`GraClo` Module Documentation 1.0

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September 2020

General Usage

This package is for finding the independent components, their number, and equations relating components of a given tensor, and corresponding equations and symmetries, in a certain dimension.

Installation

1. Download the `GraClo.mpl` file.

2. Place the file in the Maple directory you are working in.

3. Open Maple.

4. Read the package into Maple by using `> read("GraClo.mpl")`.

5. To use the package, as with every Maple package, then enter `> with(GraClo)`. This will allow you to use the commands with no prefix.

There is an alternative to the final step: you can enter `GraClo[command](...)` or `GraClo:-command(...)`.

Ancillary Functions

SYM

`SYM` is a procedure for creating a sum of permutations, described mathematically by

$$\Psi_{x_1 x_2 \ldots x_n (x_3 x_4 \ldots x_N)} = \frac{1}{k!} \sum_{\substack{all\, permutations \\ i_1 \ldots i_k}} \Psi_{x_1 x_2 \ldots x_n (x_{i_1} x_{i_2} \ldots x_{i_k})}$$

where in this case, Ψ and its associated indices is an arbitrary tensor. Equivalently, this is equally valid for contravariant tensor components.

`SYM` takes two arguments: `condition`, and `tenss`[1]. To begin with the latter, `tenss` is the tensor over which you are performing the symmetry operation. `condition` is the (you guessed it), symmetry condition, input as a list. For example, suppose we want to find the symmetry of a, b, c, in the tensor Ψ_{abcd}:

> `SYM([a,b,c],Psi[a,b,c,d]);`

$$\frac{\Psi_{a,b,c,d}}{6} + \frac{\Psi_{a,c,b,d}}{6} + \frac{\Psi_{b,a,c,d}}{6} + \frac{\Psi_{b,c,a,d}}{6} + \frac{\Psi_{c,a,b,d}}{6} + \frac{\Psi_{c,b,a,d}}{6}$$

ASYM

Similar to the above, `ASYM` is a procedure for finding the anti-symmetric sum of permutations, described mathematically by

$$\Psi_{x_1 x_2 \ldots x_n (x_3 x_4 \ldots x_N)} = \frac{1}{k!} \sum_{\substack{all\, permutations \\ i_1 \ldots i_k}} \epsilon_{i_1 i_2 \ldots i_k} \Psi_{x_1 x_2 \ldots x_n (x_{i_1} x_{i_2} \ldots x_{i_k})}$$

where ϵ is the Levi-Civita Symbol, being 1 for even permutations, and -1 for odd permutations. Consider for this case, the Riemann Curvature tensor, R_{abcd} with the asymmetry on b, c, d

> `ASYM([b,c,d],R[a,b,c,d])`

$$\frac{R_{a,b,c,d}}{6} - \frac{R_{a,b,d,c}}{6} - \frac{R_{a,c,b,d}}{6} + \frac{R_{a,c,d,b}}{6} + \frac{R_{a,d,b,c}}{6} - \frac{R_{a,d,c,b}}{6}$$

As in `SYM`, the condition is input as a list.

Primary Functions

Each of the following functions take the same arguments: `tenss`, `basisEquations`, and `dim`, in that order. As in the `SYM` and `ASYM` commands, `tenss` is the tensor in question (e.g., $T^{\mu\nu} \to$ `T[mu,nu]`). This is necessary currently to extract the rank, indices, and base. `basisEquations` takes a list of equations, written in free index form. For instance, using the Lanczos tensor's symmetries [1] as an example, they would be entered as a list as follows:

> `[H[a,b,c] + H[b,a,c] = 0, H[a,b,c] + H[b,c,a] + H[c,a,b] = 0]`

$$[H_{a,b,c} + H_{b,a,c} = 0, H_{a,b,c} + H_{b,c,a} + H_{c,a,b} = 0]$$

or equivalently, using the symmetry commands

> `[SYM([a,b],H[a,b,c]) = 0, ASYM([a,b,c],H[a,b,c]) = 0]`.

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[1]It is called `tenss` because that is what I wrote at the time, so as not to confuse myself with other appearances of the word 'tensor'.

$$[\frac{H_{a,b,c}}{2} + \frac{H_{b,a,c}}{2} = 0, \frac{H_{a,b,c}}{6} - \frac{H_{a,c,b}}{6} - \frac{H_{b,a,c}}{6} + \frac{H_{b,c,a}}{6} + \frac{H_{c,a,b}}{6} - \frac{H_{c,b,a}}{6} = 0]$$

`dim` is, somewhat intuitively, the dimension of the system. In this system, we consider the dimensions running from $(0, 1, ..., \texttt{dim} - 1)$. Due to the limitations of `convert`, this means that we require $\texttt{dim} \geq 2$.

If you do not input any equations into `basisEquations`, but simply enter `[]`, then you will be dealing with all the components of the tensor.

IndependentComponents

As the name would suggest, this solves for, and returns a list of the independent components of a given tensor, given the equations governing the symmetry, and the dimension. For a full worked example, let us consider finding the components of the Riemann Curvature tensor, in **five** dimensions.

```
> with(GraClo):
> eqns := [R[a,b,c,d] = -R[b,a,c,d],R[a,b,c,d] = -R[a,b,d,c],
R[a,b,c,d] = R[c,d,a,b], ASYM([b,c,d],R[a,b,c,d]) = 0];
```

$$eqns := [R_{a,b,c,d} = -R_{b,a,c,d}, R_{a,b,c,d} = -R_{a,b,d,c}, R_{a,b,c,d} = R_{c,d,a,b},$$
$$\frac{R_{a,b,c,d}}{6} - \frac{R_{a,b,d,c}}{6} - \frac{R_{a,c,b,d}}{6} + \frac{R_{a,c,d,b}}{6} + \frac{R_{a,d,b,c}}{6} - \frac{R_{a,d,c,b}}{6} = 0]$$

```
> IndependentComponents(R[a,b,c,d],eqns,5);
```

$[R_{0,2,1,0}, R_{0,3,1,0}, R_{0,3,2,0}, R_{0,4,1,0}, R_{0,4,2,0}, R_{0,4,3,0}, R_{1,0,0,1}, R_{1,2,1,0}, R_{1,3,1,0},$
$R_{1,3,2,1}, R_{1,4,1,0}, R_{1,4,2,1}, R_{1,4,3,1}, R_{2,0,0,2}, R_{2,1,1,2}, R_{2,1,2,0}, R_{2,1,3,0}, R_{2,1,4,0},$
$R_{2,3,2,0}, R_{2,3,2,1}, R_{2,4,2,0}, R_{2,4,2,1}, R_{2,4,3,2}, R_{3,0,0,3}, R_{3,1,1,3}, R_{3,1,2,0}, R_{3,1,3,0},$
$R_{3,1,4,0}, R_{3,2,2,3}, R_{3,2,3,0}, R_{3,2,3,1}, R_{3,2,4,0}, R_{3,2,4,1}, R_{3,4,3,0}, R_{3,4,3,1}, R_{3,4,3,2},$
$R_{4,0,0,4}, R_{4,1,1,4}, R_{4,1,2,0}, R_{4,1,3,0}, R_{4,1,4,0}, R_{4,2,2,4}, R_{4,2,3,0}, R_{4,2,3,1}, R_{4,2,4,0},$
$R_{4,2,4,1}, R_{4,3,3,4}, R_{4,3,4,0}, R_{4,3,4,1}, R_{4,3,4,2}]$

NumberOfIndependentComponents

As (again) the name suggests, this just returns the number of independent components, if it is not necessary to know which components they are. This is the equivalent of course, of just performing `nops` or `numelems` on the list of independent components. Continuing our code from above,

```
> NumberOfIndependentComponents(R[a,b,c,d],eqns,5);
```

$$50$$

Gratefully, this it what we expect from the Riemann tensor in five dimensions. As proven algebraically in [2], there are $\frac{n^2}{12}(n^2 - 1)$ independent components in an n dimensional Riemann Curvature tensor.

```
Equations
```

This name is slightly more abstract, but it returns a list of the other components, in terms of the independent components. Since there are a large number of non-zero components, in this instance, I will reduce the dimension to 3, but keep the equations the same

```
> Equations(R[a,b,c,d],eqns,3);
```

$[R_{0,1,0,1} = -R_{1,0,0,1}, R_{0,1,0,2} = -R_{0,2,1,0}, R_{0,1,1,0} = R_{1,0,0,1}, R_{0,1,1,2} = -R_{1,2,1,0},$
$R_{0,1,2,0} = R_{0,2,1,0}, R_{0,1,2,1} = R_{1,2,1,0}, R_{0,2,0,1} = -R_{0,2,1,0}, R_{0,2,0,2} = -R_{2,0,0,2},$
$R_{0,2,1,2} = R_{2,1,2,0}, R_{0,2,2,0} = R_{2,0,0,2}, R_{0,2,2,1} = -R_{2,1,2,0}, R_{1,0,0,2} = R_{0,2,1,0},$
$R_{1,0,1,0} = -R_{1,0,0,1}, R_{1,0,1,2} = R_{1,2,1,0}, R_{1,0,2,0} = -R_{0,2,1,0}, R_{1,0,2,1} = -R_{1,2,1,0},$
$R_{1,2,0,1} = -R_{1,2,1,0}, R_{1,2,0,2} = R_{2,1,2,0}, R_{1,2,1,2} = -R_{2,1,1,2}, R_{1,2,2,0} = -R_{2,1,2,0},$
$R_{1,2,2,1} = R_{2,1,1,2}, R_{2,0,0,1} = R_{0,2,1,0}, R_{2,0,1,0} = -R_{0,2,1,0}, R_{2,0,1,2} = -R_{2,1,2,0},$
$R_{2,0,2,0} = -R_{2,0,0,2}, R_{2,0,2,1} = R_{2,1,2,0}, R_{2,1,0,1} = R_{1,2,1,0}, R_{2,1,0,2} = -R_{2,1,2,0},$
$R_{2,1,1,0} = -R_{1,2,1,0}, R_{2,1,2,1} = -R_{2,1,1,2}]$

In this instance, it may not have been that illustrative the usefulness of this equation, since there is only one term on each side.

Limitations

- Cannot work with ODEs or PDEs (e.g., $\partial_\mu \zeta^{\mu\nu}_{\rho\kappa} = 0$ or anything remotely like that)

- Cannot work with multiple, different tensors (e.g., for some system such as $T_{\mu\nu} + P_{ab}$ with those being arbitrary tensors, and some associated symmetries, the system cannot find independent components of both tensors)

- The dimension must be two or higher

- The code determining the oscillating sum for the ASYM command is 'unstable'. It relies on the way Maple orders terms. If that were to be changed, then I believe it would break

- There is no way to work explicitly with covariant and contravariant indices.

- It cannot work with components explicitly stated as functions (e.g., $\gamma^{\mu\nu}(\sigma)$) as Maple seems to always try to solve for the independent variable, even when it is specified not to.

References

[1] Agacy RL. LETTER: A Note on the Algebraic Symmetries of the Riemann and Lanczos Tensors. General Relativity and Gravitation.

1999 Feb;31(2):219–222. Available from: `https://doi.org/10.1023/A:1018844227423`.

[2] Bergmann PG. Introduction to the Theory of Relativity. New York; 1976.