Knots, SU(2), and the pillowcase

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Lecture 1 SU(2) representations

M a (smooth, oriented, connected) compact 3-manifold

Y a $(\cdots)$ closed 3-mfd

π_1 fundamental group

$SU(2) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.\ AA^* = Id, \det A = 1 \right\}$

Topologically, $SU(2) \cong S^3$

It is the simplest compact nonabelian Lie group

Poincaré conj (proved by geometrization)

If $Y \not\cong S^3$, then $\pi_1(Y) \neq \{e\}$

Kirby Problem 3.105(A)

If $M \not\cong S^3$, then $\exists$ homomorphism $\rho: \pi_1(M) \to SU(2)$ s.t. $\mathrm{Im}\rho \neq \{Id\}$

If $\pi_1(M)$ is nonabelian, when $\mathrm{Im}\rho$ is nonabelian

Def A homomorphism $\rho: \pi_1(M) \to SU(2)$ is called a SU(2) representation of M

It is called nontrivial if $\mathrm{Im}\rho \neq \{Id\}$

central if $\mathrm{Im}\rho \subset \{\pm Id\}$

reducible/abelian if $\mathrm{Im}\rho$ is abelian

irreducible/nonabelian if $\mathrm{Im}\rho$ is nonabelian

abelian means $\mathrm{Im}\,\rho \subset$ a copy of $S^1 \subset SU(2)$

i.e. $\left\{ A \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix} A^{-1} \right\} \subset SU(2)$

M is called <u>$SU(2)$ nonabelian</u>

if $\exists$ nonabelian $SU(2)$ rep of M

M is called <u>$SU(2)$ abelian</u>

if any $SU(2)$ rep of M is abelian

<u>Rem</u>
- Kirby prob $\Rightarrow$ Poincaré conj
- $\exists M \not\cong S^3$, but M is $SU(2)$ abelian

  e.g. mentioned after Kirby 3.105 (A)

  $\frac{37}{2}$-surgery on pretzel knot $P(-2,3,7)$

  which is diffeo to $(S^3 \setminus\setminus T_{2,3}) \cup_{T^2} (S^3 \setminus\setminus T_{2,-3})$

  meridian $\quad \mu \longmapsto -6\mu + \lambda$ (Seifert fiber)

  longitude $\quad \lambda \longmapsto 37\mu - 6\lambda$

  Notations will be introduced later
- $\pi_1(L(p,q)) \cong \mathbb{Z}/p$ for lens space

  $\pi_1(S^1 \times S^2) \cong \mathbb{Z} \quad \pi_1(T^3) \cong \mathbb{Z}^3$

  By geometrization, those are all

  closed <u>oriented</u> 3-mfds with π_1 abelian

  nonorientable examples:

  $\cdot\ \pi_1(S^1 \tilde{\times} S^2) \cong \mathbb{Z}$, $\pi_1(S^1 \times \mathbb{R}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2$

  open examples $\quad \pi_1(S^2 \times \mathbb{R}) = 0 \quad \pi_1(T^2 \times \mathbb{R}) = \mathbb{Z}^2$
- abelianization $Ab(\pi_1(M)) \cong H_1(M;\mathbb{Z})$

  If $H_1(M;\mathbb{Z}) \neq 0$, there always exists

  abelian $SU(2)$ rep of M:

  $$H_1(M;\mathbb{Z}) \cong \mathbb{Z}^k \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_l$$

$$\rho : \pi_1(M) \longrightarrow H_1(M;\mathbb{Z}) \longrightarrow S^1 \longrightarrow SU(2)$$

$\mathbb{Z} \mapsto S^1 \qquad 1 \longmapsto e^{i\alpha} \quad \alpha \in [0,2\pi) \to \begin{bmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{bmatrix}$

$\mathbb{Z}/p \longmapsto S^1 \qquad 1 \longmapsto e^{\frac{2\pi k i}{p}} \quad k = 1, \dots, p-1 \to \begin{bmatrix} e^{\frac{2\pi k i}{p}} & \\ & e^{\frac{-2\pi k i}{p}} \end{bmatrix}$

Def: Y is called an
integral homology sphere ($\mathbb{Z}HS^3$)
if $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$
rational homology sphere ($\mathbb{Q}HS^3$)
if $H_*(Y;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$

Rem By universal coeff thm + Poincaré duality
$H_1(Y;\mathbb{Z}) \cong H_1(S^3;\mathbb{Z}) = 0$
$\Rightarrow H^1, H_2, H^2(Y;\mathbb{Z}) = 0$
The same holds over $\mathbb{Q}$

Kirby problem, v2
If $Y \not\cong S^3$ is a $\mathbb{Z}HS^3$
then there exists a nontrivial $SU(2)$
rep of Y ($H_1(Y;\mathbb{Z})=0$ means all real rep are trivial)

Thm (Kronheimer–Mrowka '04)
For a nontrivial knot $K \subset S^3$,
rational $r = p/q \in [0,2]$, the surgery
manifold $S^3_r(K)$ is $SU(2)$
nonabelian.

Thm (Zentner '18)
For two nontrivial knots $K_1, K_2 \subset S^3$
the splicing of knot complements
is $SU(2)$ nonabelian.

$$Y(K_1,K_2) = (S^3 \setminus \mathring{\nu}(K_1)) \cup_{T^2} (S^3 \setminus \mathring{\nu}(K_2))$$
$$\mu \longmapsto \lambda$$
$$\lambda \longmapsto \mu$$

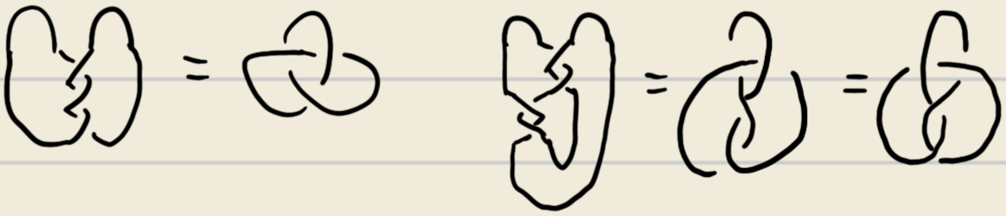
Rem $S^3_{1/q}(K)$, $Y(K_1,K_2)$ are $\mathbb{Z}HS^3$

Basic knot theory

K a knot (an embedding $S^1 \hookrightarrow S^3$ / isotopy)

unknot U

left-handed trefoil 3_1, $T_{2,-3}$

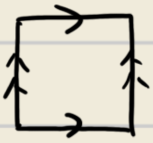


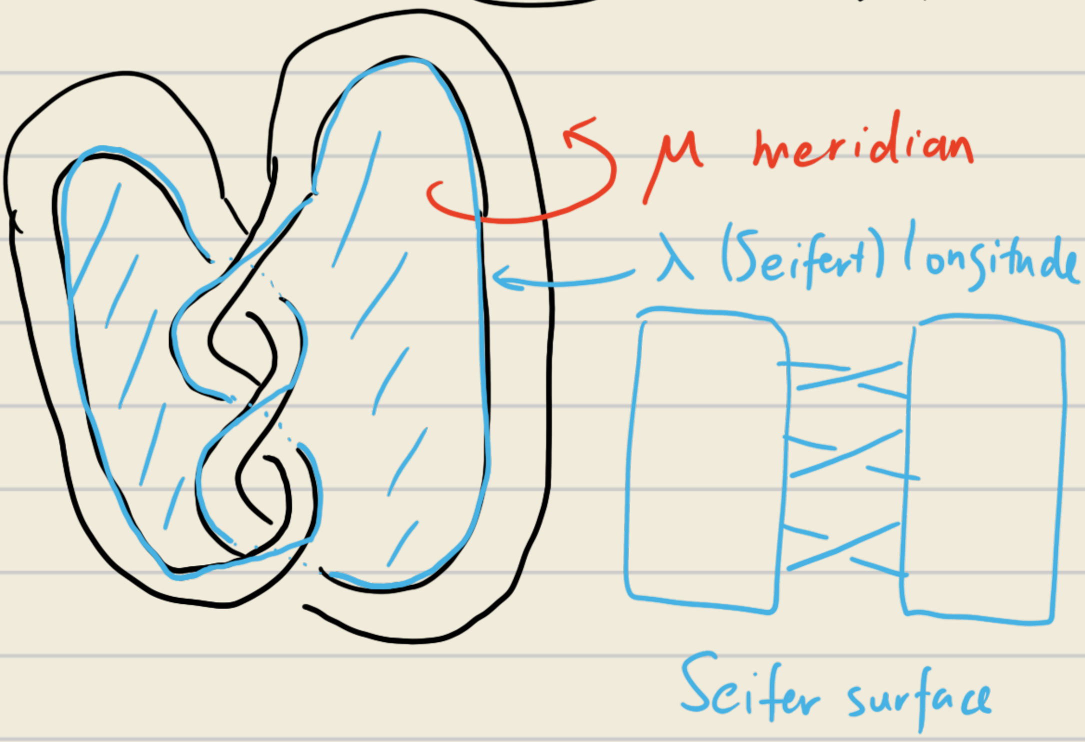
right-handed trefoil $\overline{3}_1 = \text{mirror}(3_1)$, $T_{2,3}$

figure-8 knot $4_1 = \overline{4}_1$

$$M_K = S^3 \setminus\setminus K = S^3 \setminus \mathring{\nu}(K)$$

knot complement $\partial M_K = T^2$

T^2 a 2-torus  or 



$H_1(M_K) \cong \mathbb{Z}$

$H_1(\partial M_K = T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

$\mathrm{Ker}\big(H_1(T^2) \to H_1(\nu(K) \cong S^1 \times D^2) \cong \mathbb{Z}\big)$

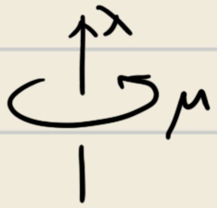
$= \mathbb{Z}\langle \mu \rangle$

$\mathrm{Ker}\big(H_1(T^2) \to H_1(M_K)\big) = \mathbb{Z}\langle \lambda \rangle$

Choose orientations of μ, λ

so that $\mu \cdot \lambda = -1$

(point inside M_K)



If K is oriented, then the orientation

of λ is also determined.

and also μ.

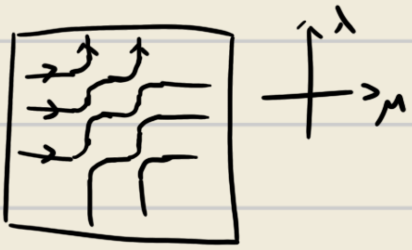
Hence $H_1(T^2)$ has a standard basis

There is a 1-1 correspondence

$\mathbb{Q} \cup \{\infty\} \longleftrightarrow \{$simple closed curve on $T^2\}$

$p/q \longleftrightarrow$ curve with homology class $p[\mu] + q[\lambda]$

e.g. $3\mu + 2\lambda$



$$S^3_{p/q}(K) = S^1 \times D^2 \cup_{T^2} M_K$$

$$pt \times \partial D^2 \longmapsto p\mu + q\lambda$$

we don't need to specify the image of $S^1 \times pt$ because different choices are related by diffeomorphism (twist on ∂D^2)

Def $S^3_{p/q}(K)$ is called the (Dehn) surgery manifold from K with slope p/q.

Rem $H_1(M_K) \cong \mathbb{Z}$

$$H_1(S^3_{p/q}(K)) \cong H_1(M_K)/\mathrm{Im}(pt \times \partial D^2) \cong \mathbb{Z}/p$$

When $p=1$, $S^3_{1/q}(K)$ is a $\mathbb{Z}HS^3$

When $p \neq 0$, $S^3_{p/q}(K)$ is a $\mathbb{Q}HS^3$

Def A Seifert surface of $K \subset S^3$ is a properly embedded, oriented, connected surface $S \subset S^3 \setminus K$ s.t. $\partial S = K$

The genus $g(K)$ of K is the minimal genus of the possible Seifert surface of K

K is called fibered if there exists a Seifert surface S s.t. $S^3 \setminus \nu(S) \cong S \times I$

i.e. there exists fibration $\begin{array}{ccc} S & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^1 \end{array}$ s.t. the boundary of any fiber is the knot K.

Rem $g=0 \Rightarrow K = U$

$g=1$ + fibered $\Rightarrow K = 3_1, \bar{3}_1, 4_1$

Let's come back to KM and Zenter's results.

For nontrivial knot K (i.e. $\neq U$)

KM $\Rightarrow$ $S^3_1(K)$ is not $SU(2)$ abelian

$\Rightarrow$ $\pi_1(S^3_1(K)) \neq 0$

so called Property P conj

(Gorden-Luecke '89. $S^3_{p/q}(K) \neq S^3$)

Zenter + geometrization

$\Longrightarrow$ Any $\mathbb{Z}HS^3$ $Y \neq S^3$

has a nonabelian (irreducible)

$SL(2,\mathbb{C})$ representation

Idea of pf:

geometrization + Boileau-Rubinstein-Wang '14

$\Rightarrow$ Any $\mathbb{Z}HS^3$ $Y \neq S^3$ has a degree $\underline{1}$

map to Y' that is either

hyperbolic, Seifert fibered

or some splicing $Y(K_1, K_2)$

- hyperbolic $\Rightarrow$ metric implies $SL(2,\mathbb{C})$ rep
- Splincing $\Rightarrow$ by Zenter's result + $SU(2) \hookrightarrow SL(2,\mathbb{C})$
- Seifert fibered $\Rightarrow$ $Y' = \Sigma_2(K)$ double branched cover of a Montesino's knot K always has nonabelian $SU(2)$ through Fintushel-Stern '90 or Kronheimer-Mrowka '10

<u>Rem</u> All above existence results about $SU(2)$ are from <u>instanton Floer homology</u>

Pillowcase

Def For a manifold Z, we write

$$R(Z) = \mathrm{Hom}(\pi_1(Z), SU(2))$$

called the <u>$SU(2)$ representation variety</u>

$$X(Z) = R(Z)/SU(2) \text{ conj}$$

called the <u>$SU(2)$ character varitey</u>

$SU(2)$ acts by conjugation, i.e.

$A \in SU(2)$ $\quad \rho \in R(Z)$ $\quad x \in \pi_1(Z)$

$(A\rho)(x) = A\rho(x)A^{-1}$ $\quad \pm Id$ acts trivially

So it descends to $SU(2)/\pm Id \cong SO(3)$

R^{red} R^{irr} X^{red} $X^{irr}(Z)$ denote

reducible and irreducible subsets

(abelian) $\qquad$ (nonabelian)

Ex. $Z = T^2$ $\quad \pi_1(Z) \cong \mathbb{Z} \oplus \mathbb{Z}$ abelian

$$X(Z) = X^{red}(Z)$$
$$= \mathrm{Hom}(\mathbb{Z} \oplus \mathbb{Z}, U(1))/SU(2)$$

Let μ, λ be generators of $\pi_1(Z)$

$[\rho] \in X(Z)$ is determined by

$$\rho(\mu) = \begin{pmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{pmatrix} \quad \rho(\lambda) = \begin{pmatrix} e^{i\beta} & \\ & e^{-i\beta} \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{R}/2\pi\mathbb{Z}$

Note that $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{bmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}^{-1}$

$$= \begin{bmatrix} & -e^{-i\alpha} \\ e^{i\alpha} & \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$$

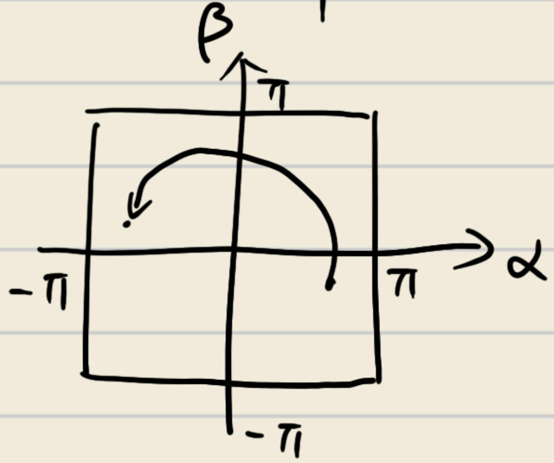
$$= \begin{bmatrix} e^{-i\alpha} & \\ & e^{i\alpha} \end{bmatrix}$$

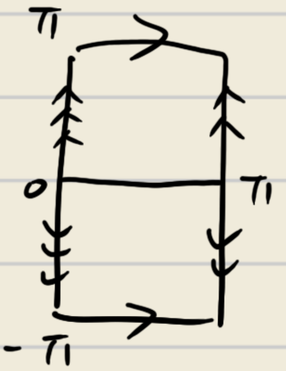
and also $\mathrm{tr}(A\rho(x)) = \mathrm{tr}(\rho(x))$

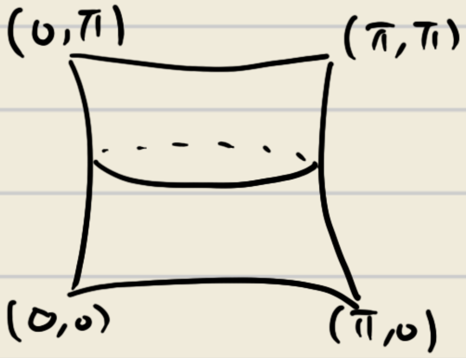
So $(\alpha,\beta) \sim (-\alpha,-\beta)$ is the only relation under the action

$$X(T^2) = \left(\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}\right)\Big/_{(\alpha,\beta)\sim(-\alpha,-\beta)}$$

$= $ pillowcase P







Sometimes we will rescale the pillowcase so that π becomes 1 i.e. $P = (\mathbb{R}/\mathbb{Z})^2/\sim$

For a knot $K \subset S^3$

write $X(K)$ for $X(S^3\setminus K)$

The inclusion $i : T^2 = \partial(S^3\setminus K) \hookrightarrow S^3\setminus K$

induces $i^* : X(K) \longrightarrow X(T^2)$

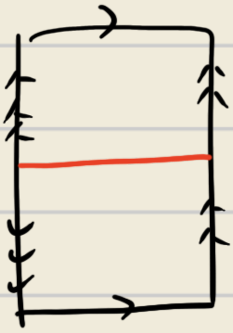
$$\begin{tikzcd}
\pi_1(S^3\setminus K) \arrow[r, "\rho"] & SU(2) \\
\pi_1(T^2) \arrow[u, "i"] \arrow[ur, dotted, "i^*\rho = \rho\circ i"'] &
\end{tikzcd}$$

i^* is not necessarily an inclution

Since $H_1(S^3\setminus K) \cong \mathbb{Z}\langle\mu\rangle$

$$X^{red}(K) = \left(\mathbb{R}/2\pi\mathbb{Z}\right)\Big/_{\pm 1} = [0,\pi]\times\{0\}$$

$i^* X^{red}(K) =$



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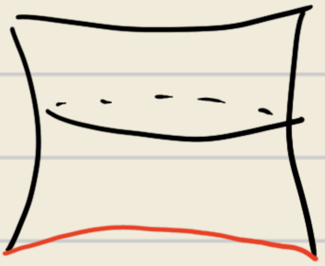
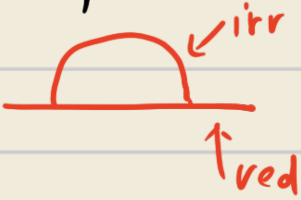
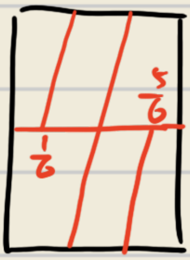


The irreducible part is more complicated

Ex $X(3_1) = X($  $) =$



$i^* X(3_1) =$



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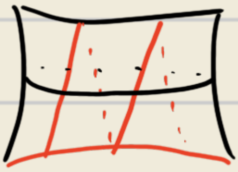
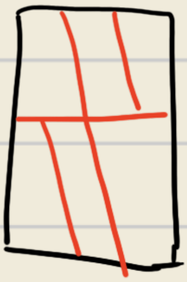


$i^* X(\bar{3}_1) =$



because $\pi_1(S^3 \setminus\setminus 3_1) = \pi_1(S^3 \setminus\setminus \bar{3}_1)$

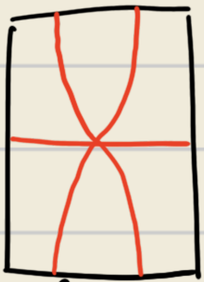
and (μ, λ) for 3_1 is $(-\mu, \lambda)$ for $\bar{3}_1$

by $\mu \cdot \lambda = -1$

$S^3 \setminus\setminus \bar{3}_1$ is the orientation reversal of $S^3 \setminus\setminus 3_1$, so boundary orientation is opposite

$X(4_1) =$

$i^* X(4_1) =$



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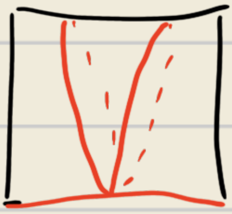


Moreover, X for torus knots and 2-bridge knots are computatable

Lec2 the pillowcase

We first introduce torus knots and 2-bridge knots as digression

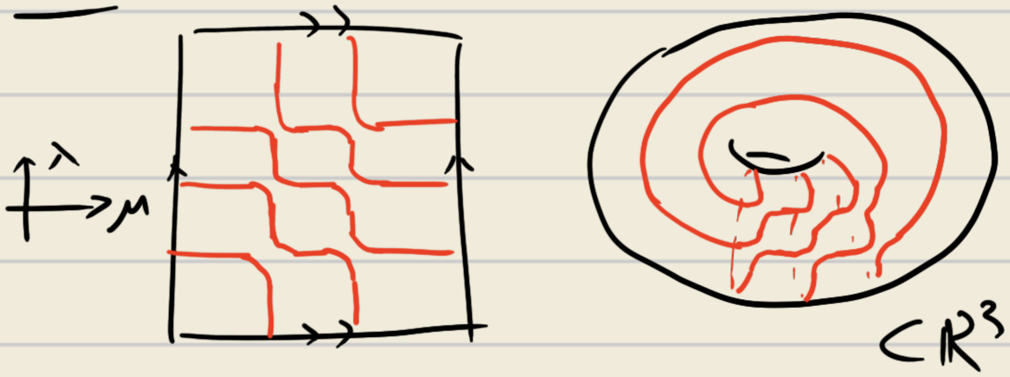
<u>torus knots</u> $T_{p,q}$ for $\gcd(p,q)=1$

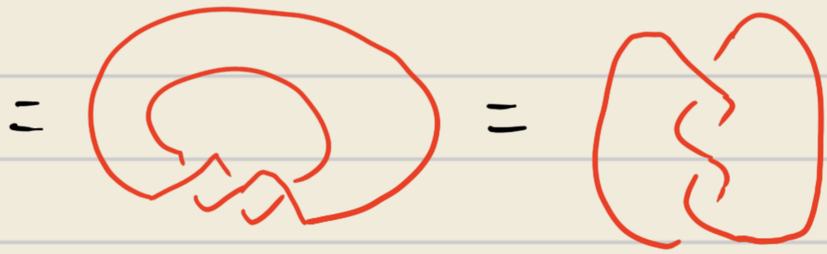
the curve $p\mu + q\lambda \subset T^2$

under the standard embedding

$T^2 \hookrightarrow \mathbb{R}^3$

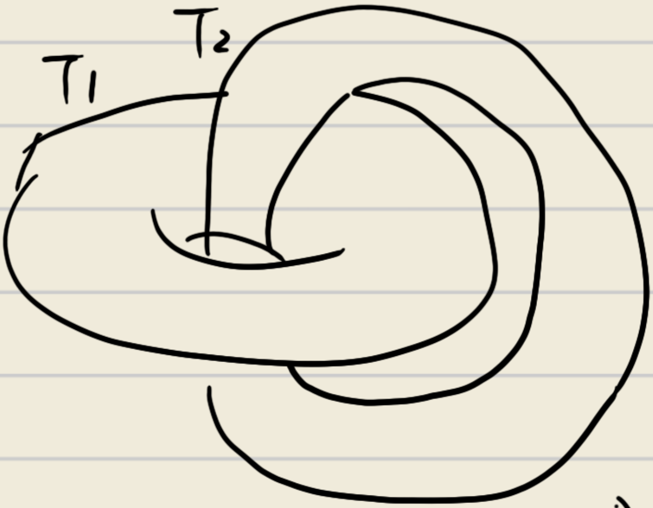
<u>Ex</u> $T_{2,-3}$





$T_{p,q} = T_{q,p}$ because

$S^3 = \mathbb{R}^3 \cup \{\infty\} = S^1 \times D^2 \cup_{T^2} S^1 \times D^2$

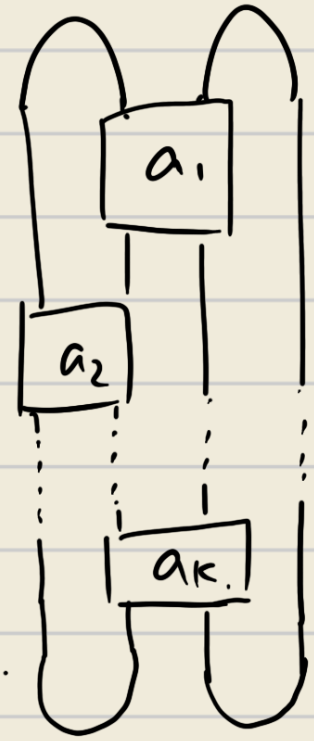


$p\mu + q\lambda$ on T_1 = $q\lambda + p\mu$ on T_2

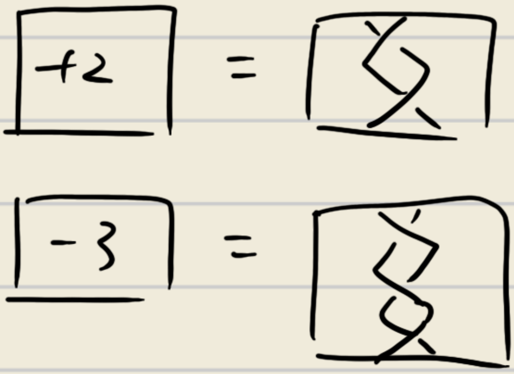
T_1 and T_2 are isotopic

2-bridge knot $B_{p,q}$

also called rational knots
because it only depends on p/q



a_i denote positive/negative half twists

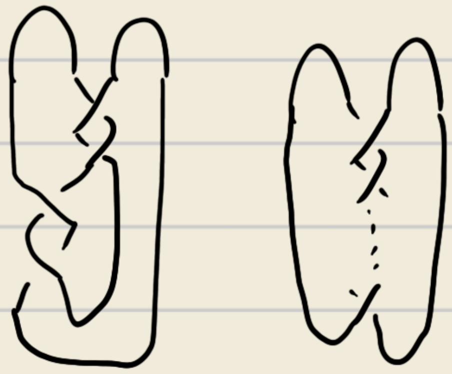


The knot is only determined by the continued function

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \cdots}}$$

Ex $\frac{5}{2} = 2 - \frac{1}{-2}$ $B_{5,2} = 4_1$

$B_{n,1} = T_{2,n}$



Rem • when p even $B_{p,q}$ is a 2-component link

i.e an embedding $S^1 \amalg S^1 \hookrightarrow S^3$

when p is odd, $B_{p,q}$ is a knot

• The double branched cover of $B_{p,q}$ is the lens space $L(p,q)$

Since $L(p_1,q_1) \cong L(p_2,q_2)$ iff $p_1 = p_2 = p$ and $q_1 \equiv q_2^{\pm 1} \bmod p$

we also have $B_{p_1,q_1} = B_{p_2,q_2}$ iff this relation holds

Let's come back to the pillowcase

For $Y = M_1 \cup_{T^2} M_2$, $i_j : T^2 \to M_j$

Seifert-van-Kampan thm implies

$$\pi_1(Y) = \frac{\pi_1(M_1) * \pi_1(M_2)}{\langle i_1\gamma = i_2\gamma \mid \gamma \in \pi_1(T^2)\rangle}$$

$$R(Y) = R(M_1) \times_{R(T^2)} R(M_2)$$

$$= \left\{ \begin{array}{l} (\rho_1, \rho_2) \in R(M_1) \times R(M_2) \\ \rho_1|_{i_1\pi_1(T^2)} = \rho_2|_{i_2\pi_2(T^2)} \end{array} \right\}$$

we have commutative diagram

$$\begin{tikzcd} R(Y) = R(M_1) \times_{R(T^2)} R(M_2) \arrow[r, "p_1"] \arrow[dr, "p_3"'] & X(Y) \arrow[d, "p_2"] \\ & X(M_1) \times_{X(T^2)} X(M_2) \end{tikzcd}$$

For $\rho \in R(Z)$, define the stabilizer

$$\mathrm{Stab}(\rho) = \{ A \in SU(2) \mid \rho(x) = A\rho(x)A^{-1} \text{ for all } x \}$$

For ρ irr, $\mathrm{Stab}(\rho) = \{\pm Id\}$

noncentral red, $\mathrm{Stab}(\rho) = U(1)$

central, $\mathrm{Stab}(\rho) = SU(2)$

[Hedden-Herald-Kirk Lem 4.2]

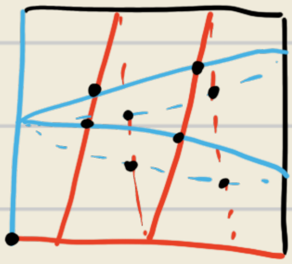
Lem For $(\rho_1, \rho_2) \in R(M_1) \times_{R(T^2)} R(M_2)$ and $([\rho_1], [\rho_2]) = p_3(\rho_1, \rho_2)$

The fiber is $p_2^{-1}([\rho_1], [\rho_2]) \cong$

$$\mathrm{Stab}(\rho_1) \backslash \mathrm{Stab}(\rho_1|_{i_1\pi_1(T^2)}) / \mathrm{Stab}(\rho_2)$$

By this lem, we can compute $X(Y)$ by the intersection points of $i_1^* X(M_1)$ and $i_2^* X(M_2)$ in $X(T^2) = P$

Ex $Y = Y(3_1, 4_1)$, i.e. the splicing of $M_1 = S^3 \setminus 3_1$, $M_2 = S^3 \setminus 4_1$, gluing map $\mu \mapsto \lambda$ $\lambda \mapsto \mu$



For each nonsingular intersection pt $\mathrm{Stab}(\rho_1|_{i_1\pi_1(T^2)}) = U(1)$ because any nonsingular pt in P is nontrivial red

$\mathrm{Stab}(\rho_j) = \{\pm Id\}$ because ρ_j is irr

So fiber is $\{\pm Id\} \backslash U(1) / \{\pm Id\} = U(1) = S^1$

There is an singular intersection in $X^{red}(M_1) \cap X^{red}(M_2)$

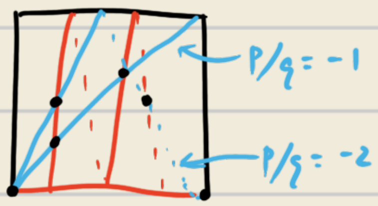
Stabilizers are all $SU(2)$, so fiber is one pt

So $X(Y) = pt \amalg 8S^1$

<u>Ex 2</u> $Y = S^3_{p/q}(3_1) = S^3 \setminus\setminus 3_1 \cup_{T^2} S^1 \times D^2$

$p\mu + q\lambda \longleftarrow pt \times \partial D^2$

$\pi_1(Y) = \pi_1(S^3 \setminus\setminus 3_1) / p\mu+q\lambda = Id$



Now for nonsingular intersection pt, $Stab(p_2) = U(1)$

So the fiber is

$\{\pm Id\} \setminus U(1) / U(1) = pt$

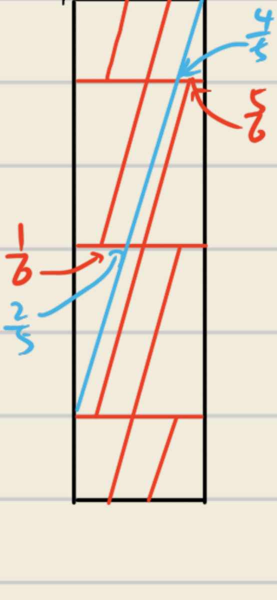
Hence

$X(S_{-1}(3_1)) = \underline{1}$ trivial rep $\cup$ 2 irr reps

$X(S_{-2}(3_1)) = \underline{1}$ trivial rep $\cup$ $\underline{1}$ central rep $\cup$ 2 irr reps

Note that $S^3_{-1}(3_1)$ is the mirror of Poincaré sphere

Moreover, we consider $p/q = -5$



$S^3_{-5}(3_1) = L(5,1)$

There is no irr. $SU(2)$ rep

one can try $p/q = -6 + \frac{1}{n}$ for $n \in \mathbb{Z}$ ($n=0$, $p/q = 6$)

There is no intersection pts

Indeed, the surgery mfds are all lens spaces for $p/q \neq 6$

From Sivek-Zenter'22, we know those are the only slopes when surgery mfds are $SU(2)$ abelian. One may also check by intersection pts

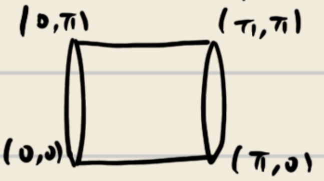
Now we sketch the idea in KM and Zenter's proofs

The main ingredient from instanton Floer homology is the following thm (we omit the pf)

Consider $P' = P \setminus$ arcs from $(0,0)$ to $(0,\pi)$

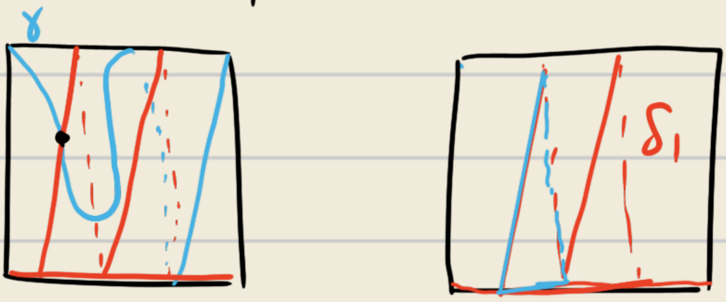
$(\pi,0)$ to (π,π)

$P' \cong S^1 \times I$



<u>Thm</u> [Zentner Thm 7.2] For a nontrivial knot $K \subset S^3$, any embedded path γ from $(0,\pi)$ to (π,π) in P' has an intersection pt with $i^* X(K)$.

<u>Ex</u>.



<u>Thm</u> [Zenter Thm 7.1] $i^* X(K)$ contains a topologically embedded curve δ s.t. $[\delta] \neq 0 \in H_1(P') \cong \mathbb{Z}$

<u>Rem</u> $i^* X(K)$ is an embedded finite graph

<u>Thm</u> (Zenter '18) For two nontrivial knots K_1, K_2, the splicing $Y(K_1, K_2)$

$$= S^3 \setminus\setminus K_1 \cup_{T^2} S^3 \setminus\setminus K_2$$

$$\mu \longmapsto \lambda$$

$$\lambda \longmapsto \mu$$

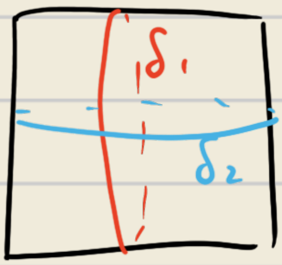
has an irreducible $SU(2)$ rep

$\boxed{\text{Proof Sketch}}$: Let δ_1, δ_2 be the curves from the above cor, then δ_1 must intersect δ_2 in

$P'' = P' \setminus$ arcs from $(0,0)$ to $(\pi,0)$

$(0,\pi)$ to (π,π)

which corresponds to an irr $SU(2)$ rep



$\square$

To prove KM's result, we need the following property

Consider the rotation ι around $(\frac{\pi}{2}, 0)$ on $X(T^2)$

or equivalently, rotation around $(\frac{\pi}{2}, \pi)$

<u>Prop</u> $i^*(X(K))$ is invariant under ι

pf: Define

$\phi: \pi_1(S^3 \setminus K) \to H_1(S^3 \setminus K) \cong \mathbb{Z}$

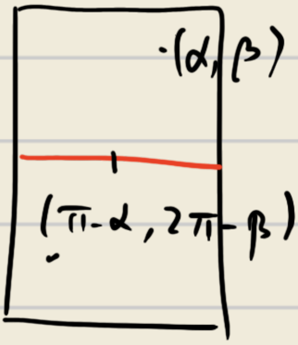
For $[\rho] \in X(K)$

$\phi^*[\rho](x) = [\rho(x)(-Id)^{\phi(x)}]$

ϕ^* sends $(\alpha, \beta) \in X(T^2)$ to

$(\alpha + \pi, \beta) \sim (\pi - \alpha, 2\pi - \beta)$

$(e^{i\pi} = -1)$



$\phi^* = \iota$ on $i^*(X(K))$

Thm (KM'04) For a nontrivial $K \subset S^3$, a rational $r = p/q \in [0,2]$ $S^3_r(K)$ is $SU(2)$ nonabelian

$\boxed{\text{Proof Sketch}}$: $i^* X(K)$ intersect the arc A_1 from $(0,0)$ to $(0,\pi)$ only at $(0,0)$

because $i^* X(K) \cap A_1$ corresponds to rep of S^3

By above prop, $i^* X(K)$ intersects the arc A_2 from $(\pi, 0)$ to (π,π) only at $(\pi, 0)$

If $S^3_r(K)$ is $SU(2)$ abelian, for $r \in [0,2]$, then $i^* X(K) \cap L_r = \phi$

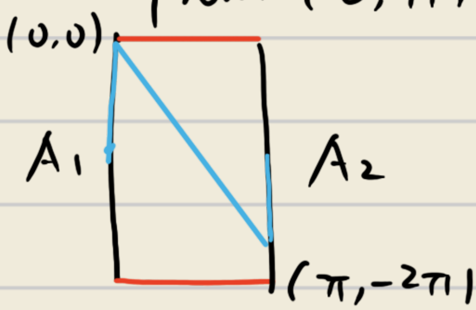
L_r is the arc of slope $-r$ going through $(0,0)$

In summary

$i^* X(K) \cap (A_1 \cup A_2 \cup L_r) = \phi$

But for $r \in [0,2]$

$A_1 \cup A_2 \cup L_r$ contains a path from $(0,\pi)$ to (π,π)



which contradicts with the ingredient from instanton Floer homology □

Since $S^3_r(\bar{K}) = -S^3_{-r}(K)$
and $\pi_1(-Y) = \pi_1(Y)$
for orientation reversal
we also know $S^3_{-r}(K)$ is
SU(2) nonabelian for $r \in [0,2]$
Later, we only consider positive slopes

Also, note that

$$S^3_{6-\frac{1}{n}}(\bar{3}_1) = -S^3_{-6+\frac{1}{n}}(3_1)$$

is SU(2)-abelian,
In particular, $S^3_5(\bar{3}_1) = -L(5,1)$

KM asked in their paper that
if $S^3_3(K)$, $S^3_4(K)$ are SU(2)-nonabelian
We collect related results

<u>Def</u> A slope r for a knot K is
called <u>SU(2)-nonabelian</u> if
$S^3_r(K)$ is SU(2)-nonabelian

<u>Thm</u> Let x be a prime number
and e be a natural number
A slope r is SU(2)-nonabelian
in the following cases

- $r \in [0,2]$ KM'04
- $r = 4$, $\frac{x^e}{2} \in (2,3)$ Baldwin-Sivek'23

- $r = 3, \frac{2^e}{q} \in (3,7)$,
  and some slopes of from $\frac{x^e}{q} \in (3,5)$
  Baldwin-Li-Sivek-Ye '24
- $r = \frac{x^e}{q} \in (2,5)$ Farber-Reinoso-Wang '24

- $r = 5, \frac{11}{2}$ when $K \neq \bar{3}_1$ Li-Ye '25a
- $r = \frac{x^e}{q}, \frac{2x^e}{q} \in (2,6)$ when $K \neq \bar{3}_1$
  Li-Ye '25b
- $r = \frac{x^e}{q}, \frac{2x^e}{q} \in (2,8]$ when $K \neq \bar{3}_1$

  Ghosh-Miller-Eismeier '25

  (depending on unpublished work)

<u>Rem</u> $S^3_{10-\frac{1}{n}}(T_{2,5})$ is $SU(2)$-abelian

<u>Conj</u>: The $SU(2)$-nonabelian slope can be extended to $r = \frac{x^e}{q}, \frac{2x^e}{q} \in (2,12)$ when $K \neq \bar{3}_1, T_{2,5}$

The condition $r = \frac{x^e}{q}, \frac{2x^e}{q}$

is related to some nondegeneracy condition in instanton Floer homology

Note $12 = 2 \times 2 \times 3$ is the first positive integer that is

not of the form $\frac{x^e}{q}, \frac{2x^e}{q}$

If one uses techniques about pillowcase as KM did, possibly one can extend the SU(2) nonabel. slope to [0, 12] when $K \neq 3_1, T$

Lec3 SU(2) nonabelian slopes

Last time, we end up with the following thm based on work of many people

<u>Thm</u> A slope $r = \frac{x^e}{q}, \frac{2x^e}{q} \in (2, 6)$ is SU(2) nonabelian except for $K = 3_1$, $r = 6 - \frac{1}{n}$. where x^e is a prime power

The idea behind all those results is to consider a version of instanton Floer homology called <u>framed instanton homology</u> $I^{\#}(Y)$ defined for <u>ANY</u> closed 3-mfd Y,

which is constructed by KM

Note that the original instanton Floer homology (by Floer) is only for $\mathbb{Z}HS^3$ Y or some "admissible" 3-mfd Y with $b_1 > 0$

Again, we won't define $I^{\#}(Y)$ explicitly, but only describe its relation to SU(2) rep

Rem. The notation $\#$ is called "sharp"

Sometimes $I^{\natural}$ also appears,

where $\natural$ is called "natural".

These are both accidentals (变音符)

in music. There is one more

accidental $\flat$ called "flat",

but no $I^{\flat}$

In some sense, one may also

regard $I^{\#}(Y)$ as $I(Y \# T^3)^{\phi}_{\omega}$

for some $\omega = S^1 \hookrightarrow T^3$ indicates

bundle data and ϕ indicates

the quotient of some involution.

Roughly. $I^{\#}(Y)$ NOT $X(Y)$

$$= H_*(C_{Morse}(R(Y)), d_{Morse} + d_{instanton})$$

where Morse denotes the

Morse chain complex and

instanton means solutions of

ASD (anti-self-dual) equation

$$F_A^+ = 0 \quad \text{on } (Y \# T^3) \times \mathbb{R}$$

In particular, if $R(Y)$ is Morse-Bott

(the nondegeneracy condition related to $\frac{x^e}{q}, \frac{2x^e}{q}$)

then there exists a spectral sequence

$$H_*(R(Y)) \Rightarrow I^{\#}(Y)$$

In particular, $\mathrm{rk}\, R(Y) \geq \mathrm{rk}\, I^{\#}(Y)$

We may also consider $I^{\#}(Y, w)$
for some unoriented 1-submfd $w \subset Y$
indicating the bundle data.

Define $R(Y,w) = \{ \rho : \pi_1(Y \setminus w) \to SU(2) \}$
$\rho(m_w) = -Id$ for any meridian of w

Then we also have similar relation
to $I^{\#}(Y,w)$ if the nondegeneracy holds

$I^{\#}(Y, w=\phi) = I^{\#}(Y)$

Note that $I^{\#}$ doesn't have
$\mathbb{Z}$-homological grading, but only
$\mathbb{Z}/4$-grading, so we usually consider
the direct sum of all gradings and
omit $*$ in $I^{\#}_{*}$

If one needs a $\mathbb{Z}$-homological grading.
One may consider $I^{\#}_{5} = I^{\#}_{1}$ and so on

Indeed, instanton Floer homology is
conceptually related to some stable
homotopy type (spectra), so somehow
the $\mathbb{Z}/4$-grading is similar to
the $\mathbb{Z}/2$ periodicity in K-theory.
or $\mathbb{Z}/8$ for KO-theory.

If $Y = S^3_{p/q}(K)$ is $SU(2)$-abelian
any rep $\rho : \pi_1(Y) \to SU(2)$
factors through $H_1(Y) \cong \mathbb{Z}/p$

Let $\rho_k : \mathbb{Z}/p \to U(1)$ $\quad k = 0, \cdots, p-1$

$$1 \mapsto \begin{bmatrix} e^{\frac{2\pi k i}{p}} & \\ & e^{-\frac{2\pi k i}{p}} \end{bmatrix}$$

The action of $\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$ preserves ρ_0

sends ρ_k to ρ_{p-k} for $k=1,\cdots,p-1$

$Stab(\rho_0) = SU(2)$

$Stab(\rho_k) = U(1)$ for $k=1,\cdots,p-1$ and p odd

$Stab(\rho_{\frac{p}{2}}) = SU(2)$ for p even

$$R(Y) = SU(2)/SU(2) \cup \frac{p-1}{2} SU(2)/U(1)$$

$$= pt \cup \frac{p-1}{2} S^2 \text{ for } p \text{ odd}$$

$R(Y) = 2pts \cup (\frac{p}{2}-1) S^2$ for p even

In both cases, $H_*(R(Y)) \cong \mathbb{Z}^{|p|}$

Meanwhile, we have the following lems

<u>Lem</u> (Baldwin-Sivek'23)

If $p = x^e$ or $2x^e$ for x prime, $e \in \mathbb{N}$.

then $R(S^3_{p/q}(K))$ is Morse-Bott

in the sense of instanton Floer homology

<u>Rem</u> The case $2x^e$ was noticed

by Li-Ye'25, but implicitly in BS'23

<u>Lem</u> (Scaduto'15)

$$\chi(I^\#(Y)) = \begin{cases} |H_1(Y;\mathbb{Z})| & Y \text{ is } \mathbb{Q}HS^3 \\ 0 & \text{otherwise} \end{cases}$$

where χ is the Euler characteristic

defined by $\mathbb{Z}/4$-grading of $I^\#$.

By above lems, if $Y = S^3_{p/q}(K)$ is

$SU(2)$-abelian and $p = x^e$ or $2x^e$, then

$$\begin{aligned} p = rk\, H_*(R(Y)) &\geq rk\, I^\#(Y) \\ &\geq |H_1(Y;\mathbb{Z})| \\ &= p \end{aligned}$$

Since $H_*(R(Y))$ is free,
we have $I^{\#}(Y) \cong \mathbb{Z}^p$
By universal coeff thm,
$I^{\#}(Y;R) \cong R^p$ for any
coefficient ring R.

<u>Def</u> A $\mathbb{Q}HS^3$ Y is called
an <u>instanton L-space</u>
if $\operatorname{rk} I^{\#}(Y) = |H_1(Y;\mathbb{Z})|$
or equivalently
$\dim I^{\#}(Y;\mathbb{C}) = |H_1(Y;\mathbb{Z})|$
A knot $K \subset S^3$ is called
an <u>instanton L-space knot</u>
if for some slope $r \geq 0$,
$S^3_r(K)$ is an instanton L-space

<u>Cor</u>: $(S^3_{p/q}(K)$ $SU(2)$-abelian$) + (p = x^e, 2x^e)$
$\Rightarrow K$ is an instanton L-space knot

<u>Rem</u> Lens spaces are instanton L-spaces
torus knots are instanton L-space knots.

<u>Thm</u> (BS'23) If $S^3_{p/q}(K)$ is an instanton
L-space for some $p/q \geq 0$, then
$S^3_{r'}(K)$ is an instanton L-space
iff $r' \in [2g(K)-1, +\infty)$

<u>Thm</u> The only instanton L-space knot
of genus-1 is $\bar{3}_1 = T_{2,3}$ (BS'23)
of genus-2 is $T_{2,5}$ (BLSY'24, FRW'24)

<u>Cor</u> $r = \frac{x^e}{q}, \frac{2x^e}{q}$ is $SU(2)$ nonabelian for $r \in (2,5)$

Pf: We know K is an instanton L-space knot and $5 > r \geq 2g(K)-1$

$g(K) \leq 2$ $K = T_{2,3}, T_{2,5}$

By Sivek-Zentner '22, for $r \in (2,5)$ and those two knots, $S^3_r(K)$ is $SU(2)$ nonabelian $\square$

For slope $r > 5$, the above strategy needs a classification of instanton L-space knots of genus 3 or higher, which is very hard

However, one can consider $I^\#$ over other coeff. say $\mathbb{F}_2 = \mathbb{Z}/2$ to obtain more information

A conceptual reason to consider $\mathbb{F}_2$ is because for irr rep $\rho \in R(Y)$

$Stab(\rho) = \{\pm Id\}$. its orbit in $R(Y)$ is $SU(2)/\{\pm Id\} \cong SO(3)$ and $H_*(SO(3)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$.

<u>Def</u> A $\mathbb{Q}HS^3$ Y is called an <u>instanton L-space over $\mathbb{F}_2$</u> if $\dim I^\#(Y; \mathbb{F}_2) = |H_1(Y; \mathbb{Z})|$

We can define <u>instanton L-space knot over $\mathbb{F}_2$</u> similarly

Thm (Li-Ye '25) If $S^3_{p/q}(K)$ is an instanton L-space over $\mathbb{F}_2$ for some $p/q \geq 0$, then $S^3_{r}(K)$ is an instanton L-space over $\mathbb{F}_2$ for $[p/q, +\infty)$, and $p/q \geq 2g(K)$

Cor $r = \frac{x^e}{q}, \frac{2x^e}{q}$ is SU(2) nonabelian for $r \in (2,6)$ except

$$K = T_{2,3} \quad r = 6 - \frac{1}{n} \quad n \in \mathbb{N}$$

Pf: Similar to the case of $(2,5)$

Conj One can extend $[2g(K), +\infty)$ to $[4g(K), +\infty)$, so that $(2,6)$ extends to $(2,12)$

Moreover, $S^3_{4g(K)}(K)$ can't be instanton L-space over $\mathbb{F}_2$

Rem Ghosh-Miller-Eismeier '25 extends $[2g(K)-1, +\infty)$ to $\left(4\left\lceil \frac{g(K)}{2} \right\rceil, +\infty\right)$

where $\lceil x \rceil$ is the minimal integer no less than x.

So $(2,6)$ extends to $(2,8]$

But it depends on unpublished work.

We don't state it as a thm.

Finally, we explain why the slopes for instanton L-space is an interval, usually called <u>instanton L-space interval</u>

More generally, we have a dimension formula

Thm (Li–Ye '25b) For any field $\mathbb{K}$, and a knot $K \subset S^3$, there exists two integers $\nu^\#_{\mathbb{K}}(K)$, $r_{\mathbb{K}}(K)$ s.t. $\dim I^\#(S^3_{p/q}(K);\mathbb{K}) =$

$$q \cdot r_{\mathbb{K}}(K) + |p - q\cdot \nu^\#_{\mathbb{K}}(K)|$$

for $q \geq 1$, $\gcd(p,q)=1$, $p/q \neq \nu^\#_{\mathbb{K}}(K)$

If $p/q = \nu^\#_{\mathbb{K}}(K)$, then $\dim I^\#(S^3_{p/q}(K);\mathbb{K})) \in$

$$\{ r_{\mathbb{K}}(K), r_{\mathbb{K}}(K)+2 \} \quad \cdots (*)$$

<u>Rem</u>
- Over $\mathbb{K} = \mathbb{C}$, this is proved by Baldwin–Sivek '21, 24
- Over $\mathbb{K} = \mathbb{F}_2$, independently by Ghosh–Miller–Eismeier
  They also show the case $r_{\mathbb{F}_2}(K)$ in $(*)$ won't happen, and $\nu^\#_{\mathbb{F}_2}$, $r_{\mathbb{F}_2}$ are both divisible by 4
- We also have a similar result for $\dim I^\#(S^3_{p/q}(K), \omega; \mathbb{K})$
  Moreover, when $p/q = \nu^\#_{\mathbb{K}}(K)$, $\dim I^\#(S^3_{p/q}(K), \mu; \mathbb{K})$ $\neq \dim I^\#(S^3_{p/q}(K);\mathbb{K})$

  Note that $\dim I^\#(Y,\omega)$ only depends on $[\omega] \in H_1(Y;\mathbb{F}_2)$

- For $n = \nu^\sharp_{\mathbb{K}}(K) \pm 1$, $\chi(S^3_n(K)) = |n|$

  The dim formula implies $r_{\mathbb{K}}(K) \geq |\nu^\sharp_{\mathbb{K}}(K)|$
- $\nu^\sharp_{\mathbb{K}}(\bar{K}) = -\nu^\sharp_{\mathbb{K}}(K)$, $r_{\mathbb{K}}(\bar{K}) = r_{\mathbb{K}}(K)$

  for the mirror knot

<u>Cor</u> The slopes r with $S^3_r(K)$ is an instanton L-space over $\mathbb{K}$ forms an interval

<u>Pf</u>: Suppose for $r = p/q \neq \nu^\sharp_{\mathbb{K}}(K)$

$S^3_r(K)$ is an instanton L-space over $\mathbb{K}$.

then $q \cdot r_{\mathbb{K}}(K) + |p - q \cdot \nu^\sharp_{\mathbb{K}}(K)| = |p|$

$$
\begin{aligned}
q\, r_{\mathbb{K}}(K) &= |p| - |p - q\,\nu^\sharp_{\mathbb{K}}(K)| \\
&\leq \big|\, |p| - |p - q\,\nu^\sharp_{\mathbb{K}}(K)| \,\big| \\
&\leq |p - (p - q\,\nu^\sharp_{\mathbb{K}}(K))| \\
&= |q\,\nu^\sharp_{\mathbb{K}}(K)| \\
&\leq q\, r_{\mathbb{K}}(K)
\end{aligned}
$$

So all equalities hold.

In particular, $|\nu^\sharp_{\mathbb{K}}(K)| = r_{\mathbb{K}}(K)$

By taking mirror knot if necessary, we assume $\nu^\sharp_{\mathbb{K}}(K) \geq 0$.

then $\dim I^\sharp(S^3_{p'/q'}(K); \mathbb{K})$

$$
= \begin{cases} p' & \text{if } p'/q' > \nu^\sharp_{\mathbb{K}}(K) \\ 2q'\nu^\sharp_{\mathbb{K}}(K) - p' & \text{if } p'/q' < \nu^\sharp_{\mathbb{K}}(K) \end{cases}
$$

The case $p/q = \nu^\sharp_{\mathbb{K}}(K)$ is similar $\square$

Finally, we provide a conceptual idea of the dimension formula via the pillowcase.

Even through $I^{\#}(Y)$ is related to $R(Y)$ instead of $X(Y)$, we can represent $R(Y)$ by $X(Y \# T^3)^{\phi}_{w=S^1}$

i.e. $X(T^3)^{\phi}_{w=S^1}$ contains one pt with stabilizer $\{\pm Id\}$

Then similar to the case of T^2

$X(Y \# T^3)^{\phi}_{w=S^1}$ is a fiber space over $X(Y) \times_{X(S^2)} X(T^3)^{\phi}_{w=S^1}$.

with fiber $\mathrm{Stab}(\rho_1) \backslash \mathrm{Stab}(\rho_1|_{i_* \pi_1(S^2)}) / \mathrm{Stab}(\rho_2)$

$$= \mathrm{Stab}(\rho_1) \backslash SU(2) / \{\pm Id\}$$

$$= SO(3) / \mathrm{Stab}(\rho_1)$$

Which is the same as $R(Y)$

For $Y = S^3_r(K)$, we can also consider either $(S^3 \backslash\backslash K) \# T^3 \cup_{T^2} S^1 \times D^2$

or $S^3 \backslash\backslash K \cup_{T^2} (S^1 \times D^2) \# T^3$

We consider the first decomposition

Then $X(Y \# T^3)^{\phi}_{w}$

$$= X(S^3 \backslash\backslash K \# T^3)^{\phi}_{w} \times_{X(T^2)} X(S^1 \times D^2)$$
$$= R(S^3 \backslash\backslash K) \times_{X(T^2)} X(S^1 \times D^2)$$

where the second inequality is from the fact that $\mathrm{Stab}(\rho_1|_{i_* \pi_1(T^2)}) / \mathrm{Stab}(\rho_2) = pt$ (both $U(1)$ or $SU(2)$)

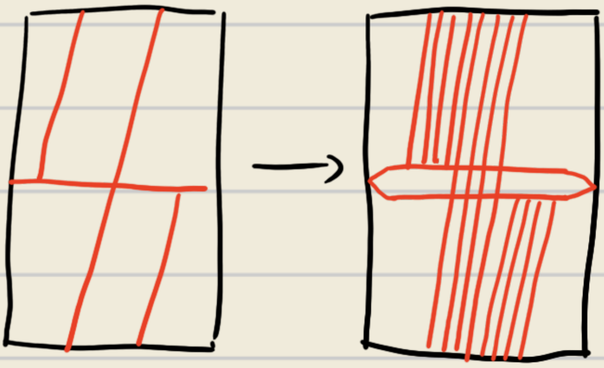
Note that the orbit of a rep ρ is

$$\begin{array}{ll} pt & \text{if } \rho \text{ is central} \\ S^2 & \text{if } \rho \text{ is noncentral red} \\ SO(3) & \text{if } \rho \text{ is irr} \end{array}$$

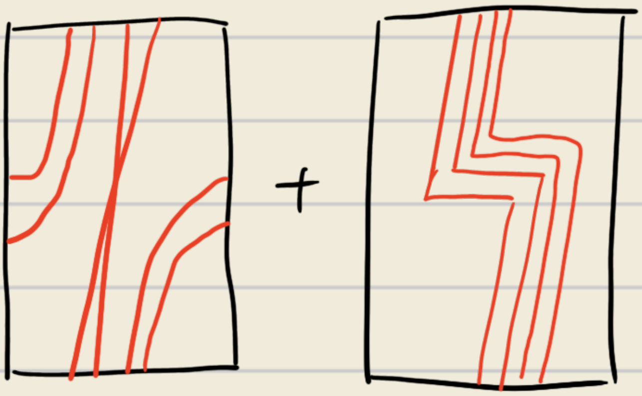
The Morse complexes of S^2 and $SO(3)$ can have 2 pts and 4 pts ($SO(3) \cong \mathbb{RP}^3$)

Conceptually, we define $i^* X^\#(K)$ by replacing pts in $i^* X(K)$ with 1, 2, 4 pts if they are central, noncentral red, irr, resp. and smooth the intersection pts

Ex.



$i^* X(3,1)$



$i^* X^\#(3,1)$

Then we can choose the chain complex

$$CI^\#(S^3_r(K)) = \mathbb{Z}\langle \text{pts in } i^* X^\#(K) \cap L_r \rangle$$

where L_r is an arc of slope $-r$ going through $(0,0)$

<u>Conj</u> (Some version of Atiyah-Floer conj)

There exists an immersed compact curve $\gamma_{IK}(K)$ in $X(T^2) = P$ s.t.

$$\dim I^{\#}(S^3_r(K)) = |\gamma_{IK}(K) \cap L_r|$$

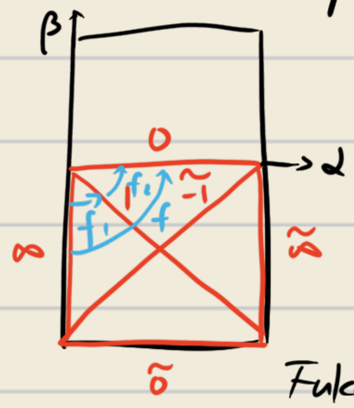
where $|\cdot|$ is the geometric intersection number, i.e. cancelling all bigons)(

More generally, γ_{IK} is defined for any 3-mfd M with $\partial M = T^2$, s.t.

$$I^{\#}(M_1 \cup_\phi M_2) = |\gamma_{IK}(M_1) \cap \phi_*(\gamma_{IK}(M_2))|$$

where $\phi : \partial M_1 \to \partial M_2$ is the gluing map and $\phi_* : P_{M_1} \to P_{M_2}$ is the induced map on the pillowcase.

The proof of the dim formula is motivated by the following correspondence

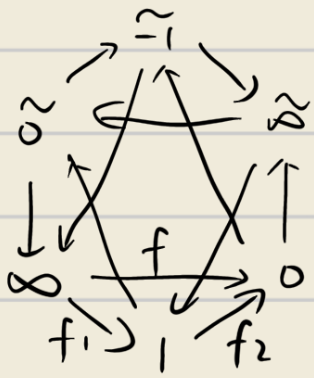


Let $\tilde{L}_r$ be the arc $\{p\alpha + q\beta = \pi\}$

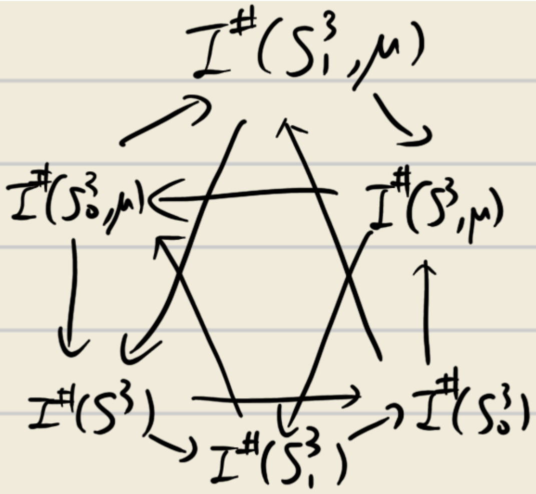
We use r to denote L_r

$\tilde{r}$ to denote $\tilde{L}_r$

Conceptually, in some version of Fukaya category of pillowcase, we have the following octahedral diagram where all maps correspond to angles between arcs



What we really have is an octahedral diagram for $I^{\#}$ where all maps are cobordism maps and all commutative diagrams hold up to sign



$L_r \to S^3_r = S^3_r(K)$

$\tilde{L}_r \to (S^3_1, \mu)$

We may replace $(\infty, 0, 1, -1)$ by other slopes by changing the basis of T^2 and obtain a lot of octahedral diagrams

If we omit the difference of L_r and $\tilde{L}_r$ and also arrows in exact triangles, then we get the Farey tessellation of the hyperbolic upper plane $\mathbb{H}^2$ (or the hyperbolic 2-disk if we add ∞