Vv256 Honors Calculus IV (Fall 2023)

Assignment 5

Date Due: See canvas
This assignment has a total of **(22 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**.

Exercise 5.1 (2 pts) Given $A' \in M_{(n-1)\times n}(\mathbb{C})$, $B' \in M_{n\times(n-1)}(\mathbb{C})$, show that

$$\det(A'B') = \sum_{j=1}^{n} \det(M_j)\det(N_j)$$

where $M_j \in M_{n-1}(\mathbb{C})$ is A' with j-th column deleted, and $N_j \in M_{n-1}(\mathbb{C})$ is B' with j-th row deleted. (You do not need fancy theorems to do this. Applying heavy machinery (without proving it first) surely increases your mathematical knowledge, which is better than nothing, but it does not enhance your mathematical skills or ability.)

Exercise 5.2 (10 pts) Given a matrix $A \in M_n(\mathbb{C})$, the *adjugate* of A denoted by $\operatorname{adj} A \in M_n(\mathbb{C})$ is given by

$$(\operatorname{adj} A)_{ij} \coloneqq (-1)^{i+j}\det(a_{mk})_{m\neq j, k\neq i}$$

Show that

(i) `(2 pts)` Let $A \in M_n(\mathbb{C})$, then $A(\operatorname{adj} A) = (\operatorname{adj} A)A = (\det A)I_n$.
(ii) `(2 pts)` Let $A, B \in M_n(\mathbb{C})$, then $\operatorname{adj}(AB) = \operatorname{adj}(B)\operatorname{adj}(A)$.
(iii) `(2 pts)` If $X \in M_n(\mathbb{C})$ is invertible, then $\operatorname{adj}(X^{-1}YX) = X^{-1}\operatorname{adj}(Y)X$, for all $Y \in M_n(\mathbb{C})$.
(iv) `(2 pts)` Given $A \in M_n(\mathbb{C})$ with $\operatorname{rank}(A) \leq n-2$, then $\operatorname{adj} A \equiv 0 \in M_{n-1}(\mathbb{C})$.
(v) `(2 pts)` Given $A \in M_n(\mathbb{C})$ with $\operatorname{rank}(A) = n-1$, then $\operatorname{adj} A = \alpha v w^\top$ for some scalar α, where $v \in \ker A$, and $w \in \ker A^\top$.

Exercise 5.3 (2 pts) Given an invertible matrix function $A(t) \in M_n(\mathbb{R})$ with differentiable entries, verify that

$$\frac{\mathrm{d}}{\mathrm{d}t}[A(t)^{-1}] = -A(t)^{-1}\frac{\mathrm{d}A(t)}{\mathrm{d}t}A(t)^{-1}$$

(i) `(1 pt)` by using the identity $\mathbf{D}\iota(A)\cdot H = -A^{-1}HA^{-1}$, where $\iota(A) = A^{-1}$.
(ii) `(1 pt)` by differentiating $A(t)A(t)^{-1} = I_n$.

Exercise 5.4 (2 pts) Let $A \in M_n(\mathbb{R})$, let $f_k(A) = A^k$, $k \in \mathbb{N}$. Show that

$$\mathbf{D}\operatorname{tr}(f_k(A))\cdot H = k\operatorname{tr}(A^{k-1}H)$$

Exercise 5.5 (6 pts) Recall that

$$\exp(A) \coloneqq \sum_{n\geq 0}\frac{A^n}{n!}$$

for a square matrix A.

(i) `(2 pts)` show that

$$\mathbf{D}\exp(A)\cdot H = \sum_{n\geq 0}\frac{1}{(n+1)!}\sum_{k=0}^{n} A^k H A^{n-k}$$

(ii) `(2 pts)` In particular, show that

$$\operatorname{tr}(\mathbf{D}\exp(A)\cdot H) = \operatorname{tr}(\exp(A)H)$$

(iii) `(2 pts)` Show that

$$\mathbf{D}\exp(A)\cdot H = \int_0^1 e^{(1-t)A}He^{tA}\,\mathrm{d}t$$

Exercise 5.6 (0 pts) Visit `https://en.wikipedia.org/wiki/Matrix_calculus` and practice the identities thereof (in ways that you are comfortable with).