MATH214
Linear Algebra

Homework 1
Manuel — UM-JI (Spring 2023)

Reminders

- Write in a neat and legible handwriting or use $\LaTeX$
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Morphisms*

1. Show that the composition of two homomorphisms is a homomorphism.
2. Show that the inverse of an isomorphism is an isomorphism.
3. Consider the following functions and say whether they are homomorphisms, isomorphisms, endomorphisms, or automorphisms.
   a) The logarithm function from $(\mathbb{R}_+^*, \times)$ into $(\mathbb{R}, +)$.
   b) The exponential function from $(\mathbb{R}, +)$ into $(\mathbb{R}, \times)$.
   c) The identity function.
   d) The function $f_x : \mathbb{Z} \to \mathbb{Z}$, $n \mapsto nx$, with $x \in \mathbb{Z}$, for $(\mathbb{Z}, +)$.
   e) The function $f_x : (\mathbb{N}, +) \to (\mathbb{Z}, \times)$, $n \mapsto x^n$, with $x \in \mathbb{Z}$.

Ex. 2 — *Function space*

In this exercise we want to understand the basics of functional spaces.

* 1. Make sure you fully understand the examples (1.17|1.37).

* 2. Knowing that $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is a vector space,
   a) what is the **0** vector?
   b) which of the following functions are vectors?
      i – x; ii – x^2; iii – $\log x$; iv – $\exp x$;

Which of the following sets are vector subspaces of $(\mathcal{F}(\mathbb{R}, \mathbb{R}), +, \cdot)$.

3. The set of the function differentiable in 0.
4. The set of the monotonic functions on $\mathbb{R}$.
5. The set of the function taking the value 0 or 1.
6. The set of the functions f which are twice differentiable and such that $f'' + 2f' - 3f = 0$.

Ex. 3 — *Vector spaces*

1. Is $V = \{(x, y, z) \in \mathbb{R}^3, x - 2y + 3z = 0\}$ a vector subspace of $\mathbb{R}^3$.

Let V be a $\mathbb{K}$-vector space and A, B, and C be three subspaces of V.

* 2. What is the difference between a set and a vector space?

3. If A, B and C are such that (i) $A \cap B = A \cap C$, (ii) $A + B = A + C$, and (iii) $B \subset C$, then show that $B = C$.

   Hint for students with basics in linear algebra. Can an argument based on the dimension be used?

4. Show that $A \cap (B + (A \cap C)) = (A \cap B) + (A \cap C)$.

5. Show that $A \cap (B + C) \neq (A \cap B) + (A \cap C)$.

Ex. 4 — *Kernel and image*

* 1. We want to understand the concepts of kernel.

   a) Let $f(x_1, x_2, x_3) = (x_1 + 2x_2, x_3)$ be a function from $\mathbb{R}^3$ into $\mathbb{R}^2$. Show that f is a linear map and determine its kernel.

   b) Let $f(P) = P'$ be the function which given a polynomial returns its differential. Show that f is a linear map an determine its kernel.

2. Let V be a $\mathbb{K}$-vector spaces and $f \in \mathcal{L}(V, V)$. Show that $\ker f^2 = \ker f$ if and only if $\operatorname{im} f \cap \ker f = \{0\}$.

* **Ex. 5** — *Challenging problem*

Let V be a finite dimensional $\mathbb{K}$-vector space, where $\mathbb{K}$ is a field with infinitely many elements.

1. Show that it is not possible to have $V = \bigcup_{i=1}^{n} W_i$, where the W_i are strict vector subspaces of V. Does it change anything if $\mathbb{K}$ is finite?

2. Let W_i, $1 \leq i \leq n$, be n subspaces of V, such that they all have the same finite dimension. Show that there exists G such that for all $1 \leq i \leq n$, $G \oplus W_i = V$.