MATH2140-Homework3

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1 Ex.1

1 We can find that $X^2+X+1+2X = X^2+3X+1$. But the set of $\{X^2+3X+$ 1,2X,$X^3+3\}$ including three elements of different degree cannot be expressed as the linear combination of others. Thus rank=card$\{\mathcal{B}\}$=3

3 Suggest that a(x+z)+b(-x+2y)+d(y+t)+c(x+y-z+t)=0, then we have

$$\begin{cases} a-b+c=0 \\ 2b+d+c=0 \\ a-c=0 \\ d+c=0 \end{cases} \Rightarrow \begin{cases} 2a-b=0 \\ 2b+d+a=0 \\ d+c=0 \end{cases} \Rightarrow \begin{cases} d+5c=0 \\ d+c=0 \end{cases} \Rightarrow a,b,c,d=0$$

Thus it is linearly independent and rank = 4.

2 Ex.2

As (2,1,0,2) and (-1,-2,3,1) are in the kernal of two linear forms suggested as (x,y,z,w). We have $\begin{cases} 2x+y+2w=0 \\ -x-2y+3z+w=0 \end{cases} \Rightarrow \begin{cases} -3y+6z+4w=0(annihilate\ x) \\ 3x+3z+5w=0(annihilate\ y) \end{cases} \Rightarrow$.
Then take $\begin{cases} z=1 \\ w=0 \end{cases}$ and $\begin{cases} z=0 \\ w=3 \end{cases}$,we get u=(-1,2,1,0) and v=(-5,4,0,3). (2,1,0,2),(-1,-2,3,1)∈ker u∩ker v. Thus $\forall a$ ∈the subspace, as (2,1,0,2) and (-1,-2,3,1) are linearly independent, we have $a = \alpha(2,1,0,2)+\beta(-1,-2,3,1)$ ∈ker u∩ker v.

3 Ex.3

im u:It is a space spanned by (1,-1,0),(1,2,3),(1,-2,-1). However, as (1,2,3)+3(1,-2,-1)=4(1,-2,0), it is not linearly independent, and {(1,2,3) , (1,-2,-1)} is linearly independent. So $\mathcal{B}_{imu} = \{(1,2,3),(1,-2,-1)\}$.

ker u: $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -2 \\ 0 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$,and the $\mathcal{B}_{keru} = \{(-4, 1, 3)\}$

4 Ex.4

1 $\Rightarrow: (ld_V - p)^2 = ld_V - p - p \circ (ld_V - p) = ld_V - 2p + p^2 = ld_V - p$.
$\Leftarrow$: Let's define $q = ld_V - p$, $((ld_V - p)^2 = ld_V - p \Rightarrow p^2 = p) \Leftrightarrow (q^2 = q \Rightarrow (ld_V - q)^2 = ld_V - q)$, which is correct. Thus it is proved.

2 To prove $im(ld_V - p) = ker\ p$, we need to prove$\forall x \in V,\ p(x - p(x)) = 0$. Firstly endomorphism implies linearity. So $p(x - p(x)) = p(x) - p^2(x) = 0$, and $im(ld_V - p) \subset ker\ p$. If we plug in $ld_V - p$ to substitute p, $im\ p \subset ker(ld_V - p)$. Then let's prove $ker(ld_V - p) \subset im\ p$. $\forall x \in V$ s.t. $x - p(x) = 0$, we have $x \in ker(ld_V - p)$, then $x = p(x)$, so $x \in im\ p$. So $ker(ld_V - p) \subset im\ p$, and we have $ker(ld_V - p) = im\ p$, and similarly, $im(ld_V - p) = ker\ p$,

3 Firstly let's prove $im\ p \cap ker\ p = 0$, we have $\forall p(v) \subset im\ p \cap ker\ p, p(p(v)) = 0 = p(v)$. Then $\forall v \in V$, we have $v - p(v) = (ld_V - p)v = w \in kerp$. Thus $\exists p(v) \in im\ p$ and $w \in ker\ p$, and we have $im\ p + ker\ p = V$. Thus $im\ p \oplus ker\ p = V$.

4 The meaning of the question is: $u \circ p = p \circ u \Leftrightarrow u(ker\ p) \subset ker\ p \vee u(im\ p) \subset im\ p$.
$\Rightarrow$: $\forall p(m) \in im\ p,\ u(p(m)) = p(u(m)) \in im\ p$. $\forall v \in ker\ p, p(v) = 0, u(p(v)) = 0 = p(u(v))$, Thus $u(v) \in ker\ p$.
$\Leftarrow$: We need to prove $u(ker\ p) \subsetneq ker\ p$, if it is not commutative. $p \circ u \neq u \circ p$, $\exists a \in ker\ p,\ u(p(a)) = 0\ but\ p(u(a)) \neq 0$. $u(a) \notin ker\ p$.
Thus $u \circ p = p \circ u \Leftrightarrow u(ker\ p) \subset ker\ p \vee u(im\ p) \subset im\ p$.

5 Ex.5

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$(iii) \to (i)$ $\forall v \in ker\ u^2,\ u(u(v)) = 0,\ u(v) \in ker\ u$, and obviously $u(v) \in im\, u$. As $ker\ u \oplus im\, u = V,\ u(v) = 0,\ v \in ker\ u$. Also, obviously $ker\ u \subset ker\ u^2$. Then $ker\ u = ker\ u^2$.

$(iii) \to (ii)$ It is obvious that $im\ u^2 \subset im\ u$, so we just need to prove $im\ u \subset im\ u^2$. If $\exists u(p) \in im\ u,\ u(p) \notin im\ u^2$, then $p \notin im\ u$. Let $p = y + z$ s.t. $y \in im\ u,\ z \in ker\ u$. Then $u(p) = u(y) + u(z) = u(y)$. However, $u(y) \in im\ u^2$. So it is contradictive. So we have $im\ u = im\ u^2$.

$(ii) \to (iii)$ Firstly I shall prove if $V \neq ker\ u + im\ u$, $im\ u = im\ u^2$ will be wrong. If $V = (ker\ u + im\ u) \oplus S$, $\forall m \in V$, $m = x + y + z$ s.t. $x \in ker\ u; y \in im\ u; z \in S; z \neq 0$. $u(m) = u(y) + u(z)$ s.t. $u(y) \in im\ u^2$ and $u(z) \in im\ u$, but $u(z) \notin im\ u^2$. Thus $u(m) \in im\ u$, but $u(m) \notin im\ u^2$. So it is contradictive.
Then I shall prove $im\ u \cap ker\ u = 0$. Let's suggest $v = x_1 + y_1 = x_2 + y_2$, where $(x_1, y_1) \neq (x_2, y_2)$ and $x_{1,2} \in S$ and $y_{1,2} \in ker\ u$ s.t. $ker\ u \oplus S = V$. $x_1 - x_2 + y_1 - y_2 = 0$, and $u(x_1 - x_2) + u(y_1 - y_2) = u(x_1 - x_2) = 0$. $x_1 - x_2 \in ker\ u$ and $x_1 - x_2 \in S$, so $x_1 = x_2$, and such x and y are unique. So $ker\ u \cap im\ u = 0$. $ker\ u \oplus im\ u = V$.

$(i) \to (iii)$ $\forall u(w) \in ker\ u \cap im\ u$, $w \in ker\ u^2$, $w \in ker\ u$, $u(w) = 0$, $ker\ u \cap im\ u = 0$. Then I shall prove if $V \neq ker\ u + im\ u$, $ker\ u = ker\ u^2$ will be wrong. If $V = (ker\ u + im\ u) \oplus S$, as $ker\ u \cap im\ u = 0$, $\forall m \in ker\ u$, $m = x + z$ s.t. $x \in ker\ u; z \in S; z \neq 0$. $u(m) = u(x) + u(z) = 0$ s.t. $x \in ker\ u^2$ and $z \in ker\ u$, but $z \notin ker\ u^2$. Thus $m \in ker\ u$, but $u(m) \notin ker\ u^2$. So it is contradictive, and $ker\ u \oplus im\ u = V$.

2 (x,y)→(x,0)

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