Project 1: Empirical Observation of Satellite Orbits

[redacted]
MAE 341
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1 Introduction

The purpose of this project is analyze the orbit of a satellite of our choice, based on observational data that we collect through a smart-phone application. The analysis portion includes the determination of orbital parameters, prediction of future satellite location, time variation of orbital parameters, and published data comparison.

1.1 Satellite Selection

The satellite that I selected for this project was the Hubble Space Telescope (HST), for several reasons.

[redacted]

1.2 Satellite History

[redacted]

[redacted]

Figure 1: Hubble Telescope Images [2]

2 Data

2.1 Procedure

In order to gather the necessary observational data, I used a combination of two apps that were available on an iPhone: Satellite Tracker [3] and Satellite Tracker Pro [4]. The first app facilitated a qualitative observation of the HST's position throughout its visible trajectory. The app labelled known planets and stars in the surrounding field of view, in order to identify the satellite's location relative to other celestial bodies. The second app enabled a more quantitative approach to satellite observation, providing real-time values of the HST's azimuth, elevation angle, velocity, altitude, and distance. By combining these two apps, it was possible to obtain both a qualitative and quantitative understanding of the HST's overhead trajectory.

In order to accurately capture real-time data, I used the screenshot and screen record feature of the iPhone. By capturing all of the provided observational values in an instant through screenshotting, I was able to eliminate the deviation that would've occurred in the seconds-long period required to take manual measurements. To collect the necessary data, I recorded the observational values at different points throughout it's trajectory, being sure to record the elapsed time as well. I repeated this process over several days and have summarized the raw observational data below.

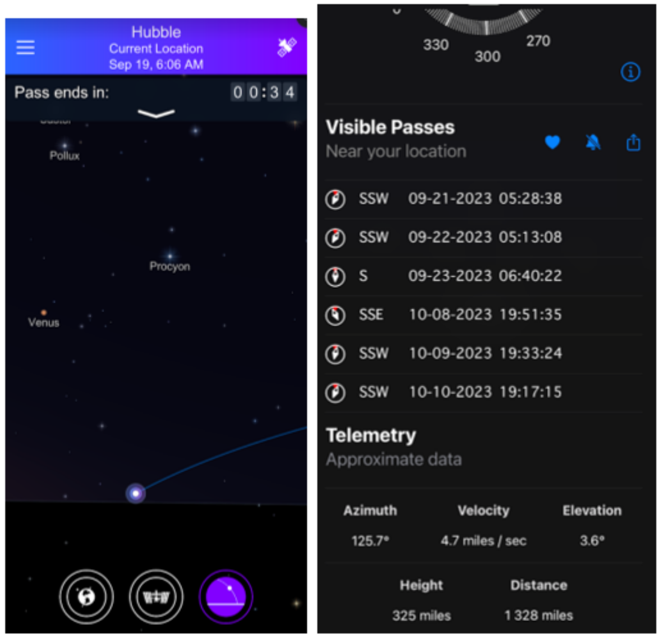

Figure 2: Data Collection Apps [3, 4]

In order to convert from the azimuth and elevation angle (which are taken from the surface of the Earth) to the right ascension α and declination δ (which are taken from the barycenter of the Earth, a python program provided by the course preceptor was used (the previous version was sued, so there may be some inaccuracies in the conversion). The latter two angles are more useful in the calculation of subsequent values as they locate the satellite relative to the equatorial coordinate system, and they have been included in the table below.

Date	**Time**	**Azimuth** (°)	**Elevation Angle** (°)	**Distance** (km)	**Right Ascension** (°)	**Declination** (°)
9/19	6:01:03 AM	[redacted]				
*9/19	6:01:11 AM					
9/19	6:02:41 AM					
9/19	6:03:07 AM					
*9/19	6:04:01 AM					
9/20	5:45:33 AM					
9/20	5:46:54 AM					
9/20	5:48:24 AM					
9/20	5:49:50 AM					
9/21	5:31:11 AM					
9/21	5:32:00 AM					
9/21	5:33:00 AM					
9/21	5:36:13 AM					
9/21	5:37:20 AM					

Table 1: Raw Observational Data and Coordinate-Shifted Data

3 Analysis

To calculate the orbital elements, two data points were used, which were taken on the same day and during the same overhead passage, several minutes apart. For clarity, the data points used in the example calculations below are marked with asterisks (*) in the table. We start by determining the time and distance elapsed between the two points.

Several key assumptions were made within the following calculations. First, it is assumed that the Earth is stationary. Given the relatively short nature of the HST's overhead trajectory on any given night, it is reasonable to assume that the Earth moves minimally in the brief passage of time required for the HST to pass through the sky. It is also assumed that the any precessions and/or orbital perturbations are negligible, and that the satellite is much smaller in mass than the Earth. Further assumptions are described within their respective sections.

3.1 Constants

$$Gravitational\ Constant \text{ [redacted]}$$

$$Mass\ of\ Earth: \text{ [redacted]}$$

$$Radius\ of\ Earth \text{ [redacted]}$$

$$Right\ Ascension \equiv RA$$

$$Declination \equiv \delta$$

3.2 Circular Orbit Assumption

In order to simplify the orbital parameter calculations, one can initially assume a circular orbit for the satellite. Documentation shows that this is a reasonable assumption, and one that would not significantly affect the accuracy of the results. This assumption implies that the eccentricity $e = 0$ and that the true anomaly is equal to the eccentric anomaly. The calculations of the orbital parameters in this section rely on this assumption.

3.2.1 Displacement Between Points

$$Elapsed\ Time \text{ [redacted]}$$

$$Angular\ Displacement\ Between\ Points\ (\psi) = \sqrt{(\Delta RA)^2 + (\Delta\delta)^2}$$

$$= \text{[redacted]}$$

$$= \text{[redacted]}$$

The value above is reasonable for two points that are relatively close to one another, as it applies Pythagorean's Theorem, which ignores the curvature of the Earth. In order to make this estimation more accurate, we can introduce the equation for the angular displacement between two points on a sphere [5]. In subsequent calculations, the more accurate value for the angular displacement, determined below, will be used.

$$\begin{aligned}&\textit{Angular Displacement Between Points } (\psi)\\&= \arccos\left[\sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos\left(RA_1 - RA_2\right)\right]\end{aligned}$$

[redacted]

3.2.2 Angular Velocity

$$\omega = \frac{\psi}{t} =$$ [redacted]

3.2.3 Period

$$\tau = \frac{2\pi}{\omega} =$$ [redacted]

3.2.4 Semi-Major Axis

$$Kepler's\ Third\ Law:\ \tau^2 = \frac{4\pi^2 a^3}{Gm_E}$$

$$\Downarrow$$

$$\sqrt[3]{\frac{Gm_E\tau^2}{4\pi^2}} =$$ [redacted]

Using the equations above, the angular velocity, period, and semi-major axis were evaluated based on each consecutive pair of data points. The results are plotted in the Orbital Parameter Evolution section and the data is summarized in the Appendix.

3.3 Lambert's Theorem

In the above section, several assumptions were made to simplify the calculations for angular displacement and period. In order to yield more accurate results, we can instead apply Lambert's Theorem which eliminates the need to assume that the orbit is circular (ie. that the orbit's eccentricity as 0). Lambert's Theorem enables the determination of orbital parameters based solely on two positional measurements of the satellite's orbit. In order to determine the orbital radius at the time of observation, the altitude measurement from the satellite tracker app was used, in addition to the time, azimuth, and elevation angle measurements presented above.

3.3.1 Displacement Between Points

$$Altitude\ Measurement\ [redacted]$$

* Note: The altitude measurement remained constant across all of the collected data. This supports the initial assumption of a roughly circular orbit, but the application of Lambert's Theorem is still described below.

$$Angular\ Displacement\ Between\ Points\ (\psi)$$
$$= \arccos\left[\sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(RA_1 - RA_2)\right]$$
$$= [redacted]$$

$$r_1 = r_2 = r_E + altitude\ [redacted]$$

$$d = \sqrt{(r_1^2 + r_2^2 - 2r_1 r_2 cos\psi} = [redacted]$$

3.3.2 Semi-Major Axis

$$\alpha = \arccos\left(1 - \frac{r_1 + r_2 + d}{2a}\right)$$
$$\beta = \arccos\left(1 - \frac{r_1 + r_2 - d}{2a}\right)$$

$$Lambert's\,Theorem : TOF = t_B - t_A = \sqrt{\frac{a^3}{Gm_E}}[(\alpha - sin\alpha) - (\beta - sin\beta)]$$

$$\Downarrow$$

$$a = \sqrt[3]{Gm_E\left(\frac{TOF}{(\alpha - sin\alpha) - (\beta - sin\beta)}\right)^2}$$

Since this equation has no closed form solution, MATLAB was used to implement a bisection algorithm to approximate the solution, which yielded a semi-major axis of $a =$ [redacted]. Since this is less than the radius of the Earth, this value can be deemed unreasonable, and subsequent calculations will use the value for the semi-major axis obtained in the first section instead. For the period calculation below, the semi-major axis obtained through Lambert's is used, which yields a value that is, again, less accurate than the method of assumptions

3.3.3 Period

$$Kepler'sThirdLaw: \ \tau = \sqrt{\frac{4\pi^2 a^3}{Gm_E}} = $$ [redacted]

3.4 Orbital Parameters

Both of the methods above (Lambert's Theorem vs. method of assumptions) yield values for the semi-major axis and period (as well as angular velocity and radius depending on assumptions). Using this information, it is possible to calculate the remaining orbital parameters. For the purposes of demonstration, the values that were obtained through method of assumptions are used below, as those were deemed to be more accurate.

3.4.1 Normal Vector of the Orbital Plane

First, the normal vector to the orbital plane is found, from which useful values like the angular momentum, velocity vector, line of nodes, and more, can be found. To find the direction of the angular momentum vector, the cross product of two radii are taken, the both raddii vectors would lie in the orbital plane. Then, the unit direction is multiplied by the magnitude of the angular momentum to obtain the angular momentum vector

$$\vec{k_O} = (r_1\hat{r} + RA_1\hat{\theta} + \delta_1\hat{\psi}) \times (r_2\hat{r} + RA_2\hat{\theta} + \delta_2\hat{\psi})$$

= [redacted]

$$\Downarrow Convert\, to\, Cartesian\, Coordinates$$

= [redacted]

$$Unit\ Normal: \hat{k_O} = \frac{\vec{k_O}}{|k_O|} = $$ [redacted]

3.4.2 Line of Nodes

The calculation for the magnitude of the angular momentum relies on the assumption that the orbit is roughly circular. By taking the cross product of the vector normal to the orbital and equatorial plane, the line of nodes is obtained.

$$h = \sqrt{a(1-e^2)Gm_E} = $$ [redacted]

$$\vec{h} = h\vec{\hat{k_O}} = $$ [redacted]

$$Line\ of\ Nodes: \vec{n} \equiv \hat{k} \times \hat{k_O} = $$ [redacted]

3.4.3 Inclination

$$\cos i = \frac{h_z}{h} = \text{[redacted]}$$

3.4.4 Longitude of the Ascending Node

$$\Omega = \arccos\left(\frac{n_x}{n}\right) = \text{[redacted]}$$

3.4.5 Velocity Vector

Knowing two radius vectors and elapsed time, it is possible to estimate the velocity vector at the time of observation. The following calculation for the velocity vector relies on the assumption that the orbit is roughly circular, as it applies the logic that the velocity vector is normal to the angular momentum and radius vector.

$$v = r\psi = \text{[redacted]}$$

$$\hat{v} = \frac{\hat{k_O} \times \vec{r}}{|\hat{k_O} \times \vec{r}|}$$

$$= \text{[redacted]}$$

$$\vec{v} = v * \hat{v} = \text{[redacted]}$$

3.4.6 Eccentricity

$$\vec{e} = \frac{1}{Gm_E}(\vec{v} \times \vec{h}) - \frac{\vec{r}}{r}$$

$$= \frac{1}{Gm_E}\left(\text{[redacted]}\right)$$

$$- \text{[redacted]}$$

$$= \text{[redacted]}$$

$$e = |\vec{e}| \text{[redacted]}$$

3.4.7 Argument of Periapse

$$\omega = \arccos\left(\frac{\vec{n}\cdot\vec{e}}{ne}\right) = \arccos$$ [redacted]

$$= \arccos$$ [redacted]

3.4.8 True Anomaly

$$\theta = \arccos\left(\frac{\vec{e}\cdot\vec{r}}{er}\right) = \arccos$$ [redacted]

$$= \arccos$$ [redacted]

3.4.9 Location Prediction

With the orbital period calculated above and the assumption that the orbit is roughly circular, it is possible to use Kepler's Equation to predict a future location of the period. Below, we confirm this by "predicting" the location of the HST at a future time, for which we are able to observe the true location of the satellite. The point that was chosen was the last observation on the second day.

$$Kepler's\ Equation : 2\pi\left(\frac{t-t_0}{\tau}\right) = \epsilon - e\sin\epsilon$$

$$\Downarrow Plot$$

$$0$$ [redacted]

True Value:

$$\epsilon = \theta = \arccos\left(\frac{\vec{e}\cdot\vec{r}}{er}\right) = \arccos$$ [redacted]

$$= \arccos$$ [redacted]

3.5 Orbital Parameter Evolution

By determining the orbital parameters for each of the collected data points, it is possible to analyze how the orbital parameters change over time. The following plots show the time variation of several orbital parameters over the three day period of data collection. In particular, the eccentricity, RA of the ascending node, and argument of periapse are plotted below, with data taken from sources online [8]. Since this data is taken from sources online, it is possible to plot the parameters over a longer period than the observation. This allows for an interesting analysis of the orbital parameters, as it is possible to see how all three values cycle over time. This cycling effect can likely be attributed to the procession of the Earth's axis and perturbations within the orbit.

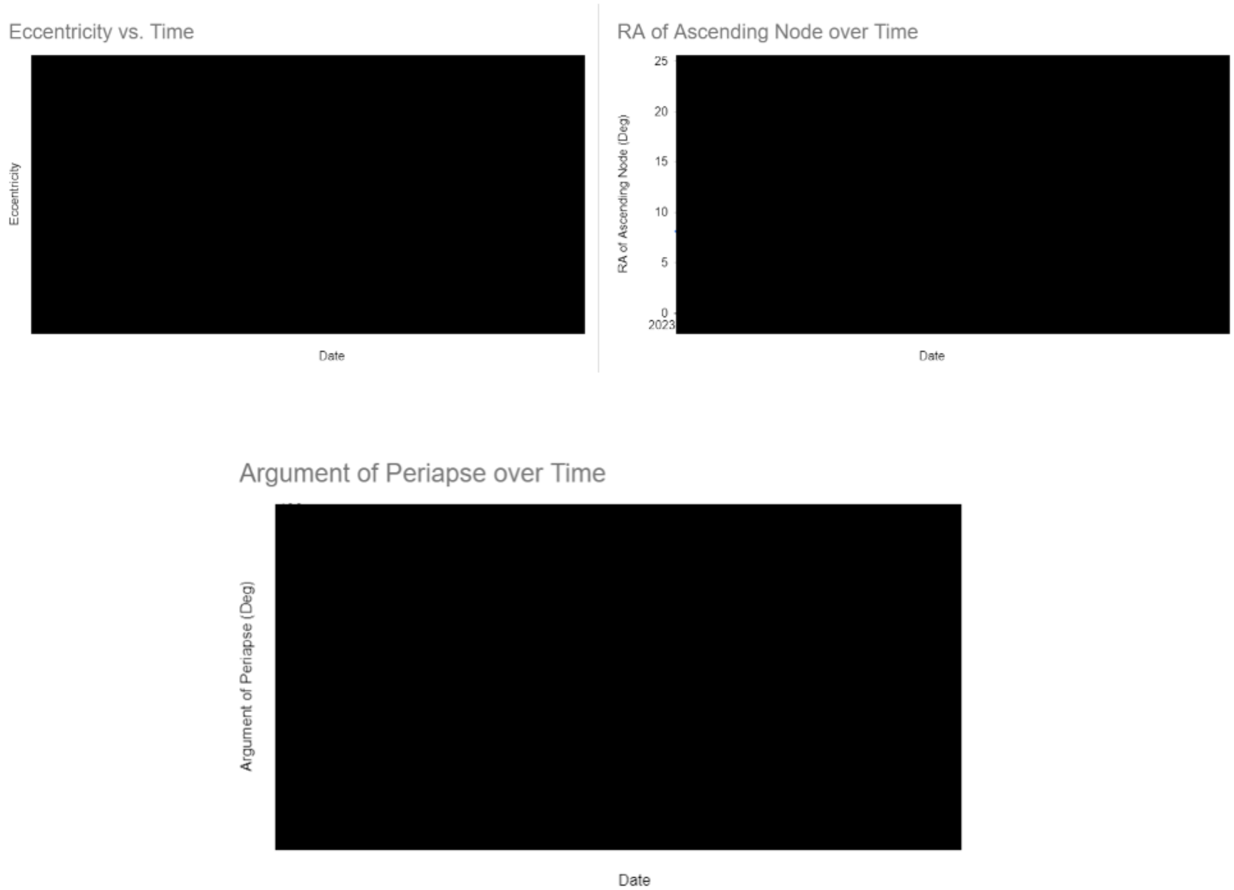

Figure 3: Orbital Parameters Over Time

In addition, the angular velocity, period, and semi major axis values that were obtained through the initial calculations are plotted below. These plots demonstrate the high degree of error, unreliability, and fluctuations that were present in the observed data

Figure 4: Observed Orbital Parameter Variation

4 Discussion

4.1 Error Analysis

There are several sources of error to consider in the analysis of our calculations.

[redacted]

4.2 Data Comparison

According to the online documentation, the true orbital parameters of the HST are as follows [7, 8]:

Parameter	Documented Value	Calculated Value
Altitude (km)	[redacted]	
Semi-Major Axis (km)		
Angular Speed (rad/s)		
Speed (km/s)		
Period (min)		
Eccentricity		
Inclination (°)		
Right ascension of ascending node (°)		
Argument of perigee (°)		
Perigee height (km)		
Apogee height (km)		

Table 2: Orbital Parameters

While the calculated values for period, angular velocity, speed, eccentricity, and semi-major axis are reasonable accurate, the values for Inclination, RA of the ascending node, and argument of perigee have significant deviations. This different can likely be attributed to measurement error and inaccuracies that result from assumptions about the orbit.

5 Conclusion

[redacted]

References

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https://in-the-sky.org/spacecraft-elements.php?id=20580startday=19startmonth=9startyear=2023endday=22endmonth=9endyear=2023

6 Appendix

Angular Displacement (rad)	dT (s)	Angular Velocity (rad/s)	Period (min)	Semi-Major Axis (m)
[redacted]				

Table 3: Angular Velocity, Period, and Semi-Major Axis from Observed Data