MATH 230A Assignment 2

Problem 1: Let $S^{n-1} = \{x \in \mathbb{R}^n \Big| \, |x| = 1\}$ and $\mathbb{RP}^{n-1} = S^{n-1}/(-x \sim x)$. Let

$$E_1 = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n \Big| \, |x| = 1 \text{ and } x \cdot v = 0\}, \quad \pi_1 : E_1 \to S^{n-1}$$

and

$$E_2 = \{(x, v) \in \mathbb{R}^n \times \mathbb{R}^n | \, |x| = 1 \text{ and } x \text{ is parallel to } v\}/((-x, v) \sim (x, v)), \quad \pi_2 : E_2 \to \mathbb{RP}^{n-1},$$

where π_1 and π_2 are projections on the first coordinates. Prove the following.

1. E_1 and E_2 are vector bundles. You may check by either the local product structure, or the bundle transition functions. (If you use the bundle transition functions, you may notice that E_1 is the tangent bundle of S^{n-1}. The bundle E_2 is called the **tautological bundle** over $\mathbb{RP}^2$.)
2. S^1 is diffeomorphic to $\mathbb{RP}^1$.
3. The bundle E_1 over S^1 is orientable, but the bundle E_2 over $\mathbb{RP}^1$ is not orientable.

Problem 2: Let M be a smooth manifold. The space of smooth functions on M is denoted by $C^\infty(M; \mathbb{R})$, which is an algebra (you can add or multiply any two elements). A **derivation** $\mathcal{L}$ of this algebra is a map from $C^\infty(M; \mathbb{R})$ to itself such that

1. $\mathcal{L}(f + g) = \mathcal{L}(f) + \mathcal{L}(g)$;
2. $\mathcal{L}(r) = 0$ for any constant function r;
3. $\mathcal{L}(f \cdot g) = \mathcal{L}(f) \cdot g + f \cdot \mathcal{L}(g)$ (Leibniz rule).

A **vector field** $s : M \to TM$ is a section of the tangent bundle TM. Show that any vector field defines a derivation as follows.

1. Let $U \subset M$ be a chart and let $\phi_U : U \to \mathbb{R}^n$ and $\phi_{U*} : TM|_U \to U \times \mathbb{R}^n$ be the diffeomorphisms. For a smooth function $f : M \to \mathbb{R}$, define $f_U = f \circ \phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$. Suppose
$$\phi_{U*} \circ s|_U = (\mathrm{Id}_U, v_1, \ldots, v_n),$$
where $v_k : U \to \mathbb{R}$. Define
$$\mathcal{L}_s(f)|_U = \sum_{k=1}^n \frac{\partial f_U}{\partial x^k} \cdot v_k : U \to \mathbb{R}.$$
If $p \in U$ is also in another chart V, show that
$$\mathcal{L}_s(f)|_U(p) = \mathcal{L}_s(f)|_V(p),$$
i.e., the construction of $\mathcal{L}_s(f)|_U$ is independent of the choice of U.
2. Use partitions of unity (see problem 5 in assignment 1) to construct the function $\mathcal{L}_s(f) : M \to \mathbb{R}$ so that locally it is defined as above.

(*The converse is also true: any derivation on M is given by a vector field.)

Problem 3: Suppose $V = \mathbb{R}^n$. Compute the dimensions of $V^{\otimes k}$, $\mathrm{Sym}^k V$, and $\bigwedge^k V$.

Problem 4: Let τ_1, τ_2, τ_3 denote the Pauli matrices (see page 63 of Cliff's book). Let M be a smooth manifold and let $S^2 \subset \mathbb{R}^3$ as defined in Problem 1. Let $f : M \to S^2$ be a smooth map and we write it as

$$(f_1, f_2, f_3) : M \to \mathbb{R}^3 \text{ with } f_1^2 + f_2^2 + f_3^3 = 1.$$

Define the map $F : M \to \mathbb{M}(2, \mathbb{C})$ that sends $p \in M$ to $\sum_{k=1}^{3} f_k(p) \cdot \tau_k$, where $\mathbb{M}(2, \mathbb{C})$ is the space of 2×2 complex matrices.

1. Prove that $F(p)^2 = -\mathrm{Id}$ for any $p \in M$.
2. Let $E \subset M \times \mathbb{C}^2$ be the set
$$\{(p, v) \Big| \, F(p)v = iv, \; i = \sqrt{-1} \in \mathbb{C}\}.$$
Prove that E is a complex vector bundle over M with fiber $\mathbb{C}$.
3. Let $E_0 \to S^2$ be the complex vector bundle obtain by setting $M = S^2$ and $f(x) = x$. For any general manifold M and any map f, prove that E is isomorphic to the pull-back bundle $f^* E_0$.

Problem 5: Read Sections 5.4-5.7 in Cliff's book about Lie group. You don't need to write down anything for this problem.