AN INTEGRAL SURGERY FORMULA FOR FRAMED INSTANTON HOMOLOGY I

ZHENKUN LI AND FAN YE

Abstract. In our previous work, we developed a large surgery formula in instanton theory. In this paper, we propose a strategy to prove an integral surgery formula for rationally null-homologous knots in arbitrary 3-manifolds, and realize it for knots in S^3. Our argument is based on interpreting framed instanton homology $I^\sharp$ via sutured instanton homology SHI, and realizing desired formula in the sutured setup. We discover some new exact triangles relating the sutured instanton homology of different sutured knot complements which are of independent interest. We also introduce a new concordance invariant ν_I and describe its role in computing the dimension of framed instanton homology of Dehn surgeries and its relation to Baldwin-Sivek's $\nu^\sharp$ invariant.

Contents

1. Introduction

1.1. Statement of the results.

Since instanton homology theory was introduced by Floer [Flo88], there have been four major Floer theories developed: instanton Floer homology, Monopole Floer homology by Kronheimer-Mrowka [KM07], Heegaard Floer homology by Ozsváth-Szabó [OS04b], and embedded contact homology by Hutchings [HS06]. Among these four known branches of Floer theories, the latter three have been known to be isomorphic by work of Kutluhan-Lee-Taubes [KLT10] and subsequent papers, or Colin-Ghiggini-Honda [CGH11], Taubes [Tau10], and subsequent papers. However, instanton theory is still isolated.

Framed instanton homology was introduced by Kronheimer-Mrowka [KM11] and is conjectured to be isomorphic to the hat version of Heegaard Floer homology [KM10]. This conjecture is still widely open and, due to the computational difficulty of instanton Floer homology, not many examples are known. In recent years, many people have done computations of the framed instanton homology special families of 3-manifolds, see for example [LPCS20, BS19, BS20]. Yet more is focused on computing the dimension and many techniques are case-specific, while a general structure theory that relates the framed instanton homology of Dehn surgeries to the information from the knot complement still remains elusive. In [LY21], the authors of this paper proved a large surgery formula for framed instanton homology which led to a series of applications in computing the framed instanton homology and studying the representations of the fundamental groups of Dehn surgeries of some families of knots. However, in the previous work, the Dehn surgery slope must be large (at least $2g+1$ where g is the Seifert genus of the knot), and thus still not much is known about the framed instanton homology of small Dehn surgery slopes. In this paper, we propose a strategy to prove an integral surgery formula for knots inside a general 3-manifold, which is based on the sutured instanton homology SHI [KM10] of different sutured knot complements. Furthermore, we realized this strategy for the case of knots in S^3 and prove the following.

Theorem 1.1. *Suppose $K \subset S^3$ is a knot and let $\Gamma_{\frac{q}{p}}$ denote the suture on $S^3(K)$ consisting of two curves of slope $-\frac{q}{p}$. Given an integer m, for any large enough integer k, there exists an exact triangle*

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}}) \arrow[rr, "\pi_{m,k}"] & & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m+2k-1}) \arrow[dl] \\ & I^{\sharp}(-S^3_{-m}(K)) \arrow[ul] & \end{tikzcd} \tag{1.1}$$

where $\underline{\mathrm{SHI}}$ is the twisted refinement of sutured instanton homology SHI by Baldwin-Sivek [Gar19] (as a vector space, we have $\underline{\mathrm{SHI}} \cong SHI$), and the minus signs before manifolds denote the orientation reversal for technical reasons.

Note that there are $\mathbb{Z}$-gradings (see Section 2.1 for details) on

$$\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}}) \text{ and } \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m+2k-1})$$

as in [Li19, Section 3]. From [LY22, Theorem 1.12] (see [LY22, Proposition 4.26] for details), we have

$$\bigoplus_{i=-g(K)}^{g(K)} \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}},i) \cong I^{\sharp}(S^3_{-m-k}(K)), \tag{1.2}$$

where the left is the direct summand of consecutive grading parts. Also, From [LY22, Lemma 4.20 and Proposition 4.26], we have that for $|i| \leqslant \frac{m+2k-2}{2} - g(K)$,

$$\underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{m+2k-1}, i) \cong I^\sharp(-S^3). \tag{1.3}$$

Hence with a more careful study of the map $\pi_{m,k}$ in (1.1), we can truncate the above exact triangle and obtain the following.

Corollary 1.2. *Suppose $K \subset S^3$ is a knot and m is a fixed integer. Then for any large enough integer k, there exists an exact triangle*

$$\begin{tikzcd} I^\sharp(-S^3_{-m-k}(K)) \arrow[rr] & & \bigoplus_{i=1}^k I^\sharp(-S^3) \arrow[dl] \\ & I^\sharp(-S^3_{-m}(K)) \arrow[ul] & \end{tikzcd} \tag{1.4}$$

Remark 1.3. The analogous result of the exact triangle (1.4) in Heegaard Floer theory was proved by Ozsváth-Szabó [OS08], and is a crucial step towards the integral surgery formula in their setups.

In order to make use of Theorem 1.1, we need not only the existence of such an exact triangle, but also an explicit description of the map $\pi_{m,k}$. In most of the cases we could understand the map $\pi_{m,k}$ that we will now describe. In [BS18], Baldwin-Sivek introduced the bypass maps and the bypass exact triangle in sutured instanton theory. On knot complements, especially, we obtain the following bypass maps for any $n \in \mathbb{Z}$:

$$\psi_{\pm,n}^{\frac{2n-1}{2}} : \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{\frac{2n-1}{2}}) \to \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_n),$$

$$\psi_{\pm,n+1}^{n} : \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_n) \to \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{n+1}),$$

where the signs in the subscripts denote different choices of bypass arcs. We can then define the follow two compositions:

$$\pi_{m,k}^{\pm} = \psi_{\pm,m+2k}^{m+2k-1} \circ ... \circ \psi_{\pm,m+k+1}^{m+k} \circ \psi_{\mp,m+k}^{\frac{2m+2k-1}{2}}.$$

Then we have the following.

Proposition 1.4. *Suppose $K \subset S^3$ is a knot and $m \in \mathbb{Z}$ is a nonzero integer. Then for any large enough k, the map $\pi_{m,k}$ in the exact triangle (1.6) can be written as*

$$\pi_{m,k} = \pi_{m,k}^{+} + \pi_{m,k}^{-}.$$

Remark 1.5. Note the maps $\pi_{m,k}^{\pm}$ are closely related to each other: if we reverse the orientation of the knot, then the positive and negative bypasses switch between each other and so do $\pi_{m,k}^{\pm}$.

The maps $\pi_{m,k}^{\pm}$ still seem elusive, but they are expected to be closely related to the natural cobordism map between $I^\sharp(-S^3_{-m-k}(K))$ and $I^\sharp(-S^3)$. In particular, we make the following conjecture.

Conjecture 1.6. Suppose $K \subset S^3$ is a knot, $m, k \in \mathbb{Z}$ and k is large enough. Let $W = W_{m,k}$ be the natural cobordism from $-S^3_{-m-k}$ that is associated to the $-(m+k)$-surgery along K. Let S be a Seifert surface of K and $\bar{S}$ be the closed surface inside W by capping off S with the core of the

2-handle associated to the Dehn surgery. From [BS19, Section 1.2], there exists a decomposition of the cobordism map $I^\sharp(W)$ along basic classes

$$I^\sharp(W)=\sum_{s\in\mathbb{Z}} I^\sharp(W,[s]),$$

where $[s]\in H^2(W)$ denote the class that satisfies the equality

$$[s]([\bar{S}])=2s.$$

Let

$$\pi^{+,i}_{m,k}:\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}},i)\to\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m+2k-1},i+\frac{m}{2})$$

be the restriction

$$\pi^{+,i}_{m,k}=\pi^{+}_{m,k}|_{\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}},i)} \tag{1.5}$$

that is zero on all other gradings of $\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}})$. Then the following diagram commutes for any $s\in[-g(K),g(K)]$

$$\begin{tikzcd}
\bigoplus_{i=-g(K)}^{g(K)}\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}},s) \arrow[r, "\cong"] \arrow[d, "\pi^{+,s}_{m,k}"] & I^\sharp(-S^3_{-m-k}(K)) \arrow[d, "I^\sharp(W,[s])"] \\
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m+2k-1},s+\frac{m}{2}) \arrow[r, "\cong"] & I^\sharp(-S^3)
\end{tikzcd}$$

Remark 1.7. In [LY21, Section 1.2 and Section 3.5], the authors of this paper constructed differentials $d_\pm$ on the instanton knot homology

$$KHI(-S^3,K):=\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{1}{0}})=\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_\mu)$$

and used those differentials to construct bent complexes following the ideas from Ozsváth-Szabó [OS04a, OS08]. It is worth mentioning that the maps $\pi^{\pm}_{m,k}$ are also closely related to the hat versions of the projection maps $\hat{v}_s$ and $\hat{h}_s$ in [OS08]. Indeed, we can also restate Theorem 1.1 and Theorem 1.14 below by bent complexes constructed in [LY21]; see Section 2.4 for more details. Then the relation between the results in this paper and [OS08, Section 4.3] would be more clear. In Section 2.5, we also state a formula of the instanton knot homology of the dual knots in the integral Dehn surgeries by bent complexes, which is analogous to results by Eftekhary [Eft18, Proposition 1.5] for Heegaard Floer theory.

Although we do not know whether Conjecture 1.6 is true or false yet, we can derive some basic properties of the maps $\pi^{\pm,i}_{m,k}$.

Lemma 1.8. *Suppose $K\subset S^3$ is a knot. Let $\tau=\tau_I(K)$ be the tau invariant as defined in [Li19] (see also (3.9)). For any fixed integer m and large enough integer k, let $\pi^{\pm,i}_{m,k}$ be the restriction as in (1.5). Then we have the following properties.*

1. *When $i>\tau$, $\pi^{+,i}_{m,k}=0$. When $i<-\tau$, $\pi^{-,i}_{m,k}=0$.*
2. *When $i<\tau$, $\pi^{+,i}_{m,k}\neq 0$. When $i>-\tau$, $\pi^{-,i}_{m,k}\neq 0$.*
3. *When $i\leqslant -g(K)$, $\pi^{+,i}_{m,k}$ is an isomorphism. When $i\geqslant g(K)$, $\pi^{-,i}_{m,k}$ is an ismorphism.*

This lemma enables us to define a new concordance invariant as follows.

Definition 1.9. Suppose $K\subset S^3$ is a knot and let $\pi^{\pm,i}_{m,k}$ be the restriction as in (1.5). Suppose k is large enough. Then define

$$\nu_I(K)=\begin{cases}\tau_I(K)+1 & \text{if } \pi^{+,\tau_I(K)}_{m,k}\neq 0,\\ \tau_I(K) & \text{if } \pi^{+,\tau_I(K)}_{m,k}=0.\end{cases}$$

Proposition 1.10. *Suppose two knots $K_1,K_2\subset S^3$ are concordant to each other. Then*

$$\nu_I(K_1)=\nu_I(K_2).$$

As an application of the integral surgery formula described by Theorem 1.1 and Proposition 1.4, we can compute the dimension of the framed instanton homology of integral Dehn surgeries along the knot.

Proposition 1.11. *For any knot $K\subset S^3$ such that $\tau_I(K)\neq 0$, we have*

$$\dim_{\mathbb{C}} I^\sharp(S^3_n(K))=\dim_{\mathbb{C}} I^\sharp(S^3_{2\nu_I(K)-1}(K))+|n+1-2\nu_I(K)|.$$

We can compare the above formula with the one obtained by Baldwin-Sivek [BS20]. In particular, their formula depends on a concordance invariant $\nu^\sharp(K)$ and thus we conclude the following.

Corollary 1.12. *For any knot $K\subset S^3$ such that $\tau_I(K)\neq 0$, we have*

$$\nu^\sharp(K)=2\nu_I(K)-1.$$

Remark 1.13. With a little more work, we could also understand the case of $\tau_I=0$ and then recover the fact that $\nu^\sharp(K)$ is either odd or zero, which is proved independently by Baldwin-Sivek [BS].

It is worth mentioning that there is a scalar ambiguity in sutured instanton theory. According to work of Baldwin-Sivek [BS15], sutured instanton homology $\underline{\mathrm{SHI}}$ is only well-defined up to the multiplication of a nonzero complex number. Hence all the maps between sutured instanton homology are also only well-defined up to a scalar. Thus we must be very careful when writing down the equation

$$\pi_{m,k}=\pi^+_{m,k}+\pi^-_{m,k}.$$

Here $\pi^+_{m,k}$ and $\pi^-_{m,k}$ are both up to scalars, so the map $\pi_{m,k}$ is even not well-defined up to a scalar. However, from the construction we know $\pi^\pm_{m,k}$ shift the grading by $\pm\frac{m}{2}$, so when $m\neq 0$, $\pi^+_{m,k}$ and $\pi^-_{m,k}$ have different grading shifts. As a result, the dimensions of the kernel and the cokernel of $\pi_{m,k}$ are both fixed for any choices of nonzero scalars of $\pi^\pm_{m,k}$, and hence Proposition 1.4 is valid whenever $m\neq 0$. On the other hand, there are potential problems for the case $m=0$. When $m=0$, $\pi^+_{m,k}$ and $\pi^-_{m,k}$ are both grading preserving. So when $m=0$ and $\pi^+_{m,k}$ and $\pi^-_{m,k}$ are both nonzero, the scalar ambiguity becomes an essential obstacle for generalizing Proposition 2.23. Note for the case $m=0$, Theorem 1.1 is still valid, but we no longer have an explicit description of the map $\pi_{m,k}$. At end of Section 1.3, we will further present an example that this kind of ambiguity indeed exists in instanton theory.

However, one can still understand a large part of $I^\sharp(-S^3_0(K))$, *i.e.*, the case $m=0$. In $S^3_0(K)$, the Seifert surface S of the knot can be capped off by the Dehn surgery to become a closed surface $\widehat{S}\subset S^3_0(K)$. This surface induces a $\mathbb{Z}$-grading on $I^\sharp(S^3_0(K))$ by taking the generalized eigenspaces of the action $\mu(\widehat{S})$ on $I^\sharp(S^3)$ associated to $\widehat{S}$ (*c.f.* [BS19, Section 2.6]):

$$I^\sharp(S^3_0(K))\cong\bigoplus_{i=1-g(K)}^{g(K)-1} I^\sharp(S^3_0(K),i).$$

In this case, Theorem 1.1 can be stated grading-wise. Moreover, one can make use of Lemma 1.8 to get rid of the scalar ambiguity in most of the cases.

Theorem 1.14. *Suppose $K \subset S^3$ is a knot. For any large enough integer $k \in \mathbb{Z}$ and a nonzero integer i, there exists an exact triangle*

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{\frac{2k-1}{2}}, i) \arrow[rr, "\pi^i_{0,k}"] & & \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{2k-1}, i) \arrow[dl] \\
& I^\sharp(-S^3_0(K), i) \arrow[ul] &
\end{tikzcd} \tag{1.6}$$

Moreover, if $\tau_I(K) \leqslant 0$ and $i \in \mathbb{Z}\backslash\{0\}$, the map $\pi^i_{0,k}$ in the above exact triangle can be chosen as

$$\pi^i_{0,k} = \pi^{+,i}_{0,k} + \pi^{-,i}_{0,k}.$$

Remark 1.15. Note in Theorem 1.14, in order to describe $\pi^i_{0,k}$, we need to make an extra assumption that $\tau_I(K) \leqslant 0$. However, for any knot K, from [GLW19] we know that $\tau_I(\bar{K}) = -\tau_I(K)$, where $\bar{K}$ is the mirror of K. Hence we can always pass to the mirror of K if necessary.

1.2. **Strategy of the proofs.**
To simplify the presentation, in this subsection we ignore the ambiguity of scalars and deal with the issue more carefully in Section 2.3. In order to prove Theorem 1.1 for a knot $K \subset S^3$, we want to apply the octahedron axiom to the following diagram (we write Γ_μ for $\Gamma_{\frac{1}{0}}$):

$$\begin{tikzcd}
& & \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{\frac{2m+2k-1}{2}}) \arrow[drr, dotted] \arrow[dll] & & \\
I^\sharp(-S^3_n(K)) \arrow[urr, dotted] \arrow[d] & & & & \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{m-1+2k}) \arrow[llll, dotted] \arrow[ddll] \\
\underline{\mathrm{SHI}}(-S^3(K), -\Gamma_\mu) \arrow[rrrr] \arrow[drr] & & & & \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{m-1+k}) \otimes \mathbb{C}^2 \arrow[u] \arrow[uull] \\
& & \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{m-1}) \arrow[uull] \arrow[urr] & &
\end{tikzcd} \tag{1.7}$$

To show the dotted exact triangle exist, We need to prove the following three exact triangles

$$\begin{tikzcd}
I^\sharp(-S^3_n(K)) \arrow[d] & \\
\underline{\mathrm{SHI}}(-S^3(K), -\Gamma_\mu) \arrow[dr] & \\
& \underline{\mathrm{SHI}}(-S^3(K), -\Gamma_{m-1}) \arrow[uul]
\end{tikzcd} \tag{1.8}$$

$$
\begin{tikzcd}
 & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}}) \arrow[dl] & \\
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\mu}) \arrow[rr] & & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+k})\otimes\mathbb{C}^2 \arrow[ul]
\end{tikzcd}
\tag{1.9}
$$

$$
\begin{tikzcd}
 & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+2k}) \arrow[ddl] \\
 & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+k})\otimes\mathbb{C}^2 \arrow[u] \\
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1}) \arrow[ur] &
\end{tikzcd}
\tag{1.10}
$$

and the following commutative diagram

$$
\begin{tikzcd}
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\mu}) \arrow[rr] \arrow[dr] & & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+k})\otimes\mathbb{C}^2 \\
 & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1}) \arrow[ur] &
\end{tikzcd}
\tag{1.11}
$$

Then the octahedral axiom will imply the existence of the dotted triangle and ensure that all diagrams in (1.7) other than exact triangles commute.

To identify the map $\pi_{m,k}$ with $\pi^+_{m,k}+\pi^-_{m,k}$ as in Proposition 1.4, we also need the following two extra commutative diagrams, where the maps other than $\pi^+_{m,k}+\pi^-_{m,k}$ come from (1.7).

$$
\begin{tikzcd}
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}}) \arrow[dr, "\pi^+_{m,k}+\pi^-_{m,k}"] & \\
 & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+2k}) \\
 & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+k})\otimes\mathbb{C}^2 \arrow[u] \arrow[uul]
\end{tikzcd}
\tag{1.12}
$$

(1.13)
$$\begin{tikzcd}
& \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\frac{2m+2k-1}{2}}) \arrow[dl] \arrow[dr, "\pi^+_{m,k}+\pi^-_{m,k}"] & \\
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{\mu}) \arrow[dr] & & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1+2k}) \arrow[dl] \\
& \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{m-1}) &
\end{tikzcd}$$

The triangle (1.8) is simply the surgery exact triangle associated to a (framed) curve on $\partial S^3(K)$ of slope $-m$ (*c.f.* [LY22, Lemma 4.9]).

The triangle (1.9) first appeared in the proof of the large surgery formula that was proved by another octahedral diagram (*c.f.* the proof of [LY21, Theorem 3.23]). Since we need to understand more about maps in this triangle to show the commutative diagrams (1.12) and (1.13), we give an alternative proof of (1.9) in this paper by diagram chasing. It is worth mentioning that this triangle holds for arbitrary positive integer k, without assuming k to be large.

The triangle (1.10) is essentially a new triangle. Note the suture Γ_{m-1} is the meridional suture of the dual knot for the $1-m$ Dehn surgery along K. Since m is an arbitrary integer, the exact triangle (1.10) can also be used to deduce a formula for the instanton knot homology of dual knots of the integral Dehn surgeries mentioned in Remark 1.7. We prove this triangle by constructing all three maps explicitly and verify the exactness at each one of the three instanton homology groups.

The commutative diagram (1.11) depends on a reinterpretation of the (-1) surgery map along a curve in terms of the two bypass maps associated to the curve. This relation will be stated explicitly in the next subsection (*c.f.* Proposition 1.16), which might be of independent interest. The commutative diagram (1.12) is entirely based on the commutativity of bypass maps that the authors already proved in [LY22, Section 4.4]. The last commutative diagram (1.13) is essentially a new one. We prove the commutativity up to a scalar by identifying the kernels of the two composition maps and then prove that these two maps have the same 1-dimensional image.

1.3. **(-1)-Dehn surgery and bypasses.**

As mentioned in the last subsection, the triangle (1.8) comes from the surgery exact triangle along a (framed) curve on $\partial S^3(K)$ of slope $-m$. The map corresponding to the (-1)-surgery (with respect to the framing) is then involved in the diagram (1.11). In order to prove the commutativity, we need to re-interpret the map associated to the (-1)-surgery using bypasses. This result holds for a general balanced sutured manifold, and we think it might be of independent interest, so we present the result in this separate subsection.

Suppose (M,γ) is a balanced sutured manifold and $\alpha\subset\partial M$ is a connected simple closed curve that intersects the suture γ twice. There are two natural bypass arcs associated to α, which lead to two bypass triangles (*c.f.* [BS18, Section 4])

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[rr, "\psi_\pm"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl] \\ & \underline{\mathrm{SHI}}(-M,-\gamma_3) \arrow[ul] & \end{tikzcd}$$

where γ_2 and γ_3 are the sutures coming from bypass attachments. Note the two bypass exact triangles involve the same set of balanced sutured manifolds but with different maps between them. Let (M_0,γ_0) be obtained from (M,γ) by attaching a contact 2-handle along α. From [BS16, Section 3.3], it is shown that a closure of $(-M_0,-\gamma_0)$ coincides with a closure of the sutured manifold obtained from $(-M,-\gamma)$ by 0-surgery along α with respect to the surface framing. Hence there is also a surgery exact triangle (*c.f.* [LY22, Lemma 3.21])

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[rr, "H_\alpha"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl] \\ & \underline{\mathrm{SHI}}(-M_0,-\gamma_0) \arrow[ul] & \end{tikzcd}$$

The map H_α is related to the bypass maps $\psi_\pm$ as follows:

Proposition 1.16. *There exist* $c_1,c_2\in\mathbb{C}\backslash\{0\}$*, so that*

$$H_\alpha = c_1\psi_+ + c_2\psi_-.$$

Remark 1.17. The proof of Proposition 1.16 is obtained during the discussion with John A. Baldwin and Steven Sivek.

We prove Proposition 1.16 by a local argument: we can give a collar of ∂M a product contact structure with γ being the dividing set and dig out a contact neighborhood of the curve α. Then the surgeries and two bypasses all happen inside this small neighborhood of α. On this small neighborhood, the map associated to (-1)-surgery and the two bypass maps can be described explicitly so the relation in the statement of Proposition 1.16 is achieved within the neighborhood. Then we use the gluing theorem developed by the first author [Li18] to extend this relation from the small neighborhood of α to the original sutured manifold (M,γ).

At the end of Section 1.1, we discussed the ambiguity coming from scalars. It is worth to mention that such ambiguity already exists in instanton theory. For example, if M is the complement of a knot $K\subset S^3$ and γ consists of two meridians of the knot, which we denote by Γ_μ in latter sections, we can pick α to be a curve on $\partial S^3(K)$ of slope $-n$. Then we have a surgery triangle:

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\Gamma_\mu) \arrow[rr, "H_n"] & & \underline{\mathrm{SHI}}(-M,-\Gamma_{n-1}) \arrow[dl] \\ & I^\sharp(-S^3_{-n}(K)) \arrow[ul] & \end{tikzcd}$$

Note this triangle is not the one from Floer's original exact triangle, but the one with slight modification on the choice of 1-cycles inside the 3-manifold that represents the second Stiefel-Whitney class of the relevant $SO(3)$-bundle; see [BS20, Section 2.2] for more details. Floer's original exact triangle, on the other hand, yields a different triangle

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_\mu) \arrow[rr, "H'_n"] & & \underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{n-1}) \arrow[dl] \\
& I^\sharp(-S^3_{-n}(K),\mu) \arrow[ul] &
\end{tikzcd}$$

where $\mu\subset -S^3_{-n}(K)$ denotes a meridian of the knot. Note the difference between H_α and H'_α is that they come from the same cobordism but the $SO(3)$-bundles over the cobordism are different. The local argument to prove Proposition 1.16 works for both H_α and H'_α. Hence there exists non-zero complex numbers c_1,c_2,c'_1,c'_2 so that

$$H_\alpha=c_1\psi^\mu_{+,n}+c_2\psi^\mu_{-,n-1} \text{ and } H'_\alpha=c'_1\psi^\mu_{+,n}+c'_2\psi^\mu_{-,n-1}$$

where the maps

$$\psi^\mu_{\pm,n-1}:\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_\mu)\to\underline{\mathrm{SHI}}(-S^3(K),-\Gamma_{n-1})$$

are the two related bypass maps. When $n\neq 0$, these two bypass maps have different grading shifting behavior, so different choice of non-zero coefficients does not change the dimension of kernel and cokernel of the map. Hence we conclude that for $n\neq 0$,

$$I^\sharp(-S^3_{-n}(K),\mu)\cong I^\sharp(-S^3_{-n}(K)).$$

However, when $n=0$, the two bypass maps $\psi^\mu_{\pm,n-1}$ are both grading preserving, so the coefficients matters, *i.e.*, $I^\sharp(-S^3_0(K),\mu)$ and $I^\sharp(-S^3_0(K))$ might have different dimensions. Indeed, it is observed by Baldwin-Sivek [BS20] that for what they called as W-shaped knots (which is clearly a non-empty class), these two framed instanton homology groups have dimensions differed by 2.

Organization. The paper is organized as follows. In Section 2, we introduce some notations for knots in general 3-manifolds and state a generalization of the strategy in Section 1.2, taking the case of general 3-manifolds and scalar ambiguity into consideration. (Though in this paper we only realize the strategy for knots in S^3.) In Section 3, we prove the exact triangles and commutative diagrams mentioned in Section 1.2, some of which are only for S^3 and others are for general 3-manifolds. In particular, we prove Proposition 1.16 in Section 3.1. In Section 4, we study properties of $\pi^\pm_{m,k}$ and the 0-surgery. In particular, we prove Lemma 1.8, Proposition 1.10, Proposition 1.11, and Theorem 1.14.

Convention. If it is not mentioned, all manifolds are smooth, oriented, and connected. Homology groups and cohomology groups are with $\mathbb{Z}$ coefficients. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{F}_2$ for the field with two elements.

A knot $K\subset Y$ is called **null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Z})$, while it is called **rationally null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Q})$.

For a knot K in a closed 3-manifold Y, we write $Y\backslash K$ for the knot complement $Y\backslash \mathrm{int}N(K)$ and then $\partial Y\backslash K\cong T^2$. This is different from the notation $Y\backslash K$ we used in previous papers. We write $Y_r(K)$ for the manifold obtained from Y by a r-surgery (with respect to some given basis of $H_1(\partial Y\backslash K;\mathbb{Z})$).

For any oriented 3-manifold M, we write $-M$ for the manifold obtained from M by reversing the orientation. For any surface S in M and any suture $\gamma\subset\partial M$, we write S and γ for the same surface and suture in $-M$, without reversing their orientations. For a knot K in a 3-manifold Y, we write $(-Y,K)$ for the induced knot in $-Y$ with induced orientation, called the **mirror knot** of K. The corresponding balanced sutured manifold is $(-Y\backslash K,-\gamma_K)$.

Acknowledgement. The authors thank John A. Baldwin and Steven Sivek for the discussion on the proof of Proposition 1.16, and thank Zekun Chen and Linsheng Wang for the discussion on homological algebra. The authors also thank Sudipta Ghosh, Jianfeng Lin, Yi Xie and Ian Zemke for valuable discussions. The second author would like to thank his supervisor Jacob Rasmussen for patient guidance and helpful comments and thank his parents for support and constant encouragement. The second author is also grateful to Yi Liu for inviting him to BICMR, Peking University.

2. Integral surgery formulae

Suppose $\widehat{K}$ is a (framed) rationally null-homologous knot in a closed 3-manifold $\widehat{Y}$. Given $m\in\mathbb{Z}$, suppose $\widehat{K}_{-m}$ is the dual knot in the manifold $\widehat{Y}_{-m}(\widehat{K})$ obtained from $\widehat{Y}$ by $(-m)$-surgery along $\widehat{K}$. In this section, we propose a conjectural formula calculating $I^\sharp(-\widehat{Y}_{-m}(\widehat{K}))$ analogous to the integral surgery formula in Heegaard Floer theory [OS08, OS11]. We also propose a strategy to prove this formula, but we could only realize the proof for $\widehat{Y}=S^3$ in Section 3 because we have to use the facts that $\dim_{\mathbb{C}} I^\sharp(S^3)=1$ and S^3 has an orientation-reversing involution in some steps.

2.1. Basic setups.

In this subsection, we review notations in [LY21, Section 3] and omit definitions and detailed discussions. Also, we introduce more notations for convenience.

For any **balanced sutured manifold** (M,γ) [Juh06, Definition 2.2], Kronheimer-Mrowka [KM10, Section 7] constructed an isomorphism class of $\mathbb{C}$-vector spaces $SHI(M,\gamma)$. Later, Baldwin-Sivek [BS15, Section 9] dealt with the naturality issue and constructed a projectively transitive system (twisted version) related to $SHI(M,\gamma)$ which we write as $\underline{\mathrm{SHI}}(M,\gamma)$ and call **sutured instanton homology**. In practice, when considering maps between sutured instanton homology, we can regard them as linear maps between actual vector spaces, at the cost that equations (or commutative diagrams) between maps only hold up to a scalar due to the projectivity.

Suppose K is a knot in a closed 3-manifold Y. Let $Y(1):=Y\backslash B^3$ and let δ be a simple closed curve on $\partial Y(1)\cong S^2$. Suppose Γ_μ is two copies of the meridian of K with opposite orientations. Define

$$I^\sharp(Y):=\underline{\mathrm{SHI}}(Y(1),\delta)\text{ and }\underline{\mathrm{KHI}}(Y,K):=\underline{\mathrm{SHI}}(Y\backslash K,\Gamma_\mu).$$

Given an **admissible surface** $S\subset(M,\gamma)$ [GL19, Definition 2.25], there is a $\mathbb{Z}$-grading on $\underline{\mathrm{SHI}}(M,\gamma)$ associated to S [Li19, GL19] which we write as $\underline{\mathrm{SHI}}(M,\gamma,S,i)$. When $M=Y\backslash K$ for a rationally null-homologous knot $K\subset Y$ and $\gamma\subset\partial M$ consists of two parallel copies of curves, we can construct an admissible surface S from a minimal genus (rational) Seifert surface of K [LY22, Definition 4.10]. Explicitly, suppose S has $2q$ intersection with γ. When q is odd, then S is admissible and induces a $\mathbb{Z}$-grading. When q is even, we need to perform either a positive stabilization or negative stabilization on S to make it admissible and induce a $\mathbb{Z}$-grading, and the two gradings induced by positive and negative stabilizations are differed by an overall grading shift of 1. To get rid of stabilizations, we can equivalently think that when q is even, the surface S induces a $(\mathbb{Z}+\frac{1}{2})$-grading. If it is not mentioned, we will consider such $\mathbb{Z}$ or $(\mathbb{Z}+\frac{1}{2})$-grading on $\underline{\mathrm{SHI}}(Y\backslash K,\gamma)$ associated to S.

Moreover, there is a relative $\mathbb{Z}_2$-grading on $\underline{\mathrm{SHI}}(M,\gamma)$ obtained from the construction of sutured instanton homology, which we consider as a **homological grading**.

Suppose $K\subset Y$ is a null-homologous knot. We will always assume that the knot complement $Y\backslash K$ is irreducible due to the discussion in [LY21, Section 3.2]. Let λ be the boundary of the Seifert surface of K and let μ be the meridian of K so that $\mu\cdot\lambda=-1$. Given $p,q\in\mathbb{Z}$ satisfying $\gcd(p,q)=1$ and $q>0$ or $(p,q)=(1,0)$, let $\widehat{Y}=\widehat{Y}_{q/p}$ be the 3-manifold obtained from Y by a q/p surgery along K and let $\widehat{K}=\widehat{K}_{q/p}$ be the dual knot, *i.e.* the core of the Dehn filling solid torus. Given a simple closed curve $\alpha\subset\partial Y\backslash K$, we also write $(\widehat{Y}_\alpha,\widehat{K}_\alpha)$ for the corresponding Dehn filling manifold and core knot. In particular, $(\widehat{Y},\widehat{K})=(\widehat{Y}_{\hat{\mu}},\widehat{K}_{\hat{\mu}})$.

The meridian of $\widehat{K}$ is written as $\hat{\mu}=q\mu+p\lambda$. We introduce a canonical way to find the corresponding longitude $\hat{\lambda}$ of K so that $\hat{\mu}\cdot\hat{\lambda}=-1$ as follows.

If $p=0$, then $q=1$ and $\hat{\mu}=\mu$. We can take $\hat{\lambda}=\lambda$. If $(q,p)=(0,1)$, then we take $\hat{\lambda}=-\mu$. If $p,q\neq 0$, then we take $\hat{\lambda}=q_0\mu+p_0\lambda$, where (q_0,p_0) is the unique pair of integers so that the following conditions are true.

(1) $0\leqslant|p_0|<|p|$ and $p_0p\leqslant 0$.
(2) $0\leqslant|q_0|<|q|$ and $q_0q\leqslant 0$.
(3) $p_0q-pq_0=1$.

In particular, if $(q,p)=(n,1)$, then $\hat{\lambda}=-\mu$.

For a homology class $x\lambda+y\mu$, let $\gamma_{x\lambda+y\mu}$ be the suture consisting of two disjoint simple closed curves representing $\pm(x\lambda+y\mu)$ on $\partial Y\backslash K$. Since the two components of the suture must be given opposite orientations, the notations $\gamma_{x\lambda+y\mu}$ and $\gamma_{-x\lambda-y\mu}$ represent the same suture. For $n\in\mathbb{Z}$, define

$$\widehat{\Gamma}_n(q/p)=\gamma_{\hat{\lambda}-n\hat{\mu}}=\gamma_{(p_0-np)\lambda+(q_0-nq)\mu},\ \text{and}\ \widehat{\Gamma}_\mu(q/p)=\gamma_{\hat{\mu}}=\gamma_{p\lambda+q\mu}.$$

When emphasizing the choice of $\hat{\mu}$, we also write $\widehat{\Gamma}_n(\hat{\mu})$ and $\widehat{\Gamma}_\mu(\hat{\mu})$. When $\hat{\lambda}$ and $\hat{\mu}$ are given, we omit the slope q/p and simply write $\widehat{\Gamma}_n$ and $\widehat{\Gamma}_\mu$. When $(q,p)=(1,0)$, we write Γ_n and Γ_μ instead.

Since we will use bypass maps, it is more convenient to consider $\underline{\mathrm{SHI}}(-M,-\gamma)$ instead of $\underline{\mathrm{SHI}}(M,\gamma)$. For simplicity, when the knot and the 3-manifold are understood, we will use the bolded notation for sutures to denote the sutured instanton homology. In particular, we write

$$\widehat{\mathbf{\Gamma}}_n=\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_n)\ \text{and}\ \widehat{\mathbf{\Gamma}}_\mu=\underline{\mathrm{SHI}}(-Y\backslash K,-\widehat{\Gamma}_\mu).$$

Similar notations apply to other sutures on $\partial Y\backslash K$. With this simplified notation, the bypass exact triangles in [LY22, Proposition 4.14] are written as

$$\begin{tikzcd}[column sep=small] \widehat{\mathbf{\Gamma}}_n \arrow[rr, "\psi^n_{+,n+1}"] & & \widehat{\mathbf{\Gamma}}_{n+1} \arrow[dl, "\psi^{n+1}_{+,\mu}"] \\ & \widehat{\mathbf{\Gamma}}_\mu \arrow[ul, "\psi^\mu_{+,n}"] & \end{tikzcd} \text{ and } \begin{tikzcd}[column sep=small] \widehat{\mathbf{\Gamma}}_n \arrow[rr, "\psi^n_{-,n+1}"] & & \widehat{\mathbf{\Gamma}}_{n+1} \arrow[dl, "\psi^{n+1}_{-,\mu}"] \\ & \widehat{\mathbf{\Gamma}}_\mu \arrow[ul, "\psi^\mu_{-,n}"] & \end{tikzcd} \tag{2.1}$$

and the commutative diagrams in [LY22, Lemma 4.33 and Lemma 4.34] (which holds up to multiplication by some $c \in \mathbb{C}\backslash\{0\}$) are written as

(2.2)
$$\begin{tikzcd}
\widehat{\boldsymbol{\Gamma}}_n \arrow[r, "\psi^n_{-,n+1}"] \arrow[d, "\psi^n_{+,n+1}"'] & \widehat{\boldsymbol{\Gamma}}_{n+1} \arrow[d, "\psi^{n+1}_{+,n+2}"] \\
\widehat{\boldsymbol{\Gamma}}_{n+1} \arrow[r, "\psi^{n+1}_{-,n+2}"] & \widehat{\boldsymbol{\Gamma}}_{n+2}
\end{tikzcd}$$

(2.3)
$$\begin{tikzcd}
\widehat{\boldsymbol{\Gamma}}_n \arrow[rr, "\psi^n_{\mp,n+1}"] \arrow[dr, "\psi^n_{\pm,\mu}"'] & & \widehat{\boldsymbol{\Gamma}}_{n+1} \arrow[dl, "\psi^{n+1}_{\pm,\mu}"] \\
& \widehat{\boldsymbol{\Gamma}}_\mu &
\end{tikzcd}
\text{ and }
\begin{tikzcd}
\widehat{\boldsymbol{\Gamma}}_n \arrow[rr, "\psi^n_{\mp,n+1}"] & & \widehat{\boldsymbol{\Gamma}}_{n+1} \\
& \widehat{\boldsymbol{\Gamma}}_\mu \arrow[ul, "\psi^\mu_{\pm,n}"] \arrow[ur, "\psi^\mu_{\pm,n+1}"'] &
\end{tikzcd}$$

From [LY22, Proposition 4.14], bypass maps in those triangles and commutative diagrams are homogeneous with respect to the $\mathbb{Z}$-grading and the grading shifts can be calculated explicitly. These maps are also homogeneous with respect to the $\mathbb{Z}_2$-grading.

In this paper, we usually consider the total sutured instanton homology, but the reader should keep in mind that most diagrams of spaces and maps can be split into $\mathbb{Z}$-gradings. When it is necessary, we write $(\widehat{\boldsymbol{\Gamma}}_n, i)$ for the grading i summand in the $\mathbb{Z}$-grading. From [LY22, Lemma 4.12], the maximal and the minimal nontrivial gradings of $\widehat{\boldsymbol{\Gamma}}_n$ are

$$\hat{i}^n_{max} = \frac{|q_0 - nq| - 1}{2} + g(K) \text{ and } \hat{i}^n_{min} = -\frac{|q_0 - nq| - 1}{2} - g(K) \tag{2.4}$$

and those for $\widehat{\boldsymbol{\Gamma}}_\mu$ are

$$\hat{i}^\mu_{max} = \frac{q - 1}{2} + g(K) \text{ and } \hat{i}^\mu_{min} = -\frac{q - 1}{2} - g(K). \tag{2.5}$$

Note that we will consider a $(\mathbb{Z} + \frac{1}{2})$-grading if q is even.

Also for simplicity, we introduce the following notation for compositions of bypass maps.

$$\Psi^n_{\pm,n+k} := \psi^{n+k-1}_{\pm,n+k} \circ \psi^{n+k-2}_{\pm,n+k-1} \circ \cdots \circ \psi^n_{\pm,n+1}. \tag{2.6}$$

We can put commutative diagrams from (2.2) and (2.3) for different n together to obtain commutative diagrams for $\Psi^n_{\pm,n+k}$.

From [LY22, Lemma 4.9], we also have the following exact triangle and commutative diagrams (which holds up to multiplication by some $c \in \mathbb{C}\backslash\{0\}$) involving framed instanton homology

(2.7)
$$\begin{tikzcd}
\widehat{\boldsymbol{\Gamma}}_n \arrow[rr, "H_n"] & & \widehat{\boldsymbol{\Gamma}}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \widehat{\mathbf{Y}}_{\hat{\mu}} \arrow[ul, "G_n"] &
\end{tikzcd}$$

$$\begin{tikzcd} \widehat{\mathbf{\Gamma}}_n \arrow[rr, "\psi^n_{\pm,n+1}"] \arrow[dr, "F_n"'] & & \widehat{\mathbf{\Gamma}}_{n+1} \arrow[dl, "F_{n+1}"] \\ & \widehat{\mathbf{Y}}_{\widehat{\mu}} & \end{tikzcd} \text{ and } \begin{tikzcd} \widehat{\mathbf{\Gamma}}_n \arrow[rr, "\psi^n_{\pm,n+1}"] & & \widehat{\mathbf{\Gamma}}_{n+1} \\ & \widehat{\mathbf{Y}}_{\widehat{\mu}} \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] & \end{tikzcd} \tag{2.8}$$

where we write $\widehat{\mathbf{Y}}_{\widehat{\mu}}$ for $I^\sharp(-\widehat{Y}_{\widehat{\mu}})$. The similar notations apply to $\widehat{\mathbf{Y}}_\alpha$ for $\alpha \subset \partial Y \backslash K$. Note that the maps H_n, G_n, F_{n+1} are obtained from surgery cobordisms and not homogeneous with respect to $\mathbb{Z}$-gradings in general.

2.2. **Algebraic lemmas.**

In this subsection, we introduce some lemmas in homological algebra. All graded vector spaces in this subsection are finite dimensional and over $\mathbb{C}$ and all maps are complex linear maps. For convenience, we will switch freely between long exact sequences and exact triangles.

From Section 2.1, we know the sutured instanton homology is usually $\mathbb{Z} \oplus \mathbb{Z}_2$-graded, where we regard the $\mathbb{Z}_2$-grading as a homological grading. Many results in this subsection come from properties of the derived category of vector spaces over $\mathbb{C}$, for which people usually consider cochain complexes. However, for a $\mathbb{Z}_2$-graded space there is no difference between the chain complex and the cochain complex. Hence by saying a **complex** we mean a $\mathbb{Z}_2$-graded (co)chain complex, though all results apply to $\mathbb{Z}$-graded cochain complexes verbatim.

For a complex C and an integer n, we write C^n for its grading n part (under the natural map $\mathbb{Z} \to \mathbb{Z}_2$). With this notation, we suppose the differential d_C on C sends C^n to C^{n+1}. For any integer k, we write $C\{k\}$ for the complex obtained from C by the grading shift $C\{k\}^n = C^{n+k}$. We write $H(C, d_C)$ or $H(C)$ for the homology of a complex C with differential d_C. A $\mathbb{Z}_2$-graded vector space is regarded as a complex with the trivial differential.

For a chain map $f : C \to D$, we write $\mathrm{Cone}(f)$ for the **mapping cone** of f, *i.e.*, the complex consisting of the space $D \oplus C\{1\}$ and the differential

$$d_{\mathrm{Cone}(f)} := \begin{bmatrix} d_D & -f \\ 0 & -d_C \end{bmatrix}.$$

Then there is a long exact sequence

$$\cdots \to H(C) \xrightarrow{f} H(D) \xrightarrow{i} H(\mathrm{Cone}(f)) \xrightarrow{p} H(C)\{1\} \to \cdots$$

where i sends $x \in D$ to $(x, 0)$ and p sends $(x, y) \in D \oplus C\{1\}$ to $-y$. If differentials of C and D are trivial, then we know

$$H(\mathrm{Cone}(f)) \cong \mathrm{Ker}(f) \oplus \mathrm{Coker}(f). \tag{2.9}$$

Remark 2.1. Our definitions about mapping cones follow from [Wei94], which are different from those in [OS08, OS11].

Note that the derived category is a triangulated category, so it satisfies the octahedral axiom (for example, see [Wei94, Proposition 10.2.4]).

Lemma 2.2 (Octahedral axiom)**.** *Suppose X,Y,Z,X',Y',Z' are $\mathbb{Z}_2$-graded vector spaces satisfying the following long exact sequences*

$$\cdots \to X \xrightarrow{f} Y \xrightarrow{h} Z' \to X\{1\} \to \cdots$$

$$\cdots \to X \xrightarrow{g\circ f} Z \xrightarrow{j} Y' \to X\{1\} \to \cdots$$

$$\cdots \to Y \xrightarrow{g} Z \to X' \xrightarrow{l} Y\{1\} \to \cdots$$

Then we have the fourth long exact sequence

$$\cdots \to Z' \xrightarrow{\psi} Y' \xrightarrow{\phi} X' \xrightarrow{h\{1\}\circ l} Z'\{1\} \to \cdots$$

such that the following diagram commutes

(2.10)
$$\begin{tikzcd}
& Y' \arrow[dr, dotted, "\phi"] \arrow[ddl] & \\
Z' \arrow[ur, dotted, "\psi"] \arrow[d] & & X' \arrow[ll, dotted, "h\{1\}\circ l"] \arrow[ddl] \\
X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \arrow[u] \arrow[uul] \\
& Y \arrow[ur, "g"'] \arrow[uul] &
\end{tikzcd}$$

where we omit the grading shifts and the notations for maps h,l,j. We can also write (2.10) in another form so that there is enough room to write the maps

(2.11)
$$\begin{tikzcd}
& & Z' \arrow[rr] \arrow[dr, "\psi"] & & X\{1\} \arrow[ddr, "f\{1\}"] & & \\
& Y \arrow[ur, "h"] \arrow[dr, "g"] & & Y' \arrow[ur] \arrow[ddrr, "\phi"] & & & \\
& & Z \arrow[ur, "j"] \arrow[drrr] & & & Y\{1\} \arrow[dr, "h\{1\}"] & \\
X \arrow[uur, "f"] \arrow[urr, "g\circ f"] & & & & & X' \arrow[u, "l"] \arrow[r, "h\{1\}\circ l"] & Z'\{1\}
\end{tikzcd}$$

The map ψ and ϕ in (2.11) can be written explicitly as follows. By the long exact sequences in the assumption of Lemma 2.2, we know that Z',X',Y' are chain homotopic to the mapping cones $\mathrm{Cone}(f),\mathrm{Cone}(g),\mathrm{Cone}(g\circ f)$, respectively. Under such homotopies, we can write

$$\psi : Y\oplus X\{1\} \to Z\oplus X\{1\}$$
$$\psi(y,x) \mapsto (g(y),x)$$

and

$$\phi : Z\oplus X\{1\} \to Z\oplus Y\{1\}$$
$$\phi(z,x) \mapsto (z,f\{1\}(x))$$

However, the chain homotopies are not canonical, and hence the maps ψ and ϕ are also not canonical. Thus, usually we cannot identify them with other given maps. Fortunately, with an extra $\mathbb{Z}$-grading, we may identify $H(\mathrm{Cone}(\phi))$ with $H(\mathrm{Cone}(\phi'))$ for another map $\phi' : Y' \to X'$.

We introduce the following lemma to deal with the projectivity of bypass maps (*i.e.* they are only well-defined up to a scalar). Note that the $\mathbb{Z}$-grading in the following lemma is not the homological grading used before.

Lemma 2.3. *Suppose X and Y are $\mathbb{Z}$-graded vector spaces and suppose $f, g : X \to Y$ are homogeneous maps with different grading shifts k_1 and k_2. Then $H(\mathrm{Cone}(f+g))$ is isomorphic to $H(\mathrm{Cone}(c_1 f + c_2 g))$ for any $c_1, c_2 \in \mathbb{C}\backslash\{0\}$.*

Proof. For simplicity, we can suppose $k_1 = 0$ and $k_2 = 1$. The proof for the general case is similar. For $i, j \in \mathbb{Z}$, we write (X, i) and (Y, j) for grading summands of X and Y, respectively. Suppose T is a automorphism of $X \oplus Y$ that acts by

$$\frac{c_2^i}{c_1^{i+1}}\mathrm{Id} \text{ on } (X,i) \text{ and } \frac{c_2^j}{c_1^j}\mathrm{Id} \text{ on } (Y,j).$$

Then T is a chain homotopy from $\mathrm{Cone}(f+g)$ and $\mathrm{Cone}(c_1 f + c_2 g)$ and induces an isomorphism between $H(\mathrm{Cone}(f+g))$ and $H(\mathrm{Cone}(c_1 f + c_2 g))$. $\square$

Then we state the lemma to relate ϕ in Lemma 2.2 to another map ϕ' constructed explicitly.

Lemma 2.4. *Suppose X, Y, Z, X', Y' are $\mathbb{Z} \oplus \mathbb{Z}_2$-graded vector spaces satisfying the following horizontal exact sequences.*

$$\begin{tikzcd}
Z \arrow[r, "j"] \arrow[d, "="] & Y' \arrow[r, "l'"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\phi'=a'+b'"] & X\{1\} \arrow[d, "f\{1\}=a+b"] \\
Z \arrow[r, "\phi\circ j=\phi'\circ j"] & X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}$$

where the shift $\{1\}$ is for the $\mathbb{Z}_2$-grading. Suppose $\phi : Y' \to X'$ satisfies the two commutative diagrams and suppose $\phi' : Y' \to X'$ satisfies the left commutative diagram. Suppose l and l' are homogeneous with respect to the $\mathbb{Z}$-grading. Suppose $f\{1\} = a + b$ and $\phi' = a' + b'$ are sums of homogeneous maps with different grading shifts with respect to the $\mathbb{Z}$-grading. Moreover, suppose the following diagrams hold up to scalars.

$$\begin{tikzcd}
Y' \arrow[r, "l'"] \arrow[d, "a'"] & X\{1\} \arrow[d, "a"] \\
X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y' \arrow[r, "l'"] \arrow[d, "b'"] & X\{1\} \arrow[d, "b"] \\
X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}$$

Then there is an isomorphism between $H(\mathrm{Cone}(\phi))$ and $H(\mathrm{Cone}(\phi'))$.

Proof. Since ϕ and ϕ' share the same domain and codomain, it suffices to show that they have the same rank. Fixing inner products on Y' and X' so that we have orthogonal decompositions

$$Y' = \mathrm{Im}(j) \oplus Y'' \text{ and } X' = \mathrm{Im}(\phi \circ j) \oplus X''.$$

By commutativity, we know both ϕ and ϕ' send $\mathrm{Im}(j)$ onto $\mathrm{Im}(\phi\circ j)$. Hence if we choose basis with respect to the decompositions so that linear maps are represented by matrices (we use row vectors), then we have

$$\phi=\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}\text{ and }\phi'=\begin{bmatrix}A' & B'\\ 0 & C'\end{bmatrix},$$

where $A=A':\mathrm{Im}(j)\to\mathrm{Im}(\phi\circ j)$ has full row rank. Then it suffices to show C and C' have the same row rank.

By the exactness at Y' and X', we know the restriction of l' on Y'' is an isomorphism between Y'' and $\mathrm{Im}(l')$ and the restriction of l on X'' is an isomorphism between X'' and $\mathrm{Im}(l)$. By commutativity, we know that both a and b send $\mathrm{Im}(l')$ to $\mathrm{Im}(l)$ and

$$\mathrm{rowrank}(C)=\mathrm{rank}(f\{1\}|_{\mathrm{Im}(l')})\text{ and }\mathrm{rowrank}(C')=\mathrm{rank}((c_1a+c_2b)|_{\mathrm{Im}(l')})$$

for some $c_1,c_2\in\mathbb{C}\backslash\{0\}$. Since l and l' are homogeneous, there exist induced $\mathbb{Z}$-gradings on $\mathrm{Im}(l)$ and $\mathrm{Im}(l')$. The maps a and b are still homogeneous with different grading shifts with respect to these induced gradings. Then we can apply Lemma 2.3 to obtain that the ranks of the restrictions of $f\{1\}=a+b$ and c_1a+c_2b on $\mathrm{Im}(l')$ are the same. $\square$

2.3. A strategy to prove the integral surgery formula.

In this subsection, we propose a conjectural integral surgery formula for a null-homologous knot $\widehat{K}\subset\widehat{Y}$ that generalizes Theorem 1.1 and Proposition 1.4. We also propose a strategy to prove this formula that is realized in later sections for the case $\widehat{Y}=S^3$. We point out the difficulties for a general $\widehat{Y}$ and hope to realize this strategy in general in future work.

Suppose K is a rationally null-homologous knot in a closed 3-manifold Y. Let $\hat{\mu}=q\mu+p\lambda$ with $q>0$ and $\hat{\lambda}=q_0\mu+p_0\lambda$ as chosen in Section 2.1. Let $(\widehat{Y},\widehat{K})$ be the dual knot with respect to $\hat{\mu}$. From now on, let $m\in\mathbb{Z}$ be fixed.

First, we introduce some notations that were first used in the proofs of [LY21, Theorem 3.23 and Theorem 3.31]. Define

$$\hat{\mu}':=m\hat{\mu}-\hat{\lambda}=(mq-q_0)\mu+(mp-p_0)\lambda$$

and

$$\hat{\lambda}':=\hat{\lambda}-(m-1)\hat{\mu}=(q_0-(m-1)q)\mu+(p_0-(m-1)p)\lambda.$$

Note that if $m>1$, then this choice of $(\hat{\mu}',\hat{\lambda}')$ satisfies the conditions for $(\hat{\mu},\hat{\lambda})$ in Section 2.1, *i.e.* we have

$$\begin{aligned}&mq-q_0>0,\\&0\leqslant|p_0-(m-1)p|\leqslant|mp-p_0|,(p_0-(m-1)p)(mp-p_0)\leqslant0,\\&0\leqslant|q_0-(m-1)q|\leqslant|mq-q_0|,(q_0-(m-1)q)(mq-q_0)\leqslant0,\\&(p_0-(m-1)p)(mq-q_0)-(q_0-(m-1)q)(mp-p_0)=1.\end{aligned}\tag{2.12}$$

However, even if $m\leqslant1$, we could still use $(\hat{\mu}',\hat{\lambda}')$ as a new basis of the boundary torus. The basis is not canonical in the sense of Section 2.1 since equations in (2.12) do not always hold, but we can still adapt the notations mentioned below.

By direct calculation, we have

$$\widehat{\Gamma}_\mu(\hat{\mu}')=\gamma_{\hat{\mu}'}=\widehat{\Gamma}_m,\quad\widehat{\Gamma}_{-1}(\hat{\mu}')=\gamma_{\hat{\lambda}'+\hat{\mu}'}=\widehat{\Gamma}_\mu,$$
$$\widehat{\Gamma}_0(\hat{\mu}')=\gamma_{\hat{\lambda}'}=\widehat{\Gamma}_{m-1},\text{ and }\widehat{\Gamma}_1(\hat{\mu}')=\gamma_{\hat{\lambda}'-\hat{\mu}'}=\gamma_{2\hat{\lambda}-(2m-1)\hat{\mu}}.$$

We write $\psi^{*}_{\pm,*}(\hat{\mu}')$ for the bypass maps between spaces

$$\widehat{\mathbf{\Gamma}}_n(\hat{\mu}') := \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_n(\hat{\mu}')) \text{ and } \widehat{\mathbf{\Gamma}}_\mu(\hat{\mu}') := \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_\mu(\hat{\mu}')).$$

Note that n can be any integer. Then we can identify

$$\psi^{-1}_{\pm,0}(\hat{\mu}') = \psi^{\mu}_{\mp,m-1} := \psi^{\mu}_{\mp,m-1}(\hat{\mu}), \psi^{\mu}_{\pm,-1}(\hat{\mu}') = \psi^{m}_{\mp,\mu}, \psi^{0}_{\pm,\mu}(\hat{\mu}') = \psi^{m}_{\mp,m-1}. \tag{2.13}$$

For any n, we also write $F_n(\hat{\mu}'), G_{n+1}(\hat{\mu}')$, and $H_n(\hat{\mu}')$ for maps in (2.7) when replacing $\hat{\mu}$ by $\hat{\mu}'$.

When a large integer k is fixed, we also define $\hat{\mu}'' = (m+k)\hat{\mu} - \hat{\lambda}$ and adapt other notations similarly. Note that $m+k>0$ when k is large enough so we do not have the issue mentioned after (2.12).

Then we can state the conjectural integral surgery formula.

Conjecture 2.5. Suppose $\widehat{K}$ is a rationally null-homologous knot in $\widehat{Y}$. Given an integer m, for any large enough integer k, there exists an exact triangle

$$\begin{tikzcd}
\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \arrow[rr, "\pi_{m,k}(\hat{\mu})"] & & \widehat{\mathbf{\Gamma}}_{m-1+2k} \arrow[dl] \\
& \widehat{\mathbf{Y}}_{\hat{\lambda}-m\hat{\mu}} = I^{\sharp}(-\widehat{Y}_{-m}(\widehat{K})) \arrow[ul] &
\end{tikzcd} \tag{2.14}$$

Moreover, if $\hat{\lambda} - m\hat{\mu}$ is not the Seifert longitude λ, then we can identify $\pi_{m,k}(\hat{\mu})$ with the following sum of maps

$$\Psi^{m+k}_{+,m-1+2k} \circ \psi^{1}_{-,\mu}(\hat{\mu}'') + \Psi^{m+k}_{-,m-1+2k} \circ \psi^{1}_{+,\mu}(\hat{\mu}'')$$

It is straightforward to generalize the strategy in Section 1.2 to a strategy to prove Conjecture 2.5. However, we have not yet found proofs of generalizations of the triangle (1.10) and the commutative diagrams (1.12) (1.13).

In the rest of this subsection, we will focus on the following octahedral diagram, where the constants c_i will be chosen carefully later. Note this the same octahedral diagram as (1.7), but with S^3 replaced by a general 3-manifold $\widehat{Y}$ and taking scalar ambiguities into account.

(2.15)

$$\begin{tikzcd}
& & \widehat{\mathbf{Y}}_{\hat{\lambda}-m\hat{\mu}} \arrow[rr, "F_{-1}(\hat{\mu}')"] \arrow[dr, "\psi"] & & \widehat{\mathbf{\Gamma}}_{\mu} \arrow[ddr, "c_1\psi^{\mu}_{+,m-1}+ c_2\psi^{\mu}_{-,m-1}"] & \\
& \widehat{\mathbf{\Gamma}}_{m-1} \arrow[ur, "G_0(\hat{\mu}')"] \arrow[dr, "(c_3\Psi^{m-1}_{+,m-1+k}{,}\ c_4\Psi^{m-1}_{-,m-1+k})"'] & & \boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \arrow[ur, "l'"] \arrow[ddrr, "\phi"] & & \\
& & \widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \arrow[ur, "c_7\psi^{0}_{-,1}(\hat{\mu}'')+ c_8\psi^{0}_{+,1}(\hat{\mu}'')"'] \arrow[drrr, "c_9\Psi^{m-1+k}_{-,m-1+2k}+ c_{10}\Psi^{m-1+k}_{+,m-1+2k}"'] & & & \widehat{\mathbf{\Gamma}}_{m-1} \\
\widehat{\mathbf{\Gamma}}_{\mu} \arrow[uur, "c_1\psi^{\mu}_{+,m-1}+ c_2\psi^{\mu}_{-,m-1}"] \arrow[urr, "(c_5\psi^{\mu}_{-,m-1+k}{,}\ c_6\psi^{\mu}_{+,m-1+k})"'] & & & & & \widehat{\mathbf{\Gamma}}_{m-1+2k} \arrow[u, "l"]
\end{tikzcd}$$

The reader can compare (2.15) with (2.10) and (2.11). We omit the term corresponding to $Z'\{1\}$ because there is no enough room and the maps involving it are not important in our proof.

Then we consider the exact sequences and commutative diagrams in (2.15). If we do not mention the assumption $\widehat{Y} = S^3$ or k is large enough, then the results apply to a general setup.

The first exact sequence in (2.15)

$$\widehat{\mathbf{\Gamma}}_\mu \xrightarrow{c_1\psi^\mu_{+,m-1}+c_2\psi^\mu_{-,m-1}} \widehat{\mathbf{\Gamma}}_{m-1} \xrightarrow{G_0(\hat{\mu}')} \widehat{\mathbf{Y}}_{\hat{\lambda}-m\hat{\mu}} \xrightarrow{F_{-1}(\hat{\mu}')} \widehat{\mathbf{\Gamma}}_\mu \tag{2.16}$$

is a generalization of (1.8) for rationally null-homologous knots in general 3-manifolds. It follows directly from (2.7) and the following special case of Proposition 1.16. Note that Proposition 1.16 will be proved in Section 3.1.

Proposition 2.6. *There exist constants $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ such that*

$$H_{-1}(\hat{\mu}') = c_1\psi^\mu_{+,m-1} + c_2\psi^\mu_{-,m-1},$$

where $H_{-1}(\hat{\mu}')$ is the map in (2.7) induced by a (-1)-surgery.

Then by (2.3), for any $c_3, c_4 \in \mathbb{C}\backslash\{0\}$, there exist $c_5, c_6 \in \mathbb{C}\backslash\{0\}$ such that

$$(c_3\Psi^{m-1}_{+,m-1+k}, c_4\Psi^{m-1}_{-,m-1+k}) \circ H_{-1}(\hat{\mu}') = (c_5\psi^\mu_{-,m-1+k}, c_6\psi^\mu_{+,m-1+k}), \tag{2.17}$$

i.e., the commutative diagram in the assuption of Lemma 2.2 (that is a generalization of (1.11)) holds up to rescaling bypass maps.

The exactness at

$$\widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \cong \widehat{\mathbf{\Gamma}}_{m-1+k} \otimes \mathbb{C}^2$$

in the second and the third exact sequences are both special cases of the following proposition, which will be proved in Section 3.2 by diagram chasing.

Proposition 2.7. *For any $n \in \mathbb{Z}, k_0 \in \mathbb{N}_+, c_1, c_2 \in \mathbb{C}\backslash\{0\}$, there exist constants $c_3, c_4 \in \mathbb{C}\backslash\{0\}$ so that the following sequence is exact*

$$\widehat{\mathbf{\Gamma}}_n \xrightarrow{(c_1\Psi^n_{+,n+k_0}, c_2\Psi^n_{-,n+k_0})} \widehat{\mathbf{\Gamma}}_{n+k_0} \oplus \widehat{\mathbf{\Gamma}}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0}+c_4\Psi^{n+k_0}_{+,n+2k_0}} \widehat{\mathbf{\Gamma}}_{n+2k_0}$$

Remark 2.8. The exactness at the direct summand for the second exact sequence (the one involving $\widehat{\mathbf{\Gamma}}_{\hat{\mu}}$) might not be so clear from Proposition 2.7. Explicitly, we apply the proposition to

$$\hat{\mu}'' = (m+k)\hat{\mu} - \hat{\lambda}$$

and set $n = -1, k_0 = 1$. Then

$$\widehat{\mathbf{\Gamma}}_n(\hat{\mu}'') = \widehat{\mathbf{\Gamma}}_\mu \text{ and } \widehat{\mathbf{\Gamma}}_{n+2k_0}(\hat{\mu}'') = \boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}}$$

by direct calculation.

The remaining exactness at $\widehat{\mathbf{\Gamma}}_\mu$ and $\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}}$ in the second exact sequence

$$\widehat{\mathbf{\Gamma}}_\mu \xrightarrow{(c_5\psi^\mu_{-,m-1+k}, c_6\psi^\mu_{+,m-1+k})} \widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \xrightarrow{c_7\psi^0_{-,1}(\hat{\mu}'')+c_8\psi^0_{+,1}(\hat{\mu}'')} \boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \xrightarrow{l'} \widehat{\mathbf{\Gamma}}_\mu \tag{2.18}$$

in (2.15) will also be proved by diagram chasing. Note that the second exact sequence is a generalization of (1.9). We can construct the map l' explicitly by the composition of bypass maps

$$l' := \psi^{m+k}_{-,\mu} \circ \psi^1_{+,\mu}(\hat{\mu}'') = \psi^\mu_{+,-1}(\hat{\mu}'') \circ \psi^1_{+,\mu}(\hat{\mu}''),$$

where the last equation follows from (2.13). By (2.2), there exists $c_0 \in \mathbb{C}\backslash\{0\}$ so that

$$l' = c_0\psi^{m+k}_{+,\mu} \circ \psi^1_{-,\mu}(\hat{\mu}'') = c_0\psi^{\mu}_{-,-1}(\hat{\mu}'') \circ \psi^1_{-,\mu}(\hat{\mu}'').$$

The following proposition will be proved in Section 3.3 by diagram chasing.

Proposition 2.9. *Suppose l' is constructed as above. For any $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$, the following sequence is exact*

$$\widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \xrightarrow{c_3\psi^0_{-,1}(\hat{\mu}'')+c_4\psi^0_{+,1}(\hat{\mu}'')} \boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \xrightarrow{l'} \widehat{\mathbf{\Gamma}}_{\mu} \xrightarrow{(c_1\psi^{\mu}_{-,m-1+k}, c_2\psi^{\mu}_{+,m-1+k})} \widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k}$$

Remark 2.10. In the proof of [LY21, Theorem 3.23], we obtained a long exact sequence

$$\widehat{\mathbf{\Gamma}}_{\mu} \xrightarrow{(\psi^{\mu}_{+,n-1}, \psi^{\mu}_{-,n-1})} \widehat{\mathbf{\Gamma}}_{n-1} \oplus \widehat{\mathbf{\Gamma}}_{n-1} \to \boldsymbol{\gamma}_{2\hat{\lambda}-(2n-1)\hat{\mu}} \to \widehat{\mathbf{\Gamma}}_{\mu}$$

by the octahedral axiom. However, we did not know the two maps involving $\boldsymbol{\gamma}_{2\hat{\lambda}-(2n-1)\hat{\mu}}$ explicitly. Thus, the second exact sequence here is stronger than the one from octahedral axiom.

The exactness at $\widehat{\mathbf{\Gamma}}_{m-1}$ and $\widehat{\mathbf{\Gamma}}_{m-1+2k}$ in the third exact sequence

(2.19)
$$\widehat{\mathbf{\Gamma}}_{m-1} \xrightarrow{(c_3\Psi^{m-1}_{+,m-1+k}, c_4\Psi^{m-1}_{-,m-1+k})} \widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \xrightarrow{c_9\Psi^{m-1+k}_{-,m-1+2k}+c_{10}\Psi^{m-1+k}_{+,m-1+2k}} \widehat{\mathbf{\Gamma}}_{m-1+2k} \xrightarrow{l} \widehat{\mathbf{\Gamma}}_{m-1}$$

in (2.15) is harder to prove since the map l cannot be constructed by bypass maps. We expect that there are many equivalent constructions of l and we will use the one for which the exactness is easiest to prove. Even so, we only prove the exactness with the assumption that k is large and $\widehat{Y} = S^3$ (we use the fact that $\dim_{\mathbb{C}} I^{\sharp}(-S^3) = 1$ and S^3 has an orientation-reversing involution). We expect that there exists a good construction of l so that the exactness can be proved for any $\widehat{Y}$ and $k \in \mathbb{N}_+$. See Section 3.4 for more details.

Proposition 2.11. *Suppose $\widehat{Y} = S^3$. Suppose $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$ and suppose k_0 is a large integer. For any $n \in \mathbb{Z}$, there exists a map l such that the following sequence is exact*

$$\widehat{\mathbf{\Gamma}}_{n+k_0} \oplus \widehat{\mathbf{\Gamma}}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0}+c_4\Psi^{n+k_0}_{+,n+2k_0}} \widehat{\mathbf{\Gamma}}_{n+2k_0} \xrightarrow{l} \widehat{\mathbf{\Gamma}}_{n} \xrightarrow{(c_1\Psi^{n}_{+,n+k_0}, c_2\Psi^{n}_{-,n+k_0})} \widehat{\mathbf{\Gamma}}_{n+k_0} \oplus \widehat{\mathbf{\Gamma}}_{n+k_0}$$

Remark 2.12. The reason that Proposition 2.11 holds for any choices of c_1, c_2, c_3, c_4 is because

$$\mathrm{Ker}((c_1\Psi^n_{+,n+k_0}, c_2\Psi^n_{-,n+k_0})) = \mathrm{Ker}(c_1\Psi^n_{+,n+k_0}) \cap \mathrm{Ker}(c_2\Psi^n_{-,n+k_0})$$

and

$$\mathrm{Im}(c_3\Psi^{n+k_0}_{-,n+2k_0} + c_4\Psi^{n+k_0}_{+,n+2k_0}) = \mathrm{Im}(c_3\Psi^{n+k_0}_{-,n+2k_0}) + \mathrm{Im}(c_4\Psi^{n+k_0}_{+,n+2k_0}),$$

where the right hand sides of the equations are independent of constants.

Next, we consider modifications of commutative diagrams (1.12) and (1.13) to apply Lemma 2.4. We set

$$\phi'(c_{11}, c_{12}) := c_{11}\Psi^{m+k}_{-,m-1+2k} \circ \psi^1_{+,\mu}(\hat{\mu}'') + c_{12}\Psi^{m+k}_{+,m-1+2k} \circ \psi^1_{-,\mu}(\hat{\mu}'').$$

The following proposition follows directly from (2.1), (2.3), and (2.13). It is a generalization of (1.12).

Proposition 2.13. *For any $c_7, c_8, c_9, c_{10} \in \mathbb{C}\backslash\{0\}$, there exist $c_{11}, c_{12} \in \mathbb{C}\backslash\{0\}$ so that the following diagram commutes*

$$\begin{tikzcd}
\widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \arrow[r, "c_7\psi^0_{-,1}(\hat{\mu}'')+c_8\psi^0_{+,1}(\hat{\mu}'')"] \arrow[d, "="] & \boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \arrow[d, "\phi'(c_{11},c_{12})"] \\
\widehat{\mathbf{\Gamma}}_{m-1+k} \oplus \widehat{\mathbf{\Gamma}}_{m-1+k} \arrow[r, "c_9\Psi^{m-1+k}_{-,m-1+2k}+c_{10}\Psi^{m-1+k}_{+,m-1+2k}"] & \widehat{\mathbf{\Gamma}}_{m-1+2k}
\end{tikzcd}$$

The following proposition will be proved in Section 3.5. It is a modification of (1.13).

Proposition 2.14. *If $\widehat{Y} = S^3$, then there are two commutative diagrams up to scalars.*

$$\begin{tikzcd}
\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \arrow[r, "l'"] \arrow[d, "\Psi^{m+k}_{+,m-1+2k}\circ\psi^1_{-,\mu}(\hat{\mu}'')"] & \widehat{\mathbf{\Gamma}}_{\mu} \arrow[d, "\psi^{\mu}_{+,m-1}"] \\
\widehat{\mathbf{\Gamma}}_{m-1+2k} \arrow[r, "l"] & \widehat{\mathbf{\Gamma}}_{m-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}} \arrow[r, "l'"] \arrow[d, "\Psi^{m+k}_{-,m-1+2k}\circ\psi^1_{+,\mu}(\hat{\mu}'')"] & \widehat{\mathbf{\Gamma}}_{\mu} \arrow[d, "\psi^{\mu}_{-,m-1}"] \\
\widehat{\mathbf{\Gamma}}_{m-1+2k} \arrow[r, "l"] & \widehat{\mathbf{\Gamma}}_{m-1}
\end{tikzcd}$$

Finally, we put all propositions together to prove the following special case of the integral surgery formula, which is exactly the combination of Theorem 1.1 and Proposition 1.4.

Theorem 2.15. *If $\widehat{Y} = S^3$, then Conjecture 2.5 holds.*

Proof. We choose the constants in (2.15) carefully as follows. First, we fix the constants c_1, c_2 by Proposition 2.6. We choose c_3, c_4 arbitrarily and then c_5, c_6 are fixed by (2.17). From Proposition 2.7, the constants c_7, c_8 are determined by c_5, c_6 and the constants c_9, c_{10} are determined by c_3, c_4.

Then by the discussion in this subsection, if $\widehat{Y} = S^3$, then three sequences (2.16), (2.18), and (2.19) in (2.15) satisfy the assumption of the octahedral axiom (Lemma 2.2). Hence there exist maps ψ and ϕ so that the fourth sequence in (2.15) is exact and all diagram commute. In particular, we have

$$\widehat{\mathbf{Y}}_{\hat{\lambda}-m\hat{\mu}} = I^{\sharp}(-\widehat{Y}_{\hat{\lambda}-m\hat{\mu}}) \cong H(\mathrm{Cone}(\phi)).$$

When $\hat{\lambda} - m\hat{\mu}$ is not Seifert longitude (*i.e.* $m \neq 0$ since $\widehat{Y} = S^3$), the grading shifts of $\psi^{\mu}_{\pm,m-1}$ are different. Also, in such case, the grading shifts of

$$\pi^{\pm}_{m,k} = \Psi^{m+k}_{\pm,m-1+2k} \circ \psi^1_{\mp,\mu}(\hat{\mu}'')$$

are different. From Proposition 2.13, Proposition 2.14, and the commutative diagrams in (2.15) involving ϕ, we know the maps ϕ and $\phi'(c_{11}, c_{12})$ satisfy the assumption of Lemma 2.4 for some constants c_{11}, c_{12}. Hence by Lemma 2.4, we have

$$H(\mathrm{Cone}(\phi)) \cong H(\mathrm{Cone}(\phi'(c_{11}, c_{12})).$$

When the grading shifts of $\pi^{\pm}_{m,k}$ are different, Lemma 2.3 implies

$$H(\mathrm{Cone}(\phi'(c_{11}, c_{12})) \cong H(\mathrm{Cone}(\phi'(1,1))) = H(\mathrm{Cone}(\pi^{+}_{m,k} + \pi^{-}_{m,k})).$$

□

2.4. An equivalent version from bent complexes.

In this subsection, we restate Conjecture 2.5 by maps from bent complexes. The readers who are not familiar with the constructions in [LY21] can safely skip this subsection. Suppose K is a null-homologous knot in a closed 3-manifold Y. Let $\hat{\mu} = q\mu + p\lambda$ with $q > 0$ and $\hat{\lambda} = q_0\mu + p_0\lambda$ as chosen in Section 2.1. Let $(\widehat{Y}, \widehat{K})$ be the dual knot with respect to $\hat{\mu}$.

Putting bypass triangles in (2.1) for different n together, we obtain the following diagram:

(2.20)

$$
\begin{tikzcd}
\cdots & \widehat{\mathbf{\Gamma}}_{n+1} \arrow[l] \arrow[dr, "\psi^{n+1}_{+,\mu}"'] & & \widehat{\mathbf{\Gamma}}_{n} \arrow[ll, "\psi^{n}_{+,n+1}"'] \arrow[dr, "\psi^{n}_{+,\mu}"'] & & \widehat{\mathbf{\Gamma}}_{n-1} \arrow[ll, "\psi^{n-1}_{+,n}"'] \arrow[dr, "\psi^{n-1}_{+,\mu}"'] & & \widehat{\mathbf{\Gamma}}_{n-2} \arrow[ll, "\psi^{n-2}_{+,n-1}"'] & \cdots \arrow[l] \\
& \cdots & \widehat{\mathbf{\Gamma}}_{\mu} \arrow[ur, "\psi^{\mu}_{+,n}"] \arrow[dl, "\psi^{\mu}_{-,n-2}"'] & & \widehat{\mathbf{\Gamma}}_{\mu} \arrow[ur, "\psi^{\mu}_{+,n-1}"] \arrow[dl, "\psi^{\mu}_{-,n-1}"'] & & \widehat{\mathbf{\Gamma}}_{\mu} \arrow[ur, "\psi^{\mu}_{+,n-2}"] \arrow[dl, "\psi^{\mu}_{-,n}"'] & \cdots & \\
\cdots \arrow[r] & \widehat{\mathbf{\Gamma}}_{n-2} \arrow[rr, "\psi^{n-2}_{-,n-1}"'] & & \widehat{\mathbf{\Gamma}}_{n-1} \arrow[rr, "\psi^{n-1}_{-,n}"'] \arrow[ul, "\psi^{n-1}_{-,\mu}"'] & & \widehat{\mathbf{\Gamma}}_{n} \arrow[rr, "\psi^{n}_{-,n+1}"'] \arrow[ul, "\psi^{n}_{-,\mu}"'] & & \widehat{\mathbf{\Gamma}}_{n+1} \arrow[r] \arrow[ul, "\psi^{n+1}_{-,\mu}"'] & \cdots
\end{tikzcd}
$$

where the $\mathbb{Z}$-grading shift of $\psi^{\mu}_{\pm,k} \circ \psi^{k}_{\pm,\mu}$ is $\pm q$ for any $k \in \mathbb{Z}$. From (2.20), we constructed in [LY21, Section 3.4] two spectral sequences $\{E_{r,+}, d_{r,+}\}_{r\geqslant 1}$ and $\{E_{r,-}, d_{r,-}\}_{r\geqslant 1}$ from $\widehat{\mathbf{\Gamma}}_{\mu} := \underline{\mathrm{SHI}}(-Y\backslash K, -\widehat{\Gamma}_{\mu})$ to $\widehat{\mathbf{Y}}_{\hat{\mu}} := I^{\sharp}(-\widehat{Y}_{\hat{\mu}})$, where $d_{r,\pm}$ is roughly

$$\psi^{k}_{\pm,\mu} \circ (\Psi^{k}_{\pm,k+r})^{-1} \circ \psi^{\mu}_{\pm,k+r} \text{ for any } k \in \mathbb{Z}. \tag{2.21}$$

The independence of k follows from (2.3) and the composition with the inverse map is well-defined on the r-th page. The $\mathbb{Z}$-grading shift of $d_{r,\pm}$ is $\pm rq$. By fixing an inner product on $\widehat{\mathbf{\Gamma}}_{\mu}$, we then lifted those spectral sequences to two differentials d_+ and d_- on $\widehat{\mathbf{\Gamma}}_{\mu}$ so that

$$H(\widehat{\mathbf{\Gamma}}_{\mu}, d_+) \cong H(\widehat{\mathbf{\Gamma}}_{\mu}, d_-) \cong I^{\sharp}(-\widehat{Y}_{\hat{\mu}}).$$

In such way, the inverses of $\Psi^{k}_{\pm,k+r}$ are also well-defined, which we will use freely later.

Then we propose an integral surgery formula for $I^{\sharp}(-\widehat{Y}_{\hat{\lambda}-m\hat{\mu}})$ using differentials d_+ and d_- on $\widehat{\mathbf{\Gamma}}_{\mu}$. To state the formula, we introduce the following notations.

Definition 2.16 ([LY21, Construction 3.27 and Definition 5.12])**.** Suppose $Y, K, \hat{\mu}$ are given as above. For any integer s, define complexes

$$B^{\pm}(s) := (\bigoplus_{k\in\mathbb{Z}} (\widehat{\mathbf{\Gamma}}_{\mu}, s+kq), d_{\pm}), \quad B^{+}(\geqslant s) := (\bigoplus_{k\geqslant 0} (\widehat{\mathbf{\Gamma}}_{\mu}, s+kq), d_{+}),$$

$$\text{and } B^{-}(\leqslant s) := (\bigoplus_{k\leqslant 0} (\widehat{\mathbf{\Gamma}}_{\mu}, s+kq), d_{-}).$$

Define

$$I^{+}(s) : B^{+}(\geqslant s) \to B^{+}(s) \text{ and } I^{-}(s) : B^{-}(\leqslant s) \to B^{-}(s)$$

to be the inclusion maps. We also write the same notation for the induced map on homology.

Remark 2.17. Note that $\hat{i}^{\mu}_{max}$ and $\hat{i}^{\mu}_{min}$ are finite. Then for any large enough integer s_0 such that $s - s_0 q \leqslant \hat{i}^{\mu}_{min}$ and $s + s_0 q \geqslant \hat{i}^{\mu}_{max}$, we have

$$B^{+}(s) = B^{+}(\leqslant s - s_0 q) \text{ and } B^{-}(s) = B^{-}(\geqslant s + s_0 q).$$

In such case, $I^+(s-s_0q)$ and $I^-(s+s_0q)$ are isomorphisms.

By splitting the diagram (2.20) into $\mathbb{Z}$-gradings, we can calculate homologies of complexes defined in Definition 2.16.

Proposition 2.18. *Suppose $n\in\mathbb{N}_+$ and $i\in\mathbb{Z}$. If $i>\hat{i}^{\mu}_{max}-nq$, then there exists a canonical isomorphism*

$$H(B^+(\geqslant i))\cong(\widehat{\mathbf{\Gamma}}_n, i+\hat{i}^n_{max}-\hat{i}^{\mu}_{max}).$$

If $i<\hat{i}^{\mu}_{min}+nq$, then there exists a canonical isomorphism

$$H(B^-(\leqslant i))\cong(\widehat{\mathbf{\Gamma}}_n, i+\hat{i}^n_{min}-\hat{i}^{\mu}_{min}).$$

Proof. The proof is similar to that of [LY21, Lemma 5.13]. Following the notations in [LY21, (3.9) and (3.10)], if $i>\hat{i}^{\mu}_{max}-nq$, then $\widehat{\mathbf{\Gamma}}^{i,+}_0=0$ (the corresponding grading summand of $\widehat{\mathbf{\Gamma}}_0$) and the isomorphism follows from the convergence theorem of the unrolled spectral squence [LY21, Theorem 2.4] (see also [Boa99, Theorem 6.1]). The other statement holds by the similar reason. $\square$

Definition 2.19 ([LY21, Construction 3.27 and Definition 5.12])**.** Suppose $Y,K,\hat{\mu}$ are given as above. For any integer s define the **bent complex**

$$A(s):=(\bigoplus_{k\in\mathbb{Z}}(\widehat{\mathbf{\Gamma}}_\mu, s+kq),d_s),$$

where for any element $x\in(\widehat{\mathbf{\Gamma}}_\mu,s+kq)$,

$$d_s(x)=\begin{cases}d_+(x) & k>0,\\ d_+(x)+d_-(x) & k=0,\\ d_-(x) & k<0.\end{cases}$$

Define

$$\pi^+(s):A(s)\to B^+(s)\text{ and }\pi^-(s):A(s)\to B^-(s)$$

by

$$\pi^+(s)(x)=\begin{cases}x & k>0,\\ 0 & k\leqslant 0,\end{cases}\text{ and }\pi^-(s)(x)=\begin{cases}0 & k\geqslant 0,\\ 0 & k<0,\end{cases}$$

where $x\in(\widehat{\mathbf{\Gamma}}_\mu,s+kq)$. Define

$$\pi^{\pm}:\bigoplus_{s\in\mathbb{Z}}A(s)\to\bigoplus_{s\in\mathbb{Z}}B^{\pm}(s)$$

by putting $\pi^{\pm}(s)$ together for all s . We also use the same notation for the induced map on homology.

Remark 2.20. Note that $\hat{i}^{\mu}_{max}$ and $\hat{i}^{\mu}_{min}$ are finite. Then for any large enough integer s_0 such that $s_0\geqslant\hat{i}^{\mu}_{max}$ and $-s_0\leqslant\hat{i}^{\mu}_{min}$, we have

$$A(s_0)=B^-(s_0)\text{ and }A(-s_0)=B^+(-s_0).$$

In such case, $\pi^-(s_0)$ and $\pi^+(-s_0)$ are isomorphisms.

Now we state the general integral surgery formula in the above setup.

Conjecture 2.21. Suppose $\hat{\mu} = q\mu + p\lambda$ with $q > 0$ and suppose $\hat{\lambda} = q_0\mu + p_0\lambda$ as chosen in Section 2.1. For any $m \in \mathbb{Z}$ so that $\hat{\lambda} - m\hat{\mu}$ is not the Seifert longitude λ, there exists an isomorphism

$$\Xi_m : \bigoplus_{s\in\mathbb{Z}} H(B^+(s)) \xrightarrow{\cong} \bigoplus_{s\in\mathbb{Z}} H(B^-(s + mq - q_0))$$

so that $I^\sharp(-\widehat{Y}_{\hat{\lambda}-m\hat{\mu}})$ is isomorphic to the homology of the mapping cone of the map

$$\pi^- + \Xi_m \circ \pi^+ : \bigoplus_{s\in\mathbb{Z}} H(A(s)) \to \bigoplus_{s\in\mathbb{Z}} H(B^-(s)).$$

Though we cannot prove this conjecture, we can reduce it to Conjecture 2.5. Then Theorem 2.15 implies Conjecture 2.21 holds for $\widehat{Y} = S^3$.

Theorem 2.22. *Conjecture 2.5 implies Conjecture 2.21.*

Proof. By Remark 2.20, we only need to consider the maps $\pi^\pm(s)$ for $|s|$ smaller than a fixed integer. For such s, we can apply the following proposition.

Proposition 2.23 ([LY21, Proposition 3.28])**.** *Fix $m \in \mathbb{Z}$ and integer $s \in [\hat{i}^\mu_{min}, \hat{i}^\mu_{max}]$. For any large integer k, there exist $s_1, s_2^+, s_2^-, s_3^+, s_3^- \in \mathbb{Z}$ so that the following diagram commutes*

$$\begin{tikzcd}
H(A(s)) \arrow[rr, "\pi^{\pm}(s)"] \arrow[d, "\cong"] & & H(B^{\pm}(s)) \arrow[d, "\cong"] \\
(\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}}, s_1) \arrow[r, "\psi^1_{\mp,\mu}(\hat{\mu}'')"] & (\widehat{\boldsymbol{\Gamma}}_{m+k}, s_2^{\pm}) \arrow[r, "\Psi^{m+k}_{\pm,m-1+2k}"] & (\widehat{\boldsymbol{\Gamma}}_{m-1+2k}, s_3^{\pm})
\end{tikzcd}$$

Remark 2.24. The maps $\pi^\pm(s)$ factor through $I^\pm(s)$ constructed in Definition 2.16. We write

$$\pi^\pm(s) = I^+(s) \circ \pi^{\pm,\prime}(s).$$

This corresponds to the factorization about $(\widehat{\boldsymbol{\Gamma}}_{m+k}, s_2^\pm)$ in Proposition 2.23 (*c.f.* Proposition 2.18), *i.e.*, the following diagrams commute

$$\begin{tikzcd}
H(A(s)) \arrow[r, "\pi^{+,\prime}(s)"] \arrow[d, "\cong"] & H(B^+(\geqslant s)) \arrow[r, "I^+(s)"] \arrow[d, "\cong"] & H(B^+(s)) \arrow[d, "\cong"] \\
(\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}}, s_1) \arrow[r, "\psi^1_{-,\mu}(\hat{\mu}'')"] & (\widehat{\boldsymbol{\Gamma}}_{m+k}, s_2^+) \arrow[r, "\Psi^{m+k}_{+,m-1+2k}"] & (\widehat{\boldsymbol{\Gamma}}_{m-1+2k}, s_3^+)
\end{tikzcd}$$

$$\begin{tikzcd}
H(A(s)) \arrow[r, "\pi^{-,\prime}(s)"] \arrow[d, "\cong"] & H(B^-(\leqslant s)) \arrow[r, "I^-(s)"] \arrow[d, "\cong"] & H(B^-(s)) \arrow[d, "\cong"] \\
(\boldsymbol{\gamma}_{2\hat{\lambda}-(2m+2k-1)\hat{\mu}}, s_1) \arrow[r, "\psi^1_{+,\mu}(\hat{\mu}'')"] & (\widehat{\boldsymbol{\Gamma}}_{m+k}, s_2^-) \arrow[r, "\Psi^{m+k}_{-,m-1+2k}"] & (\widehat{\boldsymbol{\Gamma}}_{m-1+2k}, s_3^-)
\end{tikzcd}$$

Remark 2.25. From the calculation in [LY21, Remark 3.29] (we replace n and l there by $m+k$ and $k-1$, and note that there is a typo about sign in version 1 of [LY21]), the difference of the grading shifts is

$$s_3^+ - s_3^- = (m + k - (k-1) - 1)q - q_0 = mq - q_0.$$

Then we can construct the isomorphism

$$\Xi_m : \bigoplus_{s\in\mathbb{Z}} H(B^+(s)) \xrightarrow{\cong} \bigoplus_{s\in\mathbb{Z}} H(B^-(s+mq-q_0))$$

by identifying both $H(B^+(s))$ and $H(B^-(s+mq-q_0))$ with $(\widehat{\mathbf{\Gamma}}_{m-1+2k}, s_3^+)$. Then we have

$$H(\mathrm{Cone}(\pi^- + \Xi_m \circ \pi^+)) \cong H(\mathrm{Cone}(\Psi^{m+k}_{-,m-1+2k} \circ \psi^1_{+,\mu}(\hat{\mu}'') + \Psi^{m+k}_{+,m-1+2k} \circ \psi^1_{-,\mu}(\hat{\mu}''))).$$

□

2.5. **An integral surgery formula for instanton knot homology.**

In this subsection, for any $m \in \mathbb{Z}$, we provide a integral surgery formula of

$$\underline{\mathrm{KHI}}(-S^3_m(K), K_{-m}) := \underline{\mathrm{SHI}}(-S^3\backslash K, -\widehat{\Gamma}_m) = \widehat{\mathbf{\Gamma}}_m$$

using differentials d_+ and d_- on $\widehat{\mathbf{\Gamma}}_\mu$. Similar to the last subsection, the readers who are not familiar with the constructions in [LY21] can safely skip this subsection. The formula is inspired by Eftekhary's formula for knot Floer homology $\widehat{HFK}$ [Eft18, Proposition 1.5] (see also [HL21]).

Theorem 2.26. *Suppose $\widehat{Y} = S^3$ in Section 2.1. Let $m, j \in \mathbb{Z}$. Then there exist integers*

$$j^+ = j - \hat{i}^m_{max} + \hat{i}^\mu_{max} \text{ and } j^- = j - \hat{i}^m_{min} + \hat{i}^\mu_{min}$$

and an isomorphism

$$\Xi'_m : H(B^+(j^+)) \xrightarrow{\cong} H(B^-(j^-))$$

so that the space $(\widehat{\mathbf{\Gamma}}_m, j)$ is isomorphic to the homology of the mapping cone of the map

$$(I^-(j^-), \Xi'_m \circ I^+(j^+)) : H(B^-(\leqslant j^-)) \oplus H(B^+(\geqslant j^+)) \to H(B^-(j^-)).$$

Proof. By the third exact sequence (2.19) in Section 2.3, we have

$$\widehat{\mathbf{\Gamma}}_m \cong H(\mathrm{Cone}((c_3\Psi^{m+k}_{-,m+2k}, c_4\Psi^{m+k}_{+,m+2k})))$$

for any $k \in \mathbb{N}_+$. By Lemma 2.3, it suffices to calculate the mapping cone when $c_3 = c_4 = 1$.

Since bypass maps are homogeneous, the above mapping cone splits into $\mathbb{Z}$-gradings. By the grading shift formulae in [LY22, Proposition 4.13], when calculating $(\widehat{\mathbf{\Gamma}}_m, j)$, the corresponding spaces are

$$(\widehat{\mathbf{\Gamma}}_{m+k}, j - \hat{i}^m_{min} + i^{m+k}_{min}) \oplus (\widehat{\mathbf{\Gamma}}_{m+k}, j - \hat{i}^m_{max} + i^{m+k}_{max})$$

and

$$(\widehat{\mathbf{\Gamma}}_{m+2k}, j - \hat{i}^m_{min} + i^{m+k}_{min} - \hat{i}^{m+k}_{max} - i^{m+2k}_{max}) = (\widehat{\mathbf{\Gamma}}_{m+2k}, j - \hat{i}^m_{max} + i^{m+k}_{max} - \hat{i}^{m+k}_{min} - i^{m+2k}_{min}).$$

By Proposition 2.18, we know that

$$(\widehat{\mathbf{\Gamma}}_{m+k}, j + i^{m+k}_{min} - \hat{i}^m_{min}) \cong H(B^-(\leqslant j - \hat{i}^m_{min} + \hat{i}^\mu_{min})) = H(B^-(\leqslant j^-)) \text{ for } j < \hat{i}^m_{min} + (m+k)q$$

and

$$(\widehat{\mathbf{\Gamma}}_{m+k}, j + \hat{i}^{m+k}_{max} - i^m_{max}) \cong H(B^+(\geqslant j - \hat{i}^m_{max} + \hat{i}^\mu_{max})) = H(B^-(\geqslant j^+)) \text{ for } j > \hat{i}^m_{max} - (m+k)q.$$

Since m is fixed, when k is large enough, we know that any $j \in [\hat{i}^m_{min}, \hat{i}^m_{max}]$ satisfies the above inequalities. By Proposition 2.18 again and Remark 2.17, for k large enough, we know that

$$(\widehat{\mathbf{\Gamma}}_{m+2k}, j - \hat{i}^m_{min} + i^{m+k}_{min} - \hat{i}^{m+k}_{max} - i^{m+2k}_{max}) \cong H(B^-(j^-)) \cong H(B^+(j^+)).$$

By unpackaging the construction of differentials d_+ and d_- in [LY21, Section 3.4], we know that and the restrictions of maps $\Psi^m_{-,m+2k}$ and $\Psi^m_{+,m+2k}$ on the corresponding gradings coincide the maps induced by the inclusions $I^-(j^-)$ and $I^+(j^+)$ under the canonical isomorphisms, respectively.

Suppose

$$\Xi'_m : H(B^+(j^+)) \xrightarrow{\cong} H(B^-(j^-))$$

is the canonical isomorphism obtained from identifying both spaces to the corresponding grading summand of $\widehat{\mathbf{\Gamma}}_{m+2k}$. Then we know that

$$(\widehat{\mathbf{\Gamma}}_m, j) \cong H(\mathrm{Cone}(\Psi^{m+k}_{-,m+2k}, \Psi^{m+k}_{+,m+2k})) \cong H(\mathrm{Cone}((I^-(j^-), \Xi'_m \circ I^+(j^+)))).$$

□

3. Exact triangles and commutative diagrams

In this section, we prove propositions in Section 2.3.

3.1. **Identifying the surgery map with the sum of bypass maps.**

In this subsection, we prove Proposition 2.6 by proving a stronger result, namely Proposition 1.16, which applies to any balanced sutured manifold.

Recall the setups in Section 1.3: Given an arbitrary balanced sutured manifold (M, γ) and a curve $\alpha \subset \partial M$ that intersects γ twice. The (-1)-surgery along α with respect to the surface framing is equivalent to a Dehn twist along α. Let γ_2 be the resulting suture of γ under this Dehn twist. There is a natural map associated to the surgery:

$$H_\alpha : \underline{\mathrm{SHI}}(-M, -\gamma) \to \underline{\mathrm{SHI}}(-M, -\gamma_2).$$

On the other hand, there are two natural bypass maps related to α, and their effect on γ are both the same Dehn twist along α, *i.e.*, resulting in the same suture γ_2. The two bypasses lead to two bypass maps

$$\psi_\pm : \underline{\mathrm{SHI}}(-M, -\gamma) \to \underline{\mathrm{SHI}}(-M, -\gamma_2)$$

We will prove that there exists non-zero constants c_1 and c_2 so that

$$H_\alpha = c_1\psi_+ + c_2\psi_-.$$

Proof of Proposition 1.16. Let $A \subset \partial M$ be a tubular neighborhood of $\alpha \subset \partial M$. Pushing the interior of A into the interior of M to make it a properly embedded surface. By a standard argument in [Hon00], we can assume that a collar of ∂M is equipped with a product contact structure so that γ is (isotopic to) the dividing set, α is a Legendrian curve, A is in the contact collar, and A is a convex surface with Legendrian boundary that separates a standard contact neighborhood of α off M. The convex decomposition of M along A yields two pieces

$$M = M' \underset{A}{\cup} V,$$

where M' is diffeomorphic to M and V is the contact neighborhood of α. It is straightforward to check that after rounding the conners the contact structure near the boundary of M' is still a product contact structure with $\partial M'$ a convex boundary. Let γ' be the dividing set on $\partial M'$. Also, after rounding the conners the contact structure on $V \cong S^1 \times D^2$, we suppose ∂V is a convex surface with dividing set being the union of two connected simple closed curve on ∂V of slope -1. When viewing V as the complement of an unknot in S^3, the dividing set coincides with the suture $\Gamma_1 \subset V$, so from now on we call it Γ_1. By the construction of gluing map in [Li18], there exists a map

$$G_1 : \underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_1) \to \underline{\mathrm{SHI}}(-M, -\gamma).$$

As in [Li18], the map G_1 comes from attaching contact handles to $(M',\gamma')\sqcup(V,\Gamma_1)$ to recover the gluing along A. From [Li19, Proposition 1.4], we know that

$$\underline{\mathrm{SHI}}(-V,-\Gamma_1)\cong\mathbb{C}.$$

Note that M' and M are both equipped with the product contact structure near the boundary. From the functoriality of the contact gluing map in [Li18], we know that G_1 is an isomorphism. Now both the (-1)-surgery along a push off of α and the bypass attachments can be thought of as happening in the piece V. Note that the result of both (-1)-surgery and the bypass attachments for Γ_1 is Γ_2. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-M',-\gamma')\otimes\underline{\mathrm{SHI}}(-V,-\Gamma_1) \arrow[r, "G_1"] \arrow[d, "\mathrm{Id}\otimes\widehat{H}_\alpha"'] & \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[d, "H_\alpha"] \\
\underline{\mathrm{SHI}}(-M',-\gamma')\otimes\underline{\mathrm{SHI}}(-V,-\Gamma_2) \arrow[r, "G_2"] & \underline{\mathrm{SHI}}(-M,-\gamma_2)
\end{tikzcd} \tag{3.1}$$

where $\widehat{H}_\alpha$ denotes the surgery map for the manifold V and G_2 is the gluing map obtained by attaching the same set of contact handles as G_1. A similar commutative diagram holds when replacing H_α and $\widehat{H}_\alpha$ by $\psi_\pm$ and

$$\hat{\psi}_\pm:\underline{\mathrm{SHI}}(-V,-\Gamma_1)\to\underline{\mathrm{SHI}}(-V,-\Gamma_2)$$

in (3.1), respectively.

Since G_1 is an isomorphism, to obtain a relation between H_α and $\psi_\pm$, it suffices to understand the relation between $\widehat{H}_\alpha$ and $\hat{\psi}_\pm$. From [Li19, Proposition 1.4], we know that

$$\underline{\mathrm{SHI}}(-V,-\Gamma_2)\cong\mathbb{C}^2.$$

Moreover, the meridian disk of V induces a $(\mathbb{Z}+\frac{1}{2})$ grading on $\underline{\mathrm{SHI}}(-V,-\Gamma_2)$ and we have

$$\underline{\mathrm{SHI}}(-V,-\Gamma_2)\cong\underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2})\oplus\underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2}),$$

with

$$\underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2})\cong\underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2})\cong\mathbb{C}.$$

Let

$$\mathbf{1}\in\underline{\mathrm{SHI}}(-V,-\Gamma_1)\cong\mathbb{C}$$

be a generator. In [Li19, Section 4.3] it is shown that

$$\hat{\psi}_-(\mathbf{1})\in\underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2})\text{ and }\hat{\psi}_+(\mathbf{1})\in\underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2})$$

are nonzero. Also, when viewing V as the complement of the unknot U, there is an exact triangle

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-V,-\Gamma_1) \arrow[rr, "\widehat{H}_\alpha"] & & \underline{\mathrm{SHI}}(-V,-\Gamma_2) \arrow[dl, "F_2"] \\
& I^\sharp(-S^3) \arrow[ul, "G_1"] &
\end{tikzcd} \tag{3.2}$$

as in (2.7). Comparing the dimensions of the spaces in (3.2), we have $G_1 = 0$ and $\widehat{H}_\alpha$ is injective. From the fact that $\tau_I(U) = 0$, we know from [GLW19, Corollary 3.5] that

$$F_2\big|_{\underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2})} \neq 0 \text{ and } F_2\big|_{\underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2})} \neq 0,$$

By the exactness in (3.2), we have $\mathrm{Ker}(F_2) = \mathrm{Im}(\widehat{H}_\alpha)$ and then $\widehat{H}_\alpha(\mathbf{1})$ is not in $\underline{\mathrm{SHI}}(-V,-\Gamma_2,\pm\frac{1}{2})$, *i.e.*, it is a linear combination of generators of $\underline{\mathrm{SHI}}(-V,-\Gamma_2,\pm\frac{1}{2})$. Hence we know that there are $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ so that

$$\widehat{H}_\alpha(\mathbf{1}) = c_1\hat{\psi}_+(\mathbf{1}) + c_2\hat{\psi}_-(\mathbf{1}).$$

Then the proposition follows from the commutative diagram (3.1). □

3.2. **The exactness at the direct summand.**

In this subsection, we prove Proposition 2.7 by diagram chasing. For convenience, we restate it as follows, which is a little stronger than the previous version.

Proposition 3.1. *For any $n \in \mathbb{Z}, k_0 \in \mathbb{N}_+, c_1, c_2 \in \mathbb{C}\backslash\{0\}$, there exist constants $c_3, c_4 \in \mathbb{C}\backslash\{0\}$ so that the following sequence is exact*

$$\widehat{\mathbf{\Gamma}}_n \xrightarrow{(c_1\Psi^n_{+,n+k_0}, c_2\Psi^n_{-,n+k_0})} \widehat{\mathbf{\Gamma}}_{n+k_0} \oplus \widehat{\mathbf{\Gamma}}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0} + c_4\Psi^{n+k_0}_{+,n+2k_0}} \widehat{\mathbf{\Gamma}}_{n+2k_0}$$

Moreover, if $c_5 \in \mathbb{C}\backslash\{0\}$ is the constant such that the following equation holds

$$\Psi^{n+k_0}_{+,n+2k_0} \circ \Psi^n_{-,n+k_0} = c_5\Psi^{n+k_0}_{-,n+2k_0} \circ \Psi^n_{+,n+k_0},$$

then we can choose any c_3, c_4 satisfying the following equation

$$c_1c_3 = -c_2c_4c_5. \tag{3.3}$$

Conversely, if c_3, c_4 are given, then we can also choose any c_1, c_2 satisfying (3.3) so that the sequence is exact.

Proof. For simplicity, we only prove the proposition for $n = 0$. The proof for any general n is similar (replacing all $\widehat{\mathbf{\Gamma}}_m$ below by $\widehat{\mathbf{\Gamma}}_{n+m}$ and modifying the notations for bypass maps). We prove the proposition by induction on k_0.

First, we assume $k_0 = 1$. The proposition reduces to

$$\mathrm{Ker}(c_3\psi^1_{-,2} + c_4\psi^1_{+,2}) = \mathrm{Im}((c_1\psi^0_{+,1}, c_2\psi^0_{-,1}))$$

for some c_3, c_4 (given c_1, c_2) or some c_1, c_2 (given c_3, c_4). By (2.2), there exists $c_5 \in \mathbb{C}\backslash\{0\}$ so that

$$\psi^1_{+,2} \circ \psi^0_{-,1} = c_5\psi^1_{-,2} \circ \psi^0_{+,1}. \tag{3.4}$$

Hence by setting $c_1c_3 = -c_2c_4c_5$, we have

$$(c_3\psi^1_{-,2} + c_4\psi^1_{+,2}) \circ (c_1\psi^0_{+,1}, c_2\psi^0_{-,1}) = c_2c_4\psi^1_{+,2} \circ \psi^0_{-,1} + c_1c_3\psi^1_{-,2} \circ \psi^0_{+,1} = 0,$$

i.e.,

$$\mathrm{Ker}(c_3\psi^1_{-,2} + c_4\psi^1_{+,2}) \supset \mathrm{Im}((c_1\psi^0_{+,1}, c_2\psi^0_{-,1})).$$

Then we prove

$$\mathrm{Ker}(c_3\psi^1_{-,2} + c_4\psi^1_{+,2}) \subset \mathrm{Im}((c_1\psi^0_{+,1}, c_2\psi^0_{-,1})).$$

Suppose

$$(x_1, x_2) \in \mathrm{Ker}(c_3\psi^1_{-,2} + c_4\psi^1_{+,2}), \textit{i.e.}, c_3\psi^1_{-,2}(x_1) + c_4\psi^1_{+,2}(x_2) = 0.$$

By (2.3), we have

$$\psi^1_{+,\mu}(x_1) \doteq \psi^2_{+,\mu} \circ \psi^1_{-,2}(x_1) \doteq \psi^2_{+,\mu} \circ \psi^1_{+,2}(x_2) = 0,$$

where $\doteq$ means the equation holds up to a scalar and the last equation follows from the exactness in (2.1). Then by the exactness in (2.1) again, there exists $y \in \widehat{\mathbf{\Gamma}}_0$ so that $c_1\psi^0_{+,1}(y) = x_1$. By (3.4), we have

$$c_1\psi^1_{+,2} \circ \psi^0_{-,1}(y) = c_1c_5\psi^1_{-,2} \circ \psi^0_{+,1}(y) = c_5\psi^1_{-,2}(x_1).$$

Then we have

$$c_4\psi^1_{+,2}(x_2 - c_2\psi^0_{-,1}(y)) = c_4\psi^1_{+,2}(x_2 + \frac{c_1c_3}{c_4c_5}\psi^0_{-,1}(y)) = -c_3\psi^1_{-,2}(x_1) + c_3\psi^1_{-,2}(x_1) = 0,$$

where we use the setting $c_1c_3 = -c_2c_4c_5$. By the exactness in (2.1) again, there exists $z \in \widehat{\mathbf{\Gamma}}_\mu$ so that

$$\psi^\mu_{+,1}(z) = x_2 - c_2\psi^0_{-,1}(y).$$

By (2.3), there exists $c_6 \in \mathbb{C}\backslash\{0\}$ so that

$$c_6\psi^0_{-,1} \circ \psi^\mu_{+,0}(z) = \psi^\mu_{+,1}(z).$$

Let

$$y' = y + \frac{c_6}{c_2}\psi^\mu_{+,0}(z).$$

Then

$$c_1\psi^0_{+,1}(y') = c_1\psi^0_{+,1}(y) = x_1$$

and

$$c_2\psi^0_{-,1}(y') = c_2\psi^0_{-,1}(y) + c_6\psi^0_{-,1} \circ \psi^\mu_{+,0}(z) = c_2\psi^0_{-,1}(y) + \psi^\mu_{+,1}(z) = x_2,$$

which concludes the proof for $k_0 = 1$.

Suppose the proposition holds for $k_0 = k$. We prove it also holds for $k_0 = k + 1$. The proof is similar to the case for $k_0 = 1$. To be clear, we use c_1, c_2, c_3, c_4, c_5 to denote the constants in the induction hypothesis and use $c'_1, c'_2, c'_3, c'_4, c'_5$ to denote the constants for $k_0 = k + 1$.

By (2.2), there exists $c'_5 \in \mathbb{C}\backslash\{0\}$ so that

$$\Psi^{k+1}_{+,2k+2} \circ \Psi^0_{-,k+1} = c'_5\Psi^{k+1}_{-,2k+2} \circ \Psi^0_{+,k+1}. \tag{3.5}$$

Hence by setting $c'_1c'_3 = -c'_2c'_4c'_5$, we have

$$(c'_3\Psi^{k+1}_{-,2k+2} + c'_4\Psi^{k+1}_{+,2k+2}) \circ (c'_1\Psi^0_{+,k+1}, c'_2\Psi^0_{-,k+1}) = c'_2c'_4\Psi^{k+1}_{+,2k+1} \circ \Psi^0_{-,k+1} + c'_1c'_3\Psi^{k+1}_{-,2k+2} \circ \Psi^0_{+,k+1} = 0,$$

i.e.,

$$\operatorname{Ker}(c'_3\Psi^{k+1}_{-,2k+2} + c'_4\Psi^{k+1}_{+,2k+2}) \supset \operatorname{Im}((c'_1\Psi^0_{+,k+1}, c'_2\Psi^0_{-,k+1})).$$

Then we prove

$$\operatorname{Ker}(c'_3\Psi^{k+1}_{-,2k+2} + c'_4\Psi^{k+1}_{+,2k+2}) \subset \operatorname{Im}((c'_1\Psi^0_{+,k+1}, c'_2\Psi^0_{-,k+1})).$$

Suppose

$$(x_1, x_2) \in \operatorname{Ker}(c'_3\Psi^{k+1}_{-,2k+2} + c'_4\Psi^{k+1}_{+,2k+2}), \textit{i.e.}, c'_3\Psi^{k+1}_{-,2k+2}(x_1) + c'_4\Psi^{k+1}_{+,2k+2}(x_2) = 0.$$

By (2.3) and (2.1), we have

$$\psi^{k+1}_{+,\mu}(x_1) \doteq \psi^{2k+2}_{+,\mu} \circ \Psi^{k+1}_{-,2k+2}(x_1) \doteq \psi^{2k+2}_{+,\mu} \circ \Psi^{k+1}_{+,2k+2}(x_2) = 0.$$

Then by the exactness in (2.1), there exists $y_1 \in \widehat{\mathbf{\Gamma}}_k$ so that $c'_1\psi^k_{+,k+1}(y_1) = x_1$. By a similar reason, there exists $y_2 \in \widehat{\mathbf{\Gamma}}_k$ so that $c'_2\psi^k_{-,k+1}(y_2) = x_2$. The goal is to prove

$$\Psi^k_{-,2k}(y'_1) \doteq \Psi^k_{+,2k}(y'_2)$$

for some modifications of y_1 and y_2 as for y' in the case of $k_0 = 1$. Then the induction hypothesis will imply that there exists $w \in \widehat{\mathbf{\Gamma}}_0$ so that

$$\Psi^0_{+,k}(w) \doteq y'_1 \text{ and } \Psi^0_{-,k}(w) \doteq y'_2.$$

Hence we will have

$$\Psi^0_{+,k+1}(w) \doteq \psi^k_{+,k+1}(y'_1) \doteq x_1 \text{ and } \Psi^0_{-,k+1}(w) \doteq \psi^k_{-,k+1}(y'_2) \doteq x_2.$$

By checking the constants carefully, we will conclude the proof for $k_0 = k+1$.

Now we start to construct y'_1. By (2.2), there exists $c'_6 \in \mathbb{C}\backslash\{0\}$ so that

$$c'_1\psi^{2k+1}_{+,2k+2} \circ \Psi^k_{-,2k+1}(y_1) = c'_1c'_6\Psi^{k+1}_{-,2k+2} \circ \psi^k_{+,k+1}(y_1) = c'_6\Psi^{k+1}_{-,2k+2}(x_1). \tag{3.6}$$

Then we have

$$\begin{aligned} c'_4\psi^{2k+1}_{+,2k+2}(\Psi^{k+1}_{+,2k+1}(x_2) - \frac{c'_2c'_5}{c'_6}\Psi^k_{-,2k+1}(y_1)) &= c'_4\psi^{2k+1}_{+,2k+2}(\Psi^{k+1}_{+,2k+1}(x_2) + \frac{c'_1c'_3}{c'_4c'_6}\Psi^k_{-,2k+1}(y_1)) \\ &= c'_4\Psi^{k+1}_{+,2k+2}(x_2) + \frac{c'_1c'_3}{c'_6}\psi^{2k+1}_{+,2k+2} \circ \Psi^k_{-,2k+1}(y_1) \\ &= -c'_3\Psi^{k+1}_{-,2k+2}(x_1) + c'_3\Psi^{k+1}_{-,2k+2}(x_1) \\ &= 0. \end{aligned}$$

By the exactness in (2.1) again, there exists $z_1 \in \widehat{\mathbf{\Gamma}}_\mu$ so that

$$\psi^\mu_{+,2k+1}(z_1) = \Psi^{k+1}_{+,2k+1}(x_2) - \frac{c'_2c'_5}{c'_6}\Psi^k_{-,2k+1}(y_1).$$

By (2.3), there exists $c'_7 \in \mathbb{C}\backslash\{0\}$ so that

$$c'_7\Psi^k_{-,2k+1} \circ \psi^\mu_{+,k}(z_1) = \psi^\mu_{+,2k+1}(z_1).$$

Let

$$y'_1 = y_1 + \frac{c'_6c'_7}{c'_2c'_5}\psi^\mu_{+,k}(z_1).$$

Then

$$c'_1\psi^k_{+,k+1}(y'_1) = c'_1\psi^k_{+,k+1}(y_1) = x_1$$

and

$$\begin{aligned} \frac{c'_2c'_5}{c'_6}\Psi^k_{-,2k+1}(y'_1) &= \frac{c'_2c'_5}{c'_6}\Psi^k_{-,2k+1}(y_1) + c'_7\Psi^k_{-,2k+1} \circ \psi^\mu_{+,k}(z_1) \\ &= \frac{c'_2c'_5}{c'_6}\Psi^k_{-,2k+1}(y_1) + \psi^\mu_{+,2k+1}(z_1) \\ &= \Psi^{k+1}_{+,2k+1}(x_2), \end{aligned}$$

Then to start to construct y'_2. By (2.2), there exists $c'_8 \in \mathbb{C}\backslash\{0\}$ so that

$$c'_2\psi^{2k}_{-,2k+1} \circ \Psi^k_{+,2k}(y_2) = c'_2c'_8\Psi^{k+1}_{+,2k+1} \circ \psi^k_{-,k+1}(y_2) = c'_8\Psi^{k+1}_{-,2k+1}(x_2). \tag{3.7}$$

Then we have

$$\begin{aligned}
-\frac{c_2'c_5'}{c_6'}\psi^{2k}_{-,2k+1}(\Psi^k_{-,2k}(y_1')-\frac{c_6'}{c_5'c_8'}\Psi^k_{+,2k}(y_2)) &= -\frac{c_2'c_5'}{c_6'}\Psi^k_{-,2k+1}(y_1')+\frac{c_2'}{c_8'}\psi^{2k}_{-,2k+1}\circ\Psi^k_{+,2k}(y_2)\\
&= -\frac{c_2'c_5'}{c_6'}\Psi^k_{-,2k+1}(y_1')+\Psi^{k+1}_{-,2k+1}(x_2)\\
&= 0.
\end{aligned}$$

By the exactness in (2.1) again, there exists $z_2\in\widehat{\mathbf{\Gamma}}_\mu$ so that

$$\psi^\mu_{-,2k}(z_2)=\Psi^k_{-,2k}(y_1')-\frac{c_6'}{c_5'c_8'}\Psi^k_{+,2k}(y_2).$$

By (2.3), there exists $c_9'\in\mathbb{C}\backslash\{0\}$ so that

$$c_9'\Psi^k_{+,2k}\circ\psi^\mu_{-,k}(z_2)=\psi^\mu_{-,2k}(z_2).$$

Let

$$y_2'=y_2+\frac{c_5'c_8'c_9'}{c_6'}\psi^\mu_{-,k}(z_2).$$

Then

$$c_2'\psi^k_{-,k+1}(y_2')=c_2'\psi^k_{-,k+1}(y_2)=x_2$$

and

$$\begin{aligned}
\frac{c_6'}{c_5'c_8'}\Psi^k_{+,2k}(y_2') &= \frac{c_6'}{c_5'c_8'}\Psi^k_{+,2k}(y_2)+c_9'\Psi^k_{+,2k}\circ\psi^\mu_{-,k}(z_2)\\
&= \frac{c_6'}{c_5'c_8'}\Psi^k_{+,2k}(y_2)+\psi^\mu_{-,2k}(z_2)\\
&= \Psi^k_{-,2k}(y_1'),
\end{aligned}$$

By the induction hypothesis, the constant c_5 is chosen so that

$$\Psi^k_{+,2k}\circ\Psi^0_{-,k}=c_5\Psi^k_{-,2k}\circ\Psi^0_{+,k}. \tag{3.8}$$

Omitting the constants, we have the following commutative diagrams

$$\begin{tikzcd}
 & & x_1 \in \widehat{\mathbf{\Gamma}}_{k+1} \arrow[ddrrr, "-"] & & & \\
 & y_1' \in \widehat{\mathbf{\Gamma}}_k \arrow[ur, "+"] \arrow[drr, "-"] & & & & \\
\widehat{\mathbf{\Gamma}}_0 \arrow[ur, "+"] \arrow[ddr, "-"'] & & & \widehat{\mathbf{\Gamma}}_{2k} \arrow[dr, "-"] & & \widehat{\mathbf{\Gamma}}_{2k+2} \\
 & & & & \widehat{\mathbf{\Gamma}}_{2k+1} \arrow[ur, "+"] & \\
 & y_2' \in \widehat{\mathbf{\Gamma}}_k \arrow[uurr, "+"] \arrow[dr, "-"] & & & & \\
 & & x_2 \in \widehat{\mathbf{\Gamma}}_{k+1} \arrow[uurr, "+"] & & &
\end{tikzcd}$$

Comparing (3.5), (3.6), (3.7), and (3.8), we have

$$c_5'c_8' = c_5c_6'.$$

We can set

$$c_1 = 1, c_2 = 1, c_3 = c_5'c_8', c_4 = -c_6'$$

so that

$$c_1c_3 = -c_2c_4c_5.$$

By the induction hypothesis, there exists $w \in \widehat{\mathbf{\Gamma}}_0$ so that

$$\Psi^0_{+,k}(w) = y_1' \text{ and } \Psi^0_{-,k}(w) = y_2'.$$

Hence we have

$$c_1'\Psi^0_{+,k+1}(w) = c_1'\psi^k_{+,k+1}(y_1') = x_1 \text{ and } c_2'\Psi^0_{-,k+1}(w) = c_2'\psi^k_{-,k+1}(y_2') = x_2,$$

which concludes the proof for $k_0 = k+1$. $\square$

Remark 3.2. By similar arguments, we can prove the following sequence is exact for any $k_0, k_1 \in \mathbb{N}_+$

$$\widehat{\mathbf{\Gamma}}_n \xrightarrow{(c_1\Psi^n_{+,n+k_0}, c_2\Psi^n_{-,n+k_1})} \widehat{\mathbf{\Gamma}}_{n+k_0} \oplus \widehat{\mathbf{\Gamma}}_{n+k_1} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+k_0+k_1} + c_4\Psi^{n+k_1}_{+,n+k_0+k_1}} \widehat{\mathbf{\Gamma}}_{n+k_0+k_1},$$

where the constants are determined by $c_1c_3 = -c_2c_4c_5$ and c_5 comes from the following equation

$$\Psi^{n+k_1}_{+,n+k_0+k_1} \circ \Psi^n_{-,n+k_1} = c_5\Psi^{n+k_0}_{-,n+k_0+k_1} \circ \Psi^n_{+,n+k_0}.$$

3.3. **The remaining exactness in the second exact triangle.**

In this subsection, we prove Proposition 2.9 by diagram chasing. For convenience, we restate it as follows, which is a little stronger than the previous version. Note that replacing $\hat{\mu}$ by $\hat{\mu}''=(m+k)\hat{\mu}-\hat{\lambda}$ and set $n=-1$ will recover Proposition 2.9.

Proposition 3.3. *Suppose*
$$l'=\psi^{\mu}_{+,n-1}\circ\psi^{n+1}_{+,\mu}=c_0\psi^{\mu}_{-,n-1}\circ\psi^{n+1}_{-,\mu}$$
for some $c_0\in\mathbb{C}\backslash\{0\}$. Then For any $c_1,c_2,c_3,c_4\in\mathbb{C}\backslash\{0\}$, the following sequence is exact
$$\widehat{\mathbf{\Gamma}}_n\oplus\widehat{\mathbf{\Gamma}}_n\xrightarrow{c_3\psi^n_{-,n+1}+c_4\psi^n_{+,n+1}}\widehat{\mathbf{\Gamma}}_{n+1}\xrightarrow{l'}\widehat{\mathbf{\Gamma}}_{n-1}\xrightarrow{(c_1\psi^{n-1}_{-,n},c_2\psi^{n-1}_{+,n})}\widehat{\mathbf{\Gamma}}_n\oplus\widehat{\mathbf{\Gamma}}_n.$$

Proof. We prove the exactness at $\widehat{\mathbf{\Gamma}}_{n-1}$ first. By (2.1), we have
$$\psi^{n-1}_{\pm,n}\circ l'\doteq\psi^{n-1}_{\pm,n}\circ\psi^{\mu}_{\pm,n-1}\circ\psi^{n+1}_{\pm,\mu}=0,$$
where $\doteq$ means the equation holds up to a scalar. Hence
$$\mathrm{Ker}((c_1\psi^{n-1}_{-,n},c_2\psi^{n-1}_{+,n}))\supset\mathrm{Im}(l').$$
Then we prove
$$\mathrm{Ker}((c_1\psi^{n-1}_{-,n},c_2\psi^{n-1}_{+,n}))\subset\mathrm{Im}(l').$$
Suppose
$$x\in\mathrm{Ker}((c_1\psi^{n-1}_{-,n},c_2\psi^{n-1}_{+,n}))=\mathrm{Ker}(\psi^{n-1}_{-,n})\cap\mathrm{Ker}(\psi^{n-1}_{+,n}).$$
By the exactness in (2.1), there exists $y\in\widehat{\mathbf{\Gamma}}_{\mu}$ so that $\psi^{\mu}_{+,n-1}(y)=x$. By (2.3), we have
$$\psi^{\mu}_{+,n}(y)\doteq\psi^{n-1}_{-,n}\circ\psi^{\mu}_{+,n-1}(y)=\psi^{n-1}_{-,n}(x)=0.$$
By the exactness in (2.1), there exists $z\in\widehat{\mathbf{\Gamma}}_{n+1}$ so that $\psi^{n+1}_{+,\mu}(z)=y$. Thus, we have $l'(z)=x$, which concludes the proof for the exactness at $\widehat{\mathbf{\Gamma}}_{n-1}$.

Then we prove the exactness at $\widehat{\mathbf{\Gamma}}_{n+1}$. Similarly by (2.1), we have
$$\mathrm{Ker}(l')\supset\mathrm{Im}(c_3\psi^n_{-,n+1}+c_4\psi^n_{+,n+1})=\mathrm{Im}(\psi^n_{-,n+1})+\mathrm{Im}(\psi^n_{+,n+1}).$$
Suppose $x\in\mathrm{Ker}(l')$. If $\psi^{n+1}_{+,\mu}(x)=0$, then by the exactness in (2.1), we know $x\in\mathrm{Im}(\psi^n_{+,n+1})$. If $\psi^{n+1}_{+,\mu}(x)\neq0$, then by the exactness in (2.1), there exists $y\in\widehat{\mathbf{\Gamma}}_n$ so that
$$\psi^n_{+,\mu}(y)=\psi^{n+1}_{+,\mu}(x).$$
By (2.3), there exists $c_1\in\mathbb{C}\backslash\{0\}$ so that
$$\psi^n_{+,\mu}(y)=c_1\psi^{n+1}_{+,\mu}\circ\psi^n_{-,n+1}(y).$$
Then we know
$$x-c_1\psi^n_{-,n+1}(y)\in\mathrm{Ker}(\psi^{n+1}_{+,\mu})=\mathrm{Im}(\psi^n_{+,n+1}).$$
Thus, we have
$$x\in\mathrm{Im}(\psi^n_{-,n+1})+\mathrm{Im}(\psi^n_{+,n+1}),$$
which concludes the proof for the exactness at $\widehat{\mathbf{\Gamma}}_{n+1}$. ∎

3.4. The remaining exactness in the third exact triangle.

In this subsection, we construct the map l and prove Proposition 2.11. For simplicity, define

$$\Phi^n_{n+k} := (c_1\Psi^n_{+,n+k}(x), c_2\Psi^n_{-,n+k}(x)) : \widehat{\mathbf{\Gamma}}_n \to \widehat{\mathbf{\Gamma}}_{n+k} \oplus \widehat{\mathbf{\Gamma}}_{n+k}$$

and

$$\Phi^{n+k}_{n+2k} := c_3\Psi^{n+k}_{-,n+2k}(x) + c_4\Psi^{n+k}_{+,n+2k}(y) : \widehat{\mathbf{\Gamma}}_{n+k} \oplus \widehat{\mathbf{\Gamma}}_{n+k} \to \widehat{\mathbf{\Gamma}}_{n+k},$$

where $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$ are chosen as in Proposition 3.1 so that

$$\mathrm{Ker}(\Phi^{n+k}_{n+2k}) = \mathrm{Im}(\Phi^n_{n+k}).$$

From the grading shifting behavior in [LY22, Proposition 4.14], we know that the maps

$$\Psi^{n+k}_{\pm,n+2k} \circ \Psi^n_{\mp,n+k}$$

are grading preserving. This motivates the following construction. Let

$$F_{n+2k} : \widehat{\mathbf{\Gamma}}_{n+2k} \to \widehat{\mathbf{Y}} \text{ and } G_n : \widehat{\mathbf{Y}} \to \widehat{\mathbf{\Gamma}}_n$$

be the maps in (2.7) and let

$$\Pi^i_n : \widehat{\mathbf{\Gamma}}_n \to (\widehat{\mathbf{\Gamma}}_n, i)$$

be the projection. Define $\Phi^{n+2k}_n : \widehat{\mathbf{\Gamma}}_{n+2k} \to \widehat{\mathbf{\Gamma}}_n$ by

$$\Phi^{n+2k}_n\big|_{(\widehat{\mathbf{\Gamma}}_{n+2k},i)} = \Pi^i_n \circ G_n \circ F_{n+2k}.$$

Remark 3.4. For $i \notin [\hat{i}^n_{min}, \hat{i}^n_{max}]$, the restrictions of Φ^n_{n+k} and Φ^{n+2k}_n on $(\widehat{\mathbf{\Gamma}}_n, i)$ vanish since $(\widehat{\mathbf{\Gamma}}_n, i) = 0$. From [LY22, Lemma 4.16], when k is large, the restriction of Φ^{n+k}_{n+2k} on the subspace corresponding to $(\widehat{\mathbf{\Gamma}}_n, i)$ is an isomorphism. Thus, when considering the long exact sequence, it suffices to consider the exactness for subspaces corresponding to $i \in [\hat{i}^n_{min}, \hat{i}^n_{max}]$.

From now on, we suppose $\widehat{Y} = S^3$ and $(q,p) = (1,0)$ in Section 2.1. To distinguish this special case, we write $\mathbf{\Gamma}_n, \mathbf{\Gamma}_\mu$, and $\mathbf{\Gamma}_{\frac{2n-1}{2}}$ instead of $\widehat{\mathbf{\Gamma}}_n, \widehat{\mathbf{\Gamma}}_\mu$, and $\boldsymbol{\gamma}_{2\hat{\lambda}-2(n-1)\hat{\mu}}$, respectively. We still use the same notations for maps. Then we can use Φ^{n+2k}_n as a construction of l in Proposition 2.11. We prove the exactness for this construction of l in the rest of this subsection.

First, we state the grading shifting behavior of bypass maps in [LY22, Proposition 4.14] explicitly. Note that we consider either a $\mathbb{Z}$-grading or a $(\mathbb{Z}+\frac{1}{2})$-grading associated to a fixed Seifert surface.

Lemma 3.5. *For a knot $K \subset S^3$ and $n \in \mathbb{Z}$, there are two graded bypass exact triangles*

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i+\frac{1}{2}) \arrow[rr, "\psi^{n-1}_{+,n}"] & & (\mathbf{\Gamma}_n, i) \arrow[dl, "\psi^n_{+,\mu}"] \\
& (\mathbf{\Gamma}_\mu, i-\frac{n-1}{2}) \arrow[ul, "\psi^\mu_{+,n-1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i-\frac{1}{2}) \arrow[rr, "\psi^{n-1}_{-,n}"] & & (\mathbf{\Gamma}_n, i) \arrow[dl, "\psi^n_{-,\mu}"] \\
& (\mathbf{\Gamma}_\mu, i+\frac{n-1}{2}) \arrow[ul, "\psi^\mu_{-,n-1}"] &
\end{tikzcd}$$

Lemma 3.6. *For a knot $K \subset S^3$ and $n \in \mathbb{Z}$, there are two graded bypass exact triangles*

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i+\frac{n}{2}) \arrow[rr, "\psi^{n-1}_{+,\frac{2n-1}{2}}"] & & (\mathbf{\Gamma}_{\frac{2n-1}{2}}, i) \arrow[dl, "\psi^{\frac{2n-1}{2}}_{+,n}"] \\
& (\mathbf{\Gamma}_n, i-\frac{n-1}{2}) \arrow[ul, "\psi^{n}_{+,n-1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i-\frac{n}{2}) \arrow[rr, "\psi^{n-1}_{-,\frac{2n-1}{2}}"] & & (\mathbf{\Gamma}_{\frac{2n-1}{2}}, i) \arrow[dl, "\psi^{\frac{2n-1}{2}}_{-,n}"] \\
& (\mathbf{\Gamma}_n, i+\frac{n-1}{2}) \arrow[ul, "\psi^{n}_{-,n-1}"] &
\end{tikzcd}$$

The following lemma is from the computation in the proof of [GLW19, Proposition 3.3].

Lemma 3.7. *Suppose $K \subset S^3$ is a knot and $n \geqslant 2g(K)+1$. Then we know that for $|i| \leqslant \frac{n-1}{2} - g(K)$,*

$$(\mathbf{\Gamma}_n, i) \cong \mathbb{C}.$$

From [GLW19], we write

$$\tau_I(K) := \max\{\, i \mid F_k|_{(\mathbf{\Gamma}_k, i)} \neq 0\} - \frac{n-1}{2} = \frac{k-1}{2} - \min\{\, i \mid F_k|_{(\mathbf{\Gamma}_k, i)} \neq 0\} \tag{3.9}$$

for any $k \geqslant 2g(K)+1$. The definition of $\tau_I(K)$ is independent of the choice of k. When the knot K is fixed, we write τ for $\tau_I(K)$ for short. We have the following lemma.

Lemma 3.8. *Suppose $K \subset S^3$ is a knot and $n \in \mathbb{Z}$.*

(1) $F_n \neq 0$ if and only if $n > -2\tau$ and $G_n \neq 0$ if and only if $n < -2\tau$.

(2) If $n > -2\tau$, then

$$F_n|_{(\mathbf{\Gamma}_n, i)} \neq 0 \text{ if and only if } |i| \leqslant \frac{2\tau + n - 1}{2}.$$

(3) If $n < -2\tau$, then

$$\Pi_n^i \circ G_n \neq 0 \text{ if and only if } |i| \leqslant \frac{-2\tau - n - 1}{2}.$$

Proof. By definition of τ, we can pick any large enough k,

$$x \in (\mathbf{\Gamma}_k, \tau + \frac{k-1}{2}), \text{ and } y \in (\mathbf{\Gamma}_k, -\tau - \frac{k-1}{2}),$$

so that

$$F_k(x) = F_k(y) \neq 0 \in I^\sharp(-S^3).$$

From (2.8), we know that

$$F_{k+2\tau-1} \circ \Psi^k_{+,k+2\tau-1}(x) \doteq F_k(x) = F_k(y) \doteq F_{k+2\tau-1} \circ \Psi^k_{-,k+2\tau-1}(y) \neq 0.$$

From Lemma 3.7 we know that

$$(\mathbf{\Gamma}_{k+2\tau-1}, 0) \cong \mathbb{C}.$$

Hence we know

$$\Psi^k_{+,k+2\tau-1}(x) \doteq \Psi^k_{-,k+2\tau-1}(y),$$

since both elements are nonvanishing and live in $(\mathbf{\Gamma}_{k+2\tau-1},0)$ by the grading shifting behavior in (3.5). By Proposition 3.1, there exists

$$z \in (\mathbf{\Gamma}_{-2\tau+1},0) \tag{3.10}$$

so that

$$x \doteq \Psi^{-2\tau+1}_{-,k}(z) \text{ and } y \doteq \Psi^{-2\tau+1}_{+,k}(z).$$

Hence from (2.8), we have

$$F_{-2\tau+1}(z) \doteq F_k \circ \Psi^{-2\tau+1}_{-,k}(z) \doteq F_k(x) \neq 0$$

Hence $F_{-2\tau+1} \neq 0$. From (2.8) again we know $F_n \neq 0$ for all $n > -2\tau$. Exactness in (2.7) then implies that $G_n = 0$ for all $n \geqslant -2\tau$.

For any $n \in \mathbb{Z}$ and i so that $|i| > \frac{2\tau+n-1}{2}$, we want to argue that

$$F_n\big|_{(\mathbf{\Gamma}_n,i)} = 0.$$

If not, suppose $x \in (\mathbf{\Gamma}_n, i)$ so that $F_n(x) \neq 0$. We know from Lemma 3.5 that for a large enough k, either

$$\mathrm{gr}(\Psi^n_{+,n+k}(x)) > \tau + \frac{n+k-1}{2} \text{ and } F_{n+k} \circ \Psi^n_{+,n+k}(x) \neq 0$$

or

$$\mathrm{gr}(\Psi^n_{-,n+k}(x)) < -\tau - \frac{n+k-1}{2} \text{ and } F_{n+k} \circ \Psi^n_{-,n+k}(x) \neq 0.$$

Both cases contradict with the definition of τ.

Note when $n \leqslant -2\tau$, we have $\frac{2\tau+n-1}{2} < 0$, so $F_n = 0$ and hence from the exactness in (2.7), we know $G_n \neq 0$. This finishes the proof of part (1).

Then we prove part (2). For any $n \in \mathbb{Z}$ such that $n > -2\tau$ and $i \in \mathbb{Z}$ such that $|i| \leqslant \frac{2\tau+n-1}{2}$, as in (3.10), there exists $z \in (\mathbf{\Gamma}_{-2\tau+1},0)$ so that $F_{-2\tau+1}(z) \neq 0$. Then for the given n and i, from Lemma 3.5, we can find $k \in \mathbb{Z}$ so that

$$x \doteq \Psi^k_{-,n} \circ \Psi^{-2\tau+1}_{+,k}(z) \in (\mathbf{\Gamma}_n, i).$$

Hence by (2.8)

$$F_n(x) \doteq F_n \circ \Psi^k_{-,n} \circ \Psi^{-2\tau+1}_{+,k} \doteq F_{-2\tau+1}(z) \neq 0.$$

For part (3), we can study the mirror knot $\bar{K}$. There is an orientation reversing diffeomorphism from (S^3,K) to $(S^3,\bar{K})$, which sends the suture Γ_n for K to Γ_{-n} for $\bar{K}$. From [GLW19, Section 5.1], we know $\tau_I(\bar{K}) = -\tau$. Hence part (3) follows from the duality of sutured instanton homology (*c.f.* [LY21, Propositon 3.33]). □

Then we start to prove the exactness in Proposition 2.11. For simplicity, we write $g = g(K)$.

Proposition 3.9. *For $\widehat{Y} = S^3$ and any large enough $k \in \mathbb{Z}$, we have*

$$\mathrm{Ker}(\Phi^{n+2k}_n) = \mathrm{Im}(\Phi^{n+k}_{n+2k}).$$

Proof. From Lemma 3.7, we know

$$(\mathbf{\Gamma}_{n+2k}, i) \cong \mathbb{C} \text{ for } |i| \leqslant \frac{n+2k-1}{2} - g.$$

On such a grading, the restriction of F_{n+2k} is nonzero if and only if it is an isomorphism. Then from Lemma 3.8, when k is large so that

$$\frac{n+2k-1}{2} - g > \frac{-2\tau-n-1}{2},$$

we know that

$$\operatorname{Ker}(\Phi_n^{n+2k}) = \bigoplus_{|i| > \frac{-2\tau-n-1}{2}} (\mathbf{\Gamma}_{n+2k}, i). \tag{3.11}$$

Then we prove $\operatorname{Im}(\Phi_{n+2k}^{n+k})$ also consists of those subspaces. By Remark 3.4, it suffices the prove the exactness for $i \in [\hat{i}_{min}^n, \hat{i}_{max}^n]$. We write out the arguments in Remark 3.4 explicitly. By (2.5), we know $(\mathbf{\Gamma}_\mu, i) = 0$ when $|i| > g$. By Lemma 3.5, we know that for any $r \in \mathbb{Z}$, the bypass map $\psi_{-,r+1}^r$ is an isomorphism

$$\psi_{-,r+1}^r : (\mathbf{\Gamma}_r, i) \xrightarrow{\cong} (\mathbf{\Gamma}_{r+1}, i + \frac{1}{2})$$

for any $i \geqslant g - \frac{r-1}{2}$ and the bypass map $\psi_{+,r+1}^r$ is an isomorphism

$$\psi_{+,r+1}^r : (\mathbf{\Gamma}_r, i) \xrightarrow{\cong} (\mathbf{\Gamma}_{r+1}, i + \frac{1}{2})$$

for any $r \leqslant \frac{r-1}{2} - g$. Hence we know that the map

$$\Psi_{-,n+2k}^{n+k} : (\mathbf{\Gamma}_{n+k}, i - \frac{k}{2}) \to (\mathbf{\Gamma}_{n+2k}, i)$$

is an isomorphism for $i \geqslant g - \frac{n-1}{2}$ and the map

$$\Psi_{+,n+2k}^{n+k} : (\mathbf{\Gamma}_{n+k}, i + \frac{k}{2}) \to (\mathbf{\Gamma}_{n+2k}, i)$$

is an isomorphism for $i \leqslant -g + \frac{n-1}{2}$. As a result we have

$$(\mathbf{\Gamma}_{n+2k}, i) \subset \operatorname{Im}(\Phi_{n+2k}^{n+k})$$

for $i \geqslant g - \frac{n-1}{2}$ or $i \leqslant -g + \frac{n-1}{2}$.

For $\frac{-2\tau-n-1}{2} < i < g - \frac{n-1}{2}$, we know from Lemma 3.7 that

$$(\mathbf{\Gamma}_{n+2k}, i) \cong \mathbb{C}.$$

Furthermore, by Lemma 3.8, we know that there exists $x \in (\mathbf{\Gamma}_{n+k}, i - \frac{k}{2})$ so that

$$F_{n+k}(x) \neq 0.$$

From (2.8) we know that

$$F_{n+2k} \circ \Psi_{-,n+2k}^{n+k}(x) \doteq F_{n+k}(x) \neq 0.$$

Hence we conclude that

$$\mathbb{C} \cong (\mathbf{\Gamma}_{n+2k}, i) \subset \operatorname{Im}(\Psi_{-,n+k}^{n+2k}) \subset \operatorname{Im}(\Phi_{n+2k}^{n+k}).$$

Using a similar argument, we can show that if $i < -\frac{-2\tau-n-1}{2}$, then

$$(\mathbf{\Gamma}_{n+2k}, i) \subset \operatorname{Im}(\Psi_{+,n+k}^{n+2k}) \subset \operatorname{Im}(\Phi_{n+2k}^{n+k}).$$

It remains to check the grading i for $|i| \leqslant \frac{-2\tau-n-1}{2}$. For such i, from Lemma 3.7 we know that

$$(\mathbf{\Gamma}_{n+2k}, i) \cong \mathbb{C}.$$

From Lemma 3.8, we know that

$$F_{n+2k}\big|_{(\mathbf{\Gamma}_{n+2k}, i)} \neq 0.$$

Now suppose $(\mathbf{\Gamma}_{n+2k},i)\subset\operatorname{Im}(\Phi^{n+k}_{n+2k})$. Then there exists $x\neq 0\in(\mathbf{\Gamma}_{n+2k},i)$ so that either $x\in\operatorname{Im}(\Psi^{n+k}_{-,n+2k})$ or $x\in\operatorname{Im}(\Psi^{n+k}_{+,n+2k})$. If the first case happens, then there exists $y\in(\mathbf{\Gamma}_{n+k},i-\frac{k}{2})$, so that $x=\Psi^{n+k}_{-,n+2k}(y)$. Then by (2.8)

$$F_{n+k}(y)\doteq F_{n+2k}\circ\Psi^{n+k}_{-,n+2k}(y)=F_{n+2k}(x)\neq 0,$$

which contradicts the definition of τ in (3.9) with the given range of i. There is a similar contradiction when the second case happens. Hence we conclude that

$$\operatorname{Im}(\Phi^{n+k}_{n+2k})=\bigoplus_{|i|>\frac{-2\tau-n-1}{2}}(\mathbf{\Gamma}_{n+2k},i)=\operatorname{Ker}(\Phi^{n+2k}_{n}).$$

∎

Proposition 3.10. *For $\widehat{Y}=S^3$ and any large enough $k\in\mathbb{Z}$, we have*

$$\operatorname{Ker}(\Phi^{n}_{n+k})=\operatorname{Im}(\Phi^{n+2k}_{n}).$$

Proof. First, we deal with the constant issue in Proposition 1.16 and commutative diagrams (2.2) and (2.8). Note that we can choose different representatives of sutured instanton homology and maps (*i.e.* replacing a space X and a map f by cX and cf for some $c\in\mathbb{C}\backslash\{0\}$, respectively). Explicitly, we make the following claim.

Claim 1. For $n\leqslant r\leqslant n+k$ we can choose representatives of $\mathbf{\Gamma}_r$, $I^\sharp(-S^3)$, $\psi^r_{\pm,r+1}$, and G_r so that the following conditions hold.

$$\psi^{r+1}_{-,r+2}\circ\psi^r_{+,r+1}=\psi^{r+1}_{+,r+2}\circ\psi^r_{-,r+1}\tag{3.12}$$

$$\psi^{r+1}_{+,r}\circ G_r=G_{r+1}\text{ and }\psi^{r+1}_{-,r}\circ G_r=G_{r+1}\tag{3.13}$$

$$\operatorname{Ker}(H_r)=\operatorname{Ker}(\psi^r_{+,r+1}-\psi^r_{-,r+1}).\tag{3.14}$$

Proof of Claim 1. We pick representatives inductively. First, we pick representatives of $\mathbf{\Gamma}_n$, $\mathbf{\Gamma}_{n+1}$ $I^\sharp(-S^3)$, G_n, and $\psi^n_{+,n+1}$ arbitrarily. Then we pick a representative of G_{n+1} so that

$$\psi^{r+1}_{+,r}\circ G_r=G_{r+1}$$

and pick a representative of $\psi^n_{-,n+1}$ so that

$$\psi^{r+1}_{-,r}\circ G_r=G_{r+1}.$$

Now for $r\geqslant n+2$, suppose we have already picked the representatives of $\mathbf{\Gamma}_{r-2}$, $\mathbf{\Gamma}_{r-1}$, $\psi^{r-2}_{+,r-1}$, $\psi^{r-1}_{-,r}$, G_{n-2}, G_{n-1}. Then we can pick representatives of $\mathbf{\Gamma}_r$ and $\psi^{r-1}_{+,r}$ arbitrarily and pick a representative of $\psi^{r-1}_{-,r}$ so that

$$\psi^{r-1}_{-,r}\circ\psi^{r-2}_{+,r-1}=\psi^{r-1}_{+,r}\circ\psi^{r-2}_{-,r-1}.$$

Next, pick a representative of G_r so that

$$\psi^{r-1}_{+,r}\circ G_{r-1}=G_r.$$

It remains to show that

$$\psi^{r-1}_{-,r}\circ G_{r-1}=G_r.$$

Note that $I^\sharp(-S^3)\cong\mathbb{C}$, so we can fix a generator $\mathbf{1}\in I^\sharp(-S^3)$. We know

$$\begin{aligned}G_r(\mathbf{1}) &= \psi^{r-1}_{+,r}\circ G_{r-1}(\mathbf{1})\\ &= \psi^{r-1}_{+,r}\circ\psi^{r-2}_{-,r-1}\circ G_{r-2}(\mathbf{1})\\ &= \psi^{r-1}_{-,r}\circ\psi^{r-2}_{+,r-1}\circ G_{r-2}(\mathbf{1})\\ &= \psi^{r-1}_{-,r}\circ G_{r-1}(\mathbf{1}).\end{aligned}$$

It remains to verify (3.14). Note by Proposition 1.16, there exists $c_r\in\mathbb{C}\backslash\{0\}$ so that

$$\operatorname{Ker}(H_r)=\operatorname{Ker}(\psi^r_{+,r+1}+c_r\psi^r_{-,r+1}).$$

It remains to show that $c_r=-1$ (or we can choose $c_r=-1$). If $G_r(\mathbf{1})\neq 0$, we know from (2.7) that $G_r(\mathbf{1})\in\operatorname{Ker}(H_r)$. As a result we know that

$$\begin{aligned}0&=H_r\circ G_r(\mathbf{1})\\ &=(\psi^r_{+,r+1}+c_r\psi^r_{-,r+1})\circ G_r(\mathbf{1})\\ &=G_r(\mathbf{1})+c_rG_{r+1}(\mathbf{1})\end{aligned}$$

and we conclude that $c_r=-1$. If $G_r(\mathbf{1})=0$, since we know

$$\operatorname{Ker}(\psi^r_{+,r+1}+c_r\psi^r_{-,r+1})=\operatorname{Ker}(H_r)=\operatorname{Span}(G_r(\mathbf{1}))=0.$$

It remains to show that

$$\operatorname{Ker}(\psi^r_{+,r+1}-\psi^r_{-,r+1})=0.$$

Suppose the contrary, *i.e.*, there exists an element

$$0\neq z=\sum_i z_i,$$

where $z_i\in(\mathbf{\Gamma}_r,i)$, so that

$$(\psi^r_{+,r+1}-\psi^r_{-,r+1})(z)=0.$$

Since the maps $\psi^r_{+,r+1}$ and $\psi^r_{-,r+1}$ shift the gradings as in Lemma 3.5, it is straightforward to check that if we pick

$$0\neq z'=\sum_i(-c_r)^i\cdot z_i,$$

then

$$\psi^r_{+,r+1}+c_r\psi^r_{-,r+1}(z')=0,$$

which is a contradiction. □

From now on, we work with the particular representatives chosen as in Claim 1. Note $\operatorname{Im}(\phi_n^{n+2k})$ can be described as follows. Since $I^\sharp(-S^3)\cong\mathbb{C}$, we can pick a generator $\mathbf{1}\in I^\sharp(-S^3)$ and write

$$G_r(\mathbf{1})=\sum_i z_i^r,$$

where $z_i^r\in(\mathbf{\Gamma}_r,i)$. We can define

$$Z_r=\operatorname*{Span}_i(z_i^r)\subset\mathbf{\Gamma}_r.$$

Note from Lemma 3.8, we know that $z_i^r\neq 0$ if and only if

$$|i|\leqslant\frac{-2\tau-n-1}{2}.$$

Also, from (3.13) and Lemma 3.6, it is straightforward to show that

$$\psi^{r+1}_{+,r}(z^r_i) = z^{r+1}_{i-\frac{1}{2}} \text{ and } \psi^{r+1}_{-,r}(z^r_i) = z^{r+1}_{i+\frac{1}{2}} \tag{3.15}$$

Then we know

$$\mathrm{Im}(\Phi^{n+2k}_n) = Z_r.$$

To understand $\mathrm{Ker}(\Phi^n_{n+k})$, we make the following claims.

Claim 2. Suppose $x \in (\mathbf{\Gamma}_r, i)$ so that $\psi^r_{+,r+1}(x), \psi^r_{-,r+1}(x) \in Z_{r+1}$. Then $x \in Z_r$.

Proof of Claim 2. Assume

$$\psi^r_{+,r+1}(x) = c_1 \cdot z^{r+1}_{i-\frac{1}{2}} + c_2 \cdot z^{r+1}_{i+\frac{1}{2}}.$$

We can pick

$$x_j = \begin{cases} c_2 \cdot z^r_j & j > i \\ x & j = i \\ c_1 \cdot z^r_j & j < i \end{cases}$$

It is straightforward to check

$$(\psi^r_{+,r+1} - \psi^r_{-,r+1})(\sum_j x_j) = 0.$$

Hence the claim follows from (3.14) and (2.7). □

Claim 3. When $n + k > -2\tau$, we have

$$\mathrm{Ker}(\Phi^n_{n+k}) = \mathrm{Ker}(\Phi^n_{n+k-1}).$$

Proof of Claim 3. By definition we have

$$\mathrm{Ker}\,\Phi^n_{n+k-1} \subset \mathrm{Ker}\,\Phi^n_{n+k}.$$

Hence it remains to show that

$$\mathrm{Ker}\,\Phi^n_{n+k} \subset \mathrm{Ker}\,\Phi^n_{n+k-1}.$$

Now suppose $x \in \mathrm{Ker}\,\Phi^n_{n+k}$. Since $\Psi^n_{+,n+k}$ and $\Psi^n_{-,n+k}$ have different grading shifting behavior according to Lemma 3.5, it suffices to study homogeneous elements, *i.e.*, we may assume $x \in (\mathbf{\Gamma}_n, i)$ for some grading i. Let

$$y = \sum_{n \leqslant l \leqslant n+k-1} \Psi^r_{+,n+k-1} \circ \Psi^n_{-,r}(x) \in \mathbf{\Gamma}_{n+k-1},$$

where we regard $\Psi^n_{-,n}$ and $\Psi^{n+k-1}_{+,n+k-1}$ as identity maps on the corresponding spaces. Since

$$x \in \mathrm{Ker}(\Phi^n_{n+k}) = \mathrm{Ker}(\Psi^n_{+,n+k}) \cap \mathrm{Ker}(\Psi^n_{-,n+k}).$$

By (3.12), it is straightforward to check that

$$(\psi^{n+k-1}_{+,n+k} - \psi^{n+k-1}_{-,n+k})(y) = 0.$$

Hence by (3.14) and Lemma 3.8, we know that $y = 0$. Since $\Psi^*_{+,*}$ and $\Psi^*_{-,*}$ have pre-scribed grading shifting behavior according to Lemma 3.5, we conclude that

$$\Psi^n_{+,n+k-1}(x) = 0 \text{ and } \Psi^n_{-,n+k-1}(x) = 0$$

and hence

$$x \in \mathrm{Ker}(\Phi^n_{n+k-1})$$

which concludes the proof of the claim. □

Using Claim 3, we know that either $n \geqslant -2\tau$ and

$$\mathrm{Ker}(\Phi^n_{n+k}) = \mathrm{Ker}(\Phi^n_{n+1}) = \mathrm{Ker}(\psi^n_{+,n+1}) \cap \mathrm{Ker}(\psi^n_{-,n+1}) \subset \mathrm{Im}(G_n) = 0 = \mathrm{Im}(\Phi^{n+2k}_n),$$

or $n < -2\tau$ and

$$\mathrm{Ker}(\Phi^n_{n+k}) = \mathrm{Ker}(\Phi^n_{-2\tau+2}).$$

Hence we can apply a backward induction on n. Suppose we have already shown that

$$\mathrm{Ker}(\Phi^{n+1}_{n+1+k}) = \mathrm{Ker}(\Phi^{n+1}_{-2\tau+2}) = \mathrm{Im}(\Phi^{n+1+2k}_{n+1}) = Z_{n+1}.$$

To deal with the case of n. On the one hand, it is straightforward to check that

$$Z_n \subset \mathrm{Ker}(\Phi^n_{n+k})$$

by (3.15) and the fact that $Z_n = 0$ for $n \geqslant -2\tau$ from Lemma 3.8. On the other hand, suppose $x \in (\mathbf{\Gamma}_n, i)$ so that

$$x \in \mathrm{Ker}(\Phi^n_{n+k}) = \mathrm{Ker}(\Phi^n_{-2\tau+2}) = \mathrm{Ker}(\Psi^n_{+,-2\tau+2}) \cap \mathrm{Ker}(\Psi^n_{-,-2\tau+2}).$$

Let

$$y = \sum_{n \leqslant r \leqslant -2\tau+1} \Psi^r_{+,-2\tau+1} \circ \Psi^n_{-,r}(x) \in \mathbf{\Gamma}_{-2\tau+1}.$$

It is straightforward to check that

$$(\psi^{-2\tau+1}_{+,-2\tau+2} - \psi^{-2\tau+1}_{-,-2\tau+2})(y) = 0.$$

Hence by (3.14) and Lemma 3.8, we know that $y = 0$. By (3.12), we know that

$$\psi^n_{+,n+1}(x), \psi^n_{-,n+1}(x) \in \mathrm{Ker}(\Psi^{n+1}_{+,-2\tau+1}) \cap \mathrm{Ker}(\Psi^{n+1}_{-,-2\tau+1}) = \mathrm{Ker}(\Psi^{n+1}_{+,n+1+k}) \cap \mathrm{Ker}(\Psi^{n+1}_{-,n+1+k}) = Z_{n+1}.$$

Hence by Claim 2 we conclude that $x \in Z_n$. $\square$

3.5. **Commutative diagrams.**

In this subsection, we prove Proposition 2.14. For convenience, we restate them as follows. We still assume $\widehat{Y} = S^3$, adapt notations in Section 3.4, and write $\tau = \tau_I(K), g = g(K)$.

Proposition 3.11. *There are two commutative diagrams up to scalars.*

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k-1}{2}} \arrow[r, "\psi^{n+k}_{+,\mu}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}"] \arrow[d, "\Psi^{n+k}_{+,n+2k-1}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}"'] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{+,n-1}"] \\
\mathbf{\Gamma}_{n-1+2k} \arrow[r, "\Phi^{n+2k-1}_{n-1}"] & \mathbf{\Gamma}_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k-1}{2}} \arrow[r, "\psi^{n+k}_{-,\mu}\circ\psi^{\frac{2n+2k-1}{2}}_{+,n+k}"] \arrow[d, "\Psi^{n+k}_{-,n+2k-1}\circ\psi^{\frac{2n+2k-1}{2}}_{+,n+k}"'] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{-,n-1}"] \\
\mathbf{\Gamma}_{n-1+2k} \arrow[r, "\Phi^{n+2k-1}_{n-1}"] & \mathbf{\Gamma}_{n-1}
\end{tikzcd}$$

We start with the following lemma.

Lemma 3.12. *Suppose $n \in \mathbb{Z}$. Then*

(1) When $n \neq -2\tau$, we have

$$\mathrm{Im}(\psi^n_{+,\mu}) = \mathrm{Im}(\psi^{n+1}_{+,\mu}) \text{ and } \mathrm{Im}(\psi^n_{-,\mu}) = \mathrm{Im}(\psi^{n+1}_{-,\mu}).$$

(2) When $n = -2\tau$, we have

$$\mathrm{Im}(\psi^{n+1}_{+,\mu})/\mathrm{Im}(\psi^{n}_{+,\mu}) \cong \mathbb{C} \text{ and } \mathrm{Im}(\psi^{n+1}_{-,\mu})/\mathrm{Im}(\psi^{n}_{-,\mu}) \cong \mathbb{C}.$$

Moreover, we have

$$\dim_{\mathbb{C}}(\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, \tau)) = \dim_{\mathbb{C}}(\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, \tau)) + 1$$

and

$$\dim_{\mathbb{C}}(\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, -\tau)) = \dim_{\mathbb{C}}(\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, -\tau)) + 1.$$

Proof. We only prove the case for the positive bypass, and the proof for the negative bypass is similar. First, from (2.3), we know that

$$\mathrm{Im}(\psi^{n}_{+,\mu}) \subset \psi^{n+1}_{+,\mu}$$

for all $n \in \mathbb{Z}$. Now suppose $y \in \mathbf{\Gamma}_n$ so that

$$x := \psi^{n+1}_{+,\mu}(y) \in \mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, i)$$

for some $i \in \mathbb{Z}$. We know from Lemma 3.5 that

$$z := \psi^{\mu}_{+,n-1}(x) \in (\mathbf{\Gamma}_{n-1}, i + \frac{n}{2}). \tag{3.16}$$

Now from the exactness in (2.1) we know $\psi^{n-1}_{+,n}(z) = 0$. From the exactness again and (2.3) we know

$$\begin{aligned}
\psi^{n-1}_{-,n}(z) &= \psi^{n-1}_{-,n} \circ \psi^{\mu}_{+,n-1}(x) \\
&\doteq \psi^{\mu}_{+,n}(x) \\
&= \psi^{\mu}_{+,n} \circ \psi^{n+1}_{+,\mu}(y) \\
&= 0.
\end{aligned}$$

As a result, by Proposition 1.16 we know

$$z \in \mathrm{Ker}(\psi^{n-1}_{+,n}) \cap \mathrm{Ker}(\psi^{n-1}_{-,n}) \subset \mathrm{Ker}(H_{n-1}) = \mathrm{Im}(G_{n-1}).$$

When $n > -2\tau$, we know that $G_{n-1} = 0$ and hence $z = 0$, which means $x \in \mathrm{Im}(\psi^{n}_{+,\mu})$ by the exactness. When $n < -2\tau$, we know that every element in $\mathrm{Im}(G_{n-1})$ is not homogeneous by Lemma 3.8. Hence again we conclude $z = 0$ which means $x \in \mathrm{Im}(\psi^{n}_{+,\mu})$ by the exactness. For $n = -2\tau$, we know from Lemma 3.8 that

$$\mathrm{Im}(G_{n-1}) \subset (\mathbf{\Gamma}_{n-1}, 0).$$

Hence we know from (3.16) that

$$\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, i) = \mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, i)$$

unless $i = \tau$. It remains to show that for the case of $i = \tau$, we have

$$\dim_{\mathbb{C}}(\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, \tau)) = \dim_{\mathbb{C}}(\mathrm{Im}(\psi^{n+1}_{+,\mu}) \cap (\mathbf{\Gamma}_\mu, \tau)) + 1.$$

To verify this, note as above we have already shown that

$$\psi^{\mu}_{+,n-1}(\mathrm{Im}(\psi^{n+1}_{+,\mu})) \subset \mathrm{Im}(G_{n-1}) \cong \mathbb{C}.$$

By exactness we already know that $\mathrm{Ker}(\psi^{\mu}_{+,n-1}) = \mathrm{Im}(\psi^{n}_{+,\mu}) \subset \mathrm{Im}(\psi^{n+1}_{+,\mu})$, so it remains to show that

$$\psi^{\mu}_{+,n-1} : \mathrm{Im}(\psi^{n+1}_{+,\mu}) \to \mathrm{Im}(G_{n-1}) \cong \mathbb{C}$$

is surjective. Indeed, let $\mathbf{1} \in \mathrm{Im}(G_{n-1})$ be a generator. By Lemma 3.8 and (2.8), we know that $\psi_{+,n}^{n-1}(\mathbf{1}) = 0$ and $\psi_{-,n}^{n-1}(\mathbf{1}) = 0$. The fact that $\psi_{+,n}^{n-1}(\mathbf{1}) = 0$ implies that we can find $x \in \mathbf{\Gamma}_\mu$ such that $\psi_{+,n-1}^{\mu}(x) = \mathbf{1}$. Then we know from (2.3) that

$$\begin{aligned}\psi_{+,n}^{\mu}(x) &\doteq \psi_{-,n}^{n-1} \circ \psi_{+,n-1}^{\mu}(x)\\ &= \psi_{-,n}^{n-1}(\mathbf{1})\\ &= 0.\end{aligned}$$

Hence $x \in \mathrm{Ker}(\psi_{+,n}^{\mu}) = \mathrm{Im}(\psi_{+,\mu}^{n+1}) = \mathrm{Im}(\psi_{+,\mu}^{n})$. □

Lemma 3.13. *Suppose $n < -2\tau$ and $x \in \mathrm{Im}(\psi_{+,\mu}^{-2\tau+1}) \cap (\mathbf{\Gamma}_\mu, \tau)$, then*

$$\psi_{+,n}^{\mu}(x) = 0 \text{ if and only if } \psi_{+,-2\tau+1}^{\mu}(x) = 0.$$

Proof. Suppose $x = \psi_{+,\mu}^{-2\tau+1}(y)$. From (2.1) and (2.3), we know that

$$\Psi_{+,-2\tau}^{n} \circ \psi_{+,n}^{\mu}(x) \doteq \Psi_{+,-2\tau}^{n+1} \circ \psi_{+,n+1}^{n} \circ \psi_{+,n}^{\mu}(x) = 0,$$

and

$$\Psi_{-,-2\tau}^{n} \circ \psi_{+,n}^{\mu}(x) \doteq \psi_{+,-2\tau}^{\mu}(x) = \psi_{+,-2\tau}^{\mu} \circ \psi_{+,\mu}^{-2\tau+1}(y) = 0.$$

Hence, by Proposition 3.10 we know

$$\psi_{+,n}^{\mu}(x) = \mathrm{Ker}(\Psi_{+,-2\tau}^{n}) \cap \mathrm{Ker}(\Psi_{-,-2\tau}^{n}) = \mathrm{Ker}(\Phi_{-2\tau}^{n}) = Z_n,$$

where Z_n is the space defined in the proof of Proposition 3.10. Note that we use Claim 1 in the proof of Proposition 3.10 to fix constants. Note that

$$\psi_{+,n}^{\mu}(x) \in (\mathbf{\Gamma}_n, \tau + \frac{n+1}{2}) \text{ and } Z_n \cap (\mathbf{\Gamma}_n, \tau + \frac{n+1}{2}) \cong \mathbb{C}.$$

Hence the lemma follows from (3.15). □

Proof of Proposition 3.11. We prove the first commutative diagram and the proof for the second one is similar. We study the commutativity for each graded part of $\mathbf{\Gamma}_{\frac{2n+2k-1}{2}}$. For any grading i, we know that

$$\Phi_{n-1}^{n+2k-1} \circ \Psi_{+,n+2k-1}^{n+k} \circ \psi_{-,n+k}^{\frac{2n+2k-1}{2}}((\mathbf{\Gamma}_{\frac{2n+2k-1}{2}}, i)) \subset (\mathbf{\Gamma}_{n-1}, i + \frac{n}{2})$$

and

$$\psi_{+,n-1}^{\mu} \circ \psi_{+,\mu}^{n+k} \circ \psi_{-,n+k}^{\frac{2n+2k-1}{2}}((\mathbf{\Gamma}_{\frac{2n+2k-1}{2}}, i)) \subset (\mathbf{\Gamma}_{n-1}, i + \frac{n}{2}).$$

When $i < \tau$, we know from Lemma 3.8 that

$$\Pi_{n-1}^{i+\frac{n}{2}} \circ G_n = 0.$$

Hence

$$\Phi_{n-1}^{n+2k-1} \circ \Psi_{+,n+2k-1}^{n+k} \circ \psi_{-,n+k}^{\frac{2n+2k-1}{2}} = 0.$$

When $i > \tau$, from Lemma 3.5 we have

$$\psi_{+,n+k}^{\frac{2n+2k-1}{2}}((\mathbf{\Gamma}_{\frac{2n+2k-1}{2}}, i)) \subset (\mathbf{\Gamma}_{n+k}, i + \frac{n+k-1}{2}).$$

By (2.8), we have for grading i,

$$\begin{aligned}\Phi_{n-1}^{n+2k-1} \circ \Psi_{+,n+2k-1}^{n+k-1} \circ \psi_{-,n+k}^{\frac{2n+2k-1}{2}} &= \Pi_{n-1}^{i} \circ G_{n-1} \circ F_{n+2k-1} \circ \Psi_{+,n+2k-1}^{n+k} \circ \psi_{-,n+k}^{\frac{2n+2k-1}{2}}\\ &= \Pi_{n-1}^{i} \circ G_{n-1} \circ F_{n+k} \circ \psi_{-,n+k}^{\frac{2n+2k-1}{2}}\end{aligned} \tag{3.17}$$

From Lemma 3.8, we have

$$F_{n+k}\big|_{(\mathbf{\Gamma}_{n+k},i+\frac{n+k-1}{2})}=0,$$

which implies

$$\Phi_{n-1}^{n+2k-1}\circ\Psi_{+,n+2k-1}^{n+k-1}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}}=0$$

as well. In summary, whenever $i\neq\tau$, we know that

$$\Phi_{n-1}^{n+2k-1}\circ\Psi_{+,n+2k-1}^{n+k-1}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}}=0.$$

To compare with, we know from Lemma 3.12 that whenever $i\neq k$

$$\operatorname{Im}(\psi_{+,\mu}^{n+k})\cap(\mathbf{\Gamma}_\mu,i)=\operatorname{Im}(\psi_{+,\mu}^{n})\cap(\mathbf{\Gamma}_\mu,i).$$

Hence we conclude that

$$\psi_{+,n-1}^{\mu}\circ\psi_{+,\mu}^{n+k}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}}=0=\Phi_{n-1}^{n+2k-1}\circ\Psi_{+,n+2k-1}^{n+k-1}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}},$$

i.e., the diagram commutes trivially. When $i=\tau$, we know from the construction of Φ-maps and the proof of Lemma 3.12 that

$$\Phi_{n-1}^{n+2k-1}\circ\Psi_{+,n+2k-1}^{n+k-1}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}}((\mathbf{\Gamma}_{\frac{2n+2k-1}{2}},\tau))\subset(\mathbf{\Gamma}_{n-1},\tau+\frac{n}{2})\cap Z_n\cong\mathbb{C}$$

and

$$\psi_{+,n-1}^{\mu}\circ\psi_{+,\mu}^{n+k}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}}((\mathbf{\Gamma}_{\frac{2n+2k-1}{2}},\tau))\subset(\mathbf{\Gamma}_{n-1},\tau+\frac{n}{2})\cap Z_n\cong\mathbb{C}.$$

Hence it remains to show that, for any element $x\in(\mathbf{\Gamma}_{\frac{2n+2k-1}{2}},\tau)$,

$$\Phi_{n-1}^{n+2k-1}\circ\Psi_{+,n+2k-1}^{n+k-1}\circ\psi_{-,n+k}^{\frac{2n+2k-1}{2}}(x)=0\text{ if and only if }\psi_{+,n-1}^{\mu}\psi_{+,\mu}^{n+k}\psi_{-,n+k}^{\frac{2n+2k-1}{2}}(x)=0.\tag{3.18}$$

First, we make the following claim. Recall

$$\psi_{+,n+k}^{\frac{2n+2k-1}{2}}((\mathbf{\Gamma}_{\frac{2n+2k-1}{2}},\tau))\subset(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2}).$$

Claim 4. We have

$$\operatorname{Im}(\psi_{+,\mu}^{-2\tau})\cap(\mathbf{\Gamma}_\mu,\tau)=\psi_{+,\mu}^{n+k}(\operatorname{Ker}(F_{n+k})\cap(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2}))$$

Proof of Claim 4. Suppose $y\in(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2})$ so that there exists $z\in\mathbf{\Gamma}_{-2\tau}$ with

$$\psi_{+,\mu}^{-2\tau}(z)=\psi_{+,\mu}^{n+k}(y).$$

From (2.3), we know that

$$\begin{aligned}\psi_{+,\mu}^{n+k}(y)&=\psi_{+,\mu}^{-2\tau}(z)\\&c\psi_{+,\mu}^{n+k}\circ\Psi_{-,n+k}^{-2\tau}(z)\end{aligned}$$

for some $c\in\mathbb{C}\backslash\{0\}$. From the exactness in (2.1), we know that there exists $w\in(\mathbf{\Gamma}_{n+k-1},\tau+\frac{n+k}{2})$ so that

$$y=c\Psi_{-,n+k}^{-2\tau}(z)+\psi_{+,n+k}^{n+k-1}(w).$$

Hence by (2.8) and Lemma 3.8, we know that

$$F_{n+k}(y)=0,$$

which implies that

$$\operatorname{Im}(\psi_{+,\mu}^{-2\tau})\cap(\mathbf{\Gamma}_\mu,\tau)\subset\psi_{+,\mu}^{n+k}(\operatorname{Ker}(F_{n+k})\cap(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2})).$$

From Lemma 3.12, it remains to show that

$$\psi^{n+k}_{+,\mu}(\mathrm{Ker}(F_{n+k})\cap(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2}))\neq \mathrm{Im}(\psi^{n+k}_{+,\mu})\cap(\mathbf{\Gamma}_{\mu},\tau).$$

Indeed, from Lemma 3.8, there exists $u\in(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2})$ so that $u\notin\mathrm{Ker}(F_{n+k})$.

Suppose there exists $v\in(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2})\cap\mathrm{Ker}(F_{n+k})$ so that

$$\psi^{n+k}_{+,n}(u)=\psi^{n+k}_{+,n}(v).$$

From the exactness in (2.1), there exist $w'\in(\mathbf{\Gamma}_{n+k-1},\tau+\frac{n+k}{2})$ so that

$$u=v+\psi^{n+k-1}_{+,n+k}(w').$$

Hence once again we conclude $F_{n+k}(u)=0$ which is a contradiction. $\square$

Recall $x\in(\mathbf{\Gamma}_{\frac{2n+2k-1}{2}},\tau)$, and for this particular grading, $\Pi^{\tau+\frac{n}{2}}_{n-1}\circ G_{n-1}\neq 0$. We apply Claim 4 and Lemma 3.13) to obtain

$$\begin{aligned}
&\Phi^{n+2k-1}_{n-1}\circ\Psi^{n+k-1}_{+,n+2k-1}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}(x)=0\\
\Leftrightarrow\ &\Pi^{i}_{n-1}\circ G_{n-1}\circ F_{n+k}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}(x)=0\\
\Leftrightarrow\ &F_{n+k}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}(x)=0\\
\Leftrightarrow\ &\psi^{n+k}_{+,\mu}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}(x)\in\psi^{n+k}_{+,\mu}(\mathrm{Ker}(F_{n+k})\cap(\mathbf{\Gamma}_{n+k},\tau+\frac{n+k-1}{2}))\\
\Leftrightarrow\ &\psi^{n+k}_{+,\mu}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}(x)\in\mathrm{Im}(\psi^{-2\tau}_{+,\mu})\\
\Leftrightarrow\ &\psi^{\mu}_{+,n-1}\circ\psi^{n+k}_{+,\mu}\circ\psi^{\frac{2n+2k-1}{2}}_{-,n+k}(x)=0
\end{aligned}$$

Hence we prove (3.18) and this concludes the proof of Proposition 3.11. $\square$

4. Properties and applications

In this section, we prove the remain results in the introduction. We adapt notations in Section 3.4 and write $\mathbf{\Gamma}_n,\mathbf{\Gamma}_\mu$, and $\mathbf{\Gamma}_{\frac{2n-1}{2}}$ instead of $\widehat{\mathbf{\Gamma}}_n,\widehat{\mathbf{\Gamma}}_\mu$, and $\boldsymbol{\gamma}_{2\hat{\lambda}-2(n-1)\hat{\mu}}$, respectively. Moreover, we write

$$\pi^+_{m,k}=\Psi^{m+k}_{+,m+2k-1}\circ\psi^{\frac{2m+2k-1}{2}}_{-,m+k}:\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}\to\mathbf{\Gamma}_{m+2k-1},$$

and

$$\pi^-_{m,k}=\Psi^{m+k}_{-,m+2k-1}\circ\psi^{\frac{2m+2k-1}{2}}_{+,m+k}:\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}\to\mathbf{\Gamma}_{m+2k-1}$$

as in the introduction.

Proof of Lemma 1.8. For part (1), we only prove the statement regarding $\pi^+_{m,k}$. The statement regarding $\pi^-_{m,k}$ follows from the symmetry between K and $-K$, where $-K$ is the orientation reversal of K (*c.f.* Remark 1.5). Note when we switch the orientation of the knot, the tau invariant remains the same, $\pi^\pm$ switches with each other, and the grading induced by the Seifert surface becomes the additive inverse. Let

$$\psi^{\frac{2m+2k-1}{2},i}_{-,m+k}=\psi^{\frac{2m+2k-1}{2}}_{-,m+k}|_{(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}},i)}.$$

We know that

$$\pi^{+,i}_{m,k}=\Psi^{m+2k-1}_{+,m+k}\circ\psi^{\frac{2m+2k-1}{2},i}_{-,m+k}.$$

From Lemma 3.6 we know

$$\mathrm{Im}(\psi_{-,m+k}^{\frac{2m+2k-1}{2},i}) \subset (\mathbf{\Gamma}_{m+k}, i+\frac{m+k-1}{2}).$$

When k is large enough so that $m+k$ is large, the map $\Psi^{m+k}_{+,m+2k-1}$ corresponds to the composition of $(k-1)$ many U-actions as in the construction of $\underline{\mathrm{KHI}}^-$ in [Li19]. By the definition of τ in [Li19], we immediately conclude that

$$\Psi^{m+k}_{+,m+2k-1}|_{(\mathbf{\Gamma}_{m+k},j)} = 0$$

whenever $j > \tau + \frac{m+k-1}{2}$. Hence as a result we have

$$\pi^{+,i}_{m,k} = 0$$

when $i > \tau$.

For part (2), again we only prove the statement involving $\pi^{+}_{m,k}$. By the definition of τ in [Li19] and the correspondence between $\Psi^{m+k}_{+,m+2k-1}$ and U^{k-1} on $\underline{\mathrm{KHI}}^-$, we know that when k is large enough, there exists

$$x \in (\mathbf{\Gamma}_{m+k-\tau+i}, \tau+\frac{m+k-\tau+i-1}{2})$$

so that

$$\Psi^{m+k-\tau+i}_{+,m+2k-1}(x) \neq 0.$$

Take

$$y = \Psi^{m+k-\tau+i}_{+,m+k}(x) \in (\mathbf{\Gamma}_{m+k}, i),$$

we know that

$$\Psi^{m+k}_{+,m+2k}(y) = \Psi^{m+k-\tau+i}_{+,m+2k-1}(x) \neq 0.$$

So it remains to show that $y \in \mathrm{Im}(\psi_{-,m+k}^{\frac{2m+2k-1}{2},i})$. Indeed, from the construction of y we know that

$$\psi^{m+k}_{+,\mu}(y) = 0.$$

Then from the commutative diagram (2.2), we know that

$$\psi^{m+k}_{-,m+k-1}(y) \doteq \psi^{\mu}_{-,m+k} \circ \psi^{m+k}_{+,\mu}(y) = 0.$$

Hence by Lemma 3.6 we have

$$y \in \mathrm{Ker}(\psi^{m+k}_{-,m+k-1}) = \mathrm{Im}(\psi_{-,m+k}^{\frac{2m+2k-1}{2},i}).$$

Part (3) follows from a straightforward combination of Lemma 3.5, Lemma 3.6, Formula (2.4), and Formula (2.5). □

4.1. **A concordance invariant.**
According to Lemma 1.8, we can define a new invariant $\nu_I(K)$ as in Definition 1.9. We first show that it is a concordance invariant as in Proposition 1.10.

Proof of Proposition 1.10. Suppose K_1 and K_2 are concordance to each other and A is the concordance between them. According to [GLW19, Proposition 1.12], we can write

$$\tau = \tau_I(K_1) = \tau_I(K_2).$$

Moreover, as in the proof of the proposition, there are two commutative diagrams:

$$\begin{tikzcd} SHI(-S^3(K_1),-\Gamma_{\frac{2m+2k-1}{2}},\tau) \arrow[r, "F_{A,\frac{2m+2k-1}{2}}"] \arrow[d, "\pi^{+,\tau}_{m,k}(K_1)"] & SHI(-S^3(K_2),-\Gamma_{\frac{2m+2k-1}{2}},\tau) \arrow[d, "\pi^{+,\tau}_{m,k}(K_2)"] \\ SHI(-S^3(K_1),-\Gamma_{m+2k-1},\tau+\frac{m}{2}) \arrow[r, "F_{A,m+2k-1}"] & SHI(-S^3(K_2),-\Gamma_{m+2k-1},\tau+\frac{m}{2}) \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd} SHI(-S^3(K_1),-\Gamma_{m+2k-1},\tau+\frac{m}{2}) \arrow[r, "F_{A,m+2k-1}"] \arrow[d, "C_{h,m+2k-1}"] & SHI(-S^3(K_2),-\Gamma_{m+2k-1},\tau+\frac{m}{2}) \arrow[d, "C_{h,m+2k-1}"] \\ I^{\sharp}(-S^3) \arrow[r, "\mathrm{Id}"] & I^{\sharp}(-S^3) \end{tikzcd} \tag{4.2}$$

Here $F_{A,\frac{2m+2k-1}{2}}$ and $F_{A,m+2k-1}$ are maps coming from the concordance, with notations adopted from [GLW19, Proposition 1.12]. The reason for the first diagram (4.1) to commute is similar to the proof of Claim 1 in the proof of [GLW19, Proposition 1.12]: the horizontal and vertical maps can both be interpreted as cobordism maps between suitable closures of related sutured manifolds. But the 4-dimensional handles composing these cobordisms are attached in disjoint regions, and hence the corresponding cobordism maps commute with each other. The second diagram (4.2) is exactly [GLW19, Equation (3.20)]. From the proofs of [GLW19, Proposition 1.12 and Proposition 1.13], we know

$$F_{A,m+2k-1}: SHI(-S^3(K_1),-\Gamma_{m+2k-1},\tau+\frac{m}{2})\cong\mathbb{C}\to SHI(-S^3(K_2),-\Gamma_{m+2k-1},\tau+\frac{m}{2})\cong\mathbb{C}$$

is an isomorphism. Then applying (4.1), we know that if $\pi^{+,\tau}_{m,k}(K_1)$ is nonvanishing, then $\pi^{+,\tau}_{m,k}(K_2)$ is also nonvanishing. Since the concordance relation is a symmetric relation, we conclude $\pi^{+,\tau}_{m,k}(K_1)$ is nonvanishing if and only if $\pi^{+,\tau}_{m,k}(K_2)$ is nonvanishing. This means $\nu_I(K_1)=\nu_I(K_2)$ and ν_I is a concordance invariant. $\square$

In [BS20], Baldwin-Sivek introduced the concordance invariant $\nu^{\sharp}(K)$ to describe the framed instanton homology of Dehn surgeries of knots. Explicitly, they proved the following theorem.

Theorem 4.1 ([BS20, Theorem 1.1])**.** *For every knot $K\subset S^3$, there is an integer $r_0(K)\geqslant|\nu^{\sharp}(K)|$ such that*

$$\dim_{\mathbb{C}} I^{\sharp}(S^3_{p/q}(K))=q\cdot r_0(K)+|p-q\cdot\nu^{\sharp}(K)| \tag{4.3}$$

for all nonzero rational p/q with p and q relatively prime and $q\geqslant 1$. If $\nu^{\sharp}(K)\neq 0$, then the same is true for $p/q=0/1$.

Using the integral surgery formula for knots in S^3, we can compute $I^{\sharp}(S^3_{-m}(K)))$ for any $m\in\mathbb{Z}\backslash\{0\}$ by the concordance invariant $\nu_I(K)$. Comparing the result with (4.3), we can understand the relation between $\nu^{\sharp}$ and ν_I as in Corollary 1.12.

Proof of Proposition 1.11 and Corollary 1.12. For simplicity, let $\tau=\tau_I(K)$ and $g=g(K)$. When $\tau\neq 0$, we can switch to the mirror of K if necessary, to assume that $\tau<0$. Since we pick k after fixing m, we can make $m+k$ large enough. From [GLW19] and [LY21], we know the structures of $\mathbf{\Gamma}_{m+2k-1}$ and $\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}$ with respect to the $\mathbb{Z}$-gradings are as follows:

$$(\mathbf{\Gamma}_{m+2k-1}, i) \cong \mathbb{C} \text{ for } |i| \leqslant \frac{m}{2} + k - 1 - g. \tag{4.4}$$

$$(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, i) \cong \begin{cases} \mathbb{C} & g \leqslant |i| \leqslant m+k-1-g \\ A_i & |i| < g \end{cases}, \tag{4.5}$$

where A_i for $|i| \leqslant g$ is independent of $m+k$ as long as $m+k$ is large enough. In the following, we compute $I^\sharp(S^3_{-m}(K)))$ and compare the result with (4.3).

Case 1. Suppose $m > 0$. From Lemma 1.8, we know that when $i > m+g$, we have

$$\pi^{+,i}_{m,k} = 0 \text{ and } \pi^{-,i}_{m,k} \text{ is an isomorphism.}$$

Also, when $i < -m-g$, we have

$$\pi^{-,i}_{m,k} = 0 \text{ and } \pi^{+,i}_{m,k} \text{ is an isomorphism.}$$

As a result, Theorem 1.1 (with Proposition 1.4) can be reformulated as follows:

$$I^\sharp(S^3_{-m}(K)) \cong H(\mathrm{Cone}(\sum_{|i| \leqslant m+g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}))). \tag{4.6}$$

Note again from Lemma 1.8, we know

$$\mathrm{Im} \sum_{|i| \leqslant m+g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}) \subset \bigoplus_{|i| \leqslant \frac{m}{2}+g} (\mathbf{\Gamma}_{m+2k-1}, i).$$

Hence from Formulae (4.4), (4.5), and (4.6), we know that

$$\dim_{\mathbb{C}} I^\sharp(S^3_{-m}(K)) = \sum_{|i|<g} \dim_{\mathbb{C}} A_i + 2m + 2 + m + 2g + 1 - 2 \cdot \dim_{\mathbb{C}} \mathrm{Im} \sum_{|i| \leqslant m+g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}).$$

It remains to understand the the last term in the above formula.

Case 1.1 Suppose $\nu_I(K) = \tau$, *i.e.*, $\pi^{+,i}_{m,k} \neq 0$ for $i = \tau$ (the only missing case in Lemma 1.8). By the symmetry between K and $-K$, we know that $\pi^{-,i}_{m,k} \neq 0$ for $i = -\tau$. Since we have assumed that $\tau < 0$, we know that when $-\tau \leqslant i \leqslant m+g$, we have

$$\pi^{+,i}_{m,k} = 0 \text{ and } \pi^{-,i}_{m,k} \neq 0.$$

When $\tau < i < -\tau$, we have

$$\pi^{+,i}_{m,k} = 0 \text{ and } \pi^{-,i}_{m,k} = 0.$$

When $-m-g \leqslant i \leqslant -\tau$, we have

$$\pi^{+,i}_{m,k} \neq 0 \text{ and } \pi^{-,i}_{m,k} = 0.$$

Note we already know that

$$\mathrm{Im}(\pi^{\pm,i}_{m,k}) \subset (\mathbf{\Gamma}_{m+2k-1}, i \pm \frac{m}{2}),$$

and

$$(\mathbf{\Gamma}_{m+2k-1}, j) \cong \mathbb{C}$$

for $|j| \leqslant \frac{m}{2} + g$. Hence for $|i| \leqslant m + g$, $\pi^{\pm,i}_{m,k} \neq 0$ implies it is surjective. As a result, we conclude that

$$\dim_{\mathbb{C}} \operatorname{Im} \sum_{|i| \leqslant m+g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}) = \begin{cases} 2m + 2g + 2\tau + 2 & m \leqslant -2\tau - 1 \\ m + 2g + 1 & m > -2\tau - 1 \end{cases}.$$

and

$$\dim_{\mathbb{C}} I^{\sharp}(S^3_{-m}(K)) = \begin{cases} \sum_{|i|<g} \dim_{\mathbb{C}} A_i - 2g - 4\tau - 1 - m & m \leqslant -2\tau - 1 \\ \sum_{|i|<g} \dim_{\mathbb{C}} A_i - 2g + 1 + m & m > -2\tau - 1 \end{cases}$$

Note $\dim_{\mathbb{C}} A_i$ for $|i| < g$ is independent of m. Then Formula (4.3) implies

$$\nu^{\sharp}(K) = 2\tau + 1 = 2\nu_I(K) - 1.$$

Case 1.2. Suppose $\nu_I(K) = \tau$, *i.e.*, $\pi^{+,i}_{m,k} = 0$ for $i = \tau$. By the symmetry between K and $-K$, we know that $\pi^{-,i}_{m,k} = 0$ for $i = -\tau$. A similar argument as above shows that

$$\dim_{\mathbb{C}} \operatorname{Im} \sum_{|i| \leqslant m+g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}) = \begin{cases} 2m + 2g + 2\tau & n \leqslant -2\tau + 1 \\ m + 2g + 1 & n > -2\tau + 1 \end{cases}.$$

As a result, we conclude

$$\dim_{\mathbb{C}} I^{\sharp}(S^3_{-m}(K)) = \begin{cases} \sum_{|i|<g} \dim_{\mathbb{C}} A_i - 2g - 4\tau + 3 - m & m \leqslant -2\tau + 1 \\ \sum_{|i|<g} \dim_{\mathbb{C}} A_i - 2g + 1 + m & m > -2\tau + 1 \end{cases}$$

Then Formula (4.3) implies

$$\nu^{\sharp} = 2\tau - 1 = 2\nu_I - 1.$$

Case 2. Suppose $m < 0$. As in case 1, we can compute

$$I^{\sharp}(S^3_{-m}(K)) \cong H(\operatorname{Cone}(\sum_{|i| \leqslant g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}))). \tag{4.7}$$

Hence from Formulae (4.4), (4.5), and (4.7), we know that

$$\dim_{\mathbb{C}} I^{\sharp}(S^3_{-m}(K)) = \sum_{|i|<g} \dim_{\mathbb{C}} A_i + 2 - m + 2g + 1 - 2 \cdot \dim_{\mathbb{C}} \operatorname{Im} \sum_{|i| \leqslant g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}).$$

Case 2.1 Suppose $\nu_I(K) = \tau + 1$, *i.e.*, $\pi^{+,i}_{m,k} \neq 0$ for $i = \tau$. By the symmetry between K and $-K$, we know that $\pi^{-,i}_{m,k} \neq 0$ for $i = -\tau$. Since we have assumed that $\tau < 0$, we know that when $-\tau \leqslant i \leqslant g$, we have

$$\pi^{+,i}_{m,k} = 0 \text{ and } \pi^{-,i}_{m,k} \neq 0.$$

When $\tau < i < -\tau$, we have

$$\pi^{+,i}_{m,k} = 0 \text{ and } \pi^{-,i}_{m,k} = 0.$$

When $-g \leqslant i \leqslant -\tau$, we have

$$\pi^{+,i}_{m,k} \neq 0 \text{ and } \pi^{-,i}_{m,k} = 0.$$

Note we already know that

$$\operatorname{Im}(\pi^{\pm,i}_{m,k}) \subset (\mathbf{\Gamma}_{m+2k-1}, i \pm \frac{m}{2}),$$

and

$$(\mathbf{\Gamma}_{m+2k-1}, j) \cong \mathbb{C}$$

for $|j| \leqslant -\frac{m}{2}+g$. Hence for $|i| \leqslant m+g$, $\pi^{\pm,i}_{m,k} \neq 0$ implies it is surjective. As a result, we conclude that

$$\dim_{\mathbb{C}} \operatorname{Im} \sum_{|i| \leqslant g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}) = 2g + 2\tau + 2$$

and

$$\dim_{\mathbb{C}} I^{\sharp}(S^3_{-m}(K)) = \sum_{|i|<g} \dim_{\mathbb{C}} A_i - 2g - 4\tau - 1 - m.$$

Case 2.2. Suppose $\nu_I(K) = \tau$, *i.e.*, $\pi^{+,i}_{m,k} = 0$ for $i = \tau$. By the symmetry between K and $-K$, we know that $\pi^{-,i}_{m,k} = 0$ for $i = -\tau$. A similar argument as above shows that

$$\dim_{\mathbb{C}} \operatorname{Im} \sum_{|i| \leqslant g} (\pi^{+,i}_{m,k} + \pi^{-,i}_{m,k}) = 2g + 2\tau.$$

As a result, we conclude

$$\dim_{\mathbb{C}} I^{\sharp}(S^3_{-m}(K)) = \sum_{|i|<g} \dim_{\mathbb{C}} A_i - 2g - 4\tau + 3 - m.$$

So in Case 2 we could only conclude that $\nu^{\sharp} \leqslant 0$, which reduces to Case 1. □

4.2. **The 0-surgery formula.**

The main obstruction to apply the proof of the integral surgery formula in Section 2.3 to the 0-surgery is that $\pi^{+}_{m,k}$ and $\pi^{-}_{m,k}$ have the same grading shift. Then the homology of the mapping cone

$$H(\operatorname{Cone}(c_{11}\pi^{+}_{m,k} + c_{12}\pi^{-}_{m,k}))$$

may depend on the coefficients. If either map vanishes, then the homology is still independent of the coefficients. However, this is not true in general. Fortunately, we can make use of the $\mathbb{Z}$-grading on $I^{\sharp}(S^3_0(K))$ mentioned in the introduction. Note that one of the restrictions of $\pi^{\pm}_{m,k}$ on a single grading vanishes.

Theorem 4.2 (0-surgery formula)**.** *Suppose $K \subset S^3$ is a knot. Suppose $A(s), B^{\pm}(s)$ and $\pi^{\pm}(s) : A(s) \to B^{\pm}(s)$ are complexes and maps constructed in Section 2.4 for K. For any $s \in \mathbb{Z}\backslash\{0\}$, there exists an isomorphism*

$$\Xi_{0,s} : H(B^{+}(s)) \to H(B^{-}(s))$$

so that $I^{\sharp}(-S^3_0(K), s)$ is isomorphic to the homology of the mapping cone of the map

$$\pi^{-}(s) + \Xi_{0,s} \circ \pi^{+}(s) : H(A(s)) \to H(B^{-}(s)).$$

If $\tau_I(K) \neq 0$ or $\tau_I(K) = \nu_I(K) = 0$, then the same result also applies to $s = 0$.

Proof. Following the construction of the gradings, the maps in the long exact sequence

$$\cdots \to \mathbf{\Gamma}_{\mu} \xrightarrow{c_1\psi_+ + c_2\psi_-} \mathbf{\Gamma}_{-1} \to I^{\sharp}(-S^3_0(K)) \to \mathbf{\Gamma}_{\mu} \to \cdots$$

are all grading preserving with respect to the gradings induced by the Seifert surface. Since all other maps in the octahedral diagram (2.15) are from bypass maps that are also homogeneous, we could decompose the diagram into different grading parts.

When $\tau_I(K) \neq 0$, we can switch to the mirror of K if necessary, to assume that $\tau_I(K) < 0$. Then from Lemma 1.8, we know for any $i \in \mathbb{Z}$, either $\pi^{+,i}_{0,k}$ or $\pi^{-,i}_{0,k}$ vanishes, and hence $H(\operatorname{Cone}(c_{11}\pi^{+,i}_{0,k} +$

$c_{12}\pi_{0,k}^{-,i}))$ is independent of the coefficients. Following the same strategy in the proof of Theorem 2.15, we obtain for any $s \in \mathbb{Z}$,

$$\begin{aligned} I^\sharp(-S_0^3(K),s) &\cong H(\mathrm{Cone}(c_{11}\pi_{0,k}^{+,i} + c_{12}\pi_{0,k}^{-,i})) \\ &\cong H(\mathrm{Cone}(\pi_{0,k}^{+,i} + \pi_{0,k}^{-,i})) \\ &\cong H(\mathrm{Cone}(\pi^-(s) + \Xi_{0,s} \circ \pi^+(s))), \end{aligned}$$

where $\Xi_{0,s}$ is constructed similarly to Ξ_m for $m \neq 0$.

When $\tau_I = 0$, the above argument still holds for $s \neq 0$. If furthermore $\nu_I(K) = 0$, the by definition $\pi_{0,k}^{+,i}$ vanishes $\pi_{0,k}^{+,i}$. By the symmetry between K and $-K$, the other map $\pi_{0,k}^{-,i}$ also vanishes. Then the above argument also holds for $s = 0$. $\square$

Remark 4.3. The proof of Theorem 4.2 also implies Theorem 1.14.

Baldwin-Sivek also studied framed instanton homology with twisted bundle for 0-surgery, which is denoted by $I^\sharp(S_0^3(K),\mu)$, where μ is the meridian of the knot. There is also a $\mathbb{Z}$-grading on this homology induced by the Seifert surface and we also have a long exact sequence

$$\cdots \to \mathbf{\Gamma}_\mu \xrightarrow{c_1'\psi_+ + c_2'\psi_-} \mathbf{\Gamma}_{-1} \to I^\sharp(-S_0^3(K),\mu) \to \mathbf{\Gamma}_\mu \to \cdots$$

so that all maps are grading preserving (the coefficients c_1' and c_2' may be different from c_1 and c_2). Thus, we can use the similar splitted octahedral diagram to prove the result in Theorem 4.2 when replacing $I^\sharp(-S_0^3(K))$ by $I^\sharp(-S_0^3(K),\mu)$. As a result, we obtain the following corollary. We also write $I^\sharp(-S_0^3(K),\mu,i)$ for the grading summand of $I^\sharp(-S_0^3(K),\mu)$.

Corollary 4.4. *Suppose $K \subset S^3$ is a knot. For any $s \in \mathbb{Z}\backslash\{0\}$, we have*

$$I^\sharp(-S_0^3(K),s) \cong I^\sharp(-S_0^3(K),\mu,s).$$

If $\tau_I(K) \neq 0$ or $\tau_I(K) = \nu_I(K) = 0$, then the above equation also holds for $s = 0$.

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Department of Mathematics, Stanford University
Email address: zhenkun@stanford.edu

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge
Email address: fy260@cam.ac.uk