

SINGULAR INSTANTON HOMOLOGY OF DUAL KNOTS

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ABSTRACT. We establish a dimension formula for the unreduced singular instanton homology of dual knots $\tilde{K}_{p/q} \subset S^3_{p/q}(K)$ for a knot $K \subset S^3$:

$$\dim I^\sharp(S^3_{p/q}(K), \tilde{K}_{p/q}, \omega; \mathbb{K}) = 2q \cdot r_{\mathbb{K}}(K) + 2|p - q \cdot \nu_{\mathbb{K}}^\sharp(K)| \text{ for } p/q \neq \nu_{\mathbb{K}}^\sharp(K),$$

where $\omega \subset S^3 \setminus K$ is any unoriented 1-submanifold as the bundle set, $r_{\mathbb{K}}(K)$ and $\nu_{\mathbb{K}}^\sharp(K)$ are integers from the dimension formula of $I^\sharp(S^3_{p/q}(K); \mathbb{K})$ for a field \mathbb{K} defined by Li and the author. In particular, when \mathbb{K} is the two-element field \mathbb{F}_2 , the reduced singular instanton homology satisfies

$$\dim I^\sharp(S^3_{p/q}(K), \tilde{K}_{p/q}, \omega; \mathbb{F}_2) = \dim I^\sharp(S^3_{p/q}(K); \mathbb{F}_2) \text{ for } p/q \neq \nu_{\mathbb{F}_2}^\sharp(K).$$

As an application, for a determinant-one knot $K \subset S^3$ other than the unknot and the torus knots $T_{2,3}, T_{2,5}$ and a rational $p/q \in (0, 6)$ with p odd prime power, the surgery manifold $\hat{Y}_{p/2q}(\hat{K})$ is not $SU(2)$ -abelian for the double branched cover $\hat{Y} = \Sigma(S^3, K)$ and the preimage $\hat{K} \subset \hat{Y}$ of K . We also obtain non-abelian results for $SU(2)$ representations of the knot complement that send the curves of some fixed slope in $(0, 6)$ to traceless elements.

1. INTRODUCTION

The reduced and unreduced singular instanton homologies $I^\sharp(Y, K)$ and $I^\sharp(Y, K)$ for a closed oriented connected 3-manifold Y and an unoriented knot $K \subset Y$ are introduced by Kronheimer–Mrowka [KM11] via singular instantons on orbifolds. The “reduced” and “unreduced” correspond to the notions in Khovanov homology, as there are spectral sequences from Khovanov homology of a knot in S^3 to instanton homology of the mirror knot with the same variant.

In this paper, we study the singular instanton homology of the dual knot in a surgery manifold for a knot $K \subset S^3$, i.e. the core of the Dehn filling solid torus. For $p/q \in \mathbb{Q}$, let $S^3_{p/q}(K)$ denote the manifold obtained from S^3 by Dehn surgery along K with coefficient p/q and let $\tilde{K}_{p/q} \subset S^3_{p/q}(K)$ denote the dual knot. Let $\omega \subset S^3 \setminus K$ be an unoriented 1-submanifold so that it also lies in $S^3_{p/q}(K)$. Baldwin–Sivek [BS21, BS22] studied the dimension of the framed instanton homology $I^\sharp(S^3_{p/q}(K), \omega; \mathbb{C})$ over \mathbb{C} . Later, Li and the author [LY25c] extend the results to any coefficient field \mathbb{K} . More precisely, we have the following theorem.

Theorem 1.1 ([LY25c, Theorem 1.1 and Remark 1.2]). *Suppose $K \subset S^3$ is a knot and suppose μ is the meridian of K . Suppose p and q are co-prime integers with $q \geq 1$. Then there exist a concordance invariant $\nu_{\mathbb{K}}^\sharp(K) \in \mathbb{Z}$ satisfying $\nu_{\mathbb{K}}^\sharp(\bar{K}) = \nu_{\mathbb{K}}^\sharp(K)$ for the mirror knot \bar{K} . Define*

$$M = \nu_{\mathbb{K}}^\sharp(K) \text{ and } R = r_{\mathbb{K}}(K) = \min \left\{ \dim I^\sharp(S^3_M(K); \mathbb{K}), \dim I^\sharp(S^3_M(K), \mu; \mathbb{K}) \right\}.$$

We have $R = |M| + 2h$ for some $h \in \mathbb{Z}_+$ depending on \mathbb{K} . Moreover, we have

$$(1.1) \quad \dim I^\sharp(S_{p/q}^3(K); \mathbb{K}) = \dim I^\sharp(S_{p/q}^3(K), \mu; \mathbb{K}) = qR + |p - qM|.$$

except possibly when $p/q = M$ and M is even. In the exceptional case, we have

$$(1.2) \quad \{\dim I^\sharp(S_M^3(K); \mathbb{K}), \dim I^\sharp(S_M^3(K), \mu; \mathbb{K})\} = \{R, R + 2\}.$$

Remark 1.2. Note that for a general choice of ω , we know $I^\sharp(S_M^3(K), \omega; \mathbb{K})$ is isomorphic to either $I^\sharp(S_M^3(K); \mathbb{K})$ or $I^\sharp(S_M^3(K), \mu; \mathbb{K})$. In a recent work of Ghosh–Miller–Eismeier [GME25], they could prove that when $\mathbb{K} = \mathbb{F}_2$, the integers $M_2 = \nu_{\mathbb{F}_2}^\sharp(K)$ and $R_2 = r_{\mathbb{F}_2}(K)$ are both divisible by 4. Moreover, they could show that

$$\dim I^\sharp(S_{M_2}^3(K); \mathbb{F}_2) = R_2 + 2 \text{ and } \dim I^\sharp(S_{M_2}^3(K), \mu; \mathbb{F}_2) = R_2.$$

The main theorem of this paper is the following dimension formula for dual knots.

Theorem 1.3. Suppose $K \subset S^3$ is a knot and suppose $\omega \subset S^3 \setminus K$ is an unoriented 1-submanifold. Suppose p and q are co-prime integers with $q \geq 1$. Let

$$M = \nu_{\mathbb{K}}^\sharp(K) \text{ and } r_2 = r_{\mathbb{K}}(K)$$

be the integers from Theorem 1.1. Then the isomorphism class of $I^\sharp(S_{p/q}^3(K), \tilde{K}_{p/q}, \omega; \mathbb{K})$ does not depend on ω and

$$\dim I^\sharp(S_{p/q}^3(K), \tilde{K}_{p/q}, \omega; \mathbb{K}) = \begin{cases} 2qR + 2|p - qM| & \text{if } p/q \neq M; \\ 2R \text{ or } 2R + 2 & \text{if } p/q = M. \end{cases}$$

When $\mathbb{K} = \mathbb{F}_2$, the case of $2R + 2$ will not happen and

$$\dim I^\sharp(S_{p/q}^3(K), \tilde{K}_{p/q}, \omega; \mathbb{F}_2) = qR + |p - qM|.$$

Remark 1.4. Based on an on-going project of Bhat, Li, and the author, we expect that when $\mathbb{K} = \mathbb{C}$, the case of $2R + 2$ in Theorem 1.3 happens for the right-handed trefoil $T_{2,3}$ and the case of $2R$ happens for the torus knot $T_{2,5}$, or possibly other instanton L-space knots of genera at least 2. On the other hand, the invariants $\nu_{\mathbb{C}}^\sharp$ and $r_{\mathbb{C}}$ were calculated for many knots by Baldwin–Sivek [BS21].

Remark 1.5. In [GLW24, §5.3] (see also [LY25b, Lemmas 4.3 and 4.4] and §2 for the identification of I^\sharp and KHI), Ghosh–Li–Wong proved that

$$\dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{C}) = R' + |n - 2\tau_I(K)|$$

for some integer R' (possibly different from $r_{\mathbb{C}}(K)$) and the instanton tau invariant $\tau_I(K)$. However, the author of this paper is not sure if the dimension formula over \mathbb{C} can be extended to rational slopes.

Remark 1.6. From [LY22, Theorem 1.2] and [LY21, Theorem 3.20], for $p/q \in \mathbb{Q} \setminus \{0\}$, there is a dimension inequality

$$\dim I^\sharp(S_{p/q}^3(K), \tilde{K}_{p/q}; \mathbb{C}) \geq \dim I^\sharp(S_{p/q}^3(K); \mathbb{C})$$

that is indeed from a spectral sequence. However, from Theorems 1.1 and 1.3, we have

$$\dim I^\sharp(S_{p/q}^3(K), \tilde{K}_{p/q}; \mathbb{F}_2) = \dim I^\sharp(S_{p/q}^3(K); \mathbb{F}_2) \text{ for } p/q \neq \nu_{\mathbb{F}_2}^\sharp(K).$$

Furthermore, Ghosh–Miller–Eismeier’s work mentioned in Remark 1.2 implies that

$$\dim I^\sharp(S^3_{p/q}(K), \tilde{K}_{p/q}; \mathbb{F}_2) = \dim I^\sharp(S^3_{p/q}(K); \mathbb{F}_2) - 2,$$

which implies that there cannot be an analogous spectral sequence over \mathbb{F}_2 if $\nu_{\mathbb{F}_2}^\sharp(K) \neq 0$. As the construction of the spectral sequence is highly depends on the existence of the Alexander grading and the two kinds of bypass maps with different Alexander grading shifts, the above discussion indicates that either the Alexander grading, or the difference between the two bypass maps does not exist over \mathbb{F}_2 .

The singular instanton homology of knots is closely related to $SU(2)$ representations of fundamental groups, in particular for the knot complements whose images of the meridian are traceless. To state the results more precisely, we introduce the following definitions.

Definition 1.7. Suppose K is a framed knot in a closed oriented connected 3-manifold Y . We write $Y \setminus K = Y \setminus \text{int } N(K)$ for the knot complement and write (μ, λ) for the meridian-longitude basis of $\partial(Y \setminus K) \cong T^2$ with $\mu \cdot \lambda = -1$. Recall that Y is called $SU(2)$ -abelian if all representations

$$\rho : \pi_1(Y) \rightarrow SU(2)$$

have abelian images, i.e. factor through a copy of $U(1) \subset SU(2)$.

If all meridian-traceless $SU(2)$ representations

$$(1.3) \quad \rho : \pi_1(Y \setminus K) \rightarrow SU(2) \text{ with } \rho(\mu) = \mathbf{i}$$

have abelian images, then (Y, K) is called *meridian-traceless $SU(2)$ -abelian*. Here $\mathbf{i} \in SU(2)$ denotes the diagonal matrix with entries $i, -i$. We know any traceless element in $SU(2)$ is conjugate to \mathbf{i} and any $SU(2)$ representation ρ' with $\rho'(\mu)$ traceless is conjugate to one satisfying (1.3).

Similarly, if all representations

$$\rho : \pi_1(Y \setminus K) \rightarrow SU(2) \text{ with } \rho(\mu^p \lambda^q) = \mathbf{i}$$

have abelian images for some co-prime integers p and q , then (Y, K) is called p/q -traceless $SU(2)$ -abelian. Note that $\rho(\mu^p \lambda^q) = \mathbf{i}$ implies that $\rho(\mu^{-p} \lambda^{-q}) = -\mathbf{i}$, so p/q -traceless condition is equivalent to $(-p)/(-q)$ -traceless condition, which justifies the rational notation.

Recall from [BLSY24, FRW24], the only instanton L-space knots of genera at most 2 are the unknot U and the torus knots $T_{2,3}$ and $T_{2,5}$. We have the following theorems about $SU(2)$ -representations for other knots.

Theorem 1.8. Suppose $K \subset S^3$ is a knot that is not $U, T_{2,3}, T_{2,5}$ and suppose $\det(K) = |\Delta_K(-1)|$ is the determinant of K , where $\Delta_K(t)$ is the Alexander polynomial of K . Then for any slope $p/q \in (0, 6)$ such that p is an odd prime power and p does not divides $\det(K)$, we know that (S^3, K) is not p/q -traceless $SU(2)$ -abelian.

Theorem 1.9. Suppose $K \subset S^3$ is a knot that is not $U, T_{2,3}, T_{2,5}$. Suppose $\hat{Y} = \Sigma(S^3, K)$ is the double branched cover of K and $\hat{K} \subset \hat{Y}$ is the preimage of K . If $\det(K) = 1$, or equivalently \hat{Y} is an integral homology sphere, then for any slope $p/q \in (0, 6)$ such that p is an odd prime power, we know that $\hat{Y}_{p/2q}(\hat{K})$ is not $SU(2)$ -abelian.

Remark 1.10. The condition for p is from some non-degeneracy condition [BLSY24, §4]. Since $\det(K)$ does not depend on the choice of p/q , there is only finitely many choices of p that divide $\det(K)$. For the cases of $T_{2,3}$ and $T_{2,5}$, one may compute the $SU(2)$ representations directly by [HHK14, §5.1] and [SZ22].

Remark 1.11. Together with Ghosh–Miller–Eismeier’s results [GME25] mentioned in Remark 1.2, we could replace the assumption $p/q \in (0, 6)$ in Theorems 1.8 and 1.9 by $p/q \in (0, 8)$.

Organization. In §2, we review many kinds of instanton homologies and study their relation and their dependence of the bundle sets. In §3, we prove Theorem 1.3 by considering the integral case and the rational case separately. In §4, we describe the relation between singular instanton homology and $SU(2)$ representations, and then prove Theorems 1.8 and 1.9.

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2. INSTANTON KNOT HOMOLOGY

In this section, we review many kinds of instanton homologies for knots and their relation. We also study the dependence of the bundle sets carefully, similar to the case of framed instanton homology as in [LY25c, §2].

Recall from [LY25a, §2.1] that there are three approaches to define the framed instanton homology $I^\sharp(Y, \omega)$ for a closed connected oriented 3-manifold Y and an unoriented 1-submanifold $\omega \subset Y$ (called the *bundle set*). We extend the discussion to an unoriented knot K in Y that is disjoint from ω and describe the three approaches to define the (*reduced*) instanton knot homology $I^\sharp(Y, K, \omega)$, with the correspondence

$$(2.1) \quad I^\sharp(Y, \omega) = I^\sharp(Y, U, \omega)$$

for the unknot $U \subset Y$ that bounds a trivial disk. We also add the fourth approach which is only available over \mathbb{F}_2 or coefficient rings with $4 = 0$.

- (1) From [KM11, §4.3], we take the singular instanton homology

$$I(Y, K \# H, \omega \sqcup \alpha),$$

where H is a standard Hopf link in S^3 and α is a standard arc connecting two components of H . Here we need to pick a basepoint on K to specify the position of connected sum and a tangent vector at the basepoint to specify the arc α . Note that this is the original definition of $I^\sharp(Y, K, \omega)$ when $\omega = 0$ (the emptyset).

- (2) From [Flo90] and [KM11, §5.4], we take the original instanton homology

$$I(T_K^3, \omega \sqcup S_3^1)^\psi.$$

Here we write

$$T^3 = S_1^1 \times S_1^2 \times S_1^3 \text{ and } T_K^3 = (Y \setminus K) \cup (T^3 \setminus S_1^1)$$

via gluing the longitudes of K (for some chosen framing) to the meridians of S_1^1 and vice versa. The superscript $\psi \subset H^1(T_K^3; \mathbb{F}_2)$ is the 2-element subgroup generated by

the torus $R = S^1_1 \times S^1_2 \subset T_K^3$, and we take the quotient of the usual configuration space by this extra 2-element group when constructing the instanton homology.

- (3) From [KM10, §7.6], we take the sutured instanton knot homology

$$KHI(Y, K, \omega) = SHI(Y \setminus K, \gamma_K = \mu \cup (-\mu), \omega),$$

for an oriented meridian μ of K . Here we need to choose two basepoints on K to specify the positions of the meridians. This is only defined over \mathbb{C} and on the homology level since the construction uses the generalized eigenspace decomposition of the μ operators on the ordinary instanton homology. It is only an isomorphism class from [KM10, §7], or a projectively transitive system by Baldwin–Sivek’s naturality result [BS15, §9].

- (4) From [KM19a] and [KM21, §2.5], we take the singular instanton homology for webs

$$I(Y, K \# \Theta, \omega),$$

where Θ is a standard theta graph in S^3 . This is only defined over \mathbb{F}_2 or coefficient rings with $4 = 0$ because of the existence of trivalent vertices indicates that the square of the differential ∂^2 is a multiple of 4. Here we need to pick a basepoint on K and an oriented basis $(\epsilon_1, \epsilon_2, \epsilon_3)$ of tangent vector at the basepoint with ϵ_1 pointing along K to specify the position of the connected sum.

All definitions can be generalized to a link $L \subset Y$ if we choose one component of L to place the basepoint or carry out the gluing construction (in the second approach, we need to use singular instanton homology and in the third approach, we need to use sutured instanton homology for singular tangles by Xie–Zhang [XZ25]). If we take $L = K \sqcup U$ for a split unknot U with basepoint, then the corresponding instanton homology

$$I^\natural(Y, K \sqcup U, \omega) = I^\natural(Y, K, \omega)$$

is called the *unreduced instanton knot homology*. Note that K could be empty in the unreduced variant, for which we will omit it from the notation and obtain the framed instanton homology $I^\natural(Y, \omega)$. When $\omega = 0$, we also omit it from the notation.

From [KM19b, KM21], there are further constructions about local coefficients for the fourth approach. Moreover, from [KM11], in the first and the fourth approaches, we allow ω to include arcs with endpoints on K . Indeed, we may also extend arcs into circles in the second and the third approaches and allow ω to include arcs. However, we will not use those extended constructions and the corresponding results for the main results of this paper.

The constructions in the first two approaches inherit $\mathbb{Z}/4$ homological grading, while ones in the last two only inherits a $\mathbb{Z}/2$ homological grading. Here the homological grading can be absolute or relative, depending on ω and the approaches.

From [KM11, §5.4], the first two definitions are isomorphic via Floer’s excision cobordism, and hence the isomorphism intertwines with any cobordism map supported in $Y \setminus K$ naturally. The first and the last definitions are isomorphic via another excision cobordism [KM19b, §3.3], and hence also natural for cobordism maps supported in $Y \setminus K$.

Also from [KM11, §5.4], the second definition is isomorphic to the third definition via a *special* choice of closure in the construction of $KHI(Y, K, \omega)$, which is exactly $(T_K^3, R, \omega \sqcup S^1_3)$. Since $g(R) = 1$, by [BS23, Theorem 2.5] (see also [Fy02, Lemma 4]), we know $\mu(\text{pt})^2 = 4$ for the μ map in the construction of $KHI(Y, K, \omega)$, and then the isomorphism can be made to intertwine with the cobordism map. Indeed, as in [LY25c, §8], one can replace \mathbb{C} by any field \mathbb{K} with $\text{char } \mathbb{K} \neq 2$ and the isomorphism still holds.

Note that Baldwin–Sivek’s naturality result [BS15] only works for closures of genera larger than one, while the relation between closures of genus one and larger genera remains to be isomorphism rather than *canonical* isomorphism. Hence, the best result we can state about the cobordism map is that, for any *fixed* closure and any *fixed* isomorphism between the special closure above and the fixed closure, the isomorphism intertwines the cobordism map supported in $Y \setminus\setminus K$. Similar isomorphism results hold for the unreduced variant by Floer’s excision theorem (for sutured version, see [XZ25, Remark 7.8]). All isomorphisms above respect the homological gradings that defined on both sides.

Then we list some relation between I^\natural and I^\sharp as follows, though we only use some of them. Let \mathcal{R} denote a general coefficient ring and let \mathbb{K} denote a general coefficient field.

From [KM19a, Lemma 7.7], we have

$$(2.2) \quad \dim I^\sharp(Y, K, \omega; \mathbb{F}_2) = 2 \dim I^\natural(Y, K, \omega; \mathbb{F}_2).$$

The equation might not hold over a general ring \mathcal{R} . Indeed, by moving the earring (the meridian and the arc) in [KM11, Fig. 13] to the disjoint unknot in the third picture, we obtain a skein exact triangle

$$(2.3) \quad \begin{array}{ccc} I^\natural(Y, K, \omega; \mathcal{R}) & \xrightarrow{f} & I^\natural(Y, K, \omega; \mathcal{R}) \\ & \searrow & \swarrow \\ & I^\sharp(Y, K, \omega; \mathcal{R}) & \end{array}$$

Hence we have for any field \mathbb{K} ,

$$(2.4) \quad \dim I^\sharp(Y, K, \omega; \mathbb{K}) = 2 \dim I^\natural(Y, K, \omega; \mathbb{K}) - 2 \operatorname{rank} f \leq 2 \dim I^\natural(Y, K, \omega; \mathbb{K}).$$

Meanwhile, from [LY25a, Lemma B.1], we have

$$(2.5) \quad \dim I^\sharp(Y, K, \omega; \mathbb{C}) \geq \dim I^\natural(Y, K, \omega; \mathbb{C}).$$

For more relation, see [Xie21].

On the other hand, the flip symmetry in [XZ25, §2.3] implies that

$$(2.6) \quad I^\sharp(Y, K, \omega; \mathcal{R}) \cong I^\sharp(Y, K, \omega \cup K; \mathcal{R})$$

for the first definition because K is a singular knot disjoint from the bundle set $\omega \cup \alpha$. Combining (2.2) and (2.6), we obtain

$$(2.7) \quad \dim I^\natural(Y, K, \omega; \mathbb{F}_2) = \dim I^\natural(Y, K, \omega \cup K; \mathbb{F}_2).$$

Finally, we study the dependence of ω . Since ω is a geometric representative of Poincaré dual of the second Stiefel–Whitney class of the $SO(3)$ -bundle over manifolds (or orbifolds in some approaches), the isomorphism classes of $I^\natural(Y, K, \omega)$ and $I^\sharp(Y, K, \omega)$ only depend on the homology class $[\omega] \in H_1(Y \setminus\setminus K; \mathbb{F}_2)$. Indeed, only the class $[\omega] \in H_1(Y; \mathbb{F}_2)$ matters (or $[\omega, \partial\omega] \in H_1(Y, K; \mathbb{F}_2)$ if ω contains arcs with endpoints on K); see the discussion after [GM23, Proposition 3.3] for the case when $Y = S^3$.

Similar to the case of $I(Y, \omega)$ in [LY25c, §2], we study the concrete isomorphisms induced by (instanton) cobordism maps. We take the first approach (or the fourth approach over \mathbb{F}_2) and allow ω to contain arcs with endpoints on K . From [KM11, §4.2] (or [KM21, §2.4]), a cobordism

$$(W, \Sigma, \nu) : (Y_1, K_1, \omega_1) \rightarrow (Y_2, K_2, \omega_2)$$

consists of

- an oriented 4-manifold W with boundary as a cobordism of 3-manifolds $W : Y_1 \rightarrow Y_2$,
- an possibly nonorientable surface Σ with boundary as a cobordism of unoriented knots $\Sigma : K_1 \rightarrow K_2$.
- a possibly nonorientable surface or a 2-simplicial complex ν with boundary (called a *bundle set* on W) as a cobordism of the bundle sets $\nu : \omega_1 \rightarrow \omega_2$, which can have boundaries on Σ and some transverse intersection points with $\text{int } \Sigma$.

We have the following lemmas about the dependence of the bundle sets, similar to the discussion in [LY25c, §2].

Lemma 2.1. *Let*

$$(W, \Sigma, \nu_i) : (Y_1, K_1, \omega_1) \rightarrow (Y_2, K_2, \omega_2) \text{ for } i = 1, 2$$

be two cobordisms with different bundle sets satisfying

$$(2.8) \quad \Sigma \cap \partial\nu_1 = \Sigma \cap \partial\nu_2.$$

Note that the transverse intersection points between $\text{int } \Sigma$ and ν_i could be different. If

$$(2.9) \quad [\nu_1 \cup \nu_2] = 0 \in H_2(W; \mathbb{F}_2),$$

then for any coefficient ring \mathcal{R} , we have

$$I^\natural(W, \Sigma, \nu_1) = \pm I^\natural(W, \Sigma, \nu_2) : I^\natural(Y_1, K_1, \omega_1; \mathcal{R}) \rightarrow I^\natural(Y_2, K_2, \omega_2; \mathcal{R}),$$

$$\text{and } I^\sharp(W, \Sigma, \nu_1) = \pm I^\sharp(W, \Sigma, \nu_2) : I^\sharp(Y_1, K_1, \omega_1; \mathcal{R}) \rightarrow I^\sharp(Y_2, K_2, \omega_2; \mathcal{R}).$$

Remark 2.2. When $\Sigma \cap \partial\nu_1$ and $\Sigma \cap \partial\nu_2$ are not identical but have the same homology class in $H_1(W, \Sigma; \mathbb{F}_2)$, one might have the same results. The author does not state this stronger result because he is not sure if some similar issues as in [LY25c, Remark 2.2] would happen. We will mainly focus on the cobordisms with $\Sigma \cap \partial\nu_i = \emptyset$ as ω_i do not contain arcs. Hence we do not need the strongest result.

Proof of Lemma 2.1. Following [KM11, §2.2-2.3], let W_Δ^h be obtained from W by gluing $W \setminus \Sigma$ and the double cover $\tilde{\nu}_\Delta$ of the tubular neighborhodd $\nu = N(\Sigma)$ along $\partial\nu$ using the 2-to-1 map $\partial\tilde{\nu}_\Delta \rightarrow \partial\nu$. Here Δ is some local system on Σ with structure group $\{\pm 1\}$, or equivalently some double cover Σ_Δ of Σ . There is also a non-Hausdorff space W_Δ obtained from $W \setminus \Sigma$ and $\tilde{\nu}_\Delta$ by identifying each point in $\tilde{\nu}_\Delta \setminus \Sigma_\Delta$ with its image in $X \setminus \Sigma$.

From [KM11, §4.2], we construct double cover Σ_{Δ_i} of Σ for $i = 1, 2$ by taking $(\Sigma \setminus \omega) \times \{\pm 1\}$ and identifying across the cut with an interchange of the two sheets. In particular, if $\partial\nu_i \cap \Sigma = \emptyset$, then Σ_{Δ_i} is just the trivial double cover. The assumption (2.8) implies that $\Sigma_{\Delta_1} = \Sigma_{\Delta_2}$ and hence

$$W_{\Delta_1}^h = W_{\Delta_2}^h \text{ and } W_{\Delta_1} = W_{\Delta_2}.$$

Hence we omit the subscript of Δ_i .

Furthermore, let $\Sigma_{\pm, i}$ be the closure of $(\Sigma \setminus \partial\nu_i) \times \{\pm 1\}$ in W_Δ and take

$$\nu'_i = \pi^{-1}(\nu_i) \cup \Sigma_{-, i},$$

where the map $\pi : W_\Delta \rightarrow W$. Let ν_i^h be the inverse images of ν'_i in W_Δ^h . Note that a transverse intersection point of $\text{int } \Sigma$ and ν_i contributes two points in $\pi^{-1}(\nu_i)$, in which one point cancels with a point in $\Sigma_{-, i}$ (in mod 2 sense). Hence the intersection contributes to one point in $\Sigma_{+, i}$.

By [KM11, Proposition 2.6], we use ν_i^h to determine singular bundle data \mathbf{P}_i on X_Δ^h up to isomorphism and the addition of instantons and monopoles, which is used to construct the instanton cobordism maps.

The assumption (2.8) implies that $\Sigma_{-,1} = \Sigma_{-,2}$. The assumption (2.9) and [KM11, Lemma 2.4 and Proposition 2.6] imply that

$$[\nu_1^h \cup \nu_2^h] = 0 \in H_2(W_\Delta^h; \mathbb{F}_2),$$

and we know that the corresponding singular bundle data \mathbf{P}_i are isomorphic up to the addition of instantons and monopoles. Hence the corresponding instanton cobordism maps equal up to sign. \square

Based on Lemma 2.1, we have the similar result as [LY25c, Lemmas 2.3 and 2.5]. The proof is also similar and we omit it.

Lemma 2.3. *Suppose Y is a closed connected oriented 3-manifold and suppose $K \subset Y$ is an unoriented knot. Suppose $\omega_1, \omega_2 \subset Y \setminus K$ are two disjoint unoriented 1-submanifolds such that $[\omega_1] = [\omega_2] \in H_1(Y; \mathbb{F}_2)$. Then there exists an embedded, possibly non-orientable surface $S \subset Y$ with $\partial S = \omega_1 \cup \omega_2$. Pushing S into $Y \times I$, we obtain cobordisms*

$$(Y \times I, K \times I, S) : (Y, K, \omega_1) \rightarrow (Y, K, \omega_2)$$

$$\text{and } (Y \times I, K \times I, S') : (Y, K, \omega_2) \rightarrow (Y, K, \omega_1).$$

We write the corresponding instanton cobordism maps by \mathbb{I}_S and \mathbb{I}'_S , respectively, for either I^\sharp or $I^\#$. Then we have the following.

- For any choice of S , the maps \mathbb{I}_S and \mathbb{I}'_S are isomorphisms over any ring \mathcal{R} .
- If $H_2(Y; \mathbb{F}_2) = 0$ (or equivalently $b_1(Y) = 0$), then, up to sign, \mathbb{I}_S is independent of the choice of S . As a consequence, up to sign, $I(Y, \omega_1; \mathcal{R})$ and $I(Y, \omega_2; \mathcal{R})$ are canonically isomorphic.

3. DIMENSION FORMULA

In this section, we prove Theorem 1.3 by considering the integral case and the rational case separately. The strategy is similar, both relying on Bhat's triangle stated as follows.

Lemma 3.1 ([Bha24, Theorem 1.1]). *Suppose $K \subset Y$ is a framed knot in a closed oriented connected 3-manifold Y and suppose $\omega \subset Y \setminus K$ is an unoriented 1-submanifold. Then for any coefficient ring \mathcal{R} , there is an exact triangle*

$$\begin{array}{ccc} I^\sharp(Y_0(K), \omega; \mathcal{R}) & \longrightarrow & I^\sharp(Y_2(K), \omega; \mathcal{R}) \\ & \searrow & \swarrow \\ & I^\sharp(Y, K, \omega; \mathcal{R}) & \end{array}$$

We fix a coefficient field \mathbb{K} and let

$$M = \nu_{\mathbb{K}}^\sharp(K) \text{ and } R = r_{\mathbb{K}}(K)$$

be the integers from Theorem 1.1.

Proposition 3.2. Suppose $K \subset S^3$ is a knot and $n \in \mathbb{Z}$. Then we have

$$\dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}) = \begin{cases} 2R + 2|n - M| & \text{if } n \neq M; \\ 2R \text{ or } 2R + 2 & \text{if } n = M, \end{cases}$$

When $\mathbb{K} = \mathbb{F}_2$, the case $2R + 2$ will not happen and

$$\dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{F}_2) = R + |n - M|.$$

Proof. We only consider I^\sharp , and the result of I^\sharp is then obtained from (2.2). For the mirror knot \bar{K} of K , we have $-S_{-n}^3(\bar{K}) \cong S_n^3(K)$ and

$$\dim I^\sharp(S_{-n}^3(\bar{K}), \widetilde{\bar{K}}_{-n}; \mathbb{K}) = \dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}).$$

By taking mirror knots, we can assume $n \geq M$. Suppose μ is the meridian of K and λ is the Seifert longitude of K with $\mu \cdot \lambda = -1$. We take $(Y, K) = (S_n^3(K), \tilde{K}_n)$ in Lemma 3.1 with the meridian $\tilde{\mu} = n\mu + \lambda$ and the framed longitude

$$\tilde{\lambda} = -\mu + k\tilde{\mu} = (nk - 1)\mu + k\lambda$$

for some $k \in \mathbb{Z}$. Then we have

$$\tilde{\lambda} + 2\tilde{\mu} = (n(k + 2) - 1)\mu + (k + 2)\lambda.$$

Taking $k = -1, -2$, we obtain the exact triangles

$$\begin{array}{ccccc} I^\sharp(S_{n+1}^3(K); \mathbb{K}) & \xrightarrow{\hspace{2cm}} & I^\sharp(S_{n-1}^3(K); \mathbb{K}) & & \\ \searrow & & \swarrow & & \\ & I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}) & & & \\ \\ I^\sharp(S_{(2n+1)/2}^3(K); \mathbb{K}) & \xrightarrow{\hspace{2cm}} & I^\sharp(S^3; \mathbb{K}) & & \\ \searrow & & \swarrow & & \\ & I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}) & & & \end{array}$$

Then we have the following dimension inequalities

$$(3.1) \quad \begin{aligned} \dim I^\sharp(S_{n+1}^3(K); \mathbb{K}) + \dim I^\sharp(S_{n-1}^3(K); \mathbb{K}) &\geq \dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}), \\ \dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}) + \dim I^\sharp(S^3; \mathbb{K}) &\geq \dim I^\sharp(S_{(2n+1)/2}^3(K); \mathbb{K}). \end{aligned}$$

We split the proof into the following three cases and apply Theorem 1.1 in each case.

Case 1. $n \geq M + 2$. We have

$$\begin{aligned} \dim I^\sharp(S_{n+1}^3(K); \mathbb{K}) &= R + n + 1 - M, \quad \dim I^\sharp(S_{n-1}^3(K); \mathbb{K}) = R + n - 1 - M, \\ \text{and } \dim I^\sharp(S_{(2n+1)/2}^3(K); \mathbb{K}) &= 2R + 2n + 1 - 2M. \end{aligned}$$

Together with $\dim I^\sharp(S^3; \mathbb{K}) = 1$, the dimension inequalities (3.1) imply

$$\dim I^\sharp(S_n^3(K), \tilde{K}_n; \mathbb{K}) = 2(R + n - M).$$

Case 2. $n = M$. We have

$$\begin{aligned} \dim I^\sharp(S_{n+1}^3(K); \mathbb{K}) &= \dim I^\sharp(S_{n-1}^3(K); \mathbb{K}) = R + 1, \\ \text{and } \dim I^\sharp(S_{(2n+1)/2}^3(K); \mathbb{K}) &= 2R + 1. \end{aligned}$$

the dimension inequalities (3.1) imply

$$2R + 2 \geq \dim I^\#(S_n^3(K), \tilde{K}_n; \mathbb{K}) \geq 2R.$$

The parity result in (2.4) implies that the only possibilities of the dimension are $2R$ and $2R + 2$. Then we exclude $2R + 2$ when $\mathbb{K} = \mathbb{F}_2$.

From the exact triangles in [LY24, Lemmas 2.6 and 2.16], we know

$$\begin{aligned} \chi(I^\#(S_n^3(K), \tilde{K}_n; \mathbb{C})) &= \pm \chi(I^\#(S^3, K; \mathbb{C}) \pm \chi(I^\#(S_{n-1}^3(K), \tilde{K}_{n-1}; \mathbb{C})) \\ &= \chi(I^\#(S_n^3(K); \mathbb{C})) \pmod{2} \end{aligned}$$

where we use the identification of $I^\#$ and KHI from §2 and do not consider the Alexander garding on KHI . Since the Euler characteristics are independence of the coefficients, we have

$$\chi(I^\#(S_n^3(K), \tilde{K}_n; \mathbb{K})) = \chi(I^\#(S_n^3(K); \mathbb{K})) \pmod{2}.$$

From

$$\dim I^\#(S_n^3(K); \mathbb{K}) \in \{R, R + 2\},$$

we know that

$$\dim I^\#(S_n^3(K), \tilde{K}_n; \mathbb{K})$$

has the same parity as R . Hence we again conclude the result by (2.2). Note that the proof only works over \mathbb{F}_2 because we do not know if the parity of rank of the map f in (2.3).

Case 3. $n = M + 1$. Theorem 1.1 again implies that

$$2R + 4 \geq \dim I^\#(S_n^3(K), \tilde{K}_n; \mathbb{K}) \geq 2R + 2.$$

By adding the meridian μ of the knot K to all manifolds, we also obtain

$$2R + 4 \geq \dim I^\#(S_n^3(K), \tilde{K}_n, \mu; \mathbb{K}) \geq 2R + 2.$$

Since one of

$$\dim I^\#(S_M^3(K); \mathbb{K}) \text{ and } \dim I^\#(S_M^3(K), \mu; \mathbb{K})$$

equals to R , we know that one of

$$\dim I^\#(S_n^3(K), \tilde{K}_n; \mathbb{K}) \text{ and } I^\#(S_n^3(K), \tilde{K}_n, \mu; \mathbb{K})$$

equals to $2R + 2$. By (2.6) and the fact that $\tilde{K}_n = \mu \subset S_n^3(K)$, we conclude that

$$\dim I^\#(S_n^3(K), \tilde{K}_n; \mathbb{K}) = 2R + 2.$$

□

Similar to the proof in [LY25c, §6], the proof of the rational case relies on the following lemma.

Lemma 3.3 ([BS21, §4] and [LY25c, Lemma 6.1]). *Suppose p_0 and q_0 are co-prime integers satisfying $p_0 \neq 0$ and $|q_0| > 1$. Suppose $r_0 = p_0/q_0 \in (k, k+1)$ for some integer k . Then there exist $r_i = p_i/q_i$ for $i = 1, 2$ that satisfy the following conditions.*

- For $i = 1, 2$, p_i and q_i are co-prime, possibly zero integers, such that p_i and q_i have the same signs with p_0 and q_0 , respectively, when they are not zero.
- $r_1, r_2 \in [k, k+1]$.
- $p_1 + p_2 = p_0$ and $q_1 + q_2 = q_0$.

- (r_0, r_1, r_2) fits into a slope triad, or more precisely i.e.

$$p_0(-q_1) - (-p_1)q_0 = (-p_1)(-q_2) - (-p_2)(-q_1) = (-p_2)q_0 - p_0(-q_2) = 1.$$

Note that this implies that

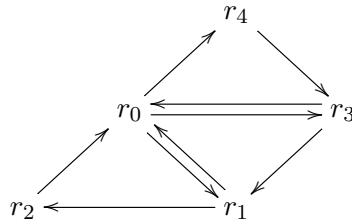
$$\frac{p_1}{q_1} > \frac{p_0}{q_0} > \frac{p_2}{q_2}.$$

We can further define relative prime integers p_i, q_i and $r_i = p_i/q_i$ for $i = 3, 4$ by

$$p_3 = p_0 + p_1, \quad q_3 = q_0 + q_1, \quad p_4 = p_0 + p_3, \quad \text{and} \quad q_4 = q_0 + q_3.$$

Then (r_0, r_3, r_1) and (r_0, r_4, r_3) are also slope triads, drawn as in

(3.2)



Remark 3.4. Although $n/1$ does not satisfies the assumption of Lemma 3.3, the case

$$(r_0, r_1, r_2, r_3, r_4) = \left(\frac{n}{1}, \frac{1}{0}, \frac{n-1}{1}, \frac{n+1}{1}, \frac{2n+1}{2}\right)$$

satisfies the slope triads in (3.2). Those slopes are used in the proof of Proposition 3.2, and motivates the choice of r_i in Lemma 3.3.

Proposition 3.5. Suppose $K \subset S^3$ is a knot and suppose p and q are co-prime integers with $q > 1$. Then we have

$$\dim I^\sharp(S_{p/q}^3(K), \tilde{K}_{p/q}; \mathbb{K}) = 2qR + 2|p - qM|.$$

$$\text{and } \dim I^\natural(S_{p/q}^3(K), \tilde{K}_{p/q}; \mathbb{F}_2) = qR + |p - qM|.$$

Proof. We only focus on I^\sharp . The result of I^\natural then follows from (2.2). Again by taking mirror knots, we can assume $p/q > M$. We apply Lemma 3.3 to $(p, q) = (p_0, q_0)$ to obtain the slopes r_i for $i = 1, 2, 3, 4$. Then Lemma 3.1 with suitable choices of framed knots implies the following two exact triangles.

$$\begin{array}{ccc}
I^\sharp(S_{r_3}^3(K); \mathbb{K}) & \longrightarrow & I^\sharp(S_{r_2}^3(K); \mathbb{K}) \\
& \searrow & \swarrow \\
& I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) & \\
\\
I^\sharp(S_{r_4}^3(K); \mathbb{K}) & \longrightarrow & I^\sharp(S_{r_1}^3(K); \mathbb{K}) \\
& \searrow & \swarrow \\
& I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) &
\end{array}$$

Then we have dimension inequalities

$$(3.3) \quad \begin{aligned} \dim I^\sharp(S_{r_3}^3(K); \mathbb{K}) + \dim I^\sharp(S_{r_2}^3(K); \mathbb{K}) &\geq I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}), \\ \dim I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) + \dim I^\sharp(S_{r_1}^3(K); \mathbb{K}) &\geq \dim I^\sharp(S_{r_4}^3(K); \mathbb{K}). \end{aligned}$$

From the choice of the slopes in Lemma 3.3, we have

$$r_1 > r_3 > r_4 > r_0 > r_2 \geq [r_0],$$

where $[r_0]$ is the maximal integer less than r_0 . Since we assume that $r_0 > M$, we know $r_2 \geq M$. We split the proof into the following two cases and apply Theorem 1.1 in either case. We assume $q_i \geq 1$ for $i = 1, 2, 3, 4$.

Case 1. $r_2 > M$. We have

$$\dim I^\sharp(S_{r_i}^3(K); \mathbb{K}) = q_i R + p_i - q_i M$$

the dimension inequalities (3.3) imply

$$\begin{aligned} \dim I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) &\leq (q_3 R + p_3 - q_3 M) + (q_2 R + p_2 - q_2 M), \\ \dim I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) &\geq (q_4 R + p_4 - q_4 M) - (q_1 R + p_1 - q_1 M). \end{aligned}$$

Since

$$p_4 = p_0 + p_3 = 2p_0 + p_1 = p_0 + p_1 + p_2,$$

$$\text{and } q_4 = q_0 + q_3 = 2q_0 + q_1 = q_0 + q_1 + q_2,$$

we have

$$\begin{aligned} \dim I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) &= (q_3 R + p_3 - q_3 M) + (q_2 R + p_2 - q_2 M) \\ &= 2(q_0 R + p_0 - q_0 M). \end{aligned}$$

Case 2. $r_2 = M$. Similar to Case 1, we have

$$2(q_0 R + p_0 - q_0 M) + 2 \geq \dim I^\sharp(S_{r_0}^3(K), \tilde{K}_{r_0}; \mathbb{K}) \geq 2(q_0 R + p_0 - q_0 M).$$

We use the same argument as in Case 3 in the proof of Proposition 3.2 to conclude the result. Note that

$$[\tilde{K}_{p/q}] = [\mu] \in H_1(S_{p/q}^3(K); \mathbb{F}_2)$$

and we apply Lemma 2.3. □

Proof of Theorem 1.3. The case $\omega = 0$ follows directly from Propositions 3.2 and 3.5. The case $\omega = \tilde{K}_{p/q}$ follows from (2.7).

For general choice of ω , by Lemma 2.3, we know the dimension only depends on the homology class $[\omega] \in H_1(S_{p/q}^3(K); \mathbb{F}_2)$. Note that

$$(3.4) \quad H_1(S_{p/q}^3(K); \mathbb{F}_2) = \begin{cases} 0 & \text{when } p \text{ odd;} \\ \mathbb{F}_2\langle[\mu]\rangle & \text{when } p \text{ even,} \end{cases}$$

where μ is the meridian of K . Moreover, we have $[\tilde{K}_{p/q}] = [\mu] \in H_1(S_{p/q}^3(K); \mathbb{F}_2)$. Hence the cases $\omega = 0$ and $\omega = \tilde{K}_{p/q}$ imply the case of general ω . □

4. $SU(2)$ -REPRESENTATIONS

In this section, we describe the relation between $I^\natural(Y, K)$ and $SU(2)$ representation, and then prove Theorems 1.8 and 1.9.

Lemma 4.1 ([BLSY24, §4]). *Suppose Y is a rational homology sphere with $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}/p$ for some odd prime power p . Suppose $K \subset Y$ is a knot such that $H_1(Y \setminus K; \mathbb{Z}) \cong \mathbb{Z}$. If either of the following condition holds, then*

$$I^\natural(Y, K; \mathbb{Z}) \cong \mathbb{Z}^p.$$

- The double branched cover $\Sigma(Y, K)$ is $SU(2)$ -abelian and $\det(K) = 1$.
- (Y, K) is meridian-traceless $SU(2)$ -abelian and p does not divides $\det(K)$.

In such cases, for any field \mathbb{K} , we have

$$I^\natural(Y, K; \mathbb{K}) = \dim I^\natural(Y; \mathbb{C}) = p.$$

In particular, we know Y is an instanton L-space and K is an instanton Floer simple knot.

Proof. The result for the first condition follows from [BLSY24, Theorem 4.1 and Lemma 4.4]. Note that we use the notation (Y, K) instead of (L, J) , and the conditions $H_1(Y; \mathbb{Z}) \cong \mathbb{Z}/p$ and $H_1(Y \setminus K; \mathbb{Z}) \cong \mathbb{Z}$ imply that $[K]$ generates $H_1(Y; \mathbb{Z})$, for which K is called *primitive*. We also replace KHI with I^\natural by the discussion in §2. Indeed, we use the second approach to consider the representation variety, which is defined over \mathbb{Z} instead of just \mathbb{C} .

The result for the second condition follows from the last paragraph in the proof of [BLSY24, Proposition 4.5], [BLSY24, Proposition 4.9], and the arguments in the proof of [BLSY24, Proposition 4.1] about cyclotomic polynomials.

The last equations follow from the universal coefficient theorem, [KM11, Proposition 5.7], [LY22, Theorem 1.2], and [Sca15, Corollary 1.4], i.e.

$$\dim I^\natural(Y, K; \mathbb{C}) = \dim KHI(Y, K) \geq \dim I^\natural(Y; \mathbb{C}) \geq |\chi(I^\natural(Y; \mathbb{C}))| = |H_1(Y; \mathbb{Z})| = p.$$

□

Proof of Theorems 1.8 and 1.9. We apply Lemma 4.1 to the case

$$(Y, K) = (S_{p/q}^3(K'), \tilde{K}'_{p/q}).$$

In this case, by the proof of [BLSY24, Lemma 4.4], we have

$$H_1(Y; \mathbb{Z}) = |p| \text{ and } \det(K) = |\Delta_K(-1)| = |\Delta_{K'}(-1)| = \det(K').$$

By the proof of [BLSY24, Lemma 4.2], we have

$$\Sigma(S_{p/q}^3(K'), \tilde{K}'_{p/q}) \cong Y'_{p/2q}(\tilde{K}'),$$

where $\hat{Y} = \Sigma(S^3, K)$ and \hat{K}' is the preimage of K' in \hat{Y} . Moreover, since $Y \setminus K \cong S^3 \setminus K'$, we know that (Y, K) is meridian-traceless $SU(2)$ -abelian if and only if (S^3, K') is p/q -traceless $SU(2)$ -abelian.

Then it suffices to show that for $p/q \in (0, 6)$, we have

$$\dim I^\natural(S_{p/q}^3(K'), \tilde{K}'_{p/q}; \mathbb{K}) > p$$

for some field \mathbb{K} . We take $\mathbb{K} = \mathbb{F}_2$, then Theorems 1.1, 1.3, and the proof of [LY25c, Theorem 7.3] imply that the only possibilities are instanton L-space knots of genera at most 2, namely $U, T_{2,3}, T_{2,5}$ [BLSY24, FRW24].

□

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