MATH2140-Homework2

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1 Ex.1

1 To check whether it is linearly independent, we have$\mathbf{a}a + \mathbf{b}b + \mathbf{c}c = \mathbf{d}d$

$$\begin{cases} a+2b+c=d \\ 2a+3b+3c=2d \\ -a-c=d \\ -2a-b=4d \end{cases} \Rightarrow \begin{cases} 2a+2b+2c=0 \\ 4a+3b+5c=0 \\ 2a-b+4c=0 \end{cases} \Rightarrow \begin{cases} c-b=0 \\ 2c-3b=0 \end{cases} \Rightarrow a=b=$$

$c=d=0$

Therefore it is injective. As a vector space spanned by $\mathcal{B}$,it is also surjective. So it is bijective and $\mathcal{B}$ is a basis.

2 We have:

$$\begin{cases} a+2b+c+d=7 \\ 2a+3b+3c+2d=14 \\ -a-c+d=-1 \\ -2a-b+4d=2 \end{cases} \Rightarrow \begin{cases} 2b+2d=6 \\ 3b+c+4d=12 \\ -b+2c+2d=4 \end{cases} \Rightarrow \begin{cases} 4c+6d=14 \\ 7c+10d=24 \end{cases} \Rightarrow$$

$$\begin{cases} c=2 \\ d=1 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=2 \\ c=2 \\ d=1 \end{cases}$$

So the cardinal is (0,2,2,1).

2 Ex.2

1 To check whether it is linearly independent, we have $a\mathbf{a} + b\mathbf{b} = c\mathbf{c}$

$$\begin{cases} a+b=c\alpha \\ a+b\alpha=c \\ a\alpha+b=c \end{cases} \Rightarrow \begin{cases} a-a\alpha+b-b\alpha^2=0 \\ a-a\alpha^2+b-b\alpha=0 \end{cases} \Rightarrow \alpha(\alpha-1)(b-a)=0$$

$$\begin{cases} \alpha=0, \begin{cases} a=b=c \\ a+b=0 \end{cases}, \ a=b=c=0, \ linearly\ independent. \\ \alpha=1, \forall a+b=c\ is\ valid,\ it\ is\ not\ linearly\ independent. \\ b=a, \begin{cases} 2b=c\alpha \\ (1+\alpha)b=c \end{cases} \Rightarrow (\alpha^2+\alpha-2)b=0.\ If\ b\neq 0,\ \alpha=1\ or\ -2. \\ In\ other\ condition, we\ only\ have\ a=b=c=0,\ so\ it\ is\ linearly\ independent. \end{cases}$$

Concludingly, it is linearly dependent when $\alpha = 1$ or $\alpha = -2$. Else it is independent.

2 We have $\begin{cases} 0+0=0 \\ 0+1=1 \\ 1+0=1 \\ 1+1=0 \end{cases}$ and $\begin{cases} 0\cdot 0=0 \\ 0\cdot 1=0 \\ 1\cdot 0=0 \\ 1\cdot 1=1 \end{cases}$.

1.Thus ($\mathbb{Z}/2\mathbb{Z}$,+) is commutative. a+(b+c)=(a+b+c)(mod 2)=(a+b)+c, so it is associative. The unit element is 0 and the inverse is itself. Thus ($\mathbb{Z}/2\mathbb{Z}$,+) is a abelian group.
2.($\mathbb{Z}/2\mathbb{Z}\backslash\{0\}$,·) only have an element 1.So it is obviously commutative and associative. And the unit element and inverse are also 1.
3.We also have 0≠1.
4.(a+b)·0=0=a·0+b·0;(a+b)·1=a+b=a·1+b·1. So · distributes on +.
Concludingly, ($\mathbb{Z}/2\mathbb{Z}$,+,·) is a field.

3 If α=1, a=b=c, it is obvious that it is not a basis.
If α=0, we have (1,1,0)+(1,0,1)=(0,1,1), so it is not too.
So it is not, concludingly.

3 Ex.3

$\forall v \in V$, we have f(v)$\in V_2$. As p is surjective, $\forall f(v) \in V_2, \exists$u such that $p(u) \in V_2$ $\varphi(v) = u$ is such a $V \mapsto V_1$ map. Among such map we can find a linear one s.t. $p \circ \varphi = f$. According to a previous homework and theorem 2.82, the f is linear if p and φ are linear.

4 Ex.4

1

(a) To prove $x \in V$, we just need to prove y ∈S, which is trival, and $u(z)$ ∈im u. $\forall z \in V$, we have $z = z' + z_0$ s.t. $z' \in im\ u$ and $z_0 \in S$. Then we have $u(z) = u(z' + z_0) = u(z_0)$. Thus $\forall z \in V, \exists z_0 \in S, u(z_0) = u(z) \in im\ u$. We therefore proves the existence. ∀z∈S, if z=0, we have u(z)=0. Then we shall prove the unicity. There exists unique y and u(z), as u is an endomorphism and is linear, z is unique. Therefore there exists a unique pair (y,z).

(b),(c) $\alpha x_1 + \beta x_2 = \alpha(y_1 + u(z_1)) + \beta(y_2 + u(z_2)) = (\alpha y_1 + \beta y_2) + (\alpha u(z_1) + \beta u(z_2))$. Also, as (y,z) is unique, it is the only valid consequence. Thus, $v(\alpha x_1 + \beta x_2) = \alpha y_1 + \beta y_2$, and $u(w(\alpha x_1 + \beta x_2)) = \alpha u(z_1) + \beta u(z_2)$. Therefore both v and $u \circ w$ are linear. As u is endomorphism and is linear, then w is linear.

$u \circ v(x) + v \circ u(x) = v(u(y) + u(u(x))) + u(z) = v(u(y)) + u(z)$. As $u(y) \in$im u and im u$\cap$ S=$\{0\}$, so $u(y) = y_1 + u(z_1)$ s.t. $y_1 = 0$, so $v(u(y)) = y$. And $u \circ v(x) + v \circ u(x) = y + u(z) = x$, and $u \circ v + v \circ u = Id$.

$u \circ w(x) + w \circ u(x) = w \circ u(y) + u(y)$. Similarly, $w \circ u(y) = 0$, then$0 + u(y) = u(x - u(z)) = u(x)$.

2 $\forall v \in$im u, u(v)=0,,v$\in$ker u. So im u $\in$ ker u. Then we need to prove ker u $\in$ im u. $\forall a \in$ker u, we have v(u(a))+u(v(a))=a, where u(a)=0. Thus u(v(a))=a. Therefore $a \in$ im u. In conclusion, we have im u=ker u.

3 Let's suggest u is a map from (x,y,z) to (y,0,0). Then im u is x-axis, and ker u is xOz plane. And when w is a map from (x,y,z) to 0.5(x,y,z), we have $u \circ w + w \circ u = u$. Also we have $u^2 = 0$, but ker u $\neq$im u.

4 $u^2 = 0 \not\Leftrightarrow ker\ u = im\ u$, it can only guarantee that im u$\subset$ker u. Then difference is generated here.

5 Ex.5

1. if $V_1 + V_2 \subsetneqq V$, card $\{\mathcal{B}_{V_1+V_2}\}$<card $\mathcal{B}_V$. Let $S \oplus (V_1 + V_2) = V$ Then $\mathcal{B}_V = \mathcal{B}_S \cup \mathcal{B}_{V_1+V_2}$. Let $\{\mathcal{B}_{v_i}\}$ be all basis vectors of V_1+V_2, there $\exists \mathbf{s} \in \mathcal{B}_S$,$\mathcal{B}_S \cup \{\mathcal{B}_{v_i} + \mathbf{s}\}$ is a group of basis spanning V which contains no element in $V_1 + V_2$.
2. if $V_1+V_2 = V$, according to Ex 5.1 in hw1, $V_1+V_2 \neq V$. So $\exists \mathbf{v} \in V \backslash \{V_1 \cup V_2\}$. Also, $\exists V' \subset V_2$ s.t.$V_1 \oplus V' = V$. We have a $\mathcal{B}_V = \mathcal{B}_{V_1} + \mathcal{B}_{V'}$ s.t. $\forall \mathbf{u} \in \mathcal{B}_V, \mathbf{u} \in \{V_1 \cap V_2\}$. Then $\mathcal{B}'_V = \{\mathcal{B}_{V_i} + \mathbf{u}\}$ is an basis with no element in $V_1 \cup V_2$.