MATH214
Linear Algebra

Homework 2
Manuel — UM-JI (Spring 2023)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Vectors and bases*

We consider the vector space $\mathbb{R}^4$ and define the vectors $a = (1, 2, -1, -2)$, $b = (2, 3, 0, -1)$, $c = (1, 3, -1, 0)$, and $d = (1, 2, 1, 4)$ in its canonical basis.

1. Show that $\mathcal{B} = \{a, b, c, d\}$ is a basis for $\mathbb{R}^4$.
2. Find the coordinates of $(7, 14, -1, 2)$ in the basis $\mathcal{B}$.

Ex. 2 — *Importance of the base field*

Let $a = (1, 1, \alpha)$, $b = (1, \alpha, 1)$, and $c = (\alpha, 1, 1)$ be three vectors over a field $\mathbb{K}$. Let $S = \{a, b, c\}$.

1. If $\mathbb{K} = \mathbb{R}$, is S a basis for $\mathbb{K}^3$?

* 2. We denote by $\mathbb{Z}/2\mathbb{Z}$ the set composed of the two elements $\{0, 1\}$. Prove that this $(\mathbb{Z}/2\mathbb{Z}, +, \cdot)$ is a field.

3. If $\mathbb{K} = \mathbb{Z}/2\mathbb{Z}$, is S a basis for $\mathbb{K}^3$?

Ex. 3 — *Equality of functions*

Let V_1 and V_2 be two vector spaces over $\mathbb{K}$ and p be a surjective linear map from V_1 to V_2. If V is a finite dimensional vector space over $\mathbb{K}$, and f is a linear map from V to V_2, then show the existence of a linear map φ defined from V to V_1 and such that $f = p \circ \varphi$.

Hint: use remark (2.22|2.84).

Ex. 4 — *Image and kernel*

Let V be a vector space.

1. Let u be an endomorphism of V such that $\ker u = \operatorname{im} u$ and S be such that $V = \operatorname{im} u \oplus S$.
   a) Prove that for any $x \in V$, there is a unique pair $(y, z) \in S^2$, such that $x = y + u(z)$. Let $z = v(x)$ and $y = w(x)$.
   b) Show that v is linear and determine $u \circ v + v \circ u$.
   c) Show that w is linear and determine $u \circ w + w \circ u$.
2. Let u be an endomorphism of V such that $u^2 = 0$. Show that if there exists $v \in \mathcal{L}(V)$ such that $u \circ v + v \circ u = \mathrm{Id}_V$, then $\ker u = \operatorname{im} u$.

3. Let $u \in \mathcal{L}(V)$, such that $u^2 = 0$ and $u \neq 0$. Show that even if there exists $w \in \mathcal{L}(V)$ such that $u \circ w + w \circ u = u$, then $\ker u \neq \operatorname{im} u$.

   Hint. What is the most simple way to disprove a result?

4. Explain the difference between the results from questions 2 and 3.

* **Ex. 5** — *Challenging problem*

Let V be a finite dimensional $\mathbb{R}$-vector space and V_1 and V_2 two subspaces of V which are distinct from V and $\{0\}$. Prove the existence of a basis of V containing no element from V_1 or V_2.