KNOT SURGERY FORMULAE FOR INSTANTON FLOER HOMOLOGY I: THE MAIN THEOREM

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Abstract. We prove an integral surgery formula for framed instanton homology $I^\sharp(Y_m(K))$ for any knot K in a 3-manifold Y with $[K]=0\in H_1(Y;\mathbb{Q})$ and $m\neq 0$. Although the statement is similar to Ozsváth-Szabó's integral surgery formula for Heegaard Floer homology, the proof is new and based on sutured instanton homology SHI and the octahedral lemma in the derived category. As byproducts, we obtain a formula computing instanton knot homology of the dual knot analogous to Eftekhary's and Hedden-Levine's work, and also an exact triangle between $I^\sharp(Y_m(K))$, $I^\sharp(Y_{m+k}(K))$ and k copies of $I^\sharp(Y)$ for any $m\neq 0$ and large k. In the proof of the formula, we discover many new exact triangles for sutured instanton homology and relate some surgery cobordism map to the sum of bypass maps, which are of independent interest. In a companion paper, we derive many applications and computations based on the integral surgery formula.

Contents

1. Introduction

The framed instanton homology $I^\sharp(Y)$ for a closed 3-manifold Y was introduced by Kronheimer and Mrowka in [KM11] and has been conjectured to be isomorphic to the hat version of Heegaard Floer homology $\widehat{HF}(Y)$. This conjecture is still widely open and, due to the computational difficulty of instanton Floer homology, not many examples has been known. In recent years, many people have done computations of the framed instanton homology of special families of 3-manifolds, see for example [LPCS22, BS23, BS21]. Yet, most results have focused on computing the dimension of framed instanton Floer homology and many techniques only work for S^3 or rational homology spheres, however, a general structural theorem that relates the framed instanton homology of Dehn surgeries to the information from the knot complement still remains elusive.

In [LY21], the authors of the current paper proved a large surgery formula for framed instanton homology which led to a series of applications in computing the framed instanton homology and studying the representations of the fundamental groups of Dehn surgeries of some families of knots. However, in that work, the Dehn surgery slope must be large (at least $2g+1$ where g is the Seifert genus of the knot), and thus still not much is known about the framed instanton homology of small Dehn surgery slopes. In this paper, we further prove an integral surgery formula for rationally null-homologous knots, inspired by Ozsváth-Szabó's surgery formula for Heegaard Floer homology [OS08, OS11]. For simplicity, in the introduction, we present only the discussions and results for (integral) null-homologous knots (*e.g.* knots in S^3) and leave the general setup to Section 3.3.

First, let us recall the results from [LY21]. Suppose $K\subset Y$ is a null-homologous knot. Let $Y\backslash N(K)$ be the knot complement, and let Γ_μ be the union of two oppositely oriented meridians of the knot on $\partial(Y\backslash N(K))$. Let $SHI(-Y\backslash N(K),-\Gamma_\mu)$ be the corresponding sutured instanton homology introduced by Kronheimer-Mrowka [KM10], where the minus sign denotes orientation reversal for technical needs (note that $SHI(-M,-\gamma)\cong SHI(M,\gamma)$ and in particular $I^\sharp(-Y_{-m}(K))\cong I^\sharp(Y_{-m}(K))$). A Seifert surface of K induces a $\mathbb{Z}$-grading on $SHI(-Y\backslash N(K),-\Gamma_\mu)$. In [LY21], we constructed a set of differentials on $SHI(-Y\backslash N(K),-\Gamma_\mu)$

$$d_j^i: SHI(-Y\backslash N(K),-\Gamma_\mu,i)\to SHI(-Y\backslash N(K),-\Gamma_\mu,j)$$

for any gradings $i\neq j\in\mathbb{Z}$. We then constructed bent complexes

$$A_s=\left(SHI(-Y\backslash N(K),-\Gamma_\mu),\sum_{s\leqslant i<j}d_j^i+\sum_{s\geqslant i>j}d_j^i\right),$$

$$B^+=\left(SHI(-Y\backslash N(K),-\Gamma_\mu),\sum_{i<j}d_j^i\right),\text{ and }B^-=\left(SHI(-Y\backslash N(K),-\Gamma_\mu),\sum_{i>j}d_j^i\right).$$

From [LY21], the homologies of these complexes are related to the Dehn surgeries of K as follows:

$$H(B^+)\cong H(B^-)\cong I^\sharp(-Y), \tag{1.1}$$

$$I^\sharp(Y_{-m}(K))\cong\bigoplus_{s=\lfloor\frac{1-m}{2}\rfloor}^{\lfloor\frac{m-1}{2}\rfloor}H(A_s)\text{ for any integer }m\geqslant 2g(K)+1. \tag{1.2}$$

To state the integral surgery formula, we introduce more notations. For $s \in \mathbb{Z}$, let $B_s^{\pm}$ be identical copies of $B^{\pm}$. Define chain maps
$$\pi^{\pm,s}: A_s \to B_s^{\pm}$$
as follows. For $x \in (SHI(-Y\backslash N(K), -\Gamma_\mu), i)$,
$$\pi^{+,s}(x) = \begin{cases} x & i \geqslant s, \\ 0 & i < s, \end{cases} \text{ and } \pi^{-,s}(x) = \begin{cases} x & i \leqslant s, \\ 0 & i > s. \end{cases}$$

Let $\pi^{\pm}$ denote the direct sum of all $\pi^{\pm,s}$. While this slightly abuses the notation, we also use them to denote the induced maps on the homologies. The main result of the paper is the following.

Theorem 1.1 (Integral surgery formula). *Suppose $K \subset Y$ is a null-homologous knot. Let A_s, $B_s^{\pm}$, and $\pi^{\pm}$ be defined as above. Then for any $m \in \mathbb{Z}\backslash\{0\}$, there exists an isomorphism*
$$\Xi_m : \bigoplus_{s\in\mathbb{Z}} H(B_s^+) \xrightarrow{\cong} \bigoplus_{s\in\mathbb{Z}} H(B_{s+m}^-)$$
as the direct sum of isomorphisms
$$\Xi_{m,s} : H(B_s^+) \xrightarrow{\cong} H(B_{s+m}^-)$$
such that
$$I^\sharp(-Y_{-m}(K)) \cong H\left(\operatorname{Cone}(\pi^- + \Xi_m \circ \pi^+ : \bigoplus_{s\in\mathbb{Z}} H(A_s) \to \bigoplus_{s\in\mathbb{Z}} H(B_s^-))\right).$$

Remark 1.2. As an analog of the surgery formula in Heegaard Floer homology, the map π^- is related to the vertical projection map v, and the map $\Xi_m \circ \pi^+$ is related to the map h, which is defined by the composition of the horizontal projection and a chain homotopy equivalence between $C\{j \geqslant 0\}$ to $C\{i \geqslant 0\}$. Here we bend the horizontal part of the hook complex to become vertical, so the differentials go upwards. The homotopy equivalence in Heegaard Floer homology depends on many auxiliary choices (*c.f.* Construction before [HL21, Lemma 2.16]). The same situation applies to Ξ_m. Hence we only state the existence of the isomorphism.

Remark 1.3. The hypothesis of Theorem 1.1 excludes the case where $m = 0$. This is due to the sign ambiguity in the definition of sutured instanton homology. The original version of sutured instanton homology defined by Kronheimer-Mrowka [KM10] was only well-defined up to isomorphisms, and then Baldwin-Sivek [BS15] proved that they are well-defined up to a scalar in $\mathbb{C}$. As a result, all related maps are only well-defined up to scalars. When $m \neq 0$, the maps $\pi^{+,s}$ and $\Xi_m \circ \pi^{-,s}$ have distinct target spaces, namely B_s and B_{s+m}. As a result, the scalar ambiguity for individual maps does not influence the dimension of the homology of the mapping cone. However, when $m = 0$, different scalars would indeed make differences. For an example of this subtlety, see the end of Section 4.

Remark 1.4. We also obtain a formula computing instanton knot homology $KHI(-Y_{-m}(K), K_{-m})$ of the dual knot K_{-m} inside the resulting manifold in Theorem 3.22, which is analogous to the results by Eftekhary for knot Floer homology $\widehat{HFK}$ [Eft18, Proposition 1.5] and by Hedden-Levine [HL21]. In this formula, we may assume $m = 0$ because the scalar issue in Remark 1.3 does not appear.

With the isomorphisms in (1.2) and (1.1), we can truncate the above formula for $I^\sharp(Y_{-m}(K))$ to obtain the following exact triangle.

Corollary 1.5 (Generalized surgery exact triangle)**.** *Suppose $K \subset Y$ is a null-homologous knot, and m is a fixed non-zero integer. Then for any sufficiently large integer k, there exists an exact triangle*

$$\begin{tikzcd} I^\sharp(-Y_{-m-k}(K)) \arrow[rr] & & \bigoplus_{i=1}^k I^\sharp(-Y) \arrow[dl] \\ & I^\sharp(-Y_{-m}(K)) \arrow[ul] & \end{tikzcd} \tag{1.3}$$

Remark 1.6. The analogous result of the exact triangle (1.3) in Heegaard Floer theory was proved by Ozsváth-Szabó [OS08] using twisted coefficients, which is a crucial step towards proving the integral surgery formula in their setup. The proof cannot be applied to instanton theory directly. Thus, in this paper, we adopt a reversed approach: we use sutured instanton theory to prove Theorem 1.1 and derive Corollary 1.5 as a direct application. The strategy to prove Theorem 1.1 can be found in Section 3.1 and Section 3.2.

The analogs of $\pi^{\pm,s}$ in Heegaard Floer theory can be interpreted as cobordism maps associated to some particular spin^c structures. In instanton theory, there is a decomposition of cobordism maps along basic classes. However, currently such a decomposition is only known to exist for cobordisms whose first Betti number is zero. So for the moment let us assume the ambient 3-manifold Y is a rational homology sphere. For any integer m, there is a natural cobordism W_m from $-Y^3_{-m}(K)$ to $-Y^3$. From [BS23, Section 1.2], there exists a decomposition of the cobordism map $I^\sharp(W_m)$ along basic classes

$$I^\sharp(W_m) = \sum_{s\in\mathbb{Z}} I^\sharp(W_m, [s]),$$

where $[s] \in H^2(W)$ denotes the class that satisfies the equality

$$[s]([\bar{S}]) = 2s - m.$$

We make the following conjecture.

Conjecture 1.7. Suppose $K \subset Y$ is a null-homologous knot. Suppose $b_1(Y) = 0$ and $m \in \mathbb{Z}$ with $m \geqslant 2g(K) + 1$. Let A_s, B_s^-, $\pi^{\pm,s}$, W_m, $I^\sharp(W_m, [s])$ be defined as above. Then for any $s \in [\lfloor \frac{m-1}{2} \rfloor, \lfloor \frac{1-m}{2} \rfloor] \cap \mathbb{Z}$, there are commutative diagrams

$$\begin{tikzcd} H(A_s) \arrow[r, hook] \arrow[d, "\pi^{-,s}"] & I^\sharp(-Y_{-m}(K)) \arrow[d, "I^\sharp(W_m,[s])"] \\ H(B_s) \arrow[r, "\cong"] & I^\sharp(-Y) \end{tikzcd} \qquad \begin{tikzcd} H(A_s) \arrow[r, hook] \arrow[d, "\pi^{+,s}"] & I^\sharp(-Y_{-m}(K)) \arrow[d, "I^\sharp(W_m,[s+m])"] \\ H(B_s) \arrow[r, "\cong"] & I^\sharp(-Y) \end{tikzcd}$$

Remark 1.8. In Heegaard Floer theory, the large surgery formula in [OS04, Theorem 4.1] states that the homology of the bent complex A_s is isomorphic to the Heegaard Floer homology of $Y_{-m}(K)$ together with a spin^c structure specified by s. In instanton theory, we do not have the spin^c structures in the construction of instanton Floer homology but a similar decomposition was introduced in [LY22a, LY23]. However, involving the spin^c-type decomposition in the statement of Conjecture 1.7 would make the statement more complicated. So here we only write the top horizontal map in each commutative diagram as an inclusion.

The obstacle to obtaining a decomposition of the instanton cobordism map, in general, is one of the difficulties in exporting the original proof of the integral surgery formula in Heegaard Floer theory to the instanton setup. To overcome this problem, we need to work with a suitable setup for which some kind of decompositions do exist. A good candidate is sutured instanton theory. In sutured instanton theory, properly embedded surfaces induce $\mathbb{Z}$-gradings on the homology, and bypass maps relating different sutures are homogeneous with respect to such gradings. We have already used this setup to construct spin^c-like decompositions for the framed instanton homology of Dehn surgeries of knots, construct bent complexes in instanton theory, and establish a large surgery formula in our previous work [LY22a, LY23, LY21].

In this paper, to prove the integral surgery formula, we further study the relations between different sutures on the knot complement and establish some new exact triangles and commutative diagrams that may be of independent interest. Then these new and old algebraic structures relating to different sutures enable us to apply the octahedral lemma to prove the desired integral surgery formula. It is worth mentioning that ultimately the whole proof in the current paper depends only on some most fundamental properties of Floer theory: the surgery exact triangle, the functoriality of the cobordism maps, and the adjunction inequality. This implies that the existence of the surgery formula is an inherent property of Floer theory.

The surgery formula developed in the current paper is a powerful tool to study the Dehn surgeries along knots. It enables us to do explicit computations in many cases, even when the ambient 3-manifold has a positive first Betti number. In a companion paper [LY22b], we will use the surgery formula and the techniques developed in this paper to derive many new applications and computations. We sketch the results as follows.

(1) We study the behavior of the integral surgery formula under the connected sum with a core knot in a lens space (whose complement is a solid torus) and then derive a rational surgery formula for framed instanton homology.
(2) We study the 0-surgery on a knot K inside S^3. We derive a formula computing the non-zero grading part of $I^\sharp(S_0(K))$ with respect to the grading induced by the Seifert surface.
(3) We study non-zero integral surgeries on Boromean knots inside $Y = \#^{2g} S^1 \times S^2$, which gives nontrivial circle bundles over surfaces. In this case the bent complexes A_s and $B_s^\pm$ can be computed directly and the maps $\pi^\pm$ between them can be fixed with the help of the $\Lambda^* H_1(Y;\mathbb{C})$-action. Moreover, we show $\dim_{\mathbb{C}} I^\sharp(Y) = \dim_{\mathbb{F}_2} \widehat{HF}(Y)$ for most Seifert fibered manifolds Y with non-zero orbifold degrees.
(4) We study a family of alternating knots. Using an inductive argument by the oriented skein relation, we can describe their bent complexes explicitly and then the surgery formula applies routinely.
(5) Using the same technique as above, we also study the non-zero integral surgery of twisted Whitehead doubles. The results for Whitehead doubles can also tell us the framed instanton Floer homology of the splicing of two knot complements in S^3, where one knot is a twist knot, *i.e.* the Whitehead double of the unknot.
(6) We study almost L-space knots, *i.e.*, a non-L-space knot K such that there exists $n \in \mathbb{N}_+$ with $\dim I^\sharp(S^3_n(K)) = n + 2$ (see [BS22a] for the results in Heegaard Floer theory). We prove a genus one almost L-space knot is either the figure-eight or the mirror of the knot 5_2. We also show that almost L-space knots of genus at least 2 are fibered and strongly quasi-positive.

Organization. The paper is organized as follows. In Section 2, we introduce basic setup, the notations in sutured instanton homology, and deal with the scalar ambiguity mentioned in Remark 2.4. We also present some algebraic lemmas including the octahedral lemma in the derived category that are used in latter sections. In Section 3, we present the strategy to prove the integral surgery formula. We first restate the integral surgery formula using sutured instanton homology, and explain how to apply the octahedral lemma to prove it. Then we explain how to translate the integral surgery formula from the language of sutured instanton theory to the language of bent complexes, which coincides with the discussions in the introduction. All the rest of the sections are devoted to prove the three exact triangles and three commutative diagrams that are involved in the octahedral lemma, *i.e.*, Equation (3.2) to Equation (3.7). In Section 4, we study the relation between the (-1)-Dehn surgery map associated to a curve intersecting the suture twice and the two natural bypass maps associated to that curve. This helps us to prove Equation (3.2) and Equation (3.5). In Section 5, Equation (3.3), Equation (3.6) and part of Equation (3.4) are proved. The last two sections are devoted to prove Equation (3.4) and Equation (3.7), which is the most technical part of the paper. In Section 6 we prove some technical lemmas that are finally used in Section 7 to finish the proof.

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2. Basic setup

2.1. **Conventions.** If it is not mentioned, all manifolds are smooth, oriented, and connected. Homology groups and cohomology groups are with $\mathbb{Z}$ coefficients. We write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{F}_2$ for the field with two elements.

A knot $K \subset Y$ is called **null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Z})$, while it is called **rationally null-homologous** if it represents the trivial homology class in $H_1(Y;\mathbb{Q})$.

For any oriented 3-manifold M, we write $-M$ for the manifold obtained from M by reversing the orientation. For any surface S in M and any suture $\gamma \subset \partial M$, we write S and γ for the same surface and suture in $-M$, without reversing their orientations. For a knot K in a 3-manifold Y, we write $(-Y, K)$ for the induced knot in $-Y$ with induced orientation, called the **mirror knot** of K. The corresponding balanced sutured manifold is $(-Y\backslash N(K), -\gamma_K)$.

2.2. **Sutured instanton homology.** For any **balanced sutured manifold** (M,γ) [Juh06, Definition 2.2], Kronheimer-Mrowka [KM10, Section 7] constructed an isomorphism class of $\mathbb{C}$-vector spaces $SHI(M,\gamma)$. Later, Baldwin-Sivek [BS15, Section 9] dealt with the naturality issue and constructed (untwisted and twisted vesions of) projectively transitive systems related to $SHI(M,\gamma)$. We will use the twisted version, which we write as $\underline{\mathrm{SHI}}(M,\gamma)$ and call **sutured instanton homology**.

In this paper, when considering maps between sutured instanton homology, we can regard them as linear maps between actual vector spaces, at the cost that equations (or commutative diagrams) between maps only hold up to a non-zero scalar due to the projectivity. A more detailed discussion on the projectivity can be found in the next subsection.

Moreover, there is a relative $\mathbb{Z}_2$-grading on $\underline{\mathrm{SHI}}(M,\gamma)$ obtained from the construction of sutured instanton homology, which we consider as a **homological grading** and use to take Euler characteristic.

Definition 2.1. Suppose K is a knot in a closed 3-manifold Y. Let $Y(1) := Y\backslash B^3$ and let δ be a simple closed curve on $\partial Y(1) \cong S^2$. Let $Y\backslash N(K)$ be the knot complement and let Γ_μ be two oppositely oriented meridians of K on $\partial(Y\backslash N(K)) \cong T^2$. Define

$$I^\sharp(Y) := \underline{\mathrm{SHI}}(Y(1),\delta) \text{ and } \underline{\mathrm{KHI}}(Y,K) := \underline{\mathrm{SHI}}(Y\backslash N(K),\Gamma_\mu).$$

Remark 2.2. By the naturality results, we should specify the places of the removing ball, the neighborhood of the knot, and the sutures to define $I^\sharp(Y)$ and $\underline{\mathrm{KHI}}(Y,K)$. These data can be fixed by choosing a basepoint in Y or K. For simplicity, we omit those choices in the notations.

From now on, we will suppose $K \subset Y$ is a rationally null-homologous knot and fix some notations. Let μ be the meridian of K and pick a longitude λ (such that $\lambda\cdot\mu = 1$) to fix a framing of K. We will always assume $Y\backslash N(K)$ is irreducible, but many results still hold due to the following connected sum formula of sutured instanton homology [Li20, Remark 1.6]:

$$\underline{\mathrm{SHI}}(Y'\#Y\backslash N(K),\gamma) \cong I^\sharp(Y') \otimes \underline{\mathrm{SHI}}(Y\backslash N(K),\gamma).$$

Given coprime integers r and s, let $\Gamma_{r/s}$ be the suture on $\partial(Y\backslash N(K))$ consists of two oppositely oriented, simple closed curves of slope $-r/s$, with respect to the chosen framing (*i.e.* the homology of the curves are $\pm(-r\mu + s\lambda) \in H_1(\partial N(K))$). Pick S to be a minimal genus Seifert surface of K, with genus $g = g(S)$. Note that ∂S may have multiple components.

Convention. For a fixed pair (λ,μ) as above, we write $p = \partial S\cdot\mu$ and $q = \partial S\cdot\lambda$. Note that when an orientation of the knot K is chosen, the orientation of S is induced by the knot. The orientation of μ is chosen such that $p > 0$ and the orientation of λ is chosen such that $\lambda\cdot\mu = 1$. Note that p is the order of $[K] \in H_1(Y)$, *i.e.*, p is the minimal positive integer satisfying $p[K] = 0 \in H_1(Y)$. When K is null-homologous, we always choose the Seifert framing $\lambda = \partial S$. In such case, we have $(p,q) = (1,0)$.

Remark 2.3. The meanings of p and q above are different from our previous papers [LY22a, LY21]. Before, we used $\hat{\mu}$ and $\hat{\lambda}$ to denote the meridian of the knot K and the preferred framing. In particular, the framing is fixed by [LY22a, Definition 4.2]. Note that in that case, we assume that ∂S is connected, and hence it is the same as the homological longitude (with the notation λ in previous papers, while we use μ to denote the homological meridian). Also, the numbers p and q in this paper should be q and q_0 in previous papers.

For simplicity, we use the bold symbol of the suture to represent the sutured instanton homology of the balanced sutured manifold with the reversed orientation:

$$\mathbf{\Gamma}_{r/s} := \underline{\mathrm{SHI}}(-(Y\backslash N(K)),-\Gamma_{r/s}).$$

When $(r,s) = (\pm1,0)$, we write $\Gamma_{r/s} = \Gamma_\mu$. When $s = \pm1$, we write $\Gamma_n = \Gamma_{n/1} = \Gamma_{(-n)/(-1)}$. We also write $\mathbf{\Gamma}_\mu$ and $\mathbf{\Gamma}_n$ for the corresponding sutured instanton homologies.

Remark 2.4. Strictly speaking, the sutures corresponding to $(r,s) = (1,0)$ and $(-1,0)$ are not identical because the orientations are opposite. Since both sutures are on $\partial(Y\backslash N(K))$ of the same slope, they are isotopic. Moreover, we can choose a canonical isotopy by rotating the suture along the direction specified by the orientation of the knot. Due to discussion in Heegaard Floer theory [Sar15, Zem19] and the conjectural relation between Heegaard Floer theory and instanton theory

[KM10], it is expected that rotating the suture back to the original position induces a nontrivial isomorphism of the sutured instanton homology. So we pick the canonical isotopy to be the minimal rotation of the suture. Hence we can abuse notations and write Γ_μ for both sutures. The same discussion also applies to the relation between $\Gamma_{n/1}$ and $\Gamma_{(-n)/(-1)}$.

We always assume that ∂S has minimal intersections with $\Gamma_{r/s}$, i.e. $|\partial S \cap \gamma| = 2|rp - sq|$. When the intersection number $rp - sq$ is odd, then S induces a $\mathbb{Z}$-grading on $\mathbf{\Gamma}_{r/s}$. When $rp - sq$ is even, we need to perform either a positive stabilization or negative stabilization on S to induce a $\mathbb{Z}$-grading, and the two gradings are related by an overall grading shift of 1. To get rid of stabilizations, we can equivalently regard that, in this case, the surface S induces a $(\mathbb{Z} + \frac{1}{2})$-grading. We write the graded part of $\mathbf{\Gamma}_{r/s}$ as

$$(\mathbf{\Gamma}_{r/s}, i) := \underline{\mathrm{SHI}}(-(Y \backslash N(K)), -\Gamma_{r/s}, S, i)$$

with $i \in \mathbb{Z}$ or $i \in \mathbb{Z} + \frac{1}{2}$, depending on the parity of the intersection number. From the construction of the grading in [Li21b], we have the following vanishing theorem due to the adjunction inequality.

Lemma 2.5. *We have* $(\mathbf{\Gamma}_{r/s}, i) = 0$ *when*

$$|i| > \frac{|rp - sq| - \chi(S)}{2}.$$

Proof. This follows from [LY22a, Theorem 2.21 (1)] (which is ultimately based on [KM10, Proposition 7.5]) and a direct computation in the new notations. $\square$

The bypass exact triangle for sutured instanton homology was introduced by Baldwin-Sivek in [BS22c, Section 4]. In [LY22a, Section 4.2], we applied the triangle to sutures on knot complements and computed the grading shifts. We restate the results in the notation introduced before.

Lemma 2.6. *For any $n \in \mathbb{Z}$, there are two graded bypass exact triangles*

$$\begin{tikzcd}
(\mathbf{\Gamma}_n, i + \frac{p}{2}) \arrow[rr, "\psi^{n}_{+,n+1}"] & & (\mathbf{\Gamma}_{n+1}, i) \arrow[dl, "\psi^{n+1}_{+,\mu}"] \\
& (\mathbf{\Gamma}_\mu, i - \frac{np-q}{2}) \arrow[ul, "\psi^{\mu}_{+,n}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{\Gamma}_n, i - \frac{p}{2}) \arrow[rr, "\psi^{n}_{-,n+1}"] & & (\mathbf{\Gamma}_{n+1}, i) \arrow[dl, "\psi^{n+1}_{-,\mu}"] \\
& (\mathbf{\Gamma}_\mu, i + \frac{np-q}{2}) \arrow[ul, "\psi^{\mu}_{-,n}"] &
\end{tikzcd}$$

where the maps are homogeneous with respect to the homological $\mathbb{Z}_2$-gradings.

Proof. This is [LY22a, Proposition 4.14] in the new notations. The idea of the proof can be found in [LY22a, Lemma 3.18] (see also [LY22a, Remark 3.19]). Roughly, we perturb the surface S by stabilizations so that its boundary is disjoint from the bypass arc. Then the grading shifts are obtained by counting the number of positive or negative stabilizations.

Unlike the setup in [LY22a, Section 4], here K is not necessarily a dual knot of the Dehn surgery on a null-homologous knot, so we adopt the remarks in the beginning of [LY22a, Section 5]. For example, when n is large enough so that $np-q\geqslant 0$ and [LY22a, Proposition 4.14 (1)] applies, we have

$$\hat{i}^n_{\max}=\frac{np-q-\chi(S)}{2},\ \hat{i}^n_{\min}=-\frac{np-q-\chi(S)}{2},\ \hat{i}^\mu_{\max}=\frac{p-\chi(S)}{2},\ \hat{i}^\mu_{\min}=-\frac{p-\chi(S)}{2},$$

where we omit $\lceil\cdot\rceil$ since we think about $\mathbb{Z}+\frac{1}{2}$ if necessary. Then

$$\hat{i}^{n+1}_{\min}-\hat{i}^n_{\min}=-\frac{p}{2}\text{ and }\hat{i}^{n+1}_{\max}-\hat{i}^\mu_{\max}=\frac{np-q}{2}$$

An easy way to understand the grading shift was described in [LY22a, Remark 4.15]. Note that the grading shift of a map between two spaces equals half of the intersection number between ∂S and the curve corresponding to the third space up to the sign, while the sign depends on the choice of the sign in the bypass map. For example, we have $\partial S\cdot\mu=p$, so the grading shifts of $\psi^n_{\pm,n+1}$ are $\mp p/2$.

□

Remark 2.7. The reason to use balanced sutured manifolds with reversed orientation is because of the above bypass exact triangles.

Remark 2.8. If we do not mention gradings, the above triangles and the results in the rest of this subsection (except Corollary 2.9 and Lemma 2.19 since the statements involve gradings) also hold without the assumption that K is rationally null homologous since the proofs only involve the neighborhood of $\partial(-Y\backslash N(K))$.

Corollary 2.9. *For any sufficiently large integer n, we have the following properties for restrictions of maps.*

(1) The map $\psi^n_{+,n+1}|_{(\mathbf{\Gamma}_n,i)}$ is an isomorphism when

$$i<\frac{1}{2}\Big(np-q+\chi(S)\Big).$$

(2) The map $\psi^n_{-,n+1}|_{(\mathbf{\Gamma}_n,i)}$ is an isomorphism when

$$i>-\frac{1}{2}\Big(np-q+\chi(S)\Big)$$

(3) For any grading i such that

$$-\frac{1}{2}\Big(np-q+\chi(S)\Big)<i<\frac{1}{2}\Big((n-2)p-q+\chi(S)\Big),$$

there is an isomorphism

$$(\psi^n_{+,n+1})^{-1}\circ\psi^n_{-,n+1}:(\mathbf{\Gamma}_n,i)\xrightarrow{\cong}(\mathbf{\Gamma}_n,i+p).$$

(4) The map $\psi^{-n}_{-,1-n}|_{(\mathbf{\Gamma}_{-n},i)}$ is an isomorphism when

$$i<\frac{1}{2}\Big((n-2)p+q+\chi(S)\Big)$$

(5) The map $\psi^{-n}_{+,1-n}|_{(\mathbf{\Gamma}_{-n},i)}$ is an isomorphism when

$$i>-\frac{1}{2}\Big((n-2)p+q+\chi(S)\Big)$$

(6) For any grading i such that

$$-\frac{1}{2}\Big((n-2)p+q+\chi(S)\Big) < i < \frac{1}{2}\Big((n-4)p+q+\chi(S)\Big)$$

there is an isomorphism

$$(\psi^{-n}_{+,1-n})^{-1}\circ\psi^{-n}_{-,1-n} : (\mathbf{\Gamma}_{-n}, i) \xrightarrow{\cong} (\mathbf{\Gamma}_{-n}, i+p).$$

Proof. It is a combination of Lemma 2.5 and Lemma 2.6. □

Definition 2.10. The maps in Lemma 2.6 are called **bypass maps**. The ones with subscripts $+$ and $-$ are called **positive** and **negative bypass maps**, respectively. We will use $\pm$ to denote one of the bypass maps. For any integer n and any positive integer k, define

$$\Psi^n_{\pm,n+k} := \psi^{n+k-1}_{\pm,n+k}\circ\cdots\circ\psi^n_{\pm,n+1} : \mathbf{\Gamma}_n \to \mathbf{\Gamma}_{n+k}.$$

In [LY22a, Section 4.4], we proved many commutative diagrams for bypass maps, which we restate as follows by notations introduced before.

Lemma 2.11. *For any $n\in\mathbb{Z}$, we have the following commutative diagrams up to scalars.*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[r, "\psi^n_{+,n+1}"] \arrow[d, "\psi^n_{-,n+1}"'] & \mathbf{\Gamma}_{n+1} \arrow[d, "\psi^{n+1}_{-,n+2}"] \\
\mathbf{\Gamma}_{n+1} \arrow[r, "\psi^{n+1}_{+,n+2}"] & \mathbf{\Gamma}_{n+2}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{n+2} \arrow[r, "\psi^{n+2}_{+,\mu}"] \arrow[d, "\psi^{n+2}_{-,\mu}"'] & \mathbf{\Gamma}_{\mu} \arrow[d, "\psi^{\mu}_{+,n}"] \\
\mathbf{\Gamma}_{\mu} \arrow[r, "\psi^{\mu}_{-,n}"] & \mathbf{\Gamma}_{n}
\end{tikzcd}$$

Proof. The first diagram follows from [LY22a, Lemma 4.33]. Note that the proof only used the functionality of the contact gluing map and did not depend on the assumption that K is rationally null-homologous. The second diagram is obtained from the first diagram by changing the choice of the framed knot. Explicitly, let K' be the dual knot corresponding to $\mathbf{\Gamma}_{n+1}$. Let $\mu' = -(n+1)\mu+\lambda$ denote its meridian. Then $\lambda' = -\mu$ is a framing of K'. Applying the first diagram to K', we will obtain the second diagram for the original K. Note that the sign of the bypass map may switch when we regard it as the bypass map for the original knot. That is the reason for the signs in the second diagram. This can be double-checked by keeping track of the grading shifts. □

Lemma 2.12. *For any $n\in\mathbb{Z}$, we have the following commutative diagrams up to scalars*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{+,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{\Gamma}_{\mu} \arrow[ul, "\psi^{\mu}_{-,n}"] \arrow[ur, "\psi^{\mu}_{-,n+1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{\Gamma}_{\mu} \arrow[ul, "\psi^{\mu}_{+,n}"] \arrow[ur, "\psi^{\mu}_{+,n+1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{+,n+1}"] \arrow[dr, "\psi^n_{-,\mu}"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\psi^{n+1}_{-,\mu}"] \\
& \mathbf{\Gamma}_{\mu} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"] \arrow[dr, "\psi^n_{+,\mu}"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\psi^{n+1}_{+,\mu}"] \\
& \mathbf{\Gamma}_{\mu} &
\end{tikzcd}$$

There are more bypass triangles involving more complicated sutures, which are obtained from changing the choice of the framed knot as in the proof of Lemma 2.11.

Lemma 2.13. *For a knot $K \subset Y$ and $n \in \mathbb{Z}$, there are two graded bypass exact triangles*

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i + \frac{np-q}{2}) \arrow[rr, "\psi^{n-1}_{+,\frac{2n-1}{2}}"] & & (\mathbf{\Gamma}_{\frac{2n-1}{2}}, i) \arrow[dl, "\psi^{\frac{2n-1}{2}}_{+,n}"] \\
& (\mathbf{\Gamma}_n, i - \frac{(n-1)p-q}{2}) \arrow[ul, "\psi^{n}_{+,n-1}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathbf{\Gamma}_{n-1}, i - \frac{np-q}{2}) \arrow[rr, "\psi^{n-1}_{-,\frac{2n-1}{2}}"] & & (\mathbf{\Gamma}_{\frac{2n-1}{2}}, i) \arrow[dl, "\psi^{\frac{2n-1}{2}}_{-,n}"] \\
& (\mathbf{\Gamma}_n, i + \frac{(n-1)p-q}{2}) \arrow[ul, "\psi^{n}_{-,n-1}"] &
\end{tikzcd}$$

Lemma 2.14. *For a knot $K \subset Y$ and $n \in \mathbb{Z}$, there are commutative diagrams up to scalars*

$$\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[r, "\psi^{\mu}_{+,n-1}"] \arrow[d, "\psi^{\mu}_{-,n-1}"'] & \mathbf{\Gamma}_{n-1} \arrow[d, "\psi^{n-1}_{+,\frac{2n-1}{2}}"] \\
\mathbf{\Gamma}_{n-1} \arrow[r, "\psi^{n-1}_{-,\frac{2n-1}{2}}"] & \mathbf{\Gamma}_{\frac{2n-1}{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n-1}{2}} \arrow[r, "\psi^{\frac{2n-1}{2}}_{+,n}"] \arrow[d, "\psi^{\frac{2n-1}{2}}_{-,n}"'] & \mathbf{\Gamma}_{n} \arrow[d, "\psi^{n}_{-,\mu}"] \\
\mathbf{\Gamma}_{n} \arrow[r, "\psi^{n}_{+,\mu}"] & \mathbf{\Gamma}_{\mu}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr, "\psi^{\mu}_{+,n-1}"] & & \mathbf{\Gamma}_{n-1} \\
& \mathbf{\Gamma}_n \arrow[ul, "\psi^{n}_{-,\mu}"] \arrow[ur, "\psi^{n}_{+,n-1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr, "\psi^{\mu}_{-,n-1}"] & & \mathbf{\Gamma}_{n-1} \\
& \mathbf{\Gamma}_n \arrow[ul, "\psi^{n}_{+,\mu}"] \arrow[ur, "\psi^{n}_{-,n-1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_{n-1} \arrow[rr, "\psi^{\frac{2n-1}{2}}_{+,n-1}"] \arrow[dr, "\psi^{n-1}_{+,n}"'] & & \mathbf{\Gamma}_{\frac{2n-1}{2}} \arrow[dl, "\psi^{\frac{2n-1}{2}}_{-,n}"] \\
& \mathbf{\Gamma}_n &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{n-1} \arrow[rr, "\psi^{\frac{2n-1}{2}}_{-,n-1}"] \arrow[dr, "\psi^{n-1}_{-,n}"'] & & \mathbf{\Gamma}_{\frac{2n-1}{2}} \arrow[dl, "\psi^{\frac{2n-1}{2}}_{+,n}"] \\
& \mathbf{\Gamma}_n &
\end{tikzcd}$$

Remark 2.15. The choices of positive and negative bypass maps in Lemma 2.14 seem to be different from Lemma 2.11 and Lemma 2.12. But indeed they are the same up to changing the framed knot. In particular, the grading shifts match. Note that similar to the second diagram in Lemma 2.11, we always use the notations of the bypass maps for the original knot, while the signs may change if the maps are regarded as the bypass maps of the dual knot.

Suppose α is a connected non-separating simple closed curve on $\partial(Y\backslash N(K))$. We can push α into the interior of $Y\backslash N(K)$. For any fixed suture on $\partial(Y\backslash N(K))$ and a closure of the sutured manifold, the push-off of α is inside the closure, which is a closed 3-manifold. We can then take the framing on α induced by the surface $\partial(Y\backslash N(K))$ and there is an exact triangle associated to the instanton Floer homology of the (-1)- 0- and ∞-surgeries along the push-off of α. Since the push-off of α is disjoint from $\partial(Y\backslash N(K))$, the exact triangle descends to one between corresponding sutured instanton Floer homologies.

According to [BS16a, Section 4], when α intersects the suture at two points, the 0-surgery along the push-off of α (with framing induced by $\partial(Y\backslash N(K))$) corresponds to a 2-handle attachment along α. Note that attaching a 2-handle along $\alpha\subset\partial(Y\backslash N(K))$ will change the 3-manifold from $Y\backslash N(K)$ to $Y_\alpha(K)\backslash B^3$, where $Y_\alpha(K)$ is the Dehn surgery along K with slope specified by α. We write

$$\mathbf{Y}_{\mathbf{r/s}}:=I^\sharp(-Y_{-r/s}(K)),$$

and in particular

$$\mathbf{Y}_n:=I^\sharp(-Y_{-n}(K))\text{ and }\mathbf{Y}:=I^\sharp(-Y).$$

Lemma 2.16 ([LY22a, Lemma 3.21])**.** *For any $n\in\mathbb{Z}$, we have the following exact triangles.*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "H_n"] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \mathbf{Y} \arrow[ul, "G_n"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr, "A_{n-1}"] & & \mathbf{\Gamma}_{n-1} \arrow[dl, "B_{n-1}"] \\
& \mathbf{Y}_n \arrow[ul, "C_n"] &
\end{tikzcd}$$

Proof. To obtain the first exact triangle, we can take the sutured manifold $(-(Y\backslash N(K)),-\Gamma_n)$, and take a meridian $\alpha\subset\partial(Y\backslash N(K))$. As explained before the lemma, there is a surgery exact triangle associated to the sutured instanton Floer homology of the three sutured manifolds obtained by taking (-1)-, 0-, and ∞-surgeries [Sca15, Theorem 2.1]; see also [Flo90] for the original construction and [BS22c, Proof of Theorem 1.21, especially (16)-(19)] for the resolution of the subtlety of the bundle data.

The ∞-surgery will keep the manifold $(-(Y\backslash N(K)),-\Gamma_n)$. The (-1)-surgery changes the framing and hence we obtain $(-(Y\backslash N(K)),-\Gamma_{n+1})$. The 0-surgery, as discussed above, gives rise to the manifold $Y_\alpha(K)\backslash B^3$ which is $Y\backslash B^3$ since α is the meridian. Hence we obtain the expected triangle. The second exact triangle in the statement of the lemma is obtained similarly by taking α to be a curve on $\partial(Y\backslash N(K))$ having slope $-n$ instead of a meridian. $\square$

Remark 2.17. From [BS16a, Section 4], we know the 0-surgery corresponds to a 2-handle attachment and a 1-handle attachment. Hence $\mathbf{Y}$ in the above lemma is indeed $\underline{\mathrm{KHI}}(-Y,U)$, where U is the unknot in $-Y$ bounding an embedded disk. By [BS16a, Section 4], a 1-handle attachment does not change the closure of the balanced sutured manifold, and then there is a canonical identification between $\underline{\mathrm{KHI}}(-Y,U)$ and $I^\sharp(-Y)$. Hence we can abuse the notations. The same discussion also applies to $\mathbf{Y}_n$.

Furthermore, we proved the following properties in [LY22a]. Note that the assumption that K is the dual knot of a null-homologous knot in that paper is inessential by remarks in the beginning of [LY22a, Section 5]. The inequalities of the gradings are from Corollary 2.9.

Lemma 2.18 ([LY22a, Lemma 3.21 and Lemma 4.9])**.** *For any $n\in\mathbb{Z}$, we have the following commutative diagrams up to scalars*

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{+,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{Y} \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"] & & \mathbf{\Gamma}_{n+1} \\
& \mathbf{Y} \arrow[ul, "G_n"] \arrow[ur, "G_{n+1}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{+,n+1}"] \arrow[dr, "F_n"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \mathbf{Y} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_n \arrow[rr, "\psi^n_{-,n+1}"] \arrow[dr, "F_n"'] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "F_{n+1}"] \\
& \mathbf{Y} &
\end{tikzcd}$$

Lemma 2.19 ([LY22a, Lemma 4.17, Proposition 4.26, Lemma 4.29 and Proposition 4.32])**.** *Let F_n and G_n be defined as in Lemma 2.16. Then for any sufficiently large integer n, we have the following properties.*

(1) The map G_{n-1} is zero and F_n is surjective. Moreover, for any grading

$$-(np-q+\chi(S))/2 < i_0 < (np-q+\chi(S))/2 - p + 1,$$

the restriction of the map

$$F_n : \bigoplus_{i=0}^{p-1} (\mathbf{\Gamma}_n, i_0 + i) \to \mathbf{Y}$$

is an isomorphism.

(2) The map F_{-n+1} is zero and G_{-n} is injective. Moreover, for any grading

$$-((n-2)p+q+\chi(S))/2 < i_0 < ((n-2)p+q+\chi(S))/2 - p + 1,$$

the map

$$\mathrm{Proj} \circ G_{-n} : \mathbf{Y} \to \bigoplus_{i=0}^{p-1} (\mathbf{\Gamma}_{-n}, i_0 + i),$$

is an isomorphism, where

$$\mathrm{Proj} : \mathbf{\Gamma}_{-n} \to \bigoplus_{i=0}^{p-1} (\mathbf{\Gamma}_{-n}, i_0 + i)$$

is the projection.

The following lemma is a special case of Proposition 4.1, which we will prove later.

Lemma 2.20. *For any $n \in \mathbb{Z}$, let the maps H_n and $\psi^n_{\pm,n+1}$ be defined as in Lemma 2.16 and Lemma 2.6 respectively. Then there exist $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ such that*

$$H_n = c_1 \psi^n_{+,n+1} + c_2 \psi^n_{-,n+1}$$

2.3. **Fixing the scalars.** By construction, sutured instanton homology forms a projectively transitive system, which means all the spaces and maps between spaces are well-defined only up to non-zero scalars. When the balanced sutured manifold is obtained from a framed knot as in the last subsection, we can make some canonical choices to reduce the projective ambiguities.

Suppose $K \subset Y$ is a framed knot with the meridian μ and the framing λ. Fix a knot complement $Y\backslash N(K)$ and the suture Γ_μ. We fix a special choice of a (marked odd) closure of $(Y\backslash N(K), \Gamma_\mu)$ following the construction in [KM10, Formula (18)].

Let F be a closed, oriented, connected surface of genus at least 2. Suppose c_0 is a non-separating curve in F. Let $c = \mathrm{pt}\times c_0 \subset S^1\times F$ and let (μ_c,λ_c) be the meridian and the longitude of c (the latter comes from the surface framing). Let

$$(\bar{Y}_\mu, R) = (S^1\times F\backslash N(c)\cup_{(\mu_c,\lambda_c)\sim(\lambda,\mu)} Y\backslash N(K), \mathrm{pt}'\times F), \tag{2.1}$$

where pt' is a point different from pt. We can pick $\alpha = S^1\times \mathrm{pt}''$ and $\eta\subset R$ be a curve intersecting $\mathrm{pt}'\times c_0$ once. Since $\alpha\cdot R = 1$ and $\eta\cdot R = 0$, the pair $(\bar{Y}_\mu, \alpha\cup\eta)$ defines an instanton Floer homology in the setting of [KM10, Section 7.1]. Moreover, $\mathcal{D}_\mu = (Y, R, \eta, \alpha)$ forms a marked odd closure in [BS15, Definition 9.2], which was used in the naturality result [BS15, Theorem 9.17]. The reason for $g(F)\geqslant 2$ is to apply the naturality result (*c.f.* [BS15, Remark 9.4]).

Similarly, for $(Y\backslash N(K),\Gamma_n)$ and $(Y\backslash N(K),\Gamma_{\frac{2n-1}{2}})$, we fix closures $\mathcal{D}_n$ and $\mathcal{D}_{\frac{2n-1}{2}}$ as in (2.1), except replacing the gluing map $(\mu_c,\lambda_c)\sim(\lambda,\mu)$ by $(\mu_c,\lambda_c)\sim(-\mu,-n\mu+\lambda)$ and $(\mu_c,\lambda_c)\sim(-n\mu+\lambda,(1-2n)\mu+2\lambda)$, respectively.

For the sutured manifold $(Y\backslash B^3,\delta)$, we regard it as $(Y\backslash N(U),\Gamma_{\mu,U})$ by Remark 2.17, where U is the unknot and $\Gamma_{\mu,U}$ is meridian suture on the unknot complement. Then we apply the above construction to obtain a special closure of $(Y\backslash B^3,\delta)$. We reverse the orientations of the chosen closures when the orientations of the sutured manifolds are reversed. Note that we do not choose canonical closures for $(Y_n(K)\backslash B^3,\delta)$ since we only care about the dimension of its framed instanton homology.

After fixing the choices of closures, we can view $\mathbf{\Gamma}_n$ and $\mathbf{Y}$ as actual vector spaces, and then the elements in them are well-defined. Strictly speaking, we also need to choose extra data such as the metric and the perturbation on the closure to define the instanton Floer homology of the closure, but different choices of metrics and perturbations now induce a transitive system of vector spaces, from which we can construct an actual vector space. So, we omit the discussion on those extra data.

The construction of bypass maps and surgery maps may not be realized as cobordism maps between the chosen closures, but the construction of the projectively transitive system (*c.f.* [BS15, Definition 9.18]) guarantees the existence of such maps up to scalars. Now we make (non-canonical) choices of the maps to get rid of the scalar ambiguities in the commutative diagrams mentioned in the last subsection.

We first assume that $I^\sharp(Y)\neq 0$. When $I^\sharp(Y)=0$, the first exact triangle in Lemma 2.16 is trivial for any n and hence the maps F_n and G_n that play an important role in later sections are both zero. This makes fixing the scalars a somewhat straightforward job: we just need to fix the scalars for the bypass maps. Also, it is worth mentioning that, the Euler characteristic result in [Sca15, Corollary 1.4] implies that $I^\sharp(Y)\neq 0$ for any rational homology sphere Y and, up to author's knowledge, there is no known closed oriented 3-manifold Y with $I^\sharp(Y)=0$.

To help us fixing the scalars, suppose the maps F_n and G_n are defined as in the proof of Lemma 2.16, and we define

$$n_G = \min\{n\in\mathbb{Z}\mid G_n = 0\} \text{ and } n_F = \max\{n\in\mathbb{Z}\mid F_n = 0\}. \tag{2.2}$$

We have the following basic properties for these indices.

Lemma 2.21. *Assuming $I^\sharp(Y)\neq 0$. Suppose n_G and n_F are defined as in Equation (2.2). Then we have*

$$-\infty < n_F \leqslant n_G < \infty.$$

Moreover, we have $G_n = 0$ if and only if $n\geqslant n_G$ and $F_n = 0$ if and only if $n\leqslant n_F$.

Proof. The fact that $-\infty < n_F < \infty$ and $-\infty < n_G < \infty$ follow from Lemma 2.19 and the fact that they fits into an exact triangle as in Lemma 2.16. Next, the commutative diagrams in Lemma

2.18 implies that $G_{n+1}=0$ whenever $G_n=0$ and thus we know $G_n=0$ if and only if $n\geqslant n_G$. The argument for F_n is similar. Finally, by definition we know $G_{n_G}=0$ and hence from the exact triangle we know that

$$\operatorname{Im}F_{n+1}=\ker G_{n_G}=I^\sharp(Y)\neq 0.$$

Hence we conclude that $n_F\leqslant n_G$. □

By Lemma 2.19, we can pick a sufficiently large integer n_0 such that $-n_0<n_F\leqslant n_G$. Pick arbitrary representatives of the maps

$$G_{-n_0},\psi_{+,\mu}^{-n_0},\psi_{-,\mu}^{-n_0},\psi_{+,-n_0}^{\mu},\psi_{-,-n_0}^{\mu}$$

and we also pick arbitrary representatives of the maps

$$\psi_{+,n+1}^{n},\psi_{+,\frac{2n-1}{2}}^{n-1}$$

for all $n\in\mathbb{Z}$.

Now we explain how to fix the scalars for maps with $n\geqslant n_0$. Note that we have already chosen a representative for $\psi_{+,-n_0+1}^{-n_0}$ and G_{-n_0}. From Lemma 2.18, we have

$$G_{-n_0+1}\doteq\psi_{+,-n_0+1}^{-n_0}\circ G_{-n_0}$$

where $\doteq$ means commutative up to a non-zero scalar. We can choose a representative of G_{-n_0+1} to obtain an equality

$$G_{-n_0+1}=\psi_{+,-n_0+1}^{-n_0}\circ G_{-n_0}$$

We then choose a representative of $\psi_{-,-n_0+1}^{-n_0}$ so that

$$G_{-n_0+1}=\psi_{-,-n_0+1}^{-n_0}\circ G_{-n_0}.$$

Next, we pick representatives of the maps $\psi_{-,n+1}^{n}$ inductively, with the base case $n=-n_0$ constructed above, so that

$$\psi_{-,n+1}^{n}\circ\psi_{+,n}^{n-1}=\psi_{+,n+1}^{n}\circ\psi_{-,n}^{n-1}$$

hold for all $n\geqslant -n_0+1$. If the compositions happen to be zero, we could pick an arbitrary representative since the diagram will be trivially satisfied. We will discuss the ambiguity arising from the possibility that $\psi_{+,n+1}^{n}\circ\psi_{-,n}^{n-1}=0$ more carefully later.

Similarly, we pick the maps $\psi_{+,\mu}^{n}$ inductively to satisfy the commutative diagram

$$\psi_{+,\mu}^{n}\circ\psi_{-,n}^{n-1}=\psi_{+,\mu}^{n-1}.$$

We can choose representatives of $\psi_{-,\mu}^{n}$, $\psi_{+,n}^{\mu}$, and $\psi_{-,n}^{\mu}$ in a similar manner.

Furthermore, the representatives of

$$\psi_{-,n}^{\frac{2n-1}{2}},\psi_{+,n-1}^{n},\psi_{-,n-1}^{n},\psi_{-,\frac{2n-1}{2}}^{n-1},\psi_{+,n}^{\frac{2n-1}{2}}$$

can be chosen according to Lemma 2.14. As mentioned in Remark 2.15, we always use the notations of the bypass maps for the original knot even though we consider some dual knots in the proofs. Hence here we first fix the knot $K\subset Y$ and then fix the representatives, while we do not fix the representatives by any commutative diagrams for the dual knot in the proofs.

Next, we deal with maps G_n and F_n in Lemma 2.16. We choose representatives of the maps G_n inductively so that

$$G_{n+1}=\psi_{+,n+1}^{n}\circ G_n$$

is satisfied for all $n \geqslant -n_0$. We pick arbitrary representatives of the map F_{n_F+1} and then pick F_n inductively so that

$$F_{n+1} \circ \psi^n_{+,n+1} = F_n$$

is satisfied for all $n \geqslant n_F + 1$. We can then use induction to prove the following two equalities.

- $G_{n+1} = \psi^n_{-,n+1} \circ G_n$ for all $n \geqslant -n_0$, and
- $F_{n+1} \circ \psi^n_{-,n+1} = c \cdot F_n$ for a non-zero scalar c that is independent of n with $n \geqslant n_F + 1$.

We verify the equality for G first. The base case $n = n_0$ is by construction. Assuming we have already established the equality for n, from Lemma 2.11 and Lemma 2.18, we have

$$\begin{aligned} G_{n+2} &= \psi^{n+1}_{+,n+2} \circ G_{n+1} \\ \text{(Inductive hypothesis)} &= \psi^{n+1}_{+,n+2} \circ \psi^n_{-,n+1} \circ G_n \\ \text{(Lemma 2.11)} &= \psi^{n+1}_{-,n+2} \circ \psi^n_{+,n+1} \circ G_n \\ &= \psi^{n+1}_{-,n+2} \circ G_{n+1}. \end{aligned}$$

The argument for F_n is similar, once we take $c \neq 0$ to be the complex number such that

$$F_{n_F+2} \circ \psi^{n_F+1}_{-,n_F+2} = c \cdot F_{n_F+1}. \tag{2.3}$$

The remaining issues are summarized as follows:

(i) When choosing representatives of $\psi^n_{-,n+1}$, we use the commutative diagram

$$\psi^n_{-,n+1} \circ \psi^{n-1}_{+,n} \doteq \psi^n_{+,n+1} \circ \psi^{n-1}_{-,n}$$

However, when $\psi^n_{+,n+1} \circ \psi^{n-1}_{-,n} = 0$, there is no unique choice of $\psi^n_{-,n+1}$ and one might worry that different choices of $\psi^n_{-,n+1}$ may affect the commutative diagrams in Lemma 2.18.

(ii) By Proposition 4.1, we can assume that the map H_n in the exact triangle in Lemma 2.16 to have the form

$$H_n = \psi^n_{+,n+1} - c_n \cdot \psi^n_{-,n+1}.$$

We want to pin down the values of c_n.

(iii) We want to get rid of the scalar c in Equation 2.3.

We treat these issues in several different cases.

Case 1. $n_F + 3 \leqslant n_G$. In this case, we know that

$$\psi^n_{+,n+1} \circ \psi^{n-1}_{-,n} \neq 0$$

for any integer n. Indeed, if $n + 1 < n_G$, we know from Lemma 2.18 that

$$\psi^n_{+,n+1} \circ \psi^{n-1}_{-,n} \circ G_{n-1} = G_{n+1} \neq 0.$$

If $n + 1 \geqslant n_G \geqslant n_F + 3$, instead of the above equation involving G, we have

$$F_{n+1} \circ \psi^n_{+,n+1} \circ \psi^{n-1}_{-,n} = c \cdot F_{n-1} \neq 0.$$

This implies that inductively we can fix a unique representative of $\psi^n_{-,n+1}$ for all $n \geqslant -n_0 + 1$.

For any integer n with $-n_0 \leqslant n < n_G - 1$, we have $G_{n+1} \neq 0$. Then we take an element $\alpha \in \mathbf{Y}$ so that $G_{n+1}(\alpha) \neq 0$. Then we can solve the scalar c_n as follows.

$$\begin{aligned} 0 &= H_n \circ G_n(\alpha) \\ &= (\psi^n_{+,n+1} - c_n \cdot \psi^n_{-,n+1}) \circ G_n(\alpha) \\ &= \psi^n_{+,n+1} \circ G_n(\alpha) - c_n \cdot \psi^n_{-,n+1} \circ G_n(\alpha) \\ &= (1 - c_n) \cdot G_{n+1}(\alpha) \end{aligned}$$

Hence we conclude that $c_n = 1$. In particular, we can take $n = n_F + 1 < n_G - 1$. Note $F_n \neq 0$ so we can take $x \in \mathbf{\Gamma}_n$ so that $F_n(x) \neq 0$. Then we have

$$\begin{aligned} 0 &= F_{n+1} \circ H_n(x) \\ &= F_{n+1} \circ (\psi^n_{+,n+1} - \psi^n_{-,n+1})(x) \\ &= F_{n+1} \circ \psi^n_{+,n+1}(x) - F_{n+1} \circ \psi^n_{-,n+1}(x) \\ &= (1-c) \cdot F_n(x) \end{aligned}$$

This implies that $c = 1$ as well. Now for any $n \geqslant n_G - 1 \geqslant n_F + 2$, we can take $x \in \mathbf{\Gamma}_n$ such that $F_n(x) \neq 0$. we have

$$\begin{aligned} 0 &= F_{n+1} \circ H_n(x) \\ &= F_{n+1} \circ (\psi^n_{+,n+1} - c_n \cdot \psi^n_{-,n+1})(x) \\ &= (1-c_n) \cdot F_n(x) \end{aligned}$$

Hence we conclude that $c_n = 1$. In summary, in Case 1, we have the following.

- We can fix a unique representative of $\psi^n_{-,n+1}$ for any $n \geqslant -n_0$.
- We have $c_n = 1$ for any $n \geqslant -n_0$.
- We have $c = 1$.

Case 2. $n_G = n_F + 2$. Note that some arguments in Case 1 still apply. We summarize as follows.

- For $n < n_G - 1$, we have $G_{n+1} \neq 0$ so there is a unique choice of $\psi^n_{-,n+1}$.
- For $n \geqslant n_G = n_F + 2$, we have $F_{n-1} \neq 0$ so again there is a unique choice of $\psi^n_{-,n+1}$.
- For $n < n_G - 1$, we have $G_{n+1} \neq 0$ so $c_n = 1$ in the expression of H_n.
- For $n \geqslant n_G - 1 = n_F + 1$, we have $F_n \neq 0$ so $c_n = c^{-1}$.

To resolve the issue (i), the only nonfixed index is $n = n_F + 1 = n_G - 1$. In case

$$\psi^n_{+,n+1} \circ \psi^{n-1}_{-,n} \neq 0,$$

there is a unique choice of $\psi^n_{-,n+1}$ so that

$$\psi^n_{-,n+1} \circ \psi^{n-1}_{+,n} = \psi^n_{+,n+1} \circ \psi^{n-1}_{-,n}.$$

Otherwise, we just fix any representative of $\psi^n_{-,n+1}$.

To resolve issues (ii) and (iii), we rescale the maps F_n according to the grading on $\mathbf{\Gamma}_n$. To do this, for an integer n and a grading i, define

$$j(n,i) = \lfloor \frac{i}{p} - \frac{n}{2} \rfloor.$$

It is straightforward that

$$j(n+1, i+\frac{p}{2}) = j(n,i) \text{ and } j(n+1, i-\frac{p}{2}) = j(n,i) - 1.$$

Hence we define

$$\widetilde{F}_n = \sum_i c^{j(n,i)} \cdot F_n \circ \mathrm{Proj}^i_n, \tag{2.4}$$

where the map

$$\mathrm{Proj}^i_n : \mathbf{\Gamma}_n \to (\mathbf{\Gamma}_n, i)$$

is the projection. Equivalently, if we have an element $x \in (\mathbf{\Gamma}_n, i)$, we take

$$\widetilde{F}_n(x) = c^{j(n,i)} \cdot F_n(x).$$

We then need to verify the following two equalities for $\widetilde{F}$:

- $\widetilde{F}_{n+1} \circ \psi^n_{+,n+1} = \widetilde{F}_n$, and
- $\widetilde{F}_{n+1} \circ \psi^n_{-,n+1} = \widetilde{F}_n$.

For the first one, note that $\psi^n_{+,n+1}$ increases the grading by $p/2$ from Lemma 2.6. So assume $x \in (\mathbf{\Gamma}_n, i)$, we have

$$\begin{aligned}\widetilde{F}_{n+1} \circ \psi^n_{+,n+1}(x) &= c^{j(n+1,i+\frac{p}{2})} \cdot F_{n+1}(x) \circ \psi^n_{+,n+1} \\ &= c^{j(n,i)} \cdot F_n(x) \\ &= \widetilde{F}_n(x)\end{aligned}$$

On the other hand, the map $\psi^n_{-,n+1}$ decreases the grading by $p/2$ from Lemma 2.6, and so for $x \in (\mathbf{\Gamma}_n, i)$,

$$\begin{aligned}\widetilde{F}_{n+1} \circ \psi^n_{-,n+1}(x) &= c^{j(n+1,i-\frac{p}{2})} \cdot F_{n+1}(x) \circ \psi^n_{-,n+1} \\ &= c^{j(n,i)-1} \cdot c \cdot F_n(x) \\ &= \widetilde{F}_n(x)\end{aligned}$$

Hence we can use $\widetilde{F}_n$ instead of F_n to get rid of the scalar c. However, changing F_n to $\widetilde{F}_n$, we might break the exact triangle in Lemma 2.16. Hence we need to alter the definition of H_n as well. We define

$$\widetilde{H}_n = \psi^n_{+,n+1} - \psi^n_{-,n+1} \tag{2.5}$$

and it remains to establish the following exact triangle:

$$\begin{tikzcd} \mathbf{\Gamma}_n \arrow[rr, "\widetilde{H}_n"] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\widetilde{F}_{n+1}"] \\ & \mathbf{Y} \arrow[ul, "G_n"] & \end{tikzcd}$$

When $n < n_G - 2 = n_F$, the triangle automatically holds. Indeed, for such n, we have $c_n = 1$ so $\widetilde{H}_n = H_n$; the fact $F_{n+1} = 0$ implies that $\widetilde{F}_{n+1} = 0$, so the new exact triangle is exactly the original one in Lemma 2.16.

When $n = n_F$, we still have $c_n = 1$ and hence $\widetilde{H}_n = H_n$. So the exactness at $\mathbf{\Gamma}_n$ is from Lemma 2.16. By construction we have $\operatorname{Im} \widetilde{F}_{n+1} = \operatorname{Im} F_{n+1}$ for any n so we conclude the exactness at $\mathbf{Y}$. This also implies that

$$\dim \ker \widetilde{F}_{n+1} = \dim \ker F_{n+1} = \dim \operatorname{Im} H_n.$$

Hence to show the exactness at $\mathbf{\Gamma}_{n+1}$, it remains to show that

$$\operatorname{Im} H_n \subset \ker \widetilde{F}_{n+1}.$$

For any $x \in \mathbf{\Gamma}_n$, we have

$$\begin{aligned}\widetilde{F}_{n+1} \circ H_n(x) &= \\ &= \widetilde{F}_{n+1} \circ (\psi^n_{+,n+1} - \psi^n_{-,n+1})(x) \\ &= \widetilde{F}_{n+1} \circ \psi^n_{+,n+1}(x) - \widetilde{F}_{n+1} \circ \psi^n_{-,n+1}(x) \\ &= \widetilde{F}_n(x) - \widetilde{F}_n(x) \\ &= 0.\end{aligned}$$

Hence we are done.

Finally, we verify the exact triangle for $n \geqslant n_F + 1$. Define a homomorphism

$$\iota_n : \mathbf{\Gamma}_n \to \mathbf{\Gamma}_n$$

as

$$\iota_n(x) = \sum_i c^{-j(n,i)} \mathrm{Proj}_n^i(x).$$

It is clear that ι is an isomorphism as its inverse is

$$\iota_n^{-1}(x) = \sum_i c^{j(n,i)} \mathrm{Proj}_n^i(x),$$

and from the construction of $\widetilde{F}$ in Equation (2.4), we have

$$\iota_{n+1}(\ker \widetilde{F}_{n+1}) = \ker F_{n+1}.$$

From the fact that $H_n = \psi^n_{+,n+1} - c^{-1} \cdot \psi^n_{-,n+1}$ (we have $c_n = c^{-1}$), $\widetilde{H}_n = \psi^n_{+,n+1} - \psi^n_{-,n+1}$, and that $\psi^n_{\pm,n+1}$ shift the grading by $\mp p/2$, we conclude that

$$\iota_{n+1}(\mathrm{Im}\, \widetilde{H}_n) = \mathrm{Im}\, H_n.$$

As a result, we conclude the exactness at Γ_{n+1}. The exactness at $\mathbf{Y}$ holds as above, since we still have

$$\mathrm{Im}\, \widetilde{F}_{n+1} = \mathrm{Im}\, F_{n+1} = \ker G_n.$$

Dimension counting similar to the above argument then implies that

$$\dim \ker \widetilde{H}_n = \dim \mathrm{Im}\, G_n.$$

We then verify that

$$\mathrm{Im}\, G_n \subset \ker \widetilde{H}_n.$$

For any $\alpha \in \mathbf{Y}$, we have

$$\begin{aligned} \widetilde{H}_n \circ G_n(\alpha) &= (\psi^n_{+,n+1} - \psi^n_{-,n+1}) \circ G_n(\alpha) \\ &= \psi^n_{+,n+1} \circ G_n(\alpha) - \psi^n_{-,n+1} \circ G_n(\alpha) \\ &= G_{n+1}(\alpha) - G_{n+1}(\alpha) \\ &= 0. \end{aligned}$$

In summary, in Case 2, we did the following (extra things):

- We choose representatives of $\psi^n_{-,n+1}$ for all $n \geqslant -n_0 + 1$ so that

$$\psi^n_{-,n+1} \circ \psi^{n-1}_{+,n} = \psi^n_{+,n+1} \circ \psi^{n-1}_{-,n}.$$

- We define new maps $\widetilde{F}_n$ for all n so that

$$\widetilde{F}_n = \widetilde{F}_{n+1} \circ \psi^n_{\pm,n+1}. \tag{2.6}$$

- We define new maps $\widetilde{H}_n$ for all n so that we have the following exact triangle for all n.

$$\begin{tikzcd} \mathbf{\Gamma}_n \arrow[rr, "\widetilde{H}_n"] & & \mathbf{\Gamma}_{n+1} \arrow[dl, "\widetilde{F}_{n+1}"] \\ & \mathbf{Y} \arrow[ul, "G_n"] & \end{tikzcd} \tag{2.7}$$

Case 3. $n_G = n_F + 1$ or $n_G = n_F$. The situation and argument are similar to those in Case 2. We summarize the differences here:

- In Case 3, the composition $\psi^n_{+,n+1} \circ \psi^{n-1}_{-,n}$ could only be zero when $n_G - 1 \leqslant n \leqslant n_F + 1$. In case the composition is indeed 0, we choose an arbitrary representative of the map $\psi^n_{-,n+1}$.
- We still define the maps $\widetilde{F}_n$ as in Equation (2.4) and the maps $\widetilde{H}_n$ as in Equation (2.5), and can verify the Equation (2.6) and the exact triangle (2.7) as in Case 2.

Note from Lemma 2.21, we must have $n_G \geqslant n_F$, so the above three cases cover all situations.

Finally, we could extend the choice of representatives for all relevant maps for the indices $n < -n_0$. Note that when $n < -n_0$, we have that G_n is injective and $F_n = 0$. Hence we do not need to worry about the issues (i), (ii), and (iii).

Convention. From now on, we write the maps $\widetilde{H}_n$ and $\widetilde{F}_n$ simply as H_n and F_n, respectively. From the above discussion, when $K \subset Y$ is a fixed rationally null-homologous knot, we can assume the first commutative diagram in Lemma 2.11 and all commutative diagrams in Lemma 2.12, Lemma 2.14 and Lemma 2.18 hold without introducing scalars.

2.4. **Algebraic lemmas.** In this subsection, we introduce some lemmas in homological algebra. All graded vector spaces in this subsection are finite dimensional and over $\mathbb{C}$ and all maps are complex linear maps. For convenience, we will switch freely between long exact sequences and exact triangles.

From Section 2.2, we know the sutured instanton homology is usually $\mathbb{Z} \oplus \mathbb{Z}_2$-graded, where we regard the $\mathbb{Z}_2$-grading as a homological grading. Many results in this subsection come from properties of the derived category of vector spaces over $\mathbb{C}$, for which people usually consider cochain complexes. However, for a $\mathbb{Z}_2$-graded space there is no difference between the chain complex and the cochain complex. Hence by saying a **complex** we mean a $\mathbb{Z}_2$-graded (co)chain complex, though all results apply to $\mathbb{Z}$-graded cochain complexes verbatim.

For a complex C and an integer n, we write C^n for its grading n part (under the natural map $\mathbb{Z} \to \mathbb{Z}_2$). With this notation, we suppose the differential d_C on C sends C^n to C^{n+1}. For any integer k, we write $C\{k\}$ for the complex obtained from C by the grading shift $C\{k\}^n = C^{n+k}$. We write $H(C, d_C)$ or $H(C)$ for the homology of a complex C with differential d_C. A $\mathbb{Z}_2$-graded vector space is regarded as a complex with the trivial differential.

For a chain map $f : C \to D$, we write $\mathrm{Cone}(f)$ for the **mapping cone** of f, *i.e.*, the complex consisting of the space $D \oplus C\{1\}$ and the differential

$$d_{\mathrm{Cone}(f)} := \begin{bmatrix} d_D & -f \\ 0 & -d_C \end{bmatrix}.$$

Then there is a long exact sequence

$$\cdots \to H(C) \xrightarrow{f} H(D) \xrightarrow{i} H(\mathrm{Cone}(f)) \xrightarrow{p} H(C)\{1\} \to \cdots$$

where i sends $x \in D$ to $(x, 0)$ and p sends $(x, y) \in D \oplus C\{1\}$ to $-y$. If the differentials of C and D are trivial, then we know

$$H(\mathrm{Cone}(f)) \cong \ker(f) \oplus \mathrm{coker}(f). \tag{2.8}$$

Remark 2.22. Our definitions about mapping cones follow from [Wei94], which are different from those in [OS08, OS11].

Note that the derived category is a triangulated category, so it satisfies the octahedral lemma (for example, see [Wei94, Proposition 10.2.4]).

Lemma 2.23 (octahedral lemma). *Suppose X, Y, Z, X', Y', Z' are $\mathbb{Z}_2$-graded vector spaces satisfying the following long exact sequences*

$$\cdots \to X \xrightarrow{f} Y \xrightarrow{h} Z' \to X\{1\} \to \cdots$$

$$\cdots \to X \xrightarrow{g\circ f} Z \xrightarrow{h'} Y' \xrightarrow{l'} X\{1\} \to \cdots$$

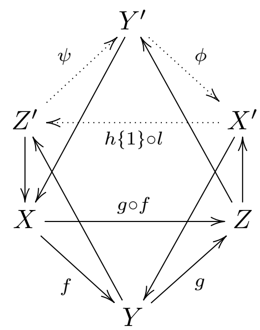
$$\cdots \to Y \xrightarrow{g} Z \to X' \xrightarrow{l} Y\{1\} \to \cdots$$

Then we have the fourth long exact sequence

$$\cdots \to Z' \xrightarrow{\psi} Y' \xrightarrow{\phi} X' \xrightarrow{h\{1\}\circ l} Z'\{1\} \to \cdots$$

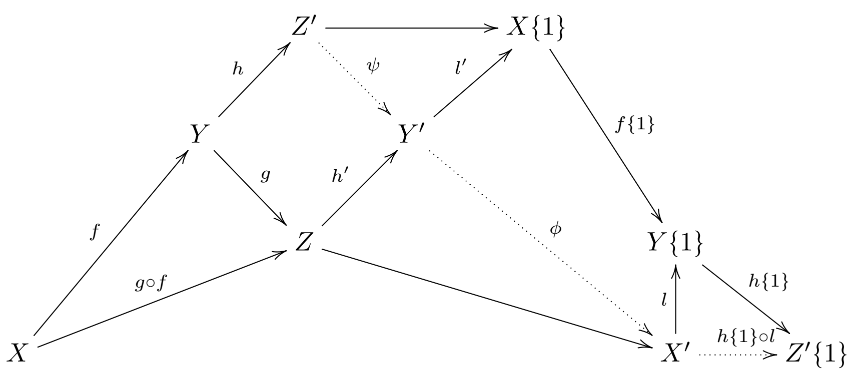
such that the following diagram commutes

(2.9)



where we omit the grading shifts and the notation for the maps h, l, j. We can also write (2.9) in another form so that there is enough room to write the maps

(2.10)



The maps ψ and ϕ in (2.10) can be written explicitly as follows. By the long exact sequences in the assumption of Lemma 2.23, we know that Z', X', Y' are chain homotopic to the mapping cones $\mathrm{Cone}(f), \mathrm{Cone}(g), \mathrm{Cone}(g \circ f)$, respectively. Under such homotopies, we can write

$$\psi : Y \oplus X\{1\} \to Z \oplus X\{1\}$$
$$\psi(y, x) \mapsto (g(y), x)$$

and

$$\phi : Z \oplus X\{1\} \to Z \oplus Y\{1\}$$
$$\phi(z, x) \mapsto (z, f\{1\}(x))$$

However, the chain homotopies are not canonical, and hence the maps ψ and ϕ are also not canonical. Thus, usually we cannot identify them with other given maps. Fortunately, with an extra $\mathbb{Z}$-grading, we may identify $H(\mathrm{Cone}(\phi))$ with $H(\mathrm{Cone}(\phi'))$ for another map $\phi' : Y' \to X'$.

First, we introduce the following lemma to deal with the projectivity of maps (*i.e.* maps well-defined only up to scalars). Note that the $\mathbb{Z}$-grading in the following lemma is not the homological grading used before.

Lemma 2.24. *Suppose X and Y are $\mathbb{Z}$-graded vector spaces and suppose $f, g : X \to Y$ are homogeneous maps with different grading shifts k_1 and k_2. Then* $\mathrm{Cone}(f+g)$ *is isomorphic to* $\mathrm{Cone}(c_1 f + c_2 g)$ *for any $c_1, c_2 \in \mathbb{C}\backslash\{0\}$.*

Proof. For simplicity, we can suppose $k_1 = 0$ and $k_2 = 1$. The proof for the general case is similar. For $i, j \in \mathbb{Z}$, we write (X, i) and (Y, j) for grading summands of X and Y, respectively. Suppose T is an automorphism of $X \oplus Y$ that acts by

$$\frac{c_2^i}{c_1^{i+1}}\mathrm{Id} \text{ on } (X,i) \text{ and } \frac{c_2^j}{c_1^j}\mathrm{Id} \text{ on } (Y,j).$$

Then T is an isomorphism between $\mathrm{Cone}(f+g)$ and $\mathrm{Cone}(c_1 f + c_2 g)$. □

Then we state the lemma that relates the map ϕ in Lemma 2.23 to another map ϕ' constructed explicitly.

Lemma 2.25. *Suppose X, Y, Z, X', Y' are $\mathbb{Z} \oplus \mathbb{Z}_2$-graded vector spaces satisfying the following horizontal exact sequences.*

$$\begin{tikzcd}
Z \arrow[r, "h'"] \arrow[d, "="] & Y' \arrow[r, "l'"] \arrow[d, shift right, "\phi"'] \arrow[d, shift left, "\phi'=a'+b'"] & X\{1\} \arrow[d, "f\{1\}=a+b"] \\
Z \arrow[r, "\phi\circ h'=\phi'\circ h'"] & X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}$$

where the shift $\{1\}$ is for the $\mathbb{Z}_2$-grading. Suppose $\phi : Y' \to X'$ satisfies the two commutative diagrams and suppose $\phi' : Y' \to X'$ satisfies the left commutative diagram. Suppose l and l' are homogeneous with respect to the $\mathbb{Z}$-grading. Suppose $f\{1\} = a + b$ and $\phi' = a' + b'$ are sums of homogeneous maps with different grading shifts with respect to the $\mathbb{Z}$-grading. Moreover, suppose the following diagrams hold up to scalars.

$$\begin{tikzcd}
Y' \arrow[r, "l'"] \arrow[d, "a'"] & X\{1\} \arrow[d, "a"] \\
X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
Y' \arrow[r, "l'"] \arrow[d, "b'"] & X\{1\} \arrow[d, "b"] \\
X' \arrow[r, "l"] & Y\{1\}
\end{tikzcd}$$

Then there is an isomorphism between $H(\mathrm{Cone}(\phi))$ and $H(\mathrm{Cone}(\phi'))$.

Proof. Since ϕ and ϕ' share the same domain and codomain, it suffices to show that they have the same rank. Fix inner products on Y' and X' such that we have orthogonal decompositions

$$Y' = \mathrm{Im}(h') \oplus Y'' \text{ and } X' = \mathrm{Im}(\phi \circ h') \oplus X''.$$

By commutativity, we know both ϕ and ϕ' send $\mathrm{Im}(h')$ onto $\mathrm{Im}(\phi\circ h')$. Hence if we choose bases with respect to the decompositions so that linear maps are represented by matrices (we use row vectors), then we have

$$\phi=\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}\text{ and }\phi'=\begin{bmatrix}A' & B'\\ 0 & C'\end{bmatrix},$$

where $A=A':\mathrm{Im}(h')\to\mathrm{Im}(\phi\circ h')$ has full row rank. Then it suffices to show C and C' have the same row rank.

By the exactness at Y' and X', we know the restriction of l' on Y'' is an isomorphism between Y'' and $\mathrm{Im}(l')$ and the restriction of l on X'' is an isomorphism between X'' and $\mathrm{Im}(l)$. By commutativity, we know that both a and b send $\mathrm{Im}(l')$ to $\mathrm{Im}(l)$ and

$$\mathrm{rowrank}(C)=\mathrm{rank}(f\{1\}|\mathrm{Im}(l'))\text{ and }\mathrm{rowrank}(C')=\mathrm{rank}((c_1a+c_2b)|\mathrm{Im}(l'))$$

for some $c_1,c_2\in\mathbb{C}\backslash\{0\}$. Since l and l' are homogeneous, there exist induced $\mathbb{Z}$-gradings on $\mathrm{Im}(l)$ and $\mathrm{Im}(l')$. The maps a and b are still homogeneous with different grading shifts with respect to these induced gradings. Then we can apply Lemma 2.24 to obtain that the ranks of the restrictions of $f\{1\}=a+b$ and c_1a+c_2b on $\mathrm{Im}(l')$ are the same. □

3. Integral surgery formulae

3.1. **A formula for framed instanton homology.** In this subsection, we propose an integral surgery formula based on sutured instanton homology and package it into the language of bent complexes in a later subsection.

Suppose $K\subset Y$ is a framed rationally null-homologous knot, and we adopt the notations introduced in Section 2.2. Define

$$\pi^{\pm}_{m,k}:=\Psi^{m+k}_{\pm,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{\mp,m+k}:\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}\to\mathbf{\Gamma}_{m+2k-1}$$

and write $\pi^{\pm,i}_{m,k}$ as the restriction of $\pi^{\pm}_{m,k}$ on $(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}},i)$. From Lemma 2.13 and Lemma 2.6, we can verify that $\pi^{\pm}_{m,k}$ shifts grading by $\pm(mp-q)/2$, and then the integral surgery formula can be stated as follow.

Theorem 3.1. *Suppose m is a fixed integer such that $mp-q\neq 0$. Then for any sufficiently large integer k, there exists an exact triangle*

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[rr, "\pi^{+}_{m,k}+\pi^{-}_{m,k}"] & & \mathbf{\Gamma}_{m+2k-1} \arrow[dl] \\
& \mathbf{Y}_m \arrow[ul] &
\end{tikzcd}$$

Hence we have an isomorphism

$$\mathbf{Y}_m\cong H(\mathrm{Cone}(\pi^{+}_{m,k}+\pi^{-}_{m,k})).$$

Remark 3.2. Let μ and λ represent the meridian and the longitude of the knot K, respectively. Then, $mp-q\neq 0$ is equivalent to the fact that $-m\mu+\lambda$ is not isotopic to a connected component of the boundary of the Seifert surface. Specifically, if K is null-homologous, we must have $m\neq 0$.

In the rest of this subsection and in the next subsection, we state the strategy to prove Theorem 3.1, and defer the proofs of some propositions to the remaining sections. An important step is to apply the octahedral axiom mentioned in Section 2.4 to the following diagram:

(3.1)
$$\begin{tikzcd}
 & \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[ddl] \arrow[dr, dotted] & \\
\mathbf{Y}_m \arrow[ur, dotted] \arrow[d] & & \mathbf{\Gamma}_{m-1+2k} \arrow[ll, dotted] \arrow[ddl] \\
\mathbf{\Gamma}_\mu \arrow[rr] \arrow[dr] & & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \arrow[uul] \arrow[u] \\
 & \mathbf{\Gamma}_{m-1} \arrow[uul] \arrow[ur] &
\end{tikzcd}$$

To obtain the dotted exact triangle, we need to establish the following three exact triangles:

(3.2)
$$\begin{tikzcd}
I^\sharp(-S^3_{-m}(K)) \arrow[d] & \\
\mathbf{\Gamma}_\mu \arrow[dr] & \\
 & \mathbf{\Gamma}_{m-1} \arrow[uul]
\end{tikzcd}$$

(3.3)
$$\begin{tikzcd}
 & \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[dl] & \\
\mathbf{\Gamma}_\mu \arrow[rr] & & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \arrow[ul]
\end{tikzcd}$$

(3.4)
$$\begin{tikzcd}
 & \mathbf{\Gamma}_{m-1+2k} \arrow[ddl] \\
 & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \arrow[u] \\
\mathbf{\Gamma}_{m-1} \arrow[ur] &
\end{tikzcd}$$

and establish the following commutative diagram:

(3.5)
$$\begin{tikzcd}
\mathbf{\Gamma}_\mu \arrow[rr] \arrow[dr] & & \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \\
 & \mathbf{\Gamma}_{m-1} \arrow[ur] &
\end{tikzcd}$$

The octahedral lemma then implies the existence of the dotted triangle and ensure that all diagrams in (3.1) other than exact triangles commute.

We will then use Lemma 2.25 to identify the map coming from the octahedral lemma with $\pi^+_{m,k}+\pi^-_{m,k}$. We also require the following two extra diagrams to commute, where the maps other than $\pi^+_{m,k}+\pi^-_{m,k}$ come from (3.1).

(3.6)
$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[dr, "\pi^+_{m,k}+\pi^-_{m,k}"] & \\
& \mathbf{\Gamma}_{m-1+2k} \\
& \mathbf{\Gamma}_{m-1+k}\oplus\mathbf{\Gamma}_{m-1+k} \arrow[u] \arrow[uul]
\end{tikzcd}$$

(3.7)
$$\begin{tikzcd}
& \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[dr, "\pi^+_{m,k}+\pi^-_{m,k}"] \arrow[ddl] & \\
& & \mathbf{\Gamma}_{m-1+2k} \arrow[ddl] \\
\mathbf{\Gamma}_{\mu} \arrow[dr] & & \\
& \mathbf{\Gamma}_{m-1} &
\end{tikzcd}$$

Indeed, by applying Lemma 2.25, it suffices to prove some weaker commutative diagrams involving $\pi^{\pm}_{m,k}$ separately.

3.2. **A strategy of the proof.** In this subsection, we provide more details of the strategy mentioned in Section 3.1. For simplicity, we fix the scalar ambiguities of commutative diagrams as in Section 2.3. To write down the maps, we redraw the octahedral diagram (3.1) as follows:

(3.8)
$$\begin{tikzcd}
& & \mathbf{Y}_m \arrow[rr] \arrow[dr, dotted, "\psi"] & & \mathbf{\Gamma}_{\mu} \arrow[ddr, "\psi^{\mu}_{+,m-1}+\psi^{\mu}_{-,m-1}"] & \\
& \mathbf{\Gamma}_{m-1} \arrow[ur] \arrow[dr, "{(\Psi^{m-1}_{+,m-1+k},\ \Psi^{m-1}_{-,m-1+k})}"] & & \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[ur, "l'"] \arrow[ddrr, dotted, "\phi"] & & \\
& & \mathbf{\Gamma}_{m-1+k}\oplus\mathbf{\Gamma}_{m-1+k} \arrow[ur, "h'"] \arrow[drrr, "\Psi^{m-1+k}_{-,m-1+2k}-\Psi^{m-1+k}_{+,m-1+2k}"'] & & & \mathbf{\Gamma}_{m-1} \\
\mathbf{\Gamma}_{\mu} \arrow[uur, "\psi^{\mu}_{+,m-1}+\psi^{\mu}_{-,m-1}"] \arrow[urr, "{(\psi^{\mu}_{-,m-1+k},\ \psi^{\mu}_{+,m-1+k})}"'] & & & & & \mathbf{\Gamma}_{m-1+2k} \arrow[u, "l"']
\end{tikzcd}$$

where

$$h' = \psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} - \psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}.$$

The reader can compare (3.8) with (2.9) and (2.10). We omit the term corresponding to $Z'\{1\}$ because there is not enough room, and the maps involving it are not important in our proof.

The first exact sequence of (3.8)

$$\mathbf{\Gamma}_\mu \xrightarrow{\psi^\mu_{+,m-1}+\psi^\mu_{-,m-1}} \mathbf{\Gamma}_{m-1} \to \mathbf{Y}_m \to \mathbf{\Gamma}_\mu \tag{3.9}$$

follows from the second exact triangle in Lemma 2.16. Though the map A_{m-1} may not be the same as the sum $\psi^\mu_{+,m-1} + \psi^n_{-,m-1}$, we can use the following proposition and Lemma 2.24 (another special case of Proposition 4.1) to identify $\mathrm{Cone}(A_{m-1})$ with $\mathrm{Cone}(\psi^\mu_{+,m-1} + \psi^n_{-,m-1})$. Here we use the assumption that $mp - q \neq 0$.

Proposition 3.3. *Suppose A_{n-1} is the map in Lemma 2.16. For any integer n, there exist scalars $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ such that*

$$A_{n-1} = c_1\psi^\mu_{+,n-1} + c_2\psi^\mu_{-,n-1}.$$

The exactness at

$$\mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k}$$

in the second and the third exact sequences are both special cases of the following proposition, which will be proved in Section 5.1 by diagram chasing.

Proposition 3.4. *Fixing the scalars as in Section 2.3, and given $n \in \mathbb{Z}$ and $k_0 \in \mathbb{N}_+$. Then, for any c_1, c_2, c_3, c_4 satisfying the equation*

$$c_1c_3 = -c_2c_4,$$

the following sequence is exact

$$\mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^n_{+,n+k_0}, c_2\Psi^n_{-,n+k_0})} \mathbf{\Gamma}_{n+k_0} \oplus \mathbf{\Gamma}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0} + c_4\Psi^{n+k_0}_{+,n+2k_0}} \mathbf{\Gamma}_{n+2k_0}$$

Remark 3.5. The exactness at the direct summand for the second exact sequence (the one involving $\mathbf{\Gamma}_\mu$) might not be as clear from Proposition 3.4. Explicitly, we apply the proposition to the dual knot K' corresponding to Γ_{m+k} with framing $\lambda' = -\mu$ and $n = 0, k_0 = 1$.

The exactness at $\mathbf{\Gamma}_\mu$ and $\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}$ in the second exact sequence of (3.8)

$$\mathbf{\Gamma}_\mu \xrightarrow{(\psi^\mu_{-,m-1+k}, \psi^\mu_{+,m-1+k})} \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \xrightarrow{\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} - \psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \xrightarrow{l'} \mathbf{\Gamma}_\mu \tag{3.10}$$

will also be proved by diagram chasing. We can explicitly construct the map l' by the composition of bypass maps

$$l' := \psi^{m+k}_{-,\mu} \circ \psi^{\frac{2m+2k-1}{2}}_{+,m+k} = \psi^{m+k}_{+,\mu} \circ \psi^{\frac{2m+2k-1}{2}}_{-,m+k},$$

where the last equation follows from Lemma 2.14 and the conventions in Section 2.3. The following proposition will be proved in Section 5.2 by diagram chasing.

Proposition 3.6. *Suppose l' is constructed as above. For any $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$, the following sequence is exact*

$$\mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \xrightarrow{c_3\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}} + c_4\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \xrightarrow{l'} \mathbf{\Gamma}_\mu$$
$$\xrightarrow{(c_1\psi^\mu_{-,m-1+k}, c_2\psi^\mu_{+,m-1+k})} \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k}$$

Remark 3.7. In the proof of [LY21, Theorem 3.23], we obtained a long exact sequence

$$\mathbf{\Gamma}_\mu \xrightarrow{(\psi^\mu_{+,n-1},\psi^\mu_{-,n-1})} \mathbf{\Gamma}_{n-1}\oplus\mathbf{\Gamma}_{n-1}\to\mathbf{\Gamma}_{\frac{2n-1}{2}}\to\mathbf{\Gamma}_\mu$$

by the octahedral lemma. However, we did not know the two maps involving $\mathbf{\Gamma}_{\frac{2n-1}{2}}$ explicitly. Thus, the second exact sequence here is stronger than the one from octahedral lemma.

Remark 3.8. The reason that Proposition 3.6 holds for any choices of c_1,c_2,c_3,c_4 is because

$$\ker((c_1\psi^\mu_{-,m-1+k},c_2\psi^\mu_{+,m-1+k}))=\ker(c_1\psi^\mu_{-,m-1+k})\cap\ker(c_2\psi^\mu_{+,m-1+k})$$

and

$$\operatorname{Im}(c_3\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}}+c_4\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}})=\operatorname{Im}(c_3\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}})+\operatorname{Im}(c_4\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}),$$

where the right hand sides of the equations are independent of scalars.

The exactness at $\mathbf{\Gamma}_{m-1}$ and $\mathbf{\Gamma}_{m-1+2k}$ in the third exact sequence of (3.8)

$$\mathbf{\Gamma}_{m-1}\xrightarrow{(\Psi^{m-1}_{+,m-1+k},\Psi^{m-1}_{-,m-1+k})}\mathbf{\Gamma}_{m-1+k}\oplus\mathbf{\Gamma}_{m-1+k}\xrightarrow{\Psi^{m-1+k}_{-,m-1+2k}-\Psi^{m-1+k}_{+,m-1+2k}}\mathbf{\Gamma}_{m-1+2k}\xrightarrow{l}\mathbf{\Gamma}_{m-1}\tag{3.11}$$

is harder to prove since the map l cannot be constructed by bypass maps. We expect that there are many equivalent constructions of l and we will use the one for which the exactness is easiest to prove. Even so, we only prove the exactness with the assumption that k is large. See Section 7.2 for more details.

Proposition 3.9. *Suppose $c_1,c_2,c_3,c_4\in\mathbb{C}\backslash\{0\}$ and suppose k_0 is a large integer. For any $n\in\mathbb{Z}$, there exists a map l such that the following sequence is exact*

$$\mathbf{\Gamma}_{n+k_0}\oplus\mathbf{\Gamma}_{n+k_0}\xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0}+c_4\Psi^{n+k_0}_{+,n+2k_0}}\mathbf{\Gamma}_{n+2k_0}\xrightarrow{l}\mathbf{\Gamma}_n\xrightarrow{(c_1\Psi^n_{+,n+k_0},c_2\Psi^n_{-,n+k_0})}\mathbf{\Gamma}_{n+k_0}\oplus\mathbf{\Gamma}_{n+k_0}$$

Remark 3.10. In the first arXiv version of this paper, we only proved Proposition 3.9 for knots in S^3 because we had to use the fact that $\dim_{\mathbb{C}} I^\sharp(-S^3)=1$ and S^3 has an orientation-reversing involution. The construction of l for knots in general 3-manifolds is inspired by the original proof for S^3 and the proof in Section 7 is a generalization of the previous proof.

Remark 3.11. For the same reason as in Remark 3.8, the coefficients in Proposition 3.9 are not important.

Then we consider the commutative diagrams mentioned in Section 3.1. By Lemma 2.6 and Lemma 2.12, we have

$$(\Psi^{m-1}_{+,m-1+k},\Psi^{m-1}_{-,m-1+k})\circ(\psi^\mu_{+,m-1}+\psi^\mu_{-,m-1})=(\psi^\mu_{-,m-1+k},\psi^\mu_{+,m-1+k}),$$

which verifies the commutative diagram in the assumption of the octahedral axiom.

Define

$$\phi':=\pi^+_{m,k}+\pi^-_{m,k}=\Psi^{m+k}_{+,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{-,m+k}+\Psi^{m+k}_{-,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{+,m+k}$$

By Lemma 2.13 and Lemma 2.14 with $n=m+k$, we have

$$\begin{aligned}\phi'\circ h'=&(\Psi^{m+k}_{+,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{-,m+k}+\Psi^{m+k}_{-,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{+,m+k})\circ(\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}}-\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}})\\ =&\Psi^{m+k}_{-,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{+,m+k}\circ\psi^{m+k-1}_{-,\frac{2m+2k-1}{2}}-\Psi^{m+k}_{+,m-1+2k}\circ\psi^{\frac{2m+2k-1}{2}}_{-,m+k}\circ\psi^{m+k-1}_{+,\frac{2m+2k-1}{2}}\\ =&\Psi^{m+k}_{-,m-1+2k}\circ\psi^{m+k-1}_{-,m+k}-\Psi^{m+k}_{+,m-1+2k}\circ\psi^{m+k-1}_{+,m+k}\\ =&\Psi^{m-1+k}_{-,m-1+2k}-\Psi^{m-1+k}_{+,m-1+2k}\end{aligned}$$

This verifies the second commutative diagram mentioned in Section 3.1.

Finally, we state a weaker version of the third commutative diagram mentioned in Section 3.1, which is enough to apply Lemma 2.25. The following proposition will be proved in Section 7.4.

Proposition 3.12. *Suppose l' and l are the maps in Proposition 3.6 and Proposition 3.9. Then, there are two commutative diagrams up to scalars.*

$$\begin{tikzcd} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[r, "l'"] \arrow[d, "\pi^+_{m,k}"] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{+,m-1}"] \\ \mathbf{\Gamma}_{m-1+2k} \arrow[r, "l"] & \mathbf{\Gamma}_{m-1} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{\Gamma}_{\frac{2m+2k-1}{2}} \arrow[r, "l'"] \arrow[d, "\pi^-_{m,k}"] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^\mu_{-,m-1}"] \\ \mathbf{\Gamma}_{m-1+2k} \arrow[r, "l"] & \mathbf{\Gamma}_{m-1} \end{tikzcd}$$

Proof of Theorem 3.1. We verified all assumptions of the octahedral lemma (Lemma 2.23) for the diagram (3.8). Hence, there exists a map ϕ such that

$$\mathbf{Y}_m \cong H(\mathrm{Cone}(\phi)).$$

We also verified all assumptions of Lemma 2.25 for $\phi' = \pi^+_{m,k} + \pi^-_{m,k}$. Thus, we have

$$H(\mathrm{Cone}(\phi')) \cong H(\mathrm{Cone}(\phi)) \cong \mathbf{Y}_m.$$

Then the desired triangle in the theorem holds. □

3.3. **Reformulation by bent complexes.**

In this subsection, we restate Theorem 3.1 using the language of bent complexes introduced in [LY21]. Suppose K is a rationally null-homologous knot in a closed 3-manifold Y. We continue to adopt the notations and conventions from Section 2.2 and Section 2.3.

Putting bypass triangles in Lemma 2.6 for different n together, we obtain the following diagram:

$$\begin{tikzcd} \cdots & \mathbf{\Gamma}_{n+1} \arrow[l] \arrow[dr, "\psi^{n+1}_{+,\mu}"'] & & \mathbf{\Gamma}_n \arrow[ll, "\psi^n_{+,n+1}"'] \arrow[dr, "\psi^n_{+,\mu}"'] & & \mathbf{\Gamma}_{n-1} \arrow[ll, "\psi^{n-1}_{+,n}"'] \arrow[dr, "\psi^{n-1}_{+,\mu}"'] & & \mathbf{\Gamma}_{n-2} \arrow[ll, "\psi^{n-2}_{+,n-1}"'] & \cdots \arrow[l] \\ & \cdots & \mathbf{\Gamma}_\mu \arrow[ur, "\psi^\mu_{+,n}"] \arrow[dl, "\psi^\mu_{-,n-2}"] & & \mathbf{\Gamma}_\mu \arrow[ur, "\psi^\mu_{+,n-1}"] \arrow[dl, "\psi^\mu_{-,n-1}"] & & \mathbf{\Gamma}_\mu \arrow[ur, "\psi^\mu_{+,n-2}"] \arrow[dl, "\psi^\mu_{-,n}"] & \cdots & \\ \cdots \arrow[r] & \mathbf{\Gamma}_{n-2} \arrow[rr, "\psi^{n-2}_{-,n-1}"'] & & \mathbf{\Gamma}_{n-1} \arrow[ul, "\psi^{n-1}_{-,\mu}"'] \arrow[rr, "\psi^{n-1}_{-,n}"'] & & \mathbf{\Gamma}_n \arrow[ul, "\psi^n_{-,\mu}"'] \arrow[rr, "\psi^n_{-,n+1}"'] & & \mathbf{\Gamma}_{n+1} \arrow[ul, "\psi^{n+1}_{-,\mu}"'] \arrow[r] & \cdots \end{tikzcd} \tag{3.12}$$

where the $\mathbb{Z}$-grading shift of $\psi^\mu_{\pm,k} \circ \psi^k_{\pm,\mu}$ is $\pm p$ for any $k \in \mathbb{Z}$. From (3.12), we constructed in [LY21, Section 3.4] two spectral sequences $\{E_{r,+}, d_{r,+}\}_{r \geqslant 1}$ and $\{E_{r,-}, d_{r,-}\}_{r \geqslant 1}$ from $\mathbf{\Gamma}_\mu$ to $\mathbf{Y}$, where $d_{r,\pm}$ is roughly

$$\psi^k_{\pm,\mu} \circ (\Psi^k_{\pm,k+r})^{-1} \circ \psi^\mu_{\pm,k+r} \text{ for any } k \in \mathbb{Z}. \tag{3.13}$$

The composition with the inverse map is well-defined on the r-th page, and the independence of k (and hence n in (3.13)) follows from Lemma 2.12. The $\mathbb{Z}$-grading shift of $d_{r,\pm}$ is $\pm rp$. By fixing an inner product on $\mathbf{\Gamma}_\mu$, we then lifted those spectral sequences to two differentials d_+ and d_- on $\mathbf{\Gamma}_\mu$ such that

$$H(\mathbf{\Gamma}_\mu, d_+) \cong H(\mathbf{\Gamma}_\mu, d_-) \cong \mathbf{Y}.$$

In such way, the inverses of $\Psi^k_{\pm,k+r}$ are also well-defined, which we will use freely later.

Then we propose an integral surgery formula for $\mathbf{Y}_m$ using differentials d_+ and d_- on $\mathbf{\Gamma}_\mu$. To state the formula, we introduce the following notations.

Definition 3.13 ([LY21, Construction 3.27 and Definition 5.12])**.** For any integer s, define the complexes

$$B^{\pm}(s) := (\bigoplus_{k\in\mathbb{Z}}(\mathbf{\Gamma}_\mu, s+kp), d_\pm), \quad B^+(\geqslant s) := (\bigoplus_{k\geqslant 0}(\mathbf{\Gamma}_\mu, s+kp), d_+),$$

$$\text{and } B^-(\leqslant s) := (\bigoplus_{k\leqslant 0}(\mathbf{\Gamma}_\mu, s+kp), d_-).$$

Furthermore, define

$$I^+(s) : B^+(\geqslant s) \to B^+(s) \text{ and } I^-(s) : B^-(\leqslant s) \to B^-(s)$$

to be the inclusion maps. We also write the same notation for the induced map on homology.

Remark 3.14. By Lemma 2.5, we know that the nontrivial gradings of $\mathbf{\Gamma}_\mu$ are finite. Then, for any sufficiently large integer s_0 satisfying

$$s - s_0 p \leqslant -\frac{p-\chi(S)}{2} \text{ and } s + s_0 p \geqslant \frac{p-\chi(S)}{2},$$

we have

$$B^+(s) = B^+(\geqslant s - s_0 p) \text{ and } B^-(s) = B^-(\leqslant s + s_0 p).$$

In such case, $I^+(s - s_0 p)$ and $I^-(s + s_0 p)$ are identities.

By splitting the diagram (3.12) into $\mathbb{Z}$-gradings, we can calculate homologies of the complexes defined in Definition 3.13.

Proposition 3.15. *Suppose $n \in \mathbb{N}_+$ and i is a grading. Fix an inner product on $\mathbf{\Gamma}_n$. If $i > (p-\chi(S))/2 - np$, then there exists a canonical isomorphism*

$$H(B^+(\geqslant i)) \cong (\mathbf{\Gamma}_n, i + \frac{(n-1)p-q}{2}).$$

If $i < -(p-\chi(S))/2 + np$, then there exists a canonical isomorphism

$$H(B^-(\leqslant i)) \cong (\mathbf{\Gamma}_n, i - \frac{(n-1)p-q}{2}).$$

Proof. The proof mirrors that of [LY21, Lemma 5.13]. Following the notation in [LY21, (3.9) and (3.10)], if

$$i > \hat{i}^{\mu}_{max} - nq = (p-\chi(S))/2 - np,$$

then $\mathbf{\Gamma}_0^{i,+} = 0$ (the corresponding grading summand of $\mathbf{\Gamma}_0$) and the isomorphism follows from the convergence theorem of the unrolled spectral sequence [LY21, Theorem 2.4] (see also [Boa99, Theorem 6.1]). Note that the unrolled spectral sequence induces a filtration on $\mathbf{\Gamma}_n$, and the homology is canonically isomorphic to the direct sum of all associated graded objects of the filtration. Then we use the inner product to identify the direct sum with the total space $\mathbf{\Gamma}_n$. The other statement holds for the same reason. $\square$

Definition 3.16 ([LY21, Construction 3.27 and Definition 5.12])**.** For any integer s, define the **bent complex**

$$A(s) := (\bigoplus_{k\in\mathbb{Z}}(\mathbf{\Gamma}_\mu, s+kp), d_s),$$

where for any element $x \in (\mathbf{\Gamma}_\mu, s+kp)$,

$$d_s(x) = \begin{cases} d_+(x) & k > 0, \\ d_+(x) + d_-(x) & k = 0, \\ d_-(x) & k < 0. \end{cases}$$

Define

$$\pi^+(s) : A(s) \to B^+(s) \text{ and } \pi^-(s) : A(s) \to B^-(s)$$

by

$$\pi^+(s)(x) = \begin{cases} x & k \geqslant 0, \\ 0 & k < 0, \end{cases} \text{ and } \pi^-(s)(x) = \begin{cases} x & k \leqslant 0, \\ 0 & k > 0, \end{cases}$$

where $x \in (\mathbf{\Gamma}_\mu, s+kp)$. Define

$$\pi^\pm : \bigoplus_{s\in\mathbb{Z}} A(s) \to \bigoplus_{s\in\mathbb{Z}} B^\pm(s)$$

by putting $\pi^\pm(s)$ together for all s . We also use the same notation for the induced map on homology.

Remark 3.17. Similar to Remark 3.14, according to Lemma 2.5, there are only finitely many the nontrivial gradings of $\mathbf{\Gamma}_\mu$. Then, for any sufficiently large integer s_0 such that $s_0 \geqslant (p-\chi(S))/2$, we have

$$A(s_0) = B^-(s_0) \text{ and } A(-s_0) = B^+(-s_0).$$

In such case, $\pi^-(s_0)$ and $\pi^+(-s_0)$ are identities.

Now, we state the integral surgery formula in the above setup.

Theorem 3.18. *Suppose m is a fixed integer such that $mp-q \neq 0$. Then there exists an isomorphism*

$$\Xi_m : \bigoplus_{s\in\mathbb{Z}} H(B^+(s)) \xrightarrow{\cong} \bigoplus_{s\in\mathbb{Z}} H(B^-(s+mp-q))$$

as the direct sum of isomorphisms

$$\Xi_{m,s} : H(B^+_s) \xrightarrow{\cong} H(B^-_{s+mp-q})$$

so that

$$\mathbf{Y}_m \cong H\bigg(\operatorname{Cone}(\pi^- + \Xi_m \circ \pi^+ : \bigoplus_{s\in\mathbb{Z}} H(A(s)) \to \bigoplus_{s\in\mathbb{Z}} H(B^-(s)))\bigg).$$

Proof. According to Remark 3.17, we only need to consider the maps $\pi^\pm(s)$ for $|s|$ less than a fixed integer. For such values of s, we can apply the following proposition.

Proposition 3.19 ([LY21, Proposition 3.28])**.** *Fix $m, s \in \mathbb{Z}$ such that $|s| \leqslant (p-\chi(S))/2$. For any large integer k, fix inner products on $\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}$ and $\mathbf{\Gamma}_{m-1+2k}$. Then there exist $s_1, s_2^+, s_2^-, s_3^+, s_3^- \in \mathbb{Z}$ such that the following diagram commutes*

$$\begin{tikzcd} H(A(s)) \arrow[r, "\pi^\pm(s)"] \arrow[d, "\cong"] & H(B^\pm(s)) \arrow[d, "\cong"] \\ (\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, s_1) \arrow[r, "\pi^{\pm,s_1}_{m,k}"] & (\mathbf{\Gamma}_{m-1+2k}, s_3^\pm) \end{tikzcd}$$

where the maps $\pi^{\pm,s_1}_{m,k}$, defined in Section 3.1, factor through $(\mathbf{\Gamma}_{m+k}, s_2^\pm)$.

Remark 3.20. The maps $\pi^{\pm}(s)$ factor through $I^{\pm}(s)$ constructed in Definition 3.13. We denote

$$\pi^{\pm}(s) = I^{+}(s) \circ \pi^{\pm,\prime}(s).$$

This corresponds to the factorization about $(\mathbf{\Gamma}_{m+k}, s_2^{\pm})$ in Proposition 3.19 (we fix an inner product on $\mathbf{\Gamma}_{m+k}$ to apply Proposition 3.15), *i.e.*, the following diagrams commute

$$\begin{tikzcd}
H(A(s)) \arrow[r, "\pi^{+,\prime}(s)"] \arrow[d, "\cong"] & H(B^{+}(\geqslant s)) \arrow[r, "I^{+}(s)"] \arrow[d, "\cong"] & H(B^{+}(s)) \arrow[d, "\cong"] \\
(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, s_1) \arrow[r, "\psi_{-,m+k}^{\frac{2m+2k-1}{2}}"] & (\mathbf{\Gamma}_{m+k}, s_2^{+}) \arrow[r, "\Psi^{m+k}_{+,m-1+2k}"] & (\mathbf{\Gamma}_{m-1+2k}, s_3^{+})
\end{tikzcd}$$

$$\begin{tikzcd}
H(A(s)) \arrow[r, "\pi^{-,\prime}(s)"] \arrow[d, "\cong"] & H(B^{-}(\leqslant s)) \arrow[r, "I^{-}(s)"] \arrow[d, "\cong"] & H(B^{-}(s)) \arrow[d, "\cong"] \\
(\mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, s_1) \arrow[r, "\psi_{+,m+k}^{\frac{2m+2k-1}{2}}"] & (\mathbf{\Gamma}_{m+k}, s_2^{-}) \arrow[r, "\Psi^{m+k}_{-,m-1+2k}"] & (\mathbf{\Gamma}_{m-1+2k}, s_3^{-})
\end{tikzcd}$$

From the calculation in [LY21, Remark 3.29] (we replace n and l there by $m+k$ and $k-1$, and note that there is a typo about sign in the first arXiv version of [LY21]), the difference of the grading shifts is

$$s_3^{+} - s_3^{-} = (m+k-(k-1)-1)p - q = mp - q.$$

Note that the notations in this paper and [LY21] are different (*c.f.* Remark 2.3).

Then we can construct the isomorphism

$$\Xi_{m,s} : H(B^{+}(s)) \xrightarrow{\cong} H(B^{-}(s+mp-q))$$

for $|s| \leqslant (p-\chi(S))/2$ by identifying both $H(B^{+}(s))$ and $H(B^{-}(s+mp-q))$ with $(\mathbf{\Gamma}_{m-1+2k}, s_3^{+})$ for a sufficiently large k. *A priori*, this isomorphism depends on inner products on

$$\mathbf{\Gamma}_{\mu}, \mathbf{\Gamma}_{\frac{2m+2k-1}{2}}, \mathbf{\Gamma}_{m-1+2k} \text{ and } \mathbf{\Gamma}_{m+k}.$$

For other s, we can take any isomorphism $\Xi_{m,s}$ since the choice does not affect the computation of the mapping cone.

Consequently, we obtain

$$H(\mathrm{Cone}(\pi^{-} + \Xi_m \circ \pi^{+})) \cong H(\mathrm{Cone}(\pi^{-}_{m,k} + \pi^{+}_{m,k})) \cong \mathbf{Y}_m,$$

where the last isomorphism comes from Theorem 3.1. $\square$

Remark 3.21. Theorem 3.18 is slightly weaker than Theorem 3.1. Indeed, when we use the integral surgery formula to calculate surgeries on the Boromean knot in the companion paper [LY22b], we have to study the $H_1(Y)$ action on sutured instanton homology, where $Y = \#^{2n} S^1 \times S^2$ is the ambient manifold of the knot. This action vanishes on $\mathbf{\Gamma}_{\mu}$ so vanishes on the bent complex. But it is nonvanishing on $\mathbf{\Gamma}_{m+k}$ and $\mathbf{\Gamma}_{m-1+2k}$ and we use this information to realize the computation. This issue for the bent complex might be resolved by introducing some E_0-pages for differentials d_+ and d_- such that the action is nontrivial on E_0-pages.

3.4. **A formula for instanton knot homology.** The third exact sequence (3.11) implies

$$\mathbf{\Gamma}_{m-1} \cong H(\mathrm{Cone}(\Psi^{m-1+k}_{-,m-1+2k} - \Psi^{m-1+k}_{+,m-1+2k} : \mathbf{\Gamma}_{m-1+k} \oplus \mathbf{\Gamma}_{m-1+k} \to \mathbf{\Gamma}_{m-1+2k}))$$

for any sufficiently large integer k. Since there are two copies $\mathbf{\Gamma}_{m-1+k}$, we can always regard the grading shifts of the maps $\Psi^{m-1+k}_{-,m-1+2k}$ as different ones by rescaling the grading of the first summand from i to $2i-1$ and the second summand from i to $2i$. Hence we do not need the assumption $mp - q \neq 0$ as in the previous mapping cone formula in Theorem 3.1. By Lemma 2.24, we can replace the minus sign with any coefficient.

In this subsection, we restate this result in the language of bent complexes. The formula is inspired by Eftekhary's formula for knot Floer homology $\widehat{HFK}$ [Eft18, Proposition 1.5] (see also Hedden-Levine's work [HL21]). Since m can be any integer, we replace $m-1$ by m.

Theorem 3.22. *Suppose $m, j \in \mathbb{Z}$. Define*

$$j^+ = j - \frac{(m-1)p - q}{2} \text{ and } j^- = j + \frac{(m-1)p-q}{2}.$$

Then there exists an isomorphism

$$\Xi'_{m,j} : H(B^+(j^+)) \xrightarrow{\cong} H(B^-(j^-))$$

such that

$$(\mathbf{\Gamma}_m, j) \cong H\bigg(\mathrm{Cone}(I^-(j^-) + \Xi'_{m,j} \circ I^+(j^+) : H(B^-(\leqslant j^-)) \oplus H(B^+(\geqslant j^+)) \to H(B^-(j^-)))\bigg).$$

Proof. As mentioned before, we have

$$\mathbf{\Gamma}_m \cong H(\mathrm{Cone}(\Psi^{m+k}_{-,m+2k} - \Psi^{m+k}_{+,m+2k})) \cong H(\mathrm{Cone}(\Psi^{m+k}_{-,m+2k} + \Psi^{m+k}_{+,m+2k}))$$

for any sufficiently large integer k.

Since bypass maps are homogeneous, the above mapping cone splits into $\mathbb{Z}$-gradings (or $(\mathbb{Z}+\frac{1}{2})$-gradings). Hence we can use it to calculate $(\mathbf{\Gamma}_m, j)$. By Lemma 2.6, the corresponding spaces are

$$(\mathbf{\Gamma}_{m+k}, j - \frac{kp}{2}) \oplus (\mathbf{\Gamma}_{m+k}, j + \frac{kp}{2}) \text{ and } (\mathbf{\Gamma}_{m+2k}, j).$$

From Proposition 3.15 with $i = j \pm kp/2$, by fixing an inner product on $\mathbf{\Gamma}_{m+k}$, we know that

$$(\mathbf{\Gamma}_{m+k}, j - \frac{kp}{2}) \cong H(B^-(\leqslant j^-)) \text{ for } j - \frac{kp}{2} < -\frac{p - \chi(S)}{2} + (m+k)p$$

and

$$(\mathbf{\Gamma}_{m+k}, j + \frac{kp}{2}) \cong H(B^+(\geqslant j^+)) \text{ for } j + \frac{kp}{2} > \frac{p - \chi(S)}{2} - (m+k)p.$$

Since m is fixed, when k is sufficiently large, we know that any j with $(\mathbf{\Gamma}_m, j)$ nontrivial (i.e. $|j| \leqslant (|mp-q| - \chi(S))/2$ by Lemma 2.5) satisfies the above inequalities. By Proposition 3.15 again (fixing an inner product on $\mathbf{\Gamma}_{m+2k}$) and Remark 3.14, for k sufficiently large, we know that

$$(\mathbf{\Gamma}_{m+2k}, j) \cong H(B^-(j^-)) \cong H(B^+(j^+))$$

for such j. By unpackaging the construction of differentials d_+ and d_- in [LY21, Section 3.4], we know that the restrictions of maps $\Psi^m_{-,m+2k}$ and $\Psi^m_{+,m+2k}$ on the corresponding gradings coincide with the maps induced by the inclusions $I^-(j^-)$ and $I^+(j^+)$ under the canonical isomorphisms, respectively.

For $|j| \leqslant (|mp-q|-\chi(S))/2$, let

$$\Xi'_{m,j}: H(B^+(j^+)) \xrightarrow{\cong} H(B^-(j^-))$$

be the isomorphism obtained from identifying both spaces to the corresponding grading summand of $\mathbf{\Gamma}_{m+2k}$. Note that it depends on inner products on $\mathbf{\Gamma}_\mu, \mathbf{\Gamma}_{m+k}$ and $\mathbf{\Gamma}_{m+2k}$. For other j, we can take any isomorphism $\Xi'_{m,j}$ since the choice does not affect the computation of the mapping cone. Then we know that

$$\begin{aligned}(\mathbf{\Gamma}_m, j) \cong & H(\mathrm{Cone}(\Psi^{m+k}_{-,m+2k} + \Psi^{m+k}_{+,m+2k}|(\mathbf{\Gamma}_{m+k}, j+\frac{kp}{2}) \oplus (\mathbf{\Gamma}_{m+k}, j-\frac{kp}{2}))) \\ \cong & H(\mathrm{Cone}(I^-(j^-) + \Xi'_{m,j} \circ I^+(j^+))).\end{aligned}$$

□

4. Dehn surgery and bypass maps

In this section, we prove a generalization of Lemma 2.20 and Proposition 3.3.

Suppose (M,γ) is a balanced sutured manifold and $\alpha \subset \partial M$ is a connected simple closed curve that intersects the suture γ twice. There are two natural bypass arcs associated to α, each of which intersects the suture at three points and induces a bypass triangle (*c.f.* [BS22c, Section 4])

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[rr, "\psi_\pm"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl] \\ & \underline{\mathrm{SHI}}(-M,-\gamma_3) \arrow[ul] & \end{tikzcd}$$

where γ_2 and γ_3 are the sutures coming from bypass attachments. Note that the two bypass exact triangles involve the same set of balanced sutured manifolds but have different maps between them. Let (M_0,γ_0) be obtained from (M,γ) by attaching a contact 2-handle along α. From [BS16b, Section 3.3], it has been shown that a closure of $(-M_0,-\gamma_0)$ coincides with a closure of the sutured manifold obtained from $(-M,-\gamma)$ by 0-surgery along α with respect to the surface framing. Hence there is also a surgery exact triangle (*c.f.* [LY22a, Lemma 3.21])

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\gamma) \arrow[rr, "H_\alpha"] & & \underline{\mathrm{SHI}}(-M,-\gamma_2) \arrow[dl] \\ & \underline{\mathrm{SHI}}(-M_0,-\gamma_0) \arrow[ul] & \end{tikzcd}$$

The map H_α is related to the bypass maps $\psi_\pm$ as follows:

Proposition 4.1. *There exist $c_1, c_2 \in \mathbb{C}\backslash\{0\}$, such that*

$$H_\alpha = c_1\psi_+ + c_2\psi_-.$$

Remark 4.2. The proof of Proposition 4.1 was developed through the discussions with John A. Baldwin and Steven Sivek.

Proof of Proposition 4.1. Let $A \subset \partial M$ be a tubular neighborhood of $\alpha \subset \partial M$. Push the interior of A into the interior of M to make it a properly embedded surface. By a standard argument in [Hon00], we can assume that a collar of ∂M is equipped with a product contact structure such that γ is (isotopic to) the dividing set, α is a Legendrian curve, A is in the contact collar, and A is a convex surface with Legendrian boundary that separates a standard contact neighborhood of α off M. The convex decomposition of M along A yields two pieces

$$M = M' \underset{A}{\cup} V,$$

where M' is diffeomorphic to M and V is the contact neighborhood of α. It is straightforward to check that, after rounding the corners, the contact structure near the boundary of M' is still a product contact structure with $\partial M'$ being a convex boundary. Let γ' be the dividing set on $\partial M'$. Also, after rounding the corners, with the contact structure on $V \cong S^1 \times D^2$, we can suppose ∂V is a convex surface with dividing set being the union of two connected simple closed curves on ∂V of slope -1. When viewing V as the complement of an unknot in S^3, the dividing set coincides with the suture $\Gamma_1 \subset V$, so from now on we call it Γ_1. By the construction of the gluing map in [Li21a], there exists a map

$$G_1 : \underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_1) \to \underline{\mathrm{SHI}}(-M, -\gamma).$$

As in [Li21a], the map G_1 comes from attaching contact handles to $(M', \gamma') \sqcup (V, \Gamma_1)$ to recover the gluing along A. From [Li21b, Proposition 1.4], we know that

$$\underline{\mathrm{SHI}}(-V, -\Gamma_1) \cong \mathbb{C}.$$

Note that M' and M are equipped with the product contact structure near their boundaries. From the functoriality of the contact gluing map in [Li21a], we know that G_1 is an isomorphism. Now both the (-1)-surgery along a push off of α and the bypass attachments can be thought of as happening in the piece V. Note that the result of both (-1)-surgery and the bypass attachments for Γ_1 is Γ_2. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_1) \arrow[r, "G_1"] \arrow[d, "\mathrm{Id} \otimes \widehat{H}_\alpha"'] & \underline{\mathrm{SHI}}(-M, -\gamma) \arrow[d, "H_\alpha"] \\
\underline{\mathrm{SHI}}(-M', -\gamma') \otimes \underline{\mathrm{SHI}}(-V, -\Gamma_2) \arrow[r, "G_2"] & \underline{\mathrm{SHI}}(-M, -\gamma_2)
\end{tikzcd} \tag{4.1}$$

where $\widehat{H}_\alpha$ denotes the surgery map for the manifold V and G_2 is the gluing map obtained by attaching the same set of contact handles as G_1. A similar commutative diagram holds when replacing H_α and $\widehat{H}_\alpha$ by $\psi_\pm$ and

$$\hat{\psi}_\pm : \underline{\mathrm{SHI}}(-V, -\Gamma_1) \to \underline{\mathrm{SHI}}(-V, -\Gamma_2)$$

in (4.1), respectively.

Since G_1 is an isomorphism, to obtain a relation between H_α and $\psi_\pm$, it suffices to understand the relation between $\widehat{H}_\alpha$ and $\hat{\psi}_\pm$. From [Li21b, Proposition 1.4], we know that

$$\underline{\mathrm{SHI}}(-V, -\Gamma_2) \cong \mathbb{C}^2.$$

Moreover, the meridian disk of V induces a $(\mathbb{Z} + \frac{1}{2})$ grading on $\underline{\mathrm{SHI}}(-V, -\Gamma_2)$ and we have

$$\underline{\mathrm{SHI}}(-V, -\Gamma_2) \cong \underline{\mathrm{SHI}}(-V, -\Gamma_2, \frac{1}{2}) \oplus \underline{\mathrm{SHI}}(-V, -\Gamma_2, -\frac{1}{2}),$$

with

$$\underline{\mathrm{SHI}}(-V, -\Gamma_2, \frac{1}{2}) \cong \underline{\mathrm{SHI}}(-V, -\Gamma_2, -\frac{1}{2}) \cong \mathbb{C}.$$

Let

$$\mathbf{1}\in \underline{\mathrm{SHI}}(-V,-\Gamma_1)\cong\mathbb{C}$$

be a generator. In [Li21b, Section 4.3] it is shown that

$$\hat{\psi}_-(\mathbf{1})\in \underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2}) \text{ and } \hat{\psi}_+(\mathbf{1})\in \underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2})$$

are non-zero. Also, when viewing V as the complement of the unknot U, there is an exact triangle

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-V,-\Gamma_1) \arrow[rr, "\widehat{H}_\alpha"] & & \underline{\mathrm{SHI}}(-V,-\Gamma_2) \arrow[dl, "F_2"] \\ & I^\sharp(-S^3) \arrow[ul, "G_1"] & \end{tikzcd} \tag{4.2}$$

as in Lemma 2.16. Comparing the dimensions of the spaces in (4.2), we have $G_1=0$ and $\widehat{H}_\alpha$ is injective. From the fact that $\tau_I(U)=0$, we know from [GLW19, Corollary 3.5] that

$$F_2|\underline{\mathrm{SHI}}(-V,-\Gamma_2,\frac{1}{2})\neq 0 \text{ and } F_2|\underline{\mathrm{SHI}}(-V,-\Gamma_2,-\frac{1}{2})\neq 0,$$

By the exactness in (4.2), we have $\ker(F_2)=\mathrm{Im}(\widehat{H}_\alpha)$ and then $\widehat{H}_\alpha(\mathbf{1})$ is not in $\underline{\mathrm{SHI}}(-V,-\Gamma_2,\pm\frac{1}{2})$, *i.e.*, it is a linear combination of generators of $\underline{\mathrm{SHI}}(-V,-\Gamma_2,\pm\frac{1}{2})$. Hence we know that there are $c_1,c_2\in\mathbb{C}\backslash\{0\}$ such that

$$\widehat{H}_\alpha(\mathbf{1})=c_1\hat{\psi}_+(\mathbf{1})+c_2\hat{\psi}_-(\mathbf{1}).$$

Then the proposition follows from the commutative diagram (4.1). □

In Remark 1.3, we discussed the ambiguity arising from scalars. It is worth mentioning that such ambiguity already exists in instanton theory. For example, if M is the complement of a knot $K\subset S^3$ and γ consists of two meridians of the knot, which we denote by Γ_μ, we can choose α to be a curve on $\partial(S^3\backslash N(K))$ of slope $-n$. Then we have a surgery triangle:

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-M,-\Gamma_\mu) \arrow[rr, "H_n"] & & \underline{\mathrm{SHI}}(-M,-\Gamma_{n-1}) \arrow[dl] \\ & I^\sharp(-S^3_{-n}(K)) \arrow[ul] & \end{tikzcd}$$

Note that this triangle is not the one from Floer's original exact triangle, but rather one with a slight modification on the choice of 1-cycles inside the 3-manifold that represents the second Stiefel-Whitney class of the relevant $SO(3)$-bundle; see [BS21, Section 2.2] for more details. Floer's original exact triangle, on the other hand, yields a different triangle

$$\begin{tikzcd} \underline{\mathrm{SHI}}(-S^3\backslash N(K),-\Gamma_\mu) \arrow[rr, "H_n'"] & & \underline{\mathrm{SHI}}(-S^3\backslash N(K),-\Gamma_{n-1}) \arrow[dl] \\ & I^\sharp(-S^3_{-n}(K),\mu) \arrow[ul] & \end{tikzcd}$$

where $\mu\subset -S^3_{-n}(K)$ denotes a meridian of the knot. Note the difference between H_α and H'_α is that they come from the same cobordism but the $SO(3)$-bundles over the cobordism are different.

The local argument to prove Proposition 4.1 works for both H_α and H'_α. Hence there exists non-zero complex numbers c_1, c_2, c'_1, c'_2 such that

$$H_\alpha = c_1\psi^\mu_{+,n} + c_2\psi^\mu_{-,n-1} \text{ and } H'_\alpha = c'_1\psi^\mu_{+,n} + c'_2\psi^\mu_{-,n-1}$$

where the maps

$$\psi^\mu_{\pm,n-1} : \underline{\mathrm{SHI}}(-S^3\backslash N(K), -\Gamma_\mu) \to \underline{\mathrm{SHI}}(-S^3\backslash N(K), -\Gamma_{n-1})$$

are the two related bypass maps. When $n \neq 0$, these two bypass maps have different grading shifting behavior, so by Lemma 2.24, different choice of non-zero coefficients does not change the dimensions of kernel and cokernel of the map. Hence we conclude that for $n \neq 0$,

$$I^\sharp(-S^3_{-n}(K), \mu) \cong I^\sharp(-S^3_{-n}(K)).$$

However, when $n = 0$, the two bypass maps $\psi^\mu_{\pm,n-1}$ both preserves gradings, making the coefficients significant, *i.e.*, $I^\sharp(-S^3_0(K), \mu)$ and $I^\sharp(-S^3_0(K))$ might have different dimensions for different choices of coefficients. Indeed, it is observed by Baldwin-Sivek [BS21] that for what they called as W-shaped knots (which is clearly a non-empty class, *e.g.* the figure-8 knot [BS22b, Proposition 10.4]), these two framed instanton homologies have dimensions differing by 2.

5. Some exactness by diagram chasing

5.1. **At the direct summand.** In this subsection, we prove Proposition 3.4 by diagram chasing. We restate the result in Proposition 5.1. We also adopt the conventions for scalars from Section 2.3, and this together with Lemma 2.11 implies that

$$\Psi^{n+k_0}_{+,n+2k_0} \circ \Psi^n_{-,n+k_0} = \Psi^{n+k_0}_{-,n+2k_0} \circ \Psi^n_{+,n+k_0}.$$

for any n and k_0.

Proposition 5.1. *Given $n \in \mathbb{Z}$ and $k_0 \in \mathbb{N}_+$, for any c_1, c_2, c_3, c_4 satisfying the equation*

$$c_1c_3 = -c_2c_4,$$

the following sequence is exact

$$\mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^n_{+,n+k_0}, c_2\Psi^n_{-,n+k_0})} \mathbf{\Gamma}_{n+k_0} \oplus \mathbf{\Gamma}_{n+k_0} \xrightarrow{c_3\Psi^{n+k_0}_{-,n+2k_0} + c_4\Psi^{n+k_0}_{+,n+2k_0}} \mathbf{\Gamma}_{n+2k_0}$$

Proof. For simplicity, we only prove the proposition for $n = 0$. The proof for any general n is similar (replacing all $\mathbf{\Gamma}_m$ below by $\mathbf{\Gamma}_{n+m}$ and modifying the notations for bypass maps). Also, we only prove the case when

$$c_1 = c_2 = c_3 = 1, c_4 = -1.$$

The proof for general scalars can be obtained similarly.

We prove the proposition by induction on k_0. We will use the exactness in Lemma 2.6 and the commutative diagrams in Lemma 2.12 and Lemma 2.11 for many times. For simplicity, we will use them without mentioning the lemmas.

First, we assume $k_0 = 1$. The proposition reduces to

$$\ker(\psi^1_{-,2} - \psi^1_{+,2}) = \mathrm{Im}((\psi^0_{+,1}, \psi^0_{-,1})).$$

The commutative diagram in Lemma 2.11 implies

$$\ker(\psi^1_{-,2} - \psi^1_{+,2}) \supset \mathrm{Im}((\psi^0_{+,1}, \psi^0_{-,1})).$$

We then prove

$$\ker(\psi^1_{-,2} - \psi^1_{+,2}) \subset \mathrm{Im}((\psi^0_{+,1}, \psi^0_{-,1})).$$

Suppose

$$(x_1,x_2)\in\ker(\psi^1_{-,2}-\psi^1_{+,2}),\, i.e., \psi^1_{-,2}(x_1)-\psi^1_{+,2}(x_2)=0.$$

Then we have

$$\psi^1_{+,\mu}(x_1)=\psi^2_{+,\mu}\circ\psi^1_{-,2}(x_1)=\psi^2_{+,\mu}\circ\psi^1_{+,2}(x_2)=0.$$

By exactness, there exists $y\in\mathbf{\Gamma}_0$ such that $\psi^0_{+,1}(y)=x_1$. Then

$$\psi^1_{+,2}\circ\psi^0_{-,1}(y)=\psi^1_{-,2}\circ\psi^0_{+,1}(y)=\psi^1_{-,2}(x_1) \text{ and } \psi^1_{+,2}(x_2-\psi^0_{-,1}(y))=0.$$

By exactness, there exists $z\in\mathbf{\Gamma}_\mu$ such that

$$\psi^\mu_{+,1}(z)=x_2-\psi^0_{-,1}(y).$$

Let $y'=y+\psi^\mu_{+,0}(z)$. Then

$$\psi^0_{+,1}(y')=\psi^0_{+,1}(y)=x_1$$

and

$$\psi^0_{-,1}(y')=\psi^0_{-,1}(y)+\psi^0_{-,1}\circ\psi^\mu_{+,0}(z)=\psi^0_{-,1}(y)+\psi^\mu_{+,1}(z)=x_2,$$

which concludes the proof for $k_0=1$.

Suppose the proposition holds for $k_0=k$. We prove it also holds for $k_0=k+1$. The proof is similar to the case for $k_0=1$. Again by Lemma 2.11, we have

$$\ker(\Psi^{k+1}_{-,2k+2}-\Psi^{k+1}_{+,2k+2})\supset\operatorname{Im}((\Psi^0_{+,k+1},\Psi^0_{-,k+1})).$$

Then we prove

$$\ker(\Psi^{k+1}_{-,2k+2}-\Psi^{k+1}_{+,2k+2})\subset\operatorname{Im}((\Psi^0_{+,k+1},\Psi^0_{-,k+1})).$$

Suppose

$$(x_1,x_2)\in\ker(\Psi^{k+1}_{-,2k+2}-\Psi^{k+1}_{+,2k+2}),\, i.e., \Psi^{k+1}_{-,2k+2}(x_1)-\Psi^{k+1}_{+,2k+2}(x_2)=0.$$

Then we have

$$\psi^{k+1}_{+,\mu}(x_1)=\psi^{2k+2}_{+,\mu}\circ\Psi^{k+1}_{-,2k+2}(x_1)=\psi^{2k+2}_{+,\mu}\circ\Psi^{k+1}_{+,2k+2}(x_2)=0.$$

By exactness, there exists $y_1\in\mathbf{\Gamma}_k$ such that $\psi^k_{+,k+1}(y_1)=x_1$. By a similar reason, there exists $y_2\in\mathbf{\Gamma}_k$ such that $\psi^k_{-,k+1}(y_2)=x_2$. The goal is to prove

$$\Psi^k_{-,2k}(y_1')=\Psi^k_{+,2k}(y_2')$$

for some modifications y_1' and y_2' of y_1 and y_2 as for y' in the case of $k_0=1$. Then the induction hypothesis will imply that there exists $w\in\mathbf{\Gamma}_0$ such that

$$\Psi^0_{+,k}(w)=y_1' \text{ and } \Psi^0_{-,k}(w)=y_2'.$$

Hence we will have

$$\Psi^0_{+,k+1}(w)=\psi^k_{+,k+1}(y_1')=x_1 \text{ and } \Psi^0_{-,k+1}(w)=\psi^k_{-,k+1}(y_2')=x_2.$$

This will conclude the proof for $k_0=k+1$.

Now we start to construct y_1'. We have

$$\begin{aligned}\psi^{2k+1}_{+,2k+2}(\Psi^{k+1}_{+,2k+1}(x_2)-\Psi^k_{-,2k+1}(y_1)) &= \psi^{2k+1}_{+,2k+2}\circ\Psi^{k+1}_{+,2k+1}(x_2)-\psi^{2k+1}_{+,2k+2}\circ\Psi^k_{-,2k+1}(y_1)\\ &= \Psi^{k+1}_{+,2k+2}(x_2)-\psi^{2k+1}_{+,2k+2}\circ\Psi^k_{-,2k+1}(y_1)\\ &= \Psi^{k+1}_{-,2k+2}(x_2)-\Psi^{k+1}_{+,2k+2}(x_1)\\ &= 0.\end{aligned}$$

By exactness, there exists $z_1 \in \mathbf{\Gamma}_\mu$ such that

$$\psi^{\mu}_{+,2k+1}(z_1) = \Psi^{k+1}_{+,2k+1}(x_2) - \Psi^{k}_{-,2k+1}(y_1).$$

Let $y_1' = y_1 + \psi^{\mu}_{+,k}(z_1)$. Then

$$\psi^{k}_{+,k+1}(y_1') = \psi^{k}_{+,k+1}(y_1) = x_1$$

and

$$\begin{aligned}\Psi^{k}_{-,2k+1}(y_1') &= \Psi^{k}_{-,2k+1}(y_1) + \Psi^{k}_{-,2k+1} \circ \psi^{\mu}_{+,k}(z_1)\\ &= \Psi^{k}_{-,2k+1}(y_1) + \psi^{\mu}_{+,2k+1}(z_1)\\ &= \Psi^{k+1}_{+,2k+1}(x_2),\end{aligned}$$

Then we start to construct y_2'. We have

$$\begin{aligned}\psi^{2k}_{-,2k+1}(\Psi^{k}_{-,2k}(y_1') - \Psi^{k}_{+,2k}(y_2)) &= \Psi^{k}_{-,2k+1}(y_1') - \psi^{2k}_{-,2k+1} \circ \Psi^{k}_{+,2k}(y_2)\\ &= \Psi^{k}_{-,2k+1}(y_1') - \Psi^{k+1}_{-,2k+1}(x_2)\\ &= 0.\end{aligned}$$

By exactness, there exists $z_2 \in \mathbf{\Gamma}_\mu$ such that

$$\psi^{\mu}_{-,2k}(z_2) = \Psi^{k}_{-,2k}(y_1') - \Psi^{k}_{+,2k}(y_2).$$

Let $y_2' = y_2 + \psi^{\mu}_{-,k}(z_2)$. Then

$$\psi^{k}_{-,k+1}(y_2') = \psi^{k}_{-,k+1}(y_2) = x_2$$

and

$$\begin{aligned}\Psi^{k}_{+,2k}(y_2') &= \Psi^{k}_{+,2k}(y_2) + \Psi^{k}_{+,2k} \circ \psi^{\mu}_{-,k}(z_2)\\ &= \Psi^{k}_{+,2k}(y_2) + \psi^{\mu}_{-,2k}(z_2)\\ &= \Psi^{k}_{-,2k}(y_1'),\end{aligned}$$

Then we have the following commutative diagrams

$$\begin{tikzcd}
 & & x_1 \in \mathbf{\Gamma}_{k+1} \arrow[ddrrr, "-"] & & & \\
 & y_1' \in \mathbf{\Gamma}_k \arrow[ur, "+"] \arrow[drr, "-"] & & & & \\
\mathbf{\Gamma}_0 \arrow[ur, "+"] \arrow[ddr, "-"'] & & & \mathbf{\Gamma}_{2k} \arrow[dr, "-"] & & \mathbf{\Gamma}_{2k+2} \\
 & & & & \mathbf{\Gamma}_{2k+1} \arrow[ur, "+"] & \\
 & y_2' \in \mathbf{\Gamma}_k \arrow[uurr, "+"] \arrow[dr, "-"] & & & & \\
 & & x_2 \in \mathbf{\Gamma}_{k+1} \arrow[uurr, "+"] & & &
\end{tikzcd}$$

By the induction hypothesis, there exists $w \in \mathbf{\Gamma}_0$ such that

$$\Psi^0_{+,k}(w) = y_1' \text{ and } \Psi^0_{-,k}(w) = y_2',$$

which concludes the proof for $k_0 = k + 1$. $\square$

Remark 5.2. By similar arguments, we can prove that the following sequence is exact for any $k_1, k_2 \in \mathbb{N}_+$

$$\mathbf{\Gamma}_n \xrightarrow{(c_1\Psi^n_{+,n+k_1}, c_2\Psi^n_{-,n+k_2})} \mathbf{\Gamma}_{n+k_1} \oplus \mathbf{\Gamma}_{n+k_2} \xrightarrow{c_3\Psi^{n+k_1}_{-,n+k_1+k_2} + c_4\Psi^{n+k_2}_{+,n+k_1+k_2}} \mathbf{\Gamma}_{n+k_1+k_2},$$

where the scalars satisfies the equality $c_1c_3 = -c_2c_4$.

5.2. **The second exact triangle.** In this subsection, we prove Proposition 3.6 by diagram chasing. For convenience, we restate it as follows, which is a little stronger than the previous version. Replacing the original knot in the proposition by the dual knot in the Dehn filling of slope $-(m+k)\mu + \lambda$ with framing $-\mu$ and setting $n = -1$ will recover Proposition 3.6.

Proposition 5.3. *Suppose*

$$l' = \psi^{\mu}_{+,n-1} \circ \psi^{n+1}_{+,\mu} = \psi^{\mu}_{-,n-1} \circ \psi^{n+1}_{-,\mu}.$$

Then for any $c_1, c_2, c_3, c_4 \in \mathbb{C}\backslash\{0\}$*, the following sequence is exact*

$$\mathbf{\Gamma}_n \oplus \mathbf{\Gamma}_n \xrightarrow{c_3\psi^n_{-,n+1} + c_4\psi^n_{+,n+1}} \mathbf{\Gamma}_{n+1} \xrightarrow{l'} \mathbf{\Gamma}_{n-1} \xrightarrow{(c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})} \mathbf{\Gamma}_n \oplus \mathbf{\Gamma}_n.$$

Proof. We adopt the conventions from Section 2.3. We will use Lemma 2.6, Lemma 2.11 and Lemma 2.12 without mentioning them. We prove the exactness at $\mathbf{\Gamma}_{n-1}$ first. We have

$$\psi^{n-1}_{\pm,n} \circ l' = \psi^{n-1}_{\pm,n} \circ \psi^{\mu}_{\pm,n-1} \circ \psi^{n+1}_{\pm,\mu} = 0.$$

Hence
$$\ker((c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})) \supset \operatorname{Im}(l').$$
Then we prove
$$\ker((c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})) \subset \operatorname{Im}(l').$$
Suppose
$$x \in \ker((c_1\psi^{n-1}_{-,n}, c_2\psi^{n-1}_{+,n})) = \ker(\psi^{n-1}_{-,n}) \cap \ker(\psi^{n-1}_{+,n}).$$
By exactness, there exists $y \in \mathbf{\Gamma}_\mu$ such that $\psi^{\mu}_{+,n-1}(y) = x$. Then we have
$$\psi^{\mu}_{+,n}(y) = \psi^{n-1}_{-,n} \circ \psi^{\mu}_{+,n-1}(y) = \psi^{n-1}_{-,n}(x) = 0.$$
By exactness, there exists $z \in \mathbf{\Gamma}_{n+1}$ such that $\psi^{n+1}_{+,\mu}(z) = y$. Thus, we have $l'(z) = x$, which concludes the proof for the exactness at $\mathbf{\Gamma}_{n-1}$.

Then we prove the exactness at $\mathbf{\Gamma}_{n+1}$. Similarly by exactness, we have
$$\ker(l') \supset \operatorname{Im}(c_3\psi^{n}_{-,n+1} + c_4\psi^{n}_{+,n+1}) = \operatorname{Im}(\psi^{n}_{-,n+1}) + \operatorname{Im}(\psi^{n}_{+,n+1}).$$
Suppose $x \in \ker(l')$. If $\psi^{n+1}_{+,\mu}(x) = 0$, then by the exactness, we know $x \in \operatorname{Im}(\psi^{n}_{+,n+1})$. If $\psi^{n+1}_{+,\mu}(x) \neq 0$, then by the exactness, there exists $y \in \mathbf{\Gamma}_n$ such that
$$\psi^{n}_{+,\mu}(y) = \psi^{n+1}_{+,\mu}(x).$$
Then we know
$$x - \psi^{n}_{-,n+1}(y) \in \ker(\psi^{n+1}_{+,\mu}) = \operatorname{Im}(\psi^{n}_{+,n+1}).$$
Thus, we have
$$x \in \operatorname{Im}(\psi^{n}_{-,n+1}) + \operatorname{Im}(\psi^{n}_{+,n+1}),$$
which concludes the proof for the exactness at $\mathbf{\Gamma}_{n+1}$. □

6. Some technical constructions

6.1. **Filtrations.** In this subsection, we study some filtrations on $\mathbf{Y}$ and $\mathbf{\Gamma}_\mu$ that will be important in later sections. We continue to adopt conventions from Section 2.3. In particular, $K \subset Y$ is a rationally null-homologous knot and S is a rational Seifert surface of K.

Lemma 6.1. *The maps G_n in Lemma 2.16 lead to a filtration on $\mathbf{Y}$: for a sufficiently large integer n_0,*
$$0 = \ker G_{-n_0} \subset \cdots \subset \ker G_n \subset \ker G_{n+1} \subset \cdots \subset \ker G_{n_0} = \mathbf{Y}.$$

Proof. It follows from Lemma 2.19 that when n_0 is sufficiently large we have
$$0 = \ker G_{-n_0} \text{ and } \ker G_{n_0} = \mathbf{Y}.$$
It follows from Lemma 2.18 that for any $n \in \mathbb{Z}$,
$$\ker G_n \subset \ker G_{n+1}.$$
□

Lemma 6.2. *For any $n \in \mathbb{Z}$, the map G_n induces an isomorphism*
$$G_n : \left(\ker G_{n+1}/\ker G_n\right) \xrightarrow{\cong} \ker \psi^{n}_{+,n+1} \cap \ker \psi^{n}_{-,n+1}.$$

Proof. Suppose $x \in \ker G_{n+1}$. Then from Lemma 2.18 we know that

$$\psi^n_{\pm,n+1} \circ G_n(x) = G_{n+1}(x) = 0.$$

Hence we have

$$G_n(\ker G_{n+1})) \subset \ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}.$$

Clearly G_n is injective on $\ker G_{n+1}/\ker G_n$ so it suffices to show that the image is $\ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}$. To achieve this, for any element $x \in \ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}$, Lemma 2.20 implies that

$$x \in \ker H_n = \operatorname{Im} G_n.$$

As a result, there exists $\alpha \in \mathbf{Y}$ such that

$$x = G_n(\alpha).$$

Again from Lemma 2.18 we know that

$$G_{n+1}(\alpha) = \psi^{n+1}_{+,n} \circ G_n(\alpha) = \psi^{n+1}_{+,n}(x) = 0.$$

This implies that $\alpha \in \ker G_{n+1}$. $\square$

Lemma 6.3. *For any $n \in \mathbb{Z}$, the maps $\psi^\mu_{\pm,n}$ induce isomorphisms*

$$\psi^\mu_{+,n} : \left(\operatorname{Im} \psi^{n+2}_{+,\mu} / \operatorname{Im} \psi^{n+1}_{+,\mu}\right) \xrightarrow{\cong} \ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}$$

$$\psi^\mu_{-,n} : \left(\operatorname{Im} \psi^{n+2}_{-,\mu} / \operatorname{Im} \psi^{n+1}_{-,\mu}\right) \xrightarrow{\cong} \ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}$$

Proof. We only prove the lemma for positive bypasses. The proof for the negative bypasses is similar. Let $u \in \operatorname{Im} \psi^{n+2}_{+,\mu}$. By Lemma 2.6 and Lemma 2.12, we have

$$\psi^n_{+,n+1} \circ \psi^\mu_{+,n}(u) = 0 \text{ and } \psi^n_{-,n+1} \circ \psi^\mu_{+,n}(u) = \psi^\mu_{+,n+1}(u) = 0.$$

Hence we know

$$\psi^\mu_{+,n}(\operatorname{Im} \psi^{n+2}_{+,\mu}) \subset \ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}.$$

Since $\ker \psi^\mu_{+,n} = \operatorname{Im} \psi^{n+1}_{+,\mu}$, the map $\psi^\mu_{+,n}$ is injective on $\operatorname{Im} \psi^{n+2}_{+,\mu} / \operatorname{Im} \psi^{n+1}_{+,\mu}$. To show it is surjective as well, pick $x \in \ker \psi^n_{+,n+1} \cap \ker \psi^n_{-,n+1}$. Note $x \in \ker \psi^n_{+,n+1} = \operatorname{Im} \psi^\mu_{+,n}$ implies that there exists $u \in \mathbf{\Gamma}_\mu$ such that $\psi^\mu_{+,n}(u) = x$. Lemma 2.12 then implies that

$$\psi^\mu_{+,n+1}(u) = \psi^n_{-,n+1} \circ \psi^\mu_{+,n}(u) = \psi^n_{-,n+1}(x) = 0.$$

As a result, $u \in \ker \psi^\mu_{+,n+1} = \operatorname{Im} \psi^{n+2}_{+,\mu}$. $\square$

Corollary 6.4. *(1) For any $n \in \mathbb{Z}$, there is a canonical isomorphism*

$$\left(\ker G_{n+1} / \ker G_n\right) \cong \left(\operatorname{Im} \psi^{n+2}_{+,\mu} / \operatorname{Im} \psi^{n+1}_{+,\mu}\right) \cong \left(\operatorname{Im} \psi^{n+2}_{-,\mu} / \operatorname{Im} \psi^{n+1}_{-,\mu}\right).$$

(2) For sufficiently large n_0, there exists a (noncanonical) isomorphism

$$\mathbf{Y} \cong \left(\operatorname{Im} \psi^{n_0}_{+,\mu} / \operatorname{Im} \psi^{-n_0}_{+,\mu}\right) \cong \left(\operatorname{Im} \psi^{n_0}_{-,\mu} / \operatorname{Im} \psi^{-n_0}_{-,\mu}\right)$$

Definition 6.5. For any integer $n \in \mathbb{Z}$ and any grading i, define the map F^i_n as the restriction

$$F^i_n = F_n|(\mathbf{\Gamma}_n, i).$$

where F_n is the map from Lemma 2.16.

Lemma 6.6. *Suppose $n_0 \in \mathbb{Z}$ is small enough such that $F_{n_0} = 0$ (c.f. Lemma 2.19). Then for any integer $n \geqslant n_0$ and any grading i, we have*

$$\psi^n_{\pm,\mu}(\ker F^i_n) = \operatorname{Im}\left(\operatorname{Proj}_\mu^{i \mp \frac{(n-1)p-q}{2}} \circ \psi^{n_0}_{\pm,\mu}\right),$$

where

$$\operatorname{Proj}_\mu^{i \mp \frac{(n-1)p-q}{2}} : \mathbf{\Gamma}_\mu \to (\mathbf{\Gamma}_\mu, i \mp \frac{(n-1)p-q}{2})$$

is the projection.

Proof. We only prove the lemma for positive bypasses and the proof for negative bypasses is similar. First, suppose

$$u \in \operatorname{Im}\left(\operatorname{Proj}_\mu^{i - \frac{(n-1)p-q}{2}} \circ \psi^{n_0}_{+,\mu}\right) = \operatorname{Im}\psi^{n_0}_{+,\mu} \cap (\mathbf{\Gamma}_\mu, i - \frac{(n-1)p-q}{2}).$$

Pick $x \in (\mathbf{\Gamma}_{n_0}, i - \frac{(n-n_0)p}{2})$ such that

$$\psi^{n_0}_{+,\mu}(x) = u.$$

Taking $y = \Psi^{n_0}_{-,n}(x)$, we know from Lemma 2.6 that $y \in (\mathbf{\Gamma}_n, i)$, from Lemma 2.18 that $F_n(y) = F_{n_0}(x) = 0$, and from Lemma 2.12 that $\psi^n_{+,\mu}(y) = u$. As a result, we conclude $u \in \psi^n_{+,\mu}(\ker F^i_n)$.

Second, suppose $u \in \psi^n_{+,\mu}(\ker F^i_n)$ is non-zero. Pick $x_1 \in \ker F^i_n$ such that

$$\psi^n_{+,\mu}(x_1) = u.$$

By Lemma 2.5 and Lemma 2.6, the fact that $\psi^n_{+,\mu}(x_1) = u \neq 0$ implies that

$$-\frac{p-\chi(S)}{2} \leqslant i - \frac{(n-1)p-q}{2} \leqslant \frac{p-\chi(S)}{2}. \tag{6.1}$$

Pick a sufficiently large integer k and then take

$$x_2 = \Psi^n_{-,n+k}(x_1) \text{ and } x_3 = \Psi^{n+k}_{+,2n+2k-n_0}(x_2).$$

By Lemma 2.18 we have

$$F_{2n+2k-n_0}(x_3) = F_{n+k}(x_2) = F_n(x_1) = 0.$$

Note that the grading j of x_3 equals to

$$j = i + \frac{kp}{2} - \frac{(n+k-n_0)p}{2} = i - \frac{(n-n_0)p}{2}. \tag{6.2}$$

Combining 6.1 and 6.2, we obtain

$$\frac{(n_0-2)p-q-\chi(S)}{2} \leqslant j \leqslant \frac{n_0p-q-\chi(S)}{2}.$$

Note that we pick k to be a sufficiently large integer. In particular, we can assume

$$\frac{-(2n+2k-n_0)p+q+\chi(S)}{2} + 1 \leqslant \frac{(n_0-2)p-q-\chi(S)}{2}$$

and

$$\frac{n_0p-q-\chi(S)}{2} \leqslant \frac{(2n+2k-n_0)p-q+\chi(S)}{2}.$$

Thus j is in this range as well and then Lemma 2.19 implies that $F_{2n+2k-n_0}$ is injective on the grading j. Hence $x_3 = 0$. Then the following Lemma 6.7 applies to $(x, y) = (x_2, 0)$ and there exists $x_4 \in \mathbf{\Gamma}_{n_0}$ such that

$$\Psi^{n_0}_{-,n+k}(x_4) = x_2.$$

Thus by Lemma 2.12,

$$u = \psi^n_{+,\mu}(x_1) = \psi^{n+k}_{+,\mu}(x_2) = \psi^{n_0}_{+,\mu}(x_4) \in \operatorname{Im}\left(\operatorname{Proj}_\mu^{i-\frac{(n-1)p-q}{2}} \circ \psi^{n_0}_{+,\mu}\right).$$

□

Lemma 6.7. *Suppose $n \in \mathbb{Z}$ and $k_1, k_2 \in \mathbb{N}_+$. Suppose $x \in \mathbf{\Gamma}_{n+k_1}, y \in \mathbf{\Gamma}_{n+k_2}$ such that*

$$\Psi^{n+k_1}_{+,n+k_1+k_2}(x) = \Psi^{n+k_2}_{-,n+k_1+k_2}(y)$$

Then there exists $z \in \mathbf{\Gamma}_n$ such that

$$\Psi^n_{-,n+k_1}(z) = x \text{ and } \Psi^n_{+,n+k_2}(z) = y.$$

Proof. This is a restatement of Remark 5.2. The proof is similar to that of Proposition 5.1. □

6.2. **Tau invariants in a general 3-manifold.**

Definition 6.8. An element $\alpha \in \mathbf{Y}$ is called **homogeneous** if there exists an $n \in \mathbb{Z}$ and a grading i such that $\alpha \in \operatorname{Im} F^i_n$. Note that from 2.18 and Corollary 2.9, we know that

$$\alpha \in \operatorname{Im} F^i_n \Rightarrow \alpha \in \operatorname{Im} F^{i\pm\frac{p}{2}}_{n+1}.$$

For a homogeneous element $\alpha \in \mathbf{Y}$, we pick a sufficiently large n_0 and define

$$\tau^+(\alpha) := \max_i \{i \mid \exists x \in (\mathbf{\Gamma}_{n_0}, i),\ F_{n_0}(x) = \alpha\} - \frac{(n_0-1)p-q}{2}$$

$$\tau^-(\alpha) := \min_i \{i \mid \exists x \in (\mathbf{\Gamma}_{n_0}, i),\ F_{n_0}(x) = \alpha\} + \frac{(n_0-1)p-q}{2}$$

$$\tau(\alpha) := 1 + \frac{\tau^-(\alpha) - \tau^+(\alpha) + q}{p} = \frac{\min - \max}{p} + n_0.$$

We will prove the independence of these τ invariants about n_0 later in Lemma 6.12.

Remark 6.9. Here we fix the knot $K \subset Y$ and define the tau invariants for a homogeneous element $\alpha \in I^\sharp(Y)$. The reason why we go in this order is because (1) currently the definition of homogeneous elements depends on the choice of the knot and (2) in this paper we only focus on the Dehn surgeries of a fixed knot.

Remark 6.10. The normalization $\mp\frac{(n_0-1)p-q}{2}$ comes from the grading shifts of $\psi^{n_0}_{\pm,\mu}$ in Lemma 2.6. When K is a knot inside $Y = S^3$, we have that $\tau^\pm(\alpha)$ is equal to the tau invariant $\tau_I(K)$ defined in [GLW19], where α is the unique generator of $I^\sharp(-S^3) \cong \mathbb{C}$ up to a scalar. Then $\tau(\alpha) = 1 - 2\tau_I(K)$.

Lemma 6.11. *We have the following properties.*

(1) Suppose n_1, n_2 are two integers and i_1, i_2 are two gradings such that there exist $x_1 \in (\mathbf{\Gamma}_{n_1}, i_1)$ and $x_2 \in (\mathbf{\Gamma}_{n_2}, i_2)$ with

$$F_{n_1}(x_1) = F_{n_2}(x_2) \neq 0.$$

Then there exists an integer N such that

$$i_2 = i_1 - \frac{(n_2-n_1)p}{2} + Np$$

i.e. when we send x_1 and x_2 into the same $\mathbf{\Gamma}_{n_3}$ with $n_3 > n_1, n_2$ by bypass maps, then the difference of the expected gradings of the images is divisible by p (the grading shifts of the bypass maps $\psi^n_{\pm,n+1}$ are $\mp p/2$).

(2) *Suppose we have an integer n_1, a grading i_1, and an element $x_1 \in (\mathbf{\Gamma}_{n_1}, i_1)$. Then for any integer $n_2 \geqslant n_1$ and grading i_2 such that there exists an integer $N \in [0, n_2 - n_1]$ with*

$$i_2 = i_1 - \frac{(n_2 - n_1)p}{2} + Np,$$

there exists an element $x_2 \in (\mathbf{\Gamma}_{n_2}, i_2)$ such that

$$F_{n_1}(x_1) = F_{n_2}(x_2).$$

(3) *Suppose $n \in \mathbb{Z}$ and for $1 \leqslant j \leqslant l$ we have a grading i_j and an element $x_j \in (\mathbf{\Gamma}_n, i_j)$ such that $F_n(x_1), \dots, F_n(x_l)$ are linearly independent. Then the element*

$$\alpha = \sum_{j=1}^{l} F_n(x_j)$$

is homogeneous if and only if for any $1 \leqslant j \leqslant l$, we have

$$i_j \equiv i_1 \pmod p$$

Proof. (1). Take n_0 a sufficiently large integer. For $j = 1, 2$, take $i'_j \in (-\frac{p}{2}, \frac{p}{2}]$ to be the unique grading such that there exists an integer N_j with

$$i'_j = i_j - \frac{(n_0 - n_j)p}{2} + N_j p.$$

Take

$$x'_j = \Psi^{n_j + N_j}_{+, n_0} \circ \Psi^{n_j}_{-, n_j + N_j}(x_j).$$

From Lemma 2.18 we know that

$$x'_j \in (\mathbf{\Gamma}_{n_0}, i'_j) \text{ and } F_{n_0}(x'_1) = F_{n_1}(x_1) = F_{n_2}(x_2) = F_{n_0}(x'_2).$$

By Lemma 2.19, we know that $x'_1 = x'_2$ and in particular, $i'_1 = i'_2$. As a result, we can take $N = N_1 - N_2$ then it is straightforward to verify that

$$i_2 = i_1 - \frac{(n_2 - n_1)p}{2} + Np.$$

(2). We can take

$$x_2 = \Psi^{n_1 + N}_{-, n_2} \circ \Psi^{n_1}_{+, n_1 + N}(x_1)$$

Then it follows from Lemma 2.6 that $x_2 \in (\mathbf{\Gamma}_{n_2}, i_2)$ and follows from Lemma 2.18 that

$$F_{n_2}(x_2) = F_{n_1}(x_1).$$

(3). The proof is similar to that of (1). ☐

Lemma 6.12. *For a homogeneous element α, we have the following.*

(1) *$\tau^{\pm}(\alpha)$ and hence $\tau(\alpha)$ are well-defined. (i.e. they are independent of the choice of the large integer n_0.)*
(2) *We have $\tau(\alpha) \in \mathbb{Z}$.*
(3) *For any integer n and grading i, the following two statements are equivalent.*
   (a) *There exists $x \in (\mathbf{\Gamma}_n, i)$ such that $F_n(x) = \alpha$.*
   (b) *We have $n \geqslant \tau(\alpha)$ and there exists $N \in \mathbb{Z}$ such that $N \in [0, n - \tau(\alpha)]$ and*

$$i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (n - \tau(\alpha))p}{2} + Np.$$

(4) We have
$$\tau^+(\alpha) \geqslant -\frac{p-\chi(S)}{2} \text{ and } \tau^-(\alpha) \leqslant \frac{p-\chi(S)}{2}.$$

Proof. (1). Suppose α is a homogeneous element. Then by definition there exists $x \in (\mathbf{\Gamma}_n, i)$ for some integer n and grading i such that
$$F_n(x) = \alpha.$$
Then for sufficiently large n_0, we can take
$$y = \psi^n_{+,n_0}(x)$$
and from Lemma 2.18 implies that
$$F_{n_0}(y) = \alpha$$
and hence $\tau^{\pm}(\alpha)$ exists.

To show the value of $\tau^{\pm}(\alpha)$ is independent of n_0 as long as it is sufficiently large, a combination of Lemma 2.5 and Lemma 2.6 implies that the map
$$\psi^{n_0}_{-,n_0+1} : (\mathbf{\Gamma}_{n_0}, i) \to (\mathbf{\Gamma}_{n_0+1}, i + \frac{p}{2})$$
is an isomorphism for any $i > g - \frac{n_0p-q-1}{2}$. Then Lemma 2.18 implies that τ^+ is well-defined. The argument for τ^- is similar.

(2). It follows directly from Lemma 6.11 part (1).

(3). We first establish the following claim.

Claim. There exists an element
$$z \in (\mathbf{\Gamma}_{\tau(\alpha)}, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$$
such that
$$F_{\tau(\alpha)}(z) = \alpha.$$

Proof of the claim. Suppose $n_0 \in \mathbb{Z}$ is sufficiently large and
$$x_{\pm} \in (\mathbf{\Gamma}_{n_0}, \tau^{\pm}(\alpha) \pm \frac{(n_0-1)p-q}{2})$$
such that $F_{n_0}(x_{\pm}) = \alpha$. Note that the existence of $x_{\pm}$ follows from the definition of $\tau^{\pm}(\alpha)$. Let
$$x'_{\pm} = \Psi^n_{\pm,2n_0-\tau(\alpha)}(x_{\pm}).$$
It follows from Lemma 2.6 that
$$x'_{\pm} \in (\mathbf{\Gamma}_{2n_0-\tau(\alpha)}, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2}).$$
From Lemma 2.18 we know that
$$F_{2n_0-\tau(\alpha)}(x'_+) = \alpha = F_{2n_0-\tau(\alpha)}(x'_-).$$
By Lemma 2.19 this implies that
$$x'_+ = x'_-.$$
Hence Lemma 6.7 applies and there exists $z \in \mathbf{\Gamma}_{\tau(\alpha)}$ such that
$$\Psi^{\tau(\alpha)}_{\mp,n}(z) = x_{\pm}.$$

Again Lemma 2.6 implies that z is in the grading

$$z \in (\mathbf{\Gamma}_{\tau(\alpha)}, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$$

and Lemma 2.18 implies

$$F_{\tau(\alpha)}(z) = \alpha.$$

$\square$

Now if an integer n and a grading i satisfy statement (b), then (a) is a direct consequence of the above claim and Lemma 6.11 part (2).

It remains to show that (a) implies (b). Suppose there exists $x \in (\mathbf{\Gamma}_n, i)$ such that $F_n(x) = \alpha$. From the above claim, we already know that there exists

$$z \in (\mathbf{\Gamma}_{\tau(\alpha)}, \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$$

such that

$$F_{\tau(\alpha)}(z) = \alpha$$

Hence Lemma 6.11 part (1) implies that there exists $N \in \mathbb{Z}$ such that

$$i = \frac{\tau^+(\alpha)+\tau^-(\alpha)-(n-\tau(\alpha))p}{2} + Np.$$

If $N > n - \tau(\alpha)$, we can take a sufficiently large n_0 and

$$x' = \Psi^n_{-,n_0}(x).$$

It follows from Lemma 2.6 that

$$x' \in (\mathbf{\Gamma}_{n_0}, i') \text{ with } i' > \tau^+(\alpha) + \frac{(n_0-1)p-q}{2}.$$

Then Lemma 2.18 implies that

$$F_{n_0}(x') = \alpha$$

which contradicts the definition of τ^+ in Definition 6.8. Similarly if $N < 0$ we can take

$$x' = \Psi^n_{+,n_0}(x)$$

which would be an element contradicting the definition of τ^-. When $n < \tau(\alpha)$ we have $n - \tau(\alpha) < 0$ so there is always a contradiction by the above argument. This concludes (b).

(4). It follows from the definition of $\tau^{\pm}$ and Lemma 2.19 that F_{n_0} is an isomorphism when restricted to the direct sum of p consecutive middle gradings of $\mathbf{\Gamma}_{n_0}$ when n_0 is large. $\square$

Lemma 6.13. *For any $n \in \mathbb{Z}$ we have that*

$$\operatorname{Im} F_n = \operatorname{Span}\{\alpha \in \mathbf{Y} \mid \alpha \text{ homogeneous and } \tau(\alpha) \leqslant n\}$$

Proof. Suppose $\alpha \in \operatorname{Im} F_n$. Let

$$\alpha = \sum_i \alpha_i \text{ where } \alpha_i \in \operatorname{Im} F_n^i \text{ is homogeneous.}$$

From Lemma 6.12 we know that $\tau(\alpha_i) \leqslant n$ for all i. On the other hand, suppose

$$\alpha = \sum_i \alpha_i \text{ where } \tau(\alpha_i) \leqslant n \text{ for all } i.$$

By Lemma 6.12 part (3) we can pick $z_i \in \mathbf{\Gamma}_{\tau(\alpha_i)}$ such that

$$F_{\tau(\alpha_i)}(z_i) = \alpha_i.$$

Then from Lemma 2.18 we know

$$\alpha = F_n(\sum_i \Psi_{+,n}^{\tau(\alpha_i)}(z_i)).$$

□

6.3. **A basis for framed instanton homology.** We pick a basis $\mathfrak{B}$ for $\mathbf{Y}$ as follows. First

$$\mathfrak{B} = \bigcup_{n\in\mathbb{Z}} \mathfrak{B}_n.$$

To construct the set $\mathfrak{B}_n$, first, let $\mathfrak{B}_n = \varnothing$ if $F_n = 0$. By Lemma 2.19 this means $\mathfrak{B}_n = \varnothing$ for all small enough n. Write

$$\mathfrak{B}_{\leqslant n} = \bigcup_{k\leqslant n} \mathfrak{B}_k.$$

We pick the set $\mathfrak{B}_n$ inductively. Note that we have taken $\mathfrak{B}_n = \varnothing$ for n with $F_n = 0$. Suppose we have already constructed the set $\mathfrak{B}_{\leqslant n-1}$ that consists of homogeneous elements and is a basis of $\operatorname{Im} F_{n-1}$, we pick the set $\mathfrak{B}_n$ such that $\mathfrak{B}_n$ consists of homogeneous elements with $\tau = n$, and the set

$$\mathfrak{B}_{\leqslant n} = \mathfrak{B}_{\leqslant n-1} \cup \mathfrak{B}_n$$

forms a basis of $\operatorname{Im} F_n$. Note that Lemma 6.13 implies that $\mathfrak{B}_n$ exists and

$$|\mathfrak{B}_n| = \dim_{\mathbb{C}}\left(\operatorname{Im} F_n / \operatorname{Im} F_{n-1}\right).$$

For any $n, k \in \mathbb{Z}$ such that $k \leqslant n-2$, define maps

$$\eta_{\pm,k}^n : \mathfrak{B}_n \to \mathbf{\Gamma}_k$$

as follows: for any $\alpha \in \mathfrak{B}_n \subset \operatorname{Im} F_n$, since α is homogeneous and $\tau(\alpha) = n$, we can pick

$$z \in (\mathbf{\Gamma}_n, \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2})$$

by Lemma 6.12 part (3) such that $F_n(z) = \alpha$. Then define

$$\eta_{\pm,k}^n(\alpha) = \psi_{\pm,k}^{\mu} \circ \psi_{\pm,\mu}^n(z)$$

.

Lemma 6.14. *Suppose $n, k \in \mathbb{Z}$ such that $k \leqslant n-2$.*

(1) The maps $\eta_{\pm,k}^n$ are all well-defined.
(2) We have $\eta_{+,n-2}^n = c_n \cdot \eta_{-,n-2}^n$. for some scalar $c_n \in \mathbb{C}\backslash\{0\}$.
(3) Elements in $\operatorname{Im} \eta_{\pm,k}^n \subset \mathbf{\Gamma}_k$ are linearly independent.
(4) $\operatorname{Im} \eta_{\pm,n-2}^n$ forms a basis for $\ker \psi_{+,n-1}^{n-2} \cap \ker \psi_{-,n-1}^{n-2}$.
(5) For any $\alpha \in \mathfrak{B}_n$ we have

$$\eta_{\pm,k}^n(\alpha) \in (\mathbf{\Gamma}_k, \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2} \mp \frac{(n-2-k)p}{2}).$$

(6) We have

$$\psi_{\mp,k}^{k-1} \circ \eta_{\pm,k-1}^n = \eta_{\pm,k}^n, \text{ and } \psi_{\pm,k}^{k-1} \circ \eta_{\pm,k-1}^n = 0.$$

Proof. (1). We only work with $\eta^n_{+,k}$ and the arguments for $\eta^n_{-,k}$ are similar. Suppose there are $z_1, z_2 \in (\mathbf{\Gamma}_n, i)$ such that $F_n(z_1) = F_n(z_2) = \alpha$, where $i = \frac{\tau^+(\alpha)+\tau^-(\alpha)}{2}$. Then

$$z_1 - z_2 \in \ker F_n^i$$

and by Lemma 6.6 we have

$$\psi^n_{+,\mu}(z_1 - z_2) \in \psi^n_{+,\mu}(\ker F_n^i) \subset \operatorname{Im} \psi^{n_0}_{+,\mu} \subset \operatorname{Im} \psi^{k+1}_{+,\mu}.$$

Here $n_0 \in \mathbb{Z}$ is a small enough integer. As a result,

$$\eta^n_{+,k}(\alpha) = \psi^\mu_{+,k} \circ \psi^n_{+,\mu}(z_1) = \psi^\mu_{+,k} \circ \psi^n_{+,\mu}(z_2)$$

is well-defined.

(2). This follows directly from Lemma 2.11. Note that in Section 2.3 we do not fix the scalars of the second commutative diagram of Lemma 2.11, and hence a non-zero coefficient c_n would possibly arise.

(3). We only work with $\eta^n_{+,k}$ and the arguments for $\eta^n_{-,k}$ are similar. Suppose

$$\mathfrak{B}_n = \{\alpha_1, \ldots, \alpha_l\}, \text{ where } l = |\mathfrak{B}_n| = \dim_{\mathbb{C}}\bigg(\operatorname{Im} F_n / \operatorname{Im} F_{n-1}\bigg).$$

Suppose there exists $\lambda_1, \ldots, \lambda_l$ such that

$$\sum_{j=1}^{l} \lambda_j \cdot \eta^n_{+,k}(\alpha_j) = 0.$$

Pick $z_j \in (\mathbf{\Gamma}_n, \frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)}{2})$ such that $F_n(z_j) = \alpha_j$. Then we have

$$\psi^\mu_{+,k} \circ \psi^n_{+,\mu}\bigg(\sum_{j=1}^{l} \lambda_j z_j\bigg) = 0.$$

As a result, there exists $x \in \mathbf{\Gamma}_{k+1}$ such that

$$\psi^{k+1}_{+,\mu}(x) = \psi^n_{+,\mu}\bigg(\sum_{j=1}^{l} \lambda_j z_j\bigg).$$

Note that, from Lemma 2.12, we know

$$\psi^n_{+,\mu} \circ \Psi^{k+1}_{-,n}(x) = \psi^{k+1}_{+,\mu}(x) = \psi^n_{+,\mu}\bigg(\sum_{j=1}^{l} \lambda_j z_j\bigg)$$

so as a result there exists $y \in \mathbf{\Gamma}_{n-1}$ such that

$$\sum_{j=1}^{l} \lambda_j z_j = \Psi^{k+1}_{-,n}(x) + \psi^{n-1}_{+,n}(y).$$

Hence by Lemma 2.18 we have

$$\begin{aligned}\sum_{j=1}^{l}\lambda_j\alpha_j &= F_n(\sum_{j=1}^{l}\lambda_j z_j)\\ &= F_n\circ\Psi_{-,n}^{k+1}(x)+F_n\circ\psi_{+,n}^{n-1}(y)\\ &= F_{k+1}(x)+F_{n-1}(y)\\ &\subset \operatorname{Im}F_{n-1}.\end{aligned}$$

Since α_j form a basis of $\mathfrak{B}_n$, the sum cannot be in $\operatorname{Im}F_{n-1}$ except $\lambda_i=0$ for all i.

(4). For $\alpha\in\mathfrak{B}_n$, pick $z\in(\mathbf{\Gamma}_n,\frac{\tau^+(\alpha)+\tau^-(\alpha)}{2})$ such that $F_n(z)=\alpha$. Then by definition

$$\eta_{+,n-2}^n(\alpha)=\psi_{+,n-2}^{\mu}\circ\psi_{+,\mu}^n(z).$$

Now we can compute

$$\psi_{+,n-1}^{n-2}\circ\eta_{+,n-2}^n(\alpha)=\psi_{+,n-1}^{n-2}\circ\psi_{+,n-2}^{\mu}\circ\psi_{+,\mu}^n(z)=0,$$

and by Lemma 2.12

$$\psi_{-,n-1}^{n-2}\circ\eta_{+,n-2}^n(\alpha)=\psi_{-,n-1}^{n-2}\circ\psi_{+,n-2}^{\mu}\circ\psi_{+,\mu}^n(z)=\psi_{+,n-1}^{\mu}\circ\psi_{+,\mu}^n(z)=0.$$

Hence

$$\eta_{+,n-2}^n(\alpha)\in\ker\psi_{+,n-1}^{n-2}\cap\ker\psi_{-,n-1}^{n-2}.$$

Then (4) follows from (3), Lemma 6.2, and $\operatorname{Im}F_n=\ker G_{n-1}$.

(5). It follows directly from the construction of $\eta_{\pm,k}^n$ and Lemma 2.6.

(6). It follows from the construction of $\eta_{\pm,k}^n$, the commutativity in Lemma 2.12 and the exactness in Lemma 2.6. □

Convention. We can define

$$\tilde{\eta}_{+,k}^n=\eta_{+,k}^n \text{ and } \tilde{\eta}_{-,k}^n=c_n\cdot\eta_{-,k}^n$$

such that

$$\tilde{\eta}_{+,n-2}^n=\tilde{\eta}_{-,n-2}^n$$

and the new maps satisfy all properties in Lemma 6.14 except (2). We will use $\eta_{+,k}^n$ to denote $\tilde{\eta}_{+,k}^n$ in latter sections.

7. The map in the third exact triangle

In this section, we construct the map l in Proposition 3.9 and Proposition 3.12 and show it satisfies the exactness and the commutative diagram. We continue to adopt conventions from Section 2.3. We restate the propositions as follows and no longer use the notations l,l' for maps.

Proposition 7.1. *Suppose $n\in\mathbb{Z}$ is fixed and $k\in\mathbb{Z}$ is sufficiently large. Then there is an exact triangle*

$$\begin{tikzcd}\mathbf{\Gamma}_n \arrow[rr, "\Phi_{n+k}^n"] & & \mathbf{\Gamma}_{n+k}\oplus\mathbf{\Gamma}_{n+k} \arrow[dl, "\Phi_{n+2k}^{n+k}"]\\ & \mathbf{\Gamma}_{n+2k}. \arrow[ul, "\Phi_n^{n+2k}"] & \end{tikzcd}$$

where two of the maps are already constructed

$$\Phi_{n+k}^n:=(\Psi_{+,n+k}^n,\Psi_{-,n+k}^n):\mathbf{\Gamma}_n\to\mathbf{\Gamma}_{n+k}\oplus\mathbf{\Gamma}_{n+k}$$

$$\Phi^{n+k}_{n+2k} := \Psi^{n+k}_{-,n+2k} - \Psi^{n+k}_{+,n+2k} : \mathbf{\Gamma}_{n+k} \oplus \mathbf{\Gamma}_{n+k} \to \mathbf{\Gamma}_{n+2k}.$$

Proposition 7.2. *Suppose $n \in \mathbb{Z}$ is fixed and $k \in \mathbb{Z}$ is sufficiently large. Suppose Φ^{n+2k-1}_{n-1} is constructed in Proposition 7.1. Then, there are two commutative diagrams up to scalars.*

$$\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k+1}{2}} \arrow[r, "\psi^{n+k+1}_{+,\mu}\circ\psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"] \arrow[d, "\Psi^{n+k+1}_{+,n+2k}\circ\psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"'] & \mathbf{\Gamma}_{\mu} \arrow[d, "\psi^{\mu}_{+,n}"] \\
\mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_{n}"] & \mathbf{\Gamma}_{n}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{\Gamma}_{\frac{2n+2k+1}{2}} \arrow[r, "\psi^{n+k+1}_{-,\mu}\circ\psi^{\frac{2n+2k+1}{2}}_{+,n+k+1}"] \arrow[d, "\Psi^{n+k+1}_{-,n+2k+1}\circ\psi^{\frac{2n+2k+1}{2}}_{+,n+k+1}"'] & \mathbf{\Gamma}_{\mu} \arrow[d, "\psi^{\mu}_{-,n}"] \\
\mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_{n}"] & \mathbf{\Gamma}_{n}
\end{tikzcd}$$

7.1. **Characterizations of the kernel and the image.** Before constructing Φ^{n+2k}_n, we characterize the spaces $\ker \Phi^n_{n+k}$ and $\operatorname{Im} \Phi^{n+k}_{n+2k}$. These results will motivate the construction of Φ^{n+2k}_n to ensure that

$$\operatorname{Im} \Phi^{n+2k}_n = \ker \Phi^n_{n+k} \text{ and } \ker \Phi^{n+2k}_n = \operatorname{Im} \Phi^{n+k}_{n+2k}.$$

Since Φ^n_{n+k} and Φ^{n+k}_{n+2k} are constructed using bypass maps, it suffices to consider their restrictions on each grading.

Lemma 7.3. *Suppose $n \in \mathbb{Z}$ is fixed and $k \in \mathbb{Z}$ is sufficiently large. Let*

$$\operatorname{Proj}^i_n : \mathbf{\Gamma}_n \to (\mathbf{\Gamma}_n, i)$$

be the projection. Then we have

$$\ker \Phi^n_{n+k} \cap (\mathbf{\Gamma}_n, i) = \operatorname{Im}\left(\operatorname{Proj}^i_n \circ G_n\right).$$

Proof. We need to apply Lemma 2.20. Following conventions in Section 2.3, we have

$$H_n = \psi^n_{+,n+1} - \psi^n_{-,n+1}. \tag{7.1}$$

Suppose $x \in \operatorname{Im}\left(\operatorname{Proj}^i_n \circ G_n\right)$. Pick $\alpha \in \mathbf{Y}$ and $y \in \mathbf{\Gamma}_n$ such that

$$G_n(\alpha) = x + y$$

where $\operatorname{Proj}^i_n(y) = 0$. When k is sufficiently large, we know from Lemma 2.19 that

$$G_{n+k} \equiv 0.$$

In particular, from Lemma 2.18

$$\Psi^n_{\pm,n+k}(x) + \Psi^n_{\pm,n+k}(y) = G_{n+k}(\alpha) = 0.$$

Since the maps $\Psi^n_{\pm,n+k}$ are homogeneous, we know that

$$\Psi^n_{\pm,n+k}(x) = 0,$$

which implies that $x \in \ker \Phi^n_{n+k} \cap (\mathbf{\Gamma}_n, i)$.

Next, suppose $x \in \ker \Phi^n_{n+k} \cap (\mathbf{\Gamma}_n, i)$. We take $x^i_n = x$ and we will pick $x^j_n \in (\mathbf{\Gamma}_n, j)$ for all $j \neq i$ such that

$$\sum_j x^j_n \in \ker H_n = \operatorname{Im} G_n.$$

We will use the notation x_a^b to denote an element in $(\mathbf{\Gamma}_a, b)$. Recall that from Lemma 2.6, the grading shifts of $\psi^n_{\pm,n+1}$ are $\mp\frac{p}{2}$. Take

$$x_{n+k-1}^{i+\frac{(k-1)p}{2}} = \Psi^n_{-,n+k-1}(x) \text{ and } x_{n+k-1}^{i+\frac{(k+1)p}{2}} = 0.$$

Since $x \in \ker \Phi^n_{n+k} \cap (\mathbf{\Gamma}_n, i)$ we know that

$$\psi^{n+k-1}_{-,n+k}(x_{n+k-1}^{i+\frac{(k-1)p}{2}}) = \Psi^n_{-,n+k}(x) = 0 = \psi^{n+k-1}_{+,n+k}(x_{n+k-1}^{i+\frac{(k+1)p}{2}}). \tag{7.2}$$

Hence from Lemma 6.7, there exists

$$x_{n+k-2}^{i+\frac{kp}{2}} \in (\mathbf{\Gamma}_{n+k-2}, i+\frac{kp}{2})$$

such that

$$\psi^{n+k-2}_{-,n+k-1}(x_{n+k-2}^{i+\frac{kp}{2}}) = x_{n+k-1}^{i+\frac{(k+1)p}{2}} = 0 \text{ and } \psi^{n+k-2}_{+,n+k-1}(x_{n+k-2}^{i+\frac{kp}{2}}) = x_{n+k-1}^{i+\frac{(k-1)p}{2}}.$$

Then we can take

$$x_{n+k-2}^{i+\frac{(k+2)p}{2}} = 0 \text{ and } x_{n+k-2}^{i+\frac{(k-2)p}{2}} = \Psi^n_{-,n+k-2}(x).$$

We can apply the same argument and use Lemma 6.7 to find

$$x_{n+k-3}^{i+\frac{(k+3)p}{2}},\ x_{n+k-3}^{i+\frac{(k+1)p}{2}},\ x_{n+k-3}^{i+\frac{(k-1)p}{2}},\ x_{n+k-3}^{i+\frac{(k-3)p}{2}} \in \mathbf{\Gamma}_{n+k-3}$$

such that $\psi^{n+k-3}_{\pm,n+k-2}$ send them to corresponding elements in $\mathbf{\Gamma}_{n+k-2}$. Repeating this argument, we can obtain elements

$$x_n^{i+pj} \in (\mathbf{\Gamma}_n, i+pj) \text{ for } j \in [1,k] \cap \mathbb{Z}$$

such that $x_n^i = x$, $\psi^n_{-,n+1}(x_n^{i+pk}) = 0$, $\psi^n_{+,n+1}(x_n^{i-pk}) = 0$, and for any $j \in [1, k-1] \cap \mathbb{Z}$ we have

$$\psi^n_{-,n+1}(x_n^{i+pj}) = \psi^n_{+,n+1}(x_n^{i+p(j+1)}).$$

Note that we obtain the above x_n^{i+pj} for $j \in [1,k] \cap \mathbb{Z}$ essentially from the fact that $\Psi^n_{-,n+k}(x) = 0$ as in Equation (7.2). However, $x \in \ker\Phi^n_{n+k}$ so we have $\Psi^n_{+,n+k}(x) = 0$ as well. A similar argument as above then yields

$$x_n^{i+pj} \in (\mathbf{\Gamma}_n, i+pj) \text{ for } j \in [-k,-1] \cap \mathbb{Z}.$$

Together with $x_n^i = x$, we obtain x_n^{i+pj} for all $j \in [-k,k] \cap \mathbb{Z}$.

It is then straightforward to check that

$$y = \sum_{j=-k}^{k} x_n^{i+pj} \in \ker(\psi^n_{+,n+1} - \psi^n_{-,n+1}) = \ker H_n = \operatorname{Im} G_n.$$

□

Lemma 7.4. *Suppose* $\alpha \in \mathbf{Y}$ *is a homogeneous element and*

$$\alpha = \sum_{j=1}^{l} \lambda_j \cdot \alpha_j$$

where $\lambda_j \neq 0$ *and* $\alpha_j \in \mathfrak{B}$ *for* $1 \leqslant j \leqslant l$. *Let* n *be an integer,* i *be a grading and* k *be a sufficiently large integer. For an element* $x \in (\mathbf{\Gamma}_{n+2k}, i)$ *such that* $F_{n+2k}(x) = \alpha$, *the following is true.*

(1) We have

$$\tau^+(\alpha) = \min_{1\leqslant j\leqslant l}\{\tau^+(\alpha_j)\} \text{ and } \tau^-(\alpha) = \max_{1\leqslant j\leqslant l}\{\tau^-(\alpha_j)\}.$$

(2) We have $x \in \operatorname{Im}\Phi^{n+k}_{n+2k}$ if and only if for any $1 \leqslant j \leqslant l$, at least one of the following inequalities holds

$$i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p-q}{2} \text{ and } i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p-q}{2}.$$

(3) If $x \notin \operatorname{Im}\Phi^{n+k}_{n+2k}$ then there exists $j, N \in \mathbb{Z}$ such that $1 \leqslant j \leqslant l$, $0 \leqslant N \leqslant \tau(\alpha_j) - n - 2$, and

$$i = \tau^+(\alpha_j) + \frac{(n-1)p-q}{2} + (N+1)p.$$

Proof. (1). We only demonstrate the proof of the result for τ^+ and the proof for τ^- is similar. First, We make the following two claims.

Claim 1. For any homogeneous elements (not necessarily elements in $\mathfrak{B}$) α_1 and α_2 such that $\alpha_1 + \alpha_2$ is also homogeneous, if $\tau^+(\alpha_1) > \tau^+(\alpha_2)$ then $\tau^+(\alpha_1 + \alpha_2) = \tau^+(\alpha_2)$.

To prove Claim 1, let n_0 be sufficiently large. From Lemma 6.11 part (3) we know that

$$\tau^+(\alpha_1) \equiv \tau^+(\alpha_2) \equiv \tau^+(\alpha_1 + \alpha_2) \pmod p. \tag{7.3}$$

Assume $\tau^+(\alpha_1 + \alpha_2) > \tau^+(\alpha_2)$. Let

$$\tau^+ = \min\{\tau^+(\alpha_1), \tau^+(\alpha_1 + \alpha_2)\} > \tau^+(\alpha_2).$$

We claim that there exist

$$x_1, x_3 \in (\mathbf{\Gamma}_{n_0}, \tau^+ + \frac{(n_0-1)p-q}{2})$$

such that

$$F_{n_0}(x_1) = \alpha_1 \text{ and } F_{n_0}(x_3) = \alpha_1 + \alpha_2.$$

We prove only the existence of x_1, and the argument for the existence of x_3 is similar. By Definition 6.8, we know that

$$\tau^+(\alpha_1) + \frac{(n_0-1)p-q}{2} = \frac{\tau^+(\alpha_1) + \tau^-(\alpha_1) - (n_0 - \tau(\alpha_1))p}{2} + (n_0 - \tau(\alpha_1))p.$$

Taking

$$N = (n_0 - \tau(\alpha_1)) - \frac{1}{p}(\tau^+(\alpha_1) - \tau^+),$$

we know that

$$\tau^+ + \frac{(n_0-1)p-q}{2} = \frac{\tau^+(\alpha_1) + \tau^-(\alpha_1) - (n_0 - \tau(\alpha_1))p}{2} + Np. \tag{7.4}$$

Equation (7.3) implies that $N \in \mathbb{Z}$. The definition of τ^+ makes sure that $N \leqslant n_0 - \tau(\alpha_1)$. The fact that n_0 is sufficiently large and Lemma 6.12 part (4) implies that $N \geqslant 0$. Hence Lemma 6.12 part (3) implies the existence of x_1 such that

$$x_1 \in (\mathbf{\Gamma}_{n_0}, \tau^+ + \frac{(n_0-1)p-q}{2}) \text{ and } F_{n_0}(x_1) = \alpha_1.$$

Now the existence of x_1 and x_3 implies that

$$F_n(x_3 - x_1) = \alpha_2$$

which contradicts the definition of $\tau^+(\alpha_2)$.

Claim 2. Suppose $\alpha_1, \ldots, \alpha_u \in \mathfrak{B}$ are pairwise distinct elements in $\mathfrak{B}$ such that

$$\tau^+(\alpha_1) = \tau^+(\alpha_2) = \cdots = \tau^+(\alpha_u) = \tau^+.$$

Suppose

$$\alpha' = \sum_{i=1}^{u} \lambda_i \cdot \alpha_i$$

and suppose it is homogeneous. Then $\tau^+(\alpha') = \tau^+$.

To prove Claim 2, assume that $\tau^+(\alpha') > \tau^+$. Without loss of generality, assume that $\lambda_1 \neq 0$ and

$$\tau^-(\alpha_1) = \min_{1 \leqslant j \leqslant u} \{\tau^-(\alpha_j)\}.$$

Then a similar argument as in the proof of Claim 1 implies that

$$\tau^-(\alpha') \leqslant \tau^-(\alpha_1).$$

Note that we have assumed $\tau^+(\alpha') > \tau^+ = \tau^+(\alpha_1)$. Hence by Definition 6.8, $\tau(\alpha') < \tau(\alpha_1)$, which contradicts the construction of the set $\mathfrak{B}$.

Now we prove part (1). Suppose $\alpha_1, \ldots, \alpha_l \in \mathfrak{B}$ are pairwise distinct elements in $\mathfrak{B}$. Let

$$\alpha = \sum_{j=1}^{l} \lambda_j \cdot \alpha_j.$$

We want to show that

$$\tau^+(\alpha) = \min_{1 \leqslant j \leqslant l} \{\tau^+(\alpha_j)\}.$$

To do this, relabel the elements α_j if necessary such that

$$\tau^+(\alpha_1) = \tau^+(\alpha_2) = \cdots = \tau^+(\alpha_u) < \tau^+(\alpha_{u+1}) \leqslant \tau^+(\alpha_{u+2}) \leqslant \cdots \leqslant \tau^+(\alpha_l).$$

Since α is homogeneous, from Lemma 6.11 part (3), we know that the sum

$$\sum_{j=1}^{v} \lambda_j \cdot \alpha_j$$

is also homogeneous for any $v = 1, \ldots, l$. Applying Claim 2, we conclude that

$$\tau^+\left(\sum_{j=1}^{u} \lambda_j \cdot \alpha_j\right) = \tau^+(\alpha_1).$$

Hence we can apply Claim 1 repeatedly to conclude that

$$\tau^+\left(\sum_{j=1}^{l} \lambda_j \cdot \alpha_j\right) = \tau^+(\alpha_1) = \min_{1 \leqslant j \leqslant l} \{\tau^+(\alpha_j)\}.$$

(2). If $x \in \operatorname{Im} \Phi_{n+2k}^{n+k}$, then there exists $y \in (\mathbf{\Gamma}_{n+k}, i - \frac{kp}{2})$ and $z \in (\mathbf{\Gamma}_{n+k}, i + \frac{kp}{2})$ such that

$$x = \Psi_{-,n+2k}^{n+k}(y) + \Psi_{+,n+2k}^{n+k}(z).$$

By assumption

$$F_{n+2k}(x) = \alpha = \sum_{j=1}^{l} \lambda_j \cdot \alpha_j$$

with $\lambda_j \neq 0$ and α homogeneous. By Lemma 2.18 we have

$$\alpha = F_{n+k}(y + z).$$

Since $\mathfrak{B}$ forms a basis for $\mathbf{Y}$, we can write

$$F_{n+k}(y)=\sum_{j=1}^{l}\lambda'_j\cdot\alpha_j \text{ and } F_{n+k}(z)=\sum_{j=1}^{l}\lambda''_j\cdot\alpha_j,$$

where $l=|\mathfrak{B}|$. Then for any $1\leqslant j\leqslant l$, at least one of λ'_j and λ''_j is non-zero. Since both $F_{n+k}(y)$ and $F_{n+k}(z)$ are homogeneous, from part (1) we know

$$i-\frac{kp}{2}\geqslant\tau^-(\alpha_j)-\frac{(n+k-1)p-q}{2} \text{ when } \lambda'_j\neq 0$$

$$\text{and } i+\frac{kp}{2}\leqslant\tau^+(\alpha_j)+\frac{(n+k-1)p-q}{2} \text{ when } \lambda''_j\neq 0.$$

Conversely, suppose for any $1\leqslant j\leqslant l$ at least one of the following inequalities holds

$$i\geqslant\tau^-(\alpha_j)-\frac{(n-1)p-q}{2} \text{ and } i\leqslant\tau^+(\alpha_j)+\frac{(n-1)p-q}{2}.$$

We need to show $x\in\operatorname{Im}\Phi^{n+k}_{n+2k}$. We deal with three cases.

Case 1. The grading i satisfies

$$i\geqslant\frac{(n+2k-2)p-q+\chi(S)}{2}.$$

We want to argue that

$$\Psi^{n+k}_{-,n+2k}:(\mathbf{\Gamma}_{n+k},i-\frac{kp}{2})\to(\mathbf{\Gamma}_{n+2k},i)$$

is surjective and hence conclude that $x\in\operatorname{Im}\Phi^{n+k}_{n+2k}$. To do this, note that $\Psi^{n+k}_{-,n+2k}|_{(\mathbf{\Gamma}_{n+k},i-\frac{kp}{2})}$ is the composition of maps $\psi^{n+k+j}_{-,n+k+j+1}|_{(\mathbf{\Gamma}_{n+k+j},i-\frac{kp}{2}+\frac{jp}{2})}$ for $j=0,1,...,k-1$. With the assumption of Case 1, we have

$$i-\frac{kp}{2}+\frac{jp}{2}\geqslant\frac{(n+k+j-2)p-q+\chi(S)}{2}>-\frac{(n+k+j)p-q+\chi(S)}{2}.$$

(Note that since k is sufficiently large this is a very loose inequality.) Then Corollary 2.9 part (2) applies and we conclude that $\psi^{n+k+j}_{-,n+k+j+1}|_{(\mathbf{\Gamma}_{n+k+j},i-\frac{kp}{2}+\frac{jp}{2})}$ is an isomorphism for all $j=0,1,...,k-1$. Hence we conclude Case 1.

Case 2. The grading i satisfies

$$i\leqslant-\frac{(n+2k-2)p-q+\chi(S)}{2}.$$

The argument is similar to that for Case 1, except for using $\Psi^{n+k}_{+,n+2k}$ instead of $\Psi^{n+k}_{-,n+2k}$.

Case 3. If the grading i satisfies

$$|i|<\frac{(n+2k-2)p-q+\chi(S)}{2}$$

Under the assumption of Case 3, Lemma 2.19 part (1) implies that F_{n+2k} is injective when restricted to $(\mathbf{\Gamma}_{n+2k},i)$.

Now, for each $j=1,2,...,l$, if we have

$$i\geqslant\tau^-(\alpha_j)-\frac{(n-1)p-q}{2},$$

we claim that there exists $y_j \in (\mathbf{\Gamma}_{n+k}, i - \frac{kp}{2})$ such that

$$F_{n+k}(y_j) = \lambda_j \cdot \alpha_j.$$

If instead

$$i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p-q}{2},$$

we claim that there exists $z_j \in (\mathbf{\Gamma}_{n+k}, i + \frac{kp}{2})$ such that

$$F_{n+k}(z_j) = \lambda_j \cdot \alpha_j$$

We will verify the existence of y_j or z_j in a moment, but for now let y be the sum of all y_j's and z be the sum of all z_j's. Then from Lemma 2.18 it is straightforward to check that

$$F_{n+2k}(\Psi^{n+k}_{-,n+2k}(y) + \Psi^{n+k}_{+,n+2k}(z)) = \alpha = F_{n+2k}(x).$$

Since in Case 3 the restriction of F_{n+2k} on $(\mathbf{\Gamma}_{n+2k}, i)$ is injective, we conclude that

$$x = \Psi^{n+k}_{-,n+2k}(y) + \Psi^{n+k}_{+,n+2k}(z) \in \operatorname{Im}\Phi^{n+k}_{n+2k}.$$

It remains to show that the desired y_j or z_j exists. We only prove the existence of y_j and the argument for z_j is similar. Now assume that

$$i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p-q}{2}.$$

This implies that

$$i - \frac{kp}{2} \geqslant \tau^-(\alpha_j) - \frac{(n+k-1)p-q}{2}.$$

The hypothesis of the Lemma and the definition of $\tau^-(\alpha_j)$ in Definition 6.8, together with Lemma 6.11 part (3) imply that

$$i \equiv \tau^-(\alpha_j) - \frac{(n-1)p-q}{2} \pmod p.$$

As a result, there exists an integer $N \geqslant 0$ such that

$$i - \frac{kp}{2} = \tau^-(\alpha_j) - \frac{(n+k-1)p-q}{2} + Np.$$

Note that, from Definition 6.8, we know

$$\tau^-(\alpha_j) - \frac{(n+k-1)p-q}{2} = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (n+k-\tau(\alpha_j))p}{2}.$$

As a result, we have

$$i - \frac{kp}{2} = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (n+k-\tau(\alpha_j))p}{2} + Np.$$

The assumption in Case 3 and Lemma 6.12 part (4) then implies that $N \leqslant (n+k) - \tau(\alpha_j)$. Hence Lemma 6.12 part (3) implies the existence of y_j.

(3). If $x \notin \operatorname{Im}\Phi^{n+k}_{n+2k}$, then part (2) means that there exists some j such that

$$\tau^+(\alpha_j) + \frac{(n-1)p-q}{2} < i < \tau^-(\alpha_j) - \frac{(n-1)p-q}{2}.$$

Note that, by Lemma 6.11 part (3), we must have

$$i \equiv \tau^-(\alpha_j) - \frac{(n-1)p-q}{2} \equiv \tau^+(\alpha_j) + \frac{(n-1)p-q}{2} \pmod p.$$

By direct calculation, we have

$$(\tau^-(\alpha_j) - \frac{(n-1)p-q}{2}) - (\tau^+(\alpha_j) + \frac{(n-1)p-q}{2}) = (\tau(\alpha_j) - n)p$$

Then we can choose N with $0 \leqslant N \leqslant \tau(\alpha_j) - n - 2$ as desired. □

7.2. **The construction of the map.** Since Φ^n_{n+k} and Φ^{n+k}_{n+2k} are homogeneous, we can construct Φ^{n+2k}_n for each grading to achieve both the exactness and the commutativity. Given the grading shifts in Lemma 2.6 and Lemma 2.13, the map Φ^{n+2k}_n preserves the gradings. From Lemma 2.5, for any grading i with

$$|i| > \frac{|np-q| - \chi(S)}{2},$$

we have $(\mathbf{\Gamma}_n, i) = 0$. From Corollary 2.9, we know either $\Psi^{n+k}_{+,n+2k}$ or $\Psi^{n+k}_{-,n+2k}$ is surjective onto $(\mathbf{\Gamma}_{n+2k}, i)$ for such grading i. Thus, on such grading i, the zero map satisfies the exactness for Φ^{n+2k}_n (though we still have to verify the commutativity in Proposition 7.2).

On the other hand, from Lemma 2.19, the restriction of F_{n+2k} on the consecutive p middle gradings is an isomorphism. In particular, when $p = 1$, it is an isomorphism when restricted to each middle grading. Also from Lemma 7.3, it seems that the definition of Φ^{n+2k}_{n+k} on $(\mathbf{\Gamma}_{n+2k}, i)$ should involve $\mathrm{Proj}^i_n \circ G_n$. However, if we simply take

$$\mathrm{Proj}^i_n \circ G_n \circ F_{n+2k}$$

as the definition, the current techniques fall short of demonstrating exactness and commutativity.

We resolve this issue by introducing an isomorphism

$$I : \mathbf{Y} \xrightarrow{\cong} \mathbf{Y}$$

and define

$$\Phi^{n+2k}_n(x) = \mathrm{Proj}^i_n \circ G_n \circ I \circ F_{n+2k}(x) \text{ for } x \in (\mathbf{\Gamma}_{n+2k}, i). \tag{7.5}$$

The construction of I is noncanonical but it helps us to prove the exactness and commutativity.

Remark 7.5. In the first arXiv version of this paper, we deal with the special case $Y = S^3$. In this case $\mathbf{Y} \cong \mathbb{C}$ so up to a scalar we have $I = \mathrm{Id}$. In this special case indeed we could prove the exactness and commutativity without explicitly writing down the isomorphism I as follows.

We first define the map I on the basis

$$\mathfrak{B} = \bigcup_{n \in \mathbb{Z}} \mathfrak{B}_n$$

of $\mathbf{Y}$ chosen in Section 6.3 that consists of homogeneous elements and then extend the map on the whole space linearly. We will show it is an isomorphism.

Fix $n_0 \in \mathbb{Z}$ small enough such that Corollary 2.9 and Lemma 2.19 apply. For any $\alpha \in \mathfrak{B}_n$, there exists a grading $i(\alpha) \in (-\frac{p}{2}, \frac{p}{2}]$ such that there exists $N(\alpha) \in \mathbb{Z}$ with

$$i(\alpha) = \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2} - \frac{(\tau(\alpha) - 2 - n_0)p}{2} + N(\alpha)p.$$

Note that, from the above equality, we know that

$$N(\alpha) = \frac{\tau(\alpha) - 2 - n_0}{2} + \frac{2i(\alpha) - \tau^+(\alpha) - \tau^-(\alpha)}{2p}.$$

Note that, except for n_0, the rest of the terms are bounded, so $N(\alpha) \geqslant 0$ when n_0 is small enough. Similarly,

$$N(\alpha) + n_0 = \frac{\tau(\alpha) - 2 + n_0}{2} + \frac{2i(\alpha) - \tau^+(\alpha) - \tau^-(\alpha)}{2p}$$

so we have $N(\alpha) + n_0 \leqslant \tau(\alpha) - 2$ when n_0 is small enough. As a result, by Lemma 6.14 part (2) and (6) (and the convention after the lemma), we know that for any $\alpha \in \mathfrak{B}_n$

$$\Psi^{n_0+N(\alpha)}_{-,\tau(\alpha)-2}\left(\eta^{\tau(\alpha)}_{+,n_0+N(\alpha)}(\alpha)\right) = \eta^{\tau(\alpha)}_{+,\tau(\alpha)-2}(\alpha) = \eta^{\tau(\alpha)}_{-,\tau(\alpha)-2}(\alpha) = \Psi^{\tau(\alpha)-2-N(\alpha)}_{+,\tau(\alpha)-2}\left(\eta^{\tau(\alpha)}_{-,\tau(\alpha)-2-N(\alpha)}(\alpha)\right).$$

Then by Lemma 6.7, there exists $w \in (\mathbf{\Gamma}_{n_0}, i(\alpha))$ such that

$$\Psi^{n_0}_{+,n_0+N(\alpha)}(w) = \eta^{\tau(\alpha)}_{+,n_0+N(\alpha)}(\alpha) \text{ and } \Psi^{n_0}_{-,\tau(\alpha)-2-N(\alpha)}(w) = \eta^{\tau(\alpha)}_{-,\tau(\alpha)-2-N(\alpha)}(\alpha). \tag{7.6}$$

Let

$$\mathrm{Proj} : \mathbf{\Gamma}_{n_0} \to \bigoplus_{i \in (-\frac{p}{2}, \frac{p}{2}]} (\mathbf{\Gamma}_{n_0}, i).$$

From Lemma 2.19, we know

$$\mathrm{Proj} \circ G_{n_0} : \mathbf{Y} \to \bigoplus_{i \in (-\frac{p}{2}, \frac{p}{2}]} (\mathbf{\Gamma}_{n_0}, i)$$

is an isomorphism. Hence we define

$$I(\alpha) = (\mathrm{Proj} \circ G_{n_0})^{-1}(w).$$

The following diagram might be helpful for understanding the construction of I. (We write $n = \tau(\alpha)$, $n_1 = \tau(\alpha) - 2 - N(\alpha)$, and $n_2 = n_0 + N(\alpha)$.)

$$\begin{tikzcd}
 & \eta^n_{-,n_1}(\alpha) \in \mathbf{\Gamma}_{n_1} \arrow[dr, "\Psi^{n_1}_{+,n-2}"] & & \\
w \in \mathbf{\Gamma}_{n_0} \arrow[ur, "\Psi^{n_0}_{-,n_1}"] \arrow[dr, "\Psi^{n_0}_{+,n_2}"] & & \eta^n_{-,n-2}(\alpha) = \eta^n_{+,n-2}(\alpha) \in \mathbf{\Gamma}_{n-2} & z \in \mathbf{\Gamma}_n \arrow[dl] \arrow[dd, "F_n"] \\
 & \eta^n_{+,n_2}(\alpha) \in \mathbf{\Gamma}_{n_2} \arrow[ur, "\Psi^{n_2}_{-,n-2}"] & \mathbf{\Gamma}_\mu \arrow[l, dotted] \arrow[u, dotted] \arrow[uul, dotted] & \\
I(\alpha) \in \mathbf{Y} \arrow[uu, "G_{n_0}"] & & & \alpha \in \mathbf{Y} \arrow[lll, "I"]
\end{tikzcd}$$

Remark 7.6. For a general 3-manifold Y, our construction of I is noncanonical since there are many choices such as the basis $\mathfrak{B}$ and the element w for each $\alpha \in \mathfrak{B}$. However, one could still ask whether we could simply pick $I = \mathrm{Id}$ or not. If we take $I = \mathrm{Id}$, then Proposition 7.2 can finally be reduced to Conjecture 7.7 which we state below. We believe that the following conjecture is true, though currently, we do not find a proof for it. Hence in order to fulfill the main purpose of the paper, we introduce the isomorphism I to bypass this conjecture.

Conjecture 7.7. For any $\alpha \in \mathfrak{B}$, and any integer $n \leqslant \tau(\alpha) - 2$, we have

$$\eta_{\pm,n}^{\tau(\alpha)}(\alpha) = \mathrm{Proj}_n^{j_\pm} \circ G_n(\alpha),$$

where

$$j_\pm = \frac{\tau^+(\alpha) + \tau^-(\alpha)}{2} \mp \frac{(\tau(\alpha) - 2 - n)p}{2}$$

and

$$\mathrm{Proj}_n^{j_\pm} : \mathbf{\Gamma}_n \to (\mathbf{\Gamma}_n, j_\pm)$$

is the projection.

Lemma 7.8. *We have the following.*

1. *Suppose $\alpha \in \mathfrak{B}$ and n_0, w, $N(\alpha)$ are chosen as above. Suppose n, k are two integers such that $n_0 \leqslant k \leqslant n$. Then (a) $\Psi_{-,n}^{k} \circ \Psi_{+,k}^{n_0}(w) \neq 0$ if and only if (b) $k \leqslant n_0 + N(\alpha)$ and $n - k \leqslant \tau(\alpha) - 2 - n_0 - N(\alpha)$ (in particular, we have $n \leqslant \tau(\alpha) - 2$).*
2. *The map $I : \mathbf{Y} \to \mathbf{Y}$ is an isomorphism.*
3. *For an element $\alpha \in \mathfrak{B}$, an integer n and a grading i, the following two statements are equivalent.*
   (a) We have $\mathrm{Proj}_n^i \circ G_n \circ I(\alpha) \neq 0$.
   (b) We have $n \leqslant \tau(\alpha) - 2$ and there exists $N \in \mathbb{Z}$ such that $N \in [0, \tau(\alpha) - 2 - n]$ and
   $$i = \frac{\tau^+(\alpha) + \tau^-(\alpha) - (\tau(\alpha) - 2 - n)p}{2} + Np.$$
4. *Suppose for an integer n and a grading i we have $\alpha_1, \ldots, \alpha_L \in \mathfrak{B}$ such that $\mathrm{Proj}_n^i \circ G_n \circ I(\alpha_j) \neq 0$ for all $1 \leqslant j \leqslant L$, then $\mathrm{Proj}_n^i \circ G_n \circ I(\alpha_1), \ldots, \mathrm{Proj}_n^i \circ G_n \circ I(\alpha_L)$ are linearly independent.*
5. *Suppose $\alpha \in \mathfrak{B}$. For any $n \in \mathbb{Z}$ such that $n \leqslant \tau(\alpha) - 2$, we have*
   $$\mathrm{Proj}_n^{i_\pm} \circ G_n \circ I(\alpha) = \eta_{\pm,n}^{\tau(\alpha)}(\alpha) \text{ where } i_\pm = \frac{\tau^+(\alpha) + \tau^-(\alpha) \mp (\tau(\alpha) - 2 - n)}{2}$$

Proof. (1). First, when $k > n_0 + N(\alpha)$, from the construction of w, we know that

$$\Psi_{+,k}^{n_0}(w) = \Psi_{+,k}^{n_0+N(\alpha)} \circ \Psi_{+,n_0+N(\alpha)}^{n_0}(w) = \Psi_{+,k}^{n_0+N(\alpha)} \circ \eta_{+,n_0+N(\alpha)}^{\tau(\alpha)}(\alpha) = 0.$$

The last equality is from Lemma 6.14 part (6). Similarly, if $n - k > \tau(\alpha) - 2 - n_0 - N(\alpha)$, we know from Lemma 2.11 that

$$\begin{aligned}
\Psi_{-,n}^{k} \circ \Psi_{+,k}^{n_0}(w) &= \Psi_{+,n}^{n+n_0-k} \circ \Psi_{-,n+n_0-k}^{n_0}(w) \\
&= \Psi_{+,n}^{n+n_0-k} \circ \Psi_{-,n+n_0-k}^{\tau(\alpha)-2-N(\alpha)} \circ \Psi_{-,\tau(\alpha)-2-N(\alpha)}^{n_0}(w) \\
(\text{Definition of } w) &= \Psi_{+,n}^{n+n_0-k} \circ \Psi_{-,n+n_0-k}^{\tau(\alpha)-2-N(\alpha)} \circ \eta_{-,\tau(\alpha)-2-N(\alpha)}^{\tau(\alpha)}(\alpha) \\
(\text{Lemma 6.14 part (6)}) &= 0.
\end{aligned}$$

Next, we need to show that $\Psi_{-,n}^{k} \circ \Psi_{+,k}^{n_0}(w) \neq 0$ when $k \leqslant n_0 + N(\alpha)$ and $n - k \leqslant \tau(\alpha) - 2 - n_0 - N(\alpha)$. Again, from Lemma 2.11 we have

$$\begin{aligned}
\Psi_{+,n+N(\alpha)+n_0-k}^{n} \circ \Psi_{-,n}^{k} \circ \Psi_{+,k}^{n_0}(w) &= \Psi_{-,n_0+N(\alpha)+n-k}^{n_0+N(\alpha)} \circ \Psi_{+,n_0+N(\alpha)}^{n_0}(w) \\
(\text{Definition of } w) &= \Psi_{-,n_0+N(\alpha)+n-k}^{n_0+N(\alpha)} \circ \eta_{+,n_0+N(\alpha)}^{\tau(\alpha)}(\alpha) \\
(\text{Lemma 6.14 part (6)}) &= \eta_{+,n_0+N(\alpha)+n-k}^{\tau(\alpha)}(\alpha) \\
(\text{Lemma 6.14 part (3)}) &\neq 0.
\end{aligned}$$

(2). Suppose $\mathfrak{B} = \{\alpha_1, \dots, \alpha_L\}$ where $L = \dim_{\mathbb{C}} \mathbf{Y}$. We order the elements α_i such that

$$\tau(\alpha_i) \geqslant \tau(\alpha_{i+1}).$$

Let w_i, $N_i = N(\alpha_i)$ be the data associated to α_i as above. Since

$$w_i \in \bigoplus_{j \in (-\frac{p}{2}, \frac{p}{2}]} (\mathbf{\Gamma}_{n_0}, j)$$

for any i, and by Lemma 2.19, the map

$$\operatorname{Proj} \circ G_{n_0} : \mathbf{Y} \to \bigoplus_{j \in (-\frac{p}{2}, \frac{p}{2}]} (\mathbf{\Gamma}_{n_0}, j)$$

is an isomorphism, in order to show that I is an isomorphism, it suffices to show that $w_1, \dots, w_L$ are linearly independent.

Now suppose there are complex numbers $\lambda_1, \dots, \lambda_L$ such that

$$\sum_{i=1}^{L} \lambda_i w_i = 0.$$

Our goal is to show that all λ_i are zero. The idea is to apply various maps $\Psi^{n_0+N_1}_{+,\tau(\alpha_1)-2} \circ \Psi^{n_0}_{-,n_0+N_1}$ to filter out different indices by part (1) of the lemma.

Applying the map $\Psi^{n_0+N_1}_{+,\tau(\alpha_1)-2} \circ \Psi^{n_0}_{-,n_0+N_1}$, from the construction of w_i, the order of α_i and part (1) of the lemma, we know

$$\begin{aligned} 0 &= \Psi^{n_0+N_1}_{+,\tau(\alpha_1)-2} \circ \Psi^{n_0}_{-,n_0+N_1}(\sum_{i=1}^{L} \lambda_i w_i) \\ &= \sum_{\alpha_i} \lambda_i \eta^{\tau(\alpha_1)}_{\pm\tau(\alpha_1)-2}(\alpha_i) \end{aligned}$$

where the summation in the second line is over all α_i with

$$\tau(\alpha_i) = \tau(\alpha_1), \text{ and } N_i = N_1.$$

Note that, from Lemma 6.14 part (2) and the convention after the lemma, we know $\eta^{\tau(\alpha_1)}_{+\tau(\alpha_1)-2} = \eta^{\tau(\alpha_1)}_{-\tau(\alpha_1)-2}$. From Lemma 6.14 again, we know that $\eta^{\tau(\alpha_1)}_{\pm\tau(\alpha_1)-2}(\alpha_i)$ are linearly independent, and as a result all relevant λ_i must be zero. Suppose i_0 is the smallest index in the rest. By our choice of α_i, the element α_{i_0} has the largest τ among the rest of the α_i. Hence we can apply the map $\Psi^{n_0+N_{i_0}}_{+,\tau(\alpha_{i_0})-2} \circ \Psi^{n_0}_{-,n_0+N_{i_0}}$ to filter out α_i with smaller τ. Repeating this argument, we could prove that all λ_i must be zero.

(3). Let n_0, w, $i(\alpha)$, and $N(\alpha)$ be constructed as above. We first prove that (b) $\Rightarrow$ (a). Note that, when constructing the isomorphism I, from Corollary 2.9 and Lemma 2.18 we can take $n_0' = n_0 - 2$ and $w' = (\psi^{n_0-2}_{+,n_0-1})^{-1} \circ (\psi^{n_0-1}_{+,n_0})^{-1}(w)$ that will lead to the same I as n_0, w. (Note that, by construction, passing from n_0 to $n_0 - 2$ will increase $N(\alpha)$ by 1.)

Remark 7.9. Note that the main goal of the current paper is to derive an integral surgery formula. For n that is sufficiently large, we already know a large surgery as in [LY21]. When n is small enough, we can pass to $-n$ for the mirror of the knot. As a result, instead of changing n_0 for particular n, we could assume a universal bound for all the integers n that we care about and make n_0 universally small.

As a result, we can always assume that n_0 is small enough compared with any given n. Now recall by construction

$$i(\alpha)=\frac{\tau^+(\alpha)+\tau^-(\alpha)}{2}-\frac{(\tau(\alpha)-2-n_0)p}{2}+N(\alpha)p$$

and by the assumption in (b) we have

$$i=\frac{\tau^+(\alpha)+\tau^-(\alpha)-(\tau(\alpha)-2-n)p}{2}+Np.$$

We can assume that n_0 is small enough such that $N(\alpha)>N$. Take $k=N(\alpha)-N+n_0$. Note that, by construction we have $w\in(\mathbf{\Gamma}_{n_0},i(\alpha))$, so from Lemma 2.6 we know that

$$\Psi^k_{-,n}\circ\Psi^{n_0}_{+,k}(w)\in(\mathbf{\Gamma}_n,i(\alpha)-\frac{(k-n_0)p}{2}+\frac{(n-k)p}{2})=(\mathbf{\Gamma}_n,i).$$

As a result, we conclude from the definition of w and Lemma 2.18 that

$$\begin{aligned}\Psi^k_{-,n}\circ\Psi^{n_0}_{+,k}(w)&=\Psi^k_{-,n}\circ\Psi^{n_0}_{+,k}\circ\mathrm{Proj}^{i(\alpha)}_{n_o}\circ G_{n_0}\circ I(\alpha)\\&=\mathrm{Proj}^i_n\circ\Psi^k_{-,n}\circ\Psi^{n_0}_{+,k}\circ G_{n_0}\circ I(\alpha)\\&=\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\end{aligned}$$

Then it is straightforward to verify that $k-n_0\leqslant N(\alpha)$ and $n-k\leqslant\tau(\alpha)-2-n_0-N(\alpha)$. As a result, we conclude from part (1) that

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\neq0.$$

Next we show that (a) $\Rightarrow$ (b). Again assume that n_0 is small enough compared with the given n. Then there exists $i'\in(-\frac{p}{2},\frac{p}{2}]$ such that there exists $N'\in\mathbb{Z}$ with

$$i'=i-\frac{(n-n_0)p}{2}+N'p.$$

By Lemma 2.18, we know that

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)=\Psi^{n_0+N'}_{-,n}\circ\Psi^{n_0}_{+,n_0+N'}\circ\mathrm{Proj}^{i'}_{n_0}\circ G_{n_0}\circ I(\alpha).$$

From the construction of $I(\alpha)$ and Lemma 2.19 we know $\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\neq0$ only if $i'=i(\alpha)$, in which case

$$\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)=\Psi^{n_0+N'}_{-,n}\circ\Psi^{n_0}_{+,n_0+N'}\circ\mathrm{Proj}^{i'}_{n_0}\circ G_{n_0}\circ I(\alpha)=\Psi^{n_0+N'}_{-,n}\circ\Psi^{n_0}_{+,n_0+N'}(w).$$

Hence $\mathrm{Proj}^i_n\circ G_n\circ I(\alpha)\neq0$ implies that

$$N'\leqslant N(\alpha)\text{ and }n-N'\leqslant\tau(\alpha)-2-N(\alpha)$$

by part (1). Taking $N=N(\alpha)-N'$, it is then straightforward to check that

$$N\in[0,\tau(\alpha)-2-n]\text{ and }i=\frac{\tau^+(\alpha)+\tau^-(\alpha)-(\tau(\alpha)-2-n)p}{2}+Np.$$

(4). The proof is similar to that of (2).

(5). It follows from the proofs of parts (1) and (3). □

7.3. **The exact triangle.** In this subsection, we prove the exact triangle. Note that we choose the basis $\mathfrak{B}$ of $\mathbf{Y}$ as in Section 6.3.

Proof of Proposition 7.1. We will verify the exactness at each space of the triangle.

The exactness at $\mathbf{\Gamma}_{n+k} \oplus \mathbf{\Gamma}_{n+k}$. This follows from Proposition 5.1.

The exactness at $\mathbf{\Gamma}_n$. From Lemma 7.3 and the construction of Φ_n^{n+2k} in (7.5), we know that $\operatorname{Im}\Phi_n^{n+2k} \subset \ker \Phi_{n+k}^n$. Now pick an arbitrary

$$x \in (\mathbf{\Gamma}_n, i) \cap \ker \Phi_{n+k}^n = \operatorname{Im}\left(\operatorname{Proj}_n^i \circ G_n\right)$$

Since I is an isomorphism, we can assume that

$$x = \sum_{j=1}^{l} \operatorname{Proj}_n^i \circ G_n(\lambda_j \cdot I(\alpha_j))$$

where $\alpha_j \in \mathfrak{B}$ and $\operatorname{Proj}_n^i \circ G_n \circ I(\alpha_j) \neq 0$. From Lemma 7.8 part (3), we know that this implies that for any $j \in [1, l] \cap \mathbb{Z}$, we have $n \leqslant \tau(\alpha_j) - 2$ and there exists $N_j \in \mathbb{Z}$ such that $N_j \in [0, \tau(\alpha_j) - 2 - n]$

$$i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (\tau(\alpha_j) - 2 - n)p}{2} + N_j p.$$

Now, for k sufficiently large, we have $n + 2k > \tau(\alpha_j)$. Taking

$$N_j' = n + k + 1 - \tau(\alpha_j) + N_j \in \mathbb{Z},$$

it is straightforward to verify that when k is sufficiently large, we have

$$N_j' \in [0, n + 2k - \tau(\alpha_j)] \text{ and } \ i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (n + 2k - \tau(\alpha_j))p}{2} + N_j' p.$$

Hence by Lemma 6.12 part (3), there exists $y_j \in (\mathbf{\Gamma}_{n+2k}, i)$ such that $F_{n+2k}(y_j) = \alpha_j$. As a result, it is straightforward to check that

$$x = \Phi_n^{n+2k}\left(\sum_{j=1}^{l} \lambda_j \cdot y_j\right) \in \operatorname{Im}\Phi_n^{n+2k}.$$

The exactness at $\mathbf{\Gamma}_{n+2k}$. Suppose $x \in (\mathbf{\Gamma}_{n+2k}, i)$ and

$$F_{n+2k}(x) = \sum_{j=1}^{l} \lambda_j \cdot \alpha_j$$

with $\lambda_j \neq 0$ and $\alpha_j \in \mathfrak{B}$.

First, if $x \in \operatorname{Im}\Phi_{n+2k}^{n+k}$, then from Lemma 7.4 part (2), we know that for any $1 \leqslant j \leqslant l$, we have

$$\text{either } i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p - q}{2} \text{ or } i \leqslant \tau^+(\alpha_j) + \frac{(n-1)p - q}{2}.$$

If we write

$$i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (\tau(\alpha_j) - 2 - n)p}{2} + N_j p,$$

for some N_j then the inequality

$$i \geqslant \tau^-(\alpha_j) - \frac{(n-1)p - q}{2}$$

implies that

$$\begin{aligned} N_j &\geqslant \frac{1}{p}\left(\tau^-(\alpha_j)-\frac{(n-1)p-q}{2}-\frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)-(\tau(\alpha_j)-2-n)p}{2}\right)\\ &=\frac{1}{2}\left(1+\frac{\tau^-(\alpha_j)-\tau^+(\alpha_j)+q}{p}\right)+\frac{\tau(\alpha_j)}{2}-1-n\\ &=\tau(\alpha_j)-1-n. \end{aligned}$$

Note that the last equality uses the definition of $\tau(\alpha)$ in Definition 6.8. Similarly, we can compute that

$$i \leqslant \tau^+(\alpha_j)+\frac{(n-1)p-q}{2}$$

implies that

$$\begin{aligned} N_j &\leqslant \frac{1}{p}\left(\tau^+(\alpha_j)+\frac{(n-1)p-q}{2}-\frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)-(\tau(\alpha_j)-2-n)p}{2}\right)\\ &=\frac{1}{2}\left(-1+\frac{\tau^+(\alpha_j)-\tau^-(\alpha_j)-q}{p}\right)+\frac{\tau(\alpha_j)}{2}-1\\ &=-1. \end{aligned}$$

In summary, $x \in \operatorname{Im}\Phi^{n+k}_{n+2k}$ implies that for all $1 \leqslant j \leqslant l$, either $N_j \geqslant \tau(\alpha_j)-1-n$ or $N_j \leqslant -1$. Hence from Lemma 7.8 part (3), we know that

$$\mathrm{Proj}^i_n \circ G_n \circ I(\alpha_j) = 0$$

for all $1 \leqslant j \leqslant l$ and as a result, $\Phi^{n+2k}_n(x) = 0$.

Second, suppose $x \notin \operatorname{Im}\Phi^{n+k}_{n+2k}$. For any $1 \leqslant j \leqslant l$, we can write

$$i = \frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)-(\tau(\alpha_j)-2-n)p}{2}+N_jp$$

for some N_j. Then from Lemma 7.4 part (3) we know that there exists j such that $1 \leqslant j \leqslant l$, and

$$N_j \in [0,\tau(\alpha_j)-2-n]\cap\mathbb{Z}.$$

Hence by Lemma 7.8 part (3) and (4) we know that

$$\mathrm{Proj}^i_n \circ G_n \circ I(\alpha_j) \neq 0 \Rightarrow \Phi^{n+2k}_n(x) \neq 0.$$

Hence we conclude that

$$\operatorname{Im}\Phi^{n+k}_{n+2k} = \ker\Phi^{n+2k}_n.$$

□

7.4. **The commutative diagram.** In this subsection, we will prove the commutative diagram presented at the beginning of the section. Note that we choose the basis $\mathfrak{B}$ of $\mathbf{Y}$ as in Section 6.3.

Lemma 7.10. *Suppose $n \in \mathbb{Z}$ and i is a grading. Suppose $x \in (\mathbf{\Gamma}_n, i)$ such that*

$$F_n(x) = \sum_j^l \lambda_j\alpha_j,$$

with $\lambda_j \neq 0$ and $\alpha_j \in \mathfrak{B}$ for all $1 \leqslant j \leqslant l$. Then for any $1 \leqslant j \leqslant l$, there exists $N_j \in [0, n+1-\tau(\alpha_j)]$ such that

$$i = \frac{\tau^+(\alpha_j)+\tau^-(\alpha_j)-(n+1-\tau(\alpha_j))p}{2}+N_jp.$$

Proof. This is a combination of Lemma 6.11 part (3), Lemma 6.12 part (3), and Lemma 7.4 part (1). The proof is similar to that of Lemma 7.4 part (2). □

Proof of Proposition 7.2. We only prove the first commutative diagram

$$\begin{tikzcd} \mathbf{\Gamma}_{\frac{2n+2k+1}{2}} \arrow[r, "\psi^{n+k+1}_{+,\mu}\circ\psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"] \arrow[d, "\Psi^{n+k+1}_{+,n+2k}\circ\psi^{\frac{2n+2k+1}{2}}_{-,n+k+1}"] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^{\mu}_{+,n}"] \\ \mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_{n}"] & \mathbf{\Gamma}_n \end{tikzcd}$$

The other is similar. Note that at the end of Section 6.3, we introduce new notations of $\eta^n_{\pm,n-2}$ to remove the scalars. Then the second commutative diagram only holds up to a scalar.

First, note that the maps from $\mathbf{\Gamma}_{\frac{2n+2k+1}{2}}$ to $\mathbf{\Gamma}_\mu$ and $\mathbf{\Gamma}_{n+2k}$ both factor through $\mathbf{\Gamma}_{n+k+1}$. As a result, we only need to prove the following commutative diagram for sufficiently large k.

$$\begin{tikzcd} \mathbf{\Gamma}_{n+k+1} \arrow[r, "\psi^{n+k+1}_{+,\mu}"] \arrow[d, "\Psi^{n+k+1}_{+,n+2k}"] & \mathbf{\Gamma}_\mu \arrow[d, "\psi^{\mu}_{+,n}"] \\ \mathbf{\Gamma}_{n+2k} \arrow[r, "\Phi^{n+2k}_{n}"] & \mathbf{\Gamma}_n \end{tikzcd}$$

Now suppose $x \in (\mathbf{\Gamma}_{n+k+1}, i)$. Write

$$F_{n+k+1}(x) = \sum_{j=1}^{l} \lambda_i \cdot \alpha_j$$

with $\lambda_j \neq 0$ and $\alpha_j \in \mathfrak{B}$ for $1 \leqslant j \leqslant l$. We want to first establish an identity

$$\psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x) = \sum_{\substack{1\leqslant j\leqslant l \\ n\leqslant \tau(\alpha_j)-2 \\ N_j = n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \eta^{\tau(\alpha_j)}_{+,n}(\alpha_j) \tag{7.7}$$

and then show that the other composition has exactly the same expression.

From Lemma 7.10, we know for any $1 \leqslant j \leqslant l$, there exists $N_j \in [0, n+k+1-\tau(\alpha_j)]$ such that

$$i = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (n+k+1-\tau(\alpha_j))p}{2} + N_j p. \tag{7.8}$$

Taking $n'_j = \tau(\alpha_j)$ and $N'_j = 0$, we can apply Lemma 6.12 part (3) to find an element

$$x_j \in (\mathbf{\Gamma}_{\tau(\alpha_j)}, \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j)}{2})$$

such that

$$F_{\tau(\alpha_j)}(x_j) = \alpha_j.$$

It is then straightforward to check that

$$y_j = \Psi^{\tau(\alpha_j)+N_j}_{+,n+k+1} \circ \Psi^{\tau(\alpha_j)}_{-,\tau(\alpha_j)+N_j}(x_j) \in (\mathbf{\Gamma}_{n+k+1}, i). \tag{7.9}$$

Write

$$y = x - \sum_{j=1}^{l} \lambda_j \cdot y_j \in (\mathbf{\Gamma}_{n+k+1}, i).$$

From Lemma 2.18 we know that

$$F_{n+k+1}(y) = 0.$$

As a result, by Lemma 6.6,

$$\psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x) = \sum_{j=1}^{l} \lambda_j \cdot \psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(y_j)$$

Note that, unless $N_j = n+k+1-\tau(\alpha_j)$, we have

$$\psi^{n+k+1}_{+,\mu} \circ \Psi^{\tau(\alpha_j)+N_j}_{+,n+k+1} = 0$$

by the exactness. As a result,

$$\begin{aligned}
\psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x) &= \sum_{\substack{1\leqslant j\leqslant l \\ N_j=n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu} \circ \Psi^{\tau(\alpha_j)}_{-,\tau(\alpha_j)+N_j}(x_j) \\
\text{(Lemma 2.12)} &= \sum_{\substack{1\leqslant j\leqslant l \\ n\leqslant\tau(\alpha_j)-2 \\ N_j=n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \psi^{\mu}_{+,n} \circ \psi^{\tau(\alpha_j)}_{+,\mu}(x_j) + \sum_{\substack{1\leqslant j\leqslant l \\ n\geqslant\tau(\alpha_j)-1 \\ N_j=n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \psi^{\mu}_{+,n} \circ \psi^{\tau(\alpha_j)}_{+,\mu}(x_j) \\
\text{Equation (7.10)} &= \sum_{\substack{1\leqslant j\leqslant l \\ n\leqslant\tau(\alpha_j)-2 \\ N_j=n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \psi^{\mu}_{+,n} \circ \psi^{\tau(\alpha_j)}_{+,\mu}(x_j) \\
(\text{Definition of } \eta^{\tau(\alpha_j)}_{+,n}) &= \sum_{\substack{1\leqslant j\leqslant l \\ n\leqslant\tau(\alpha_j)-2 \\ N_j=n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \eta^{\tau(\alpha_j)}_{+,n}(\alpha_j)
\end{aligned}$$

This verifies Equation (7.7) if we show that

$$\sum_{\substack{1\leqslant j\leqslant l \\ n\geqslant\tau(\alpha_j)-1 \\ N_j=n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \psi^{\mu}_{+,n} \circ \psi^{\tau(\alpha_j)}_{+,\mu}(x_j) = 0.$$

To verify this last equality, assume that $n \geqslant \tau(\alpha_j) - 1$. Then from Lemma 2.12 and the exactness of the bypass maps we have

$$\psi^{\mu}_{+,n} \circ \psi^{\tau(\alpha_j)}_{+,\mu}(x_j) = \psi^{\mu}_{+,n} \circ \psi^{n+1}_{+,\mu} \circ \Psi^{\tau(\alpha_j)}_{-,n+1}(x) = 0. \tag{7.10}$$

Now we deal with $\Phi^{n+2k}_{n} \circ \Psi^{n+k+1}_{+,n+2k}(x)$. Since $F_{n+k+1}(y) = 0$, Lemma 2.19 implies that

$$\Psi^{n+k+1}_{+,n+2k}(y) = 0.$$

Hence

$$\Psi^{n+k+1}_{+,n+2k}(x) = \sum_{j=1}^{l} \lambda_j \cdot \Psi^{n+k+1}_{+,n+2k}(y_j)$$

where y_j is defined as in (7.9). Note that, by definition we have $y_j \in (\mathbf{\Gamma}_{n+1+k}, i)$, so from Lemma 2.6, we know

$$\Psi^{n+k+1}_{+,n+2k}(y_j) \in (\mathbf{\Gamma}_{n+2k}, i - \frac{(k-1)p}{2}).$$

Note that, by (7.9) and Lemma 2.18, we know that

$$F_{n+2k} \circ \Psi^{n+k+1}_{+,n+2k}(y_j) = F_{\tau(\alpha_j)}(x_j) = \alpha_j.$$

Hence

$$\Phi^{n+2k}_{n} \circ \Psi^{n+k+1}_{+,n+2k}(x) = \sum_{j=1}^{l} \lambda_j \cdot \mathrm{Proj}_n^{i-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j).$$

We write

$$i - \frac{(k-1)p}{2} = \frac{\tau^+(\alpha_j) + \tau^-(\alpha_j) - (\tau(\alpha_j) - 2 - n)p}{2} + N'_j p$$

Comparing the above formula with (7.8), we know

$$N'_j = N_j + \tau(\alpha_j) - n - k - 1.$$

Note that, by construction, $N_j \leqslant n + k + 1 - \tau(\alpha_j)$, which means $N'_j \leqslant 0$. Hence from Lemma 7.8 we know

$$\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j) \neq 0$$

if and only if $N'_j = 0$, *i.e.*, $N_j = n + k + 1 - \tau(\alpha)$. Also when $N'_j = 0$ from Lemma 7.8 part (5) we know

$$\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j) = \eta^{\tau(\alpha_j)}_{+,n}(\alpha_j).$$

Note that we could focus on indices j such that $\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j) \neq 0$. This is because if an index j makes $\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j) = 0$, then on one hand it does not contribute to $\Phi^{n+2k}_{n} \circ \Psi^{n+k+1}_{+,n+2k}(x)$ since the corresponding summand is 0, on the other hand, we have $N_j \neq n+k+1-\tau(\alpha)$ hence per Equation (7.7) it does not contribute to $\psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x)$, either. Also, we know from Lemma 7.8 part (3) that when $\mathrm{Proj}_n^{i-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j) \neq 0$ we must have $n \leqslant \tau(\alpha_j) - 2$. As a result, we know

$$\begin{aligned}\Phi^{n+2k}_{n} \circ \Psi^{n+k+1}_{+,n+2k}(x) &= \sum_{j=1}^{l} \lambda_j \cdot \mathrm{Proj}_n^{j-\frac{(k-1)p}{2}} \circ G_n \circ I(\alpha_j) \\ &= \sum_{\substack{1 \leqslant j \leqslant l \\ n \leqslant \tau(\alpha_j) - 2 \\ N_j = n+k+1-\tau(\alpha_j)}} \lambda_j \cdot \eta^{\tau(\alpha_j)}_{+,n}(\alpha_j) \\ \text{Equation (7.7)} &= \psi^{\mu}_{+,n} \circ \psi^{n+k+1}_{+,\mu}(x)\end{aligned}$$

□

References

[Boa99] J. Michael Boardman. Conditionally convergent spectral sequences. In *Homotopy invariant algebraic structures*, volume 239 of *Contemporary Mathematics*, page 49–84. American Mathematical Society, Providence, RI, 1999.

[BS15] John A. Baldwin and Steven Sivek. Naturality in sutured monopole and instanton homology. *J. Differ. Geom.*, 100(3):395–480, 2015.

[BS16a] John A. Baldwin and Steven Sivek. A contact invariant in sutured monopole homology. *Forum Math. Sigma*, 4:e12, 82, 2016.

[BS16b] John A. Baldwin and Steven Sivek. Instanton Floer homology and contact structures. *Selecta Math. (N.S.)*, 22(2):939–978, 2016.

[BS21] John A. Baldwin and Steven Sivek. Framed instanton homology and concordance. *J. Topol.*, 14(4):1113–1175, 2021.

[BS22a] John A. Baldwin and Steven Sivek. Characterizing slopes for 5_2. *ArXiv:2209.09805, v1*, 2022.

[BS22b] John A. Baldwin and Steven Sivek. Framed instanton Floer homology and concordance, II. *ArXiv:2206.11531, v1*, 2022.

[BS22c] John A. Baldwin and Steven Sivek. Khovanov homology detects the trefoils. *Duke Math. J.*, 174(4):885–956, 2022.

[BS23] John A. Baldwin and Steven Sivek. Instantons and L-space surgeries. *J. Eur. Math. Soc. (JEMS)*, 25(10):4033–4122, 2023.

[Eft18] Eaman Eftekhary. Bordered Floer homology and existence of incompressible tori in homology spheres. *Compositio Math.*, 154(6):1222–1268, 2018.

[Flo90] Andreas Floer. Instanton homology, surgery, and knots. In *Geometry of low-dimensional manifolds, 1 (Durham, 1989)*, volume 150 of *London Math. Soc. Lecture Note Ser.*, pages 97–114. Cambridge Univ. Press, Cambridge, 1990.

[GLW19] Sudipta Ghosh, Zhenkun Li, and C.-M. Michael Wong. Tau invariants in monopole and instanton theories. *ArXiv:1910.01758, v3*, 2019.

[HL21] Mathew Hedden and Adam Simon Levine. A surgery formula for knot Floer homology. *arXiv:1901.02488, v2*, 2021.

[Hon00] Ko Honda. On the classification of tight contact structures I. *Geom. Topol.*, 4:309–368, 2000.

[Juh06] András Juhász. Holomorphic discs and sutured manifolds. *Algebr. Geom. Topol.*, 6:1429–1457, 2006.

[KM10] Peter B. Kronheimer and Tomasz S. Mrowka. Knots, sutures, and excision. *J. Differ. Geom.*, 84(2):301–364, 2010.

[KM11] Peter B. Kronheimer and Tomasz S. Mrowka. Knot homology groups from instantons. *J. Topol.*, 4(4):835–918, 2011.

[Li20] Zhenkun Li. Contact structures, excisions and sutured monopole Floer homology. *Algebr. Geom. Topol.*, 20(5):2553–2588, 2020.

[Li21a] Zhenkun Li. Gluing maps and cobordism maps in sutured monopole and instanton Floer theories. *Algebr. Geom. Topol.*, 21(6):3019–3071, 2021.

[Li21b] Zhenkun Li. Knot homologies in monopole and instanton theories via sutures. *J. Symplectic Geom.*, 19(6):1339–1420, 2021.

[LPCS22] Tye Lidman, Juanita Pinzón-Caicedo, and Christopher Scaduto. Framed instanton homology of surgeries on L-space knots. *Indiana Univ. Math. J.*, 71(3):1317–1347, 2022.

[LY21] Zhenkun Li and Fan Ye. SU(2) representations and a large surgery formula. *ArXiv:2107.11005, v1*, 2021.

[LY22a] Zhenkun Li and Fan Ye. Instanton Floer homology, sutures, and Heegaard diagrams. *J. Topol.*, 15(1):39–107, 2022.

[LY22b] Zhenkun Li and Fan Ye. Knot surgery formulae for instanton Floer homology II: applications. *ArXiv:2209.11018, v1*, 2022.

[LY23] Zhenkun Li and Fan Ye. An enhanced Euler characteristic of sutured instanton homology. *Int. Math. Res. Not. IMRN*, page rnad066, 04 2023.

[OS04] Peter S. Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, 2004.

[OS08] Peter S. Ozsváth and Zoltán Szabó. Knot Floer homology and integer surgeries. *Algebr. Geom. Topol.*, 8(1):101–153, 2008.

[OS11] Peter S. Ozsváth and Zoltán Szabó. Knot Floer homology and rational surgeries. *Algebr. Geom. Topol.*, 11(1):1–68, 2011.

[Sar15] Sucharit Sarkar. Moving basepoints and the induced automorphisms of link Floer homology. *Algebr. Geom. Topol.*, 15(5):2479–2515, 2015.

[Sca15] Christopher Scaduto. Instantons and odd Khovanov homology. *J. Topol.*, 8(3):744–810, 2015.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[Zem19] Ian Zemke. Graph cobordisms and Heegaard Floer homology. *ArXiv: 1512.01184, v3*, 2019.

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