MATH214
Linear algebra

Homework 4

Manuel — UM-JI (Spring 2023)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

*Questions preceded by a * are optional. Although they can be skipped without any deduction, it is important to know and understand the results they contain.*

Ex. 1 — *Matrix calculations*

1. Use block multiplication to determine the product of M_1 and M_2, where

$$M_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 10 & 1 & 2 & -3 & -1 \\ 1 & 8 & -3 & -4 & -3 & 4 \\ 1 & 1 & 6 & -3 & -9 & 6 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 8 & -3 & -2 & 5 \\ 7 & 8 & 1 & 0 \\ 2 & 7 & -2 & 9 \\ 12 & 23 & -2 & -1 \\ 5 & 7 & -9 & 5 \end{pmatrix}.$$

2. If it exists, determine the inverse of the matrices

$$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 7 & 9 & 8 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 3 \\ 6 & -5 & 4 \\ 7 & 9 & -8 \end{pmatrix}, \text{ and } \begin{pmatrix} -1 & 2 & -3 \\ 6 & -5 & 4 \\ -7 & 9 & -8 \end{pmatrix}.$$

Ex. 2 — *Dual space*

Let V_1 and V_2 be two subspaces of a finite dimensional $\mathbb{K}$-vector space V.

1. Show that $(V_1 + V_2)^\perp = V_1^\perp \cap V_2^\perp$ and $(V_1 \cap V_2)^\perp = V_1^\perp + V_2^\perp$.

   Hint. Use simple words to explain what $(V_1 + V_2)^\perp$ is.

2. Conclude that if $V = V_1 \oplus V_2$, then $V^* = V_1^\perp \oplus V_2^\perp$.

   Hint. For a subspace $V_0 \subset V$, $(V_0^\perp)^\perp$.

Ex. 3 — *Symmetric matrices*

Let $M \in \mathcal{M}_{n,p}(\mathbb{K})$. Show that $MM^\top$ and $M^\top M$ are both symmetric matrices.

Hint. Think in term of matrix elements.

Ex. 4 — *Gram-Schmidt procedure*

Let V be a finite n-dimensional $\mathbb{R}$-vector space. A *symmetric bilinear form* on V is a bilinear form b such that for any $v_1, v_2 \in v$, $b(v, w) = b(w, v)$. We say that b is *positive definite* if for any $v \in V$, $b(v, v) \geq 0$, with equality if and only if $v = 0$.

1. Bilinear forms.

   a) Let $v = (v_1, \cdots, v_n)$ and $w = (w_1, \cdots, w_n) \in V$ be the representations of v and w on a basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ of V. Show that if b is a bilinear form then $b(v, w)$ can be expressed in terms of matrices as

$$b(v, w) = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}.$$

   b) Calling B the matrix of b in $\mathcal{B}$, show that B is symmetric if and only if b is symmetric.

A bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, which is symmetric and positive definite is called an *inner product*. A vector space endowed with an inner product is called an *inner product space*. A finite dimensional real inner product space is called a *Euclidean space*.[1]

A map $\| \cdot \| : V \to \mathbb{R}$ is called a *norm* if for any $v, w \in V$, (i) $\|v\| = 0$ if and only if $v = 0$, (ii) for any $a \in \mathbb{R}$, $\|av\| = |a| \|v\|$, and (iii) $\|v + w\| \leq \|v\| + \|w\|$.

2. Inner product and norm. Let $v, w \in V$.

   a) Show that V endowed with $\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i$, is an n-dimensional Euclidean space.

   b) Show that if V is an inner product space over $\mathbb{R}$, then for any $v \in V$, $\|v\| = \sqrt{\langle v, v \rangle}$ defines a norm and V is a normed vector space.

   c) Prove Cauchy-Swartz inequality, $|\langle v, w \rangle| \leq \|v\| \|w\|$.

A *unit vector* is a vector with norm 1. Two vectors w and v are said to *orthogonal* if $\langle v, w \rangle = 0$. As set of vectors $\{u_1, \cdots, u_n\}$ is said to be *orthonormal* if they all have norm 1 and for any $i, j \in [\![1, n]\!]$, $\langle u_i, u_j \rangle = \delta_{i,j}$.

3. Construction of an orthonormal basis.

   a) Show that any set of orthogonal vectors is linearly independent.

   b) Prove that for any basis $\mathcal{B}$ of V there exists an orthonormal basis $\mathcal{B}'$ with $\operatorname{span} \mathcal{B}' = \operatorname{span} \mathcal{B}$.

      Hint. Proceed by induction on the dimension n of the space.

Gram-Schmidt procedure transforms any given basis into an orthonormal basis.

* **Ex. 5** — *Challenging problem*

Let V be a $\mathbb{K}$-vector space, and $f_1, \cdots, f_p$ and g be linear forms on V. Prove that if $\displaystyle\bigcap_{i=1}^{p} \ker f_i \subset \ker g$, then $g \in \operatorname{span}\{f_1, \cdots, f_p\}$.

Hints.

- Do not forget the case $g = 0$.
- Independently consider the cases where V is a finite and an infinite dimensional vector space.
- Let L be a subspace of V^*. In infinite dimension do we have $({}^\circ L)^\perp = L$?
- For the infinite dimension case, reason by induction on p.

[1] Those definitions are only valid over $\mathbb{R}$. When working on $\mathbb{C}$, the notion of *sesquilinear form* generalises the definition of a bilinear form and allows the definition of the inner product over $\mathbb{C}$.