MATH2140-Homework1

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1 Ex.1

1

$$suggest\ the\ homomorphisms\ are\ f : (S, *) \rightarrow (S', *')\ and\ g : (S', *') \rightarrow (S'', *'').$$
$$Then\ g \circ f(x * y) = g(f(x) *' f(y)) = (g \circ f)(x) *'' (g \circ f)(y)$$
$$Thus\ g \circ f,\ as\ a\ composition\ of\ g\ and\ f,\ is\ the\ homomorphism$$

2

$$For\ an\ isomorphism,\ it\ is\ both\ surjective\ and\ injective$$
$$As\ a\ value\ in\ the\ domain\ must\ corresponds\ to\ one\ in\ the\ codomain,$$
$$the\ inverse\ is\ surjective.$$
$$Also,\ the\ injection\ also\ works\ in\ the\ inverse\ function.$$
$$Therefore\ it\ is\ bijective\ and\ thus\ an\ isomorphism.$$

3

a isomorphism: The function is increasing function so it is injective; its range is just R, so it is surjective.

b endomorphism: The function is a set of R to itself, so it is endomorphism. However, the range is just f(x)>0 but not R, thus it is not automorphism.

c automorphism: The function is increasing function so injective, has a range of R so surjective, and has a set of R to itself. Combining all above it is automorphism.

d $endomorphism(x \neq 1) or\ automorphism(x = 1)$
When x=1, it is quite similar to (c) and shares all properties above so it is automorphism. Or else it is not surjective because some of nx will not have corresponding n.

$$e.g.\ \ for\ nx = 1 \in \mathbb{Z}\ and\ x = 2 \in \mathbb{Z},\ n = 0.5 \notin \mathbb{Z}$$

e homomorphism

$$We\ have\ x^{m+n} = x^m \times x^n,\ but\ it's\ not\ surjective\ and\ the\ set\ is\ different$$

2 Ex.2

2.a

$$g(x) = \mathbf{0}\ in\ the\ function\ space\ is\ the\ unit\ element\ of\ addition.$$

$$Thus\ we\ have\ f(x) + g(x) = f(x).\ Therefore\ the\ \mathbf{0}\ vector\ is\ g(x) = 0.$$

2.b

$$i,\ ii\ and\ iv\ are\ vectors,\ but\ iii\ is\ not.$$

$$because\ the\ domain\ of\ iii\ is\ not\ \mathbb{R}, but\ (0, +\infty).$$

3

$$For\ any\ \alpha, \beta \in \mathbb{K}\ and\ f(x), g(x)\ differentiable\ in\ 0,$$

$$\alpha f(x) + \beta g(x)\ is\ differentiable\ in\ 0$$

$$Thus\ \alpha f(x) + \beta g(x)\ is\ differentiable\ in\ 0.\ And\ it\ is\ an\ vector\ subspace.$$

4

$$Let's\ suggest\ u = e^x \in F(R, R),\ and\ v = e^{-x} \in F(R, R).$$

$$Then\ u + v = e^x + e^{-x},\ which\ declines\ on\ (-\infty, 0)\ and\ increases\ on\ (0, +\infty).$$

$$(+)\ isn't\ an\ internal\ composition\ law.\ (F(R, R), +)\ isn't\ an\ abelian\ group.$$

$$Therefore,\ it\ is\ not\ a\ subspace.$$

5

$$Let's\ suggest\ f(u) = f(v) = 1.\ Then\ f(u) + f(v) = 2 \notin \{0, 1\}$$

$$(+)\ isn't\ an\ internal\ composition\ law.\ (F(R, R), +)\ isn't\ an\ abelian\ group.$$

$$Therefore,\ it\ is\ not\ a\ subspace.$$

6

$$The\ solution\ to\ the\ differential\ equation\ is\ f(x) = C_1 e^{-3x} + C_2 e^x.$$

$$The\ function\ space,\ V_{f(x)},\ is\ the\ space\ that\ all\ f(x)\ constitude.$$

$$For\ any\ \alpha, \beta \in \mathbb{K}\ and\ m(x_1, y_1), n(x_2, y_2) \in V_{f(x)},$$

$$\alpha m + \beta n = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2), which\ corresponds\ to\ the\ form\ of\ f(x)$$

$$Thus\ \alpha m + \beta n \in V_{f(x)}.\ And\ it\ is\ an\ vector\ subspace.$$

3 Ex.3

1

$$For\ any\ \alpha, \beta \in \mathbb{K}\ and\ m(x_1, y_1, z_1), n(x_2, y_2, z_2) \in V$$
$$we\ have\ \alpha m + \beta n = (x_0, y_0, z_0) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2)$$
$$x_0 - 2y_0 + 3z_0 = \alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) = 0$$
$$Thus,\ V\ is\ a\ subspace\ of\ \mathbb{R}^3$$

2 Vector space is a subset of set. All vector spaces are sets, but there are some additional requirements.

3

$$To\ prove\ B = C,\ we\ only\ need\ to\ prove\ C \subset B\ knowing\ that\ B \subset C.$$
$$As\ A + B = A + C, \forall \mathbf{c} \in C\ and\ \mathbf{a}_1 \in A, \exists \mathbf{a}_2 \in A\ and\ \mathbf{b} \in B,\ \mathbf{a}_2 + \mathbf{b} = \mathbf{a}_1 + \mathbf{c}.$$
$$Therefore,\ \mathbf{b} - \mathbf{c} = \mathbf{a}_1 - \mathbf{a}_2 \in A, and\ \mathbf{b} \in B \subset C,\ \mathbf{b} - \mathbf{c} \in A \cap C = A \cap B.$$
$$\mathbf{b} \in B,\ so\ \mathbf{c} \in B,\ and\ C \subset B,\ so\ B = C.$$

4 Firstly let's try to prove $A \cap (B + (A \cap C)) \subset (A \cap B) + (A \cap C)$
$\forall \mathbf{v} \in A \cap (B + (A \cap C)), \mathbf{v} \in A$ and $\mathbf{v} \in B + (A \cap C)$.
Let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in B$ and $\mathbf{v}_2 \in A \cap C$.
$\mathbf{v} \in A$, and$\mathbf{v}_2 \in A$, so $\mathbf{v}_1 \in A$, so $\mathbf{v}_1 \in A \cap B$, and$\mathbf{v} \in (A \cap B) + (A \cap C)$.

Then let's try to prove $A \cap (B + (A \cap C)) \supset (A \cap B) + (A \cap C)$
Let $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 \in A \cap (B + (A \cap C))$, where $\mathbf{v}_1 \in B$ and $\mathbf{v}_2 \in A \cap C$, and as $\mathbf{v}, \mathbf{v}_2 \in A, \mathbf{v}_1 \in A$. Thus $\mathbf{v} \in A \cap (B + (A \cap C)) \Leftrightarrow \mathbf{v} \in A \cap ((A \cap B) + (A \cap C))$,
Obviously, $\forall \mathbf{u}_1 \in A \cap B$ and $\mathbf{u}_2 \in A \cap C, \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 \in A$ and $\in (A \cap B) + (A \cap C)$.
Then $\mathbf{u} \in A \cap ((A \cap B) + (A \cap C)) = A \cap (B + (A \cap C))$.
Thus,$A \cap (B + (A \cap C)) = (A \cap B) + (A \cap C)$

5 Let A be the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,B be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$,and C be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Then B+C=XoY plane,$A \cap (B + C) = A$, but $A \cap B = A \cap C = A \cap B + A \cap C = \{0\}$, so $A \cap (B + C) \neq (A \cap B) + (A \cap C)$
*It seems that it only works when A,B and C are linearly independent.

4 Ex.4

1

(a) $f(ax_1+by_1, ax_2+by_2, ax_3+by_3) = (ax_1+by_1+2ax_2+2by_2, ax_3+by_3) = a(x_1+2x_2, x_3)+b(y_1+2y_2, y_3) = af(x_1,x_2,x_3)+bf(y_1,y_2,y_3)$. So it is linear. For all $\mathbf{v}(x_1,x_2,x_3) \in kerf$, $\begin{cases} x_1+2x_2=0 \\ x_3=0 \end{cases}$, So it is a vector space spanned by vector (-2,1,0).

(b) $f(aP+bQ) = (aP+bQ)' = aP'+bQ' = af(P)+bf(Q)$, so it is linear. When$P'=0$, we have $P=C\in\mathbb{K}$. So the kernel is $\mathbb{K}$.

2 a $im\ f \cap ker\ f = \{\mathbf{0}\} \Rightarrow ker\ f = ker\ f^2$
$\forall \mathbf{v} \in ker\ f, f^2(\mathbf{v}) = f(\mathbf{0}) = \mathbf{0}$, therefore $ker\ f^2 \supset ker\ f$.
$\forall \mathbf{u} \in ker\ f^2,\ f(\mathbf{u}) \in ker\ f$, also we have $\mathbf{u}\in V,\ f(u) \in im\ f$.
$f(\mathbf{u}) \in ker\ f \cap im\ f = \mathbf{0},\ f(\mathbf{u}) = \mathbf{0}$. Thus $ker\ f^2 \subset ker\ f$, and $ker\ f = ker\ f^2$.

b $im\ f \cap ker\ f = \{\mathbf{0}\} \Leftarrow ker\ f = ker\ f^2$
$\forall \mathbf{v} \in im\ f \cap ker\ f,\ f(\mathbf{v}) = \mathbf{0}$. As $\mathbf{v} \in im\ f$, let $\mathbf{v}' = f^{-1}(\mathbf{v})$.
$f^2(\mathbf{v}') = 0, \mathbf{v}' \in ker\ f^2 = ker\ f, f(\mathbf{v}') = \mathbf{0}, \mathbf{v} = 0$. Thus $im\ f \cap ker\ f = \{\mathbf{0}\}$.

5 Ex.5

1

2