MATH214
Linear algebra

Homework 3

Manuel — UM-JI (Spring 2023)

Reminders

- Write in a neat and legible handwriting or use $\LaTeX$
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Rank*

Determine the rank of the following sets.

1. $S = \{X^2 + X + 1, X^2 + 3X + 1, 2X, X^3 + 3\} \subset \mathbb{C}[X]$;

* 2. Before next question.

3. $S = \{x + z, -x + 2y, x + y - z + t, y + t\} \subset \mathcal{L}(\mathbb{R}^4, \mathbb{R})$;

   Hints.
   - What are the elements of $\mathcal{L}(\mathbb{R}^4, \mathbb{R})$ looking like?
   - Why is $S \subset \mathcal{L}(\mathbb{R}^4, \mathbb{R})$?

Ex. 2 — *Kernel of linear forms*

* 1. What is a linear form and what is its kernel?

2. In $\mathbb{R}^4$ write the subspace spanned by $\{(2, 1, 0, 2), (-1, -2, 3, 1)\}$ as the intersection of the kernel of two linear forms.

Ex. 3 — *Matrix, kernel, and image*

Let $u \in \mathcal{L}(\mathbb{R}^3)$ whose matrix in the canonical basis is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & -2 \\ 0 & 3 & -1 \end{pmatrix}$$

Determine a basis of $\ker u$ and of $\operatorname{im} u$.

Ex. 4 — *Projections*

Let V be a $\mathbb{K}$-vector space. Any endomorphism p of V such that $p^2 = p \circ p = p$ is called a *projection*.

1. Prove that p is a projection if and only if $\mathrm{Id}_V - p$ is also a projection.
2. Show that if p is a projection then $\operatorname{im}(\mathrm{Id}_V - p) = \ker p$ and $\ker(\mathrm{Id}_V - p) = \operatorname{im} p$.

   Hint: proceed by double inclusion.
3. Prove that if p is a projection then $V = \operatorname{im} p \oplus \ker p$.

4. Show that a projection p commutes with an endomorphism u of V if and only if both its kernel and image are closed under u.

   Hint: take advantage of the result from 3.

* **Ex. 5** — *Challenging problem*

Let V be a finite dimensional $\mathbb{K}$-vector space and $u \in \mathcal{L}(V)$.

1. Prove that the following properties are equivalent.
   (i) $\ker u = \ker u^2$;
   (ii) $\operatorname{im} u = \operatorname{im} u^2$;
   (iii) $V = \ker u \oplus \operatorname{im} u$;
2. Provide some examples of such endomorphisms.
3. Is this result still valid in infinite dimension?