In the search of of 2D Boussinesq solitons: a numerical approach

Christo I. Christov

Dept. of Mathematics, University of Louisiana at Lafayette

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Work done in collaboration with Dr. M. Christou and Dr. J. Choudhury

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Background

- Boussinesq's equation (BE) was the first model for surface waves in shallow fluid layer accounting for both nonlinearity and dispersion.
- The balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave.
- The above described balance is a new paradigm in physics and can be properly termed 'Boussinesq Paradigm'.
- BE appears also in a modeling elastic rods and shells.
- BE was incorrect in the sense of Hadamard. Its derivation was revisited by Benjamin, Bona and Mahony (so called RLWE or BBM).
- In a coordinate frame moving with the center of the propagating wave, BE reduces to Korteweg-de Vries which is widely studied in 1D.
- Unlike the 1D case, in 2D no analytical solution has been found so far, which necessitates developing numerical and semi-analytical solutions.
- Even the stationary propagating solitary waves have not been studied.

Challenges for the Numerical Solution

- Unboundness of the domain:
  - Posing the asymptotic boundary conditions at finite computational domain.
  - Size of the the computational domain needed to resolve the asymptotic behavior. Inefficiency of the uniform grids.
- Bifurcation nature of the localized solution: the trivial solution is always present alongside with the solitary-wave solution.

Part I

Posing the problem

Boussinesq Paradigm Equation in Two Dimensions

The general way[1] to derive the Boussinesq Model of surface waves lead to a system, but a simplification of the model known as Boussinesq Paradigm Equation (BPE)[2]

$$v_{tt} = \Delta(\gamma^2 v + \alpha v^2 + \beta_1 v_{tt} - \beta_2 \Delta v) \tag{1}$$

encompasses both BE ($\beta_1 = 0, \beta_2 < 0$ and RLWE ($\beta_2 = 0$). Eq. (1)has physical meaning for positive dispersion coefficients $\beta_1, \beta_2 > 0$.

We set the characteristic speed of the linear waves $\gamma = 1$ and the amplitude parameter $\alpha = -1$, because it can always be eliminated by rescaling the solution. We can also select $\beta_2 = 1$.

This leaves us with only one parameter, β_1 which multiplies a term that has the physical meaning of 'rotational inertia'.

[1] Christov, C. I. 2001, 'An energy-consistent Galilean-invariant dispersive shallow-water model', *Wave Motion* **34**, 161–174.

[2] Christov, C. I. 1995*a*, Conservative difference scheme for Boussinesq model of surface waves, *in*, In: 'Proc. ICFD V', Oxford University Press, pp. 343–349.

The Moving Frame

For the numerical interaction of 2D Boussinesq solitons, one needs the shape of a stationary moving solitary wave in order to construct an initial condition. To this end, introduce relative coordinates

$$\hat{x} = x - c_1 t, \quad \hat{y} = y - c_2 t, \tag{2}$$

in a frame moving with velocity (c_1, c_2). Since there is no evolution in the moving frame $v(x, y, t) = u(\hat{x}, \hat{y})$, and the following equation holds for u:

$$\begin{aligned}(c_1^2 u_{\hat{x}\hat{x}} + 2c_1c_2 u_{\hat{x}\hat{y}} + c_2^2 u_{\hat{y}\hat{y}}) &= (u_{\hat{x}\hat{x}} + u_{\hat{y}\hat{y}}) - [(u^2)_{\hat{x}\hat{x}} + (u^2)_{\hat{y}\hat{y}})] \\ &+ \beta_1[c_1^2(u_{\hat{x}\hat{x}\hat{x}\hat{x}} + u_{\hat{x}\hat{x}\hat{y}\hat{y}}) + 2c_1c_2(u_{\hat{x}\hat{x}\hat{x}\hat{y}} + u_{\hat{x}\hat{y}\hat{y}\hat{y}}) + c_2^2(u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}})] \\ &- (u_{\hat{x}\hat{x}\hat{x}\hat{x}} + 2u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}}). \end{aligned} \tag{3}$$

The so-called asymptotic boundary conditions (a.b.c.) read

$$u \to 0, \quad \text{for} \quad \hat{x} \to \pm\infty, \ \hat{y} \to \pm\infty. \tag{4}$$

Formulation as a system

The a.b.c.'s are invariant under rotation of the coordinate system, hence it is enough to consider solitary propagating along one of the coordinate axes, only. We chose this to be the y-axis, namely $c_1 = 0$, $c_2 = c \neq 0$. For the sake of convenience we introduce the following notations

$$\Delta \stackrel{\text{def}}{=} \partial_{\hat{x}\hat{x}} + \partial_{\hat{y}\hat{y}}, \quad D^2 \stackrel{\text{def}}{=} c^2 \partial_{\hat{y}\hat{y}}, \tag{5}$$

and recast Eq. (3) as the following system

$$\beta_2 \Delta u = \beta_1 D^2 u + u - u^2 - p(\hat{x}, \hat{y}), \tag{6}$$

$$\Delta p = D^2 u. \tag{7}$$

Bifurcation Problem for the 2D Shape

The trivial solution is always present for a.b.c.'s and must be avoided. In the present work we implement in a difference scheme, an idea used first in the context of spectral methods[3]. We fix the value of the function in one point in order to prevent the iterative algorithm of 'slipping' into the trivial solution. For definiteness we take $u(0,0) = \theta$ and then introduce $u = \theta \hat{u}$. Then

$$\beta_2 \Delta \hat{u} = \beta_1 D^2 \hat{u} + \hat{u} - \theta \hat{u}^2 - p(\hat{x}, \hat{y}), \tag{8a}$$

$$\Delta p = D^2 \hat{u}. \tag{8b}$$

In order not to get an overposed system, we consider θ as unknown, which is to be defined by the equation (8a) taken at the origin. Thus

$$\hat{u}(0,0) = 1, \quad \theta = \left. \frac{-\beta_2 \Delta \hat{u} + \hat{u} + \beta_1 D^2 \hat{u} - p}{\hat{u}^2} \right|_{\hat{x}=0, \hat{y}=0}. \tag{9}$$

[3]Christov, C. I. 1995*b*, 'Fourier-Galerkin algorithm for 2D localized solutions', *Annu. de l'Univ. Sofia, Livre 2 – Math. Appliqee et Informatique* **89**, 169–179.

Without fear of confusion we will reset the ‘names’ of the independent variables to x, y and will also omit the ‘hat’ over the sought function u.

Part III

Perturbation Solution for Small Phase Speeds

Regular Perturbation Expansion

The standard connection between Cartesian and polar coordinates is $x = r\cos(\theta), y = r\sin(\theta)$, where θ is the polar angle. Then

$$\frac{\partial^2}{\partial y^2} \equiv \sin^2\theta \frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}. \quad (10)$$

Begin with the case when $\varepsilon = c^2$ is a small parameter. Then one can seek for a perturbation solution

$$u(x,y) = u_0(r) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + O(\varepsilon^3), \quad r = \sqrt{x^2+y^2}. \quad (11)$$

Now, neglecting the terms of order $O(\varepsilon^3)$, we get

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\left[u_0(r) - u_0^2(r) - \frac{1}{r}\frac{d}{dr}r\frac{du_0}{dr}\right] = 0, \quad (12a)$$

$$\varepsilon\Big[-\frac{\partial^2}{\partial y^2}\Big(u_0 - \beta_1\frac{1}{r}\frac{d}{dr}r\frac{du_0}{dr}\Big) + \Delta u_1 - 2\Delta(u_0u_1) - \Delta^2 u_1\Big] = 0. \quad (12b)$$

$$\varepsilon^2\Big[-\frac{\partial^2}{\partial y^2}\big(u_1 - \beta_1\Delta u_1\big) + \Delta u_2 - \Delta u_1^2 - 2\Delta(u_0u_2) - \Delta^2 u_2\Big] = 0. \quad (12c)$$

The ‘Radial *sech*’.

One can seek for the solution in the follwing form

$$u_0(r,\theta) = F(r), \tag{13}$$

$$u_1(r,\theta) = G(r) + H(r)\cos(2\theta), \tag{14}$$

$$u_2(r,\theta) = P(r) + Q(r)\cos 2\theta + R(r)\cos 4\theta. \tag{15}$$

For the first term $u^{(0)}(r) = F(r)$ we have the following nonlinear equation

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\left[F - F^2 - \frac{1}{r}\frac{d}{dr}r\frac{dF}{dr}\right] = 0, \tag{16}$$

which can be integrated twice to obtain

$$F - F^2 - \frac{1}{r}\frac{d}{dr}r\frac{dF}{dr} = A\ln r + B. \tag{17}$$

For a localized solution, we must set $A = B = 0$. The obvious analogy of Eq. (17) to the equation for the famous *sech*-solution, hints that the sought solution does exist. We found it numerically.

Algebraic Asymptotic Decay of the Solution

After similar manipulations we get for $G(r)$:

$$-G(r) + 2F(r)G(r) + \frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} = -\frac{1}{2}\hat{F}(r), \tag{18}$$

where $\hat{F}(r) = [F(r) - \beta_1 \frac{1}{r}\frac{d}{dr} r \frac{d}{dr} F]$. G decays exponentially for $r \to \infty$. The difference comes from H, for which we get

$$\begin{aligned}(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2})\Big[- H(r) + 2F(r)H(r) + (\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2})H(r)\Big] \\ = \frac{1}{2}\Big[\frac{d^2}{dr^2}\hat{F}(r) - \frac{1}{r}\frac{d}{dr}\hat{F}(r)\Big],\end{aligned} \tag{19}$$

The Bessel operator in Eq. (19) gives that $H(r) \propto 1/r^2$ for $r \to \infty$
It can be shown that for any other of the higher-order perturbation functions u_n, the lowest asymptotic power is r^{-2}. This means that the full solution obeys the same asymptotic law.

Solution for $F(r)$. Tackling the Bifurcation

Introduce $F(r) = \theta \hat{F}(r)$ and impose the condition $\hat{F}(0) = 1$ to ensure nontrivial solution. The grid is staggered and the a.b.c is $\hat{F}(r_\infty) = 0$.

$$r_i = (i - \tfrac{1}{2})h, \quad h = r_\infty/(N - \tfrac{1}{2}).$$

r_∞ is a large number representing the "computational infinity", and N is the total number of grid points. The finite-difference scheme reads

$$\frac{r_{i-\frac{1}{2}}}{h^2 r_i}\hat{F}_{i-1}^{n+1} - \Big[\frac{r_{i-\frac{1}{2}}}{h^2 r_i} + \frac{r_{i+\frac{1}{2}}}{h^2 r_i}\Big]\hat{F}_i^{n+1} + \frac{r_{i+\frac{1}{2}}}{h^2 r_i}\hat{F}_{i+1}^{n+1} - \hat{F}_i^{n+1} = -\theta^n\big(\hat{F}_i^n\big)^2,$$
$$\text{for} \quad i = 2, \dots, N \quad \text{and} \quad \hat{F}_1 = 1, \quad \hat{F}_N = 0.$$

The behavioral b.c. is accounted for by $r_{\frac{1}{2}} = 0$ and θ is defined as

$$\theta^{n+1} = \omega\big(\hat{F}_1^{n+1}\big)^{-2}\Big[\frac{r_{\frac{3}{2}}}{h^2 r_i}\hat{F}_1^{n+1} - \frac{r_{\frac{3}{2}}}{h^2 r_1}\hat{F}_2^{n+1} + \hat{F}_1^{n+1}\Big] + (1-\omega)\theta^n.$$

where ω is a relaxation parameter introduced for convergence of iterations.

Function $G(r)$ is computed on the same grid, after the bifurcation is resolved and the nontrivial solution for $F(r)$ has been found.

Behavioral Conditions for $H(r)$

The operator of the equation for H requires two additional behavioral conditions: $H(0) = H'(0) = 0$. Introduce $H(r) = r^2 W(r)$. Then

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)H(r) = \left(r^2\frac{d^2}{dr^2} + 5r\frac{d}{dr}\right)W(r) = r^2\left[\frac{1}{r^5}\frac{d}{dr}r^5\frac{d}{dr}W(r)\right].$$

which can be rewritten in the form of the following system

$$\frac{1}{r^3}\frac{d}{dr}r^5\frac{d}{dr}S(r) = \frac{1}{2}\left[\frac{d^2}{dr^2}F(r) - \frac{1}{r}\frac{d}{dr}F(r)\right], \tag{21a}$$

$$\frac{1}{r^5}\frac{d}{dr}r^5\frac{d}{dr}W(r) - W(r) + 2F(r)W(r) = S(r). \tag{21b}$$

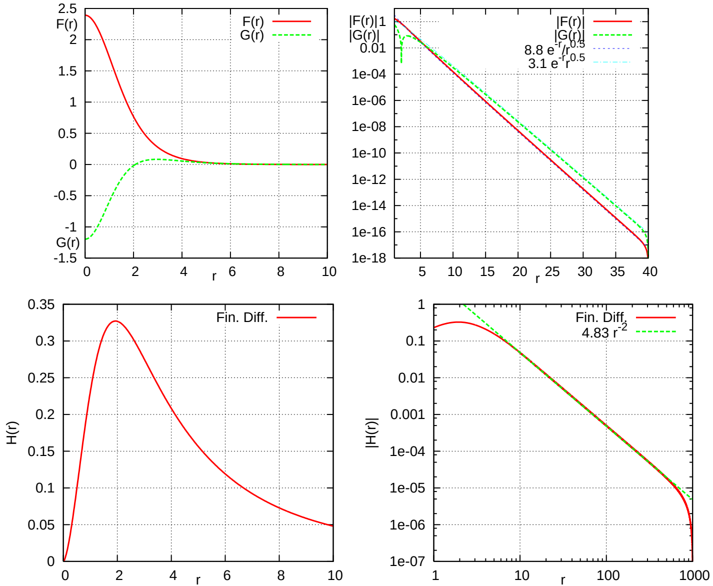
Difference Scheme for $W(r)$:

$$\frac{r^5_{i-\frac{1}{2}}}{h^2 r_i^3}S_{i-1} - \left[\frac{r^5_{i-\frac{1}{2}}}{h^2 r_i^3} + \frac{r^5_{i+\frac{1}{2}}}{h^2 r_i^3}\right]S_i + \frac{r^5_{i+\frac{1}{2}}}{h^2 r_i^3}S_{i+1} = \Gamma_i$$

$$\frac{r^5_{i-\frac{1}{2}}}{h^2 r_i^5}W_{i-1} - \left[\frac{r^5_{i-\frac{1}{2}}}{h^2 r_i^5} + \frac{r^5_{i+\frac{1}{2}}}{h^2 r_i^5}\right]W_i + \frac{r^5_{i+\frac{1}{2}}}{h^2 r_i^5}W_{i+1} - W_i + 2F_i W_i = S_i,$$

$$\Gamma_1 = 0, \quad \Gamma_i = \frac{1}{2}\left[\frac{F_{i+1} - 2F_i + F_{i-1}}{h^2} - \frac{1}{r_i}\frac{F_{i+1} - F_{i-1}}{2h}\right], \quad \text{for} \quad i = 2, \ldots, N-1.$$

Profiles of Functions F, G, H



Part III

Finite Difference Solution

Asymptotic B.C's on Truncated Region

The decay of the solution is quadratic algebraic for both functions, i.e.

$$p \simeq \frac{C_p}{r^2}, \;\; v \simeq \frac{C_v}{r^2}, \quad \text{for} \quad r \gg 1, \tag{22}$$

where C_p and C_v are some constants. We differentiate the function (say function p) and acknowledge Eq. (22), namely

$$\frac{\partial p}{\partial x} \simeq -2\frac{C_p}{r^3}\frac{\partial r}{\partial x} \simeq -\frac{2x}{r^3}p, \quad \frac{\partial p}{\partial y} \simeq -2\frac{C_p}{r^3}\frac{\partial r}{\partial y} \simeq -\frac{2y}{r^3}p, \tag{23}$$

and after obvious algebra get the following

$$x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y} \simeq -2\frac{x^2+y^2}{r^4}p = -2p, \tag{24}$$

which gives the following nonlocal boundary conditions:

$$\left(x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y}\right)\Big|_{x=\pm L_1} = -2p(\pm L_1, y), \quad \left(x\frac{\partial p}{\partial x} + y\frac{\partial p}{\partial y}\right)\Big|_{y=\pm L_2} = -2p(x, \pm L_2).$$

Nonuniform Grid

Due to the obvious symmetry of the problem, we find the solution only in the first quadrant: $x, y \in [0, L_1] \times [0, L_2]$.
We select the following rules for the non-uniform grids

$$x_i = \sinh[\hat{h}_1 i], \quad i = 0, \ldots, N_x + 1, \tag{25}$$

$$y_j = \sinh[\hat{h}_2 j], \quad j = 0, \ldots, N_y + 1, \tag{26}$$

where $\hat{h}_1 = D_1/N_x$ and $\hat{h}_2 = D_2/N_y$ and $D_{1,2}$ are selected in a manner to have large enough region. For instance, the choice $D_1 = D_2 = 9.2$ provides a box $[0, 50] \times [0, 50]$, because $L_i = \sinh(D_i)$. The numbers D_i can be used to make the scheme more efficient.
Now, for the above introduced variable spacings we have

$$h_i^x = x_{i+1} - x_i, \quad h_j^y = y_{j+1} - y_j \tag{27}$$

In order to implement the nonlocal boundary conditions, the grid is overlapping the region by one line at each boundary.

Approximations of the Differential Operators

The various difference approximations on the nonuniform grid are given by

$$\begin{aligned}\Lambda^{xx}\phi_{ij} &= \frac{2\phi_{i-1j}}{h^x_{i-1}(h^x_i + h^x_{i-1})} - \frac{2\phi_{ij}}{h^x_i h^x_{i-1}} + \frac{2\phi_{i+1j}}{h^x_i(h^x_i + h^x_{i-1})} \\ &= \left.\frac{\partial^2\phi}{\partial x^2}\right|_{ij} + O(|h^x_i - h^x_{i-1}|), \end{aligned} \tag{28a}$$

$$\begin{aligned}\Lambda^{yy}\phi_{ij} &= \frac{2\phi_{ij-1}}{h^y_{j-1}(h^y_j + h^y_{j-1})} - \frac{2\phi_{ij}}{h^y_i h^y_{i-1}} + \frac{2\phi_{ij+1}}{h^y_j(h^y_j + h^y_{j-1})} \\ &= \left.\frac{\partial^2\phi}{\partial y^2}\right|_{ij} + O(|h^y_i - h^y_{i-1}|), \end{aligned} \tag{28b}$$

$$\Lambda^{xy}\phi_{ij} = \frac{\phi_{i+1j+1} - \phi_{i-1j+1} - \phi_{i+1j-1} + \phi_{i-1j-1}}{(h^x_i + h^x_{i-1})(h^y_j - h^y_{j-1})} = \frac{\partial^2\phi}{\partial x\partial y} + O(h^x h^y). \tag{28c}$$

For smooth distribution of the nonuniform grid we have

$$O(|h^x_i - h^x_{i-1}|) \approx \frac{\partial h^x}{\partial x} O(|h_{i-1}|^2) = O(|h_{i-1}|^2). \tag{29}$$

Iterative Procedure

There is great number of different ways to create an iterative algorithm. Since, the main purpose of the present work is to investigate the proposed new boundary conditions, we chose the false transients as one of the simplest possible approaches. Then we have a parameter: the time increment, τ which can be used to manipulate the speed of convergence.

We use the following time-stepping procedure:

$$\frac{p_{ij}^{n+1} - p_{ij}^{n}}{\tau} = (\lambda^{xx} + \Lambda^{yy}) p_{ij}^{n} - c^2 \Lambda^{yy} u_{ij}^{n}, \tag{30}$$

$$\begin{aligned} \frac{u_{ij}^{n+1} - u_{ij}^{n}}{\tau} &= \beta_2 (\lambda^{xx} + \Lambda^{yy}) u_{ij} - \theta^n (u_{ij}^{n})^2 \\ &+ p_{ij}^{n+1} - u_{ij}^{n} - \beta_1 c^2 \Lambda^{yy} u_{ij}^{n}, \end{aligned} \tag{31}$$

for $i = 0, \ldots, N_x$, $j = 0, \ldots, N_y$ except the point $i = 0, j = 0$
Representing as a system reduces the condition number.

Approximation of the Asymptotic Boundary Conditions

The symmetry conditions are acknowledged by the special approximation of the main operators at the lines of symmetry $i = 0$ and $j = 0$:

$$\Lambda^{xx}\phi_{0j} = \frac{2\phi_{1j} - 2\phi_{0j}}{(h_0^x)^2}, \quad \Lambda^{yy}\phi_{i0} = \frac{2\phi_{i1} - 2\phi_{i0}}{(h_0^y)^2}, \qquad (32)$$

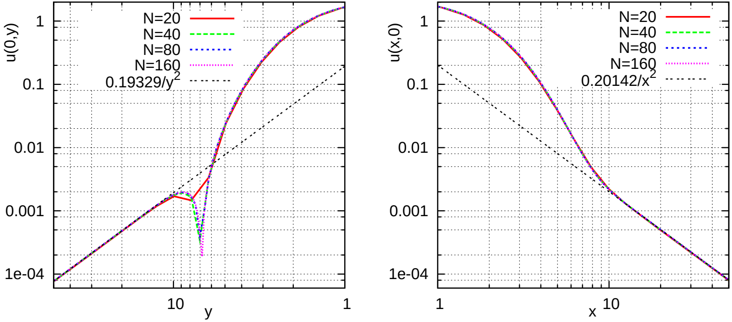
$$\Lambda^{xy}\phi_{0j} = \Lambda^{xy}\phi_{i0} = 0.$$

The conditions at the 'actual infinities' $i = N_x$ and $j = N_y$ read

$$p^{n+1}_{iN_y+1} = p^{n+1}_{iN_y-1} + \frac{h^y_{N_y} + h^y_{N_y-1}}{y_{N_y}}\Big[-2p^{n+1}_{iN_y} - \frac{x_i}{h^x_i + h^x_{i-1}}(p^{n+1}_{i+1N_y} - p^{n+1}_{i-1N_y}\Big], \quad i = 0, \ldots, N_x, \qquad (33a)$$

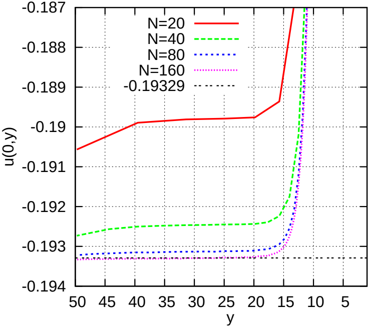
$$p^{n+1}_{N_x+1j} = p^{n+1}_{N_x-1j} + \frac{h^x_{N_x} + h^x_{N_x-1}}{x_{N_x}}\Big[-2p^{n+1}_{N_xj} - \frac{y_j}{h^y_j + h^y_{j-1}}(p^{n+1}_{N_x+1j} - p^{n+1}_{N_x-1j}\Big], \quad j =, \ldots, N_y. \qquad (33b)$$

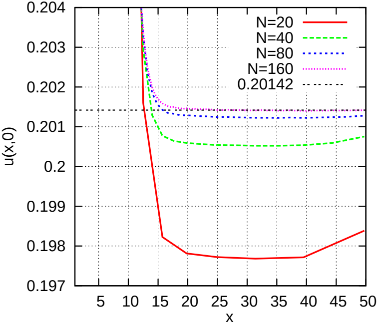
Role of resolution (Grid Size) for $\beta_1 = 1$, $c = 0.2$



Convergence of the two main cross-sections of the profile for different grid sizes, $N_x = N_y = N$.

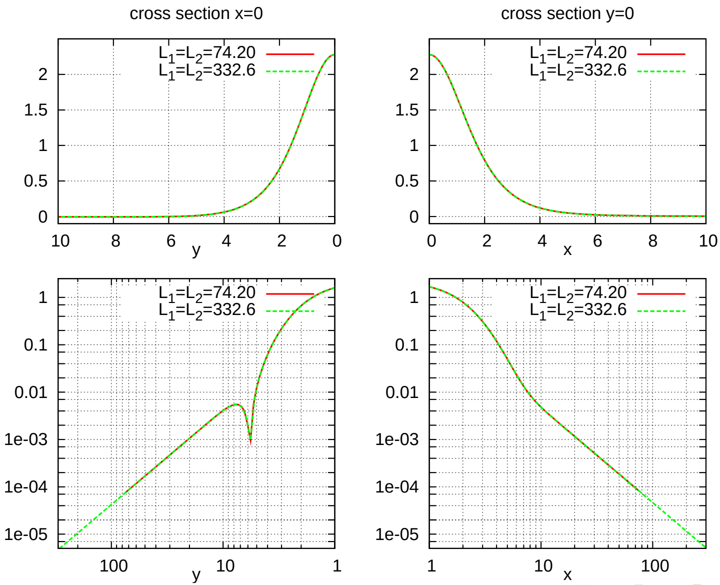
Asymptotic Behavior of 2D Soliton for $\beta_1 = 1$, $c = 0.2$



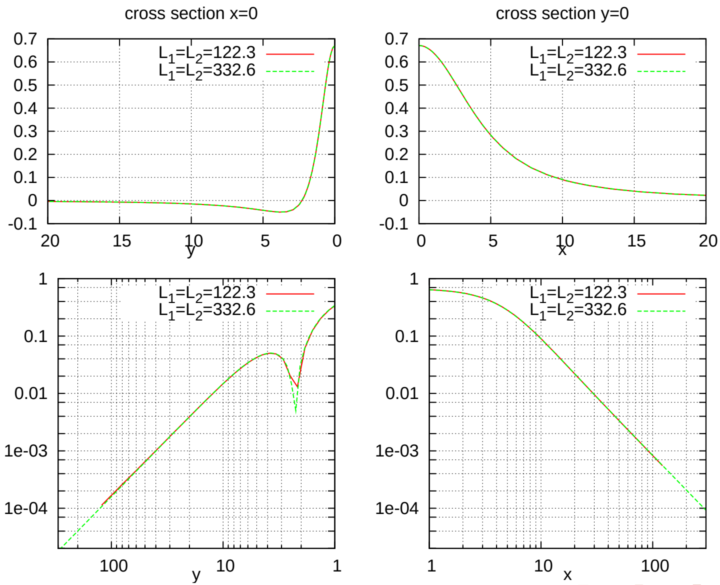


Convergence of the scaled cross-sections of the profile for different grid sizes, $N_x = N_y = N$.

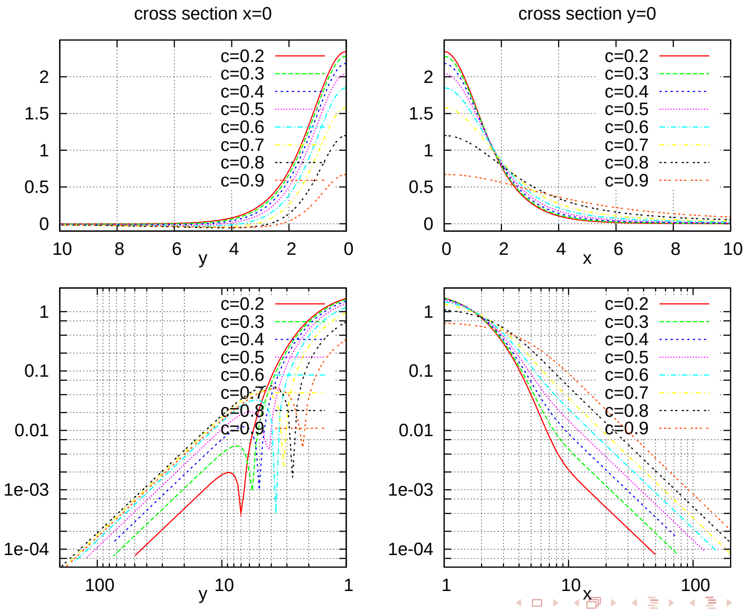
Role of the Size of Computational Box for $\beta_1 = 1$, $c = 0.3$



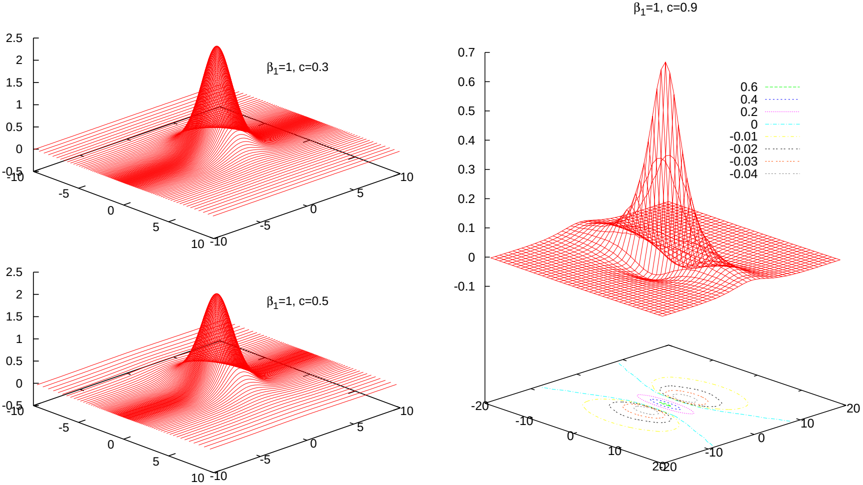
Role of the Size of Computational Boxfor $\beta_1 = 1$, $c = 0.9$



Crossections for $\beta_1 = 1$ and Different c

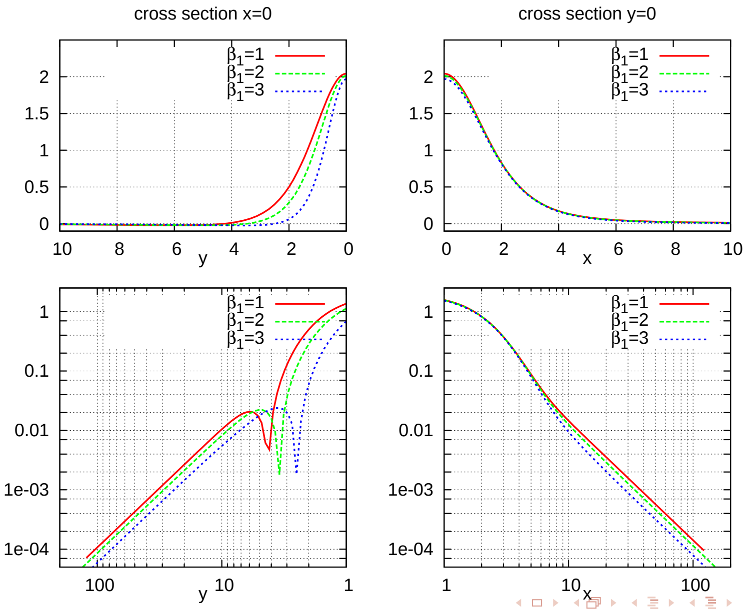


2D Profiles for $\beta_1 = 1$ and Different c

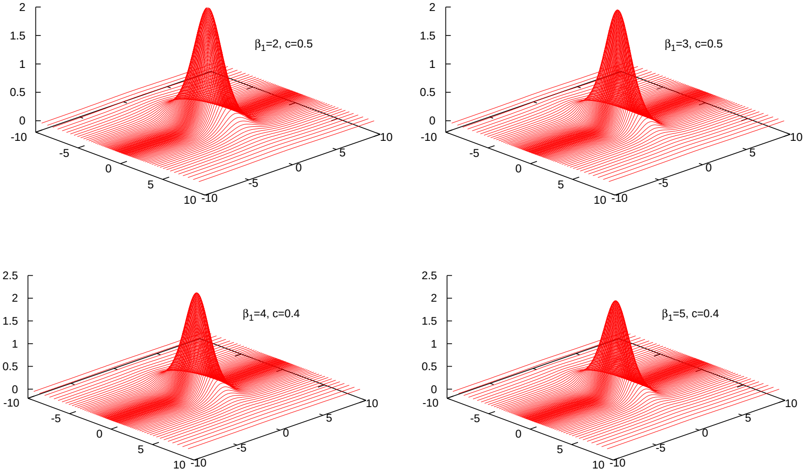


We were unable to reach convergence for supercritical solitons. The theoretical limit for the 1D subcritical solitons for $\beta_1 = \beta_2 = 1$ is $c = 1$.

Crossections for $c = 0.5$ and different β_1



2D Profiles for different β_1



Conditions for Existence of *sech*-Solitons in the 1D

In 1D, BPE reads

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - u^2 + \beta_1 \frac{\partial^2 u}{\partial t^2} - \beta_2 \frac{\partial^2 u}{\partial x^2}\right], \tag{34}$$

whose solution is

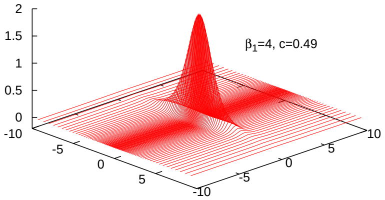
$$u = \tfrac{3}{2}(1 - c^2)\mathrm{sech}^2\left(\tfrac{1}{2}(x - ct)\sqrt{c^2 - 1}/\sqrt{\beta_1 c^2 - \beta_2}\right), \tag{35}$$

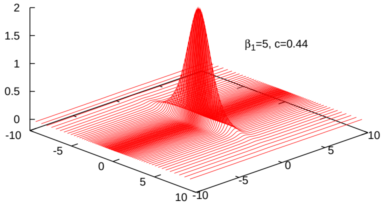
The most salient features of the *sech*-soliton (35) are:[4]

1. For $\beta_1 = \beta_2$, the solution exist for all c.
2. It exists for both subcritical $|c| \in [0, \min\{1, \sqrt{\beta_2/\beta_1}\}]$ and supecritical $|c| \in [\max\{1, \sqrt{\beta_2/\beta_1}\}, \infty]$ phase speeds;
3. In the subcritical branch, if c increases, the amplitude and suppport ('effective width') decrease;
4. In the supercritical branch, if c increases, the amplitude increases but the support decreases (the peak is sharper).

[4] Christov, C. I. 1995*a*, Conservative difference scheme for Boussinesq model of surface waves, In: 'Proc. ICFD V', Oxford University Press, pp. 343–349.

Limitation for the 2D numerical Solution





Comparison between the analytical limit of solutions in 1D with maximal c for which numerical results are obtained in 2D ($\beta_1 = 1$).

β_1	$\sqrt{\beta_2/\beta_1}$	max c reached
1	1.000	0.99
2	0.707	0.7
3	0.577	0.55
4	0.500	0.49
5	0.447	0.44

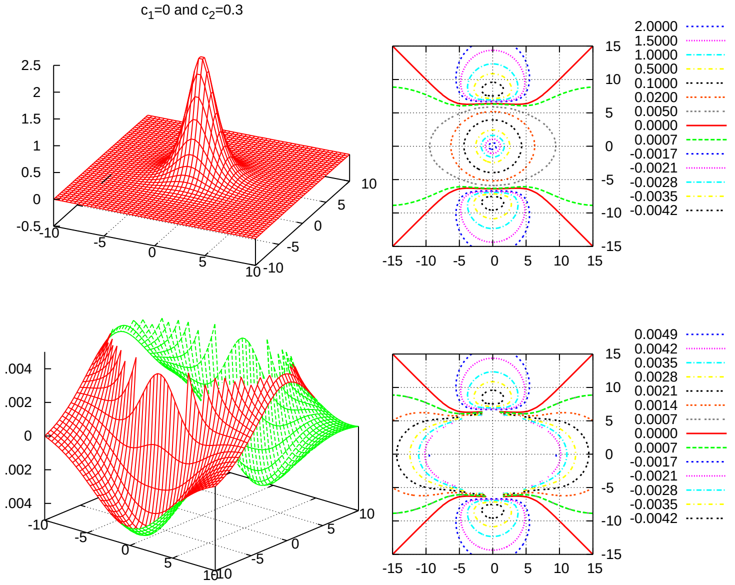
What has been done

- The solitary-wave solutions of BPE for waves on the surface of shallow inviscid layer are investigated numerically.
- The shape of stationary propagating solitary wave is a solution of a fourth-order nonlinear elliptic equation in an infinite domain.
- The bifurcation nature of the problem is addressed by a special algorithm involving scaling of the dependent variable.
- A new nonlocal approximation of the asymptotic boundary conditions is proposed at the finite boundaries of the computational box.
- Relatively sparse nonuniform grids are used in both spatial directions that allows computations in large computational domains.
- An iterative difference scheme is devised using false-transients and the stationary solutions are obtained after convergence of iterations.
- The second order approximation of the scheme is verified via computations on grids of different sizes.
- The algebraic asymptotic decay is confirmed with high accuracy.
- The solitary-wave shapes are found for different values of parameters.

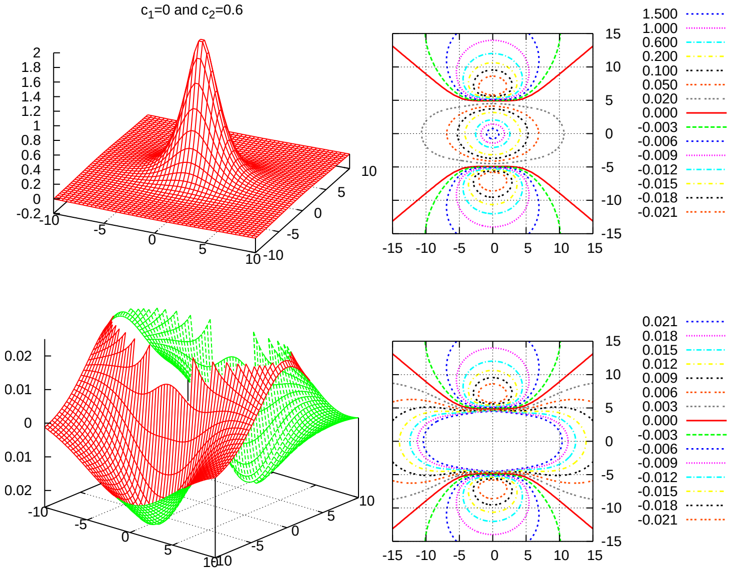
Assessment of the Asymptotic Solution

- Both the finite difference and asymptotic techniques developed here, turn out to be very good tools for investigating the shapes of the steady propagating Boussinesq waves in 2D.
- For small phase speeds $c_2 \leq 0.4.$, the results from the FD and from the first- and second-order asymptotic solutions are indistinguishable.
- For moderate phase speeds $c_2 \in (0.4, 0.6]$, the first-order is good for express assessment of the solution; the second order is in excellent agreement with the numerical solution.
- For large phase speeds $c_2 \in (0.6, 0.8]$, the asymptotic solution (as exemplified by the second order) is capable to give an 'express' qualitative assessment of the solution without resorting to numerical computations that require vast computational resources.
- For very large phase speeds $c_2 \in (0.8, 0.9]$, the second-order solution is much closer to the numerical solution than the first-order one, but it can only give a qualitative picture.

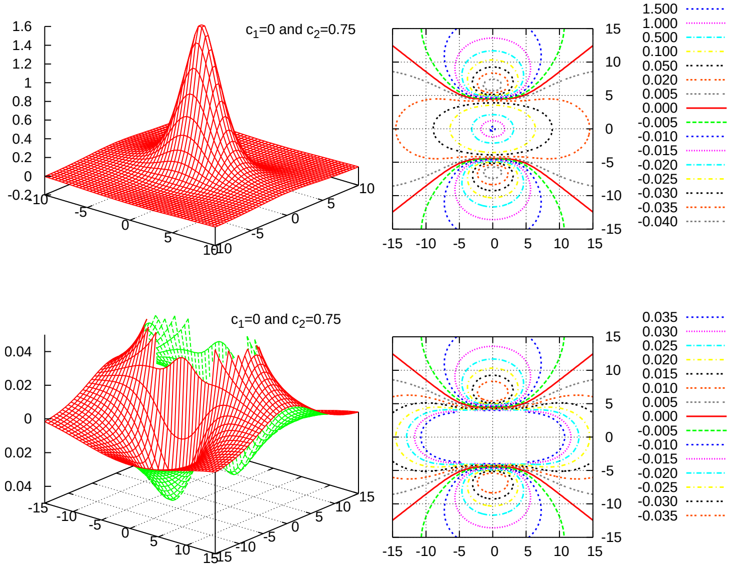
Soliton Shape for a Small Phase Speed $|c| = 0.3$



Soliton Shape for a Moderate Phase Speed $|c| = 0.6$



Soliton Shape for a Large Phase Speed $|c| = 0.75$



Part IV

Spectral Solution

Why Galerkin?

- Simpler in implementation;
- Better approximation than in the pseudo-spectral method since there is no discretization error;

What is needed for implementation?

- A CON system which has expressions for double (triple - for cubic nonlinearity) products into series with respect to the system.

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The CON System in $L^2(-\infty,\infty)$

Wiener (*circa* 1940) proposed the system

$$\rho_n = \frac{1}{\sqrt{\pi}} \frac{(ix-1)^n}{(ix+1)^{n+1}}\,, \quad n = 0, \pm 1, \pm 2, \dots\,.$$

The real-valued set was introduced in[5] and formulas for products to be expanded into series in the system

$$C_n C_k = \frac{C_{n+k+1} - C_{n+k} - C_{n-k} + C_{n-k-1}}{2\sqrt{2\pi}} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \beta_{nk,l} C_l,$$
$$S_n S_k = \frac{C_{n+k+1} - C_{n+k} + C_{n-k} - C_{n-k-1}}{2\sqrt{2\pi}} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \alpha_{nk,l} C_l,$$
$$S_n C_k = \frac{-S_{n+k+1} + S_{n+k} + S_{n-k} - S_{n-k-1}}{2\sqrt{2\pi}} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \gamma_{nk,l} S_l.$$

with properly defined α, β, and γ.

[5] Christov C. I., A complete orthonormal sequence of functions in $L^2(-^\circ, ^\circ)$ space, *SIAM J. Appl. Math*, **42** (1982), 1337–1344.

Convergence

The members of our CON system are connected to the harmonic functions through the relation

$$
\begin{aligned}
C_n(x) &= (-1)^n \frac{\cos(n+1)\theta + \cos n\theta}{\sqrt{2}}, \\
S_n(x) &= (-1)^{n+1} \frac{\sin(n+1)\theta + \sin n\theta}{\sqrt{2}}
\end{aligned}
\tag{36}
$$

where $x = \tan(\frac{\theta}{2})$ or $\theta = 2\arctan(x)$ is a transformation of the independent variable.

Since the convergence of the Fourier series is exponential, so st the convergence of the present series.

Slight mismatch of the actual decay at infinity can result in a sub-exponential (non-geometric) convergence.

Derivatives

$$\begin{Bmatrix} C_n' \\ S_n' \end{Bmatrix} = \sum_0^\infty \phi_{nm} \begin{Bmatrix} S_n' \\ -C_n' \end{Bmatrix},$$

$$\phi_{nm} = \tfrac{1}{2}[n\delta_{n,n-1} + (2n+1)\delta_{n,n-1} + (n+1)\delta_{n,n+1}] \tag{37}$$

$$\begin{Bmatrix} C_n'' \\ S_n'' \end{Bmatrix} = \sum_0^\infty \chi_{nm} \begin{Bmatrix} C_n \\ S_n \end{Bmatrix}, \quad \chi_{nm} = -\tfrac{1}{4}\left\{n(n-1)\delta_{m,n-2}\right.$$
$$-4n^2\delta_{m,n-1} + \left[n^2 + (n+1)^2 + (2n+1)^2\right]\delta_{m,n} \tag{38}$$
$$\left.-4(n+1)^2\delta_{m,n+1} + (n+1)(n+2)\delta_{m,n+2}\right\}.$$

$$\begin{Bmatrix} C_n^{(4)} \\ S_n^{(4)} \end{Bmatrix} = \sum_0^\infty \omega_{nm} \begin{Bmatrix} C_n \\ S_n \end{Bmatrix}, \qquad \omega_{nm} = \tfrac{1}{16}n(n-1)(n-2)(n-3)\delta_{m,n-4}$$
$$-\tfrac{1}{2}n(n-1)^2(n-2)\delta_{m,n-3} + \tfrac{1}{2}n^2(7n^2+5)\delta_{m,n-1}$$
$$+\tfrac{1}{4}n(n-1)(7n^2-7n+4)\delta_{m,n-2} + \tfrac{1}{8}(35n^4+70n^3+85n^2+50n+12)\delta_{m,n}$$
$$-\tfrac{1}{2}(n+1)^2[7(n+1)^2+5]\delta_{m,n+1} + \tfrac{1}{4}(n+1)(n+2)[7(n+1)^2+7(n+1)+4]\delta_{m,n+2}$$
$$-\tfrac{1}{2}(n+1)(n+2)^2(n+3)\delta_{m,n+3} + \tfrac{1}{16}(n+1)(n+2)(n+3)(n+4)\delta_{m,n+4}. \tag{39}$$

Implementation of Nonlinear Terms

For the arrangements of term in the expansion of U^2, where $U(x) = \sum_{n=0}^{\infty} u_n C_n$, we have the following expansion

$$U^2 = \sum_{n=0}^{N} \sum_{m=0}^{N} u_n u_m C_n C_m = \sum_{l=0}^{N} \left[\sum_{n=0}^{N} \sum_{m=0}^{N} \beta_{nm,l} u_n u_m \right] C_l \stackrel{\text{def}}{=} \frac{1}{2\sqrt{2\pi}} \sum_{l=0}^{N} b_l C_l,$$

$$b_l = \sum_{n=0}^{l-1} u_n u_{l-1-n} - \sum_{n=0}^{l} u_n u_{l-n} - 2 \sum_{n=l}^{N} u_n u_{n-l} + 2 \sum_{n=l+1}^{N} u_n u_{n-l-1}. \quad (40)$$

In general, to calculate the right-hand side for a specific index, requires a number of $4(N+1)^2$ operations (multiplications). The above formula reduces this number to $2N+4$. See[6]

[6] Christou M.A., Christov C.I., Fourier-Galerkin Method for 2D Solitons of Boussinesq Equation, *Math. Comp. Simulation*, **74** (2007), 82–92.

Implementation of Behavioral and Boundary Conditions

For the case of second-order equation, the “boundary conditions” are provided by the special structure of matrix χ_{mn}

$$C_0'' = -\tfrac{1}{2}C_0 + C_1 - \tfrac{1}{2}C_2, \quad C_1'' = C_0 - \tfrac{7}{2}C_1 + 4C_2 - \tfrac{3}{2}C_1,$$

For sufficiently large N we set $C_{N+1} = 0$ and $C_{N+2} = 0$, etc., and hence

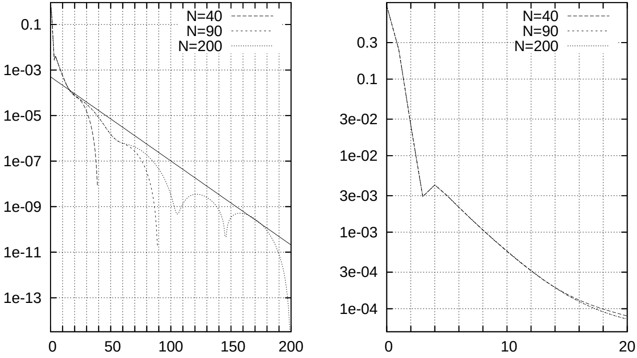
$$\begin{aligned} C_{N-1}'' &= -\tfrac{1}{4}(N-2)(N-1)C_{N-3} + (N-1)^2 C_{N-2} \\ &\quad - \tfrac{1}{2}(3N^2 - 3N + 1)C_{N-1} + n^2 C_N, \\ C_N'' &= -\tfrac{1}{4}N(N-1)C_{N-2} + N^2 C_{N-1} - \tfrac{1}{2}(3N^2 + 3N + 1)C_N. \end{aligned}$$

The above relations couple the five-diagonal system.

In a similar fashion are treated the approximations for the fourth derivatives to couple the respective nine-diagonal system.

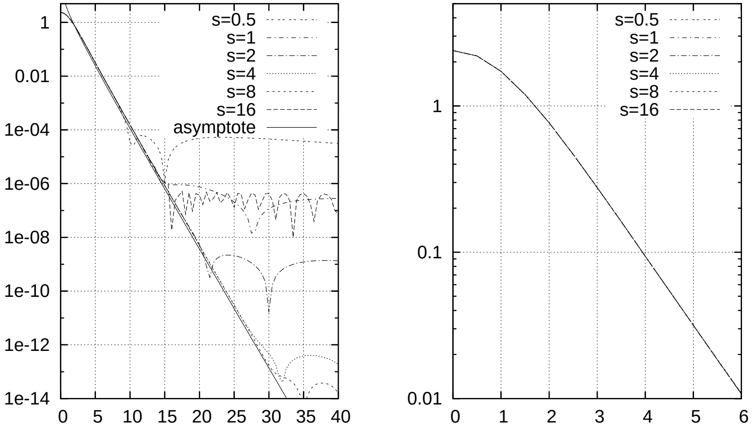
Asymptotic Rate of Convergence. BE with $\beta_1 = 0, \beta_2 = 1$

For the case $c_1 = c_2 = 0$ the scales are equal, $s = \lambda = \mu$. Consider $P_i = |p_{ii}|$, where p_{ii} are the even Galerkin coefficients.



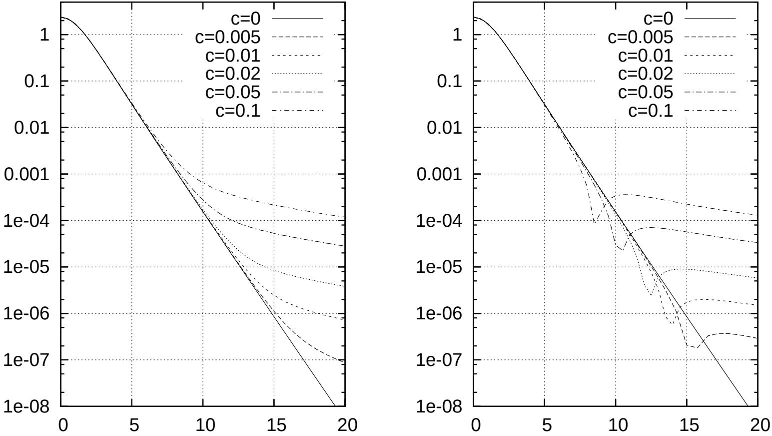
Coefficient as a function of its number. Left: - asymptotic decay of coefficients. —— $0.0005e^{-0.085x}$; right: first 21 coefficients.

Fiddling with the Scale Parameter



(a) asymptotic behavior; (b) energy containing range.
Profile of solution $u(r)$ with different scales for $N = 50$.

Impact of Phase Speed on Soliton Shape



The x- and y-cross sections of the soliton profile for different c_2.

Conclusions

• Boussinesq's model of surface waves over a shallow inviscid layer is considered. It applies also to flexural deformations of plates.

• The steady propagating solitary waves are the result of bifurcation in a fourth-order elliptic equation with asymptotic boundary conditions.

• A perturbation technique based on the asymptotic expansion for small phase speed, c, allowed obtaining solution including terms up to $O(c^4)$.

• A finite-difference iterative scheme with 'false transients' is constructed and validated with various grid parameters. Nonuniform grids are used, and a new kind of non-local condition at infinity is imposed, which 'projects' the asymptotic boundary condition at the truncated computational box.

• The results from the two techniques are in very good quantitative agreement which validates both of them.

• Novel information is obtained for the shape of a steady propagating localized wave, including for the asymptotic decay of the wave profile which turns out to be algebraic rather than exponention, as is in 1D.

• The new result are of importance both for the mathematical theory of Boussinesq solitons in multi-dimension, and for their physical applications.