Vv256 Honors Calculus IV (Fall 2023)

Assignment 3

Date Due: None
This assignment has a total of **(30 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**.

Exercise 3.1 (9 pts) The Gamma Function. [BD12, p. 316] The gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p+1) = \int_0^\infty e^{-x} x^p \, \mathrm{d}x$$

The integral converges as $x \to \infty$ for all p. For $p < 0$ it is also improper at $x = 0$, because the integrand becomes unbounded as $x \to 0$. However, the integral can be shown to converge at $x = 0$ for $p > -1$.

(i) `(2 pts)` Show that, for $p > 0$, $\Gamma(p+1) = p\Gamma(p)$.

(ii) `(2 pts)` If p is a positive integer n, use induction to show that $\Gamma(n+1) = n!$.

Since $\Gamma(p)$ is also defined when $p \notin \mathbb{N}$, this function provides an extension of the factorial function to nonintegral values of the independent variable. Note that it is also consistent to define $0! = 1$.

(iii) `(2 pts)` For $p > 0$, use induction to show that

$$\prod_{k=0}^{n-1} (p+k) = \frac{\Gamma(p+n)}{\Gamma(p)}$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length—say, $0 < p \leq 1$.

(iv) `(1 pt)` Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

(v) `(2 pts)` Find $\Gamma(\frac{3}{2})$ and $\Gamma(\frac{11}{2})$.

Exercise 3.2 (4 pts) [BD12, p. 316] Consider the Laplace transform of t^p, where $p > -1$.

(i) `(1 pt)` Show that

$$\mathcal{L}\{t^p\} = \int_0^\infty e^{-st} t^p \, \mathrm{d}t = \frac{1}{s^{p+1}} \int_0^\infty e^{-x} x^p \, \mathrm{d}x = \frac{\Gamma(p+1)}{s^{p+1}}, \qquad s > 0$$

(ii) `(1 pt)` Let $p = n \in \mathbb{N} \setminus \{0\}$, show that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad s > 0$$

(iii) `(1 pt)` Show that

$$\mathcal{L}\{t^{-\frac{1}{2}}\} = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} \, \mathrm{d}x = \sqrt{\frac{\pi}{s}}, \qquad s > 0$$

(iv) `(1 pt)` Show that

$$\mathcal{L}\{t^{\frac{1}{2}}\} = \frac{\sqrt{\pi}}{2s^{3/2}}, \qquad s > 0$$

Exercise 3.3 (5 pts) [BD12, pp. 325–6] The Laplace transforms of certain functions can be found conveniently[1] from their Taylor series expansions.

(i) `(1 pt)` Using the Taylor series for $\sin t$

$$\sin t = \sum_{n \geq 0} \frac{(-1)^n t^{2n+1}}{(2n+1)!}$$

and assuming that the Laplace transform of this series can be computed term by term, verify that

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

[1] yeah, right.

(ii) (2 pts) Let

$$f(t) = \begin{cases} (\sin t)/t, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Find the Taylor series for f about $t = 0$. Assuming that the Laplace transform of this function can be computed term by term, verify that

$$\mathcal{L}\{f(t)\} = \arctan\left(\frac{1}{s}\right), \qquad s > 1$$

(iii) (2 pts) The Bessel function of the first kind of order zero, J_0, has the Taylor series

$$J_0(t) = \sum_{n \geq 0} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2}$$

Assuming that the following Laplace transforms can be computed term by term, verify that

$$\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}, \qquad s > 1$$

and

$$\mathcal{L}\{J_0(\sqrt{t})\} = \frac{1}{s} e^{-\frac{1}{4s}}, \qquad s > 0$$

Exercise 3.4 (2 pts) [BD12, p. 327] Suppose that

$$g(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, respectively, show that

$$G(s) = \frac{F(s)}{s}$$

Exercise 3.5 (2 pts) [BD12, p. 334] Let f satisfy $f(t+T) = f(t)$ for all $t \geq 0$ and for some fixed $T > 0$; f is said to be periodic with period T on $0 \leq t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t$$

Exercise 3.6 (2 pts) [BD12, p. 342] Solve the following initial value problem.

$$y'' + y' + \frac{5}{4}y = g(t); \qquad y(0) = 0, \qquad y'(0) = 0;$$

where

$$g(t) = \begin{cases} \sin t, & 0 \leq t < \pi \\ 0, & t \geq \pi \end{cases}$$

Exercise 3.7 (2 pts) [BD12, p. 348] Solve the following initial value problem.

$$y^{(4)} - y = \delta(t-1); \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0$$

Exercise 3.8 (4 pts) Solve the following initial value problem. Solve the following **integro-differential equations**.

(a) $\phi(t) + 2\displaystyle\int_0^t \cos(t-\xi)\phi(\xi)\,\mathrm{d}\xi = e^{-t}$

(b) $\phi'(t) - \dfrac{1}{2}\displaystyle\int_0^t (t-\xi)^2\phi(\xi)\,\mathrm{d}\xi = -t$, $\phi(0) = 1$.

References

[BD12] W.E. Boyce and R.C. DiPrima. *Elementary Differential Equations*. 10th ed. Wiley, 2012 (Cited on pages 1, 2).