

## SEIFERT SURFACE COMPLEMENTS OF NEARLY FIBERED KNOTS

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ABSTRACT. In this note, we study the nearly fibered knots recently introduced by Baldwin and Sivek, *i.e.*, knots whose knot homology has top rank two. We give a topological description of the Seifert surface complement of a nearly fibered knot by showing it must fall into one of the three basic models.

## 1. INTRODUCTION

It has been known that a knot  $K \subset S^3$  is fibered if and only if the top Alexander grading summand of its knot homology in any branch of Floer theory has rank one by work of Ni [Ni07] and Kronheimer-Mrowka [KM10b]. Hence it is natural to ask what happens if the top grading summand of the knot homology has rank two. Recently, Baldwin-Sivek in [BS22] introduced the following definition.

**Definition 1.1.** A knot  $K \subset S^3$  is said to be **nearly fibered** (in the Heegaard Floer sense) if

$$\widehat{HFK}(S^3, K, g(K); \mathbb{Q}) \cong \mathbb{Q}^2.$$

Their definition is stated in Heegaard Floer theory, but we can also define nearly fibered knots in the instanton sense by requiring

$$KHI(S^3, K, g(K)) \cong \mathbb{C}^2.$$

In this note, we show that being a nearly fibered knot is a topological condition, *i.e.*, is independent of the branches of Floer theory. Moreover, we describe three models so that if a knot  $K \subset S^3$  is nearly fibered (in any sense) then the complement of its Seifert surface  $S$  in  $S^3$  must satisfy one of the three models. In particular, we prove the following theorem. Note in this paper we will only state the result and carry out the proofs in instanton setup but the same arguments apply verbatim in Heegaard Floer theory.

**Theorem 1.2.** *Suppose  $K \subset S^3$  is a knot so that*

$$KHI(S^3, K, g) \cong \mathbb{C}^2,$$

*where  $g = g(K)$  is the genus of the knot. Then, up to switching to the mirror of  $K$ , we have the following properties.*

- (1)  *$K$  has a unique minimal-genus Seifert surface up to isotopy.*
- (2) *Let  $S$  be the unique Seifert surface of  $K$ . Then there exists a balanced sutured manifold  $(M, \gamma)$ , a compact (possibly non-connected) oriented surface  $F$ , an orientation reversing diffeomorphism*

$$f : \gamma' \rightarrow \gamma''$$

*where  $\gamma' \subset \gamma$  is the union of some components of  $\gamma$  and  $\gamma'' \subset \partial F$  is the union of some components of  $\partial F$ , so that the following is true.*

- a. *The balanced sutured manifold  $(M, \gamma)$  falls into one of the following three models.*
  - (M1)  *$M$  is a solid torus and  $\gamma$  consists of four longitudes.*
  - (M2)  *$M$  is a solid torus and  $\gamma$  consists of two curves of slope 2.*
  - (M3)  *$M$  is the complement of the right handed trefoil and  $\gamma$  consists of two curves of slope  $-2$ .*
- b. *No component of  $F$  is a disk.*
- c. *There is a diffeomorphism*

$$(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S) \cong (M, \gamma) \cup_f ([-1, 1] \times F, \{0\} \times F).$$

*Remark 1.3.* The core idea of the theorem is that, if  $K$  is nearly fibered, then the complement of its Seifert surface is not too far away from being a product manifold. More precisely, we can cut off a large product piece  $[-1, 1] \times F$  off the complement of the Seifert surface and the remaining non-product piece must be one of the three basic models (M1), (M2), or (M3).

Since the same argument can also be carried out in Heegaard Floer theory, and when the Seifert surface complement do satisfy one of the above three basic models, we can argue straightforwardly that the knot homology indeed has top rank two, so we have the following corollary.

**Corollary 1.4.** *A knot is nearly fibered in the instanton sense if and only if it is nearly fibered in the Heegaard Floer sense.*

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## 2. RANK-2 TOP GRADING

In this section we prove the main theorem. The notations we use in this paper can be found in our previous papers, say [LY22].

*Proof of Theorem 1.2.* By the proof of [KM11, Proposition 7.16], we know that there is an isomorphism

$$SHI(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S) \cong KHI(S^3, K, g) \cong \mathbb{C}^2.$$

From the instanton version of [KM10b, Proposition 6.6] and [KM11, Theorem 7.18], we know that  $(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S)$  is horizontally prime. Since we have

$$b^1(S^3 \setminus [-1, 1] \times S) + 1 = 2 \cdot g(S) + 1 = 3 > 2,$$

from [GL19, Corollary 1.6], we know that the sutured manifold  $(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S)$  is not reduced. See [Juh10, Definition 2.16] for the definition of the reduced sutured manifold. From [Juh10, Lemma 2.13], there exists  $A \subset M$  the union of pair-wise disjoint essential product annuli in  $M$  so that there is a sutured manifold decomposition

$$(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S) \stackrel{A}{\rightsquigarrow} (M, \gamma) \sqcup (M_1, \gamma_1).$$

Here  $(M, \gamma)$  is reduced and  $(M_1, \gamma_1)$  is a product sutured manifold and hence can be written as  $([-1, 1] \times F, \{0\} \times \partial F)$  for some compact surface  $F$ . The fact that  $F$  does contain disk components corresponds to the fact that  $A$  contains only essential product annuli. The inverse operation of annuli decomposition is to glue sutured manifolds along identification of sutures. This identification

yields the diffeomorphism  $f$ . Now it remains to study the piece  $(M, \gamma)$ . First, from [GL19, Lemma 5.6] we can assume that  $(M, \gamma) \sqcup (M_1, \gamma_1)$  is taut and

$$\begin{aligned} \dim SHI\left((M, \gamma) \sqcup (M_1, \gamma_1)\right) &= \dim SHI(M, \gamma) \cdot \dim SHI(M_1, \gamma_1) \\ &\leq \dim SHI(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S) \\ &= 2. \end{aligned}$$

From [KM10b, Theorem 7.12] and the instanton version of [KM11, Proposition 6.7], we know that

$$1 \leq \dim SHI(M, \gamma) = \dim SHI\left((M, \gamma) \sqcup (M_1, \gamma_1)\right) \leq 2.$$

The same argument as above shows that  $(M, \gamma)$  is horizontally prime. Since by assumption  $(M, \gamma)$  is reduced, we can apply [GL19, Corollary 1.6] again and conclude that

$$g(\partial M) \leq 2 - 1 = 1.$$

If  $g(\partial M) = 0$ , since  $M$  is irreducible, we know  $M = B^3$ . Then  $(M, \gamma)$  and hence the Seifert surface complement  $(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S)$  are both product sutured manifolds, which is absurd. As a result, we must have  $\partial M \cong T^2$ . It is well-known that any torus in  $S^3$  bounds a solid torus. Hence we have two cases.

**Case 1.** When  $M$  is a solid torus. The instanton Floer homology of any sutured solid torus can be found in [Li19, Section 4.3]. So the only two possibilities are the ones as stated in part a.

**Case 2.** When  $S \setminus M$  is a solid torus, *i.e.*, there is a knot  $J \subset S^3$  so that  $M \cong S^3(J)$ . Suppose  $\gamma$  has  $2n$  components. Let  $\gamma_2$  be the union of two components of  $\gamma$ . By the proof of [KM10a, Theorem 3.1], we know that

$$SHI(M, \gamma) \cong \mathbb{C}^{2^{n-1}} \otimes SHI(M, \gamma_2).$$

Since

$$SHI(M, \gamma) \cong \mathbb{C}^2,$$

either  $n = 2$  and  $SHI(M, \gamma_2) \cong \mathbb{C}$  or  $n = 1$ . For the former case, from [KM11, Theorem 7.18] we know  $M$  must also be a solid torus which reduces to Case 1. For the later case, we further divide into two sub-cases.

**Case 2.1** Each components of  $\gamma$  represents a generator of  $\ker i_* \subset H_1(\partial M) \cong \mathbb{Z}^2$ , where

$$i_* : H_1(\partial M) \rightarrow H_1(M)$$

is the map induced by the natural inclusion

$$i : \partial M \hookrightarrow M.$$

In this case, if  $H_2(M, \partial M)$  is generated a disk, then  $(M, \gamma)$  is non-taut and  $SHI(M, \gamma) = 0$  by the adjunction inequality (*c.f.* [KM10b, Proposition 7.5]). If any properly embedded surface representing a generator of  $H_2(M, \partial M)$  has genus at least 1, then we can pick any such generator, say  $T \subset M$ , of the minimal genus. The surface  $T$  then induces a grading on  $SHI(M, \gamma)$ , and we know from [GL19, Lemma 6.2] that we have two taut decompositions

$$(M, \gamma) \overset{\pm T}{\rightsquigarrow} (M_{\pm}, \gamma_{\pm}).$$

Since  $g(T) \geq 1$ , we know from the proof of [KM10b, Proposition 7.11] that

$$SHI(M_+, \gamma_+) \oplus SHI(M_-, \gamma_-) \hookrightarrow SHI(M, \gamma).$$

Furthermore, each one of  $\gamma_+$  and  $\gamma_-$  contains at least three components that are parallel to each other, so from [KM10b, Theorem 7.12] and the proof of [KM10a, Theorem 3.1] that

$$\dim SHI(M_{\pm}, \gamma_{\pm}) \geq 2.$$

As a result we have

$$\dim SHI(M, \gamma) \geq 4.$$

which leads to a contradiction in this case.

**Case 2.2** Components of  $\gamma$  do not represent generators of  $\ker i_*$ . Let  $Y$  be the Dehn filling of  $M$  along a component of  $\gamma$ . Then [LY22, Proposition 1.4] implies that

$$\dim I^{\sharp}(Y) \leq \dim SHI(M, \gamma) = 2.$$

Note the proof of [LY22, Proposition 1.4] also implies that

$$\dim I^{\sharp}(Y) \equiv \dim SHI(M, \gamma) \pmod{2}.$$

Hence by [Sca15, Corollary 1.4], we know that

$$\dim I^{\sharp}(Y) = 2 = |H_1(Y)|.$$

We can take the mirror of the knot  $K$  if needed to assume that  $Y$  arises as a positive Dehn surgery of some knot. By [BS21, Theorem 1.10] there are only two knots whose positive Dehn surgeries can possibly yield an instanton L-space whose first homology has rank 2: the unknot and the right-handed trefoil. The case of unknot still reduces to Case 1. The case of right-handed trefoil is a new case. By [BS21, Theorem 1.1, Table 1], the surgery slope must be 2. Hence  $\gamma$  consists of curves of slope 2, which concludes the proof of Theorem 1.2.  $\square$

*Remark 2.1.* We have the following comments that can strengthen the description of the Seifert surface complements in some aspects.

- (1) We can compute the Euler characteristic in each of the three models. When

$$(M, \gamma) = (\text{Solid torus, four longitudes}),$$

results in [KM10a] show that

$$\chi(SHI(M, \gamma)) = 0.$$

As a result,

$$\chi(KHI(S^3, K, g(K))) = 0$$

and we know that the symmetrized Alexander polynomial of  $K$  has degree at most  $g(K) - 1$ . On the other hand, if  $(M, \gamma)$  is one of the rest two models, we can compute as in [LY21] that

$$\chi(SHI(M, \gamma)) = \pm 2.$$

As a result, the symmetrized Alexander polynomial of  $K$  has degree  $g(K)$  and top non-zero coefficients  $\pm 2$ .

- (2) The proof of [BS22, Lemma 3.4] implies that the Seifert surface complement  $(S^3 \setminus [-1, 1] \times S, \{0\} \times \partial S)$  admits no product annuli whose boundary has a component that is parallel to the suture  $\{0\} \times \partial S$  on the boundary of the Seifert surface complement. As a result, we can further conclude that  $\partial F$  must have one more components than  $\gamma$ , and all but one components of  $\partial F$  are glued to all of  $\gamma$ . This actually rules out one possibility in the case  $g(K) = 1$  as in the following example.

**Example 2.2.** When  $g(K) = 1$ , we know that

$$S \cong (R_+(\gamma) \cup \{1\} \times F).$$

Since in all three models we have  $\chi(R_+(\gamma)) = 0$ , we know that

$$\chi(F) = -1.$$

From part (2) of the Remark 2.1, we know that  $\partial F$  has one more component than  $\gamma$ . The Euler characteristic of  $\chi(F)$  then rules out the case that  $\gamma$  has four components. As a result, we only have two models:

- $M$  is the complement of the unknot and  $\gamma$  consists of two curves of slope 2.
- $M$  is the complement of the right handed trefoil and  $\gamma$  consists of two curves of slope 2.

Furthermore, in this case the surface  $F$  must be a pair of pants. Yet gluing such a thickened pair of pants to  $(M, \gamma)$  along two of the three boundary components is equivalent to gluing a product 1-handle to  $(M, \gamma)$ . Turning this around, we know that  $(M, \gamma)$  being one of the above two models is obtained from the complement of the Seifert surface by a disk decomposition and this coincides with the discussion in [BS22, Section 1.2] right above Theorem 5.1. Note these two models do exist: for examples, they give rise to the knot  $5_2$  in Rolfsen's table and the 2-twisted Whitehead double of the right-handed trefoil with positive clasp, respectively.

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