MATH214
Linear algebra

Homework 5

Manuel — UM-JI (Spring 2023)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Vandermonde*

We call Vandermonde determinant, the determinant

$$V(x_0, \cdots, x_n) = \begin{vmatrix} 1 & 1 & \cdots & \cdots & 1 \\ x_0 & x_1 & \cdots & \cdots & x_n \\ x_0^2 & x_1^2 & \cdots & \cdots & x_n^2 \\ \vdots & \vdots & & & \vdots \\ x_0^n & x_1^n & \cdots & \cdots & x_n^n \end{vmatrix}.$$

1. Prove by induction that $V(x_0, \cdots, x_n) = \prod_{0 \leq j < i \leq n} (x_i - x_j)$.
2. Discuss the conditions under which it is equal to zero.

Ex. 2 — *Transposes*

Let $\tau = (i, j)$ and $\tau' = (k, l)$ be two transposes of $\mathcal{S}_n$. Prove that either $\tau\tau' = \mathsf{Id}$, or $(\tau\tau')^2 = \mathsf{Id}$, or $(\tau\tau')^3 = \mathsf{Id}$.

Hint. Consider the possible cases for the intersection of the two sets $\{i, j\}$ and $\{k, l\}$.

Ex. 3 — *Determinant of a comatrix*

Let $n \in \mathbb{N} \setminus \{0, 1\}$ and $A \in \mathcal{M}_n(\mathbb{K})$. Show that $\det(\operatorname{com} A) = (\det A)^{n-1}$.

Hint. Separately consider whether or not A is invertible.

Ex. 4 — *QR factorization*

Let V be a finite n-dimensional $\mathbb{R}$-vector space. We say that a matrix M is orthogonal if its columns form an orthonormal basis for V.[1] Let $Q \in \mathcal{M}_n(\mathbb{R})$ be an orthogonal matrix.

1. Orthogonal matrix.
   a) Show that $Q^\top \cdot Q = \mathsf{I}_n$.
   b) Show that $\det Q = \pm 1$.
2. Prove that any $M \in \mathsf{GL}_n(\mathbb{R})$ can be written as the product of an orthogonal matrix with an upper

[1]Note the discrepancy in the language as the matrix is said to be **orthogonal** when its vectors are **orthonormal**.

triangular one, i.e. prove the existence of an orthogonal matrix Q and of an upper triangular matrix R such that $M = QR$. This is called the QR decomposition of M.[2]

Hint. Think of the columns of M as vectors from V and follow Gram-Schmidt procedure (homework 4, exercise 4) to build an orthonormal basis.

* 3. Find the QR decomposition of
$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix}.$$

4. Using the QR decomposition solve the linear system S defined by
$$\begin{pmatrix} 2 & 1 & 3 & 3 \\ 2 & 1 & -1 & 1 \\ 2 & -1 & 3 & -3 \\ 2 & -1 & -1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}.$$

Ex. 5 — *Solving incompatible systems*

In the lectures we restricted our attention to compatible linear systems of equations. However when a system has no solution one might still be interested in finding what could be viewed as the closest solution. In this exercise we will use the QR decomposition (exercise 4) in order to discover the closest solution to an incompatible system.

Let $A \in \mathrm{GL}_p(\mathbb{R})$, with $p < n$. We know that the vector Ax is always in the subspace $W \subset V$ spanned by the columns of A, and if S is compatible then b also belongs to W. However if S is incompatible, then we can project b on a vector $b' \in W$. To obtain an approximation $\widetilde{x}$ of x it then suffices to solve $A\widetilde{x} = b'$, which by construction is compatible. We call $\widetilde{x}$ a *least square solution* of the system.

1. Properties of Q and R when $p < n$.
   a) What are the dimensions of Q and R?[3]
   b) What can be said about $Q^\top Qv$, for $v \in \mathcal{M}_{1,p}$ and $QQ^\top v$, for $v \in \mathcal{M}_{1,n}$?
2. Let $b \in V$. Show that b', the orthogonal projection of b on W, is the closest vector of b in W, i.e. for any $w \in W$, $\|b - b'\| \leq \|b - w\|$.

   Hint. First prove that for any $w \in W$, $\|b - w\|^2 \geq \|b - b'\|^2$.
3. Taking advantage of the QR decomposition of A, show that $R\widetilde{x} = Q^\top b$.

* **Ex. 6** — *Challenging problem*

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[2] Although in this question we restricted our attention to invertible matrices, any matrix has a QR decomposition. If M is not invertible, then neither is R.

[3] A matrix Q with orthonormal column (row) vectors is sometimes said to be *column-orthogonal* (*row-orthogonal*), or *semi-orthogonal*. Notice that if Q is column-orthogonal (row-orthogonal) then we will necessarily have $p \leq n$ ($n \leq p$).

Let F and G be two non-constant polynomials in $\mathbb{C}[X]$ with degrees n and m, respectively, and

$$\begin{aligned} \Phi : \mathbb{C}_{m-1}[X] \times \mathbb{C}_{n-1}[X] &\longrightarrow \mathbb{C}_{n+m-1}[X] \\ (U, V) &\longmapsto UF + VG. \end{aligned}$$

1. Properties of Φ.
   a) Prove that Φ is well defined and linear.
   b) Using Bezout's identity, prove that Φ is injective if and only if F and G are coprime.
   c) Write the matrix M of Φ is the canonical bases.

The determinant of M is called the *resultant* of F and G. It is denoted $\mathrm{Res}(F, G)$.

2. Let $\Gamma = \{(F(t), G(t)) \in \mathbb{C}^2, t \in \mathbb{C}\}$. Show the existence of $R \in \mathbb{C}[X, Y]$, such that for all $(x, y) \in \mathbb{C}^2$, $(x, y) \in \Gamma$, if and only if $R(x, y) = 0$.

Comments on the resultant

Originally introduced in the 19th century to solve systems of polynomial equations, the resultant was "forgotten" as hard to use and calculate in practice. Only recently, over the past few decades, was it reintroduced in various fields of mathematics as it can now be calculated with the help of computers. It has in fact become a very important tool in algebra, algebraic geometry, and number theory. For instance it can used to define a norm for elements in extension fields.