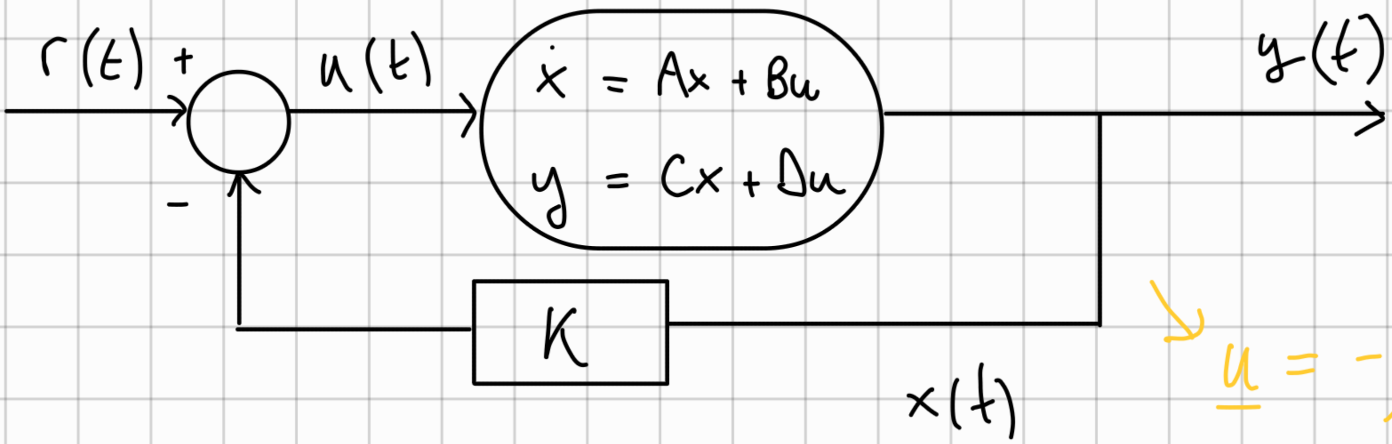
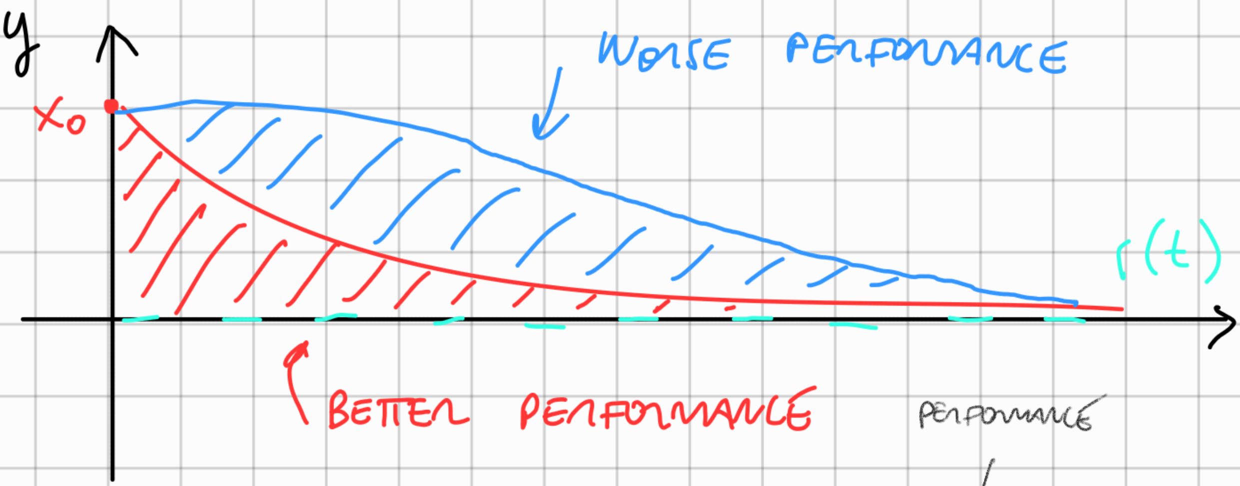
Linear Quadratic Regulator



$\underset{\sim}{u} = -\underset{\sim}{K}\,\underline{x}$

K is chosen through optimisation:



$$\min_{\underline{u}} J(\underline{x},\underline{u}) = \int_0^\infty \Big(\overbrace{\underline{x}^T \underset{\sim}{Q}\, \underline{x}}^{\text{PERFORMANCE}} + \overbrace{\underline{u}^T \underset{\sim}{R}\, \underline{u}}^{\text{ACTUATOR EFFORT}}\Big)\, dt$$

(squaring takes care of any negative values)

↳ THIS CAN BE SOLVED WITH lqr( ) IN MATLAB

$\underset{\sim}{Q}$, $\underset{\sim}{R}$ are positive definite and need to be adjusted what does lqr() do in MATLAB?

Algebraic Riccati Equation

algebraic method that substitutes brute-force algorithmic approach

or learning-based algorithms

(GRADIENT DESCENT)

introduce $\underset{\sim}{P} = \underset{\sim}{P}^T$

$$J = \underline{x}_0^T \underset{\sim}{P} \underline{x}_0 - \underline{x}_0^T \underset{\sim}{P} \underline{x}_0 + \int_0^\infty \left(\underline{x}^T \underset{\sim}{Q} \underline{x} + \underline{u}^T \underset{\sim}{R} \underline{u} \right) dt =$$

$$= \underline{x}_0^T \underset{\sim}{P} \underline{x}_0 + \int_0^\infty \left[\underbrace{\frac{d}{dt}\left(x^T P x \right)} + \underline{x}^T \underset{\sim}{Q} \underline{x} + \underline{u}^T \underset{\sim}{R} \underline{u} \right] dt =$$

STABLE SYSTEM GOES TO ZERO

$$\left[\underline{x}^T \underset{\sim}{P} \underline{x} \right]_0^\infty = \cancel{\underline{x}^T \underset{\sim}{P} \underline{x} \Big|_\infty} - \underline{x}_0^T \underset{\sim}{P} \underline{x}_0$$

$\dot{\underline{x}} = \underset{\sim}{A} \underline{x} + \underset{\sim}{B} \underline{u}$

$$\frac{d}{dt}\left(\underline{x}^T \underset{\sim}{P} \underline{x} \right) = \dot{\underline{x}}^T \underset{\sim}{P} \underline{x} + \underline{x}^T \underset{\sim}{P} \dot{\underline{x}} =$$

$$= \left(\underset{\sim}{A} \underline{x} + \underset{\sim}{B} \underline{u} \right)^T \underset{\sim}{P} \underline{x} + \underline{x}^T \underset{\sim}{P} \left(\underset{\sim}{A} \underline{x} + \underset{\sim}{B} \underline{u} \right)$$

substitute:

$$J = \underline{x}_0^T \tilde{P} \underline{x}_0 + \int_0^\infty \left[(\tilde{A}\underline{x} + \tilde{B}\underline{u})^T \tilde{P} \underline{x} + \underline{x}^T \tilde{P} (\tilde{A}\underline{x} + \tilde{B}\underline{u}) + \underline{x}^T \tilde{Q} \underline{x} + \underline{u}^T \tilde{R} \underline{u} \right] dt =$$

$\downarrow$ RE-ARRANGE

$$= \overbrace{\underline{x}_0^T \tilde{P} \underline{x}_0 + \int_0^\infty \Big[\underline{x}^T (\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{Q}) \underline{x}}^{\text{DOES NOT DEPEND ON } \underline{u}} + \boxed{\underline{u}^T \tilde{R} \underline{u} + \underline{x}^T \tilde{P} \tilde{B} \underline{u} + \underline{u}^T \tilde{B} \tilde{P} \underline{x}} \Big] dt =$$

COMPLETE THE SQUARE: $(u + ?)^2 - ?^2 = \boxed{u^2 + 2u\,?} + \cancel{?^2} - \cancel{?^2}$

So: $\underline{u}^T \tilde{R} \underline{u} + \underline{x}^T \tilde{P} \tilde{B} \underline{u} + \underline{u}^T \tilde{B} \tilde{P} \underline{x} =$

$$= (\underline{u} + \tilde{R}^{-1} \tilde{B}^T \tilde{P} \underline{x})^T \tilde{R} (\underline{u} + \tilde{R}^{-1} \tilde{B}^T \tilde{P} \underline{x}) - \underline{x}^T (\tilde{P} \tilde{B} \tilde{R}^{-1} \tilde{B}^T \tilde{P}) \underline{x}$$

substitute:

0 if $u = -R^{-1} B^T P x$

$$J = \underline{x}_0^T \tilde{P} \underline{x}_0 + \int_0^\infty \left[\underline{x}^T (\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{Q} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}) \underline{x} + (\underline{u} + \tilde{R}^{-1} \tilde{B}^T \tilde{P} \underline{x})^T \tilde{R} (\underline{u} + \tilde{R}^{-1} \tilde{B}^T \tilde{P} \underline{x}) \right] dt$$

MINIMISED BY $\underline{u} = -\tilde{K}\underline{x} = -\tilde{R}^{-1} \tilde{B}^T \tilde{P} \underline{x}$

what is $\tilde{P} = ?$

SET $\underline{x}^T (\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{Q} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}) \underline{x} = 0$

2 SOLUTIONS (1 STABLE)

ALGEBRAIC RICCATI EQUATION

$$\boxed{\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{Q} - \tilde{P}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P} = 0}$$