MATH2140-Homework5

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1 Ex.1

1 When n=1,$V(x_0,x_1)=\begin{vmatrix} 1 & 1 \\ x_0 & x_1 \end{vmatrix}$=$x_1-x_0$=$\prod\limits_{0\le j<i\le 1}(x_i-x_j)$
When n>1, if $V(x_0...x_{n-1})=\prod\limits_{0\le j<i\le n-1}(x_i-x_j)$.
$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ x_0^n & x_1^n & \cdots & x_n^n \end{vmatrix}=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_0-x_0 & x_1-x_0 & \cdots & x_n-x_0 \\ \vdots & \vdots & & \vdots \\ x_0^n-x_0^n & x_1^n-x_1^{n-1}x_0 & \cdots & x_n^n-x_n^{n-1}x_0 \end{vmatrix}.=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & x_1^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}\cdot\prod\limits_{0\le j\le n}(x_j-x_n)$
$=\prod\limits_{0\le j<i\le 1}(x_i-x_j)$

2 If any two elements among x_i is the same then it equals to 0.

2 Ex.2

As i and j is commutative, we can just consider it without its order.

(i,j)=(k,l) $\tau\tau'=ld$

(i,j)=(k,j) $\tau\tau':(x_i,x_j,x_k)\mapsto(x_k,x_i,x_j)$; $(\tau\tau')^2:(x_i,x_j,x_k)\mapsto(x_j,x_k,x_i)$ $(\tau\tau')^3:(x_i,x_j,x_k)\mapsto(x_i,x_j,x_k)$, $(\tau\tau')^3=ld$

i,j≠k,l $\tau\tau':(x_i,x_j,x_k,x_l)\mapsto(x_k,x_l,x_i,x_j)$; $(\tau\tau')^2:(x_i,x_j,x_k,x_l)\mapsto(x_i,x_j,x_k,x_l)$,$(\tau\tau')^2=ld$;
So either $\tau\tau'=ld$ or $(\tau\tau')^2=ld$ or $(\tau\tau')^3=ld$

3 Ex.3

If it is invertible, B=com A, we have $det\ A\neq 0$. $AB^T=(det\ A)I_n$, $det(AB^T)=det((det\ A)I_n)$, $det\ A\ det\ B^T=(det\ A)^n$. $det\ B=det\ B^T=(det\ A)^{n-1}$ If it is not invertible, if A=0, B=0, it is trivial, else $AB^T=0$, $im\ B^T\subset ker\ A$. $rank\ B^T<=dim\ ker\ A<n$. So B^T is not invertible and $det\ B=det\ B^T=0$. So $det\ B=(det\ A)^{n-1}$.

4 Ex.4

1 Let $Q=(q_1,q_2,\cdots,q_n)$, as it is orthogonal, we have $<q_i,q_j>=\begin{cases}1\ i=j\\0\ i\neq j\end{cases}$ While $Q^T\cdot Q=$
$\begin{pmatrix} <q_1,q_1> & <q_1,q_2> & \cdots & <q_1,q_n> \\ <q_2,q_1> & <q_2,q_2> & \cdots & <q_2,q_n> \\ \vdots & \vdots & & \vdots \\ <q_n,q_1> & <q_n,q_2> & \cdots & <q_n,q_n> \end{pmatrix}=I_n$
det($Q^T\cdot Q$)=det(Q^T)det(Q).det(Q^T)=det(Q), So det(Q)=±1

2 Let $Q^T=P=(\varphi_1,\cdots,\varphi_n)$, then we have $PP^T=\begin{pmatrix}\varphi_1 & \cdots & \varphi_n\end{pmatrix}\begin{pmatrix}\varphi_1\\ \vdots \\ \varphi_n\end{pmatrix}=I_n$. We have $P=Q^T$ is also orthogonal. Symmetrically, we can get orthogonal column ⇔ orthogonal row.
As M is linearly independent, the columns $\{m_1,\cdots,m_n\}$ forms a basis. Let $Q=\begin{pmatrix}q_1 & \cdots & q_n\end{pmatrix}$. Firstly we have

$q_1 = \frac{m_1}{||m_1||}$, and $r_{1,1} = ||m_1||$. Then $< q_1, q_1 >= 1$. And we take $q_2 = \frac{m_2 - <q_1,m_2>q_1}{||m_2 - <q_1,m_2>q_1||}$, $r_{1,2} =< q_1, m_2 >$ and $r_{2,2} = ||m_2- < q_1, m_2 > q_1||$. As $< q_1, q_2 >= 0$ and $< q_2, q_2 >= 1$, it is orthogonal. So we can have $q_n = \frac{m_n - \sum\limits_{1 \le i < n} <q_i,m_n>q_i}{||m_n - \sum\limits_{1 \le i < n} <q_i,m_n>q_i||}$, and $r_{i,n}(1 \le i < n) =< q_i, m_n >$, $r_{n,n} = ||m_n - \sum\limits_{1 \le i < n} < q_i, m_n > q_i||$. Such QR is found.

3 Following steps before, we can get $q_1 = \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{pmatrix}$, and $r_{1,1} = \sqrt{5}$. $r_{1,2} =< q_1, m_2 >= \frac{\sqrt{5}}{5}$, so $q_2 = \begin{pmatrix} -\frac{7\sqrt{30}}{90} \\ \frac{7\sqrt{30}}{45} \\ \frac{\sqrt{30}}{18} \end{pmatrix}$, and $r_{2,2} = \frac{3\sqrt{30}}{5}$. $r_{1,3} =< q_1, m_3 >= 0$, $r_{2,3} =< q_2, m_3 >= -\frac{\sqrt{30}}{2}$, $q_3 = \begin{pmatrix} -\frac{\sqrt{6}}{18} \\ \frac{\sqrt{6}}{9} \\ -\frac{7\sqrt{6}}{18} \end{pmatrix}$, $r_{3,3} = \frac{\sqrt{6}}{2}$. So $\begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 0 & 1 & -2 \end{pmatrix} =$
$\begin{pmatrix} \frac{2\sqrt{5}}{5} & -\frac{7\sqrt{30}}{90} & -\frac{\sqrt{6}}{18} \\ \frac{\sqrt{5}}{5} & \frac{7\sqrt{30}}{45} & \frac{\sqrt{6}}{9} \\ 0 & \frac{\sqrt{30}}{18} & -\frac{7\sqrt{6}}{18} \end{pmatrix} \begin{pmatrix} \sqrt{5} & \frac{\sqrt{5}}{5} & 0 \\ 0 & \frac{3\sqrt{30}}{5} & -\frac{\sqrt{30}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{pmatrix}$

4 $\begin{pmatrix} 2 & 1 & 3 & 3 \\ 2 & 1 & -1 & 1 \\ 2 & -1 & 3 & -3 \\ 2 & -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$; $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ So x= $\begin{pmatrix} \frac{5}{8} \\ \frac{1}{2} \\ -\frac{1}{4} \\ 0 \end{pmatrix}$

5 Ex.5

1 Q:n×p; R:p×p
From steps in 4.2, we can get $Q^TQ = I_n$, $Q^TQv = v$. But $QQ^T \neq I_n$, like $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2 Firstly complete Q$(q_1, \cdots, q_p)$ into Q'$(q_1, \cdots, q_n)$, we have $b = \sum\limits_{i=1}^{n} \lambda_i q_i$, $b' = \sum\limits_{i=1}^{p} \lambda_i q_i$. $||b - b'||^2 = \sum\limits_{i=p+1}^{n} \lambda_i^2 q_i^2$. $\forall w \in W, w = \sum\limits_{i=1}^{p} \mu_i q_i$, $||b - w||^2 = ||b - b'||^2 + \sum\limits_{i=p+1}^{n} (\lambda_i - \mu_i)^2 q_i^2$. So $||b - w||^2 \geq ||b - b'||^2$.

3 $A\widetilde{x} = b' \Rightarrow QR\widetilde{x} = b' \Rightarrow Q^TQR\widetilde{x} = Q^Tb' \Rightarrow R\widetilde{x} = Q^Tb'$. Let $b = \sum\limits_{i=1}^{n} \lambda_i q_i, Q^T = \begin{pmatrix} q_1^T \\ \vdots \\ q_p^T \end{pmatrix}, Q^Tb' = \sum\limits_{i=j=1}^{i \le n, j \le p} < \lambda_i q_i, q_j >$. As $< p_m, p_n >= \delta_{ij}, Q^Tb' = \sum\limits_{i=j=1}^{i,j \le p} < \lambda_i q_i, q_j >= Q^Tb$. Therefore $R\widetilde{x} = Q^Tb$.

6 Ex.6