Vv256 Honors Calculus IV (Fall 2023)

Assignment 4

Date Due: None
This assignment has a total of **(0 points)**.

Exercise 4.1 (0 pts) Describe geometrically the sets of points z in the complex plane defined by the following relations.

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$

(b) $1/z = \overline{z}$

(c) $\mathrm{Re}(z) = 3$

(d) $\mathrm{Re}(z) > c$ (resp., $\geq c$) where $c \in \mathbb{R}$

(e) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$

(f) $|z| = \mathrm{Re}(z) + 1$

(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$

Exercise 4.2 (0 pts) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{and} \qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \qquad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Exercise 4.3 (0 pts) Suppose that f is holomorphic in an open set $\Omega \subset \mathbb{C}$. Show that in any one of the following cases:

(a) $\mathrm{Re}(f)$ is constant;

(b) $\mathrm{Im}(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

Exercise 4.4 (0 pts) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ e^{-1/x^2}, & \text{if } x > 0 \end{cases}$$

Show that f is indefinitely differentiable on $\mathbb{R}$, and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Conclude that f does not have a converging power series expansion $\sum_{n\geq 0} a_n x^n$ for x near the origin. (Use induction)

Exercise 4.5 (0 pts) If f has a zero of order 2 at α, show that the residue of $1/f(z)$ at α is

$$-\frac{4\pi i}{3}\frac{f'''(\alpha)}{f''(\alpha)^2}$$

Exercise 4.6 (0 pts) Use residue calculus to verify the following definite integrals, where $a, b, c \in \mathbb{R}$.

(a) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \qquad 0 < |a| < 1.$

(b) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta\,\mathrm{d}\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right), \qquad 0 < |b| < a$

(c) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{a + b\sin^2\theta} = \frac{2\pi}{\sqrt{a}\sqrt{a+b}}, \qquad 0 < b < a$

(d) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{a\cos\theta + b\sin\theta + c} = \frac{2\pi}{\sqrt{c^2 - a^2 - b^2}}, \qquad a^2 + b^2 < c^2$

(e) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{a\cos^2\theta + b\sin^2\theta + c} = \frac{2\pi}{\sqrt{(a+c)(b+c)}}, \qquad 0 < c < a,\ c < b$

Exercise 4.7 (0 pts) Evaluate the following improper integrals by residue calculus.

(a) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+1}$ (b) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+x^2+1}$ (c) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^4+1)}$ (d) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x-i)(x+3i)}$

(e) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)^3}$ (f) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^4+1)^2}$ (g) $\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(4x^2+1)(x-i)}$ (h) $\int_{0}^{\infty} \frac{\mathrm{d}x}{x^3+1}$

Exercise 4.8 (0 pts) Evaluate the following improper integrals by residue calculus.

(a) $\int_{-\infty}^{\infty} \frac{\cos 4x}{x^2+1}\,\mathrm{d}x$ (b) $\int_{-\infty}^{\infty} \frac{\sin(\pi x/2)}{x^2+2x+4}\,\mathrm{d}x$ (c) $\int_{-\infty}^{\infty} \frac{x\sin 3x}{x^2+2}\,\mathrm{d}x$ (d) $\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2+2x+4}\,\mathrm{d}x$

(e) $\int_{-\infty}^{\infty} \frac{x^2\cos 2x}{(x^2+1)^2}\,\mathrm{d}x$ (f) $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2}\,\mathrm{d}x$ (g) $\int_{-\infty}^{\infty} \frac{\cos(a(x-b))}{x^2+c^2}\,\mathrm{d}x = \frac{\pi}{c}e^{-|a|c}\cos(ab)$, $a \in \mathbb{R}$, $c > 0$

(h) $\int_{-\infty}^{\infty} \frac{\cos(4\pi x)}{2x^2+x+1}\,\mathrm{d}x$ (i) $\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+x+9}\,\mathrm{d}x$

Exercise 4.9 (0 pts) Evaluate the Cauchy principal value following improper integrals by residue calculus. Use indented contours. ($a, b > 0$ if any)

(a) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin x\cos x}{x}\,\mathrm{d}x$ (b) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin x\cos 2x}{x}\,\mathrm{d}x$ (c) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,\mathrm{d}x$ (d) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{2x\sin x}{x^2-a^2}\,\mathrm{d}x$

(e) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x$ (f) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin ax}{x-b}\,\mathrm{d}x$ (g) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)}\,\mathrm{d}x$ (h) $\mathrm{PV}\int_{-\infty}^{\infty} \frac{\cos x}{x^2-a^2}\,\mathrm{d}x$