Vv256 Honors Calculus IV (Fall 2023)

Assignment 1

Date Due: See canvas
This assignment has a total of **(26 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**.

Exercise 1.1 (2 pts) [BD12, p. 17] The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.

Exercise 1.2 (6 pts) [BD12, p. 25] Verify that each given function is a solution of the differential equation.

(a) $t^2y'' + 5ty' + 4y = 0$, $t > 0$; $y_1 = t^{-2}$, $y_2 = t^{-2}\ln t$

(b) $y'' + y = \sec t$, $0 < t < \pi/2$; $y = (\cos t)\ln\cos t + t\sin t$

(c) $y' - 2ty = 1$; $y = e^{t^2}\int_0^t e^{-s^2}\,\mathrm{d}s + e^{t^2}$

Exercise 1.3 (8 pts) [BD12, p. 25] Verify that each given function is a solution of the partial differential equation.

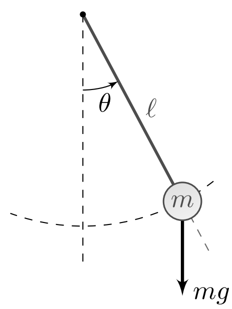
(a) $u_{xx} + u_{yy} = 0$; $u_1(x, y) = \cos x\cosh y$, $u_2(x, y) = \ln(x^2 + y^2)$

(b) $\alpha^2 u_{xx} = u_t$; $u_1(x, t) = e^{-\alpha^2 t}\sin x$, $u_2(x, t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x$, λ is a real constant

(c) $\alpha^2 u_{xx} = u_t$; $u(x, t) = (\pi/t)^{1/2}e^{-x^2/4\alpha^2 t}$, $t > 0$

(d) $a^2 u_{xx} = u_{tt}$; $u_1(x, t) = \sin\lambda x\sin\lambda at$, $u_2(x, t) = \sin(x - at)$, λ is a real constant

Exercise 1.4 (6 pts) [BD12, pp. 25–6] Consider the simple pendulum with mass m and length of the string ℓ. The string makes an angle θ with the vertical direction. It is easy to verify that the pendulum has kinetic energy



$$T = \frac{1}{2}m\ell^2\dot{\theta}^2$$

potential energy (relative to its rest position)

$$V = mg\ell(1 - \cos\theta)$$

and angular momentum

$$M = m\ell^2\dot{\theta}$$

(i) `(2 pts)` By conservation of energy, the total energy $E = T + V$ is constant. Derive the equation of motion for the pendulum by setting $\mathrm{d}E/\mathrm{d}t = 0$.

(ii) `(2 pts)` Derive the equation of motion for the pendulum by setting $\mathrm{d}M/\mathrm{d}t$ to the the moment of the gravitational force. Note that positive moments are counterclockwise.

(iii) `(2 pts)` The Lagrangian of the pendulum is given by $L = T - V$. Derive the equation of motion for the pendulum following the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

Exercise 1.5 (2 pts) Let $x : \mathbb{R} \to \mathbb{R}^n$, $n > 1$, $t \mapsto x(t) = (x_1(t), \ldots, x_n(t))$ be the solution to the differential equation $\dot{x} = Ax$, where A is an $n \times n$ constant real matrix. Suppose $x_1(t) \neq 0$, and let

$$K(t) := \begin{bmatrix} \frac{x_2(t)}{x_1(t)} & \frac{x_3(t)}{x_1(t)} & \cdots & \frac{x_n(t)}{x_1(t)} \end{bmatrix}^\top$$

Find the dynamics of $K(t)$ by partitioning A as an appropriate block matrix.

Exercise 1.6 (2 pts) Show that if $\varphi(x, t)$ is a solution to the heat equation $u_t = u_{xx}$, then so is $\psi(x, t) := e^{-\varepsilon x + \varepsilon^2 t}\varphi(x - 2\varepsilon t, t)$ for any $\varepsilon \in \mathbb{R}$.

References

[BD12] W.E. Boyce and R.C. DiPrima. *Elementary Differential Equations*. 10th ed. Wiley, 2012 (Cited on page 1).