MATH214
Linear algebra

Homework 6

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Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Rotations*

Let V be a finite n-dimensional inner-product space and $\mathcal{B}$ be a basis for V. An endomorphism on V is said to be *orthogonal*, if and only if its matrix M in $\mathcal{B}$ is orthogonal, i.e. the column vectors composing M are orthonormal. An orthogonal automorphism with determinant 1 is called a *rotation*.

1. Prove that M is orthogonal if and only if $M^\top M = \mathrm{I}_n$. Conclude that if M is orthogonal then $\det M = \pm 1$.
2. Let $\mathcal{B}$ be a positively oriented orthonormal basis. Show that $r \in \mathcal{L}(V)$ is a rotation, if and only if $r(\mathcal{B})$ is orthonormal and positively oriented.
3. Let $V = \mathbb{R}^3$, $\mathcal{B} = \{e_1, e_2, e_3\}$, and for $\theta \in \mathbb{R}$, $M = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We say that M is the rotation matrix of angle θ about the vector e_3.
   a) Determine the eigenvalues of M in $\mathbb{C}$, as well as their associated eigenspaces.
   b) Show that for any rotation r of $\mathbb{R}^3$, there exists a basis $\mathcal{B}_r$ and $\theta \in \mathbb{R}$, such that $M_{\mathcal{B}_r}(r) = M$.

Orientation of a vector space

Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases of an n-dimensional vector space V. From the construction of the determinant we know that $\det_{\mathcal{B}} B' = -\det_{\mathcal{B}'} B$. For $\mathcal{B}' = \{e_1, \cdots, e_n\}$, we consider $\mathcal{B}'' = \{-e_1, \cdots, e_n\}$, and note that for any basis $\mathcal{B}$,

$$\det_{\mathcal{B}'} \mathcal{B} = \det_{\mathcal{B}'} \mathcal{B}'' \det_{\mathcal{B}''} \mathcal{B} = -\det_{\mathcal{B}''} \mathcal{B}.$$

Hence, we can define two sets of bases: $\mathcal{O}' = \{\mathcal{B} : \det_{\mathcal{B}'} \mathcal{B} > 0\}$ and $\mathcal{O}'' = \{\mathcal{B} : \det_{\mathcal{B}''} \mathcal{B} > 0\} = \{\mathcal{B} : \det_{\mathcal{B}'} \mathcal{B} < 0\}$. Since $\mathcal{O}' \cup \mathcal{O}''$ contain all the bases of V and are disjoint, we can define the *orientation* of V with respect to them. It suffices to arbitrarily select a basis $\mathcal{B}_0$, to define an orientation. Then any basis $\mathcal{B}$, such that $\det_{\mathcal{B}_0} B > 0$, is said to be *positively oriented*. Otherwise it is said to be *negatively oriented*.

Ex. 2 — *Triangularization*

A matrix $A \in \mathcal{M}_n(\mathbb{K})$ is *triangularizable* if and only if there exists a matrix B, similar to A, which is triangular.

1. Show that if A is triangularizable, then χ_A splits on $\mathbb{K}$.

2. Use induction to show that if χ_A splits on $\mathbb{K}$, then A is triangularizable.

   Hints.

   - Use an appropriate change of basis to write $A = \begin{pmatrix} \lambda_1 & * \\ 0 & A_2 \end{pmatrix}$, where λ_1 is an eigenvalue and $A_2 \in \mathcal{M}_n(\mathbb{K})$.
   - Observe that any polynomial dividing a polynomial which splits on $\mathbb{K}$ must also split on $\mathbb{K}$.

3. Show that any endomorphism of $\mathbb{C}$ is triangularizable.

4. If possible, triangularize $A = \begin{pmatrix} -2 & 2 & -1 \\ -1 & 1 & -1 \\ -1 & 2 & -2 \end{pmatrix}$ in $\mathcal{M}_3(\mathbb{R})$.

Ex. 3 — *Proof of Cayley-Hamilton theorem*

In some textbooks Cayley-Hamilton theorem (5.46|5.292) is proven as follows.

> Taking the characteristic polynomial $\chi_A(\lambda) = \det(A - \lambda\, \mathrm{I}_n)$, set $\lambda = A$, to obtain
>
> $$\chi(A) = \det(A - A\, \mathrm{I}_n) = \det(A - A) = 0.$$

Note that based on remark (5.48|5.294) the theorem can be equivalently proven in term of matrix or endomorphism. Discuss the correctness of this alternative proof.

Ex. 4 — *Characteristic polynomial*

Let $n \in \mathbb{N}^*$, $A \in \mathcal{M}_n(\mathbb{C})$, and $P \in \mathbb{C}[X]$.

1. Show that $P(A) \in \mathrm{GL}_n(\mathbb{C})$ if and only if $\gcd(P, \chi_A) = 1$.
2. Checking your understanding of the exercise.
   a) How important or useful do you think this result is?
   b) Describe a setup, i.e. a question or an exercise, where you could use that result.

* **Ex. 5** — *Challenging problem*

Let $A \in \mathcal{M}_n(\mathbb{R})$. Solve the equation $X + X^\top = \mathrm{tr}(X)A$, where $X \in \mathcal{M}_n(\mathbb{R})$ is the unknown.

Hints.

- Apply the trace to both sides of the equation.
- Independently consider the cases where $\mathrm{tr}\, X$ is equal or different from 0.