Fundamentals of Orbital Mechanics

MAE 341
Spaceflight
Princeton University

1 The Two-Body Central Force Problem

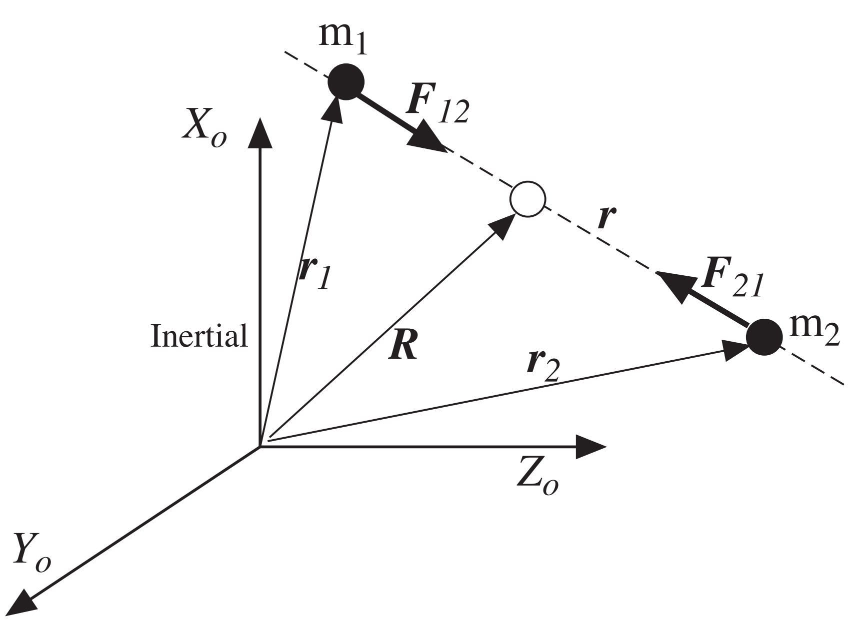

Figure 1: Geometry of the two-body problem.

1.1 Derivation of the Equations of Motion

We start our discussion of orbital mechanics by considering the two-body problem illustrated in Fig. (1) where the reference frame is inertial and the only acting forces are the mutual attractive forces $\boldsymbol{F}_{12}$ and $\boldsymbol{F}_{21}$. Since it takes three coordinates to define the position of each of the two masses, the problem has six degrees of freedom.

We can write Newton's second law as:

$$\boldsymbol{F} = -m_1 \frac{d^2\boldsymbol{r}_1}{dt^2} = m_2 \frac{d^2\boldsymbol{r}_2}{dt^2}, \tag{1}$$

where we have defined $\boldsymbol{F}$ as

$$\boldsymbol{F} \equiv \boldsymbol{F}_{21} = -\boldsymbol{F}_{12} \tag{2}$$

The relative position vector $\boldsymbol{r}$ and the vector $\boldsymbol{R}$ denoting the position of the *center of mass* are defined as

$$\boldsymbol{r} \equiv \boldsymbol{r}_2 - \boldsymbol{r}_1 \tag{3}$$

$$\boldsymbol{R} \equiv \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2}. \tag{4}$$

We also define the *equivalent mass* m as

$$m \equiv \frac{m_1 m_2}{m_1 + m_2} \tag{5}$$

Using the definitions in Eq. (3) and Eq. (4) to eliminate $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from the equations of motion, yields

$$(m_1 + m_2)\ddot{\boldsymbol{R}} = 0 \tag{6}$$

which in turn gives

$$\ddot{\boldsymbol{R}} = 0, \tag{7}$$

so that the **velocity $\dot{\boldsymbol{R}}$ of the center of mass is constant.**

Using the above definition of the equivalent mass m, we can recast the two-body equations of motion into an *equivalent one-body problem*

$$m\ddot{\boldsymbol{r}} = F\hat{\boldsymbol{r}} \tag{8}$$

which describes the motion of a particle of equivalent mass m under the action of an attractive force of magnitude F.

We are now down to three degrees of freedom. We can actually go down to two degrees of freedom without any loss in generality. We take the vector product of the above equation with $\boldsymbol{r}$

$$\boldsymbol{r} \times m\ddot{\boldsymbol{r}} = \boldsymbol{r} \times F\hat{\boldsymbol{r}} = 0 \tag{9}$$

which integrates to

$$\boldsymbol{r} \times m\dot{\boldsymbol{r}} = \text{const} \equiv \boldsymbol{H}, \tag{10}$$

where $\boldsymbol{H}$ is the angular momentum. The constant angular momentum implies that the motion occurs in a plane and we are down to *two* degrees of freedom.

Specializing now for a Newtonian gravitational force with

$$\boldsymbol{F}_g = -\frac{Gm_1m_2}{r^2}\hat{\boldsymbol{r}}, \tag{11}$$

the equation of motion in Eq. (8) becomes,

$$\ddot{\boldsymbol{r}} = -\frac{G(m_1+m_2)}{r^2}\hat{\boldsymbol{r}}. \tag{12}$$

For an artificial satellite and a planet, one mass (that of the planet) dominates.

We summarize the results obtained so far for the two-body problem:

- The center of mass moves with constant velocity.
- The relative motion occurs in a plane.
- The relative motion for any attractive central force is governed by

$$m\ddot{\boldsymbol{r}} = F\hat{\boldsymbol{r}}. \tag{13}$$

- The governing equation for the particular case of a Newtonian gravitational force is

$$\ddot{\boldsymbol{r}} = -\frac{G(m_1+m_2)}{r^2}\hat{\boldsymbol{r}}. \tag{14}$$

1.2 The Differential Equation of the Orbit

If the central force is conservative, the equation of motion in Eq. (8) can be expressed in terms of a potential V,

$$m\ddot{\boldsymbol{r}} = F\hat{\boldsymbol{r}} = -\nabla V. \tag{15}$$

Since the motion is always in a plane, we use a polar representation for the position vector,

$$\boldsymbol{r} = r\cos\theta\hat{\boldsymbol{e}}_1 + r\sin\theta\hat{\boldsymbol{e}}_2 \tag{16}$$

and the vector equation of motion becomes two scalar equations

$$m\ddot{r} - mr\dot{\theta}^2 = F = -\frac{\partial V}{\partial r}, \tag{17}$$

$$mr^2\dot{\theta} = H, \tag{18}$$

where H is the magnitude of the angular momentum defined earlier.

A single second order differential equation called the *differential equation of the orbit* can be found by rewriting the last equation as the operator

$$\frac{d}{dt} = \frac{H}{mr^2}\frac{d}{d\theta}, \tag{19}$$

and using it to eliminate time in Eq. (17), yielding

$$\frac{d^2u}{d\theta^2} + u = -\frac{mF}{H^2u^2} = -\frac{m}{H^2}\frac{\partial V}{\partial u}, \tag{20}$$

where

$$u \equiv \frac{1}{r}. \tag{21}$$

We will use this differential equation in our subsequent study of orbital mechanics.

1.3 Kepler's Laws

Kepler, in the early seventeenth century, consolidated the observations of planetary motion into three simple laws (illustrated in Fig. (2)):

1. The orbit of each planet is an ellipse with the sun at one of the foci.

2. The radius vector from the sun to a planet sweeps out equal areas in equal intervals of time.

3. A planet's orbital period is proportional to its mean distance from the sun raised to the power 3/2.

We will show that these laws follow directly from the solution to Eq. (20) and that the third law is an approximation.

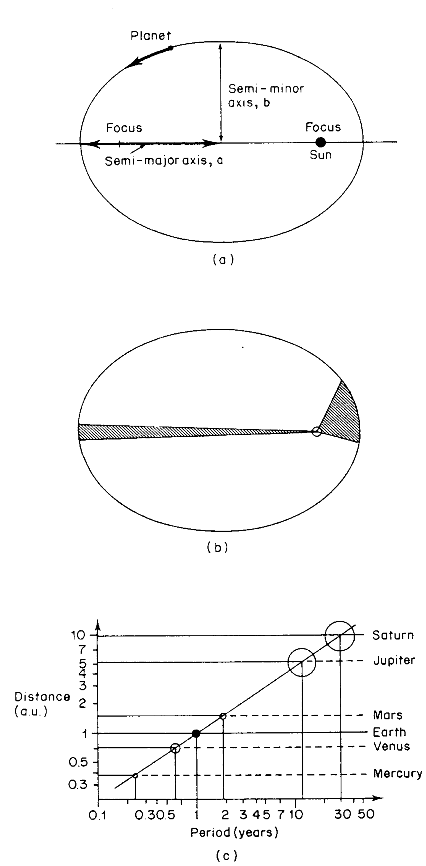

Figure 2: a) Kepler's first law: elliptical orbit; b) Kepler's second law: equal swept areas in equal times; Kepler's third law: period proportional to $a^{3/2}$. From [1].

1.4 Proving Kepler's Second Law

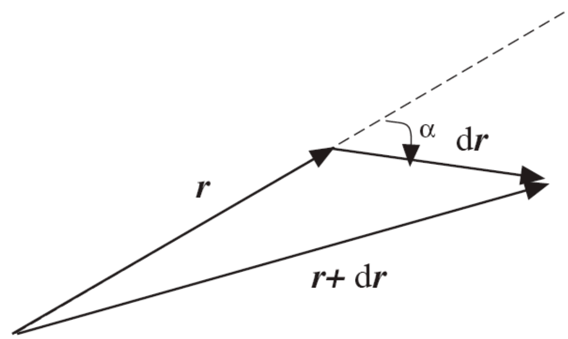

Figure 3: Areal Velocity.

The differential area dA swept by the postion vector in an increment of time dt is called the *areal velocity* and is illustrated in Fig. (3). dA is simply half the base times the height

$$dA = \frac{1}{2} r dr \sin\alpha \tag{22}$$

which can be expressed in terms of a vector product

$$dA = \frac{1}{2}|\boldsymbol{r} \times d\boldsymbol{r}| = \frac{1}{2m}|\boldsymbol{r} \times m\frac{d\boldsymbol{r}}{dt}|dt = \frac{H}{2m}dt \tag{23}$$

yielding

$$\frac{dA}{dt} = \frac{H}{2m} = \text{cont.} \tag{24}$$

This is true for all central force problems. The remaining two Kepler laws will be proven in the next subsection.

1.5 Solution of the Orbit Equation for the Case of Newtonian Gravitation.

We now study the solution of the differential equation of the orbit Eq. (20) for the case of the Newtonian gravitational force $\boldsymbol{F}_g$ (given by Eq. (11)) which we write in terms of the constant k,

$$k \equiv G m_1 m_2, \tag{25}$$

as

$$\boldsymbol{F}_g = -\frac{k}{r^2}\hat{\boldsymbol{r}}. \tag{26}$$

(The gravitational potential is therefore $V_g = -k/r$.)

The orbit equation becomes

$$\frac{d^2 u}{d\theta^2} + u = \frac{mk}{H^2} \tag{27}$$

This is equivalent to the second order nonhomogenous differential equation describing a forced oscillation of an undamped linear harmonic oscillator with a mass and spring constant equal to unity and a constant forcing function equal to mk/H^2. Consequently, we can write the solution as:

$$u = \frac{mk}{H^2}\left[1 + \left(1 + \frac{2EH^2}{mk^2}\right)^{1/2}\cos\theta\right] \tag{28}$$

where we have used the definition of the energy E,

$$E \equiv \frac{1}{2}mv^2 + V \tag{29}$$

which is a constant for the case of a conservative force whose potential is not an explicit function of time.

We now recast the solution (Eq. (28)) in terms of the radius r

$$r = \frac{H^2/mk}{1 + \left(1 + \frac{2EH^2}{mk^2}\right)^{1/2}\cos\theta} \tag{30}$$

in order to compare it to the following equation describing a conic section (defined in Fig. (4)) in polar coordinates

$$r = \frac{pe}{1 + e\cos\theta}. \tag{31}$$

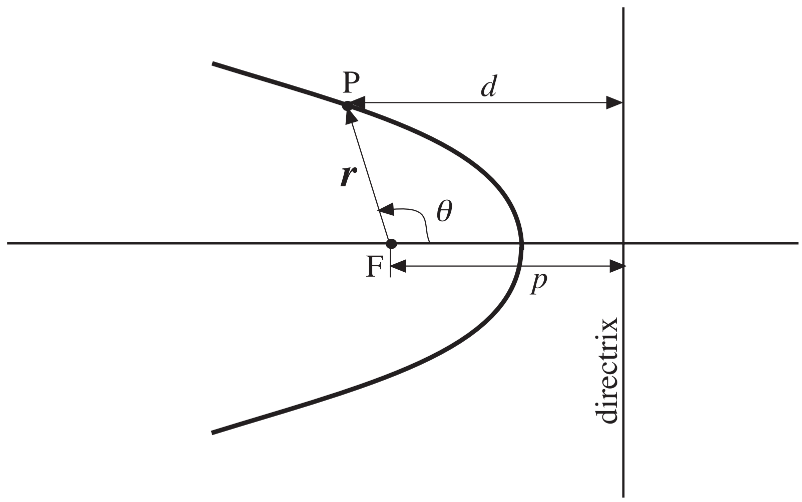

Figure 4: Generic conic section and defining parameters. F is the focus, $\boldsymbol{r}$ is the position vector, θ is called the *true anomaly* and p is the distance from the focus to the directrix. The eccentricity e is the constant ratio r/d.

The comparison yields

$$e = \left(1 + \frac{2EH^2}{mk^2}\right)^{1/2} \tag{32}$$

and

$$pe = \frac{H^2}{mk} \tag{33}$$

where e is the *eccentricity* and the length pe is called the *parameter of the conic*. For $e = 0$, $e < 1$, $e = 1$, and $e > 1$, we get a circle, an ellipse, a parabola and a hyperbola respectively.

It is now convenient to introduce a parameter a

$$a \equiv -\frac{k}{2E} \tag{34}$$

From the definition of E, we get an expression for the orbital velocity

$$v = \sqrt{\frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)} \tag{35}$$

known as the *vis-viva* equation. Substituting Eq. (33) and Eq. (34) in Eq. (32) yields

$$pe = a(1 - e^2) \tag{36}$$

so we can eliminate p in favor of a in Eq. (31) to get

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \tag{37}$$

1.5.1 Circular Orbit

If the energy E is equal to a certain constant E_c given by

$$E_c \equiv -\frac{mk^2}{2H^2} \tag{38}$$

the eccentricity, from Eq. (32), becomes zero and Eq. (37) gives

$$r_c = a_c = \text{constant} \tag{39}$$

and the orbit is a circle. The *vis-viva* equation gives a constant orbital velocity

$$v_c = \sqrt{\frac{k}{ma_c}} \tag{40}$$

and the orbital period is obtained by dividing the circumference by the velocity,

$$\tau_c = 2\pi\sqrt{\frac{ma_c^3}{k}}. \tag{41}$$

In orbital mechanics we sometimes talk of the *mean motion* which is simply the angular frequency obtained by dividing 2π by the orbital period.

1.5.2 Elliptical Orbit

If the total energy is within the following bounds

$$-\frac{mk^2}{2H^2} < E < 0, \tag{42}$$

we get from Eq. (32) an eccentricity

$$0 < e < 1 \tag{43}$$

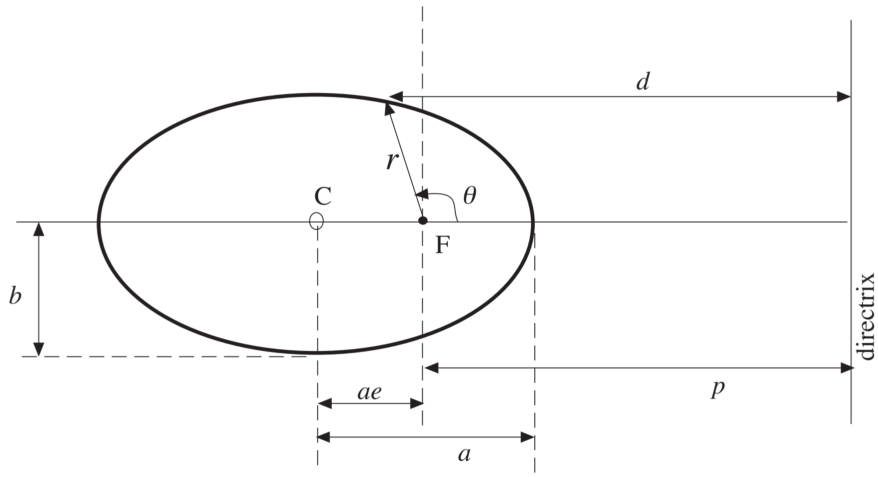

Figure 5: Elliptical orbit. C is the center F is the focus; a is the semimajor axis; and b is the semiminor axis.

and the trajectory is an ellipse as shown in Fig. (5) where a and b are the semimajor and semiminor axes related by

$$b_e = a_e\sqrt{1-e^2}. \tag{44}$$

The point on the orbit closest to the focus is called *perigee* and that farthest from the focus is called *apogee*. The values of the radius r at these two points are

$$r_p = a_e(1-e) \tag{45}$$

and

$$r_a = a_e(1+e). \tag{46}$$

The orbital period can be obtained from the above relations and is the same as that of a circular orbit with the semimajor axis a instead of the radius (exercise left to the reader).

$$\tau_e = 2\pi\sqrt{\frac{ma_e^3}{k}}. \tag{47}$$

Kepler's First and Third Laws. The elliptical solution confirms Kepler's first law. The third law can be seen by taking the ratio of the periods of two planets of masses m_1 and m_2 orbiting around the sun of mass $m_\odot$,

$$\left(\frac{\tau_1}{\tau_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3 \left(\frac{m_2 + m_\odot}{m_1 + m_\odot}\right), \tag{48}$$

Kepler, implicitly made the approximation that both m_1, m_2 are much smaller than $m_\odot$. Under this good approximation we have,

$$\left(\frac{\tau_1}{\tau_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3 \tag{49}$$

and thus all three laws can be derived from the solution to the two-body problem.

1.5.3 Time Dependence: Kepler's Equation

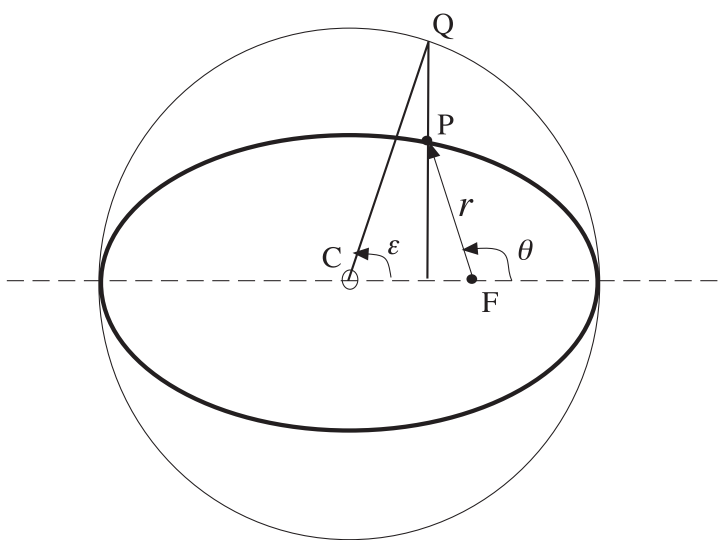

Figure 6: Eccentric anomaly for an elliptical orbit. C is the center F is the focus; P is orbital position; Q is projection on auxiliary circle; θ is true anomaly and ϵ is eccentric anomaly.

We have so far considered the solution to the differential equation of the orbit from which time has been eliminated. It is often useful in orbital tracking and analysis to relate time to position. Kepler defined a parameter called *eccentric anomaly*, ϵ_e using an auxiliary circle that circumscribes the orbit as shown in Fig. (6). As shown in that figure. ϵ_e is the angle sustained by the major axis and a line joining the center to a point Q obtained by projecting the orbital location P vertically on the auxiliary circle. From the geometry, we have the following relations between the eccentric and true anomalies,

$$\cos\theta_e = a_e \frac{(\cos\epsilon_e - e)}{r_e} \tag{50}$$

which we use to eliminate θ from equation Eq. (37) in favor of ϵ_e to yield

$$r_e = a_e(1 - e\cos\epsilon_e). \tag{51}$$

The total energy can be written as a function of the velocity expressed in terms of the time derivatives of the polar coordinates, the time derivative of θ can be eliminated using some of the equations above to get

$$kr_e^2\left(\frac{2}{r_e} - \frac{1}{a}\right) - m\dot{r}_e^2 r_e^2 = ka_e(1 - e^2) \tag{52}$$

Again, using this equation with some of the relations derived above the following differential equation is obtained

$$\dot{\epsilon}_e(1 - e\cos\epsilon_e) = \frac{2\pi}{\tau_e} \tag{53}$$

which integrates to

$$2\pi\frac{t - t_o}{\tau_e} = \epsilon_e - e\sin\epsilon_e, \tag{54}$$

where t_o is the integration constant and represents the time at perigee. The quantity $2\pi(t - t_o)/(\tau_e)$ is called the *mean anomaly* M_e and the above equation is *Kepler's equation*. For a given time (or mean anomaly) Kepler's equation can be used to obtain the eccentric anomaly which is then used in equations (50) and (51) to get the orbital position.

Exercise # 1: a) Using the equations in this document, derive Kepler's equation showing all the major steps. b) Describe how you would go about solving it for ϵ_e (given t) using a computer. c) By either developing your own computer solver or using a canned solver (not an orbital mechanics program), calculate the orbital position (radius and true anomaly) at $t = 4$ hours after perigee passage of a satellite whose orbit has a semimajor axis $a = 5R_e$ and a perigee radius $r_p = 1.5R_e$.

Kepler's equation has many applications in orbital mechanics that can be classified in two classes:

1. For a given elliptical orbit, determine the time at which the satellite will be at a specified position in the orbit.

2. Determine the position of the orbiting body at a specified time.

In Case 1, the solution is easy. ϵ_e corresponding to the specified position is calculated and Kepler's equation yields the corresponding time. An application of this case is the determination of the time at which a satellite passes from sunlight to darkness.

In Case 2, the solution is much more difficult because Kepler's equation is transcendental in the unknown variable ϵ_e. The closed form solution has so far eluded the best mathematicians. An application of this case is performing a rendez-vous with a space station where the location of at a particular time is needed.

1.5.4 Parabolic Orbit

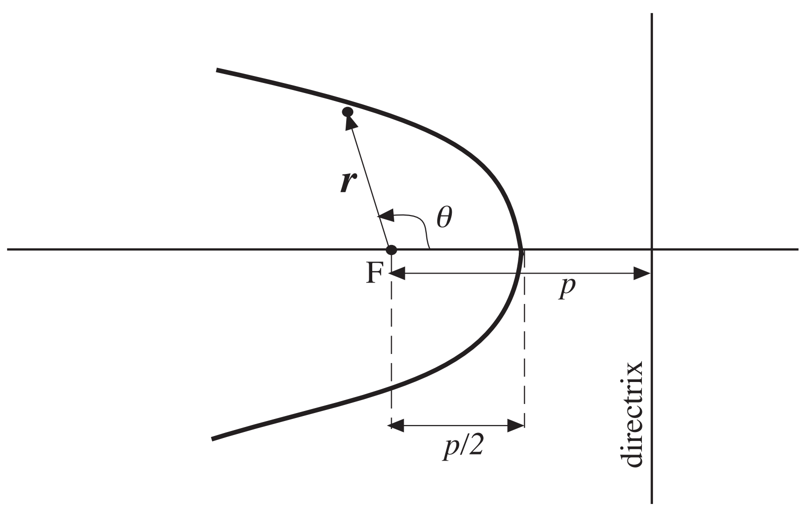

Figure 7: Parabolic orbit geometry.

If the energy E is zero (i.e. $\frac{1}{2}mv^2 = \dfrac{k}{r}$) we get

$$e_p = 1 \tag{55}$$

which describes a parabolic orbit. From Eq. (31) we get

$$r_p = \frac{p}{1 + \cos\theta} \tag{56}$$

and from the definition of a in Eq. (34) we find that a_p is infinite.

From the *vis-viva* equation, with a infinite, we get the following velocity along the orbit,

$$v_p = \sqrt{\frac{2k}{mr}}, \tag{57}$$

from which we see that v_p goes to zero as r_p goes to infinity. The parabola is therefore the minimum energy escape trajectory.

This can be used to determine the velocity needed to escape from a circular orbit of radius $a_c = p_p/2$ where the subscript p stands for "parabola". This velocity is called the *escape velocity*, and by comparing the above equation to Eq. (40) we see that it is simply a factor of $\sqrt{2}$ larger than the circular velocity at that radius.

The relation between time and position for a parabolic orbit is given by the *Barker's equation* which, like Kepler's equation, is a transcendental equation. In terms of true anomaly it is:

$$2\sqrt{\frac{k}{mp^3}}(t - t_o) = \tan\frac{1}{2}\theta + \frac{1}{3}\tan^3\frac{1}{2}\theta. \tag{58}$$

1.5.5 Hyperbolic Orbit.

If the energy is always positive, it follows from Eq. (32) that

$$e > 1 \tag{59}$$

and the resulting trajectory is a hyperbola shown in Fig. (8). We rewrite Eq. (31)

$$r_h = \frac{a_h(1 - e^2)}{1 + e\cos\theta} \tag{60}$$

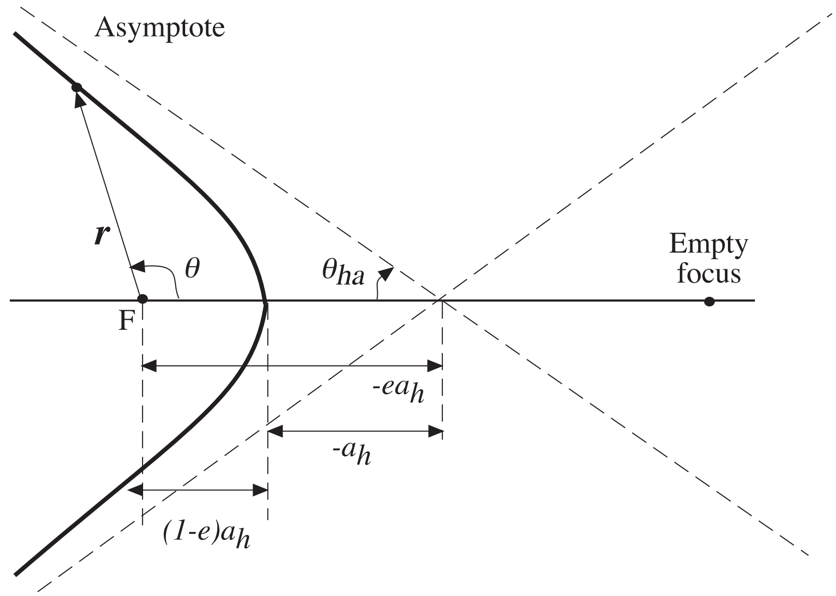

Figure 8: Hyperbolic orbit geometry

with h subscripts and note that it is convenient to treat the semimajor axis a as a negative quantity so that r_h is positive. It follows from the above relations that the distance at perigee is:

$$r_{hp} = a_h(1 - e) \tag{61}$$

and the asymptotes shown in Fig. (8) are at angle θ_{ha} (from the axis of symmetry) obtained by letting r_h go to infinity,

$$\theta_{ha} = \pm \cos^{-1}(-\frac{1}{e}). \tag{62}$$

Unlike the case of the parabolic orbit, the velocity at infinity is finite and is found from the *vis-viva* equation,

$$v_{h\infty} = \sqrt{-\frac{k}{ma_h}} \tag{63}$$

In analog to Kepler's equation, it is possible (but beyond our scope) to obtain a relation between an analog of the eccentric anomaly, ϵ_h, and true anomaly

$$\tan\frac{1}{2}\theta = \sqrt{\frac{e+1}{e-1}}\tanh\frac{1}{2}\epsilon_h \tag{64}$$

and the position can be expressed as

$$r_h = a_h(1 - e\cosh\epsilon_h) \tag{65}$$

and an analog to Kepler's equation can be written as

$$n_h(t - t_o) = e\sinh\epsilon_h - \epsilon_h, \tag{66}$$

where the characteristic frequency n_h is defined as

$$n_h \equiv \sqrt{\frac{k}{m(-a_h)^3}} \tag{67}$$

which can be used to relate the position and true anomalies to the time variable.

2 The Orbital Parameters

Six constants (six components of the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$ at a specified time) are required to completely specify the orbit of a satellite with respect to a central body. Knowing these six components at any initial time or epoch t_o, the position and velocity at any future time may be determined. Unfortunately, $\boldsymbol{r}$ and $\boldsymbol{v}$ do not directly yield much information about the orbit. For example, they do not *explicitly* tell us what type of conic the orbit represents. Another set of six constants, the *orbital elements*, is much more descriptive of the orbit. Most orbital dynamics software packages deal with one form or another of these orbital elements.

Usually, radar tracking of a satellite yields $\boldsymbol{r}$ and $\boldsymbol{v}$, the procedure described in this section allows the calculation of the six orbital elements which can be used by an orbital dynamics software to visualize and plot the orbit and its projection on a world map.

2.1 Definition of Orbital Elements

Since the angular momentum $\boldsymbol{H} = \boldsymbol{r} \times m\boldsymbol{v}$ is constant, the vector $\boldsymbol{H}$ defines the orbital plane that always contains the vectors $\boldsymbol{r}$ and $\boldsymbol{v}$.

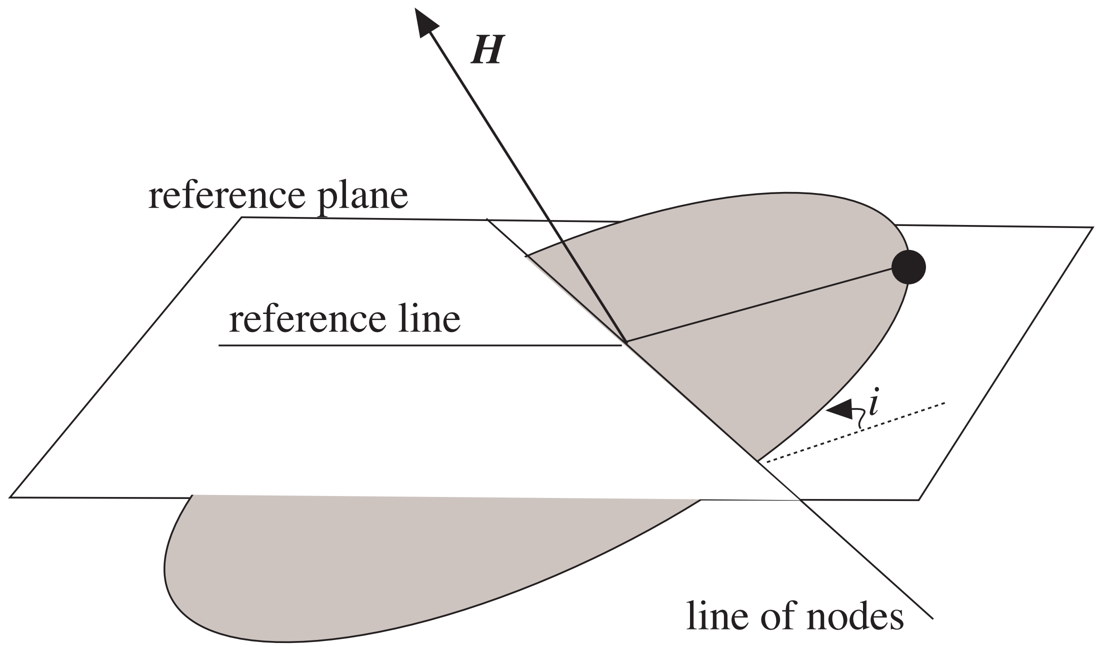

Figure 9: Orbit, reference plane and reference line

The orbital elements are defined with respect to a reference plane as shown in Figures (9) and (10). From these figures we note the following definitions:

1. The *inclination,* i, is the angle between the two planes and is the first orbital element.

2. The *line of nodes,* is the direction of the line representing the intersection between the two planes. This direction is called the *longitude of the ascending node* Ω and is defined with respect to a convenient *reference line* and represents the second orbital element. It is called the *longitude of the ascending node*, where the satellite would rise above (i.e. in a northerly direction) the reference plane.

3. The semi-major axis a and the eccentricity e, were defined previously and are the third and fourth elements.

4. The *argument of periapse*, ω gives the angular position of perigee with

respect to the line of nodes and thus unambiguously defines the orientation of the orbit in the plane.

5. The *true anomaly*, θ defined earlier, give the position of the satellite al;ong the orbit measured at epoch from perigee (periapse).

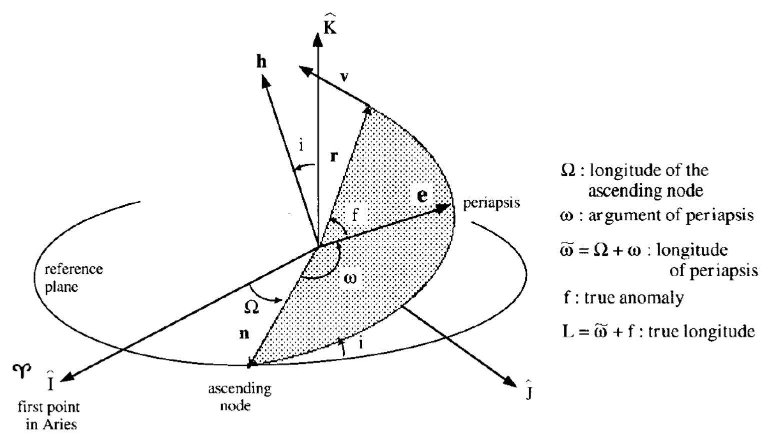

Figure 10: Definition of angular orbital parameters. From [2].

The six orbital parameters are i, Ω, ω, a, e and θ.

2.2 Definition of the Reference Line and Plane

2.2.1 The Reference Line

The customary reference line plane is the direction from the central body to a fixed point on the *celestial sphere* (the background of "fixed" stars.). This point was chosen to be the *first point in Aries*. During the vernal equinox (march 21) the sun, as seen from the Earth, will lie "over" that point which today is in the constellation Pisces as shown in Fig. (11). Since the line of intersection of the Earth's equatorial plane and the ecliptic plane (see Fig. (12)) lies in both planes and locates the first point in Aries, it is a convenient reference line from which to measure longitude, whether one is using a geocentric or heliocentric motion.

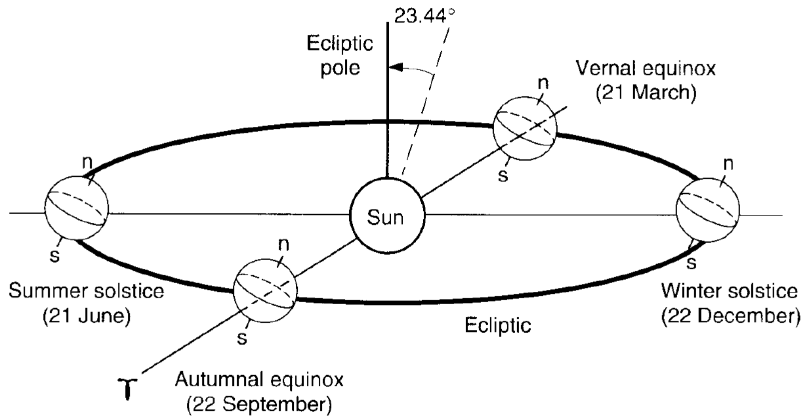

Figure 11: Heliocentric orbital motion of the Earth. From [2].

2.2.2 The Reference Plane

For a satellite of the Earth, one natural choice is the Earth's equatorial plane. However this plane precesses with a period of 26,000 years. This can be neglected without introducing much error. For very precise orbit determinations of position, longitude measurements are referred to the position of the referenec line (or vernal equinox) of a certain date or *epoch* chosen conveniently every 50 years or so (recent epochs are 1900 and 1950).

For Interplanetary trajectories, a more convenient reference plane is the *ecliptic* plane.

2.3 Determining the Orbital Parameters from the position and velocity vectors and vice versa

Depending on the practical case, there are two problems

1. *Orbit Determination* Often $\boldsymbol{r}$ and $\boldsymbol{v}$ are determined from radar tracking for a given satellite and one needs to determine the corresponding orbit.

2. *Position Prediction* Sometimes one has the orbital parameters but needs to quickly predict the position at a given time in order to know the location of the point on the celestial sphere at which to point a communication antenna.

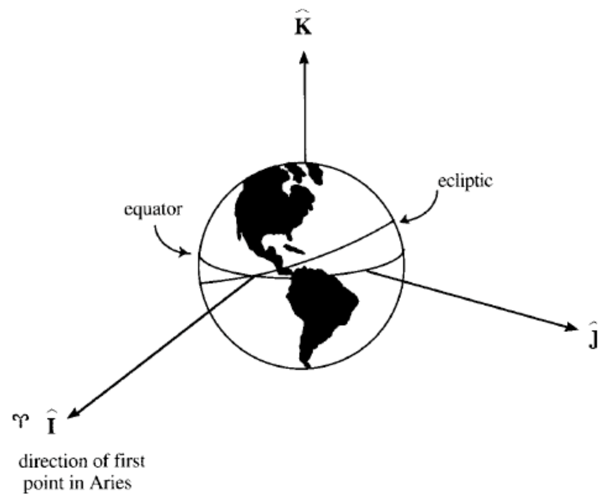

Figure 12: The Geocentric-equatorial reference frame. From [2].

2.3.1 Right Ascension and Declination

For both of these above stated purposes, it is convenient to define a commonly accepted set of coordinates for describing the position of a body on the celestial sphere.

• When it is more convenient to use the Earth's equatorial plane as a reference (e.g. Earth-bound satellites), we use the *declination*, δ, and *right ascension*, α, shown in Fig. (13). The reference $(\hat{\boldsymbol{i}} - \hat{\boldsymbol{j}})$ plane is the Earth's equatorial plane and the right ascension is measured positive eastward from the first point in Aries.

• When it is more convenient use the ecliptic plane as a reference (interplanetary trajectories) we use the *celestial longitude*, λ and *celestial latitude*, β which are analogous to α and δ respectively.

One can go from one pair of angles to the other using the following relationships:

$$\cos\delta\cos\alpha = \cos\beta\cos\lambda \tag{68}$$

$$\cos\delta\sin\alpha = \cos\beta\sin\lambda\cos\psi - \sin\beta\sin\psi \tag{69}$$

$$\sin\delta = \cos\beta\sin\lambda\sin\psi + \sin\beta\cos\psi \tag{70}$$

where ψ is the *obliquity of ecliptic* and is 23^o 27'. Knowing either (α, δ) or (λ,β) we can determine $\hat{\boldsymbol{r}}$.

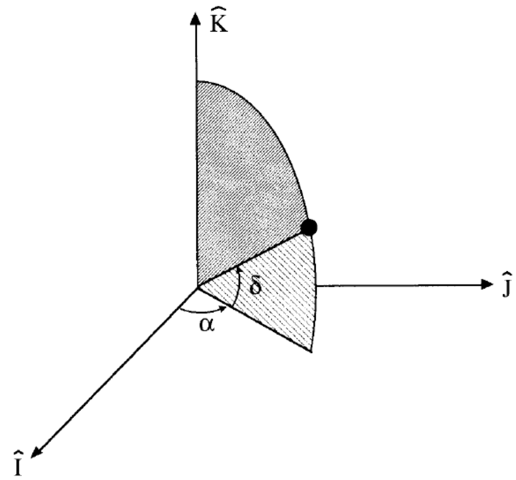

Figure 13: Right Ascension and Declination. From [2].

2.3.2 Problem 1: Getting the orbital elements from the velocity and position vectors.

• **Semimajor axis.** Starting with r and v we can get the Semimajor axis a from the *vis-viva* equation (Eq. (35)),

$$a = \frac{r}{2 - \dfrac{rv^2}{\mu}}. \tag{71}$$

where, again

$$\mu \equiv \frac{k}{m} \tag{72}$$

• **Eccentricity.** Defining the vector $\boldsymbol{h}$ as

$$\boldsymbol{h} \equiv \boldsymbol{H}/m = \boldsymbol{r} \times \dot{\boldsymbol{r}} \tag{73}$$

and recall the two-body equation of motion, Eq. (12), in the form

$$\ddot{\boldsymbol{r}} + \frac{\mu}{r^2}\hat{\boldsymbol{r}} = 0 \tag{74}$$

then taking the cross product of this equation with the constant vector $\boldsymbol{h}$, it can be put in the following form after some algebraic steps:

$$\ddot{\boldsymbol{r}} \times \boldsymbol{h} = \mu \frac{d\boldsymbol{r}/r}{dt} \tag{75}$$

which integrates to

$$\dot{\boldsymbol{r}} \times \boldsymbol{h} = \frac{\mu \boldsymbol{r}}{r} + \mu \boldsymbol{e} \tag{76}$$

where $\boldsymbol{e}$ is a dimensionless vector constant of integration. Because $\boldsymbol{e}$ is normal to $\boldsymbol{h}$ a it must lie in the plane of the orbit and can be shown (by the adventurous reader) to correspond to the *eccentricity vector*, which is a constant vector that points toward the perigee from the focus with the a magnitude equal to the orbit's eccentricity e. From the above equation we have

$$\mu \boldsymbol{e} = \boldsymbol{v} \times \boldsymbol{h} - \frac{\mu \boldsymbol{r}}{r} \tag{77}$$

which after some algebraic manipulation gives

$$\mu \boldsymbol{e} = \left[\left(v^2 - \frac{\mu}{r} \right) \boldsymbol{r} - (\boldsymbol{r}.\boldsymbol{v})\, \boldsymbol{v} \right] \tag{78}$$

from which we can get the eccentricity, $e = |\boldsymbol{e}|$, given $\boldsymbol{r}$ and $\boldsymbol{v}$.

• **Inclination.** Define the *nodal vector* $\boldsymbol{n}$ as

$$\boldsymbol{n} \equiv \hat{\boldsymbol{k}} \times \frac{\boldsymbol{h}}{h} \tag{79}$$

$\hat{k}$ is the unit vector normal to the Earth's equatorial plane as shown in Fig. (12). Since $\boldsymbol{h}$ is normal to the orbit plane, $\boldsymbol{n}$ must lie in both orbit and equatorial planes and hence is the line of nodes and points toward the ascending node. By inspection we find,

$$\cos i = \frac{\boldsymbol{h}.\hat{\boldsymbol{k}}}{h} = \frac{h_z}{h} \tag{80}$$

• **Longitude of the Ascending Node.** In a similar fashion we get

$$\cos \Omega = \frac{\boldsymbol{n}.\hat{\boldsymbol{i}}}{n} = \frac{n_x}{n} \tag{81}$$

If $\boldsymbol{n}.\hat{\boldsymbol{j}} = n_y \geq 0$, we have $0 \leq \Omega \leq \pi$. If $n_y \leq 0$, we have $\pi < \Omega < 2\pi$ and the principal value obtained by taking the inverse cosine on a calculator must be corrected by subtracting it from 2π.

• **Argument of Periapse.** By definition of $\boldsymbol{e}$, the argument of periapse is given by

$$\cos \omega = \frac{\boldsymbol{n}.\boldsymbol{e}}{ne} \tag{82}$$

where $0 \le \omega \le \pi$ if $e_z \ge 0$, and $\pi < \omega < 2\pi$ if $e_z < 0$.

- **True Anomaly.** In a similar fashion we get the true anomaly θ,

$$\cos\theta = \frac{\boldsymbol{e}.\boldsymbol{r}}{er} \tag{83}$$

where $0 \le \theta \le \pi$ if $\boldsymbol{r}.\boldsymbol{v} \ge 0$, and $\pi < \omega < 2\pi$ if $\boldsymbol{r}.\boldsymbol{v} < 0$.

2.3.3 Problem 2: Position Prediction.

To find $\boldsymbol{r}$ and $\boldsymbol{v}$ from the orbital elements we follow these steps:

1. Solve for

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \tag{84}$$

2. Use the following expression, obtained by rotating the unit vector $\boldsymbol{r}$ through the Euler angles, Ω, i and $\omega+\theta$ in succession (the math of this rotation operation is beyond our scope but can be found in a textbook of classical mechanics, e.g. Greenwood, D.T., *Principles of Dynamics*, Prentice-Hall, Chapter 2, 1988) to get an explicit expression for $\boldsymbol{r}$

$$\begin{aligned} \boldsymbol{r} &= r(\cos\Omega\cos(\omega+\theta) - \sin\Omega\sin(\omega+\theta)\cos i)\hat{\boldsymbol{\imath}} \\ &+ r(\sin\Omega\cos(\omega+\theta) + cos\Omega\sin(\omega+\theta)\cos i)\hat{\boldsymbol{\jmath}} \\ &+ r\sin(\omega+\theta)\sin i\hat{\boldsymbol{k}} \end{aligned} \tag{85}$$

3. and differentiate to get $\boldsymbol{v}$ as $\dot{\boldsymbol{r}}$,

$$\begin{aligned} \boldsymbol{v} &= -\gamma\left[\cos\Omega\left(\sin(\omega+\theta) + e\sin\omega\right) + \sin\Omega\left(\cos(\omega+\theta) + e\cos\omega\right)\cos i\right]\hat{\boldsymbol{\imath}} \\ &- \gamma\left[\sin\Omega\left(\sin(\omega+\theta) + e\sin\omega\right) - \cos\Omega\left(\cos(\omega+\theta) + e\cos\omega\right)\cos i\right]\hat{\boldsymbol{\jmath}} \\ &+ \gamma(\cos(\omega+\theta) + e\cos\omega)\sin i\hat{\boldsymbol{k}} \end{aligned} \tag{86}$$

where γ is given by

$$\gamma \equiv \left[\frac{\mu}{a(1-e^2)}\right]^{1/2}. \tag{87}$$

3 Epoch and Timekeeping

3.1 Some Definitions

Epoch refers to the time associated with an astronautical event.

Timekeeping refers to the method or clock used to describe the time elapsed between events. Einstein showed that both epoch and timekeeping is relative and is dependent on the reference frame of the observer and the gravitational potential.

In astronautics, the following four timekeeping systems are used:

1. *International Atomic Time (TAI)* is related to the frequency of light emitted during the electronic transitions taking place in an atom.

2. *Dynamical Time* is related to the motion of the planets and the moons

3. *Sidereal Time* is approximately related to the diurnal motion of the stars with respect to an observer on the Earth.

4. *Universal Time* is approximately related to the mean diurnal motion of the sun as seen by an observer on the Earth and is often referred to as *mean solar time*.

The Julian day number and Julian date count the time interval from an epoch selected to be noon of 1 January 4713 B.C., Julian proleptic calendar. The *Julian day number* is simply the number of solar days that have elapsed at Greenwich noon on the designated day since the epoch and are tabulated in *The Astronomical Almanac*. A Julian century has exactly 36,525 days. The *Julian date* refers to an instant by simply adding the Julian day number the fraction of the day elapsed since the last Greenwich noon. For instance, at midnight of of day whose Julian day number is 2451923 , the Julian date is 2451923.5.

3.2 International Atomic Time (TAI)

Since 1967 the standard defining 1 s is defined in terms of the occurrence of a specified number of cycles of radiation emitted by a specified state transition of cesium 133. The epoch for TAI was arbitrarily chosen so that it is equal to UT1 on 0^h on 1 January 1958.

3.3 Dynamical Time

refers to the family of time systems that replaced ephemeris time (ET) in 1984 as the independent argument in the ephemerides of the solar system obtained from dynamical gravitational theories. Dynamical time and ET are defined in such a way as to be independent of the variation of the rotation rate of the Earth and polar motion. Consequently they are the most uniform of the celestial-determined time systems. There are two dynamical times used today:

3.3.1 Terrestrial Dynamic Time (TDT)

is the independent argument of the apparent geocentric ephemerides of bodies in the solar system. The epoch of TDT is 1977 January 1, 0^h,0^m,0^s TAI, which is equal to 1977 January 1, 0^h,$0^m 32^s.184$ TDT exactly:

$$\mathrm{TDT} = \mathrm{TAI} + 32^s.184. \tag{88}$$

3.3.2 Barycnetric Dynamic Time (TDB)

is the independent argument of ephemerides with respect to the barycenter of the solar system. It is determined from TDT by a mathematical expression with periodic variation of amplitude less than $0^s.002$ that can be approximated by

$$\mathrm{TDB} = \mathrm{TDT} + 0^s.001658 \sin g + 0^s.000014 \sin 2g \tag{89}$$

where g is the mean anomaly of the earth given by

$$g = 357^s.53 + 0^s.98560028(\mathrm{JD} - 2451545.0) \tag{90}$$

where JD is the Julian date.

3.4 Sidereal Time

is a measure of the rotation of the Earth with respect to the equinox. It is the hour angle measured positive to the west along the celestial equator. There are 24 sidereal hours in a sidereal day for one complete rotation of the Earth with respect to the equinox. Sidereal time for the Greenwhich meridian is known as the Greenwhich sidereal time. Sidereal time for a given geographic longitude is obtained by adding the geographic longitude to Greenwhich sidereal time (longitude is always measured positive to the east.) Because of the variations in the rotation rate of the Earth, sidereal time is not uniform.

3.5 Universal Time

Universal time (UT) is related to the diurnal motion of the sun. The UT1 version is obtained from the intermediary of mean sidereal time and as such contains all the irregularities of the rotation of the Earth, including the motion of the pole. When corrections for these effects are included we talk of UT2, the most uniform of the Universal times. Provisional values of UT2 can be obtained by using a priori estimates of the corrections. Final values are available only a posteriori because of the need to average the observations. UT1 is therefore more useful but UT2 is more uniform.

3.5.1 Universal Time Coordinated (UTC)

UTC is used as the basis of civilian timekeeping and was originally defined as a smoothed version of UT2. It is now derived from TAI so that its rate is different than UT1 and UT2. UTC is maintained to within .9 s of UT1 by adding an integer second called *leap second* as necessary, only on 1 January or 1 July.

4 Orbit Perturbations

We have so far addressed orbital mechanics through the two-body problem which affords exact solutions. In practice, however, there are a number of forces in addition to the central Newtonian gravitational force that need to be considered if a precise description of the orbit is needed. The most important additional effects are the following

- Gravitational forces due to the nonsphercity of the earth (or generally the central body) and its nonuniform mass distribution.
- Aerodynamic drag forces due to motion in the upper atmosphere
- Gravitational forces due to other celestial bodies (such as the Sun and the Moon).

- Surface forces due to solar radiation pressure.

A generalized form of the equation of motion (Eq. (15)) can be written as

$$\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla} U + \boldsymbol{f_p} \tag{91}$$

where U is the gravitational potential per unit mass and $\boldsymbol{f_p}$ is the force vector per unit mass (acceleration) due to perturbing effects. A closed form solution of the above equation is not possible but there are a variety of solution methods appropriate for astronautics. These methods are called *perturbation methods*.

The Perturbation Method One method to include these effects is to make the following two assumptions:

1. the resulting additional forces are small compared to the primary central force ruling the two-body problem (i.e. their magnitude is smaller than μ/r^2),

2. the resulting orbit may be considered to have instantaneous values of the Keplerian orbital parameters (i, Ω, ω, e, a and θ). These orbital parameters vary along the path and are called *osculating elements*.

This will lead to a *perturbed* trajectory whose Kepler orbital elements are given by $i = i_o + \delta i$, $\Omega = \Omega_o + \delta\Omega$, $\omega = \omega_o + \delta\omega$, $e = e_o + \delta e$, $a = a_o + \delta a$, $\theta = \theta_o + \delta\theta$ where the osculating elements are the sum of quantities subscripted with o, which represent the unperturbed orbital parameters, and small perturbations of these parameters which depend on the time variable t.

The *Gaussian-Lagrange* planetary equations allow us to express the evolution of the osculating elements as a set of first order differential equations. Their derivation is beyond the scope of our course. The perturbing forces are represented in these equations by the three acceleration components , S (radial component), T (component in the instantaneous orbital plane), and W (normal to the orbital plane) shown in Fig. (14). In terms of derivatives with respect to the true anomaly the Gaussian-Lagrange planetary equations can be written as:

$$\frac{da}{d\theta} = \frac{2per^2}{\mu(1-e^2)^2}\left(e\sin\theta\, S + \frac{pe}{r}T\right) \tag{92}$$

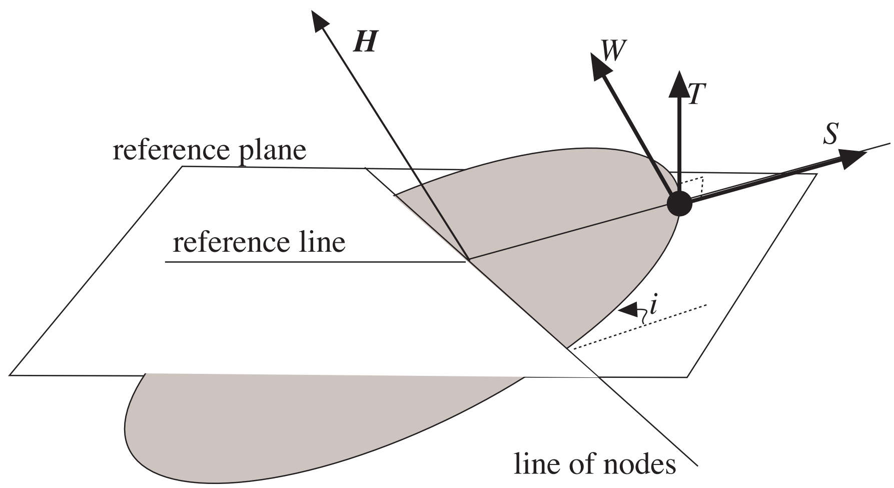

Figure 14: Acceleration components S (radial component), T (component in the instantaneous orbital plane), and W (normal to the orbital plane) for the Gaussian-Lagrange equations.

$$\frac{de}{d\theta} = \frac{r^2}{\mu}\left[\sin\theta\, S + \left(1 + \frac{r}{pe}\right)\cos\theta T + \frac{r}{p}T\right] \tag{93}$$

$$\frac{di}{d\theta} = \frac{r^3}{\mu pe}\cos(\theta+\omega)W \tag{94}$$

$$\frac{d\Omega}{d\theta} = \frac{r^3\sin(\theta+\omega)}{\mu pe\sin i}W \tag{95}$$

$$\frac{d\omega}{d\theta} = \frac{r^2}{\mu e}\left[-\cos\theta\, S + \left(1+\frac{r}{pe}\right)\sin\theta\, T\right] - \cos i\frac{d\Omega}{d\theta} \tag{96}$$

$$\frac{dt}{d\theta} = \frac{r^2}{(\mu pe)^{1/2}}\left\{1 + \frac{r^2}{\mu e}\left[\cos\theta\, S - \left(1+\frac{r}{pe}\right)\sin\theta T\right]\right\}, \tag{97}$$

where $p = a(1-e^2)/e$.

The above equations (valid for non-zero e and i) are solved numerically once the gravitational potential per unit mass U and the components of f_p have been formulated. We address these formulations in the following sections.

The Gaussian-Lagrange planetary equations are sometimes written in terms of time derivatives of the osculating elements.

4.1 The Earth's Gravitational Perturbations

The Earth is neither a perfect sphere nor has a uniform mass distribution. Its gravitational field outside its surface is most conveniently modeled as a spherical harmonic expansion (called a geoid model) of the form:

$$\begin{aligned} U_g(r,\phi,\lambda) &= \frac{\mu}{r}\left\{-1+\sum_{n=2}^{\infty}\left[\left(\frac{R_E}{r}\right)^n J_n P_{no}(\cos\phi)\right.\right. \\ &+ \left.\left.\sum_{m=1}^{n}\left(\frac{R_E}{r}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)\,P_{nm}(\cos\phi)\right]\right\} \end{aligned} \tag{98}$$

where $U_g(r,\phi,\lambda)$ is the gravitational potential per unit mass at a distance r from the center of the Earth and ϕ, λ are the latitude and longitude. P_{nm} are Legendre polynomials of order n and degree m. J_n, C_{nm} and S_{nm} are constants that depend on the mass distribution of the Earth and are called *zonal*, *tesseral* and *sectoral* harmonic coefficients respectively, and are determined from analyzing measurements of the motion of Earth-orbiting spacecraft. The first few low-order values for these constants are shown in Table (1).

$J_2 = 1082.6 \times 10^{-6}$	$C_{21} = 0$	$S_{21} = 0$
$J_3 = -2.53 \times 10^{-6}$	$C_{22} = 1.57 \times 10^{-6}$	$S_{22} = -0.904 \times 10^{-6}$
$J_4 = -1.62 \times 10^{-6}$	$C_{31} = 2.19 \times 10^{-6}$	$S_{31} = 0.27 \times 10^{-6}$
$J_5 = -.23 \times 10^{-6}$	$C_{32} = .31 \times 10^{-6}$	$S_{32} = -0.21 \times 10^{-6}$
$J_6 = 0.54 \times 10^{-6}$	$C_{33} = 0.100 \times 10^{-6}$	$S_{33} = 0.197 \times 10^{-6}$

Table 1: Magnitude of low order J, C and S values for Earth. From Lerch et al. NASA Tech. Memo 104555 (1992).

For the Earth the term J_2 represents the polar flattening of the Earth (equatorial bulge) and is some three orders of magnitude larger than the others. The equatorial bulge causes a regression of the line of nodes and a precession of the line of apsides as discussed in the following subsections.

4.1.1 Regression of the line of nodes.

The first effect of the equatorial bulge is to produce a torque which rotates the angular momentum vector. For prograde orbits ($i < 90°$) the orbit rotates in a westerly direction causing the line of nodes to regress as shown in Fig. (15).

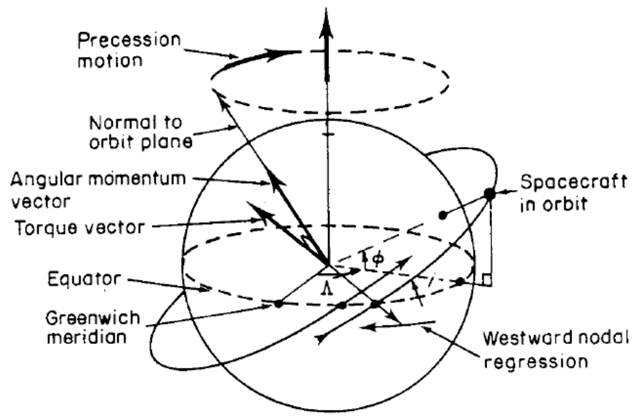

Figure 15: Regression of the line of nodes. From [1].

By neglecting all harmonic coefficients other than J_2 in Eq. (98), calculating the acceleration components of the gradient ∇U_g, introducing these components to the Gaussian-Lagrange planetary equations and integrating the resulting equation for the rate of nodal regression, the following expression for the change in the longitude of the ascending nodes can be found

$$\Omega = \Omega_o - \frac{3}{2}\frac{J_2 R_E^2}{p_e^2}\sqrt{\frac{\mu}{a^3}}t\cos i + O(J_2^2) \tag{99}$$

where $p_e = a(1-e^2)$ (not to be confused with $p = a(1-e^2)/e$).

When this regression rate is made to equal the apparent precession of the Sun, the plane of the orbit will maintain its geometry with respect to the Sun and it is known as a *Sun-Synchronous orbit.*

4.1.2 Precession of the line of apsides

The second effect of the equatorial bulge is to cause a rotation of the orbit *within* the orbit plane. This implies rotation of the semi-major axis and is called precession of the line of apsides. Following steps similar to those followed above, the secular precession of the line of apsides can be found to be:

$$\omega = \omega_0 + \frac{3}{2}\frac{J_2 R_E^2}{p_e^2}\sqrt{\frac{\mu}{a^3}}\left(2 - \frac{5}{2}\sin^2 i\right) t + O(J_2^2). \tag{100}$$

It can be verified that at an inclination of about 63.4°, the precession is zero. This is called a "Molniya" orbit.

As an example, consider a satellite in a 1000-km altitude circular orbit with an inclination of 45 degrees. The semimajor axis is 6378+1000/6378 = 1.157 R_E. It follows from the above equations that the line of nodes regresses at a rate of -4.23 deg/day and the line of apsides processes at a rate of 4.49 deg/day.

4.2 Atmospheric Drag

Low Earth satellites are subjected to an aerodynamic force that can be expressed as two orthogonal components:

- the drag applied in the direction opposite to the velocity vector
- the lift applied in the direction normal to the velocity vector.

The lift is much smaller than the drag and can be neglected. The drag force $\boldsymbol{F}_d$ on a satellite of mass m_o and cross-sectional area A is

$$\frac{\boldsymbol{F}_d}{m_o} = -\frac{C_d}{2}\left(\frac{A}{m_o}\right)\rho v_r^2 \hat{\boldsymbol{v}}_r, \tag{101}$$

where v_r is the relative velocity of the spacecraft with respect to the atmosphere, and $\hat{\boldsymbol{v}}_r$ is the unit vector in the direction of the relative velocity. Typical values for the drag coefficient C_d are close to 2.2-2.5 (2.0 being the theoretical value for a sphere in a free molecular flow). $\boldsymbol{F}_d$ is therefore large at perigee (where both the relative velocity v_r and the atmospheric density ρ are large).

The T and S components for drag are given by:

$$T = -\frac{C_d A}{2m_o}\rho v_r^2 \frac{1 + e\cos\theta}{(1 + 2e\cos\theta + e^2)^{1/2}} \tag{102}$$

$$S = -\frac{C_d A}{2m_o}\rho v_r^2 \frac{e\sin\theta}{(1 + 2e\cos\theta + e^2)^{1/2}} \tag{103}$$

($W = 0$ if we neglect the rotation of the Earth's atmosphere which actually rotates approximately synchronously with the Earth.) Substituting the above expressions in the Gaussian-Lagrange planetary equations, the following rates can be found:

$$\frac{da}{dt} = -\frac{C_d \rho A v_r^2}{n m_o \left(1-e^2\right)^{1/2}} \left(1 + 2e\cos\theta + e^2\right)^{1/2} \tag{104}$$

$$\frac{de}{dt} = -\frac{C_d \rho A v_r^2}{n m_o a} \frac{\left(1-e^2\right)^{1/2}(e + \cos\theta)}{\left(1 + 2e\cos\theta + e^2\right)^{1/2}} \tag{105}$$

$$\frac{di}{dt} = 0 \tag{106}$$

$$\frac{d\Omega}{dt} = 0, \tag{107}$$

where

$$n = \frac{\tau_e}{2\pi} = \sqrt{\frac{ma^3}{k}} = \sqrt{\frac{a^3}{\mu}} \tag{108}$$

is called the *mean motion*.

Using the above equations it can be shown that for the particular case of a circular orbit, $de/dt = 0$ (when averaged over an orbit) and the orbit remains circular but its radius will decrease. For small changes in the radius, the circular orbital period decreases according to:

$$\frac{\Delta\tau}{\tau} = -3\pi\rho r\left(\frac{AC_d}{m_0}\right) \tag{109}$$

where m_0/AC_d is called the *vehicle ballistic parameter*.

For large values of e an elliptical orbit can similarly be shown to experience reductions in both the eccentricity and the semimajor axis with the perigee remaining relatively constant while the apogee decreases. To first order, these changes are equivalent to those produced by an impulsive negative velocity increment at perigee.

4.3 Lunisolar Perturbations

It can be shown that the ratio of the perturbing acceleration a_p a third body exerts on a satellite of a central body to the acceleration a_c exerted by the central body is given by

$$\frac{a_p}{a_c} = \frac{m_p}{m_c}\left(\frac{r_s}{r_p}\right)^3\left(1 + 3\cos^2\gamma\right) \tag{110}$$

where r_s and r_p are the magnitudes of the position vectors to the satellite and perturbing bodies (with origin at the central body) and γ is the angle between these two vectors.

At geostationary orbit ($r = 42,164$ km) this ratio is 3.3×10^{-5} and 1.6×10^{-5} for the Moon and Sun respectively.

Cook (*Geophys. J.* vol. 6, p. 271 1962) found expressions for the average rates of change of the orbital elements for a single perturbing body using the Gaussian-Lagrange planetary equations. His equations are often used to model the perturbations caused by the gravitational effects of the sun and moon.

At GEO, the various resulting precession rates are of the order of .5 and .003 degrees per day from the Moon and Sun, respectively.

4.4 Solar Radiation Pressure (SRP) Perturbations

Reflection of incident electromagnetic radiation at a surface represents an exchange of momentum. Solar radiation pressure has a magnitude of

$$P_s = \frac{F_\odot}{c} \tag{111}$$

where $F_\odot$ is the solar flux (1400 W/m^2 at a distance of 1 AU from the sun) and c the speed of light. The resulting pressure is approximately $P_s = 4.7 \times 10^{-6}$ N/m^2. The magnitude of the SRP force per unit spacecraft mass (the purturbing acceleration) along the Sun-spacecraft line can be given by

$$f_{SRP} = s\frac{A}{m_o}P_s\left(\frac{a_\odot}{r_\odot}\right)^2 \tag{112}$$

where $r_\odot$ and $a_\odot$ are the spacecraft's distance and the Earth's mean distance from the Sun, respectively; s is a constant (between 0 and 2) that depends on the reflective properties of the spacecraft's surface.

The resulting perturbing accelerations are given by Aknes (*Celestial Mechanics* vol. 13, p.89 1976) in terms of S, T and W perturbing acceleration components which can be used to integrate the Gaussian-Lagrange planetary equations. An integration method is presented by Brookes and Ryland (*Celestial Mechanics* vol. 27, p. 353 1982).

SRP is important for GEO satellites with large solar array panels. Generally this leads to an increase in the eccentricity of the orbit.

4.5 Summary of Perturbing Effects.

The typical relative magnitude of the perturbing accelerations for various sources of perturbation is shown in Fig. (16).

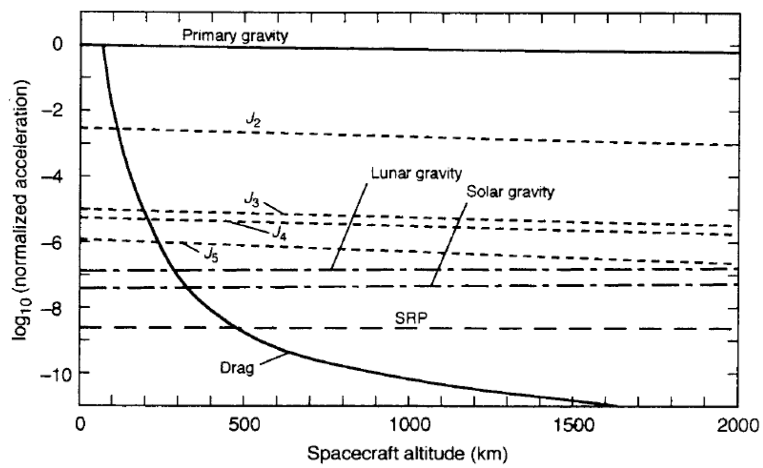

Figure 16: Comparison of the perturbing accelerations for various sources of perturbation. $A/m = .005\ \mathrm{m}^2/\mathrm{kg}$. From [1].

References

[1] P. Fortescue, G. Swinerd, and J. Stark, editors. *Spacecraft Systems Engineering*. J. Wiley & Sons, fourth edition, 2011.

[2] J. E. Prussing and B. A. Conway. *Orbital Mechanics*. Oxford University Press, New York, Oxford, 2013.