MATH2140-Homework6

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1 Ex.1

1 Let $M = \big(\varphi_1, \cdots, \varphi_n\big)$

$\Rightarrow$ Since M is orthogonal, in N=M^TM, $n_{i,j} =<\varphi_i, \varphi_j>= \begin{cases} 1(i=j) \\ 0(i \neq j) \end{cases}$. So N=$I_n$.

$\Leftarrow$ Since $M^TM = I_n$, we have $<\varphi_i, \varphi_j>= n_{i,j} = \begin{cases} 1(i=j) \\ 0(i \neq j) \end{cases}$. So it is orthogonal.

det $\quad det\ M^TM = 1, det\ M \times det\ M^T = 1, det\ M = det\ M^T$. So we have $det\ M$=± 1.

2 $\Rightarrow$: As r is an orthogonal automorphism, and $\mathcal{B}$ is orthonormal, r($\mathcal{B}$) is also orthonormal. Also, $det(r(\mathcal{B}))$=$det\ r det\mathcal{B} = 1$, so it is positively oriented.
$\Leftarrow$: $\mathcal{B}$ is a positively oriented orthonormal basis and r($\mathcal{B}$) is positively oriented orthonormal, so it is an automorphism. $det(r(\mathcal{B}))$=$det\ r det\mathcal{B} = 1$, $det\mathcal{B} = 1$, so $det\ r = 1$. So it is a rotation.

3 $(M - \lambda I_3)x = 0$, $\begin{cases} (cos\theta - \lambda)x_1 - sin\theta x_2 = 0 \\ sin\theta x_1 + (cos\theta - \lambda)x_2 = 0 \\ (1-\lambda)x_3 = 0 \end{cases}$
If $x_3 = 0, x_1x_2 \neq 0$, $\lambda^2 - 2\lambda cos\theta + 1 = 0$, $\lambda_1 = cos\theta + isin\theta$, $\lambda_2 = cos\theta - isin\theta$, and the eigenspace is spanned by $\{(i,1,0)\}$ and $\{(1,i,0)\}$
If $x_1 = x_2 = 0, x_3 \neq 0$, $\lambda = 1$, the eigenspace is the z-Axis.

4 By observation we can find that the rotation do not change the third axis, so we can get the original matrix equals to $\begin{pmatrix} x_{1,1} & x_{1,2} & 0 \\ x_{2,1} & x_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. As it is orthonormal, we have $x_{1,1}^2 + x_{2,1}^2 = x_{1,2}^2 + x_{2,2}^2 = x_{1,1}x_{2,2} - x_{1,2}x_{2,1} = 1$; $x_{1,1}x_{1,2} + x_{2,1}x_{2,2} = 0$. Then $x_{1,1} = x_{2,2} = cos\theta$, $x_{1,2} = -sin\theta$, $x_{2,1} = sin\theta$. So they are the same.

2 Ex.2

1 If A is triangularizable, we can take B similar to A and B=$\begin{pmatrix} & & \\ & * & \vdots \\ & & \\ & 0 & \lambda \end{pmatrix}$ So if we omit the last column we can get g, which restricts f to V_0. Then χ_g divides χ_f

2 When n=1, it is true. For $\mathcal{M}_{n+1}$, if we take the first column as a vector and complete it into a basis, we can have $\mathcal{M}_{n+1} \sim \begin{pmatrix} 1 & * \\ 0 & \mathcal{M}_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_n \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & \mathcal{T}_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_n^{-1} \end{pmatrix}$. So it is triangularizable.

3 According to the Fundamental Theorem of Algebra, χ_A can be split on $\mathbb{C}$. Thus it is triangularizable.

4 $det(A-\lambda I)=0$, $(\lambda+1)^3=0$, $\lambda=-1$. Its eigenvalue is -1. Then we have $\mathcal{A}_{B'}=\begin{pmatrix}-1 & a & b\\ 0 & -1 & c\\ 0 & 0 & -1\end{pmatrix}$. $B'=\{e_1,e_2,e_3\}$,we have $\begin{cases}Ae_1=-e_1\\ Ae_2=ae_1-e_2\\ Ae_3=be_1+ce_2-e_3\end{cases}$. So we firstly take $e_1=(2,1,0)\in E_{-1}(A)$. Then $(A+I_3)e_2=ae_1$. However, as the rows of $A+I_3$ is the same, we retake $e_1=(1,1,1)$, and $e_2=(1,2,3)$. Then a=0. Then $(A+I_3)e_3=be_1+ce_2$. So we have to take c=0, and b=1,$e_3=(1,1,0)$. So $\mathcal{A}_{B'}=\begin{pmatrix}-1 & 0 & 1\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$ under the basis of {(1,1,1),(1,2,3),(1,1,0)}

3 Ex.3

λ is a number while A is a matrix, so we cannot plug in A in place of λ.

4 Ex.4

1 $\Rightarrow$: $P(A)=a_k\prod_{i=0}^{k}(A-\lambda_i I_n)$. Since P(A)$\in GL_n(\mathbb{C})$,det P(A)=det $a_k\prod_{i=0}^{k}(A-\lambda_i I_n)$=$\neq$0, $\forall i$, det $A-\lambda_i I_n\neq 0$. So λ_i are not eigenvalues of A. So P and χ_A do not have common divisor, gcd(P, χ_A)=1.
$\Leftarrow$: If P(A)$\notin$GL, $\exists i$ s.t. det (A-$\lambda_i I$)=0, so λ_i is an eigenvalue of A. Thus (x-λ_i) is the common divisor. So it is contradictive.

2 Under some circumstances it is important. When the χ_A is easy to calculate but P(A) is difficult to check its determinent, we can directly work out whether it is inversible. (Why Manuel always put useful tools in the Homework while talking about useless things in the slides?)

3 Let P(x)=$\prod_{i=1}^{100}(x-i^2)$, and A=$\begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}$. Check whether P(A) is inversible.

5 Ex.5

$2X_{i,i}=tr(X)A_{i,i}$, $2\sum X_{i,i}=tr(X)\sum A_{i,i}$, $2tr(X)=tr(X)tr(A)$.
If $tr(X)=0$, $X+X^T=0$, then X is any antisymmetric matrix.
If $tr(X)\neq 0$, $tr(A)=2$, $x_{i,i}=0.5(tr(X)A_{i,i})$, and $x_{i,j}+x_{j,i}=tr(X)A_{i,j}=tr(X)A_{j,i}$. So A is symmetric.
In conclusion, if $tr(A)=2$ and A is symmetric, we have $x_{i,i}=0.5(tr(X)A_{i,i})$, and $x_{i,j}+x_{j,i}=tr(X)A_{i,j}=tr(X)A_{j,i}$.
Else, X is an antisymmetric matrix.