

## Class 8 Metrics on vector bundles (Chap 7)

Basic idea: For a real/complex vector bundle  $E \rightarrow M$ .

The fiber at each  $p \in M$  is  $\mathbb{R}^n$  or  $\mathbb{C}^n$  with vector space structures.

A metric on  $E$  is a positive definite symmetric bilinear form for each fiber

Def 1 A metric  $g$  on a real vector bundle  $E \rightarrow M$  is a section  $M \rightarrow E^* \otimes E^*$ , s.t. for each  $p \in M$

$$1) g|_p(v, v) > 0 \quad \forall v \neq 0 \in E_p \quad (\text{positive definite})$$

$$2) g|_p(v, w) = g|_p(w, v) \quad (\text{symmetric})$$

$$v, w \in E$$

Rem (By 2), we can choose a basis of  $E_p \cong \mathbb{R}^n$

s.t. the matrix representing  $g|_p$  is diagonal.

$$\text{i.e. } g|_p(v, w) = w^T \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} v = \sum_{i=1}^n \lambda_i v_i w_i$$

$$\text{By 1). } \lambda_i > 0$$

Def 2 An Hermitian metric  $g$  on a complex vector bundle  $E \rightarrow M$  is a section  $M \rightarrow \overline{E}^* \otimes E^*$  s.t. for  $p \in M$

$$1) g(v, v) \in (0, \infty) \quad \forall v \neq 0 \in E_p$$

$$2) g(v, cw) = c g(v, w)$$

$$3) g(c v, w) = \bar{c} g(v, w)$$

$$4) \overline{g(v, w)} = g(w, v)$$

Rem Similarly, at each pt, we can choose a basis s.t.

$$g|_p \text{ is diagonal } g|_p(v, w) = \sum_{i=1}^n \lambda_i \overline{v_i} w_i \quad \lambda_i \in (0, \infty)$$

Prop: Any real / cpx v.b. admits an (Hermitian) metric

We use partition of unity to construct it.

Pf. For a chart  $U \subset M$ .  $\lambda_U: \pi^{-1}(U) \rightarrow \mathbb{R}^n (\mathbb{C}^n)$

Given a map  $g_U: U \rightarrow U \times (\mathbb{R}^* \otimes \mathbb{R}^*)$  satisfying  
the positive definite, symmetric condition on each fiber.

We obtain a metric on  $U$  by

$$g_U(\lambda_U v, \lambda_U w) \quad v, w \in E|_p$$

We can choose a locally finite atlas  $\mathcal{U}$  of  $M$   
and construct the metric on each  $U \in \mathcal{U}$

To put them together, we notice that if  $g_1, g_2$  are  
two metrics, then  $s g_1 + t g_2$  is still a metric  
for  $s, t \geq 0$ .  $s, t$  are not both 0.

(Just check Condition 1) & 2))

Lem. Suppose  $\mathcal{U}$  is a locally finite atlas of  $M$ . Then there exists a set of smooth functions  $\{\chi_\alpha : M \rightarrow [0, \infty)\}_{\alpha \in \mathcal{U}}$  called a partition of unity, satisfying the following conditions

- 1)  $\chi_\alpha$  has support in  $U_\alpha$  (i.e.  $\{p \in M | \chi_\alpha(p) \neq 0\} \subset U_\alpha$ )
- 2)  $\sum_\alpha \chi_\alpha(p) = 1$  for any  $p \in M$ .

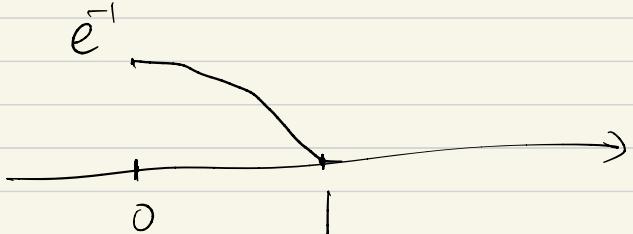
Note that the locally finite condition + 1)

implies at each  $p$ , only finitely many  $\chi_\alpha$  is nonzero.

Pf: (See Cliff's book Appendix 1.2)

First we construct a function  $\chi : [0, \infty) \rightarrow \mathbb{R}$

$$\chi(t) = \begin{cases} e^{\frac{1}{t-1}} & t < 1 \\ 0 & t \geq 1 \end{cases}$$



This function is smooth.

$$\lim_{t \rightarrow 1^-} \frac{d(e^{\frac{1}{t-1}})}{dt} = \lim_{t \rightarrow 1^-} e^{\frac{1}{t-1}} \cdot \left(-\frac{1}{(t-1)^2}\right) = 0$$

Similar for higher derivatives

Then for  $\phi_{U_\alpha} : U_\alpha \rightarrow \mathbb{R}^n$ . we construct  $\phi_\alpha : M \rightarrow \mathbb{R}$

$$\phi_\alpha(p) = \begin{cases} \chi\left(\frac{|\phi_{U_\alpha}(p)|}{r_{U_\alpha}}\right) & p \in U_\alpha \\ 0 & p \notin U_\alpha \end{cases}$$

We can choose  $r_{U_\alpha}$  s.t.  $\{\phi_{U_\alpha}^{-1}(B(r_{U_\alpha}))\}_\alpha$  covers  $M$ .

Then  $\sum_\alpha g_\alpha$  is nowhere zero

We define  $\chi_\alpha = (\sum_\alpha g_\alpha)^{-1} g_\alpha$  □

Using  $\chi_\alpha$ , we can define  $g = \sum_\alpha \chi_\alpha g_{U_\alpha}$  as a metric on  $M$  □

Recall a v.b. can be constructed by bundle transition

functions  $\{g_{UV} : U \cap V \rightarrow GL(n, \mathbb{R})\}$  with some conditions

If we have a metric, then we can choose

$Img_{UV} \subset O(n) \subset GL(n, \mathbb{R})$  as follows

First, locally we have  $\varphi_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$

and a basis of sections  $\{s_i : U \rightarrow \pi^{-1}(U)\}$

Since we have a metric on each fiber, we can use

the Gram-Schmidt procedure to obtain another basis of sections  $\{s'_i\}$  which are orthogonal with respect to

the metric at each fiber. Then we define a new diffeo

$\varphi'_U : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  by setting

$$(\varphi'_U)^{-1}(p, (v_1, \dots, v_n)) = \sum_{i=1}^n v_i s'_i(p)$$

Then the bundle transition function  $g_{VV} = \varphi_V^T \circ (\varphi_U^T)^{-1}$

Sends orthogonal basis to orthogonal basis

so  $g_{VV}: U \cap V \rightarrow O(n)$

If furthermore a real v.b.  $E$  is orientable, i.e.

$\wedge^n E$  is isomorphic to  $M \times \mathbb{R}$ , then we have

$\det g_{VV}(p) > 0$ . so  $g_{VV}: U \cap V \rightarrow SO(n)$

(The groups  $GL(n, \mathbb{R})$ ,  $O(n)$ ,  $SO(n)$  for  $g_{VV}$

are called structure groups of the bundles)

The existence of metric  $\iff$  str gp can be reduced to  $O(n)$

- - - Metric + orientation  $\iff$  str. gp - - -  $SO(n)$

- - - Hermitian metric for complex v.b.  $\iff$   $U(n) \subset GL(n, \mathbb{C})$   
(almost complex str)  $\subset GL(2n, \mathbb{R})$

Hermitian metric + orientation  $\iff$   $SU(n)$

- - - Symplectic str.  $\iff$  Some str gps.

- More facts:
- 1) a metric induces an isomorphism  $\bar{E} \xrightarrow{\cong} E^*$
  - 2) an Hermitian metric induces a  $\mathbb{C}$  anti-linear isomorphism  $\bar{E} \xrightarrow{\cong} E^*$
  - 3) a metric on  $\bar{E}$  induces a metric on  $E^*, E^{\otimes k}$   
 $\Lambda^k \bar{E}, \text{Sym}^k \bar{E}$  metrics on  $\bar{E}_1, \bar{E}_2$  induce  
 a metric on  $\bar{E}_1 \otimes \bar{E}_2 \subset \text{Hom}(\bar{E}_1, \bar{E}_2), \bar{E}_1 \otimes \bar{E}_2$
  - 4) For a smooth map  $f: M \rightarrow N$  and a pull-back  
 $f^* \bar{E} \rightarrow M$  of  $\bar{E} \rightarrow N$ , the metric on  $\bar{E}$   
 induces a metric on  $f^* \bar{E}$ .

Def. A metric on  $TU$  is called a Riemannian metric  
 also called a metric of  $M$ .

Rem. Metrics on the submfd's can be induced by the  
 metric of the ambient space.

Any  $M$  can be embedded into  $\mathbb{R}^N$ ; the metric  
 of  $\mathbb{R}^N$  also induces a metric on  $M$

## Class 9. Riemannian metric.

A metric on a (real) v.b. associates a positive definite symmetric bilinear form on each fiber. i.e.  
a section  $M \rightarrow E^* \otimes E^*$  satisfies some conditions.

### Basic properties

1) metrics always exist.

2)  $g_1, g_2$  are metrics, then  $sg_1 + tg_2$  are also metrics

$$s, t \geq 0 \quad s^2 + t^2 \neq 0$$

This implies the space of metrics is contractible because any metrics can be connected linearly by  $tg_1 + ((-t)g_2)$

3) The existence of metric reduces the str gp of the bundle transition functions from  $GL(n, \mathbb{R})$  to  $O(n)$  (Hermitian metric reduces  $GL(n, \mathbb{C})$  to  $U(n)$ )

Def. A metric on  $TM$  is called a Riemannian metric.

also called a metric of  $M$ .

Recall that we write coordinates of

$$TR^n \cong \mathbb{R}^n \times \mathbb{R}^n \text{ as } x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$$

$$\text{and } T^*R^n \cong \mathbb{R}^n \times \mathbb{R}^n \text{ as } x_1, \dots, x_n, dx_1, \dots, dx_n$$

Locally, a vector field (a section  $M \rightarrow TM$ )

is written as  $\sum_{k=1}^n v_k \frac{\partial}{\partial x^k}$  for  $v_k: U \rightarrow \mathbb{R}$

(Here we omit the function  $\varphi_U: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ )

a 1-form (a section  $M \rightarrow T^*M$ )

is written as  $\sum_{k=1}^n v_k dx^k$

Hence a metric (a section of  $T^*M \otimes T^*M$ )

can be written as  $\sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j$

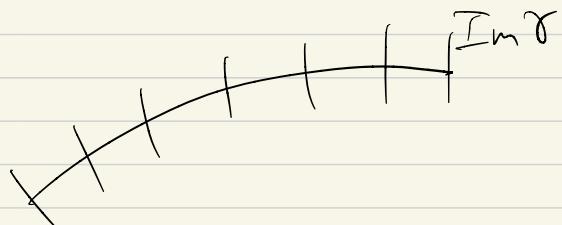
with  $\{g_{ij}(p)\}$  a sym, pos. definite metric

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$$

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Intuition of length of a path in  $\mathbb{R}^n$

Let  $\gamma: I = [0,1] \rightarrow \mathbb{R}^n$



Step 1. Break I into N segments

i.e. consider  $t_i = \frac{i}{N}$  and  $\gamma(t_i)$

Approximate  $\gamma|_{[t_i, t_{i+1}]}$  by  $|\gamma(t_{i+1}) - \gamma(t_i)|$

$$\Delta t = \frac{1}{N}$$

Step 2. Use Taylor's thm to write

$$\frac{|\gamma(t_{i+1}) - \gamma(t_i)|}{\Delta t} = \left| \frac{d}{dt} \gamma \right|_{t=t_i} + O(\Delta t)$$

$$\sum_{i=0}^{N-1} |\gamma(t_{i+1}) - \gamma(t_i)| = \left( \sum_{t=t_i} \left| \frac{d}{dt} \gamma \right| \right) \Delta t + O(\Delta t)$$

Step 3 Take the limit  $N \rightarrow \infty, \Delta t \rightarrow 0$

we get the length of  $\gamma$

$$l_\gamma := \int_I \left| \frac{d}{dt} \gamma \right| dt \quad (\text{This is a definition})$$

Def. Given a Riemannian metric  $g$  on  $M$

and a path  $\gamma: I \rightarrow M$  we define the length

of  $\gamma$  as  $l_\gamma := \int_I \left( g \left( \frac{d}{dt} \gamma, \frac{d}{dt} \gamma \right) \right)^{\frac{1}{2}} dt$

Rem.  $\frac{d}{dt} \gamma$  means the pushforward of  
(tangent map)

$$\frac{\partial}{\partial t} \in T_t I \quad \text{to} \quad \gamma_* \frac{\partial}{\partial t} \in T_{\gamma(t)} M$$

Explicitly  $U \subset I$   $V \subset M$

$$\phi_U: U \rightarrow \mathbb{R} \quad \phi_V: V \rightarrow \mathbb{R}^n$$

$$\phi_{U \times T^*I}|_U \rightarrow U \times \mathbb{R} \quad \phi_{V \times T^*M}|_V \rightarrow V \times \mathbb{R}^n$$

$$\phi_V \circ \gamma \circ \phi_U^{-1}: \mathbb{R} \rightarrow \mathbb{R}^n$$

$$\gamma \frac{\partial}{\partial t} = \phi_{V \times}^{-1} \circ (\phi_V \circ \gamma \circ \phi_U^{-1})_x \circ \phi_{U \times} \frac{\partial}{\partial t}$$

Since the notation  $\frac{\partial}{\partial t}$  already means  $\phi_{U \times}(\frac{\partial}{\partial t})$ ,  
and  $\phi_U = \text{Id}$ ,

we have

$$\begin{aligned} \frac{d}{dt} \gamma \Big|_p &= \phi_{V \times}^{-1} \circ (\phi_V \circ \gamma)_x \frac{\partial}{\partial t} \Big|_p \\ &= \phi_{V \times}^{-1}(p, \frac{d(x_1 \circ \phi_V \circ \gamma)}{dt}, \dots, \frac{d(x_n \circ \phi_V \circ \gamma)}{dt}) \end{aligned}$$

$x_i: \mathbb{R}^n \rightarrow \mathbb{R}$  coordinate map

Def The distance between  $p, q \in M$  is defined by

$d(p, q) := \inf_{\gamma} L\gamma$   $\gamma$  is a path with  $\gamma(0) = p$   
 $\gamma(1) = q$

Prop: 1)  $d(p, p) = 0$   $d(p, q) \neq 0$  for  $p \neq q$

2)  $d(p, q) = d(q, p)$

3)  $d(p, q) \leq d(p, r) + d(r, q) \quad \forall r \in M$

Rem: 1). if  $p \neq q$ , we have a nbhd of  $p$  that is disjoint from  $q$  (Haursdorff), the path must intersects the boundary of the nbhd and in the nbhd the length is nonzero

Thm (Geodesic thm)

1) There is a smooth path  $\gamma$  from  $p$  to  $q$  with  $l_\gamma = d(p, q)$ , such path is called geodesic (or its image)

2) Any geodesic is an embedded, 1-dim submfld that can be reparameterized s.t.  $g(\tau, \frac{\partial}{\partial t}, \tau, \frac{\partial}{\partial t}) = 1$

This is called a constant speed geodesic

(Repara means we get a new path by composition

$I \xrightarrow{\gamma} I \xrightarrow{\varphi} M$ )

3) A constant speed geodesic is characterized as follows

Let  $U \subset M$ ,  $\phi_U : U \rightarrow \mathbb{R}^n$   $g|_U = \phi_U^*(\sum g_{ij} dx^i \otimes dx^j)$

We write  $\gamma_U : \gamma(U) \cap I \rightarrow \mathbb{R}^n$

$$t \mapsto \phi_U \circ \gamma(t)$$

$$= (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

$$\text{Then } (\ast) \quad \frac{d^2}{dt^2} \gamma_U^i + \sum_{j,k=1}^n \Gamma_{jk}^i \frac{d}{dt} \gamma_U^j \frac{d}{dt} \gamma_U^k = 0$$

for any  $i$  where

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$  is the inverse matrix of  $\{g_{ij}\}$

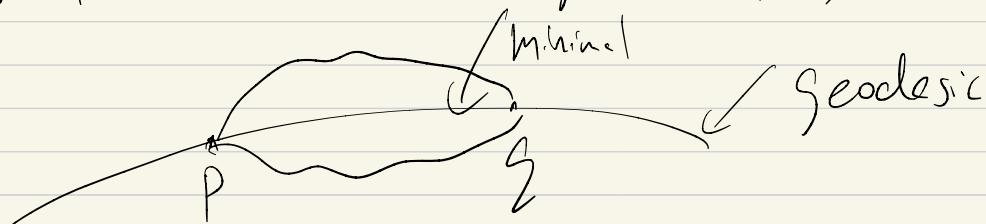
$\Gamma_{jk}^i$  is called Christoffel symbols

4) Any path satisfying  $(\ast)$  is locally length minimizing.

This means  $\exists C_0 < 1$  s.t. when  $p, q$  on the curves

with  $d(p, q) \leq C_0$ . then  $\exists$  a segment of path

connecting  $p$  and  $q$  with length  $d(p, q)$



## Class 10 Geodesic (Chap 8)

Recall a path in  $M$  is a smooth map

$\gamma: I = [0, 1] \rightarrow M$ , the length of  $\gamma$  is

$$l_\gamma = \int_I \left| \frac{d\gamma}{dt} \right| dt = \int_I g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t})^{\frac{1}{2}} dt$$

where  $\frac{\partial}{\partial t}$  is the second coordinate of  $TI \cong I \times \mathbb{R}$

and  $\gamma_*: TI \rightarrow TM$  is the tangent map

(push forward) The distance  $d(p, q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} l_\gamma$ .

Thm (Geodesic thm)  $M$  compact manifold

1)  $\exists \gamma$  achieves  $d(p, q)$ . called geodesic

2) A geodesic is embedded, and can be reparameterized

Ex. so that  $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

3) Let  $U \subset M$ ,  $\phi_U: U \rightarrow \mathbb{R}^n$   $g|_U = \phi_U^*(\sum g_{ij} dx^i \otimes dx^j)$

We write  $\gamma_U: \gamma(U) \cap I \rightarrow \mathbb{R}^n$

$$t \longmapsto \phi_U \circ \gamma(t)$$

$$= (\gamma_U^1(t), \gamma_U^2(t), \dots, \gamma_U^n(t))$$

Then 
$$(*) \quad \frac{d^2}{dt^2} \gamma_U^i + \sum_{j,k=1}^n \Gamma_{jk}^i \frac{d}{dt} \gamma_U^j \frac{d}{dt} \gamma_U^k = 0$$

for any  $i$  where geodesic equation

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$\{g^{ij}\}$  is the inverse matrix of  $\{g_{ij}\}$

$\Gamma^i_{jk}$  is called Christoffel symbols

4) Any path satisfying (\*) is locally length minimizing.

Ex

1) Geodesic in  $\mathbb{R}^n$  with respect to the standard metric:

$$\partial_i g = 0 \quad (*) \Rightarrow \frac{d^2 \sigma}{dt^2} = 0 \Rightarrow \sigma(t) = at + b$$

i.e. geodesics in  $\mathbb{R}^n$  are straight lines

2) Geodesic in  $S^2$  with the induced metric from  $\mathbb{R}^3$

the short segment of the great circle.

(see Cliff's book §4.2 for computations)

If  $p, q$  are two pole pts. then  
geodesics are not unique

Pf of Thm.

Step 1.  $\mathbb{R}^n$  with the standard metric  $g$

Consider the balls centered at the origin and their boundary spheres. Then  $g = dr \otimes dr + r^2 g_S \rightarrow \mathbb{R}^n$

$g_S$  is the induced metric on  $S^{n-1}$  with radius 1.

Let  $\gamma: I \rightarrow \mathbb{R}^n$  be a path in a ball with radius  $R$ .

$\gamma(0) = o \in \mathbb{R}^n$   $\gamma(1)$  is on the boundary

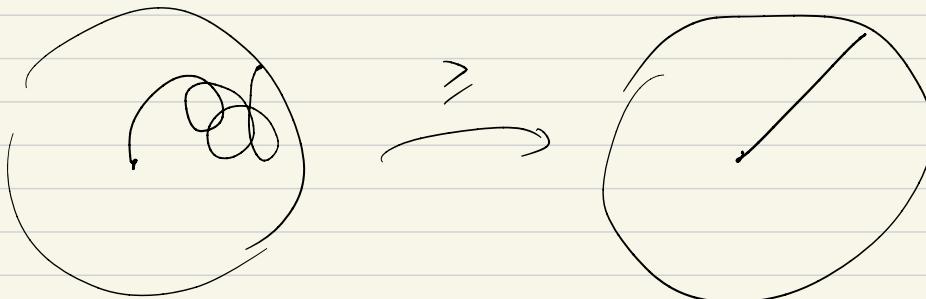
Write  $\gamma(t)$  as  $(r(t), \theta(t))$   $r: I \rightarrow \mathbb{R}_{\geq 0}$   $\theta: I \rightarrow S^{n-1}$

$$g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = |\frac{\partial r}{\partial t}|^2 + r^2 g_S(\theta_* \frac{\partial}{\partial t}, \theta_* \frac{\partial}{\partial t})$$

$$\geq |\frac{\partial r}{\partial t}|^2 \quad \text{the equation holds iff } \theta \text{ is constant}$$

$$L_\gamma \geq \int_I \left| \frac{\partial r}{\partial t} \right| dt \geq \int_I \frac{\partial r}{\partial t} dt = R$$

the equation holds when  $\gamma$  is a straight line.



Moreover, we can reparameterize  $\gamma$  s.t.  $g(\gamma_* \frac{\partial}{\partial t}, \gamma_* \frac{\partial}{\partial t}) = 1$

Step 2.

If  $M$  is compact, then  $\exists R > 0$  s.t.

Lemma (Special coordinates) For each  $p \in M$ ,  $\exists$  chart  $\phi_U: U \rightarrow \mathbb{R}^n$

s.t.  $\phi_U(p) = 0$  and on a small ball  $B_R$  of  $\mathbb{R}^n$ , the metric  $g$

on  $\phi_U^{-1}(B_R)$  is  $\phi_U^*(dr \otimes dr + r^2 g_r)$   $g_r$  metric on  $S^{n-1}$

Assuming the Lemma, we prove that for a fixed  $p \in M$  and any  $q \in M$  there exists a length minimizing curve from  $p$  to  $q$ .

a)  $d(p, q) \leq \frac{R}{2}$ , such curve exists because  $q$  is in  $B_R$  for  $p$

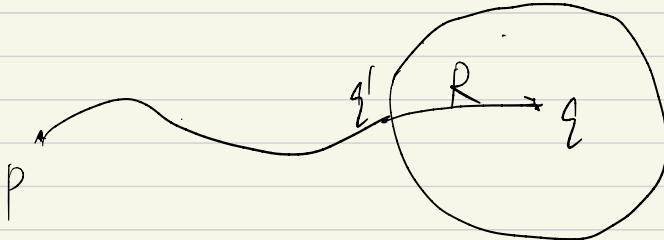
b) Suppose we already show the existence for  $q$  with

$d(p, q) \leq \frac{NR}{2}$ , we prove the existence for  $\frac{NR}{2} < d(p, q) \leq \frac{(N+1)R}{2}$

There is a path  $\gamma_0$  from  $p, q$  s.t.  $l_{\gamma_0} < d(p, q) + \frac{R}{2}$

Suppose  $q' = \text{Im } \gamma_0 \cap \partial B_R(q)$

Then  $d(p, q') \leq l_{\gamma_0} - R < d(p, q) - \frac{R}{2} \leq \frac{NR}{2}$



Let  $q'' \in \partial B_R(q)$  s.t.  $d(p, q'')$  is minimal.

We know  $d(p, q'') \leq d(p, q') < \frac{NR}{2}$

Then  $\exists$  geodesics from  $p$  to  $q'$ ,  $q''$  to  $q$ .

The union of them is the geodesic from  $p$  to  $q$

because it can't be shorten inside or outside the  $B_R(q)$

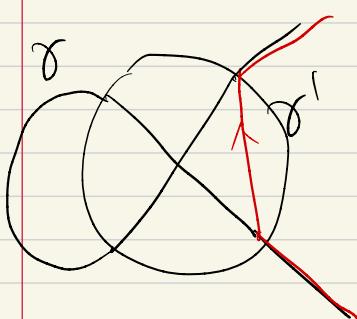
We can reparameterize it so that it has constant speed

Step 3. Nbhd of each pt on the geodesic is a straight line

some  $B_R$ , so  $\gamma$  is an immersion (locally injective)

If  $\text{Im } \gamma$  has self-intersection, then locally it is

two straight lines in  $B_R$ , if we replace  $\gamma$  by  $\gamma'$ ,



then  $l_{\gamma'} < l_{\gamma}$ . which contradicts the minimal length condition. Hence  $\gamma$  is an embedded curve

(locally injective and no self-intersection)

we explain more on Lem 1.

Thm2 (vector field flow) Let  $S: X \rightarrow TX$  be a vector field. For any  $q \in X$ .  $\exists \varepsilon > 0$  and a unique path  $6_q: (-\varepsilon, \varepsilon) \rightarrow X$  s.t.

$$1) 6_q(0) = q \quad 2) 6_q * \frac{\partial}{\partial t} = S(6_q(t))$$

Moreover, if  $U \subset X$  is an open set with compact closure,

then  $(q, t) \mapsto 6_q(t)$  is a smooth map from  $U \times [0, \varepsilon) \rightarrow X$  s.t. for each  $t$ ,  $q \mapsto 6_q(t)$  is a diffeo from  $U$  to an open set of  $X$

Pf. Cliff's book Appendix 8.1  $\square$

We take  $X = TM$  locally  $q = (p, v) \in X$

$$\text{Define } S(p, v) = (v_i, -\Gamma_{jk}^i v_j v_k)$$

Then we obtain a unique path

$$6_q: (-\varepsilon, \varepsilon) \rightarrow TM \quad \text{s.t. } 6_q(0) = q = (p, v)$$

$$6_q * \frac{\partial}{\partial t} = S(6_q(t))$$

This implies a unique path  $\gamma_{(p,v)}: (-\varepsilon, \varepsilon) \rightarrow M$

$$\text{s.t. } \gamma(0) = p \quad \dot{\gamma} * \frac{\partial}{\partial t} \Big|_{t=0} = v$$

$$\text{and } \frac{d\dot{\gamma}^2}{dt^2} + \sum_{j,k} \Gamma_{jk}^i \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

(Here we write  $\gamma = \gamma_U = \phi_U \circ \gamma_{(p,v)}: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ )

## Class 11 Geodesic thm. (Chap 9)

If  $M$  is compact, then  $\exists R > 0$  s.t.

Lem1(Special coordinates) For each  $p \in M$ ,  $\exists$  chart  $\phi_p: U \rightarrow \mathbb{R}^n$  s.t.  $\phi_p(p) = 0$  and on a small ball  $B_R$  of  $\mathbb{R}^n$ , the metric  $g$  on  $\phi_p^{-1}(B_R)$  is  $\phi_p^*(dr \otimes dr + r^2 g_r)$

Last time we use Lem2 to prove 1), 2) in Thm1  
 Today we use the following thm to prove Lem1

Thm2(Vector field thm) Let  $S: X \rightarrow TX$  be a vector field. For any  $q \in X$ .  $\exists \varepsilon > 0$  and a unique path  $\gamma_q: (-\varepsilon, \varepsilon) \rightarrow X$  s.t.

$$1) \gamma_q(0) = q \quad 2) \dot{\gamma}_q \frac{\partial}{\partial t} = S(\gamma_q(t))$$

We take  $X = TM$  locally  $q = (p, v) \in X$

Define  $S(p, v) = (v_i, -\Gamma_{jk}^i v_j v_k)$

$$\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$

$$g_{ij} = \phi_p^* (\sum g_{ij} dx^i \otimes dx^j)$$

$\{g^{ij}\}$  is the inverse matrix of  $\{g_{ij}\}$

$\Gamma_{km}^j$  is called Christoffel symbols

Then we obtain a unique path

$$G_q : (-\varepsilon, \varepsilon) \rightarrow TM \quad \text{s.t. } G_q(0) = q = (p, v)$$

$$G_q * \frac{\partial}{\partial t} = S(G_q(t))$$

This implies a unique path  $\gamma_{(p,v)} : (-\varepsilon, \varepsilon) \rightarrow M$

$$\text{s.t. } \gamma(0) = p \quad \dot{\gamma} * \frac{\partial}{\partial t} \Big|_{t=0} = v$$

$$\text{and } \frac{d^2\gamma^i}{dt^2} + \sum_j \Gamma_{jk}^i \frac{d\gamma^j}{dt} \frac{d\gamma^k}{dt} = 0$$

(Here we write  $\gamma = \gamma_U = \phi_U \circ \gamma_{(p,v)} : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^n$ )

The reason to choose such  $\gamma$  is because the following properties

Prop.

D)  $g(\dot{\gamma} * \frac{\partial}{\partial t}, \dot{\gamma} * \frac{\partial}{\partial t})$  is constant

$$\text{Pf. } \partial_t g(\dot{\gamma} * \frac{\partial}{\partial t}, \dot{\gamma} * \frac{\partial}{\partial t}) = \partial_t g_{ij} \dot{\gamma}^i \dot{\gamma}^j$$

(when an index appear twice, it means we take  $\sum$  over the index)

$$( \text{for simplicity, we write } \dot{\gamma}^i = \frac{d\gamma^i}{dt} \quad \ddot{\gamma}^i = \frac{d^2\gamma^i}{dt^2} )$$

$$\gamma_U = \phi_U \circ \gamma$$

$$\rightarrow = g_{ij} \dot{\gamma}^i \dot{\gamma}^j + g_{ij} \dot{\gamma}^i \ddot{\gamma}^j + \dot{\gamma}^k \partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

Compute  $\partial_t g_{ij} \dot{x}^i \dot{x}^j$

$$= \boxed{g_{ij} \dot{x}^i \dot{x}^j + g_{ij} \dot{x}^i \dot{x}^j} + \dot{x}^k \partial_k g_{ij} \dot{x}^i \dot{x}^j = 0$$

$$\begin{array}{l} \textcircled{1} \quad \dot{x}^i + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0 \quad \textcircled{2} \quad g_{ij} = g_{ji} \quad \mid \quad \textcircled{4} \quad f g^{il} = (g_{il}) \\ \textcircled{3} \quad \Gamma_{jk}^i = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \quad \mid \quad g_{il} g^{il} = \delta_{il} = \sum_0^1 i=l \end{array}$$

$$\textcircled{2} \rightarrow -2g_{ij} \dot{x}^i \dot{x}^j \stackrel{\textcircled{1}}{=} 2g_{ij} \Gamma_{mk}^i \dot{x}^m \dot{x}^k$$

change index

$$\begin{array}{l} m \mapsto j \\ j \mapsto m \end{array} \quad 2g_{im} \Gamma_{jk}^i \dot{x}^j \dot{x}^k \dot{x}^m$$

$$\textcircled{3} \quad g_{im} g^{ikl} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \dot{x}^j \dot{x}^k \dot{x}^m$$

$$= S_{ml} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}) \dot{x}^j \dot{x}^k \dot{x}^m$$

$$= (\partial_j g_{mk} + \partial_k g_{jm} - \partial_m g_{jk}) \dot{x}^j \dot{x}^k \dot{x}^m$$

$$= \partial_k g_{ij} \dot{x}^i \dot{x}^j \dot{x}^k$$

$$2) \quad \dot{x} \cdot \frac{\partial}{\partial t} \Big|_{t=0} = V \Rightarrow g \left( \dot{x} \cdot \frac{\partial}{\partial t}, \dot{x} \cdot \frac{\partial}{\partial t} \right) = |V|^2$$

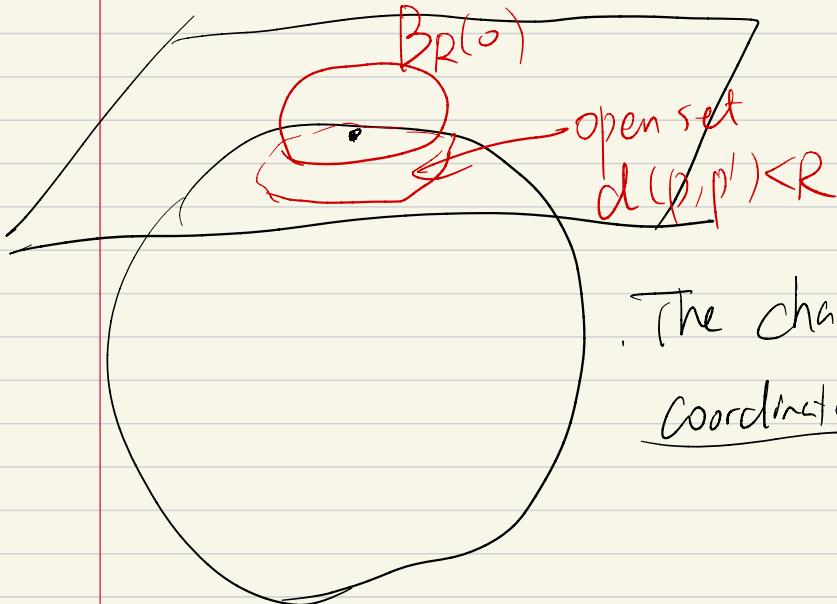
for all  $t$  (constant speed)

$$3) \quad \text{Fix } \mu > 0, \text{ let } \eta(t) = \gamma_{(p,v)}(\mu t). \text{ Then}$$

$\eta(0) = p$   $\dot{\eta} \cdot \frac{\partial}{\partial t} = \mu V$   $\eta$  also satisfy the geodesic equation.  $\eta$  is defined for  $t \in (\mu^{-\varepsilon}, \mu^\varepsilon)$

Then  $\exists R$  s.t. the map  $v \mapsto \gamma_{(p, \frac{2}{\Sigma}v)}(\frac{\varepsilon}{2})$   
 define a diffeo between  $B_R(o) \subset TM|_p \cong \mathbb{R}^n$

and the open set of pts  $p'$  in  $M$  with  
 $d(p, p') < R$ . This map is called the  
exponential map at  $p$   $\exp_p: TM|_p \rightarrow M$



Note that this is an inverse map of a coordinate chart.

The charts are called Gaussian coordinates or normal coordinates

Claim: The pullback metric  $\tilde{g} = \exp^* g$   
 has the form  $dr \otimes dr + r^2 g_r$  ← metric on  $S^{n-1}$

If  $M$  is compact, we can use finitely many Gaussian charts to cover  $M$ , hence we have a minimal value of  $\varepsilon > 0$ . This will imply Lem 1

Pf of Claim: Cliff's book section 9.3

Note that the existence of the solution of the geodesic equation only for  $t \in (-\varepsilon, \varepsilon)$ , but we may extend the solution to a larger interval.

**Def** If all geodesics are defined on  $\mathbb{R}$ ,

the manifold is called geodesically complete.

**Fact** all compact manifolds are geodesically complete

**Ex.** a manifold that is not geodesically complete

$$\mathbb{R}^2 - \{0\}$$

Understand the geodesic equation by Euler-Lagrange equation (variation principle)

Consider a functional:

$$S : \left\{ \text{the space of paths } \gamma \subset \mathbb{R}^n \right\} \rightarrow \mathbb{R}$$

with  $\gamma(0) = p, \gamma(1) = q$

$$\gamma \longmapsto \int_I g \left( \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right)^{\frac{1}{2}} dt$$

For  $s \in (-\varepsilon, \varepsilon)$ , consider the path

$$\gamma_s : I \rightarrow \mathbb{R}^n$$

$$t \mapsto \gamma(t) + s\eta(t) \quad \eta : I \rightarrow \mathbb{R}^n \text{ any path}$$

$\eta(0) = \eta(1) = 0$

$$\frac{dS(\gamma_s)}{ds} \Big|_{s=0} = \int_I \frac{dg\left(\frac{d\gamma_s}{dt}, \frac{d\gamma_s}{dt}\right)^{\frac{1}{2}}}{ds} dt \Big|_{s=0}$$

$$= \int_I \frac{d(g_{ij}(\gamma_s) \dot{\gamma}_s^i \dot{\gamma}_s^j)^{\frac{1}{2}}}{ds} dt \Big|_{s=0} \begin{pmatrix} \frac{d\gamma_s}{ds} = \eta(t) \\ \frac{d\dot{\gamma}_s}{ds} = \dot{\eta}(t) \end{pmatrix}$$

$$= \int_I \frac{1}{2} \cdot \frac{1}{\sqrt{g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j}} \left( \partial_k g_{ij} \cdot \eta^k \dot{\gamma}^i \dot{\gamma}^j + 2 g_{ij} \dot{\eta}^i \dot{\gamma}^j \right) dt$$

Integrating by parts on the second term. The boundary terms vanish because  $\eta(0) = \eta(1) = 0$

$$= \frac{1}{2} \int_I \left( \frac{\partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j}{\sqrt{g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j}} - 2 \frac{d}{dt} \left( \frac{g_{kj} \dot{\gamma}^j}{\sqrt{g_{ij}(\gamma) \dot{\gamma}^i \dot{\gamma}^j}} \right) \right) \eta^k dt$$

To obtain a minimal length path. we need

$$\frac{dS(\gamma_s)}{ds} \Big|_{s=0} = 0 \text{ for any } \eta(t).$$

$$\text{So } \frac{\partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j}{\sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}} - 2 \frac{d}{dt} \left( \frac{g_{kj} \dot{\gamma}^j}{\sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}} \right) \text{ must vanish}$$

$$\text{Let } d\tau = dt \sqrt{g_{ij} \dot{\gamma}^i \dot{\gamma}^j}$$

Then we have

$$\partial_k g_{ij} \dot{\gamma}^i \dot{\gamma}^j - 2 \frac{d}{d\tau} (g_{kj} \dot{\gamma}^j) = 0 \quad (\text{Now } \dot{\gamma}^i = \frac{d\gamma^i}{d\tau})$$

$$\Rightarrow \partial_k g_{ij} \ddot{\gamma}^i \dot{\gamma}^j - 2 \partial_i g_{kj} \dot{\gamma}^i \dot{\gamma}^j - 2 \cancel{g_{kj}} \dot{\gamma}^i \dot{\gamma}^j = 0$$

~~$2 g_{kl} \ddot{\gamma}^l$~~  (replace j by l)

$$\Rightarrow \ddot{\gamma}^l + \frac{1}{2} g^{kl} \left( \underline{2 \partial_i g_{kj}} - \partial_k g_{ij} \right) \dot{\gamma}^i \dot{\gamma}^j = 0$$

$\partial_i \cancel{g_{kj}} + \partial_j g_{ik}$  after times  $\dot{\gamma}^i \dot{\gamma}^j$

$$\Rightarrow \ddot{\gamma}^l + \sum_{ij}^l \dot{\gamma}^i \dot{\gamma}^j = 0$$

$$\sum_{ijk}^l = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$$