Two Families of Finite Difference Schemes for Multidimensional Boussinesq Paradigm Equation

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Abstract. Families of *conservative finite difference schemes* for numerical solution of the Boussinesq Paradigm equation are constructed and studied theoretically. Different approximations to the nonlinear term produce both iterative and non-iterative schemes, which are second order accurate in space and time. Our numerical experiments show clear advantage of the non-iterative schemes in precision and speed. The schemes are implemented in fast, memory efficient and numerically stable algorithms. The extensive numerical experiments show good precision in real time computations and full agreement between the theoretical results and practical evaluation for single soliton and the interaction between two solitons.

Keywords: Boussinesq Paradigm Equation, finite difference scheme, conservation law
PACS: 02.60.Lj, 02.70.Bf, 42.25.Bs

1. INTRODUCTION

In the present work we study the Cauchy problem of the Boussinesq type equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \alpha \Delta f(u), \quad x \in \mathbb{R}^d,\ t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

$$u(x,t) \to 0, \quad \Delta u(x,t) \to 0, \quad |x| \to \infty,$$

where $f(u) = u^2$, the amplitude parameter α is a real number and the dispersion parameters β_1 and β_2 are positive constants. This problem is well-posed when considered as an initial value problem. The essentials of the derivation of (1) can be found in [3], where the equation (1) is called 'Boussinesq Paradigm Equation' (BPE).

BPE (1) appears in a number of mathematical models of real processes, for example, in the modeling of surface waves in shallow water.

The available literature for physical, theoretical and numerical treatments of BPE is very large for the case of one dimensional (1D) space variable x (i.e. $d = 1$). The numerical solutions are based on finite element methods, finite difference methods, spectral and pseudo-spectral methods - [3, 5, 6]. In contrast, the study of 2D BPE (i.e. $d = 2$) is only in its initial stage (see e.g. [1, 2]).

In the present paper we exploit the finite difference method for the solution of (1). When this method is correctly applied, the constructed finite difference schemes retain

the important properties (e.g. conservation laws) of the solution to the initial BPE.

In the paper we introduce and study two families of finite difference schemes for numerical computation of the multidimensional BPE. The two families of schemes (non-iterative and iterative) differ from the approximation of the nonlinear term $\Delta f(u)$. We prove, that the solution of the iterative scheme satisfies a particular discrete conservation law, which is a proper discretization to analytic conservation law of the original problem (1). The schemes are efficient with respect to the computational complexity. Thus, the number of operation is proportional to the total number of points for discretization in space and in time for the non-iterative scheme, while for the iterative scheme the last quantity is multiplied by the number of iterations (which is not big in practice). In the final section, we demonstrate (in the 1D case) the convergence of both schemes on two basic examples – (i) one solitary wave and (ii) interaction of two solitary waves traveling with different speeds towards each other.

2. SOME PROPERTIES OF THE SOLUTION TO BOUSSINESQ EQUATION

2.1. Using the transformation $\frac{1}{\sqrt{\beta_1}}x = \xi$, $\frac{\sqrt{\beta_2}}{\beta_1}t = \theta$ equation (1) can be rewritten in the form

$$\frac{\partial^2 U}{\partial \theta^2} = \Delta U + \Delta \frac{\partial^2 U}{\partial \theta^2} - \Delta^2 U + \Delta \frac{\beta_1}{\beta_2}\left(\alpha f(U) + \left(1 - \frac{\beta_2}{\beta_1}\right)U\right),$$

with $U(\xi,\theta) = u(x,t)$. Therefore, without loss of generality, we study the following problem

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \Delta \frac{\partial^2 u}{\partial t^2} - \Delta^2 u + \Delta g(u), \;\; x \in \mathbb{R}^d, \;\; t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \;\; \frac{\partial u}{\partial t}(x,0) = u_1(x), \;\; x \in \mathbb{R}^d, \tag{3}$$

$$u(x,t) \to 0, \;\; \Delta u(x,t) \to 0, \;\; |x| \to \infty, \;\; t > 0, \tag{4}$$

where g is connected to f as follows

$$g(u) = \frac{\beta_1}{\beta_2}\left(\alpha f(u) + \left(1 - \frac{\beta_2}{\beta_1}\right)u\right).$$

We assume that the initial functions u_0, u_1, f satisfy some regularity conditions, so that a unique solution u for the problem (2), (3), (4) exists and u is smooth enough.

2.2. Essential step in studying BPE is the construction of some identities valid for its solution. These identities are called "conservation laws" because their physical meaning is usually interpreted as conservation of energy, mass, pseudomomentum, etc.

For example, a derivation of identities based on a representation of BPE as a system of two parabolic equation is carried out in [3], [5]. Here we obtain the conservation laws by following a different approach (based on [8]) – first, the inverse of the Laplace operator

is applied to (2) and, second, the resulting equation is multiplied by u_t and transformed. In this way it is straightforward to prove:

Theorem 1 (Conservation law). *For every $t \geq 0$ denote by $(Eu)(t)$ the functional*

$$(Eu)(t) = \left\| (-\Delta)^{-1/2} \frac{\partial u}{\partial t} \right\|^2 + \left\| \frac{\partial u}{\partial t} \right\|^2 + \|u\|^2 + \|\nabla u\|^2 + \int_{\mathbb{R}^d} G(u)\, dx$$

with $G(u) = \int_0^u g(s)\, ds$, where $\|\cdot\|$ stands for the standard norm in $L_2(\mathbb{R}^d)$. Then the solution u to BPE (2)–(4) satisfies the following identity

$$\frac{\partial (Eu)(t)}{\partial t} = 0, \tag{5}$$

or, equivalently, the energy $(Eu)(t)$ is conserved in time

$$(Eu)(t) = (Eu)(0). \tag{6}$$

Note that Eu here is an equivalent energy norm, rather than the physical quantity called energy.

Equalities (5) and (6) will play an important role in our further investigations. We shall obtain similar, discrete energy identities for the solutions of the finite difference schemes employed in the discretization of problem (2)–(4).

2.3. For the construction of efficient algorithms for the numerical solution of BPE (2)–(4) we present (2) in an equivalent form

$$(I - \Delta) \left(\frac{\partial^2}{\partial t^2} - \Delta \right) u = \Delta g(u), \tag{7}$$

where I is the identity operator. Representation (7) is based on the change of variables made in Subsection 2.1, which allows the above factorization of the linear operator. Without such change of variables, (1) can be factorized only in the particular case $\beta_1 = \beta_2$.

The representation of (2) as (7) together with the introduction of an auxiliary function w satisfying proper asymptotic boundary conditions allow us to split the initial BPE into a system of two problems, one of which is elliptic and the other is hyperbolic. Since the operators in the left-hand side of (7) commute, the splitting of (2) can be performed in two ways: either as

$$(I - \Delta) w = \Delta g(u), \quad \left(\frac{\partial^2}{\partial t^2} - \Delta \right) u = w, \tag{8}$$

or as

$$\left(\frac{\partial^2}{\partial t^2} - \Delta \right) w = \Delta g(u), \quad (I - \Delta) u = w. \tag{9}$$

A main advantage of splittings (8) or (9) is, that the new system of equations involves only second spatial derivatives of the solution. This will further facilitate the numerical computations, because the direct inversion of the fourth order spacial operator meets

difficulties in conditioning. We emphasize that the discretization used in obtaining our numerical schemes has to allow splittings similar to the ones given above for the continuous problem.

It can be shown after some transformations that the scheme employed in [4] is equivalent to splitting (9). In the present paper we exploit splitting (8).

3. NUMERICAL METHOD FOR THE 2D PROBLEM

3.1. The methods described here work for any space dimension. For simplicity we present them in the case $d = 2$.

Let L_1, L_2 be sufficiently large numbers. We consider the discrete problem in the computational domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, assuming that the solution with its derivatives is negligible outside this domain. We introduce a uniform grid with steps h_1, h_2 in Ω. Let τ be the time step. With $M_1 = L_1/h_1$, $M_2 = L_2/h_2$ the grid points are (x_i, y_j, t_k), where $x_i = ih_1, i = -M_1, \ldots, M_1$; $y_j = jh_2, j = -M_2, \ldots, M_2$; $t_k = k\tau, k = 0, 1, 2, \ldots$. The discrete approximation to u at mesh point (x_i, y_j, t_k) is designated $v^{(k)}_{(i,j)}$. In the following we omit the notation $^{(k)}_{(i,j)}$ for the arguments of the mesh function whenever possible.

The standard 5-point discrete Laplacian is denoted by Δ_h. The finite difference approximation to the second time derivative is $v^{(k)}_{\bar{t}t,(i,j)} = \left(v^{(k+1)}_{(i,j)} - 2v^{(k)}_{(i,j)} + v^{(k-1)}_{(i,j)}\right)/\tau^2$.

For a real parameter σ denote by v^σ the symmetric σ-weighted approximation to $v^{(k)}_{(i,j)}$ given by $v^{\sigma(k)}_{(i,j)} = \sigma v^{(k+1)}_{(i,j)} + (1-2\sigma)v^{(k)}_{(i,j)} + \sigma v^{(k-1)}_{(i,j)}$. We apply approximations with parameter σ to both purely spatial operators Δ_h and $(\Delta_h)^2$ in (2). The simplest way to approximate $g(v)$ at (x_i, y_j, t_k) is to take $g(v^{(k)}_{(i,j)})$. Thus, at interior grid points we obtain a first family of finite difference methods depending on the parameter σ

$$v_{\bar{t}t} - \Delta_h v_{\bar{t}t} - \Delta_h v^\sigma + (\Delta_h)^2 v^\sigma = \Delta_h g(v). \tag{10}$$

Another well known approximation to the nonlinear term at (x_i, y_j, t_k) is

$$g_1(v^{(k)}_{(i,j)}) = \frac{G(v^{(k+1)}_{(i,j)}) - G(v^{(k-1)}_{(i,j)})}{v^{(k+1)}_{(i,j)} - v^{(k-1)}_{(i,j)}}, \quad \text{where} \quad G(u) = \int_0^u g(s)\,ds. \tag{11}$$

Note that in the case under consideration the function g is a second degree polynomial and the anti-derivative G used in g_1 is explicitly evaluated. Hence

$$g_1(v^{(k)}_{(i,j)}) = \frac{\beta_1}{\beta_2}\alpha\frac{\left(v^{(k+1)}_{(i,j)}\right)^2 + v^{(k+1)}_{(i,j)}v^{(k-1)}_{(i,j)} + \left(v^{(k-1)}_{(i,j)}\right)^2}{3} + \left(\frac{\beta_1}{\beta_2} - 1\right)\frac{v^{(k+1)}_{(i,j)} + v^{(k-1)}_{(i,j)}}{2}.$$

Then the second family of finite difference schemes is obtained

$$v_{\bar{t}t} - \Delta_h v_{\bar{t}t} - \Delta_h v^\sigma + (\Delta_h)^2 v^\sigma = \Delta_h g_1(v). \tag{12}$$

An $O(|h|^2+\tau^2)$ approximation to the initial conditions (3) is given by

$$v^{(0)}_{(i,j)} = u_0(x_i,y_j), \tag{13}$$
$$v^{(1)}_{(i,j)} = u_0(x_i,y_j)+\tau u_1(x_i,y_j)+0.5\tau^2(I-\Delta_h)^{-1}\left(\Delta_h u_0-(\Delta_h)^2u_0+\Delta_h g(u_0)\right)(x_i,y_j).$$

In order to approximate the second boundary condition the mesh is extended outside the domain Ω by one line at each space boundary and the symmetric second-order finite difference is used for the approximation of the second equation in (4).

Equations (10) or (12) with initial conditions (13) and boundary conditions described above form two families of finite difference schemes. By Taylor series expansion at point (x_i,y_j,t_k), it is easy to conclude that for each fixed σ the local approximation error of each finite difference scheme from both families is $O(|h|^2+\tau^2)$.

3.2. For the implementation of both schemes, we use the representation $v^\sigma = v+\sigma\tau^2 v_{\bar{t}t}$ and rewrite the left hand side of (10) in the more convenient, factorized form

$$\left(I-\Delta_h-\sigma\tau^2\Delta_h+\sigma\tau^2(\Delta_h)^2\right)v_{\bar{t}t}-\Delta_h v+(\Delta_h)^2 v=(I-\Delta_h)\left(\left(I-\sigma\tau^2\Delta_h\right)v_{\bar{t}t}-\Delta_h v\right).$$

Then, similarly to (8), we split the methods (10) and (12)

$$(I-\Delta_h)w=\Delta_h g(v),\quad \left(I-\sigma\tau^2\Delta_h\right)v_{\bar{t}t}-\Delta_h v=w; \tag{14}$$

$$(I-\Delta_h)w=\Delta_h g_1(v),\quad \left(I-\sigma\tau^2\Delta_h\right)v_{\bar{t}t}-\Delta_h v=w; \tag{15}$$

respectively, using an auxiliary function w.

The representations (14) and (15) of both schemes as pairs of an elliptic and a hyperbolic discrete equations lead us to efficient evaluation of the numerical solutions. For $\sigma>1/4$ the schemes are stable for every choice of h and τ. For $\sigma\le 1/4$ the schemes are conditionally stable. Thus, for $\sigma=0$ the two schemes are stable provided $\tau<h$.

The first scheme is derived from (14). The values of the auxiliary function w are obtained from the elliptic equation with a known right-hand side $\Delta_h g(v^{(k)})$ and zero boundary conditions. The solution $v^{(k+1)}$ (corresponding to time $t^{(k+1)}$) is found from the second equation of (14) – a hyperbolic problem with zero boundary conditions. One can use efficient methods for 2D hyperbolic equations, such as ADI methods or economic factorized schemes. Thus, due to the approximation of the nonlinear term, the numerical solution at the next time step $t^{(k+1)}$ is obtained without iterations. We refer to the method (10) (or equivalently (14)) as Non-iterative Method (NM).

The numerical procedure in the second scheme (15) is similar to (14) but the values of $g_1(v^{(k)})$ depend on the unknown values $v^{(k+1)}$. Thus an iterative procedure for the evaluation of $v^{(k+1)}$ from (15) is required. Starting with $v^{(k+1)[0]}=v^{(k)}$, we proceed inductively by evaluating $g_1(v^{(k)[s]})$ from $v^{(k+1)[s]}$, solving the elliptic equation with a known right-hand side and computing the next iteration $v^{(k+1)[s+1]}$ by solving the hyperbolic equation. This procedure is repeated until $\max_{(i,j)}|v^{(k+1)[s+1]}_{(i,j)}-v^{(k+1)[s]}_{(i,j)}|<\varepsilon\max_{(i,j)}|v^{(k+1)[s]}_{(i,j)}|$, where ε is a predefined precision. Then $v^{(k+1)[s+1]}$ is taken as $v^{(k+1)}$ and the method continues with the next time step. We refer to the finite difference scheme (12) or (15) as Iterative Method (IM).

4. ANALYSIS OF THE DISCRETE EQUATION

The analysis of the discrete problems is carried out similarly to the continuous problem. For given time moment t_k we consider the space of mesh functions $v^{(k)}$ which vanish at the points on the boundary of Ω. Denote by $(v^{(k)}, w^{(k)}) = \sum_{i,j} h_1 h_2 v^{(k)}_{(i,j)} w^{(k)}_{(i,j)}$ the discrete scalar product of mesh functions $v^{(k)}, w^{(k)}$ with respect to the spatial variables.

We define the operators $A = -\Delta_h$ and $B = (I + A)(I + \sigma\tau^2 A)$, which are involved in the finite difference schemes. Note that the operator A is a self-adjoint positive definite operator, thus it has an inverse A^{-1}.

Then the equations (10) and (12) are written in the operator form

$$Bv_{\bar{t}t} + Av + A^2 v = -Ag, \quad Bv_{\bar{t}t} + Av + A^2 v = -Ag_1. \tag{16}$$

Applying A^{-1} to (16) we get the equations in equivalent forms

$$A^{-1}Bv_{\bar{t}t} + v + Av + g = 0, \quad A^{-1}Bv_{\bar{t}t} + v + Av + g_1 = 0. \tag{17}$$

Now we multiply by $v^{(k+1)} - v^{(k-1)}$ both sides of equations (17) and rearrange the terms using summation formulas as in [7], Chapter 6. From the linear part $A^{-1}Bv_{\bar{t}t} + v + Av$ of the equation we obtain at the k-th time level the functional E_h^L given by

$$(E_h^L v)^{(k)} = \left\langle A^{-1/2} v_t^{(k)}, A^{-1/2} v_t^{(k)} \right\rangle + \left\langle v_t^{(k)}, v_t^{(k)} \right\rangle + \tau^2(\sigma - 1/4) \left\langle (I + A) v_t^{(k)}, v_t^{(k)} \right\rangle$$
$$+ 1/4 \left\langle v^{(k)} + v^{(k+1)} + A(v^{(k)} + v^{(k+1)}), v^{(k)} + v^{(k+1)} \right\rangle.$$

By incorporating the non-linear term g_1 in (17) we arrive at the discrete version of the 'energy' functional

$$(E_h v)^{(k)} = (E_h^L v)^{(k)} + \left\langle G(v^{(k+1)}), 1 \right\rangle + \left\langle G(v^{(k)}), 1 \right\rangle,$$

which satisfies $(E_h v)^{(k)} = (E_h v)^{(k-1)}$. Thus we have proved the following theorem:

Theorem 2 (Discrete conservation law). *The discrete 'energy' $(E_h v)^{(k)}$ of the solution v to the IM (12) is conserved in time, i.e. it satisfies the equalities*

$$(E_h v)^{(k)} = (E_h v)^{(0)}, \qquad k = 1, 2, \ldots. \tag{18}$$

Treating the non-iterative method in the same way we obtain

Theorem 3 (Discrete identities for NM). *The solution to the NM (10) satisfies the equalities*

$$(E_h^L v)^{(k)} - (E_h^L v)^{(k-1)} + \left\langle g(v^k), v^{(k+1)} - v^{(k-1)} \right\rangle = 0, \qquad k = 1, 2, \ldots. \tag{19}$$

We see, that the discrete balance law (18) valid for the solution to IM (12) fully corresponds to equation (6) valid for the solution to the initial problem (2)–(4).

The non-iterative scheme does not have a strict conservation of the discretized energy functional $(E_h v)^{(k)}$. But if equalities (19) satisfied by the solution of (10) are divided by τ, then they will correspond to identity (5), which holds true for the solution of the continuous problem.

5. NUMERICAL RESULTS FOR 1D PROBLEM

In this section we present some numerical results concerning the convergence of the two schemes. We apply NM and IM with parameter $\sigma = 0.5$. The termination criterion for the inner iterations in IM is the relative error of two consecutive iterations to be less than $\varepsilon = 10^{-13}$. In our calculations the number of iterations for this tolerance and for $\tau = 0.05$ is approximately 5.

The 1D equation (1) admits [3] the one-parameter soliton solution. In the new coordinates introduced in Subsection 2.1 it has the following form:

$$\tilde{u}(x,t;c) = \frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{\beta_1(c^2-1)}{\beta_1 c^2-\beta_2}}\left(x-ct\sqrt{\frac{\beta_1}{\beta_2}}\right)\right),$$

where c is the speed of the localized wave. Note that the soliton $\tilde{u}(x,t;c)$ has a peak of magnitude $3(c^2-1)/(2\alpha)$, which is located at $x=0$ in the initial moment $t=0$.

We use two basic examples of solutions to BPE (2) with one and the same set of parameters $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$:

Example 1: One solitary wave. We consider the following initial data

$$u(x,0) = \tilde{u}(x,0;2), \quad \frac{\partial u}{\partial t}(x,0) = \frac{\partial \tilde{u}(x,0;2)}{\partial t},$$

which represent a single solitary wave at the initial moment. Then the exact solution is the soliton $\tilde{u}(x,t;2)$ of hight 1.5 traveling to the right with speed $c=2$.

Example 2: Interaction of two solitary waves. In this example we simulate the interaction of two solitons running in opposite directions with different speeds. The initial data are (speeds $c_1 = 2$ and $c_2 = -1.5$):

$$u(x,0) = \tilde{u}(x+40,0;2) + \tilde{u}(x-50,0;-1.5);$$
$$\frac{\partial u}{\partial t}(x,0) = \tilde{u}(x+40,0;2)_t + \tilde{u}(x-50,0;-1.5)_t.$$

Table 1. Rate of convergence and errors for one solitary wave

Example	$h=\tau$	Rate NM	Rate IM	Error NM	Error IM	IM/ NM
1	0.1	–	–	0.02559	0.32271	12.60931
	0.05	2.02762	1.87037	0.00628	0.08826	14.06140
	0.025	2.00675	1.96892	0.00156	0.02255	14.43498
	0.0125	2.00142	1.99221	0.00039	0.00567	14.52742
2	0.04	2.09561	1.97465	0.017375	0.102754	5.913796
	0.02	1.94485	1.99369	0.004207	0.026027	6.187079
	0.01	1.97704	1.99838	0.001084	0.006528	6.021106

Test I. The Example 1 in Table 1 contains the error (the maximal norm of the difference between the calculated and the exact analytical solution) for time $t = 20$ and speed $c = 2$ for both methods (NM and IM) in the case of one solitary wave. The similar quantities from Example 2 are shown in the second part of Table 1 for $t = 40$. In this case the

initial problem has no exact solution. Thus for every h the error is calculated by Runge method as $E_1^2/(E_1-E_2)$ with $E_1 = \|u_{[h]} - u_{[h/2]}\|$, $E_2 = \|u_{[h/2]} - u_{[h/4]}\|$, where $u_{[h]}$ is the calculated solution with step h and $\|\cdot\|$ is the uniform norm. The numerical rate of convergence is $(\log E_1 - \log E_2)/\log 2$.

The calculations for Example 1 and Example 2 confirm that NM and IM are of order $O(h^2+\tau^2)$. Note that for one and the same h and τ NM is about 14 times more precise than IM for Example 1 and about 6 times more precise for Example 2 (see the last column of Table 1). The better performance with respect to the error magnitude of NM compared to IM is due to the $O(\tau^2)$ difference between the two right-hand side approximations to $g(v)$.

Test II. On Table 2 the dependence of error for $t = 20$ measured both in L_2 norm and in maximal (L_∞) norm with respect to time step is shown for one solitary wave with $c = 2$. The results confirm, that the error behaves as $O(\tau^2)$. The advantage of the NM over the IM is preserved.

Table 2. Error dependence on time step with a fixed space step $h = 0.01$

	NM				IM			
τ	L_2 error	Rate	L_∞ error	Rate	L_2 error	Rate	L_∞ error	Rate
0.16	0.136372	-	0.075703	-	1.335671	-	0.677222	-
0.08	0.032492	4.19	0.017982	4.21	0.402828	3.32	0.214408	3.16
0.04	0.007996	4.06	0.004422	4.07	0.105644	3.81	0.056685	3.78
0.02	0.001956	4.09	0.001082	4.09	0.026768	3.95	0.014385	3.94
0.01	0.000450	4.34	0.000249	4.33	0.006751	3.97	0.003630	3.96
0.005	7.439e-5	6.05	4.174e-5	5.98	0.001727	3.91	0.000929	3.91

Test III. We check the error in the discrete identities (18) for IM and (19) for NM. For IM in Examples 1 and 2 at each time level the discrete 'energy' $(E_h v)^{(j)}$ is evaluated for $h = \tau = 0.025$. Up to the time $t^k = 40$ the relative errors $\max_{1\le j\le k} |(E_h v)^{(j)} - (E_h v)^{(0)}|/(E_h v)^{(0)}$ are respectively $1.294821\cdot 10^{-12}$ for one solitary wave and $1.385970\cdot 10^{-12}$ for two solitary waves. At the same time the discrete identities (19) are evaluated for NM within 10^{-15} accuracy at each time step, i.e. within the computer round-off error.

We conclude that the numerical solution of both methods satisfies the respective discrete identities with high precision.

6. CONCLUSION

In this paper two families of finite difference schemes – iterative and non-iterative – are applied for numerical solving of multidimensional BPE. The iterative FDS retain an important property – the conservation law of the solution to the initial BPE while the non-iterative schemes obey a proper balance equation. However, the non-iterative schemes demonstrate smaller approximation errors in experiments than the iterative ones. Efficient algorithms for evaluation of the numerical solutions based on the factorization of FDS as elliptic and hyperbolic discrete equations are derived. The extensive numerical experiments for the one-dimensional problem show good precision and second order of convergence for single soliton and two colliding solitary waves.

ACKNOWLEDGMENTS

The author is indebted to C. I. Christov for numerous fruitful discussions.

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