

INSTANTON DIMENSIONS OF KNOT SURGERIES OVER ARBITRARY FIELDS

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ABSTRACT. Suppose $K \subset S^3$ is a knot and suppose p and q are co-prime integers with $q \geq 1$. For any field \mathbb{K} , we establish a dimension formula for the framed instanton homology of knot surgeries:

$$\dim I^\sharp(S_{p/q}^3(K); \mathbb{K}) = q \cdot r_{\mathbb{K}}(K) + |p - q \cdot \nu_{\mathbb{K}}^\sharp(K)|$$

for certain integers $r_{\mathbb{K}}(K)$ and $\nu_{\mathbb{K}}^\sharp(K)$, except possibly when $p/q = \nu_{\mathbb{K}}^\sharp(K)$ and $\nu_{\mathbb{K}}^\sharp(K)$ is even. This formula generalizes the result of Baldwin–Sivek from the case $\mathbb{K} = \mathbb{C}$ to arbitrary fields. Based on the result for $\mathbb{K} = \mathbb{Z}/2$, we obtain that $S_{p/q}^3(K)$ is not $SU(2)$ -abelian for any knot K other than the unknot and the right-handed trefoil whenever $p/q \in [0, 6)$ and $p \in \{a^e, 2a^e\}$ for some prime number a and natural number e , thereby extending existing results for $p/q \in [0, 5]$ and $p = a^e$. A byproduct of the techniques developed in this paper is that we generalize the distance-two surgery exact triangle by Culler–Daemi–Xie and Daemi–Miller–Eismeier–Lidman from $\mathbb{Z}/2$ coefficients to any coefficient ring.

1. INTRODUCTION

For a closed connected oriented 3-manifold Y and an unoriented 1-submanifold $\omega \subset Y$, Kronheimer–Mrowka [KM10, KM11b, KM11a] constructed a relative $\mathbb{Z}/4$ -graded \mathbb{Z} -module $I^\sharp(Y, \omega)$, called the *framed instanton homology* of (Y, ω) ; see [LY25a, §2.1] for the relation between different definitions. When $\omega = 0$, the empty set, the module $I^\sharp(Y, \omega)$ can be regarded as a deformation of the homology of the representation variety (without quotient by conjugation) of Y

$$R(Y) = \mathrm{Hom}(\pi_1(Y), SU(2)),$$

and is closely related to the existence of irreducible $SU(2)$ representations of $\pi_1(Y)$ (i.e. $SU(2)$ representations with nonabelian images).

In [BS21, BS22a], Baldwin–Sivek established a dimension formula for the framed instanton homology of knot surgeries $I^\sharp(S_{p/q}^3(K), \omega; \mathbb{C})$ over \mathbb{C} , where $K \subset S^3$ is a knot and p and q are co-prime integers, which is similar to that for Heegaard Floer homology over $\mathbb{F}_2 = \mathbb{Z}/2$ by Hanselman [Han23, Proposition 15]. Baldwin–Sivek’s work depends heavily on the structure theorem and the adjunction formula for the instanton cobordism maps developed in [BS23]. Later, Deeparaj [Bha24] and the authors of this paper [LY25a, LY25b] studied these instanton homologies over \mathbb{F}_2 , which turn out to violate the adjunction formula and provides more information about irreducible $SU(2)$ representations.

In this paper, we study the framed instanton homology of knot surgeries over a general field \mathbb{K} , and obtain applications about irreducible $SU(2)$ representations by taking $\mathbb{K} = \mathbb{F}_2$. It is worth mentioning that many techniques indeed work over any coefficient ring, but we only focus on field coefficients because vector spaces over a field are easier to study. Throughout this paper, we fix a field \mathbb{K} and write \dim for $\dim_{\mathbb{K}}$ for short.

Note that $I^\sharp(Y, \omega)$ depends only on the homology class $[\omega] \in H_1(Y; \mathbb{F}_2)$ up to isomorphism, (though the isomorphism is canonical only when $b_1(Y) = 0$; see §2). A direct computation shows that

$$(1.1) \quad H_1(S_{p/q}^3(K); \mathbb{F}_2) = \begin{cases} 0 & \text{when } p \text{ odd;} \\ \mathbb{F}_2\langle [\mu] \rangle & \text{when } p \text{ even,} \end{cases}$$

where μ is the meridian of K . Hence the dimension of $I^\sharp(Y, \omega)$ is independent of ω when p is odd, and has only two possibilities $I^\sharp(Y) = I^\sharp(Y, 0)$ and $I^\sharp(Y, \mu)$ when p is even.

The main theorem of this paper is the following dimension formula for the framed instanton homology of knot surgeries.

Theorem 1.1 (Propositions 5.1 and 6.3). *Suppose $K \subset S^3$ is a knot and suppose μ is the meridian of K . Suppose p and q are co-prime integers with $q \geq 1$. Then there exists a concordance invariant $\nu_{\mathbb{K}}^\sharp(K) \in \mathbb{Z}$ satisfying $\nu_{\mathbb{K}}^\sharp(\bar{K}) = \nu_{\mathbb{K}}^\sharp(K)$ for the mirror knot \bar{K} . Moreover, for*

$$M = \nu_{\mathbb{K}}^\sharp(K) \text{ and } R = r_{\mathbb{K}}(K) = \min \{\dim I^\sharp(S_M^3(K); \mathbb{K}), \dim I^\sharp(S_M^3(K), \mu; \mathbb{K})\},$$

we have

$$(1.2) \quad \dim I^\sharp(S_{p/q}^3(K); \mathbb{K}) = \dim I^\sharp(S_{p/q}^3(K), \mu; \mathbb{K}) = qR + |p - qM|.$$

except possibly when $p/q = M$ and M is even. In the exceptional case, we have

$$(1.3) \quad \{\dim I^\sharp(S_M^3(K); \mathbb{K}), \dim I^\sharp(S_M^3(K), \mu; \mathbb{K})\} = \{R, R + 2\}.$$

Remark 1.2. From [Sca15, Corollary 1.4], we have

$$\chi(I^\sharp(S_{p/q}^3(K); \mathbb{K})) = \chi(I^\sharp(S_{p/q}^3(K), \mu; \mathbb{K})) = |p|.$$

Hence

$$\dim I^\sharp(S_{p/q}^3(K); \mathbb{K}) = |p| + 2k \text{ and } \dim I^\sharp(S_{p/q}^3(K), \mu; \mathbb{K}) = |p| + 2l$$

for some $k, l \in \mathbb{Z}_+$ depending on p/q . Taking $p/q = \nu_{\mathbb{K}}^\sharp(K)$, we obtain that

$$r_{\mathbb{K}}(K) = |\nu_{\mathbb{K}}^\sharp(K)| + 2h$$

for some $h \in \mathbb{Z}_+$.

Definition 1.3. For a knot $K \subset S^3$, it is called *V-shaped* over \mathbb{K} if either $\nu_{\mathbb{K}}^\sharp(K)$ is odd, or $\nu_{\mathbb{K}}^\sharp(K)$ is even and

$$r_{\mathbb{K}}(K) = \dim I^\sharp(S_{\nu_{\mathbb{K}}^\sharp(K)}^3(K); \mathbb{K}).$$

It is called *W-shaped* over \mathbb{K} if $\nu_{\mathbb{K}}^\sharp(K)$ is even and

$$r_{\mathbb{K}}(K) = \dim I^\sharp(S_{\nu_{\mathbb{K}}^\sharp(K)}^3(K), \mu; \mathbb{K}).$$

Remark 1.4. The case $\mathbb{K} = \mathbb{C}$ in Theorem 1.1 was proven by Baldwin–Sivek [BS21, BS22a]. They also showed that $\nu_{\mathbb{C}}^\sharp(K)$ is even only when it is zero. The case $\mathbb{K} = \mathbb{F}_2$ in Theorem 1.1 was also studied independently in upcoming work of Ghosh–Miller–Eismeier [GME]. Furthermore, they showed that K is always W-shaped over \mathbb{F}_2 and both $\nu_{\mathbb{F}_2}^\sharp(K)$ and $r_{\mathbb{F}_2}(K)$ are divisible by 4. Those results over \mathbb{C} and \mathbb{F}_2 heavily depend on the coefficient fields, while our techniques are more general and potentially applicable to all coefficient rings. In particular, our proof of (1.3) (cf. Proposition 5.10) only involves the $\mathbb{Z}/4$ -grading of instanton

homologies, which is much simpler than the proofs for $\mathbb{K} = \mathbb{C}$ in [BS22a, Theorem 6.1] and $\mathbb{K} = \mathbb{F}_2$ in [GME].

Towards proving the Property P conjecture, Kronheimer–Mrowka [KM04] showed that for any nontrivial knot K and any rational slope $r \in [0, 2]$, the surgery manifold $S_r^3(K)$ is not $SU(2)$ -abelian. Here, a closed 3-manifold Y is called $SU(2)$ -abelian if $\pi_1(Y)$ admits no irreducible representation into $SU(2)$. As $S_r^3(\bar{K}) \cong -S_{-r}^3(K)$ for the mirror knot \bar{K} of K , one can easily extend the set of non- $SU(2)$ -abelian slopes to $[-2, 2]$. Thus, we only consider positive slopes in the following discussion.

Later, the set of non- $SU(2)$ -abelian slopes for any nontrivial knot was extended by many people [BS23, BLSY24, FRW24, SZ22a], which also include all rationals $p/q \in (2, 5)$ with $p = a^e$ for some prime number a and natural number e , all rationals $p/q \in (5, 7)$ for $p = 2^e$, and all rationals greater than a fixed constant N_K that depends on the knot K . From Moser [Mos71], for the right-handed trefoil $T_{2,3}$, we have

$$S_5^3(T_{2,3}) \cong L(5, 4), \quad S_6^3(T_{2,3}) \cong L(2, 1) \# L(3, 2), \quad \text{and} \quad S_7^3(T_{2,3}) \cong L(7, 4),$$

which are all $SU(2)$ -abelian. Moreover, from Sivek–Zentner [SZ22b, Proposition 4.3], for $p/q \in [0, 8]$, the manifold $S_{p/q}^3(T_{2,3})$ is $SU(2)$ -abelian only for

$$\frac{p}{q} = \left\{ 6, 6 \pm \frac{1}{n} \right\}_{n \in \mathbb{Z}_+}.$$

Recently, the authors of this paper [LY25b] showed that $S_5^3(K)$ is not $SU(2)$ -abelian for any nontrivial knot except $T_{2,3}$.

The main applications of this paper are the following theorems. Recall from [BS23] that a knot $K \subset S^3$ is called an *instanton L-space knot* over a field \mathbb{K} if there exists $p/q \in \mathbb{Q}_+$ such that

$$\dim I^\sharp(S_{p/q}^3(K); \mathbb{K}) = |p|.$$

Theorem 1.5 (Theorem 7.2). *Suppose K is a nontrivial knot of genus $g (\geq 1)$ and suppose $p/q \in (0, \infty)$ with $q \geq 1$, $\gcd(p, q) = 1$, and $p \in \{a^e, 2a^e\}$ for some prime number a and natural number e . If $S_{p/q}^3(K)$ is $SU(2)$ -abelian, then the knot K is an instanton L-space knot over any field \mathbb{K} ,*

$$r_{\mathbb{K}}(K) = \nu_{\mathbb{K}}^\sharp(K) \geq \nu_{\mathbb{C}}^\sharp(K) = 2g - 1 \text{ and } p/q \geq \nu_{\mathbb{K}}^\sharp(K).$$

Moreover, for $\mathbb{K} = \mathbb{F}_2$, we have

$$\nu_{\mathbb{F}_2}^\sharp(K) \geq \nu_{\mathbb{C}}^\sharp(K) + 1 = 2g.$$

Theorem 1.6 (Theorem 7.3). *Suppose K is a nontrivial knot and suppose $p/q \in (2, 6)$ with $q \geq 1$, $\gcd(p, q) = 1$, and $p \in \{a^e, 2a^e\}$ for some prime number a and non-negative integer e . Then $S_{p/q}^3(K)$ is $SU(2)$ -abelian only when $K = T_{2,3}$ and*

$$\frac{p}{q} \in \left\{ 6 - \frac{1}{n} \right\}_{n \in \mathbb{Z}_+}.$$

Remark 1.7. We would like to remark that Baldwin–Sivek [BS23] only discussed the non-degeneracy for the case $p = a^e$ for some prime number a and non-negative integer e , but their argument applies verbatim to $p = 2a^e$ as well (cf. Lemma 7.1). This latter case matters now because the integral slope $6 = 2 \times 3$ is precisely of this form.

Through private communication, we were told that Ghosh and Miller-Eismeier, in their upcoming work [GME], could show that over $\mathbb{K} = \mathbb{F}_2$, all knots are W-shaped and $\nu_{\mathbb{F}_2}^\sharp$ and $r_{\mathbb{F}_2}^\sharp$ are both divisible by 4, and as an application, they conclude that for any non-trivial knot which is not $T_{2,3}$, the 7-surgery must be non- $SU(2)$ -abelian. We would like to remark that with the help of their results, the interval $(2, 6)$ in Theorem 1.6 could be further extended to $(2, 8]$, with the exception $T_{2,3}$ and $p/q \in \{6, 6 \pm 1/n\}_{n \in \mathbb{Z}_+}$. Indeed, if K were a knot and its p/q -surgery is $SU(2)$ -abelian, with $p/q \in [6, 8]$ and $p \in \{a^e, 2a^e\}$ (note 6 and 8 are both included), then Theorem 1.5 would suggest that $p/q \geq \nu_{\mathbb{F}_2}^\sharp = r_{\mathbb{F}_2}^\sharp$. Thus, divisibility by 4 would force either $\nu_{\mathbb{F}_2}^\sharp = 4$, which would imply that $K = T_{2,3}$; or $p/q = \nu_{\mathbb{F}_2}^\sharp = 8$. Then, the fact that K must be W-shaped would further exclude the possibility for 8-surgery to be $SU(2)$ -abelian.

Another application of Theorem 1.1 is to provide a better bound for the limiting slope of $SU(2)$ -averse knot, introduced by Sivek-Zentner in [SZ22a]. They showed that if a knot $K \subset S^3$ admits infinitely many $SU(2)$ -cyclic surgeries (which they call a $SU(2)$ -averse knot), then the set of such slopes has a unique limiting point which is a rational number and is denoted by $r(K)$ (cf. [SZ22a, Theorem 1.1]). Moreover, the proof of [SZ22a, Theorem 9.1] states that when $r(K) \geq 0$, there are infinitely many slopes with prime numerators in $[[r(K)] - 1, [r(K)]]$ which produces $SU(2)$ -abelian 3-manifolds, where $[x]$ denotes the minimal integer no less than x . As $r(\bar{K}) = -r(K)$ for the mirror knot K , Theorem 1.6 implies the following corollary.

Corollary 1.8. *Suppose $K \subset S^3$ is an $SU(2)$ -averse knot and suppose \mathbb{K} is an arbitrary field. Then we have*

$$|[r(K)]| \geq |\nu_{\mathbb{K}}^\sharp(K)| + 1.$$

In particular, by taking $\mathbb{K} = \mathbb{F}_2$, we have

$$|[r(K)]| \geq 2g + 1.$$

Finally, we consider the genus-one knots and propose some questions.

Proposition 1.9 (Proposition 8.4). *Suppose $K \subset S^3$ is a genus-one knot and suppose \mathbb{K} is a field with $\text{char}(\mathbb{K}) \neq 2$. Then*

$$|\nu_{\mathbb{K}}^\sharp(K)| \leq 1.$$

Example 1.10. For $n \in \mathbb{Z}_+$, let K_n be the twist knot with n positive half-twist. Note that all K_n are genus-one. Note that K_1 is the left-handed trefoil and K_2 is the figure-eight knot. From [BS21, Theorem 1.13], [Sca15, Corollaries 1.6 and 1.7], and Proposition 1.9, for any field \mathbb{K} with $\text{char}(\mathbb{K}) \neq 2$, we have

$$\begin{aligned} r_{\mathbb{C}}(K_{2n-1}) &= 2n - 1, \quad \nu_{\mathbb{C}}^\sharp(K_{2n-1}) = -1, \quad r_{\mathbb{C}}(K_{2n}) = 2n, \quad \nu_{\mathbb{C}}^\sharp(K_{2n}) = 0, \\ \left(r_{\mathbb{K}}(K_{2n-1}), \nu_{\mathbb{K}}^\sharp(K_{2n-1})\right) &\in \{(2n-1, -1), (2n, 0), (2n+1, 1)\}, \\ \text{and } \left(r_{\mathbb{K}}(K_{2n}), \nu_{\mathbb{K}}^\sharp(K_{2n})\right) &\in \{(2n-1, 1), (2n, 0), (2n+1, -1)\}. \end{aligned}$$

Motivated by the above example, it is natural to ask the following question.

Question 1.11. For any field \mathbb{K} with $\text{char}(\mathbb{K}) \neq 2$ and any knot $K \subset S^3$, do we always have the following equations?

$$r_{\mathbb{K}}(K) = r_{\mathbb{C}}(K) \text{ and } \nu_{\mathbb{K}}^\sharp(K) = \nu_{\mathbb{C}}^\sharp(K).$$

Remark 1.12. By footnote of [BS23, p. 26], the structure theorem of the cobordism map and the generalized eigenspace decomposition of the instanton homology are expected to hold over any field \mathbb{K} with $\text{char}(\mathbb{K}) = 0$. By applying techniques in [BS21, BS22a], Question 1.11 might have a positive answer for all fields with $\text{char}(\mathbb{K}) = 0$.

Motivated by the case $\mathbb{K} = \mathbb{F}_2$ in [GME] and the fact that the figure-eight knot is W-shaped over \mathbb{C} [BS22a, Theorem 10.4], we propose the following question.

Question 1.13. If $\nu_{\mathbb{K}}^\#(K)$ is even for some knot $K \subset S^3$ and some field \mathbb{K} , then do we always have that K is W-shaped over \mathbb{K} ?

1.1. Distance-two surgery exact triangle. The proof of Theorem 1.1 relies on some commutative diagrams of instanton cobordism maps from embedded spheres of self-intersection -1 and -2 , where the one for -1 follows from the blow-up formula for the Donaldson invariant [Don90, DK90, Ozs94], and the one for -2 is an ingredient in the proof of the *distance-two surgery triangle* over the coefficient field \mathbb{F}_2 by Culler–Daemi–Xie [CDX20, Theorem 1.6 for $N = 2$] and Daemi–Miller–Eismeier–Lidman [DMEL24, Theorem 1.12]. Based on those commutative diagrams, together with the original Floer’s surgery exact triangle [Flo90, Sca15], we generalize the distance-two surgery triangle to any coefficient ring. To describe the exact triangle, we first introduce the following setup.

Definition 1.14. Let Y be a closed connected oriented 3-manifold and let $\omega \subset Y$ be an unoriented 1-submanifold. Suppose $K \subset Y$ is a framed knot. We write $Y \setminus \setminus K = Y \setminus \text{int}N(K)$. Suppose μ is the meridian of K and λ is the framed longitude of K , which are both on $\partial(Y \setminus \setminus K)$ and satisfy $\mu \cdot \lambda = -1$. We call (Y, ω, K) a *surgery tuple* if either of the following conditions hold.

- (Y, ω) is *nontrivial admissible*, i.e. there exists a closed embedded oriented surface $\Sigma \subset Y$ such that the algebraic intersection number $\omega \cdot \Sigma$ is odd. The knot K is disjoint from ω and Σ . In this case, we also call (Y, ω, K) *nontrivial admissible surgery tuple*.
- (Y, ω) is *trivial admissible*, i.e. Y is a homology sphere. The knot K is framed by the boundary of a Seifert surface and $\omega \in \{0, \lambda\}$. In this case, we also call (Y, ω, K) *trivial admissible surgery tuple*.

We call the pair (Y, ω) *admissible* if it is either nontrivially or trivially admissible.

For a surgery tuple, one can consider the surgery manifold $Y_{p/q}(K)$ obtained from Y by p/q -surgery on K with respect to the basis (μ, λ) . Note that μ and λ also lie in $Y_{p/q}(K)$, and we write $\mu + \lambda$ for the curve obtained from $\mu \cup \lambda$ by resolving the unique intersection point.

Let $\tilde{K}_{p/q} \subset Y_{p/q}(K)$ be the dual knot in the surgery manifold, i.e. the core of the Dehn filling solid torus. Note that for nontrivial admissible surgery tuple (Y, ω, K) , the pair $(Y_{p/q}(K), \omega, \tilde{K}_{p/q})$ is again a nontrivial admissible surgery tuple. For a trivial admissible surgery tuple (Y, ω, K) , the pair $(Y_{1/q}(K), \omega, \tilde{K}_{1/q})$ is again a trivial admissible surgery tuple, and $(Y_0(K), \mu)$ is a nontrivial admissible pair with the surface Σ being the cap-off of the Seifert surface of K .

For surgery slopes p_1/q_1 and p_2/q_2 such that $|p_1q_2 - p_2q_1| = 1$, we write $W_{p_2/q_2}^{p_1/q_1}$ for the elementary cobordism from the surgery, called the *surgery cobordism*. Let $D_{cc,p_2/q_2}^{p_1/q_1}$ and $D_{c,p_2/q_2}^{p_1/q_1}$ be the cocore disk and the core disk in $W_{p_2/q_2}^{p_1/q_1}$.

Remark 1.15. Some authors use the notation *admissible bundles* only for nontrivial admissible pairs. Here we follow [Sca15, CDX20] and also include the case of homology spheres in the admissible pair, and call them trivial admissible pairs.

Theorem 1.16 (Theorem 4.4). *Suppose (Y, ω, K) is a surgery tuple as in Definition 1.14. Let \mathcal{R} be any coefficient ring. Then there exists an exact triangle*

$$\begin{array}{ccc} I(Y_{-1}(K), \omega; \mathcal{R}) & \xrightarrow{h} & I(Y_1(K), \omega \cup \mu; \mathcal{R}) \\ & \swarrow g_1 + g_2 & \searrow (f_1, f_2) \\ & I(Y, \omega; \mathcal{R}) \oplus I(Y, \omega \cup \lambda; \mathcal{R}) & \end{array}$$

Moreover, the maps f_1, f_2, g_1, g_2, h are instanton cobordism maps with suitable signs (we omit \mathcal{R})

$$\begin{aligned} f_1 &= I(W_\infty^1, (\omega \times I) \cup D_{cc,\infty}^1), \quad f_2 = I(W_\infty^1, (\omega \times I) \cup D_{cc,\infty}^1 \cup D_{cc,\infty}^1), \\ g_1 &= I(W_{-1}^\infty, \omega \times I), \quad g_2 = I(W_{-1}^\infty, (\omega \times I) \cup D_{c,1}^\infty), \\ \text{and } h &: I(W_1^0, (\omega \times I) \cup D_{c,1}^0 \cup D_{cc,1}^0) \circ (W_0^{-1}, (\omega \times I) \cup D_{cc,0}^{-1}). \end{aligned}$$

Remark 1.17. Our proof of Theorem 1.16 does not rely on the usual triangle detection lemma [KM11a, Lemma 7.1], but uses diagram chasing as in [LY24, §5] (especially the proof of [LY24, Proposition 5.3]). Hence we do not need maps for cobordisms with families of metrics, although they are important for the applications of the distance-two triangle in [DMEL24]. If one cares only about the existence of the maps f_1, f_2, g_1, g_2 , then there is a simpler proof based on the octahedral lemma [OSS15, Lemma A.3.10], which does not depend on the commutative diagrams about embedded spheres of self-intersection -1 and -2 . This situation is similar to that in [LY24, Remark 3.7]. Finally, note that the map h in [Sca15, CDX20] is from the cobordism map with “middle end” \mathbb{RP}^3 , which is expected to be the same as the one we used in a neck-stretching argument along \mathbb{RP}^3 and the analysis in the last paragraph of the proof of [DMEL24, Proposition 5.11].

Organization. In §2, we review the dependence of the bundle sets for instanton Floer homology. In §3, we revisit the surgery exact triangle for instanton homology, focusing on the bundle sets of the surgery cobordism maps. In §4, we consider the embedded spheres of self-intersection $0, -1, -2$ in the cobordisms and study their effects on instanton cobordism maps. We also prove Theorem 1.16 as a byproduct. The results in §2-4 work over any coefficient ring \mathcal{R} , while in remaining sections we focus on a field \mathbb{K} . In §5 and §6, we deal with the integral and rational cases of Theorem 1.1 separately. In §7, we study the connection between $SU(2)$ -representations and instanton homology, and prove Theorems 1.5 and 1.6. In §8, we fix the proof of the instanton bypass exact triangle in [BS22b, §4]. As a byproduct, we consider the genus-one knots and prove Proposition 1.9.

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2. BUNDLE SETS OF INSTANTON FLOER HOMOLOGY

In this section, we discuss the bundle sets for instanton Floer homology. By default, every manifold is smooth. We fix a commutative coefficient ring \mathcal{R} and write $\text{Mod}_{\mathcal{R}}$ for the category of \mathcal{R} -modules. We follow the setup in [KM11a, §4.1] and consider the category \mathbf{W} described as follows.

- The objects are nontrivial admissible pairs (Y, ω) from Definition 1.14, where ω is called the *bundle set* in Y .
- The morphisms are isomorphism classes of pairs (W, ν) , where W is an oriented connected cobordism between the 3-manifolds in nontrivial admissible pairs and ν is an unoriented embedded 2-dimensional simplicial complex inside W , serving as a cobordism between bundle sets in nontrivial admissible pairs. We will call (W, ν) a cobordism between nontrivial admissible pairs, or sometimes simply a *cobordism*. We also call ν the *bundle set* in W .

Instanton Floer theory induces a projective functor

$$I(-; \mathcal{R}) : \mathbf{W} \rightarrow \text{Mod}_{\mathcal{R}},$$

where *projective* means the images of objects and morphisms of the functor are only well-defined up to sign, or, equivalently, we consider the unordered pair of objects and morphisms $\{h, -h\}$ in $\text{Mod}_{\mathcal{R}}$. The module $I(Y, \omega; \mathcal{R})$ for a nontrivial admissible pair (Y, ω) is called the *instanton (Floer) homology* of (Y, ω) and the projective morphism $I(W, \nu; \mathcal{R})$ is called an *(instanton) cobordism map*. We will omit \mathcal{R} in the cobordism map but keep it for instanton homology to indicate the dependence. Here we can replace $\text{Mod}_{\mathcal{R}}$ by categories of absolute $\mathbb{Z}/2$ -graded or relative $\mathbb{Z}/8$ -graded \mathcal{R} -modules. There is also instanton homology for a trivial admissible pair from Definition 1.14, and cobordism maps between (either nontrivial or trivial) admissible pairs.

The *framed instanton homology* $I^\#(Y, \omega; \mathcal{R})$ is defined by taking the connected sum of (Y, ω) with

- either $((S^3, H), \alpha)$ for a Hopf link H as a singular set and an trivial arc α connecting two components of H ,
- or (T^3, S^1) with larger gauge group corresponding to the dual torus R of S^1 ,

where Y can be any closed oriented connected 3-manifold and ω can be any unoriented 1-submanifold; see [LY25a, §2.1] for the relation between different definitions. It also has similar properties as $I(Y, \omega; \mathcal{R})$, but only relatively $\mathbb{Z}/4$ -graded; see §5.3 for more discussion and applications.

It is worth mentioning that the construction of instanton homology for (Y, ω) needs an actual $SO(3)$ -bundle $P \rightarrow Y$, but the pair (Y, ω) only determines a transitive system of bundles: different bundles are related by canonical cobordisms over $Y \times I$, which lead to canonical isomorphisms (up to signs) for the associated instanton homology. Thus, the instanton homology (Y, ω) is only well-defined up to sign.

The sign ambiguity can be fixed by further picking a homology orientation or an almost complex structure on the cobordism between admissible pairs, but we are satisfied with the projective property. We can pick representatives in the unordered pair $\{h, -h\}$ of objects and morphisms, at the cost that all equations between cobordism maps only hold up to a sign.

Recall that ω in an admissible pair (Y, ω) is indeed a geometric representative of the Poincaré duality of the second Stiefel–Whitney class $\omega_2(P) \in H^2(Y; \mathbb{F}_2)$ of the $SO(3)$ bundle $P \rightarrow Y$. Hence the isomorphism class of $I(Y, \omega; \mathcal{R})$ depends only on the homology class $[\omega] \in H_1(Y; \mathbb{F}_2)$. However, when considering cobordism maps, we need to pin down actual modules instead of just isomorphism classes. We make explicit the effect of changing geometric representatives within the same mod 2 homology class as follows. We first deal with the dependence of the bundle sets in cobordisms.

Lemma 2.1. *Let*

$$(W, \nu_i) : (Y_1, \omega_1) \rightarrow (Y_2, \omega_2) \text{ for } i = 1, 2$$

be two cobordisms between the same admissible pairs. If

$$(2.1) \quad [\nu_1 \cup \nu_2] = 0 \in H_2(W; \mathbb{F}_2),$$

$$I(W, \nu_1) = \pm I(W, \nu_2) : I(Y_1, \omega_1; \mathcal{R}) \rightarrow I(Y_2, \omega_2; \mathcal{R}).$$

Proof. From [KM11a, §4.1], the mod 2 homology class as the dual of the (relative) second Stiefel–Whitney class determines the $SO(3)$ bundle on W up to bundle isomorphisms and addition of instantons and yields cobordism maps up to a sign. Note that the (relative) second Stiefel–Whitney class itself does not determine the bundle up to bundle isomorphisms because the first Pontryagin class could be different. \square

Remark 2.2. In [BS22b, §4] and some subsequent work (e.g. [BS23, Remark 2.11] and [ABDS22, Remark 2.5]), the authors claimed the cobordism map $I(W, \nu)$ depends only on the homology class $[\nu] \in H_2(W, \partial W; \mathbb{F}_2)$ up to sign, which is indeed not true in general; see Example 2.7. In particular, the proof of the bypass exact triangle in [BS22b, §4] needs to be fixed; see §8. From the long exact sequence associated to the pair $(W, \partial W)$

$$H_2(\partial W; \mathbb{F}_2) \rightarrow H_2(W; \mathbb{F}_2) \rightarrow H_2(W, \partial W; \mathbb{F}_2)$$

The condition (2.1) implies that $[\nu_1] = [\nu_2] \in H_2(W, \partial W; \mathbb{F}_2)$, but the converse is not true in general if $H_2(\partial W; \mathbb{F}_2) \neq 0$. Note that $H_2(\partial W; \mathbb{F}_2) = 0$ if and only if $b_1 = 0$ for both components of ∂W .

To fix the sign, one needs to choose some homology orientation of W and the bundle set ν should be oriented and represents a class $H_2(W, \partial W; \mathbb{Z})$. Similarly, the condition $[\nu_1 \cup \nu_2] = 0 \in H_2(W; \mathbb{Z})$ instead of $[\nu_1] = [\nu_2] \in H_2(W, \partial W; \mathbb{Z})$ implies the identification of the cobordism map. For ν_1 and ν_2 on the same W with the same homology orientation and $[\nu_1 \cup (-\nu_2)] = 2e \in H_2(W; \mathbb{Z})$, we have

$$I(W, \nu_1) = (-1)^{e \cdot e} I(W, \nu_2).$$

See the end of [KM22, §2.3], [Don87, Corollary (3.28)], and [BS23, Remark 6.2]. In this paper, for simplicity, we will only consider the mod 2 class and the cobordism maps up to signs.

Next, we handle the situation where the bundle set on the 3-manifolds varies within its mod 2 homology class.

Lemma 2.3. *Suppose Y is a closed connected oriented 3-manifold and $\omega \subset Y$ is unoriented 1-submanifold such that $[\omega] = 0 \in H_1(Y; \mathbb{F}_2)$. Then there exists an embedded, possibly non-orientable surface $S \subset Y$ such that $\partial S = \omega$.*

Remark 2.4. This lemma appears to be well known but we were not able to find an explicit proof in the literature to cite. For example, [Con79, Theorem 8.3] indicates the existence of a singular surface but not for embedded surfaces; [Hat07, Lemma 3.6] handles the \mathbb{Z} coefficient but not \mathbb{F}_2 ; [BCK22] states the precise result but without a proof. Hence we decide to include a proof of the lemma for the sake of completeness of the paper.

Proof of Lemma 2.3. We adapt the proof of [Hat07, Lemma 3.6]. We triangulate Y so that ω is in the 1-skeleton. The fact that $[\omega] = 0 \in H_1(Y; \mathbb{F}_2)$ then implies that there exists a 2-chain σ such that

$$\partial\sigma = \omega + 2 \cdot \theta$$

for some 1-chain θ . We then perturb σ to obtain a surface S as follows.

- For a 2-simplex f in σ , assume that its coefficient is $\lambda \neq 0$. We then take $|\lambda|$ copies of f , pushing the interiors of these copies of f disjoint from each other, while keeping the boundary (1-chain) unchanged and still coinciding with ∂f .
- For a 1-simplex e in σ , we pair the two adjacent copies of the (perturbed) 2-simplices and push the interior of the common boundary off the 1-simplex and make them pairwise disjoint. Note if e is in ω then it is involved in an odd number of 2-simplices in σ , and after pairing, only one is left. If e is not in ω , all 2-simplices having e as part of their boundary can be paired. Let the new complex after perturbations be σ' .
- For a vertex v involved in σ , we can take a small ball in Y centered at v . The boundary S^2 intersects σ' at either a (possibly disconnected) simple closed curve if v is not in ω , or a disjoint union of a simple closed curve and a connected single arc if v is in ω . A standard argument enables us to iteratively cut σ' along an innermost circle and glue back the disk it bounds (and then push the disk off the sphere S^2). The arc component will be left untouched. The final result is an embedded surface S with $\partial S = \omega$, and we are done.

□

Now, suppose we have two admissible pairs (Y, ω_1) and (Y, ω_2) such that

$$[\omega_1] = [\omega_2] \in H_1(Y; \mathbb{F}_2).$$

Then, by Lemma 2.3, there exists an embedded, possibly non-orientable surface $S \subset Y$ such that $\partial S = \omega_1 \cup \omega_2$. Inside the product cobordism $Y \times I$, we can make S into a properly embedded surface ν_S and hence obtain a cobordism

$$(Y \times I, \nu_S) : (Y, \omega_1) \rightarrow (Y, \omega_2).$$

Thus we obtain a map

$$\mathbb{I}_S = I(Y \times I, \nu_S) : I(Y, \omega_1; \mathcal{R}) \rightarrow I(Y, \omega_2; \mathcal{R}).$$

Lemma 2.5. *Under the above setup, we have the following.*

- For any choice of S , the map \mathbb{I}_S is an isomorphism.
- If $H_2(Y; \mathbb{F}_2) = 0$ (or equivalently $b_1(Y) = 0$), then, up to sign, \mathbb{I}_S is independent of the choice of S . As a consequence, up to a sign, $I(Y, \omega_1; \mathcal{R})$ and $I(Y, \omega_2; \mathcal{R})$ are canonically isomorphic.

Proof. The surface S also induces a cobordism

$$(Y \times I, \nu'_S) : (Y, \omega_2) \rightarrow (Y, \omega_1)$$

and we write \mathbb{I}'_S for the corresponding cobordism map. It is straightforward to check that

$$\omega_1 \times [0, 2] \cup \nu_S \cup \nu'_S$$

represents the trivial homology class in $H_2(Y \times [0, 2]; \mathbb{F}_2) \cong H_2(Y; \mathbb{F}_2)$. By Lemma 2.1, we know the composition

$$\mathbb{I}'_S \circ \mathbb{I}_S = \pm \text{id}.$$

Similarly, we know that

$$\mathbb{I}_S \circ \mathbb{I}'_S = \pm \text{id}.$$

Hence \mathbb{I}_S must also be an isomorphism.

If further we have $H_2(Y; \mathbb{F}_2) = 0$, then $H_2(Y \times I; \mathbb{F}_2) = 0$ and Lemma 2.1 applies again to show that, up to sign, \mathbb{I}_S is independent of the choice of S . \square

Remark 2.6. In the case of $b_1(Y) > 0$, suppose R is an embedded, possibly non-orientable surface in Y representing a nontrivial homology class in $H_2(Y; \mathbb{F}_2)$. Let $\omega \subset R$ be a separating 1-submanifold such that $R \setminus \omega = R_1 \cup R_2$. Then $[\omega] = 0 \in H_1(Y; \mathbb{F}_2)$, but the isomorphisms

$$\mathbb{I}_{R_1}, \mathbb{I}_{R_2} : I(Y, \omega; \mathcal{R}) \rightarrow I(Y, 0; \mathcal{R})$$

are not necessarily the same up to sign. The best result we obtained from Lemma 2.1 is

$$(\mathbb{I}_{R_1} \circ \mathbb{I}'_{R_2})^2 = \pm \text{id} \text{ and } (\mathbb{I}_{R_1} \circ \mathbb{I}'_{R_1})^2 = \pm \text{id},$$

where

$$\mathbb{I}'_{R_1}, \mathbb{I}'_{R_2} : I(Y, 0; \mathcal{R}) \rightarrow I(Y, \omega; \mathcal{R}).$$

We conclude this section with an example that is related to Remark 2.2.

Example 2.7. We present an explicit example for which we have two bundle sets ν_1, ν_2 in some cobordism W such that

$$[\nu_1] = [\nu_2] \in H_2(W, \partial W; \mathbb{F}_2).$$

yet they lead to non-identical cobordism maps even allowing sign ambiguity. We take $\mathcal{R} = \mathbb{C}$.

Suppose $K \subset S^3$ is the torus knot $T_{2,5}$. It is a genus-two L-space knot in the sense of [BS23]. Indeed, one may construct more examples from instanton L-space knots of genera larger than two. We consider the surgery cobordism associated to the 0-surgery on K

$$W = W_0^\infty : S^3 \rightarrow S_0^3(K)$$

and take $\tilde{\nu}_0$ from [BS22a, Formula 3.3] as the bundle set. We consider the cobordism map

$$I^\#(W, \tilde{\nu}_0) : I^\#(S^3) \rightarrow I^\#(S_0^3(K), \mu),$$

where μ is a meridian of K . Indeed, from Remark 3.8, we know that $\tilde{\nu}_0$ is the cocore disk, though here we do not use this fact.

Let $\bar{\Sigma} \subset W$ be the closed oriented surface that is the union of a genus-two Seifert surface of K and a core disk. Note that it is also isotopic to the cap-off of the Seifert surface $\bar{\Sigma}'$ in $S_0^3(K)$. Hence $\bar{\Sigma} \cdot \bar{\Sigma} = 0$, where \cdot denotes the pairing of $H_2(W, \partial W; \mathbb{Z})$ and $H_2(W; \mathbb{Z})$.

Note that $\bar{\Sigma}$ represents a generator of

$$\mathbb{Z} \cong H_2(S_0^3(K); \mathbb{Z}) \xrightarrow{\cong} H_2(W; \mathbb{Z}).$$

Let

$$\begin{aligned} s_i : H_2(W; \mathbb{Z}) &\rightarrow 2\mathbb{Z} \\ [\bar{\Sigma}] &\mapsto 2i. \end{aligned}$$

From [BS23, Theorem 1.16] and the facts that $b_1(W) = 0$, $g(\bar{\Sigma}) = 2$ and $\bar{\Sigma} \cdot \bar{\Sigma} = 0$, we obtain a decomposition of the cobordism map

$$(2.2) \quad I^\#(W, \tilde{\nu}_0) = \sum_{i=-1}^1 I^\#(W, \tilde{\nu}_0; s_i),$$

where the image of $I^\#(W, \tilde{\nu}_0; s_i)$ lies in the $(2, 2i)$ -generalized eigenspace of the actions $(\mu(\text{pt}), \mu(\bar{\Sigma}'))$. Moreover, from [BS21, Theorem 1.18] and [BS22a, Proposition 4.3], we know that each term in the decomposition (2.2) is nonvanishing.

From [BS23, Formula (7.4)], we have $\tilde{\nu}_0 \cdot \bar{\Sigma} = 1$ (the bundle set $\tilde{\nu}_0$ was written as ν_0 in [BS23, Formula (7.2)]). From [BS23, Theorem 1.16 (5)], we have

$$I^\#(W, \tilde{\nu}_0 \cup \bar{\Sigma}) = \sum_{i=-1}^1 I^\#(W, \tilde{\nu}_0 \cup \bar{\Sigma}; s_i) = \sum_{i=-1}^1 (-1)^{i+1} I^\#(W, \tilde{\nu}_0; s_i),$$

which cannot be equal to $I^\#(W, \tilde{\nu}_0)$, even up to sign.

On the other hand, from the long exact sequence of the pair $(W, \partial W)$, we have an isomorphism

$$H_2(W, \partial W; \mathbb{F}_2) \xrightarrow{\cong} H_1(\partial W; \mathbb{F}_2) \cong H_1(S_0^3(K); \mathbb{F}_2) \cong \mathbb{F}_2.$$

Thus, we conclude that

$$[\tilde{\nu}_0] = [\tilde{\nu}_0 \cup \bar{\Sigma}] \in H_2(W, \partial W; \mathbb{F}_2)$$

because they have the same boundary. The crucial point in this example is that the map

$$H_2(W; \mathbb{F}_2) \rightarrow H_2(W, \partial W; \mathbb{F}_2)$$

in the long exact sequence vanishes, though both are isomorphic to \mathbb{F}_2 .

3. SURGERY EXACT TRIANGLE REVISITED

In this section, we revisit the surgery exact triangle for instanton homology. We first focus on the general case and then specialize to the case of surgeries on knots in S^3 . Again, we fix a coefficient ring \mathcal{R} .

Proposition 3.1 ([Sca15, Theorem 2.1]). *Suppose (Y, ω, K) is a surgery tuple as in Definition 1.14. Recall that $\mu, \lambda \subset \partial(Y \setminus K)$ are the meridian and the framed longitude of K , respectively. Then there exists an exact triangle*

$$(3.1) \quad \begin{array}{ccc} I(Y, \omega; \mathcal{R}) & \xrightarrow{F_0^\infty} & I(Y_0(K), \omega \cup \mu; \mathcal{R}) \\ & \searrow^{F_\infty^1} & \swarrow^{F_1^0} \\ & I(Y_1(K), \omega; \mathcal{R}) & \end{array}$$

where the maps are cobordism maps associated to the surgery cobordisms with certain bundle sets. We will call the components in the bundle set other than ω and $\omega \times I$ the extra bundle sets.

Based on the discussion in [ABDS22, §2.3], we obtain the following result about the bundle sets.

Lemma 3.2. *The bundle sets for the cobordism maps in (3.1) are described as follows.*

- The bundle set for F_0^∞ consists of $\omega \times I$ and the cocore disk $D_{cc,0}^\infty$.
- The bundle set for F_1^0 consists of $\omega \times I$ and the core disk $D_{c,1}^0$.
- The bundle set for F_∞^1 consists of $\omega \times I$.

Proof. The first two cases follow directly from the discussion in [ABDS22, §2.3], though one needs to be careful about the issue in Remark 2.2. For the sake of completeness of the paper, we state the proof more explicitly as follows.

The first case follows from [Sca15, §3.7] directly. Note that the meridian μ is isotopic to the dual knot $\tilde{K}_0 \subset Y_0(K)$, which is the boundary of the cocore disk in the first cobordism.

In the second case, the bundle set is originally described as the union $(\omega \cup \tilde{K}_0) \times I$ and the cocore disk $D_{cc,1}^0$. Note that the boundary of the cocore disk is the dual knot $\tilde{K}_1 \subset Y_1(K)$, which is isotopic to λ in $Y_1(K)$. Let $D_1 \subset Y_0(K)$ be a properly embedded disk in the Dehn-filling solid torus. Note that the outgoing end of the bundle set is $\omega \cup \mu \cup \lambda$. To make the target of F_1^0 become $I(Y_1(K), \omega; \mathcal{R})$, we also need to add D_1 into the bundle set, which induces an isomorphism by Lemma 2.5.

We also consider disks $D_\infty \subset Y = Y_\infty(K)$ and $D_0 \subset Y_0(K)$ similarly. Then the cocore disk $D_{cc,1}^0$ is isotopic to $(\lambda \times I) \cup D_0$ and the core disk $D_{c,1}^0$ is isotopic to $((\mu + \lambda) \times I) \cup D_1 \cup D_0$, where the D_0 in the cocore disk is to make its boundary become μ instead of $\mu + \lambda$. Hence the bundle set for F_1^0 is

$$\begin{aligned} ((\omega \cup \mu) \times I) \cup D_{cc,1}^0 \cup D_1 &= ((\omega \cup \mu \cup \lambda) \times I) \cup D_0 \cup D_1 \\ &= ((\omega \cup (\mu + \lambda)) \times I) \cup D_0 \cup D_1, \\ &= (\omega \times I) \cup D_{c,1}^0. \end{aligned}$$

where we implicitly use Lemma 2.1 in the second equation.

A similar trick involving the isotopies of the cocore disk and the core disk can be applied to the third case. The bundle set in the third case is originally described as the union $(\omega \cup (\mu + \lambda)) \times I$ and the cocore disk $D_{cc,\infty}^1$, whose incoming end is $\omega \cup (\mu + \lambda)$ and outgoing end is $\omega \cup 2 \cdot (\mu + \lambda)$. To make the source and the target of F_∞^1 compatible with (3.1), we need to add D_1 for the incoming end and add a trivial annulus A with boundary $2 \cdot (\mu + \lambda)$ for the outgoing end. The cocore disk is isotopic to $((\mu + \lambda) \times I) \cup D_1$. Hence the bundle set for F_∞^1 is

$$((\omega \cup (\mu + \lambda)) \times I) \cup D_{cc,\infty}^1 \cup D_1 \cup A = (\omega \times I) \cup 2 \cdot (\mu + \lambda) \times I \cup 2 \cdot D_1 \cup A.$$

Note that $2 \cdot (\mu + \lambda) \times I \cup 2 \cdot D_1 \cup A$ bounds an embedded 3-sphere in the cobordism and hence represents the trivial mod 2 homology class. By Lemma 2.1, this union does not affect the cobordism map up to sign. Hence the bundle set for F_∞^1 is just $\omega \times I$. \square

To specify the bundle sets in the cobordism maps, we introduce the following definition.

Definition 3.3. For $\epsilon = \epsilon_1 \epsilon_2 \in \{00, 01, 10, 11\}$ and $|p_1 q_2 - p_2 q_1| = 1$, we write $F_{p_2/q_2}^{p_1/q_1}(\epsilon)$ for the cobordism map associated to the surgery cobordism from $Y_{p_1/q_1}(K)$ and $Y_{p_2/q_2}(K)$ with

certain bundle sets, where ϵ_1 and ϵ_2 denote the presence of the core disk and the cocore disk, respectively. The product $\omega \times I$ will always be included in the bundle set.

By varying the choice of ω in Proposition 3.1, we can obtain exact triangles with the same vertices and different maps. We first consider the case when (Y, ω, K) is a nontrivial admissible surgery tuple. Note that adding any unoriented 1-submanifold $\omega' \subset Y$ disjoint from Σ to ω in Definition 1.14 yields another nontrivial admissible surgery tuple $(Y, \omega \cup \omega', K)$. Since $\mu \cdot \lambda = -1$, we know that μ and λ generate

$$(3.2) \quad H_1(\partial(Y \setminus K); \mathbb{F}_2) \cong \mathbb{F}_2 \oplus \mathbb{F}_2.$$

Hence there are four natural choices of ω' that represent the four elements in (3.2), namely

$$(3.3) \quad \omega' = 0, \mu, \mu + \lambda, \lambda.$$

Hence we have the following proposition.

Proposition 3.4. *Suppose (Y, ω, K) is a nontrivial admissible surgery tuple as in Definition 1.14. Then we have the following exact triangles.*

$$(3.4) \quad \begin{array}{ccc} I(Y, \omega; \mathcal{R}) & \xrightarrow{F_0^\infty(01)} & I(Y_0(K), \omega \cup \mu; \mathcal{R}) \\ & \swarrow F_\infty^1(00) & \searrow F_1^0(10) \\ & I(Y_1(K), \omega; \mathcal{R}) & \end{array}$$

$$(3.5) \quad \begin{array}{ccc} I(Y, \omega; \mathcal{R}) & \xrightarrow{F_0^\infty(00)} & I(Y_0(K), \omega; \mathcal{R}) \\ & \swarrow F_\infty^1(10) & \searrow F_1^0(01) \\ & I(Y_1(K), \omega \cup \mu; \mathcal{R}) & \end{array}$$

$$(3.6) \quad \begin{array}{ccc} I(Y, \omega \cup \lambda; \mathcal{R}) & \xrightarrow{F_0^\infty(10)} & I(Y_0(K), \omega; \mathcal{R}) \\ & \swarrow F_\infty^1(01) & \searrow F_1^0(00) \\ & I(Y_1(K), \omega; \mathcal{R}) & \end{array}$$

$$(3.7) \quad \begin{array}{ccc} I(Y, \omega \cup \lambda; \mathcal{R}) & \xrightarrow{F_0^\infty(11)} & I(Y_0(K), \omega \cup \mu; \mathcal{R}) \\ & \swarrow F_\infty^1(11) & \searrow F_1^0(11) \\ & I(Y_1(K), \omega \cup \mu; \mathcal{R}) & \end{array}$$

If (Y, ω, K) is a trivial admissible surgery tuple as in Definition 1.14, then we only have (3.4) and (3.7) as $(Y_0(K), \omega = 0)$ is not admissible.

Remark 3.5. For triangles in Proposition 3.4, the extra bundle set in each vertex other than ω is isotopic to the dual knot in the corresponding surgery manifold. The first three triangles appeared in [Sca15, Figure 1], where the second and the third are obtained from the first one by changing the nontrivial admissible surgery tuple. Here we obtain those two triangles by modifying ω . The fourth triangle is new, but may be well-known to experts.

Proof of Proposition 3.4. We only consider the case of a nontrivial admissible surgery tuple. The case of a trivial admissible surgery tuple is similar. The first triangle follows directly from Proposition 3.1 and Lemma 3.2, with the notation from Definition 3.3. The remaining three triangles are obtained from the first one by adding ω' in (3.3) to ω and applying the identification of bundle sets in the proof of Lemma 3.2. More explicitly, we again suppose D_0, D_1, D_∞ are disks in the Dehn filling solid tori of $Y_0(K), Y_1(K), Y = Y_\infty(K)$, respectively. Then we have the following identifications for core disks and cocore disks.

$$(3.8) \quad \begin{aligned} D_{c,0}^\infty &= (\lambda \times I) \cup D_0, \quad D_{cc,0}^\infty = (\mu \times I) \cup D_\infty, \\ D_{c,1}^0 &= ((\mu + \lambda) \times I) \cup D_1 \cup D_0, \quad D_{cc,1}^0 = (\lambda \times I) \cup D_0 \cup D_1, \\ D_{c,\infty}^1 &= (\mu \times I) \cup D_\infty, \text{ and } D_{cc,\infty}^1 = ((\mu + \lambda) \times I) \cup D_1 \cup D_\infty. \end{aligned}$$

where $D_0 \subset D_{c,1}^0$, $D_1 \subset D_{cc,1}^0$, and $D_\infty \subset D_{cc,\infty}^1$ are added so that the ends of the bundle sets coincide with the dual knot representatives in

$$\tilde{K}_\infty = K = \lambda \subset Y = Y_\infty(K), \quad \tilde{K}_0 = \mu \subset Y_0(K), \quad \text{and } \tilde{K}_1 = \mu + \lambda \subset Y_1(K),$$

Finally, we apply Lemmas 2.1 and 2.5 to obtain the results. \square

Using Lemmas 2.1 and 2.5, we can further modify the exact triangles in Proposition 3.4. The “half lives, half dies” theorem shows that

$$(3.9) \quad \ker(H_1(\partial(Y \setminus K); \mathbb{F}_2) \rightarrow H_1(Y \setminus K; \mathbb{F}_2)) \cong \mathbb{F}_2.$$

Then we have the following.

- $[K] = [\lambda] \neq 0 \in H_1(Y; \mathbb{F}_2)$ if and only if $[\mu]$ generates (3.9);
- $[\tilde{K}_0] = [\mu] \neq 0 \in H_1(Y_0(K); \mathbb{F}_2)$ if and only if $[\lambda]$ generates (3.9);
- $[\tilde{K}_1] = [\mu] \neq 0 \in H_1(Y_1(K); \mathbb{F}_2)$ if and only if $[\mu + \lambda]$ generates (3.9).

In the nontrivial admissible surgery tuple case, we know that exactly one of the above cases will happen. In the trivial admissible surgery tuple case, we know that the second case happens. Without loss of generality, we assume that the second case happens. The following corollary follows directly from Lemmas 2.1 and 2.5, and from homology computations.

Corollary 3.6. *Suppose (Y, ω, K) is a nontrivial admissible surgery tuple as in Definition 1.14. Suppose $[\lambda]$ generates (3.9) and hence also trivial in $H_1(Y; \mathbb{F}_2)$ and suppose $S \subset Y$ is a surface from Lemma 2.3. Suppose $D_1 \subset Y_1(K)$ is the meridian disk and let $S + D_1$ be the surface obtained from $S \cup D_1$ by resolving intersection points. Then (3.5), (3.6), (3.7) reduce to the following triangles.*

$$(3.10) \quad \begin{array}{ccc} I(Y, \omega; \mathcal{R}) & \xrightarrow{F_0^\infty(00)} & I(Y_0(K), \omega; \mathcal{R}) \\ & \searrow F_\infty^1(10) \circ \mathbb{I}'_{S+D_1} & \swarrow \mathbb{I}_{S+D_1} \circ F_1^0(01) \\ & I(Y_1(K), \omega; \mathcal{R}) & \end{array}$$

$$(3.11) \quad \begin{array}{ccc} I(Y, \omega; \mathcal{R}) & \xrightarrow{F_0^\infty(10) \circ \mathbb{I}'_S} & I(Y_0(K), \omega; \mathcal{R}) \\ & \searrow \mathbb{I}_S \circ F_\infty^1(01) & \swarrow F_1^0(00) \\ & I(Y_1(K), \omega; \mathcal{R}) & \end{array}$$

$$(3.12) \quad \begin{array}{ccc} I(Y, \omega; \mathcal{R}) & \xrightarrow{F_0^\infty(11) \circ \mathbb{I}'_S} & I(Y_0(K), \omega \cup \mu; \mathcal{R}) \\ & \searrow \mathbb{I}_S \circ F_\infty^1(11) \circ \mathbb{I}'_{S+D_1} & \swarrow \mathbb{I}_{S+D_1} \circ F_1^0(11) \\ & I(Y_1(K), \omega; \mathcal{R}) & \end{array}$$

Moreover, we have identifications of maps

$$(3.13) \quad F_\infty^1(10) \circ \mathbb{I}'_{S+D_1} = \pm \mathbb{I}_S \circ F_\infty^1(01) \text{ and } \mathbb{I}_S \circ F_\infty^1(11) \circ \mathbb{I}'_{S+D_1} = \pm F_\infty^1(00).$$

If (Y, ω, K) is a trivial admissible surgery tuple as in Definition 1.14, then we only have (3.12) and the second equation in (3.13).

Remark 3.7. One can compare triangles in Proposition 3.4 and Corollary 3.6 with the existing surgery exact triangles in Heegaard Floer theory and monopole Floer theory. The triangles in those theories are first established over \mathbb{F}_2 [OS05, KMOS07], where an essential ingredient is that a cobordism with an embedded sphere of self-intersection -1 induces a cobordism map with a multiple of 2, which vanishes over \mathbb{F}_2 . This is not true in general in instanton theory; see Remark 4.2. This multiple of 2 cannot be canceled by altering the homology orientation on the cobordism, and one must use twisted coefficients to establish the triangle when the characteristic of the coefficient ring is not 2. Note that the twisted coefficient is roughly the analog of the extra bundle set. The case of Heegaard Floer theory was recently resolved by Abouzaid–Manolescu [AM25, §6], where they used a trick that is similar to (3.10) and (3.11), i.e. pick the twisted coefficient cleverly so that the Floer homologies remain the same but two of the cobordism maps are modified and the compositions of two consecutive cobordism maps vanish. The case of monopole Floer theory was resolved over \mathbb{Q} by Lin–Ruberman–Saveliev and over $\mathbb{Z}[i]$ by Freeman [Fre21]. Note that the one over \mathbb{Q} is obtained by adding certain signs to the components of the cobordism maps, which might be induced from some twisted coefficients. The one over $\mathbb{Z}[i]$ is similar to (3.7), where the twist coefficient is chosen for cycles from the dual knots and the union of the core and cocore disks. From (3.12), at least one of the Floer homologies is not isomorphic to those without local coefficients.

Remark 3.8. Other than (3.13), there is no further relation for cobordism maps, even when the source and the target are the same, because the difference of the bundle set represents nontrivial mod 2 homology class. Baldwin–Sivek [BS21, §2.2] used the trick of adding extra bundle sets to make all three vertices in the exact triangle have no extra bundle sets, which is indeed either (3.10) or (3.11). More explicitly, from [BS21, Equations (2.3) and (2.4)], for a knot $K \subset S^3$ and the cobordism $W_n^\infty : S^3 \rightarrow S_n^3(K)$, we have the following result.

- When n is even, we use (3.10) and (3.4) with

$$(Y, Y_0(K), Y_1(K)) = (S^3, S_n^3(K), S_{n+1}^3(K))$$

for the case where the extra bundle set is trivial and the meridian μ , respectively.

- When n is odd, we use (3.11) and (3.4) with

$$(Y, Y_0(K), Y_1(K)) = (S_n^3(K), S_{n+1}^3(K), S^3)$$

for the case where the extra bundle set is trivial and the meridian (now it is $\mu + \lambda$ instead of μ), respectively.

Hence our notation of the bundle set in Definition 3.3 is related to Baldwin–Sivek’s notation via

$$I^\sharp(X_n, \nu_n) = F_n^\infty(00) \text{ and } I^\sharp(X_n, \tilde{\nu}_n) = F_n^\infty(01) \text{ when } n \text{ is even,}$$

$$I^\sharp(X_n, \nu_n) = \mathbb{I}_{S_n} \circ F_n^\infty(01) = \pm F_n^\infty(10) \circ \mathbb{I}'_S \text{ and}$$

$$I^\sharp(X_n, \tilde{\nu}_n) = F_n^\infty(00) = \pm \mathbb{I}_{S_n} \circ F_\infty^1(11) \circ \mathbb{I}'_S \text{ when } n \text{ is odd;}$$

where S is a Seifert surface of K , S_n is obtained from S , the meridian disk $D_n \subset S_n^3(K)$, and $|n - 1/2|$ annuli with boundary $2 \cdot \mu$ by resolving intersection points. Note that $\tilde{\nu}_n$ is only defined for $n = 0, -1$ in [BS22a] but it can be extended to any integer n by considering the triangle obtained from adding the meridian (μ when n is even and $\mu + \lambda$ when n is odd) to the bundle set.

Finally, we consider surgery exact triangle for framed instanton homology of knot surgeries. We fix a knot $K \subset S^3$ and take its meridian μ and the Seifert longitude λ with $\mu \cdot \lambda = -1$. We write

$$S^3 \setminus K = S^3 \setminus \text{int}N(K)$$

for the knot complement.

For a surgery slope $r = p/q \in \mathbb{Q} \cup \{\infty = 1/0\}$, we call it *even* or *odd* if the numerator p is even or odd, respectively.

For a fixed coefficient ring \mathcal{R} , we write

$$(3.14) \quad \begin{aligned} \mathcal{H}(r, \omega) &= I^\sharp(S_r^3(K), \omega; \mathcal{R}), \\ \mathcal{H}(r) &= I^\sharp(S_r^3(K), \omega = 0; \mathcal{R}) \text{ and } \tilde{\mathcal{H}}(r) = I^\sharp(S_r^3(K), \omega = \tilde{K}_r; \mathcal{R}) \end{aligned}$$

where the dual knot \tilde{K}_r is the bundle set rather than singular locus as in [KM11a, §4.3]. If we want to specify the knot K , then we use $\mathcal{H}(K, r, \omega)$, $\mathcal{H}(K, r)$, and $\tilde{\mathcal{H}}(K, r)$ instead.

Note that the dual knot $\tilde{K}_r \subset S_r^3(K)$ represents a nontrivial mod 2 homology class if and only if r is even. Due to (1.1) and Lemma 2.5, we always have

$$[\tilde{K}_r] = [\mu] \in H_1(S_r^3(K); \mathbb{F}_2),$$

and there exists a canonical isomorphism

$$\tilde{\mathcal{H}}(r) \cong \mathcal{H}(r, \mu),$$

In particular, when $r \in \mathbb{Z}$, we pick $\tilde{K}_r = \mu$ and when $r = \infty$, we pick $\tilde{K}_\infty = \lambda$.

Then we have the following proposition analogous to Proposition 3.4 and Corollary 3.6.

Proposition 3.9 ([Sca15, §7.5]). *For $i = 0, 1, 2$, suppose p_i and q_i are co-prime integers and suppose $r_i = p_i/q_i$. If $p_i q_{i+1} - p_{i+1} q_i = 1$ for all $i \in \mathbb{Z}/3$, we call (r_0, r_1, r_2) a slope triad. In such a case, there exist exact triangles*

$$\begin{array}{ccccc} & F_{r_1}^{r_0}(01) & & F_{r_1}^{r_0}(00) & \\ \mathcal{H}(r_0) & \xrightarrow{\hspace{2cm}} & \tilde{\mathcal{H}}(r_1) & \xrightarrow{\hspace{2cm}} & \mathcal{H}(r_1) \\ F_{r_0}^{r_2}(00) \swarrow & & \searrow F_{r_2}^{r_1}(10) & & \\ & \mathcal{H}(r_2) & & & \tilde{\mathcal{H}}(r_2) \\ & \searrow F_{r_0}^{r_2}(10) & & \swarrow F_{r_2}^{r_1}(01) & \end{array}$$

$$\begin{array}{ccccc}
& & \tilde{\mathcal{H}}(r_0) & & \\
& \nearrow & \xrightarrow{F_{r_1}^{r_0}(10)} & \searrow & \\
\mathcal{H}(r_0) & & \mathcal{H}(r_1) & & \tilde{\mathcal{H}}(r_0) \\
& \swarrow & \nearrow F_{r_2}^{r_1}(00) & \searrow & \swarrow \\
& & \mathcal{H}(r_2) & & \tilde{\mathcal{H}}(r_2) \\
& \nearrow & \xrightarrow{F_{r_0}^{r_2}(11)} & \searrow & \\
& & \tilde{\mathcal{H}}(r_0) & & \tilde{\mathcal{H}}(r_1)
\end{array}$$

where the notation $F_{r_{i+1}}^{r_i}(\epsilon)$ is from Definition 3.3. Furthermore, if r_1 is even, or equivalently r_0 and r_2 are odd, then we can replace all $\tilde{\mathcal{H}}(r_0)$ and $\tilde{\mathcal{H}}(r_2)$ by $\mathcal{H}(r_0)$ and $\mathcal{H}(r_2)$, respectively and there are identifications of maps

$$(3.15) \quad F_{r_0}^{r_2}(10) = \pm F_{r_0}^{r_2}(01) \text{ and } F_{r_0}^{r_2}(00) = \pm F_{r_0}^{r_2}(11).$$

4. EMBEDDED SPHERES IN COBORDISMS

In this section, we review relations for cobordism maps when there exist embedded spheres of self-intersection $0, -1, -2$ and prove Theorem 1.16 as a byproduct. Again, we fix a coefficient ring \mathcal{R} .

Lemma 4.1. *Suppose $(W, \nu) : (Y_0, \omega_0) \rightarrow (Y_1, \omega_1)$ is a cobordism between admissible pairs. Suppose $S \subset W$ is an embedded sphere with $S \cdot S = k$ and $|\nu \cap S| = l$. Then we have the following relations for the cobordism map*

$$I(W, \nu) : I(Y_0, \omega_0; \mathcal{R}) \rightarrow I(Y_1, \omega_1; \mathcal{R}).$$

- (1) If $k = 0$ and l is odd, then $I(W, \nu) = 0$.
- (2) If $k = -1$ and l is odd, then $I(W, \nu) = 0$.
- (3) If $k = -1$ and $l = 0$, and $(W, \nu) \cong (W', \nu') \# (\overline{\mathbb{CP}^2}, 0)$ for another cobordism (W', ν') between the same admissible pairs, then $I(W, \nu) = \pm I(W', \nu')$.
- (4) If $k = -2$ and l is odd, then $I(W, \nu) = \pm I(W, \nu \cup S)$.

Proof. All terms follow from the neck-stretching argument and the analysis in the neighborhood of S . Term (1) follows directly from the proof of [LY25a, Lemma 3.10]. Term (2) follows from [Sca15, §5.2], which is an important step in the proof of the surgery exact triangle.

Term(3) follows from the blow-up formula for the Donaldson invariant (for dimension-zero part of the moduli space); see [Don90, Theorem (4.8)], [DK90, Proposition (9.3.14)], [Ozs94], and [Kro97, p. 942–943] for the closed case and [KM11b, Proposition 5.2 (2)] for the cobordism case. Note that it is not the blow-up formula for Donaldson series as in [BS23, §5.1], which only works over \mathbb{C} .

Term(4) follows from the last paragraph in the proof of [DMEL24, Proposition 5.11]; see also [CDX20, Equation (6.32)]. \square

Remark 4.2. Lemma 4.1(2)(3) over \mathbb{C} also follows from [BS23, Theorem 1.16 (4)], which provides a blow-up formula for the decomposition of the cobordism map analogous to spin c decomposition in Heegaard Floer and monopole Floer theories. The factor $1/2$ in [BS23, Theorem 1.16 (4)] makes the instanton theory different from the other two Floer theories; compare [OS06, Theorem 1.4] and [KM07, Theorem 39.3.1 and Equation (39.6)].

In our previous work [LY25a, §3.3], we use Lemma 4.1(1) for $\mathcal{R} = \mathbb{F}_2$ to derive that compositions of maps from surgery exact triangles vanish. Indeed, this vanishing result also follows from Lemma 4.1(2)(3). We describe the situation more explicitly as follows, since it

provides some clue to Theorem 1.16 and we will use the idea to study rational surgeries in §6.

We start with the setup of Proposition 3.9, i.e. for $i = 0, 1, 2$, suppose p_i and q_i are co-prime integers with

$$(4.1) \quad p_0q_1 - p_1q_0 = p_1q_2 - p_2q_1 = p_2q_0 - p_0q_2 = 1.$$

Note that $p_i/q_i = (-p_i)/(-q_i)$, but the equations in (4.1) no longer hold if we replace (p_i, q_i) by $(-p_i, -q_i)$. We have

$$(-p_2)q_1 - (-q_2)p_1 = 1,$$

and we can find another (unique) pair of co-prime integers (p_3, q_3) such that

$$p_3(-q_2) - (-p_2)q_3 = p_1q_3 - p_3q_1 = 1.$$

Take $r_i = p_i/q_i$ for $i = 0, 1, 2, 3$ (e.g. $(r_0, r_1, r_2, r_3) = (-1, 0, \infty, 1)$), we have the following surgery triads of 3-manifolds

$$(4.2) \quad \begin{array}{ccccc} & & S^3_{r_0}(K) & & \\ & \nearrow W_{r_0}^{r_2} & & \searrow W_{r_1}^{r_0} & \\ S^3_{r_2}(K) & & \xleftarrow{W_{r_2}^{r_1}} & & S^3_{r_1}(K) \\ & \swarrow W_{r_1}^{r_2} & & \nearrow W_{r_3}^{r_2} & \\ & S^3_{r_3}(K) & & & \end{array}$$

such that we can apply the exact triangles in Proposition 3.9 to either surgery triad. To obtain embedded spheres as in Lemma 4.1, we consider the compositions of two surgery cobordisms in (4.2) and take the union of the cocore disk in the first cobordism and the core disk in the second cobordism. We denote the embedded sphere by

$$(4.3) \quad S(a, b, c) = D_{cc,b}^a \cup D_{c,c}^b \subset W_c^b \circ W_b^a$$

for W_c^b and W_b^a in (4.2) and write $S(a, b, c)^2$ for its self-intersection number. Then we know from Kirby calculus (in particular, taking $(r_0, r_1, r_2, r_3) = (-1, 0, \infty, 1)$ and using the slam-dunk) that

$$(4.4) \quad \begin{aligned} 0 &= S(r_1, r_2, r_1)^2 = S(r_2, r_1, r_2)^2 \\ -1 &= S(r_0, r_1, r_2)^2 = S(r_1, r_2, r_0)^2 = S(r_2, r_0, r_1)^2 \\ -1 &= S(r_2, r_1, r_3)^2 = S(r_1, r_3, r_2)^2 = S(r_3, r_2, r_1)^2 \\ -2 &= S(r_0, r_1, r_3)^2 = S(r_3, r_2, r_0)^2. \end{aligned}$$

Moreover, we have

$$(4.5) \quad W_{r_1}^{r_2} = (W_{r_1}^{r_0} \circ W_{r_0}^{r_2}) \# \overline{\mathbb{CP}^2} \text{ and } W_{r_2}^{r_1} = (W_{r_2}^{r_3} \circ W_{r_3}^{r_1}) \# \overline{\mathbb{CP}^2}$$

If we apply the first triangle in Proposition 3.9 to the top triad of (4.2) and the fourth triangle in Proposition 3.9 to the bottom triad of (4.2), then we obtain the following.

$$(4.6) \quad \begin{array}{ccccc} & & F_{r_1}^{r_2}(11) & & \\ \widetilde{\mathcal{H}}(r_2) & \xrightarrow{\quad} & \widetilde{\mathcal{H}}(r_1) & \xrightarrow{\quad} & \mathcal{H}(r_2) \\ \swarrow F_{r_2}^{r_3}(11) & & \searrow F_{r_3}^{r_1}(11) & \nearrow F_{r_1}^{r_0}(01) & \searrow F_{r_0}^{r_2}(00) \\ & \widetilde{\mathcal{H}}(r_3) & & \mathcal{H}(r_0) & \end{array}$$

The main usage of Lemma 4.1(1) in [LY25a, §3.3] is the vanishing result of the composition

$$(4.7) \quad F_{r_2}^{r_1}(10) \circ F_{r_1}^{r_2}(11) = 0,$$

because $S(r_2, r_1, r_2)^2 = 0$ and the bundle set intersects $S(r_2, r_1, r_2)$ at one point (the intersection of the core disk and the cocore disk in $W_{r_1}^{r_2}$). Alternatively, we can derive the vanishing result from

$$(4.8) \quad F_{r_2}^{r_1}(10) = \pm F_{r_2}^{r_3}(10) \circ F_{r_3}^{r_1}(11) \text{ and } F_{r_3}^{r_1}(11) \circ F_{r_1}^{r_2}(11) = 0,$$

where the first equation is from Lemma 4.1(3) and (4.5) and the second equation follows from Lemma 4.1(2) or the exactness of the triangle. We perform careful computation of the bundle sets for the cobordisms in the first equation of (4.8) as follows.

Lemma 4.3. *Suppose (r_1, r_3, r_2) is a slope triad as in Proposition 3.9. Then we have*

$$\begin{aligned} F_{r_2}^{r_1}(00) &= \pm F_{r_2}^{r_3}(00) \circ F_{r_3}^{r_1}(00), \quad F_{r_2}^{r_1}(10) = \pm F_{r_2}^{r_3}(10) \circ F_{r_3}^{r_1}(11) \\ F_{r_2}^{r_1}(01) &= \pm F_{r_2}^{r_3}(11) \circ F_{r_3}^{r_1}(01) \text{ and } F_{r_2}^{r_1}(11) = \pm F_{r_2}^{r_3}(01) \circ F_{r_3}^{r_1}(10) \end{aligned}$$

Proof. The first equation follows directly from Lemma 4.1(3) and (4.4). The rest equations follow similarly, but need more computations on the bundle sets. We do the computation for the second equation carefully as follows. The computations for the last two equations are similar and we omit them.

Up to a change of basis for $\partial(S^3 \setminus K)$, we can assume $(r_1, r_3, r_2) = (0, 1, \infty)$ and apply the identifications in (3.8), that is

$$\begin{aligned} D_{c,r_2}^{r_1} &= (\mu \times [0, 2]) \cup D_\infty, \quad D_{c,r_2}^{r_3} = (\mu \times [1, 2]) \cup D_\infty, \\ D_{c,r_3}^{r_1} &= ((\mu + \lambda) \times [0, 1]) \cup D_1 \cup D_0, \text{ and } D_{cc,r_3}^{r_1} = (\lambda \times [0, 1]) \cup D_0 \cup D_1, \end{aligned}$$

where we replace I by actual intervals to indicate that it is in different cobordisms. Hence the bundle set in $F_{r_2}^{r_3}(10) \circ F_{r_3}^{r_1}(11)$ is

$$D_{c,r_2}^{r_3} \cup D_{c,r_3}^{r_1} \cup D_{cc,r_3}^{r_1} = \mu \times [0, 2] \cup D_\infty \cup 2 \cdot (\lambda \times [0, 1]) \cup 2 \cdot D_0 \cup 2 \cdot D_1.$$

The pieces with multiple 2 can be removed by Lemma 2.1, and the union of the rest pieces is disjoint from $S(r_1, r_3, r_2)$ and becomes $D_{c,r_2}^{r_1}$ after blow-down. \square

By (4.6) and (4.7) with $(r_0, r_1, r_2, r_3) = (-1, 0, \infty, 1)$, together with the octahedral lemma [OSS15, Lemma A.3.10], we obtain the following exact triangle

$$\begin{array}{ccc} \mathcal{H}(-1) & \xrightarrow{\quad F_1^0(11) \circ F_0^{-1}(01) \quad} & \widetilde{\mathcal{H}}(1) \\ & \swarrow g & \searrow f \\ & \mathcal{H}(\infty) \oplus \widetilde{\mathcal{H}}(\infty) & \end{array}$$

for some *abstract* maps f and g , which means those maps are not necessarily cobordism maps. Replacing Proposition 3.9 with Proposition 3.4, we obtain the exact triangle in Theorem 1.16 except the explicit descriptions of f and g . Note that the proof also works for a trivial admissible surgery pair, as we only use the first and the fourth exact triangles. This is the simpler proof mentioned in the 1.17.

To prove Theorem 1.16 with all maps identified as cobordism maps, we need to further use Lemma 4.1(4) and diagram chasing as in the proof of [LY24, Proposition 5.3].

Theorem 4.4. *Suppose (Y, ω, K) is a surgery tuple as in Definition 1.14. Then there exists an exact triangle*

$$\begin{array}{ccc} I(Y_{-1}(K), \omega; \mathcal{R}) & \xrightarrow{F_1^0(11) \circ F_0^{-1}(01)} & I(Y_1(K), \omega \cup \mu; \mathcal{R}), \\ & \swarrow F_{-1}^\infty(00) + F_{-1}^\infty(10) & \nearrow (F_\infty^1(10), F_\infty^1(11)) \\ I(Y, \omega; \mathcal{R}) \oplus I(Y, \omega \cup \lambda; \mathcal{R}) & & \end{array}$$

where the maps are from Definition 3.3, with suitable choice of signs.

Proof. Replacing Proposition 3.9 with Proposition 3.4 in (4.6) and taking

$$(r_0, r_1, r_2, r_3) = (-1, 0, \infty, 1),$$

we obtain the following two triangles (we omit \mathcal{R}). Recall that when $r \in \mathbb{Z}$, we pick $\tilde{K}_r = \mu$ and when $r = \infty$, we pick $\tilde{K}_\infty = \lambda$.

(4.9)

$$\begin{array}{ccccc} I(Y, \omega \cup \lambda) & \xrightarrow{F_0^\infty(11)} & I(Y_0(K), \omega \cup \mu) & \xrightarrow{F_\infty^0(10)} & I(Y, \omega) \\ \swarrow F_\infty^1(11) & & \nearrow F_1^0(11) & \swarrow F_0^{-1}(01) & \nearrow F_{-1}^\infty(00) \\ I(Y_1(K), \omega \cup \mu) & & I(Y_{-1}(K), \omega) & & \end{array}$$

From Lemma 4.1(4) and (4.4), we have

$$F_{-1}^\infty(00) \circ F_\infty^1(10) = \pm F_{-1}^\infty(10) \circ F_\infty^1(11).$$

By suitably choosing signs, we assume that

$$(4.10) \quad F_{-1}^\infty(00) \circ F_\infty^1(10) + F_{-1}^\infty(10) \circ F_\infty^1(11) = 0.$$

Then we use diagram chasing to prove the exactness at all three vertices. We will use the exactness in (4.9) without mentioning it.

First, we consider the vertex $I(Y, \omega) \oplus I(Y, \omega \cup \lambda)$. Suppose

$$(x, y) \in \ker(F_{-1}^\infty(00) + F_{-1}^\infty(10)),$$

$$\text{i.e. } F_{-1}^\infty(00)(x) + F_{-1}^\infty(10)(y) = 0.$$

We prove that

$$(4.11) \quad (x, y) \in \text{Im}(F_\infty^1(10), F_\infty^1(11)).$$

Indeed, we have

$$\begin{aligned} 0 &= F_0^{-1}(01)(0) \\ &= F_0^{-1}(01) \circ (F_{-1}^\infty(00)(x) + F_{-1}^\infty(10)(y)) \\ &= F_0^{-1}(01) \circ F_{-1}^\infty(10)(y) \\ &= \pm F_0^\infty(11)(y), \end{aligned}$$

where the last equation is from Lemma 4.3. Then there exists $z \in I(Y_1(K), \omega \cup \mu)$ such that

$$F_\infty^1(11)(z) = y.$$

Then we have

$$\begin{aligned} F_{-1}^\infty(00)(x - F_\infty^1(10)(z)) &= F_{-1}^\infty(00)(x) + F_{-1}^\infty(10) \circ F_\infty^1(11)(z) \\ &= F_{-1}^\infty(00)(x) + F_{-1}^\infty(10)(y) \\ &= 0, \end{aligned}$$

where the first equation is from (4.10). Then there exists $w \in I(Y_0(K), \omega \cup \mu)$ such that

$$F_\infty^0(10)(w) = x - F_\infty^1(10)(z).$$

Then we have

$$\begin{aligned} F_\infty^1(11)(z \pm F_1^0(11)(w)) &= F_\infty^1(11)(z) = y, \text{ and} \\ F_\infty^1(10)(z + F_1^0(11)(w)) &= F_\infty^1(10)(z) \pm F_\infty^0(10)(w), \end{aligned}$$

where the last equation is from Lemma 4.3. If the sign is positive, then the existence of $z + F_1^0(11)(w)$ verifies (4.11). If the sign is negative, then the existence of $z - F_1^0(11)(w)$ verifies (4.11), which concludes the proof of the exactness at $I(Y, \omega) \oplus I(Y, \omega \cup \lambda)$.

Second, we consider the exactness at $I(Y_{-1}(K), \omega)$. We will use Corollary 3.6 and Lemma 4.3 freely. We have

$$\begin{aligned} F_1^0(11) \circ F_0^{-1}(01) \circ F_{-1}^\infty(00) &= F_1^0(11)(0) = 0, \text{ and} \\ F_1^0(11) \circ F_0^{-1}(01) \circ F_{-1}^\infty(10) &= \pm F_1^0(11) \circ F_1^0(11) = 0. \end{aligned}$$

Suppose

$$u \in \ker(F_1^0(11) \circ F_0^{-1}(01)).$$

Then there exists $y \in I(Y, \omega \cup \lambda)$ such that

$$F_0^\infty(11)(y) = F_0^{-1}(01)(u).$$

Then we have

$$F_0^{-1}(01)(u + F_{-1}^\infty(10)(y)) = F_0^{-1}(01)(u) \pm F_0^\infty(11)(y) = 0.$$

Hence there exists $x \in I(Y, \omega)$ such that

$$F_{-1}^\infty(00)(x) = u + F_{-1}^\infty(10)(y) \text{ or } u - F_{-1}^\infty(10)(y),$$

which implies

$$u \in \text{Im}(F_{-1}^\infty(00) + F_{-1}^\infty(10))$$

and concludes the proof.

Finally, we consider the exactness at $I(Y_1(K), \omega \cup \mu)$. We have

$$\begin{aligned} F_\infty^1(10) \circ F_1^0(11) \circ F_0^{-1}(01) &= \pm F_\infty^0(10) \circ F_0^{-1}(01)(0) = 0, \text{ and} \\ F_\infty^1(11) \circ F_1^0(11) \circ F_0^{-1}(01) &= 0 \circ F_0^{-1}(01) = 0. \end{aligned}$$

Suppose

$$v \in \ker(F_\infty^1(10), F_\infty^1(11)).$$

Then there exists $w \in I(Y_0(K), \omega \cup \mu)$ such that

$$F_1^0(11)(w) = v.$$

Moreover, we have

$$F_\infty^0(10)(w) = \pm F_\infty^1(10) \circ F_1^0(11)(w) = \pm F_\infty^1(10)(v) = 0.$$

Then there exists $u \in I(Y_{-1}, \omega)$ such that

$$F_0^{-1}(01)(u) = w$$

and hence

$$v \in \text{Im}(F_1^0(11) \circ F_0^{-1}(01)),$$

which concludes the proof. \square

5. INTEGER SURGERIES

From now on, we consider a coefficient field \mathbb{K} instead of a general coefficient ring \mathcal{R} . In [LY25a, Proposition 1.1], we showed that for $\mathbb{K} = \mathbb{F}_2$, the sequence $\{\dim \mathcal{H}(n)\}_{n \in \mathbb{Z}}$ is either V-shaped, W-shaped, or generalized W-shaped. In this section, we first recap this result in any field \mathbb{K} , then eliminate the case of being generalized W-shaped, and finally study the cases of V-shaped and W-shaped more carefully. Following (3.14), we fix a knot $K \subset S^3$ and a field \mathbb{K} , and write

$$\begin{aligned} \mathcal{H}(r, \omega) &= \mathcal{H}(K, r, \omega) = I^\sharp(S_r^3(K), \omega; \mathbb{K}), \\ \mathcal{H}(r) &= \mathcal{H}(K, r) = I^\sharp(S_r^3(K), \omega = 0; \mathbb{K}) \\ \text{and } \tilde{\mathcal{H}}(r) &= \tilde{\mathcal{H}}(K, r) = I^\sharp(S_r^3(K), \omega = \tilde{K}_r; \mathbb{K}) \cong I^\sharp(S_r^3(K), \mu; \mathbb{K}). \end{aligned}$$

Since we only consider the case $r = n \in \mathbb{Z}$ in this section and $\tilde{K}_n = \mu$, we use $\mathcal{H}(r, \mu)$ instead of $\tilde{\mathcal{H}}(r)$.

The main result of this section is the following.

Proposition 5.1. *Suppose $K \subset S^3$ is a knot and \mathbb{K} is a field. There exists a concordance invariant $\nu_{\mathbb{K}}^\sharp(K) \in \mathbb{Z}$ satisfying $\nu_{\mathbb{K}}^\sharp(\bar{K}) = \nu_{\mathbb{K}}^\sharp(K)$ for the mirror knot \bar{K} . Moreover, for*

$$r_{\mathbb{K}}(K) = \min \left\{ \dim \mathcal{H}(\nu_{\mathbb{K}}^\sharp(K)), \dim \mathcal{H}(\nu_{\mathbb{K}}^\sharp(K), \mu) \right\},$$

we have

$$(5.1) \quad \dim \mathcal{H}(n) = \dim \mathcal{H}(n, \mu) = r_{\mathbb{K}}(K) + |n - \nu_{\mathbb{K}}^\sharp(K)|$$

for any integer $n \neq \nu_{\mathbb{K}}^\sharp(K)$. Furthermore, if $\nu_{\mathbb{K}}^\sharp(K)$ is odd, then (5.1) also holds for $n = \nu_{\mathbb{K}}^\sharp(K)$. If $\nu_{\mathbb{K}}^\sharp(K)$ is even, then we have

$$\{\dim \mathcal{H}(\nu_{\mathbb{K}}^\sharp(K)), \dim \mathcal{H}(\nu_{\mathbb{K}}^\sharp(K), \mu)\} = \{r_{\mathbb{K}}(K), r_{\mathbb{K}}(K) + 2\}.$$

Proof. This is re-statement of Propositions 5.5, 5.6, 5.10, and 5.12, with

$$\nu_{\mathbb{K}}^\sharp(K) = \frac{1}{2}(\nu_{+}^{\sharp, \mathbb{K}}(K) + \nu_{-}^{\sharp, \mathbb{K}}(K)).$$

\square

5.1. Three kinds of shapes. We first sketch the analogous results in the proof of [LY25a, Proposition 1.1] for an arbitrary field \mathbb{K} .

Lemma 5.2. *For any integer n , we have*

$$\dim \mathcal{H}(n+1, \omega) = \dim \mathcal{H}(n, \omega') \pm 1.$$

for arbitrary choices of bundle sets ω and ω' .

Proof. This follows directly from (1.1), Proposition 3.4, and the fact that

$$\dim I^\#(S^3; \mathbb{K}) = 1.$$

□

Lemma 5.3. *There exists an integer $N > 0$ such that for any fixed choice of the bundle set ω , we have the following results.*

- *For any integer $n > N$, we have*

$$\dim \mathcal{H}(n+1, \omega) = \dim \mathcal{H}(n, \omega) + 1$$

- *For any integer $n < -N$, we have*

$$\dim \mathcal{H}(n-1, \omega) = \dim \mathcal{H}(n, \omega) + 1$$

Proof. [BS21, Theorem 1.1] and [BS22a, Theorem 1.12] imply that when $\mathbb{K} = \mathbb{C}$, such a monotonicity condition holds for $|n| > N_{\mathbb{C}}$ for some sufficiently large fixed integer $N_{\mathbb{C}}$. From the universal coefficient theorem, we know that the difference

$$\dim I^\#(S_n^3(K), \omega; \mathbb{K}) - \dim_{\mathbb{C}} I^\#(S_n^3(K), \omega; \mathbb{C})$$

is a finite non-negative integer, which is non-increasing as n increases and $n > N_{\mathbb{C}}$ (resp. as n decreases and $n < -N_{\mathbb{C}}$) by Lemma 5.2. Thus, it will eventually stabilize, which concludes the proof of the lemma. □

Definition 5.4. Suppose $K \subset S^3$ is a knot. Define

$$\nu_+^{\sharp, \mathbb{K}}(K) = \min\{n \mid \forall k \geq n, \dim \mathcal{H}(k+1) = \dim \mathcal{H}(k) + 1\},$$

$$\text{and } \nu_-^{\sharp, \mathbb{K}}(K) = \max\{n \mid \forall k \leq n, \dim \mathcal{H}(k-1) = \dim \mathcal{H}(k) + 1\}.$$

Note that $\nu_{\pm}^{\sharp, \mathbb{K}}(\bar{K}) = -\nu_{\mp}^{\sharp, \mathbb{K}}(K)$ for the mirror knot \bar{K} because $\dim \mathcal{H}(\bar{K}, n) = \dim \mathcal{H}(K, -n)$.

Proposition 5.5. *The invariants $\nu_{\pm}^{\sharp, \mathbb{K}}(K)$ are concordance invariants.*

Proof. This is verbatim from the proof of [LY25a, Proposition 1.12]. □

Proposition 5.6. *For a knot $K \subset S^3$, we have the following two results regarding $\nu_{\pm}^{\sharp, \mathbb{K}}(K)$.*

- *For any $n \geq \nu_+^{\sharp, \mathbb{K}}(K)$, we have*

$$(5.2) \quad \dim \mathcal{H}(n+1) = \dim \mathcal{H}(n+1, \mu) = \dim \mathcal{H}(n) + 1 = \dim \mathcal{H}(n, \mu) + 1$$

- *For any $n \leq \nu_-^{\sharp, \mathbb{K}}(K)$, we have*

$$(5.3) \quad \dim \mathcal{H}(n-1) = \dim \mathcal{H}(n-1, \mu) = \dim \mathcal{H}(n) + 1 = \dim \mathcal{H}(n, \mu) + 1$$

Furthermore, the sequence $\{\dim \mathcal{H}(n)\}_{n \in \mathbb{Z}}$ has one of the following three shapes.

(1) *V-shaped:* we have that $\{\dim \mathcal{H}(n)\}_{n \in \mathbb{Z}}$ is unimodal, i.e. it has a unique minimum at integer $m = \nu_+^{\sharp, \mathbb{K}}(K) = \nu_-^{\sharp, \mathbb{K}}(K)$. Furthermore, we have

$$\dim \mathcal{H}(n, \mu) = \dim \mathcal{H}(n) \text{ for } n \neq m \text{ and}$$

$$\begin{cases} \dim \mathcal{H}(m, \mu) - \dim \mathcal{H}(m) \in \{0, 2\} & \text{if } m \text{ is even;} \\ \dim \mathcal{H}(m, \mu) = \dim \mathcal{H}(m) & \text{if } m \text{ is odd.} \end{cases}$$

(2) *W-shaped:* we have $\nu_+^{\sharp, \mathbb{K}}(K) = \nu_-^{\sharp, \mathbb{K}}(K) + 2$ and for $m = \nu_+^{\sharp, \mathbb{K}}(K) - 1 = \nu_-^{\sharp, \mathbb{K}}(K) + 1$, we have the following:

- If m is even and $n \neq m$, then

$$\dim \mathcal{H}(m) = \dim \mathcal{H}(m, \mu) + 2 \text{ and } \dim \mathcal{H}(n, \mu) = \dim \mathcal{H}(n)$$

- If m is odd and $n \neq m \pm 1$, then

$$\dim \mathcal{H}(m \pm 1) = \dim \mathcal{H}(m \pm 1, \mu) - 2 \text{ and } \dim \mathcal{H}(n, \mu) = \dim \mathcal{H}(n)$$

(3) *Generalized W-shaped:* we have $m = \nu_+^{\sharp, \mathbb{K}}(K) > \nu_-^{\sharp, \mathbb{K}}(K) + 2$ and the following holds.

For an even integer n such that $n \in [\nu_-^{\sharp, \mathbb{K}}(K), \nu_+^{\sharp, \mathbb{K}}(K)]$, we have

$$\begin{cases} \dim \mathcal{H}(n) = \dim \mathcal{H}(n, \mu) + 2 & \text{if } m \text{ is odd;} \\ \dim \mathcal{H}(n) = \dim \mathcal{H}(n, \mu) - 2 & \text{if } m \text{ is even,} \end{cases}$$

and for other integers n , we have

$$\dim \mathcal{H}(n) = \dim \mathcal{H}(n, \mu).$$

Proof. The proof is verbatim from that of [LY25a, Proposition 3.17], by replacing [LY25a, Lemma 3.10] with Lemma 4.1(1) for $\mathcal{R} = \mathbb{K}$. \square

Remark 5.7. From the illustration of Proposition 5.6 in [LY25a, Fig. 7-8], we have

$$\nu_+^{\sharp, \mathbb{K}}(K) - \nu_-^{\sharp, \mathbb{K}}(K) \in 2\mathbb{Z}.$$

5.2. Elimination of the third case. Next, we eliminate the case of being generalized W-shaped. We adopt the notation in Proposition 3.9 and Definition 3.3. Since different knots will be used, we add the knot into the notation:

$$\mathcal{H}(K, r, \omega), W_{r_{i+1}}^{r_i}(K) \text{ and } F_{r_{i+1}}^{r_i}(K, \epsilon).$$

From the proof of [BS21, Lemma 5.2], for any $a, b \in \mathbb{Z}$ and any two knots $K_1, K_2 \subset S^3$, there exists an embedding

$$(5.4) \quad W_{a+b}^\infty(K_1 \# K_2) \hookrightarrow W_a^\infty(K_1) \natural W_b^\infty(K_2)$$

Moreover, as in the proofs of [BS21, Lemma 5.2] and [BS22a, Lemma 8.1 and Proposition 8.2], we can consider bundle sets in 5.4 and obtain an embedding

$$(5.5) \quad (W_{a+b}^\infty(K_1 \# K_2), \nu_{K_1 \# K_2}) \hookrightarrow (W_a^\infty(K_1), \nu_{K_1}) \natural (W_b^\infty(K_2), \nu_{K_2}),$$

where

$$(5.6) \quad (\nu_{K_1}, \nu_{K_2}, \nu_{K_1 \# K_2}) \in \{(\nu_a, \nu_b, \nu_{a+b}), (\nu_0, \tilde{\nu}_0, \tilde{\nu}_0), (\tilde{\nu}_0, \nu_0, \tilde{\nu}_0), (\tilde{\nu}_0, \tilde{\nu}_0, \nu_0)\}$$

for the notation from Remark 3.8. We prefer to use the notation in Definition 3.3 because it simplifies the computation of the bundle sets via the identifications (3.8). More precisely, we obtain the following lemma.

Lemma 5.8. *Suppose*

$$(5.7) \quad \epsilon, \epsilon', \epsilon'' \in \{(00, 00, 00), (00, 01, 01), (01, 00, 01), (01, 01, 00), (10, 10, 10), (11, 11, 10)\}$$

If $F_a^\infty(K_1, \epsilon)$ and $F_b^\infty(K_2, \epsilon')$ are both injective (or equivalently, non-vanishing), so it $F_{a+b}^\infty(K_1 \# K_2, \epsilon'')$.

Remark 5.9. From Remark 3.8, the last three cases in (5.6) follow from the second, third, and fourth cases in (5.7) for $a = b = 0$, respectively. The first case in (5.6) follows from the first four cases in (5.7), depending on whether a and b are even or odd.

Proof of Lemma 5.8. It suffices to compute the bundle sets so that (5.5) holds. Suppose μ_K and λ_K are the meridian and the Seifert longitude of K . Suppose S_K is the Seifert surface of K . Suppose $D_n(K) \subset S_n^3(K)$ and $D_\infty(K) \subset S_\infty^3(K) = S^3$ are the meridian disks and suppose $D_{c,n}^\infty(K)$ and $D_{cc,n}^\infty(K)$ are the core and cocore disks in $W_n^\infty(K)$. From (3.8), we have

$$\begin{aligned} D_{c,n}^\infty(K) &= (n \cdot \mu_K + \lambda_K) \times I \cup D_n(K) \cup n \cdot D_\infty(K) \\ \text{and } D_{cc,n}^\infty(K) &= \mu_K \times I \cup D_\infty(K). \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mu_{K_1} \times I &\simeq \mu_{K_2} \times I \simeq \mu_{K_1 \# K_2} \times I, \\ (\lambda_{K_1} \times I) \cup (\lambda_{K_2} \times I) &\simeq \lambda_{K_1 \# K_2} \times I, \\ S_{K_1} \cup S_{K_2} &\simeq S_{K_1 \# K_2} \end{aligned}$$

under the embedding (5.4), where \simeq means mod 2 homologous and we apply Lemma 2.1. Hence we obtain the desired result by direct computations. \square

Proposition 5.10. *For any field \mathbb{K} and any knot $K \subset S^3$, we have*

$$\nu_+^{\sharp, \mathbb{K}}(K) - \nu_-^{\sharp, \mathbb{K}}(K) \leq 2.$$

Furthermore, if $\nu_+^{\sharp, \mathbb{K}}(K) - \nu_-^{\sharp, \mathbb{K}}(K) = 2$ then they must both be odd.

Proof. We will apply Lemma 5.8 for $K_1 = \bar{K}$ and $K_2 = K$. Note that their connected sum is slice, which is concordant to the unknot U . The unknot is W-shaped over any field \mathbb{K} with $\nu_\pm^{\sharp, \mathbb{K}} = \pm 1$ since

$$\dim I^\sharp(L(p, q); \mathbb{K}) = |p| \text{ and } \dim I^\sharp(S^1 \times S^2; \mathbb{K}) - 2 = \dim I^\sharp(S^1 \times S^2, \mu; \mathbb{K}) = 0$$

from [Sca15, Corollary 1.2 and §7.6] and Lemma 4.1(2). Thus, from Proposition 5.5, $\bar{K} \# K$ is also W-shaped over any field \mathbb{K} . Moreover, we have

$$(5.8) \quad \begin{aligned} F_n^\infty(\bar{K} \# K, 10) &\neq 0 \text{ for integer } n \in (-\infty, -2] \cap \{0\} \\ \text{and } F_n^\infty(\bar{K} \# K, 00) &\neq 0 \text{ for integer } n \leq 0. \end{aligned}$$

From Remark 5.7, we know that $\nu_\pm^{\sharp, \mathbb{K}}(K)$ have the same parity. Then we consider the following two cases.

Case 1. $\nu_\pm^{\sharp, \mathbb{K}}(K)$ are both odd. We take

$$a = \nu_+^{\sharp, \mathbb{K}}(K_1) - 1 = -\nu_-^{\sharp, \mathbb{K}}(K) - 1 \text{ and } b = \nu_+^{\sharp, \mathbb{K}}(K_2) - 1.$$

Then

$$\begin{aligned} \dim \mathcal{H}(K_1, a) &= \dim \mathcal{H}(K_1, a+1) + 1, \\ \text{and } \dim \mathcal{H}(K_2, b) &= \dim \mathcal{H}(K_2, b+1) + 1 \end{aligned}$$

by definition of $\nu_+^{\sharp, \mathbb{K}}(K)$. Then Corollary 3.6 and Remark 3.8 imply that

$$F_a^\infty(K_1, 10) \neq 0 \text{ and } F_b^\infty(K_2, 10) \neq 0$$

Then Lemma 5.8 implies that

$$F_{a+b}^\infty(\bar{K} \# K, 10) \neq 0.$$

Thus, we conclude from (5.8) that $a + b \leq 0$, which is equivalent to

$$\nu_+^{\sharp, \mathbb{K}}(K) - \nu_-^{\sharp, \mathbb{K}}(K) \leq 2.$$

Case 2. $\nu_\pm^{\sharp, \mathbb{K}}(K)$ are both even. We take

$$a = \nu_+^{\sharp, \mathbb{K}}(K_1) = -\nu_-^{\sharp, \mathbb{K}}(K) \text{ and } b = \nu_+^{\sharp, \mathbb{K}}(K_2) - 1.$$

Then

$$\begin{aligned} \dim \mathcal{H}(K_1, a, \mu) &= \dim \mathcal{H}(K_1, a+1) + 1, \\ \text{and } \dim \mathcal{H}(K_2, b) &= \dim \mathcal{H}(K_2, b+1) + 1 \end{aligned}$$

by Proposition 5.6. Then Corollary 3.6 and Remark 3.8 imply that

$$F_a^\infty(K_1, 01) \neq 0 \text{ and } F_b^\infty(K_2, 01) \neq 0.$$

Then Lemma 5.8 implies that

$$F_{a+b}^\infty(\bar{K} \# K, 00) \neq 0.$$

Thus, we conclude from (5.8) that $a + b \leq 0$, which is equivalent to

$$\nu_+^{\sharp, \mathbb{K}}(K) - \nu_-^{\sharp, \mathbb{K}}(K) \leq 1.$$

Since the two invariants have the same parity, we conclude that $\nu_+^{\sharp, \mathbb{K}}(K) = \nu_-^{\sharp, \mathbb{K}}(K)$. \square

5.3. $\mathbb{Z}/4$ -grading and the first two cases. Finally, we study the difference between $\dim \mathcal{H}(\nu_{\mathbb{K}}^{\sharp}(K))$ and $\dim \mathcal{H}(\nu_{\mathbb{K}}^{\sharp}(K), \mu)$. From (1.1), the two dimensions coincide when $\nu_{\mathbb{K}}^{\sharp}(K)$ is odd. From the first two cases in Proposition 5.6, the difference is at most two when $\nu_{\mathbb{K}}^{\sharp}(K)$ is even. In the following, we use the $\mathbb{Z}/4$ -grading on instanton homology to show that the difference must be two.

From [Sca15, §7.3], [SS18, §4] (see also [Fy02, §2.2]), for any closed oriented 3-manifold Y , there exists an *absolute* $\mathbb{Z}/4$ -grading on $I^{\sharp}(Y)$ and the grading shift of the cobordism map can be calculated by the traditional topological invariant of the cobordism via [Sca15, Equation (7.1)]. For a bundle set $\omega \subset Y$ such that $[\omega] \neq 0 \in H_1(Y; \mathbb{F}_2)$, there is only a *relative* $\mathbb{Z}/4$ -grading, and a choice of spin structure on Y determines an absolute lift.

Recall that the set of spin structures is an affine space over $H^2(Y; \mathbb{F}_2) \cong H_1(Y; \mathbb{F}_2)$. From (1.1) and [GS99, p. 189], the two spin structures on $S_{2k}^3(K)$ for $k \in \mathbb{Z}$ are characterized by the property that one \mathfrak{s}_0 extends over the surgery cobordisms

$$(5.9) \quad W_{\infty}^{2k} : S_{2k}^3(K) \rightarrow S^3 \text{ and } W_{2k}^\infty : S^3 \rightarrow S_{2k}^3(K),$$

and the other \mathfrak{s}_1 extends over the surgery cobordisms

$$(5.10) \quad W_{2k+1}^{2k} : S_{2k}^3(K) \rightarrow S_{2k+1}^3(K) \text{ and } W_{2k}^{2k-1} : S_{2k-1}^3(K) \rightarrow S_{2k}^3(K).$$

To compare with [ABDS22, Corollary 5.2], we always choose \mathfrak{s}_1 to lift the absolute $\mathbb{Z}/4$ -grading on

$$\mathcal{H}(2k, \mu) = I^{\sharp}(S_{2k}^3(K), \mu; \mathbb{K}).$$

This choice is not essential, as the absolute $\mathbb{Z}/4$ -grading from \mathfrak{s}_0 differs from that of \mathfrak{s}_1 by 2. Then the grading shifts of surgery cobordisms are calculated in the following lemma.

Lemma 5.11. Suppose $k \in \mathbb{Z}$ and suppose the absolute $\mathbb{Z}/4$ -grading on $\mathcal{H}(2k, \mu)$ is determined by \mathfrak{s}_1 from (5.10). Then the grading shifts of the surgery cobordism maps in Remark 3.8 are listed as follows, where (i) denotes the grading shift i .

$$\begin{array}{ccccc}
\mathcal{H}(2k-1) & \xrightarrow{(i_1)} & \mathcal{H}(2k) & \xrightarrow{(i_4)} & \mathcal{H}(2k+1) \\
& \searrow^{(i_3)} & \swarrow^{(i_2)} & & \\
& \mathcal{H}(\infty) & & & \mathcal{H}(\infty) \\
& & & & \\
\mathcal{H}(2k-1) & \xrightarrow{(j_1)} & \mathcal{H}(2k, \mu) & \xrightarrow{(j_4)} & \mathcal{H}(2k+1) \\
& \swarrow^{(j_3)} & \searrow^{(j_2)} & \swarrow^{(j_6)} & \searrow^{(j_5)} \\
& \mathcal{H}(\infty) & & \mathcal{H}(\infty) &
\end{array}$$

where

$$(i_1, i_2, i_3, i_4, i_5, i_6) = \begin{cases} (0, 0, 3, 0, 2, 1) & \text{when } k > 0; \\ (3, 2, 2, 2, 2, 3) & \text{when } k = 0; \\ (0, 1, 2, 0, 3, 0) & \text{when } k < 0; \end{cases}$$

$$(j_1, j_2, j_3, j_4, j_5, j_6) = \begin{cases} (0, 2, 1, 0, 0, 3) & \text{when } k > 0; \\ (3, 0, 0, 2, 0, 1) & \text{when } k = 0; \\ (0, 3, 0, 0, 1, 2) & \text{when } k < 0; \end{cases}$$

Proof. The results for i_1, \dots, i_6 are from [ABDS22, Corollary 5.2], which uses the facts that the grading shift for a cobordism with spin structure does not depend on the bundle set and the sum of grading shifts of three maps in the exact triangles is $-1 \pmod{4}$. To compute j_1, \dots, j_6 , we can again use those two facts to obtain

$$(5.11) \quad j_1 = i_1, \quad j_2 = i_2 + 2, \quad j_1 + j_2 + j_3 = -1, \quad j_4 = i_4, \quad j_5 = i_5 + 2, \quad j_4 + j_5 + j_6 = -1 \pmod{4},$$

where we use the characterization of the spin structures in (5.9) and (5.10). One can double check the results by computing j_3 and j_5 directly from [Sca15, Equation (7.1)], as there are no bundle sets on those cobordisms. \square

Proposition 5.12. If $\nu_{\mathbb{K}}^\sharp(K)$ is even, then we have

$$|\dim \mathcal{H}(\nu_{\mathbb{K}}^\sharp(K)) - \dim \mathcal{H}(\nu_{\mathbb{K}}^\sharp(K), \mu)| = 2.$$

Proof. Suppose $2k = \nu_{\mathbb{K}}^\sharp(K)$. From Propositions 5.6 and 5.10, we know that

$$(5.12) \quad \dim \mathcal{H}(2k-1) = \dim \mathcal{H}(2k) = \dim \mathcal{H}(2k) \pm 1 = \dim \mathcal{H}(2k, \mu) \pm 1.$$

From Lemma 5.11 and the fact that $\mathcal{H}(\infty) \cong \mathbb{K}$ is supported in grading 0, if the sign in (5.12) is determined, then we can calculate the dimensions of the $(0, 1, 2, 3)$ -graded summand of $\mathcal{H}(2k+1)$ from that of $\mathcal{H}(2k-1)$ by passing through either $\mathcal{H}(2k)$ or $\mathcal{H}(2k, \mu)$. It turns out that the signs for $\mathcal{H}(2k)$ and $\mathcal{H}(2k, \mu)$ must be different so that the two approaches provide the same answer. This indeed follows from (5.11).

For example, we compute explicitly to exclude the case $k > 0$ and

$$\dim \mathcal{H}(2k) = \dim \mathcal{H}(2k, \mu) = \dim \mathcal{H}(2k \pm 1) - 1.$$

Suppose the grading $(0, 1, 2, 3)$ summands of $\mathcal{H}(2k-1)$ and $\mathcal{H}(2k+1)$ are (a, b, c, d) and (a', b', c', d') , respectively. Then Lemma 5.11 implies that $(a', b', c', d') = (a, b, c+1, d-1)$ when passing through $\mathcal{H}(2k)$ and $(a', b', c', d') = (a+1, b-1, c, d)$ when passing through $\mathcal{H}(2k, \mu)$, which leads to a contradiction. \square

6. RATIONAL SURGERIES

In this section, we deal with the rational surgeries. Following (3.14), we fix a knot $K \subset S^3$ and a field \mathbb{K} , and write

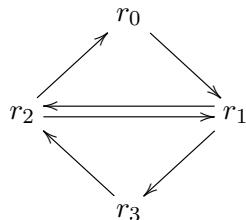
$$\begin{aligned} \mathcal{H}(r, \omega) &= \mathcal{H}(K, r, \omega) = I^\sharp(S_r^3(K), \omega; \mathbb{K}), \\ \mathcal{H}(r) &= \mathcal{H}(K, r) = I^\sharp(S_r^3(K), \omega = 0; \mathbb{K}) \\ \text{and } \tilde{\mathcal{H}}(r) &= \tilde{\mathcal{H}}(K, r) = I^\sharp(S_r^3(K), \omega = \tilde{K}_r; \mathbb{K}) \cong I^\sharp(S_r^3(K), \mu; \mathbb{K}). \end{aligned}$$

Similar to the case of $\mathbb{K} = \mathbb{C}$ as in [BS21, Theorem 4.6] and [BS22a, Theorem 7.1], an important ingredient of the proof is the following lemma from [BS21, §4].

Lemma 6.1. Suppose p_0 and q_0 are co-prime integers satisfying $p_0 \neq 0$ and $|q_0| > 1$. Suppose $r_0 = p_0/q_0 \in (k, k+1)$ for some integer k . Then there exist $r_i = p_i/q_i$ for $i = 1, 2, 3$ that satisfy the following conditions.

- For $i = 1, 2, 3$, p_i and q_i are co-prime, possibly zero integers, such that p_i and q_i have the same signs with p_0 and q_0 , respectively, when they are not zero.
 - $r_1, r_2 \in [k, k+1]$.
 - $p_1 + p_2 = p_0$ and $q_1 + q_2 = q_0$.
 - $p_3 = \text{sign}(p_0) \cdot |p_1 - p_2|$ and $q_3 = \text{sign}(q_0) \cdot |q_1 - q_2|$
 - (r_0, r_1, r_2, r_3) fits into the two slope triads as in (4.2), denoted by

(6.1)

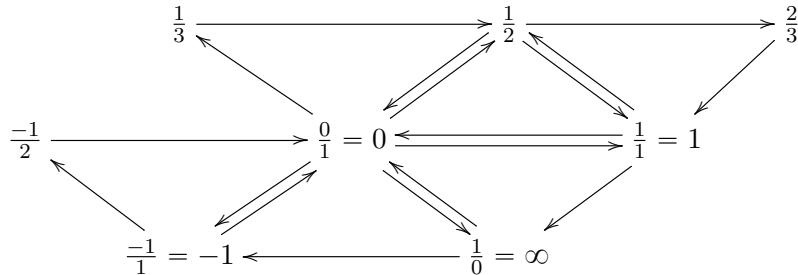


Remark 6.2. Lemma 6.1 can be illustrated by the Farey tessellation of the hyperbolic plane, where the rationals at the vertices of each triangle (with hyperbolic geodesics as edges) corresponds to one choice of r_0 , r_1 , and r_2 . Examples of (r_0, r_1, r_2, r_3) are

$$(1/2, 0/1, 1/1, 1/0), \quad (2/3, 1/1, 1/2, 0/1), \quad (1/3, 1/2, 0/1, 1/1), \quad (-1/2, 0/1, -1/1, 1/0)$$

drawn in the following diagram.

(6.2)



Note that the parallelogram with vertices $0, 1, \infty, -1$ is not an example of Lemma 6.1, but is still an example of (4.2).

The main result of this section is the following.

Proposition 6.3. *Suppose p and q are co-prime integers with $q > 1$. Suppose that $\nu_{\mathbb{K}}^{\sharp}(K)$ and $r_{\mathbb{K}}(K)$ come from Proposition 5.1. Then we have*

$$(6.3) \quad \dim \mathcal{H}(p/q, 0) = \dim \mathcal{H}(p/q, \mu) = q \cdot r_{\mathbb{K}}(K) + |p - q \cdot \nu_{\mathbb{K}}^{\sharp}(K)|.$$

Proof. The proof comes from the induction with the help of Lemma 6.1. From Proposition 5.1, the formula (6.3) also holds for $p = 0$ or $|q| = 1$ except for the case when K is W-shaped over \mathbb{K} and $p/q = \nu_{\mathbb{K}}^{\sharp}(K)$, which is the initial step in the induction. For short, we write

$$M = \nu_{\mathbb{K}}^{\sharp}(K).$$

We first deal with the case

$$(6.4) \quad p/q \notin \mathbb{Z} \text{ and } p/q \notin (M - 1, M + 1).$$

We apply Lemma 6.1 to p_0/q_0 and obtain p_i/q_i for $i = 1, 2, 3$. The induction hypothesis is that p_i/q_i for $i = 1, 2, 3$ all satisfy (6.3). Then the exact triangles in Proposition 3.9 and the dimension counting imply that the four triangles for (r_2, r_1, r_3) split and hence either

$$F_{r_3}^{r_1}(\epsilon) = 0 \text{ for all } \epsilon \text{ or } F_{r_2}^{r_3}(\epsilon) = 0 \text{ for all } \epsilon,$$

where $\epsilon \in \{00, 01, 10, 11\}$. From Lemma 4.3, we have $F_{r_2}^{r_1}(\epsilon) = 0$ for all ϵ . Thus, the four triangles in Proposition 3.9 for (r_0, r_1, r_3) also split and the dimension counting implies that p_0/q_0 also satisfies (6.3).

Then we deal with the case

$$(6.5) \quad p/q \in (M - 1, M) \cup (M, M + 1).$$

From Proposition 5.1, if M is odd, then M also satisfies (6.3) and the proof of the case in (6.4) applies. If M is even, then exactly one of $\mathcal{H}(M)$ and $\tilde{\mathcal{H}}(M)$ satisfies (6.3). The proofs for the two cases are similar by using different exact triangles. Alternatively, one can add $\mu \times I$ into the bundle sets of all cobordism maps to switch one case to the other. We only consider the case that $\tilde{\mathcal{H}}(M)$ satisfies (6.3), partly because of the expectation that any knot is W-shaped over any field \mathbb{K} in Question 1.13.

Furthermore, from the proof of the case in (6.4) and Lemma 6.1, it suffices to show that (6.3) holds for all

$$(6.6) \quad p/q \in \left\{ M \pm \frac{1}{n+1}, M \pm \frac{2}{2n+1} \right\}_{n \in \mathbb{Z}_+}$$

because then we obtain the remaining results by induction. When $M = 0$, we can consider the diagram in (6.2). In the following, we only prove the case when $M = 0$ and the signs in (6.6) are positive. The general case for M even and positive signs in (6.6) is obtained by adding M to all slopes except ∞ . The general case for M even and negative signs in (6.6) is obtained by applying the result to the mirror knot \bar{K} and by using the fact that

$$\dim \mathcal{H}(K, r) = \dim \mathcal{H}(\bar{K}, r) \text{ and } \dim \tilde{\mathcal{H}}(K, r) = \dim \tilde{\mathcal{H}}(\bar{K}, r).$$

Indeed, one may also apply the strategy for positive signs to negative signs, though the details of the proof would be different. From now on, we assume that $n \in \mathbb{Z}_+$.

From the above reduction, we assume that $\tilde{\mathcal{H}}(0)$ satisfies (6.3), and then by Proposition 5.1, we have

$$(6.7) \quad \dim \mathcal{H}(0) = \dim \tilde{\mathcal{H}}(0) + 2 = \dim \mathcal{H}(1) + 1 = \dim \tilde{\mathcal{H}}(1) + 1.$$

To be clear, we split the rest of the proofs into different cases.

Case 1. $\mathcal{H}(1/n)$ and $\tilde{\mathcal{H}}(1/n)$.

By Proposition 3.9, the following maps vanish

$$F_0^\infty(01) \text{ and } F_0^\infty(11).$$

From Lemma 4.3, we have

$$F_0^{1/n}(01) = \pm F_0^{1/(n-1)}(11) \circ F_{1/(n-1)}^{1/n}(01) \text{ and } F_0^{1/n}(11) = \pm F_0^{1/(n-1)}(01) \circ F_{1/(n-1)}^{1/n}(10)$$

Hence by induction, the following maps vanish for all $n \in \mathbb{Z}_+$

$$F_0^{1/n}(01) \text{ and } F_0^{1/n}(11).$$

By Proposition 3.9 and dimension counting, we know that $\mathcal{H}(1/n)$ and $\tilde{\mathcal{H}}(1/n)$ satisfy (6.3).

Case 2. $\mathcal{H}(2/(2n+1))$.

From Lemma 4.3, we have

$$F_{1/(n+1)}^{1/n}(10) = \pm F_{1/(n+1)}^0(10) \circ F_0^{1/n}(11) = 0.$$

By Proposition 3.9, we know that $\mathcal{H}(2/(2n+1))$ satisfies (6.3).

Case 3. $\tilde{\mathcal{H}}(2/(2n+1))$.

Note that from (1.1), we do not necessarily have

$$\dim \tilde{\mathcal{H}}(\pm 2/(2n+1)) = \dim \mathcal{H}(\pm 2/(2n+1))$$

The result that $\tilde{\mathcal{H}}(\pm 2/(2n+1))$ satisfies (6.3) needs some new ingredient from Lemma 4.1 (4). We will use exact triangles from Proposition 3.9 freely.

Case 3.1. $\tilde{\mathcal{H}}(2/3)$.

We start with the case of $2/3$ as shown in 6.2 and then prove the general case. By the triangle, it suffices to show that

$$F_{1/2}^1(00) = 0.$$

By Lemma 4.3, we have

$$\begin{aligned} F_{1/2}^1(00) &= \pm F_{1/2}^0(00) \circ F_0^1(00) = \pm F_{1/2}^0(00) \circ F_0^\infty(00) \circ F_\infty^1(00) \\ &\quad \text{and } F_0^1(10) = \pm F_0^\infty(10) \circ F_\infty^1(11). \end{aligned}$$

From (6.7), we know that

$$\operatorname{rk} F_\infty^1(00) = \operatorname{rk} F_0^\infty(00) = \operatorname{rk} F_0^\infty(10) = \operatorname{rk} F_\infty^1(11) = 1$$

$$\operatorname{Im} F_0^1(00) = \operatorname{Im} F_0^\infty(00), \text{ and } \operatorname{Im} F_0^1(10) = \operatorname{Im} F_0^\infty(10).$$

From Proposition 5.1, we know that

$$\dim \tilde{\mathcal{H}}(2) = \dim \mathcal{H}(1) + 1 \text{ and } \operatorname{rk} F_\infty^2(10) = 1.$$

Since $\dim \mathcal{H}(\infty) = 1$, we know that

$$\operatorname{rk} (F_0^\infty(00) \circ F_\infty^2(10)) = 1.$$

From Lemma 4.1 (4), we have

$$F_0^\infty(10) \circ F_\infty^2(11) = \pm F_0^\infty(00) \circ F_\infty^2(10).$$

Hence we have

$$\text{Im } F_0^\infty(10) = \text{Im } F_0^\infty(00).$$

In summary, we conclude

$$(6.8) \quad \text{Im } F_0^1(00) = \text{Im } F_0^\infty(00) = \text{Im } F_0^\infty(10) = \text{Im } F_0^1(10).$$

Thus, we have

$$\begin{aligned} \{0\} &= \text{Im} \left(F_{\frac{1}{2}}^0(00) \circ F_0^1(10) \right) \\ &= \text{Im} \left(F_{1/2}^0(00) \circ F_0^1(00) \right) \\ &= \text{Im } F_{1/2}^1(00), \end{aligned}$$

which concludes the case of 2/3.

Case 3.2. $\tilde{\mathcal{H}}(2/(2n+1))$.

For $2/(2n+1)$, we need to show that

$$F_{1/(n+1)}^{1/n}(00) = 0.$$

By applying Lemma 4.3 for many times, we have

$$\begin{aligned} F_{1/(n+1)}^{1/n}(00) &= \pm F_{1/(n+1)}^0(00) \circ F_0^{1/n}(00) \\ &= \pm F_{1/(n+1)}^0(00) \circ \left(F_0^\infty(00) \circ F_\infty^1(00) \circ \cdots \circ F_{1/(n-1)}^{1/n}(00) \right) \\ \text{and } F_0^{1/n}(10) &= \pm F_0^\infty(10) \circ \left(F_\infty^1(11) \circ \cdots \circ F_{1/(n-1)}^{1/n}(11) \right). \end{aligned}$$

The facts that $1/n$ and $1/(n+1)$ satisfy (6.3) from Case 1 and the dimension equalities in (6.7) imply that

$$\text{rk } F_0^{1/n}(00) = \text{rk } F_0^{1/n}(10) = 1.$$

Then we have

$$\begin{aligned} \{0\} &= \text{Im} \left(F_{1/(n+1)}^0(00) \circ F_0^{1/n}(10) \right) \\ &= \text{Im} \left(F_{1/(n+1)}^0(00) \circ F_0^\infty(10) \right) \\ &= \text{Im} \left(F_{1/(n+1)}^0(00) \circ F_0^\infty(00) \right) \\ &= \text{Im} \left(F_{1/(n+1)}^0(00) \circ F_0^{1/n}(00) \right) \\ &= \text{Im } F_{1/(n+1)}^{1/n}(00), \end{aligned}$$

where the third equality is from (6.8), which concludes the case of $2/(2n+1)$. \square

Remark 6.4. For the proof of the case in (6.4), one may also use Lemma 4.1(2) instead of Lemma 4.1(3) to carry out the proof. However, the authors have not found a proof of the case in (6.5) without using Lemma 4.1(4).

7. $SU(2)$ -REPRESENTATIONS

In this section, we focus on $\mathbb{K} = \mathbb{F}_2$ and study $SU(2)$ -representations on knot surgeries. The following lemma connects instanton homology to $SU(2)$ -representations.

Lemma 7.1. *Suppose $K \subset S^3$ is a knot and suppose $p/q \in \mathbb{Q} \setminus \{0\}$ for co-prime integers p and q . If $S_{p/q}^3(K)$ is $SU(2)$ -abelian and $p \in \{a^e, 2a^e\}$ for some prime number a and natural number e , then we have*

$$I^\sharp(S_{p/q}^3(K); \mathbb{Z}) \cong \mathbb{Z}^{|p|}.$$

In particular, the knot K is an instanton L-space knot over any field \mathbb{K} .

Proof. The case $p = a^e$ follows directly from [BS18, Corollary 4.8], and the case $p = 2a^e$ is a slight generalization. By [BS18, Theorem 4.6 and Proposition 4.7], this case reduces to the fact that for the Alexander polynomial $\Delta_K(t)$ of K , any $2a^e$ -th root of unity ω satisfies $\Delta_K(\omega^2) \neq 0$.

Suppose ω is a $2a^e$ -th root of unity such that $\Delta_K(\omega^2) = 0$. Let $\eta = \omega^2$. Then η is a a^e -th root of unity and $\Delta_K(\eta) = 0$. Similar to the proof of [BS23, Corollary 9.2], let $\Phi(t)$ be the cyclotomic polynomial associated to η . We have $\Phi(t) \mid \Delta_K(t)$. Hence $p = \Phi(1) \mid \Delta_K(1) = 1$, which leads to a contradiction. \square

Now we prove the main theorems.

Theorem 7.2. *Suppose K is a nontrivial knot of genus $g (\geq 1)$ and suppose $p/q \in (0, \infty)$ with $q \geq 1$, $\gcd(p, q) = 1$, and $p \in \{a^e, 2a^e\}$ for some prime number a and natural number e . If $S_{p/q}^3(K)$ is $SU(2)$ -abelian, then the knot K is an instanton L-space knot over any field \mathbb{K} ,*

$$r_{\mathbb{K}}(K) = \nu_{\mathbb{K}}^\sharp(K) \geq \nu_{\mathbb{C}}^\sharp(K) = 2g - 1 \text{ and } p/q \geq \nu_{\mathbb{K}}^\sharp(K).$$

Moreover, for $\mathbb{K} = \mathbb{F}_2$, we have

$$\nu_{\mathbb{F}_2}^\sharp(K) \geq \nu_{\mathbb{C}}^\sharp(K) + 1 = 2g.$$

Proof. Suppose $S_{p/q}^3(K)$ is $SU(2)$ -abelian with $p \in \{a^e, 2a^e\}$ and $q \geq 1$. Then Lemma 7.1 implies that

$$(7.1) \quad \dim I^\sharp(S_{p/q}^3(K); \mathbb{K}) = p$$

for any field \mathbb{K} . Hence K is an instanton L-space knot over any field \mathbb{K} . From [BS21, Theorem 1.18], we have

$$r_{\mathbb{C}}(K) = \nu_{\mathbb{C}}^\sharp(K) = 2g - 1.$$

From Theorem 1.1, if $p/q < \nu_{\mathbb{K}}^\sharp(K)$, then

$$p = q \cdot r_{\mathbb{K}}(K) - (p - q \cdot \nu_{\mathbb{K}}^\sharp(K)),$$

which implies

$$\frac{p}{q} = \frac{r_{\mathbb{K}}(K) + \nu_{\mathbb{K}}^\sharp(K)}{2} \geq \frac{|\nu_{\mathbb{K}}^\sharp(K)| + \nu_{\mathbb{K}}^\sharp(K)}{2} \geq \nu_{\mathbb{K}}^\sharp(K),$$

where the first inequality follows from Remark 1.2. This leads to a contradiction.

Thus, we obtain $p/q \geq \nu_{\mathbb{K}}^\sharp(K)$. From Theorem 1.1 and (7.1), we obtain

$$p = q \cdot r_{\mathbb{K}}(K) + p - q \cdot \nu_{\mathbb{K}}^\sharp(K),$$

or equivalently

$$(7.2) \quad r_{\mathbb{K}}(K) = \nu_{\mathbb{K}}^{\sharp}(K).$$

From the universal coefficient theorem, we have

$$\dim I^{\sharp}(S_1^3(K); \mathbb{K}) \geq \dim I^{\sharp}(S_1^3(K); \mathbb{C})$$

for any field \mathbb{K} . Applying Theorem 1.1 and (7.2) to $p/q = 1$ and $\mathbb{K} = \mathbb{C}$, we obtain that

$$\begin{aligned} \nu_{\mathbb{K}}^{\sharp}(K) + |1 - \nu_{\mathbb{K}}^{\sharp}(K)| &= |r_{\mathbb{K}}(K)| + |1 - \nu_{\mathbb{K}}^{\sharp}(K)| \\ &\geq r_{\mathbb{C}}(K) + |1 - \nu_{\mathbb{C}}^{\sharp}(K)| \\ (7.3) \quad &= \nu_{\mathbb{C}}^{\sharp}(K) + |1 - \nu_{\mathbb{C}}^{\sharp}(K)| \\ &= 2\nu_{\mathbb{C}}^{\sharp}(K) - 1 \\ &\geq 2(2g(K) - 1) - 1 \\ &= 1. \end{aligned}$$

If $\nu_{\mathbb{K}}^{\sharp}(K) \leq 1$, then all inequalities in (7.3) should be equalities and $\nu_{\mathbb{C}}^{\sharp}(K) = g(K) = 1$. If $\nu_{\mathbb{K}}^{\sharp}(K) > 1$, then the left-hand-side of (7.3) equals to $2\nu_{\mathbb{K}}^{\sharp}(K) - 1$ and (7.3) implies that

$$\nu_{\mathbb{K}}^{\sharp}(K) \geq \nu_{\mathbb{C}}^{\sharp}(K) = 2g(K) - 1.$$

Furthermore, for $\mathbb{K} = \mathbb{F}_2$, [LY25b, Theorem 1.1] implies that

$$\dim I^{\sharp}(S_1^3(K); \mathbb{F}_2) > \dim I^{\sharp}(S_1^3(K); \mathbb{C}).$$

Hence a modification of (7.3) implies that

$$\nu_{\mathbb{F}_2}^{\sharp}(K) > \nu_{\mathbb{C}}^{\sharp}(K).$$

Since both invariants are integers, we have

$$\nu_{\mathbb{F}_2}^{\sharp}(K) \geq \nu_{\mathbb{C}}^{\sharp}(K) + 1 = 2g.$$

□

Theorem 7.3. *Suppose K is a nontrivial knot and suppose $p/q \in (2, 6)$ with $q \geq 1$, $\gcd(p, q) = 1$, and $p \in \{a^e, 2a^e\}$ for some prime number a and non-negative integer e . Then $S_{p/q}^3(K)$ is $SU(2)$ -abelian only when $K = T_{2,3}$ and*

$$\frac{p}{q} \in \left\{ 6 - \frac{1}{n} \right\}_{n \in \mathbb{Z}_+}.$$

Proof. Note that K is an instanton L-space knot (over \mathbb{C}) by Theorem 7.2.

When $g \geq 3$, then Theorem 7.2 implies that

$$p/q \geq \nu_{\mathbb{F}_2}^{\sharp}(K) \geq 2g \geq 6.$$

When $g \leq 2$, the results in [BS22b, FRW24] imply that K must be either $T_{2,3}$ or the torus knot $T_{2,5}$. From [SZ22b, Proposition 4.3], only $T_{2,3}$ and $p/q \in \{6, 6 - 1/n\}_{n \in \mathbb{Z}_+}$ can be the candidate when $p/q \in (2, 6)$. □

8. BYPASS EXACT TRIANGLE AND GENUS ONE KNOTS

In this section, we fix the proof of instanton bypass exact triangle in [BS22b, §4] and then prove Proposition 1.9.

Because of Remark 2.2 and Example 2.7, one can only use Lemma 2.1 to identify cobordism maps with different bundle sets. In the proof of the instanton bypass exact triangle, especially [BS22b, Formula (19)], Baldwin–Sivek claimed the following

$$(8.1) \quad I(W_1, \nu) = I(W_1, \bar{\kappa}_1) \text{ and } I(W_2, \nu) = I(W_2, \kappa_2),$$

where $W_1 : Y_1 \rightarrow Y_2$ and $W_2 : Y_2 \rightarrow Y_3$ are two consecutive surgery cobordisms in a surgery exact triangle for some knot $K \subset Y_1$, with the extra bundle set in the incoming end of W_1 , $\nu = \omega \times I$ for some $\omega = \alpha \cup \eta$ disjoint from K , the bundle set $\bar{\kappa}_1$ is the union of some punctured torus $\hat{T} \subset Y_1$ with K framed by $\partial\hat{T}$ and the bundle set from the triangle, the bundle set $\bar{\kappa}_2$ is the one from the triangle. This is indeed the exact triangle (3.11). Thus, we know $\bar{\kappa}_1$ is the union of ν and a torus and κ_2 is just ν , which implies that the second equation in (8.1) is trivial.

Baldwin–Sivek used the fact that $b_1(Y_2) - 1 = b_1(Y_1) = b_1(Y_3)$ to obtain (8.1), which is insufficient because similar phenomenon might happen as in Example 2.7. However, with a little more work, we show that (8.1) still holds because of the existence of the punctured torus. Note that Example 2.7 is based on a Seifert surface of genus larger than 1.

Since the manifold Y_1 and the bundle set ω are obtained from a closure of balanced sutured manifold and the punctured torus \hat{T} is constructed from the fact that K intersects the suture twice [BS22b, Figure 7 and p. 921], the bundle set η intersects \hat{T} at one point and hence $|\omega \cap \hat{T}| = 1$.

Moreover, note that we only need (8.1) over the coefficient field $\mathbb{K} = \mathbb{C}$ and on the (± 2) -generalized eigenspace of $\mu(\text{pt})$. Since $\mu(\text{pt})$ is defined in [Don02, §7.3] whenever $\text{char}(\mathbb{K}) \neq 2$ (see also [DK90, KM95] for the closed case), we write

$$I(Y, \omega; \mathbb{K})_{\pm 2} = \text{colim}_N \ker((\mu(\text{pt}) \pm 2)^N) \subset I(Y, \omega; \mathbb{K})$$

for the (± 2) -generalized eigenspace of $\mu(\text{pt})$ over such a field \mathbb{K} . From [Fy02, Theorem 9], when (Y, ω) is nontrivial admissible, we know

$$(\mu(\text{pt}) - 4)^N = (u - 64)^N = 0$$

for some large integer N . From Jordan-Chevalley decomposition, since $+2 \neq -2 \in \mathbb{K}$ with $\text{char}(\mathbb{K}) \neq 2$, we have the generalized eigenspace decomposition

$$I(Y, \omega; \mathbb{K}) = I(Y, \omega; \mathbb{K})_{+2} \oplus I(Y, \omega; \mathbb{K})_{-2}.$$

Since $I(Y, \omega; \mathbb{K})$ inherits a relative $\mathbb{Z}/8$ -grading and $\mu = \mu(\text{pt})$ is degree 4, for homogeneous elements $x, y \in I(Y, \omega; \mathbb{K})$ with $\text{gr}(x) = \text{gr}(y) + 4$, we have

$$(\mu + 2)^N(x + y) = 0 \text{ if and only if } (\mu - 2)^N(x - y) = 0,$$

because both equations are equivalent to

$$(\mu^N + c_{N-2} \cdot \mu^{N-2} + \cdots)(x) + (c_{N-1} \cdot \mu^{N-1} + c_{N-3} \cdot \mu^{N-3} + \cdots)(y)$$

$$\text{and } (\mu^N + c_{N-2} \cdot \mu^{N-2} + \cdots)(y) + (c_{N-1} \cdot \mu^{N-1} + c_{N-3} \cdot \mu^{N-3} + \cdots)(x)$$

for some integers c_i . Thus, we have

$$\dim I(Y, \omega, \mathbb{K})_{\pm 2} = \frac{1}{2} \dim I(Y, \omega, \mathbb{K}).$$

Since $\mu(\text{pt})$ commutes with the cobordism map, a cobordism $(W, \nu) : (Y_0, \omega_0) \rightarrow (Y_1, \omega_1)$ between nontrivial admissible pairs induce

$$I(W, \nu)_{\pm 2} : I(Y_0, \omega_0; \mathbb{K})_{\pm 2} \rightarrow (Y_1, \omega_1; \mathbb{K})_{\pm 2}$$

$$\text{with } I(W, \nu) = I(W, \nu)_{+2} + I(W, \nu)_{-2}.$$

Then the following proposition and corollary fix the proof of the bypass exact triangle.

Proposition 8.1. *Suppose $(W, \nu) : (Y_0, \omega_0) \rightarrow (Y_1, \omega_1)$ is a cobordism between nontrivial admissible pairs. Suppose $T \subset W$ is an embedded torus such that $T \cdot T = 0$ and $l = |\nu \cap T|$ is odd. Suppose \mathbb{K} is a field with $\text{char}(\mathbb{K}) \neq 2$, then we have*

$$I(W, \nu \cup T)_{\pm 2} = c_{\pm} \cdot I(W, \nu)_{\pm 2} : I(Y_0, \omega_0; \mathbb{K})_{\pm 2} \rightarrow I(Y_1, \omega_1; \mathbb{K})_{\pm 2},$$

where $c_{\pm} \in \{\pm 1\}$.

Proof. Let $W_T = W \setminus \text{int}N(T)$ and $\nu_T = \nu \cap W_T$. Since $T \cdot T = 0$ and $l = |\nu \cap T|$ is odd. We have

$$(N(T), \nu \cap N(T)) = (T \times D^2, l \cdot D^2) \text{ and}$$

$$(W_T, \nu_T) : (Y_0, \omega_0) \sqcup (T \times S^1, l \cdot S^1) \rightarrow (Y_1, \omega_1).$$

From [Fy02, Lemma 4], we know that

$$(8.2) \quad I(T \times S^1, l \cdot S^1; \mathbb{K}) \cong \mathbb{K}\langle x, y \rangle,$$

where x and y are homogeneous generators with grading difference 4. Moreover, $\mu(\text{pt})$ sends x to $2y$ and y to $2x$. Then

$$I(T \times S^1, l \cdot S^1; \mathbb{K})_{\pm 2} = \mathbb{K}\langle x \pm y \rangle.$$

Note that $(T \times D^2, l \cdot D^2)$ and $(T \times D^2, (l \cdot D^2) \cup (T \times \{0\}))$ induce two homogeneous elements in (8.2) with a grading difference of 4 by the index formula. Without loss of generality, we assume that

$$I(T \times D^2, l \cdot D^2) = a \cdot x \text{ and } I(T \times D^2, (l \cdot D^2) \cup (T \times \{0\})) = b \cdot y$$

for some $a, b \in \mathbb{K}$. Under the pair of pants cobordism

$$(T \times P, l \cdot P) : (T \times D^2, l \cdot D^2) \sqcup (T \times D^2, l \cdot D^2) \rightarrow (T \times D^2, l \cdot D^2),$$

inserting $a \cdot x$ into one summand induces the identity map and inserting $b \cdot y$ into two summands induces $\pm a \cdot x$ by Lemma 2.1. Hence $a = \pm 1$ and $b = \pm 1$, where we implicitly use the fact that x, y and relative invariants are defined over \mathbb{Z} and hence $b^2 \geq 0$. Under the generalized eigenspace decomposition, we have

$$x = \frac{1}{2}(x + y) + \frac{1}{2}(x - y) \text{ and } y = \frac{1}{2}(x + y) - \frac{1}{2}(x - y).$$

Hence the proposition follows from the functoriality result. \square

Corollary 8.2. Suppose (Y, ω, K) is a nontrivial admissible surgery pair such that K is framed by a genus-one Seifert surface S . Suppose \mathbb{K} is a field with $\text{char}(\mathbb{K}) \neq 2$. If $|\omega \cap S|$ is odd, then the exact triangle (3.11) reduces to the following one.

$$\begin{array}{ccc} I(Y, \omega; \mathbb{K})_{\pm 2} & \xrightarrow{F_0^\infty(10) \circ \mathbb{I}'_S = \pm F_0^\infty(00)} & I(Y_0(K), \omega; \mathbb{K})_{\pm 2} \\ & \searrow \mathbb{I}_S \circ F_\infty^1(00) & \swarrow F_1^0(00) \\ & I(Y_1(K), \omega; \mathbb{K})_{\pm 2} & \end{array}$$

Proof. The union of S and the core disk in W_0^∞ is an embedded torus satisfies the assumption of Proposition 8.1. Note that, the union of S and the cocore disk in W_∞^1 has self-intersection 1 and does not satisfy the assumption of Proposition 8.1. \square

Remark 8.3. By Corollary 8.2 and the discussion at the beginning of this section, we fix the proof of bypass exact triangle by picking up suitable closure of balanced sutured manifold. Note that the sign does not matter because the kernel and image of a map are both unchanged by adding a minus sign. The idea of studying the effect of the torus as an extra bundle set is due to John Baldwin. Note that Corollary 8.2 only implies the exactness of the bypass exact triangle at one vertex, and we need to apply the naturality of the sutured instanton homology to prove the exactness at the other vertices, as in the original proof [BS22b, §4].

Proposition 8.4. Suppose $K \subset S^3$ is a genus-one knot and suppose \mathbb{K} is a field with $\text{char}(\mathbb{K}) \neq 2$. Then

$$|\nu_{\mathbb{K}}^\sharp(K)| \leq 1.$$

Proof. When $\text{char}(\mathbb{K}) \neq 2$, the discussion in [LY25a, §2.1] implies that

$$I^\sharp(Y, \omega; \mathbb{K}) \cong I((Y, \omega) \# (T^3, S^1); \mathbb{K})_{+2}$$

and the isomorphism intertwine with the cobordism map. From Proposition 3.9, we have the following exact triangle

$$\begin{array}{ccc} I^\sharp(S_1^3(K); \mathbb{K}) & \xrightarrow{F_2^1(10) \circ \mathbb{I}'_{S+D_1}} & I^\sharp(S_2^3(K); \mathbb{K}) \\ & \searrow \mathbb{I}_{S+D_1} \circ F_1^\infty(01) & \swarrow F_\infty^2(00) \\ & I^\sharp(S^3; \mathbb{K}) & \end{array}$$

From Lemma 4.3, we have

$$F_1^\infty(01) = \pm F_1^0(11) \circ F_0^\infty(01).$$

Note that the union of the genus-one Seifert surface S and the core disk in W_0^∞ is an embedded torus of self-intersection 0. Moreover, it intersects the cocore disk in W_0^∞ at one point. Then by Proposition 8.1, we have

$$F_0^\infty(01) = \pm F_0^\infty(11) \circ \mathbb{I}'_S.$$

Proposition 3.9, we have

$$\pm F_1^0(11) \circ F_0^\infty(11) = 0$$

and hence $F_1^\infty(01) = 0$. Then

$$\dim I^\sharp(S_2^3(K); \mathbb{K}) = \dim I^\sharp(S_1^3(K); \mathbb{K}) + 1.$$

Since the mirror knot \bar{K} also has a genus-one Seifert surface, the above discussion also implies that

$$\dim I^\#(S_2^3(\bar{K}); \mathbb{K}) = \dim I^\#(S_1^3(\bar{K}); \mathbb{K}) + 1.$$

Since

$$\dim I^\#(S_n^3(\bar{K}); \mathbb{K}) = \dim I^\#(S_{-n}^3(K); \mathbb{K})$$

for any integer n , by Proposition 5.1, we have $|\nu_{\mathbb{K}}^\#(K)| \leq 1$. \square

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