Model Predictive Control

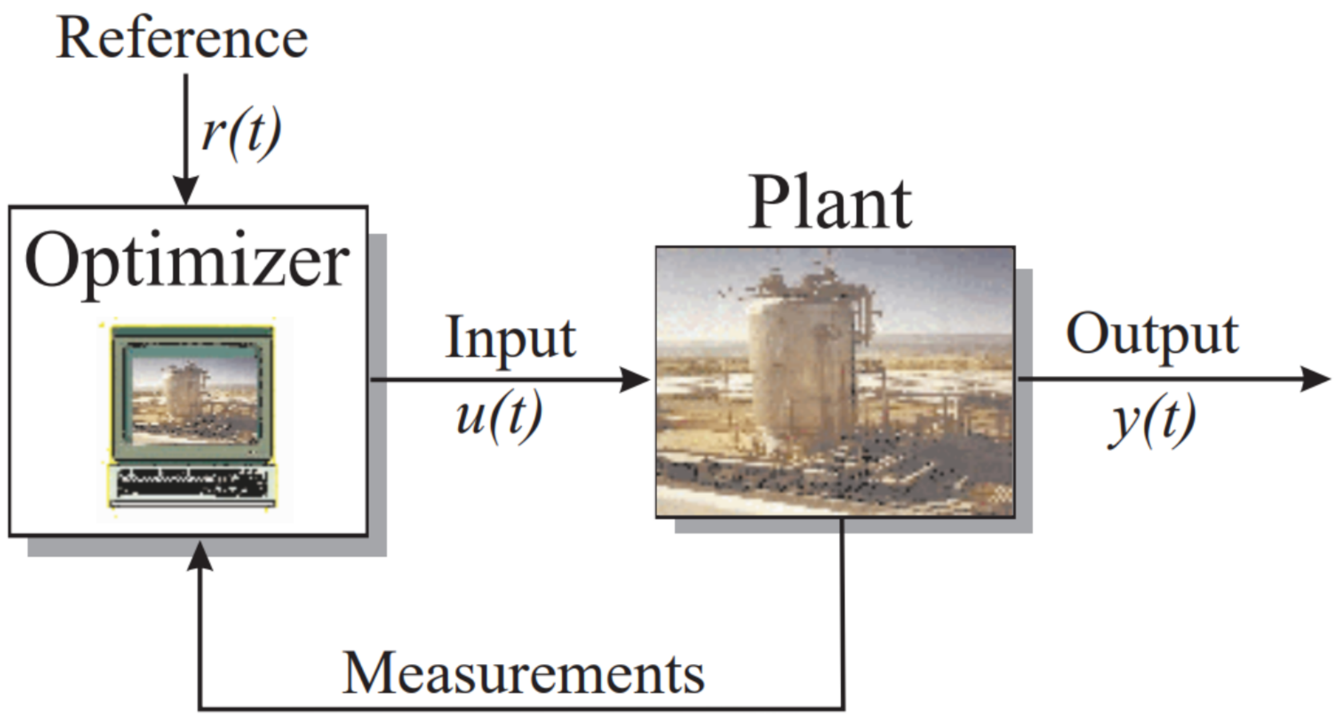
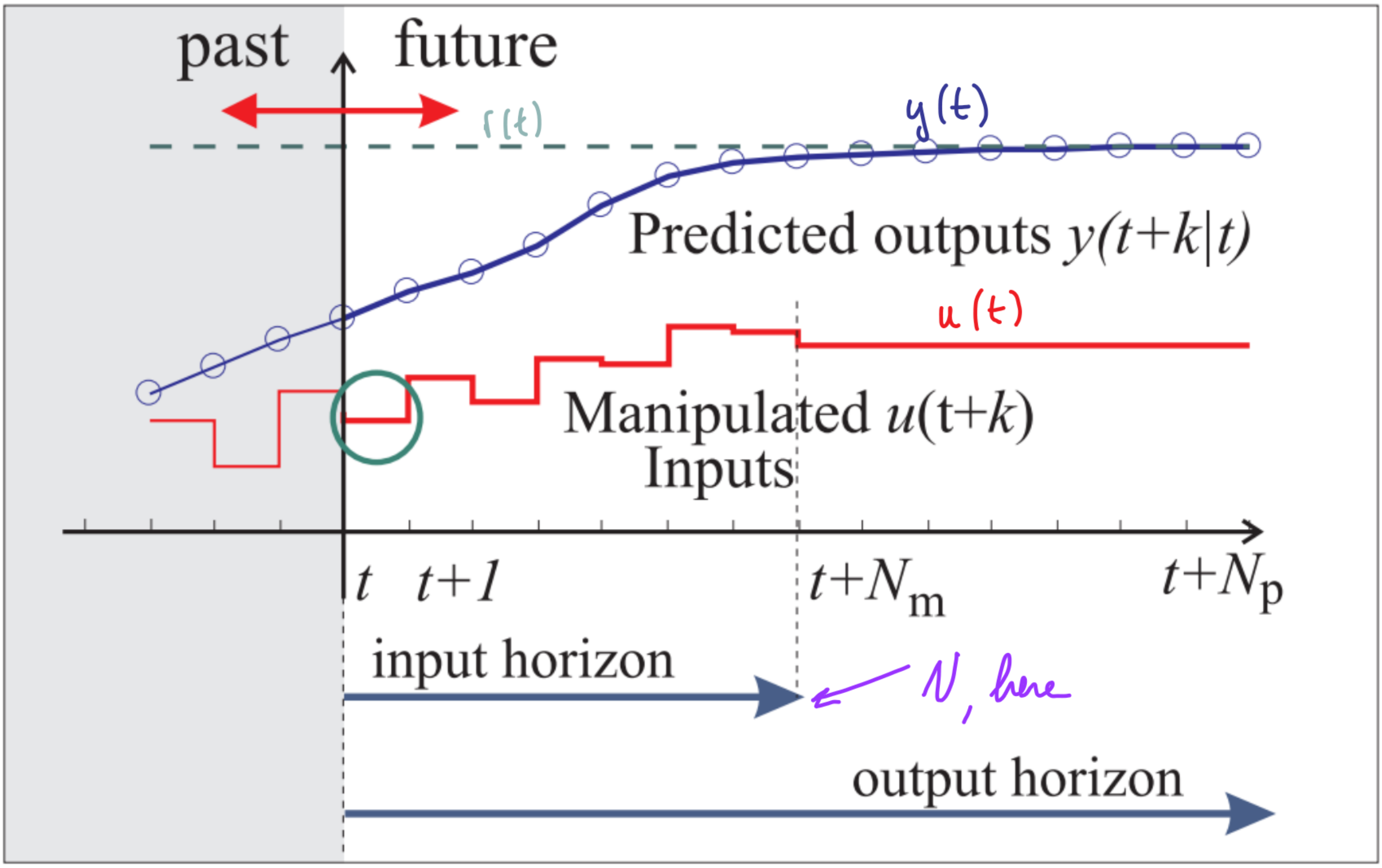


Fig. 1. Basic structure of Model Predictive Control



Typical problem layout:

$$\begin{cases} \underline{x}_{k+1} = \underset{\sim}{A}\, \underline{x}_k + \underset{\sim}{B}\, \underline{u}_k \\ \underline{y}_k = \underset{\sim}{C}\, \underline{x}_k + \underline{d}_k \end{cases}$$

with:

$\underset{\sim}{A} \in \mathbb{R}^{n_x \times n_x}$; $\underset{\sim}{B} \in \mathbb{R}^{n_x \times n_u}$; $\underset{\sim}{C} \in \mathbb{R}^{n_y \times n_x}$

and $\underline{u}_k$ subject to constraints:

$$\mathcal{U}(\underline{u}_{k-1}) := \left\{ \underline{u}_k \in \mathbb{R}^{n_u} \;\middle|\; \begin{array}{l} -\underline{\alpha} \leq \underline{u}_k \leq \underline{\alpha} \\ -\underline{\rho} \leq \underline{u}_k - \underline{u}_{k-1} \leq \underline{\rho} \end{array} \right.$$

with: $\underline{\alpha}, \underline{\rho} \in \mathbb{R}^{n_u}$

⇒ Optimizer solves an online optimization problem, usually a CQP of the form:

CONSTRAINED QUADRATIC PROGRAMMING

$$\min_{\underline{u}_{k|i}} \|\underline{x}_{k|N}\|^2_{\underset{\sim}{P}} + \sum_{i=0}^{N-1} \left(\|x_{k|i}\|^2_{\underset{\sim}{Q}} + \|u_{k|i}\|^2_{\underset{\sim}{R}} \right)$$

SUBJECT TO :

MODEL PREDICTION

$$\underline{x}_{k|i+1} = \underset{\sim}{A}\,\underline{x}_{k|i} + \underset{\sim}{B}\,\underline{u}_{k|i}$$

$$\underline{u}_{k|i} \in \mathcal{U}(\underline{u}_{k|i-1})$$

CONSTRAINTS ON INPUT

$\forall i \in [0, N-1]$

$$\underline{x}_{k|0} = \hat{\underline{x}}_k \; ; \quad u_{k|-1} = u^{*}_{k-1|0}$$

INITIAL STATE

ONLY FIRST INPUT IS APPLIED

evolution of the system is predicted as :

$$\underline{x}_{k|0} = \underline{x}_k$$

$$\underline{x}_{k|1} = \underset{\sim}{A}\,\underline{x}_k + \underset{\sim}{B}\,\underline{u}_{k|0}$$

$$\underline{x}_{k|2} = \underset{\sim}{A}^2\,\underline{x}_k + \underset{\sim}{A}\underset{\sim}{B}\,\underline{u}_{k|0} + \underset{\sim}{B}\,\underline{u}_{k|1}$$

$\vdots$

hence we can re-write in compact notation :

$$\underline{X}_k = \underset{\sim}{\mathcal{M}}\,\underline{x}_k + \underset{\sim}{\mathcal{C}}\,\underline{u}_k \quad -(1)$$

with :

$$\underset{\sim}{\mathcal{M}} = \begin{bmatrix} \underset{\sim}{I} \\ \underset{\sim}{A} \\ \underset{\sim}{A}^2 \\ \vdots \\ \underset{\sim}{A}^N \end{bmatrix} ; \quad \underset{\sim}{\mathcal{C}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \hline \underset{\sim}{B} & & & \\ \underset{\sim}{A}\underset{\sim}{B} & \underset{\sim}{B} & \ddots & \\ \vdots & \vdots & & \\ \underset{\sim}{A}^{N-1}\underset{\sim}{B} & \underset{\sim}{A}^{N-2}\underset{\sim}{B} & \cdots & \underset{\sim}{B} \end{bmatrix}$$

the cost function can then be re-written as:

$$J = \underline{X}_k^T \tilde{Q} \underline{X}_k + \underline{u}_k^T \tilde{R} \underline{u}_k \quad \text{with:}$$

$$\tilde{Q} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & P \end{bmatrix} \quad \text{and} \quad \hat{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$$

substitute (1):

$$J = (M \underline{x}_k + C \underline{u}_k)^T \tilde{Q} (M \underline{x}_k + C \underline{u}_k) + \underline{u}_k^T \tilde{R} \underline{u}_k =$$

$$= (M \underline{x}_k)^T \tilde{Q} (M \underline{x}_k) + (C \underline{u}_k)^T \tilde{Q} (C \underline{u}_k) +$$

$$+ (M \underline{x}_k)^T \tilde{Q} (C \underline{u}_k) + (C \underline{u}_k)^T \tilde{Q} (M \underline{x}_k) + \underline{u}_k^T \hat{R} \underline{u}_k =$$

BOTH SCALARS SO $(\dots)^T = (\dots)$

$$= \underline{x}_k^T M^T \tilde{Q} M \underline{x}_k + \underline{u}_k^T (\tilde{R} + C^T \tilde{Q} C) \underline{u}_k +$$

$$+ 2 \underline{x}_k^T M^T \tilde{Q} C \underline{u}_k \quad \text{which simplifies to:}$$

$$J = \underline{u}_k^T H \underline{u}_k + 2 \underline{x}_k^T F \underline{u}_k + \underline{x}_k^T G \underline{x}_k$$

for DLS-II, output terms are ignored so objective function is re-written as:

$$J = \frac{1}{2} \underline{u}^T \underset{\sim}{H} \underline{u} + \underline{q}(\hat{\underline{x}}_k, \hat{\underline{d}}_k)^T \underline{u}$$

so the CQP is reformulated as:

$$\boxed{\min_{\underline{u}_k} \left\{ \frac{1}{2} \underline{u}^T \underset{\sim}{H} \underline{u} + \underline{q}^T \underline{u} \right\} \quad \text{s.t.} \quad \underline{u} \in \mathcal{S}(\underline{u}_{-1})}$$

with $\underline{u} := (\underline{u}_{k|0}^T, \ldots, \underline{u}_{k|N-1}^T)^T \in \mathbb{R}^{N n_u}$

$$\mathcal{S}(\underline{u}_{-1}) := \mathcal{U}(\underline{u}_{k|-1}) \times \ldots \times \mathcal{U}(\underline{u}_{k|N-2})$$

$$\underset{\sim}{H} = \underset{\sim}{H}^T \in \mathbb{R}^{N n_u \times N n_u} \quad \text{(hessian)}$$

$$\underline{q}(\hat{\underline{x}}_k, \hat{\underline{d}}_k) : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \mapsto \mathbb{R}^{N n_u}$$

(affine function)