

## MATH 230A ASSIGNMENT 5

**Problem 1:** Suppose  $\nabla : C^\infty(M; E) \rightarrow C^\infty(M; E \otimes T^*M)$  is a covariant derivative. It induces a covariant derivative (which we still denote by  $\nabla$ )  $C^\infty(M; E \otimes E) \rightarrow C^\infty(M; E \otimes E \otimes T^*M)$  by Problem 4 of assignment 4. It is called **compatible** with a metric  $g \in C^\infty(M; E \otimes E)$  if

$$\nabla g = 0.$$

Suppose  $\nabla$  and  $\nabla'$  are two covariant derivative compatible with  $g$ . Suppose  $\nabla' = \nabla + \alpha$ , where  $\alpha \in C^\infty(M; \text{End}(E) \otimes T^*M)$  is a matrix-valued 1-form (the standard notation is  $\mathbf{a}$  but I use  $\alpha$  in the class for writing simplicity). Prove

$$\alpha^T \cdot g = -g \cdot \alpha$$

**Problem 2:** For simplicity, let  $M = \mathbb{R}^n$ . Let  $dx^1, \dots, dx^n$  be the basis of sections of  $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ . Let

$$\mathbb{A} : \bigotimes_{i=1}^k T^*M \rightarrow \bigwedge^k T^*M$$

be the anti-symmetrization map that sends

$$dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

to

$$dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

and extends linearly over  $C^\infty(M; \mathbb{R})$ . This map also extend to a map

$$\mathbb{A} : \bigwedge^k T^*M \otimes T^*M \rightarrow \bigwedge^{k+1} T^*M.$$

Let  $\nabla : C^\infty(M; T^*M) \rightarrow \nabla : C^\infty(M; T^*M \otimes T^*M)$  be a covariant derivative and define the induced covariant derivative on

$$\nabla : C^\infty(M; \bigwedge^k T^*M) \rightarrow C^\infty(M; \bigwedge^k T^*M \otimes T^*M)$$

inductively by

$$\nabla(\omega \wedge \eta) = (\nabla\omega \wedge \eta) + (-1)^k \omega \wedge (\nabla\eta)$$

for any  $k$ -form  $\omega$  and  $l$ -form  $\eta$  and then extending linearly. Prove the following.

1. For any  $k$ -form  $\omega$  and  $l$ -form  $\eta$ ,

$$\mathbb{A}(\nabla(\omega \wedge \eta)) = \mathbb{A}(\nabla\omega \wedge \eta) + (-1)^k \omega \wedge \mathbb{A}(\nabla\eta).$$

2. For any  $k$ -form  $\omega$ , define the **torsion tensor**  $T_\nabla\omega = \mathbb{A}(\nabla\omega) - d\omega$ . We say  $\nabla$  is **torsion-free** if  $T_\nabla\omega = 0$  for any 1-form  $\omega$ . If  $\nabla$  is torsion-free, prove  $T_\nabla\omega = 0$  for any  $k$ -form  $\omega$ .

**Problem 3:** Following Section 16.1.1 in Cliff's book, suppose  $S^n \subset \mathbb{R}^{n+1}$  is the sphere of radius  $\rho > 0$ , prove the curvature 2-form at  $y = 0$  specified by

$$R_{abcd} = \rho^{-2} \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}.$$

For simplicity, you may suppose  $n = 2$ . You have to clarify the sketch in the section from the definitions (of  $\Gamma_{bc}^a, R_{ab}$ , etc) and add details in the proof.

**Problem 4:** Following Section 16.1.3 in Cliff's book, suppose  $M \subset \mathbb{R} \times \mathbb{R}^n$  is the manifold defined by  $|x|^2 - t^2 = -\rho^2$  and  $t > 0$ , where  $t$  is the coordinate of  $\mathbb{R}$ ,  $x$  is the coordinate of  $\mathbb{R}^n$ , and  $\rho > 0$  is fixed. Prove the curvature 2-form at  $(t = \rho, x = 0)$  is specified by

$$R_{abcd} = -\rho^{-2}\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}.$$

For simplicity, you may suppose  $n = 2$ . You have to clarify the sketch in the section from the definitions (of  $\Gamma_{bc}^a, R_{ab}$ , etc) and add details in the proof.

**Problem 5:** Read Section 16 in Cliff's book about more examples of Riemannian curvature tensor. You don't need to write down anything for this problem.