MATH2140-Homework4

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1 Ex.1

1 $M_1 \times M_2 = \begin{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 10 & 1 \end{pmatrix} & \begin{pmatrix} 4 & 5 & 6 \\ 2 & -3 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 8 & -3 \\ 1 & 1 & 6 \end{pmatrix} & \begin{pmatrix} -4 & -3 & 4 \\ -3 & -9 & 6 \end{pmatrix} \end{pmatrix} \times \begin{pmatrix} \begin{pmatrix} 4 & 3 \\ 8 & -3 \\ 7 & 8 \end{pmatrix} & \begin{pmatrix} 2 & 1 \\ -2 & 5 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 2 & 7 \\ 12 & 23 \\ 5 & 7 \end{pmatrix} & \begin{pmatrix} -2 & 9 \\ -2 & -1 \\ -9 & 5 \end{pmatrix} \end{pmatrix}$

$=: \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \times \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{pmatrix}$

And:

$$A_1B_1 + A_2B_3 = \begin{pmatrix} 41 & 21 \\ 103 & -10 \end{pmatrix} + \begin{pmatrix} 98 & 185 \\ -37 & -162 \end{pmatrix} = \begin{pmatrix} 129 & 206 \\ 66 & -72 \end{pmatrix};$$
$$A_1B_2 + A_2B_4 = \begin{pmatrix} 1 & 11 \\ -11 & 54 \end{pmatrix} + \begin{pmatrix} -72 & 61 \\ 11 & 16 \end{pmatrix} = \begin{pmatrix} -71 & 72 \\ 0 & 70 \end{pmatrix};$$
$$A_3B_1 + A_4B_3 = \begin{pmatrix} 47 & -45 \\ 54 & 48 \end{pmatrix} + \begin{pmatrix} -24 & -69 \\ -84 & -186 \end{pmatrix} = \begin{pmatrix} 23 & -114 \\ -30 & -138 \end{pmatrix};$$
$$A_3B_2 + A_4B_4 = \begin{pmatrix} -17 & 41 \\ 6 & 6 \end{pmatrix} + \begin{pmatrix} -22 & -13 \\ -30 & 12 \end{pmatrix} = \begin{pmatrix} -39 & 28 \\ -24 & 18 \end{pmatrix};$$

So we have: $M_1 \times M_2 = \begin{pmatrix} 129 & 206 & -71 & 72 \\ 66 & -72 & 0 & 70 \\ 23 & -114 & -39 & 28 \\ -30 & -138 & -24 & 18 \end{pmatrix}$

2

1 $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix} \times \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = I_3$. We have $\begin{cases} a_1 + 2b_1 + 3c_1 = 1 \\ 6a_1 + 5b_1 + 4c_1 = 0 \\ 7a_1 + 9b_1 + 8c_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = \frac{4}{21} \\ b_1 = -\frac{20}{21} \\ c_1 = \frac{19}{21} \end{cases}$

$\begin{cases} a_2 + 2b_2 + 3c_2 = 0 \\ 6a_2 + 5b_2 + 4c_2 = 1 \\ 7a_2 + 9b_2 + 8c_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = \frac{11}{21} \\ b_2 = -\frac{13}{21} \\ c_2 = \frac{5}{21} \end{cases}$ $\begin{cases} a_3 + 2b_3 + 3c_3 = 0 \\ 6a_3 + 5b_3 + 4c_3 = 0 \\ 7a_3 + 9b_3 + 8c_3 = 1 \end{cases} \Rightarrow \begin{cases} a_3 = -\frac{1}{3} \\ b_3 = \frac{2}{3} \\ c_3 = -\frac{1}{3} \end{cases}$

So its inverse is $\begin{pmatrix} \frac{4}{21} & \frac{11}{21} & -\frac{1}{3} \\ -\frac{20}{21} & -\frac{13}{21} & \frac{2}{3} \\ \frac{19}{21} & \frac{5}{21} & -\frac{1}{3} \end{pmatrix}$

2 $\begin{pmatrix} -1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 9 & -8 \end{pmatrix} \times \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = I_3$. We have $\begin{cases} -a_1 + 2b_1 + 3c_1 = 1 \\ 6a_1 - 5b_1 + 4c_1 = 0 \\ 7a_1 + 9b_1 - 8c_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = \frac{4}{415} \\ b_1 = \frac{76}{415} \\ c_1 = \frac{89}{415} \end{cases}$

$\begin{cases} -a_2 + 2b_2 + 3c_2 = 0 \\ 6a_2 - 5b_2 + 4c_2 = 1 \\ 7a_2 + 9b_2 - 8c_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = \frac{43}{415} \\ b_2 = -\frac{13}{415} \\ c_2 = \frac{23}{415} \end{cases}$ $\begin{cases} -a_3 + 2b_3 + 3c_3 = 0 \\ 6a_3 - 5b_3 + 4c_3 = 0 \\ 7a_3 + 9b_3 - 8c_3 = 1 \end{cases} \Rightarrow \begin{cases} a_3 = \frac{23}{415} \\ b_3 = \frac{22}{415} \\ c_3 = -\frac{7}{415} \end{cases}$

So its inverse is $\begin{pmatrix} \frac{4}{415} & \frac{43}{415} & \frac{23}{415} \\ \frac{76}{415} & -\frac{13}{415} & \frac{22}{415} \\ \frac{89}{415} & \frac{23}{415} & -\frac{7}{415} \end{pmatrix}$

3 $\begin{pmatrix} -1 & 2 & -3 \\ 6 & -5 & 4 \\ -7 & 9 & -8 \end{pmatrix} \times \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = I_3$. We have $\begin{cases} -a_1 + 2b_1 - 3c_1 = 1 \\ 6a_1 - 5b_1 + 4c_1 = 0 \\ -7a_1 + 9b_1 - 8c_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{4}{21} \\ b_1 = -\frac{20}{21} \\ c_1 = -\frac{19}{21} \end{cases}$

$\begin{cases} -a_2 + 2b_2 - 3c_2 = 0 \\ 6a_2 - 5b_2 + 4c_2 = 1 \\ -7a_2 + 9b_2 - 8c_2 = 0 \end{cases} \Rightarrow \begin{cases} a_2 = \frac{11}{21} \\ b_2 = \frac{13}{21} \\ c_2 = \frac{5}{21} \end{cases}$ $\begin{cases} -a_3 + 2b_3 - 3c_3 = 0 \\ 6a_3 - 5b_3 + 4c_3 = 0 \\ -7a_3 + 9b_3 - 8c_3 = 1 \end{cases} \Rightarrow \begin{cases} a_3 = \frac{1}{3} \\ b_3 = \frac{2}{3} \\ c_3 = \frac{1}{3} \end{cases}$

So its inverse is $\begin{pmatrix} -\frac{4}{21} & \frac{11}{21} & \frac{1}{3} \\ -\frac{20}{21} & \frac{13}{21} & \frac{2}{3} \\ -\frac{19}{21} & \frac{5}{21} & \frac{1}{3} \end{pmatrix}$

2 Ex.2

1 $\forall \varphi \in (V_1 + V_2)^\perp$, $\forall v \in V_1 + V_2$, $\varphi(v) = 0$. So we have $\forall v \in V_1$, $\varphi(v) = 0; \forall v \in V_2$, $\varphi(v) = 0$. So $\varphi \in V_1^\perp \cap V_2^\perp$. $\forall \varphi \in V_1^\perp \cap V_2^\perp$, we have $\forall v \in V_1$, $u \in V_2$, $\varphi(v) = \varphi(u) = 0$. $\forall m \in V_1 + V_2$, $\exists v \in V_1$, $u \in V_2$ s.t. $\varphi(m) = \varphi(v+u) = \varphi(v) + \varphi(u) = 0$. So $\varphi \in (V_1 + V_2)^\perp$. So $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$.

$\forall \varphi \in V_1^\perp + V_2^\perp$, $\varphi = \varphi_1 + \varphi_2$ s.t. $\varphi_1 \in V_1^\perp, \varphi_2 \in V_2^\perp$. $\forall v \in V_1 \cap V_2$, $\varphi(v) = \varphi_1(v) + \varphi_2(v) = 0$, so $\varphi \in (V_1 \cap V_2)^\perp$, $V_1^\perp + V_2^\perp \subset (V_1 \cap V_2)^\perp$
dim $(V_1 \cap V_2)^\perp$=dim V-dim $(V_1 \cap V_2)$=dim V -dim V_1 -dim V_2 +dim $(V_1 + V_2)$.
dim $V_1^\perp + V_2^\perp$=-dim $(V_1^\perp \cap V_2^\perp)$ + dim $V_1^\perp$ + dim $V_2^\perp$ = -dim $(V_1 + V_2)^\perp$ + dim $V_1^\perp$ + dim $V_2^\perp$ =dim V -dim V_1 -dim V_2 +dim $(V_1 + V_2)$=dim $(V_1 \cap V_2)^\perp$.

So $(V_1 \cap V_2)^\perp$=$V_1^\perp + V_2^\perp$

2 Let $\mathcal{B}_V = \mathcal{B}_1 \cup \mathcal{B}_2 = \{b_{1_i}\} \cup \{b_{2_i}\}, \mathcal{B}_V^* = \{f_i\}$, $\mathcal{B}_1^* = \{f_{1_i}\}$, $\mathcal{B}_2^* = \{f_{2_i}\}$. $\forall \varphi \in V^*, \varphi = \Sigma \varphi(b_i) f_i \in \mathcal{B}_1^* + \mathcal{B}_2^*$. $\mathcal{B}_V^* = \mathcal{B}_1^* + \mathcal{B}_2^*$. So $V^* = V_1^\perp + V_2^\perp$ dim V=dim V_1+dim V_2.dim V^*= dim $(V_1 + V_2)$=dim $V_1^\perp$+dim $V_2^\perp$. So dim $V_1^\perp \cap V_2^\perp$=0. $V_1^\perp \cap V_2^\perp$=$\mathbf{0}$. So $V^* = V_1^\perp \oplus V_2^\perp$.

3 Ex.3

MM^T is a $n \times n$ matrix. Let $a_{i,j} \in M$, $b_{i,j} \in M^T$, $c_{i,j} \in MM^T$, we have $c_{p,q} = \Sigma_{i=1}^m a_{p,i} b_{i.q} = \Sigma_{i=1}^m a_{p,i} a_{q.i}$; $c_{q,p} = \Sigma_{i=1}^m a_{q,i} b_{i.p} = \Sigma_{i=1}^m a_{p,i} a_{q.i} = c_{p,q}$. So MM^T is symmetric. Similarly, if we regard M^T as M, then M is M^T. So $M^T M$ is symmetric.

4 Ex.4

1

(a) If b is a bilinear form, then it's linear form if both v and w are linear when the other is fixed. If w is fixed ,we have $\begin{pmatrix} b_{ij} \end{pmatrix} \begin{pmatrix} w_i \end{pmatrix} = \begin{pmatrix} c_{ij} \end{pmatrix}$ s.t. $\begin{pmatrix} c_{ij} \end{pmatrix}$ is a n×1 matrix. Since c(v) is linear, it is valid to express b(v,w)=$\begin{pmatrix} v_{ij} \end{pmatrix} \begin{pmatrix} c_i \end{pmatrix}$. Similarly, w can be regarded as form above. So the b can be expressed as such multiplication.

(b) $\Rightarrow$: we have $b_{i,j} = b_{j,i}$. Let's regard the conclusion b(v,w) as a 1×1 matrix. $b(v,w) = b(v,w)^T$=$w^T B^T v^T$=$w^T B v^T$=b(w,v). So it's symmetric.
$\Leftarrow$: b is symmetric, so $\forall v, w$, $b(v,w) = b(w,v)$. $\forall i, j \in [\![1, n]\!]$. If $v_m = \begin{cases} 1 & m = i \\ 0 & m \neq i \end{cases}$; $v_n = \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases}$, then $b(v,w) = v_i b_{i,j} w_j = b_{i,j}$, and $b(w,v) = w_j b_{j,i} v_i = b_{j,i}$. So $b_{i,j} = b_{j,i}$. So it is symmetric.

2

(a) $< v, w >= \Sigma_{i=1}^n v_i w_i$; $< w, v >= \Sigma_{i=1}^n w_i v_i$. So $< v, w >=< w, v >$, and the dimension is V, which is positive finite. So V is a n-dimensional Euclidean space.

(b) $||v|| = \sqrt{\Sigma_{i=1}^n v_i^2}$. If $||v|| = 0$, $\Sigma_{i=1}^n v_i^2 = 0$, $\forall i, v_i = 0$. So v=0.
$||av|| = \sqrt{\Sigma_{i=1}^n (av_i)^2} = \sqrt{a^2 \Sigma_{i=1}^n v_i^2} = a\sqrt{\Sigma_{i=1}^n v_i^2} = a||v||$.
$||v + w||^2 = \Sigma_{i=1}^n (v_i + w_i)^2$; $(||v|| + ||w||)^2 = \Sigma_{i=1}^n v_i^2 + \Sigma_{i=1}^n w_i^2 + 2\sqrt{(\Sigma_{i=1}^n v_i^2) \times \Sigma_{i=1}^n w_i^2}$.
$||v|| + ||w|| - ||v + w|| \geq 0 \Leftrightarrow \sqrt{\Sigma_{i=1}^n v_i^2 \Sigma_{i=1}^n w_i^2} \geq \Sigma_{i=1}^n v_i w_i \Leftrightarrow \Sigma_{i=1}^n v_i^2 \Sigma_{i=1}^n w_i^2 - (\Sigma_{i=1}^n v_i w_i)^2 \geq 0 \Leftrightarrow \Sigma_{i,j=1, i \neq j}^{i,j \leq n} v_i^2 w_j^2 - 2\Sigma_{i,j=1, i \neq j}^{i,j \leq n} v_i w_i v_j w_j = \Sigma_{i,j=1, i \neq j}^{i,j \leq n} (v_i w_j - v_j w_i)^2 \geq 0$, which is trivial.
So it is a normed vector space

(c) Unfortunately, I have proved it in (b).

3

(a) If it is linearly dependent, we can have $u_1 = \Sigma_{i=2}^{n}\lambda_i u_i$. Then we have $< u_1, u_1 >= 1$.(take u_1 as an example). As it is a bilinear form, $< u_1, \Sigma_{i=2}^{n}\lambda_i u_i >= \Sigma_{i=2}^{n}\lambda_i < u_1, u_i >= 0$, which is contradictive. So it is linearly independent.

(b) If dim V=1, we have $\mathcal{B} = \{v_1\}$. Then we have $\mathcal{B}' = \{\frac{v_1}{|v_1|}\}$ s.t. $|\frac{v_1}{|v_1|}| = 1$. So it is a unit vector.
$\forall\mathcal{B}'$, card $\mathcal{B}'$=dim V=n-1. $\forall V_0 \supseteq V$ s.t. dim V_0=n, we can define S s.t. $S \oplus V = V_0$, then dim S=1, whose basis is $\{v_n\}$ As $S \cap V = \{0\}$, v_n cannot be expressed as linear combination in V. So we can get the orthonormal basis of V_0, $\mathcal{B}_{\prime}{}'$=$\mathcal{B}' \cup \{\frac{v_n}{|v_n|}\}$
So by induction we can prove the theorem.

5 Ex.5