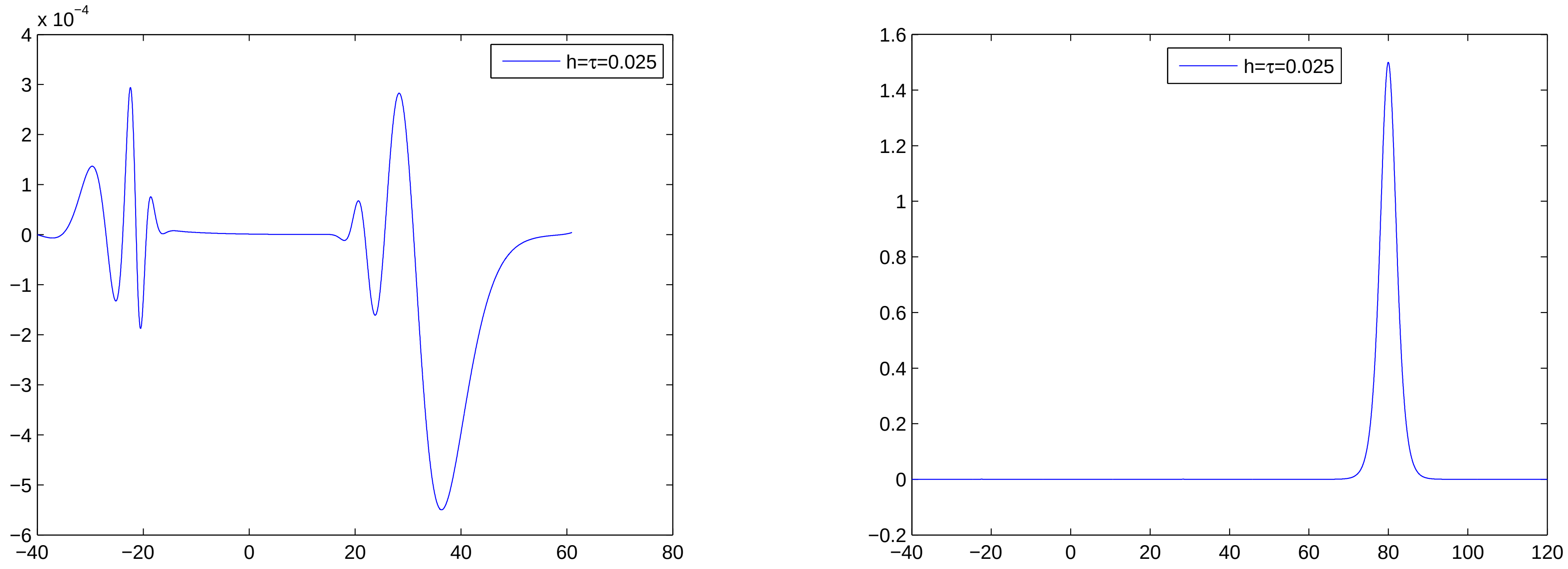


Figure 4: Method 2, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $\varepsilon = 10^{-13}$, $\sigma = s = 0.5$, $t = 40$

$$
\begin{aligned}
&E(0) \;\; = 64.232254483436, && \\
&E(10) = 64.232254263785, && E_{osc}(10) = 0.45\% E(10), \\
&E(20) = 64.232254188254, && E_{osc}(20) = 0.89\% E(20), \\
&E(30) = 64.232254258030, && E_{osc}(30) = 1.16\% E(30), \\
&E(40) = 64.232254471093, && E_{osc}(40) = 1.59\% E(40)
\end{aligned}
$$

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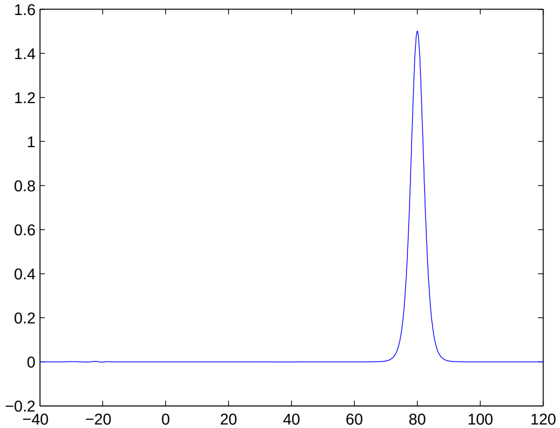
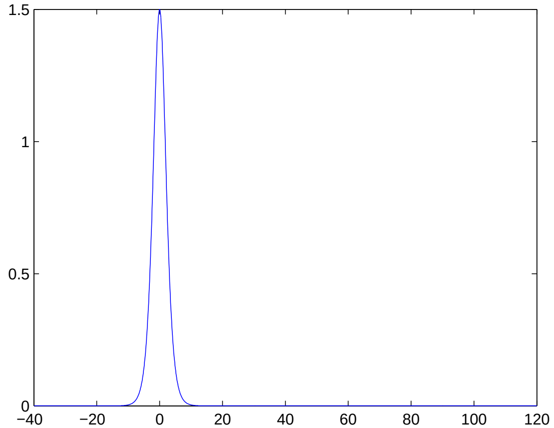
- A.G. Bratsos, A predictor-corrector scheme for the improved Boussinesq equation, *Chaos, Solitons and Fractals*, 40 (2009), 2083-2094.

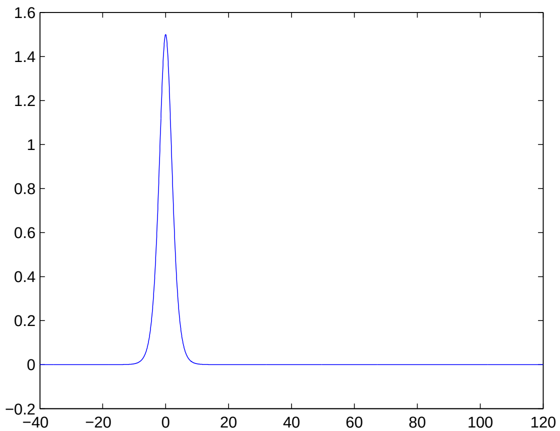
- H. El-Zoheiry, Numerical study of the improved Boussinesq equation, *Chaos, Solitons and Fractals*, 14 (2002) 377384.

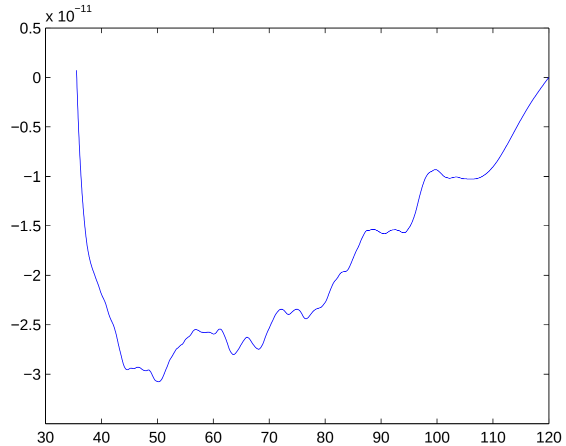


$t_0 = 0, t = 40, \tau = h = 0.1$



$t_0 = 40, t = 0, \tau = -h = -0.1$





$t_0 = 40, t = 0, \tau = -h = -0.05$

Figure 5: Method 1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$

$$\tilde{u}(x,t;c) = \frac{3}{2}\frac{(c^2-1)}{\alpha}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2-\beta_2}}(x-ct)\right)$$

$$\begin{aligned} u(x,0) &= \tilde{u}(x+40,0;c_1)+\tilde{u}(x-50,0;c_2) \\ \frac{\partial u}{\partial t}(x,0) &= \tilde{u}(x+40,0;c_1)_t+\tilde{u}(x-50,0;c_2)_t \end{aligned}$$

Example 1: $c_1 = 1.2, c_2 = -1.5,\ h = \tau = 0.025,\ 0 \le t \le 100.$

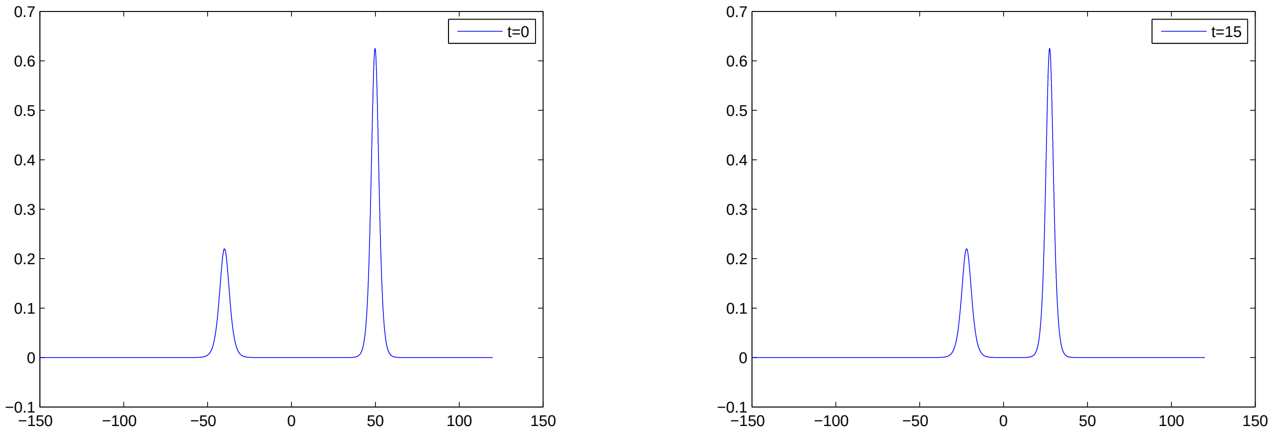


Figure 6: Method1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = 1.2$, $c_2 = -1.5$.

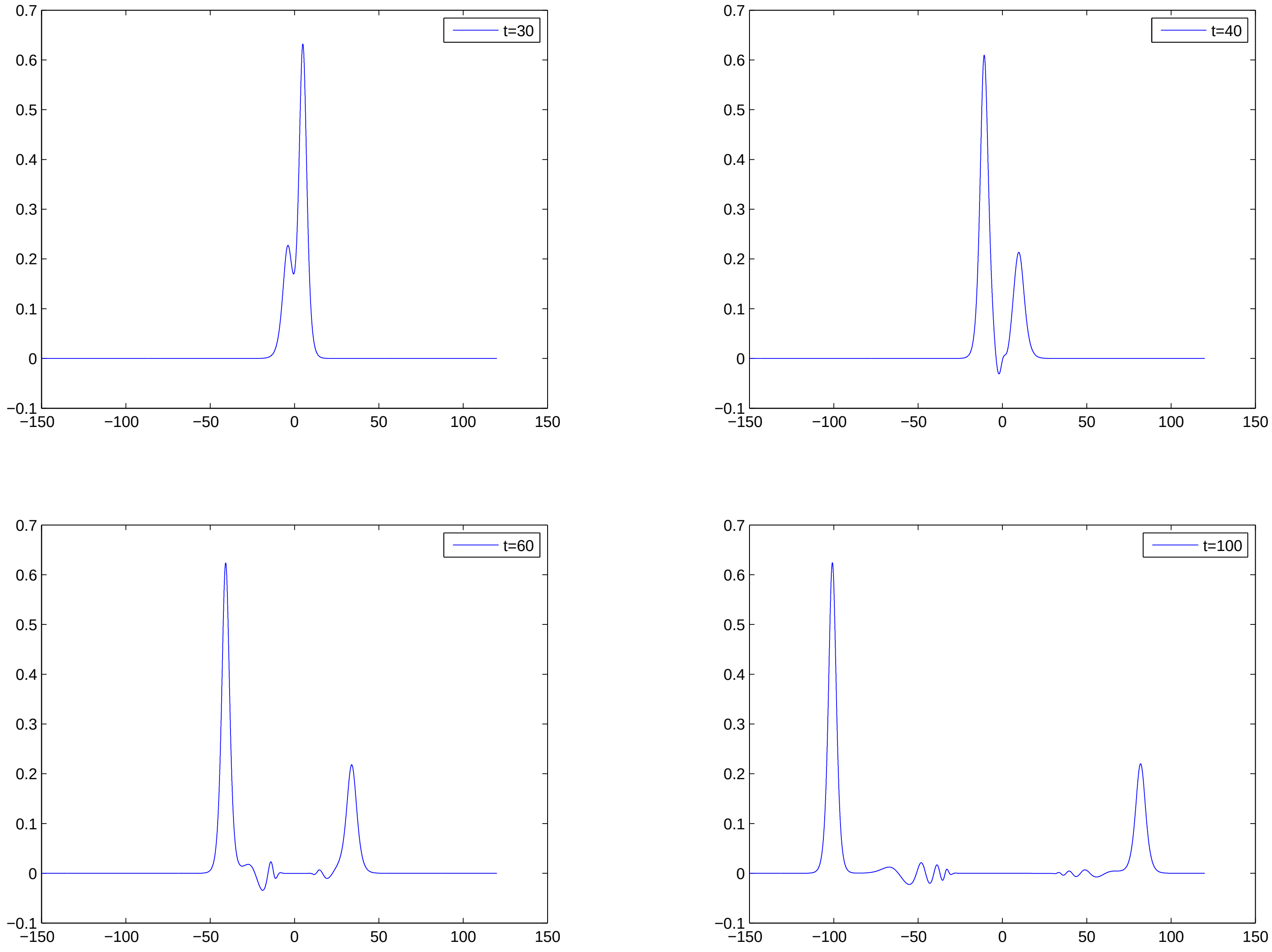


Figure 7: Method1,$\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c_1 = 1.2,\ c_2 = -1.5.$

Example 2: $c_1 = 2, c_2 = -1.5,\ h = \tau = 0.025,\ 0 \le t \le 90.$

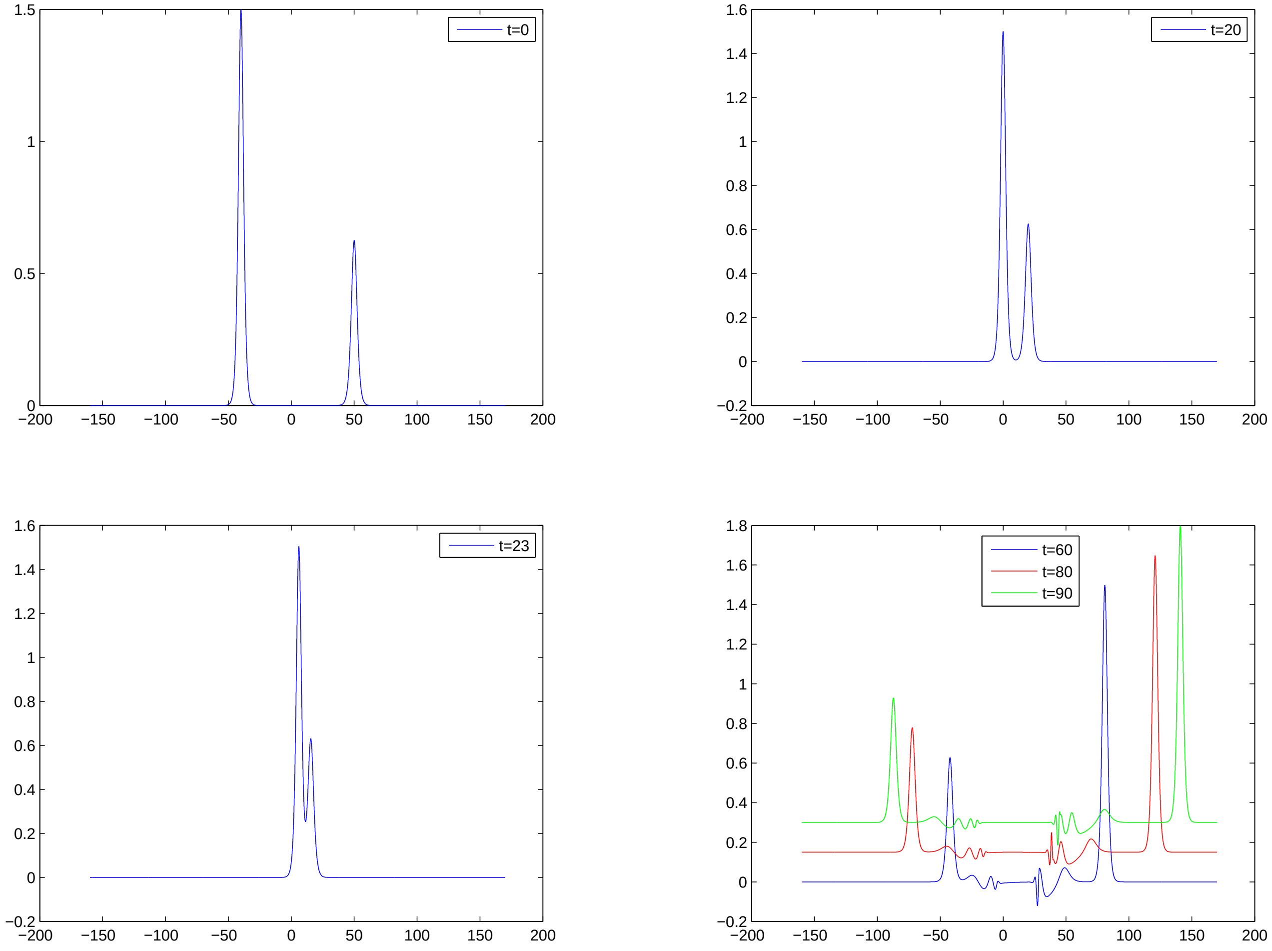


Figure 8: Method1, $\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c_1 = 2,\ c_2 = -1.5.$

Table 2: $\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c_1 = 2,\ c_2 = -1.5,\ t = 80$

$h = \tau$	Rate M1	Rate M2	Error M1	Error M2
0.1	-	-	-	-
0.05	2.1088	1.9634	0.0452458182	0.1264965870
0.025	2.0285	1.9931	0.0106743314	0.0322099785
0.0125	1.9428	2.1730	0.0026694457	0.0077848462
0.00625	-	-	-	-

$$\text{Error} = E_1^2/(E_1 - E_2), \quad E_1 = ||u_{[h]} - u_{[h/2]}||, \quad E_2 = ||u_{[h/2]} - u_{[h/4]}||,$$

$$\text{Rate} = \log_2(E_1/E_2)$$

Example 3: $c_1 = 2.2, c_2 = -2.2,\ h = \tau = 0.025,\ 0 \le t \le 27.$

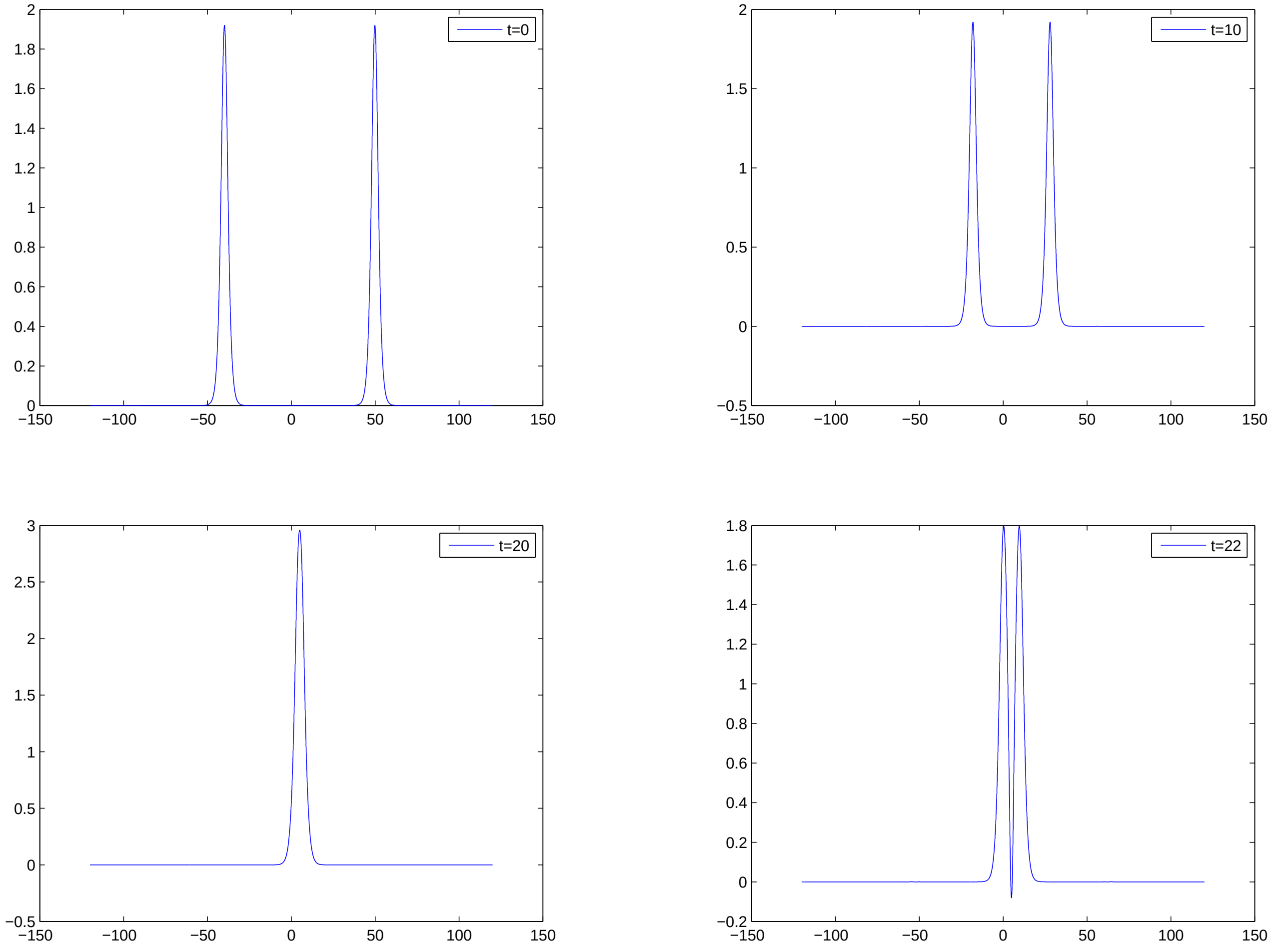


Figure 9: Method1, $\beta_1 = 1.5,\ \beta_2 = 0.5,\ \alpha = 3,\ c_1 = 2.2,\ c_2 = -2.2.$

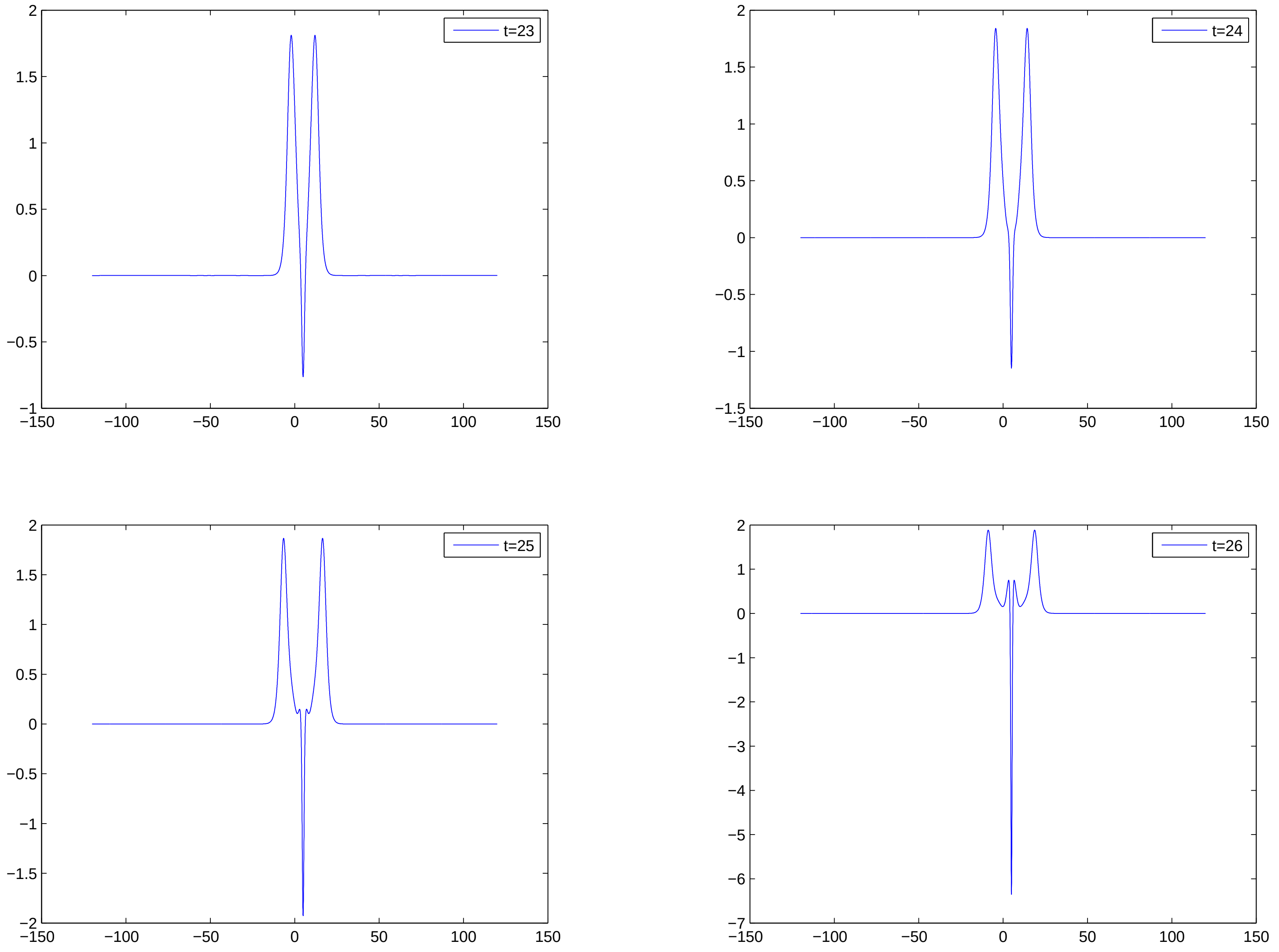


Figure 10: Method1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c_1 = 2.2$, $c_2 = -2.2$.

Arbitrary Initial Data

$$u(x,0) = \mathrm{sech}^2(0.5x), \qquad u_t(x,0) \equiv 0$$

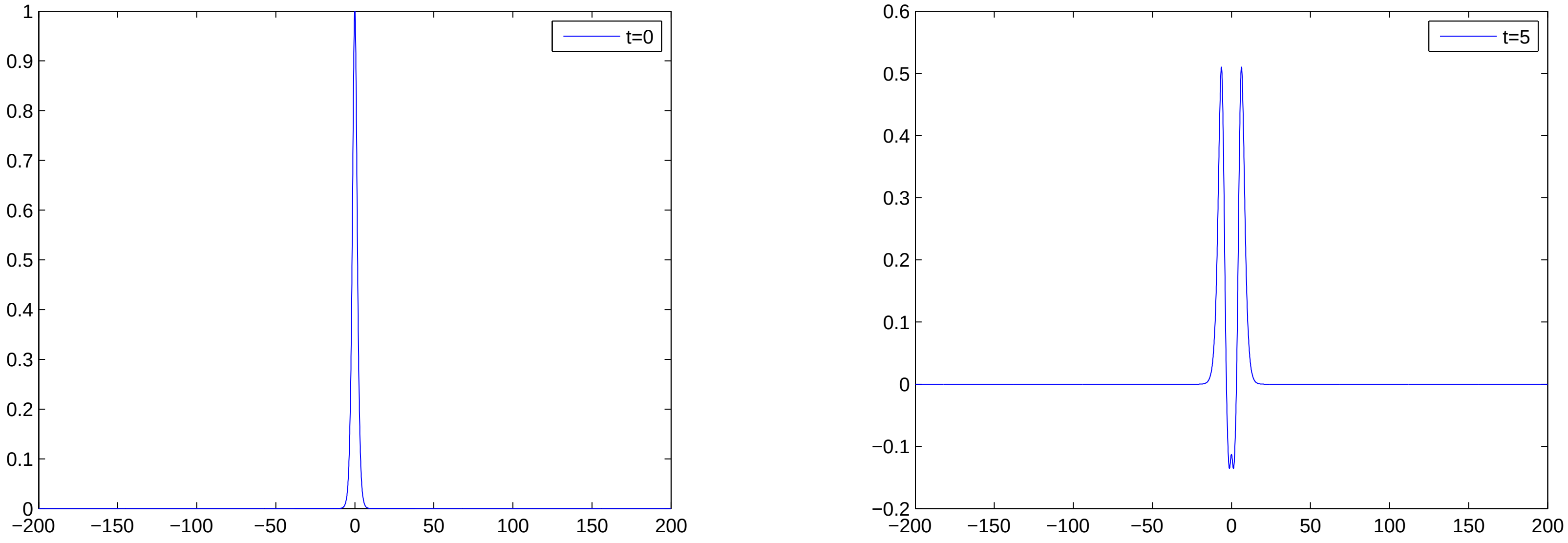


Figure 11: Method1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$.

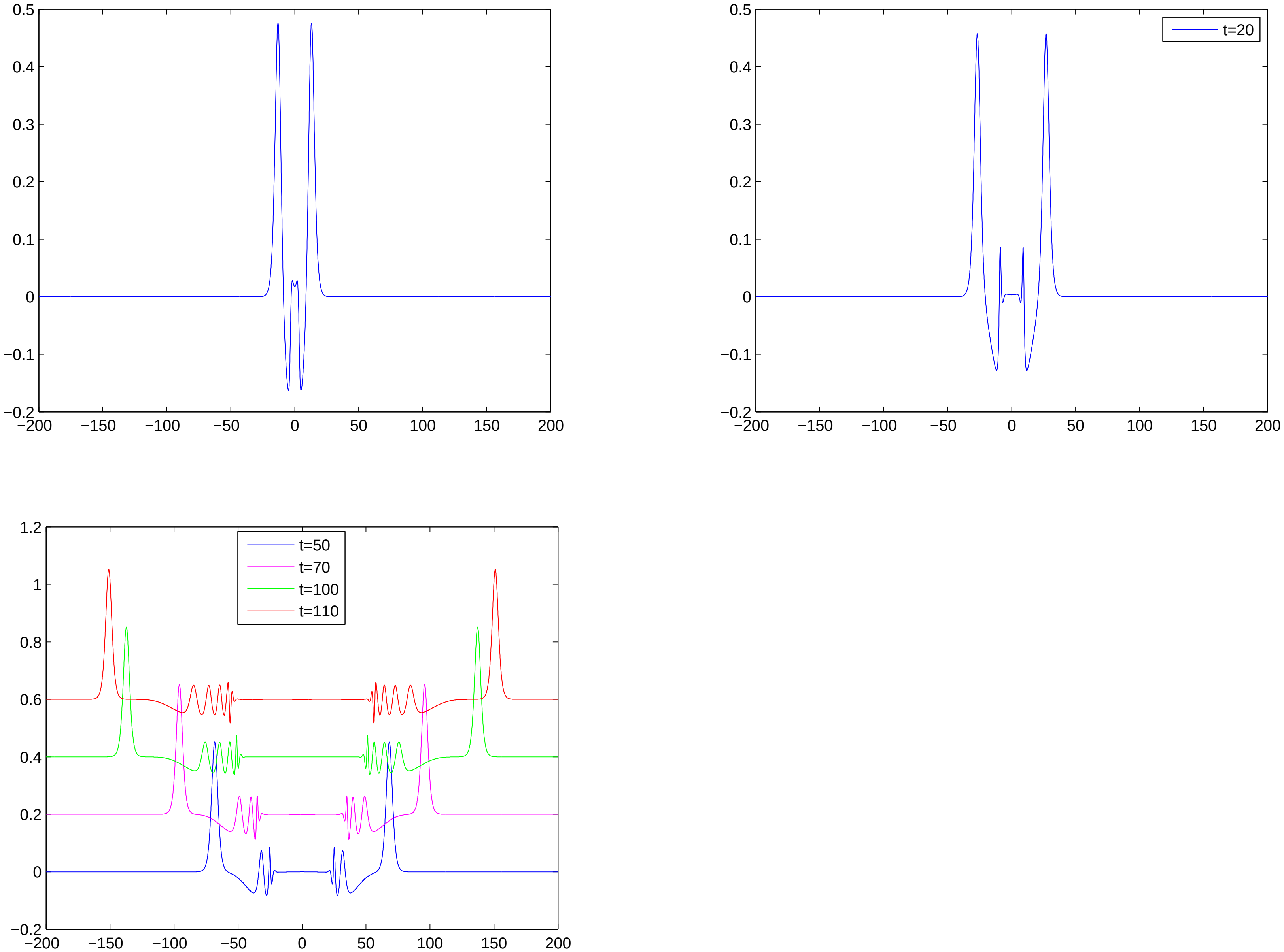
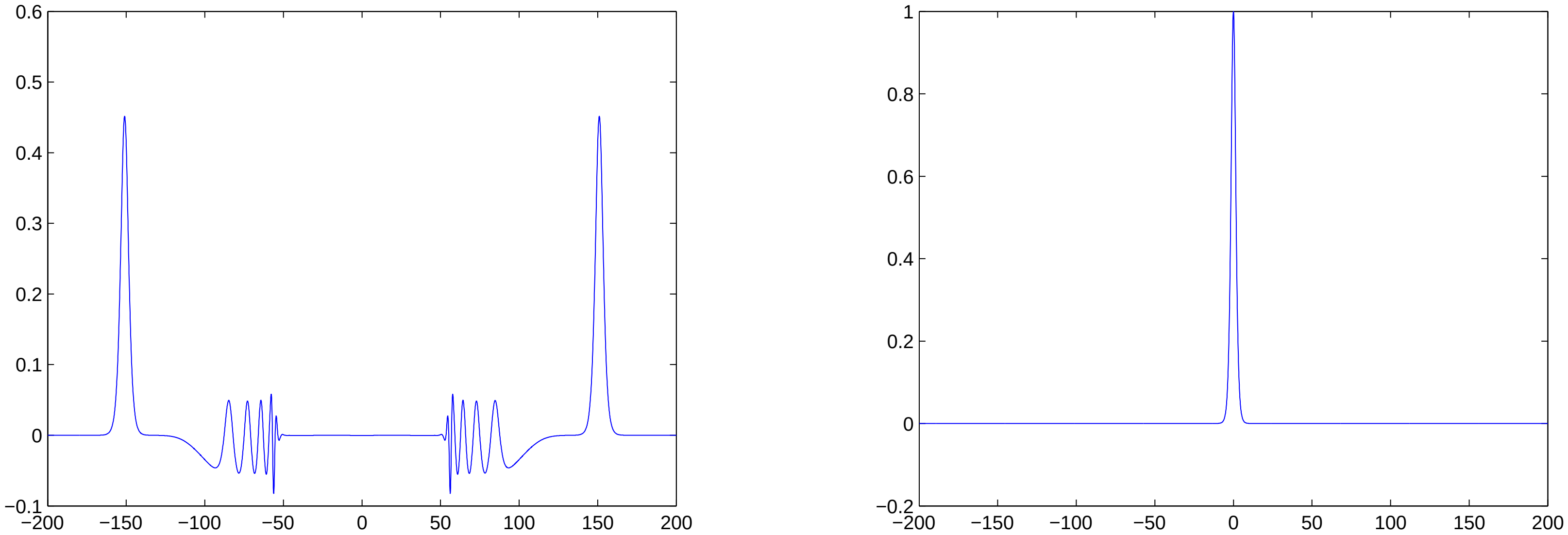


Figure 12: Method1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$.



$t_0 = 0, t = 110, \tau = h = 0.1$ $\quad$ $t_0 = 110, t = 0, \tau = -h = -0.1$

Figure 13: Method 1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$

$$u_{tt} = u_{xx} + \beta_1 u_{ttxx} - \beta_2 u_{xxxx} + \alpha (u^2)_{xx}$$

$$D\left(\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}\right) = Qu_i^n + \alpha \Lambda^{xx} (u_i^n)^2, \text{ Method 1}$$

$$D\left(\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}\right) = Qu_i^n + \alpha \Lambda^{xx} (g(u_i^n))$$

Method 2: $\quad g(u_i^n) = \dfrac{G(u_i^{n+1}) - G(u_i^{n-1})}{u_i^{n+1} - u_i^{n-1}}, \qquad G(u) = \displaystyle\int_0^u s^2 ds = \frac{u^3}{3}$

$$g(u_i^n) = \frac{(u_i^{n+1})^2 + u_i^{n+1} u_i^{n-1} + (u_i^{n-1})^2}{3}$$

Method 3: $\quad g(u_i^n) = 2\dfrac{G((u_i^{n+1} + u_i^n)/2) - G((u_i^n - u_i^{n-1})/2)}{u_i^{n+1} - u_i^{n-1}}$

$$g(u_i^n) = \frac{(u_i^{n+1} + u_i^n)^2 + (u_i^{n+1} + u_i^n)(u_i^n + u_i^{n-1}) + (u_i^n + u_i^{n-1})^2}{12}$$

Table 3: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $t = 40$

$h = \tau$	Rate M2	Rate M3	Error M2	Error M3
0.2	-	-	0.2651151649	0.1441062320
0.1	1.8836	1.9411	0.0718490891	0.0375272732
0.05	1.9720	1.9852	0.0183149097	0.0094783849
0.025	1.9929	1.9961	0.0046012352	0.0023759538
0.0125	1.9966	1.9961	0.0011529897	0.0005956124

Table 4: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $t = 40$, Method1

$h = \tau$	Rate $s = \sigma = 0.5$	Rate $s = \sigma = 0.25$	Error $s = \sigma = 0.5$	Error $s = \sigma = 0.25$
0.2	-	-	0.0111339769	0.0337359865
0.1	2.0403	2.0245	0.0027068659	0.0082917701
0.05	2.0094	2.0056	0.0006723417	0.0020648469
0.025	2.0033	2.0019	0.0001677038	0.0005155318
0.0125	2.0578	2.0287	0.0000402788	0.0001263450

$$u_{tt} = u_{xx} + \beta_1 u_{ttxx} - \beta_2 u_{xxxx} + \alpha(u^3)_{xx}$$

$$\tilde{u}(x,t;c) = \sqrt{\frac{2(c^2-1)}{\alpha}} \operatorname{sech}\left(\sqrt{\frac{c^2-1}{\beta_1 c^2 - \beta_2}}(x-ct)\right)$$

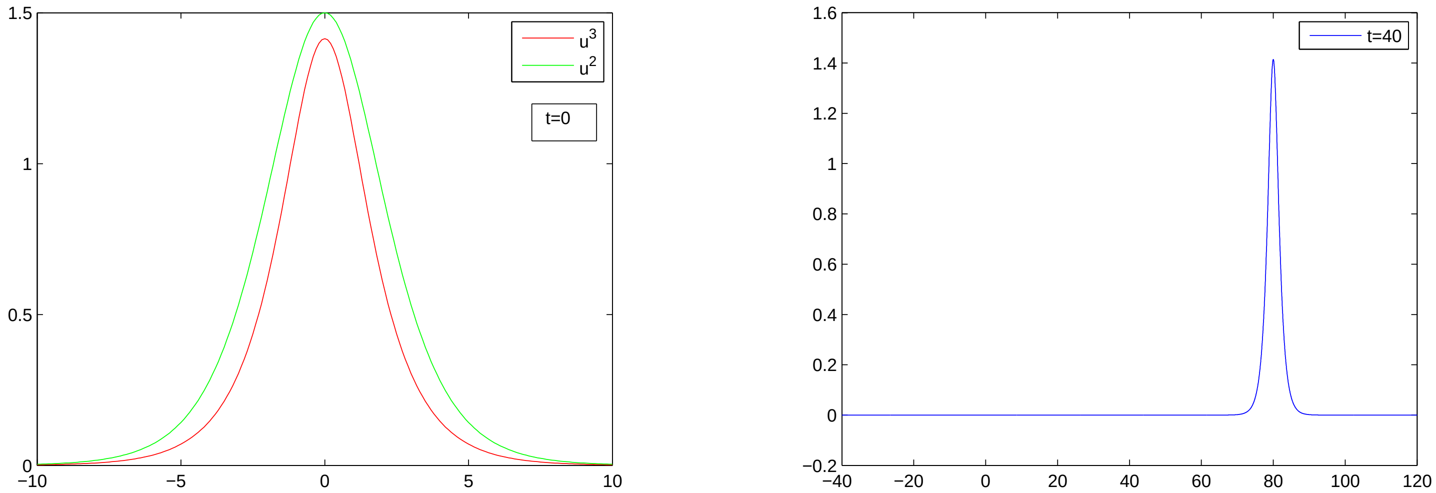


Figure 14: Method1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$.

Table 5: $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $t = 40$

$h = \tau$	Rate M1 (u^2)	Rate M1 (u^3)	Error M1 (u^2)	Error M1 (u^3)
0.2	-	-	0.0111339769	0.0944116735
0.1	2.0403	2.0535	0.0027068659	0.0227429902
0.05	2.0094	2.0126	0.0006723417	0.0056361787
0.025	2.0033	2.0034	0.0001677038	0.0014057022
0.0125	2.0578	2.0112	0.0000402788	0.0003487118
0.00625		2.8865		0.0000471583

$h=\tau$	Rate M2 (u^2)	Rate M2 (u^3)	Error M2 (u^2)	Error M2 (u^3)
0.2	-	-	0.2651151649	0.6079978518
0.1	1.8836	1.7228	0.0718490891	0.1841944426
0.05	1.9720	1.9387	0.0183149097	0.0480464800
0.025	1.9929	1.9855	0.0046012352	0.0121328769
0.0125	1.9966	1.9953	0.0011529897	0.0030430757
0.00625		1.9264		0.0008006003

$h=\tau$	$E_{error}\ (u^2)$	$E_{error}\ (u^3)$
0.2	2.09546213767$\times 10^{-12}$	1.61479218399$\times 10^{-12}$
0.1	3.92258760110$\times 10^{-11}$	3.28229802044$\times 10^{-11}$
0.05	5.45994876224$\times 10^{-10}$	4.43473518418$\times 10^{-10}$
0.025	4.60008425216$\times 10^{-9}$	3.74948329820$\times 10^{-9}$
0.0125	8.15913691568$\times 10^{-8}$	6.63400242583$\times 10^{-8}$

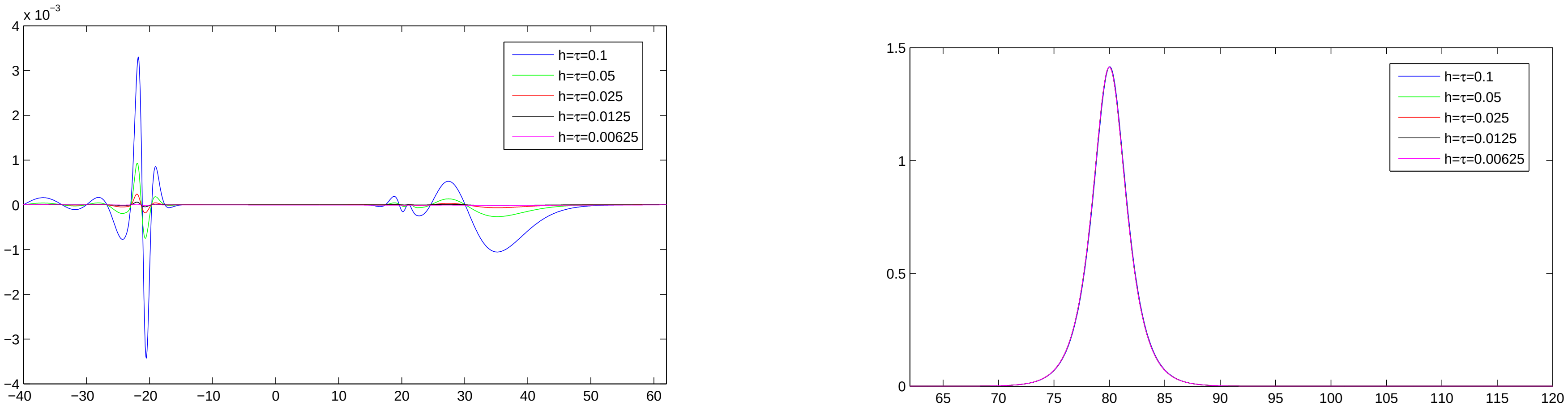


Figure 15: Method 1, $\beta_1 = 1.5$, $\beta_2 = 0.5$, $\alpha = 3$, $c = 2$, $\sigma = s = 0.5$, $t = 40$

$h = \tau$	u_{oscill} (u^2)	u_{oscill} (u^3)
0.2	$7{\times}10^{-3}$	$1{\times}10^{-2}$
0.1	$2{\times}10^{-3}$	$3.3{\times}10^{-3}$
0.05	$5{\times}10^{-4}$	$9{\times}10^{-4}$
0.025	$1.3{\times}10^{-4}$	$2.3{\times}10^{-4}$
0.0125	$3.2{\times}10^{-5}$	$6{\times}10^{-5}$
0.00625		$1.2{\times}10^{-5}$