Vv256 Honors Calculus IV (Fall 2023)

Assignment 6

Date Due: See canvas
This assignment has a total of **(50 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**.

Exercise 6.1 (2 pts) Find the flow ϕ_t of the ODE

$$\begin{aligned}\dot{x} &= x + y^2 \\ \dot{y} &= -y\end{aligned}$$

Exercise 6.2 (2 pts) Find $f : \mathbb{R}^2 \to \mathbb{R}^2$ if the flow of the ODE $\dot{\xi} = f(\xi)$, $\xi = (x, y) \in \mathbb{R}^2$ is given by

$$\phi_t(x, y) = ((x + y^2/3)e^t - y^2 e^{-2t}/3, ye^{-t})$$

Exercise 6.3 (2 pts) Find $A \in M_3(\mathbb{R})$ such that

$$\exp(At) = \begin{bmatrix} te^t + 1 & -4te^t + 3e^t - 3 & te^t - e^t + 1 \\ 1 - e^t & 4e^t - 3 & 1 - e^t \\ -te^t - 3e^t + 3 & 4te^t + 9e^t - 9 & -te^t - 2e^t + 3 \end{bmatrix}$$

Exercise 6.4 (4 pts) Given $A, E \in M_n(\mathbb{C})$,

(i) `(2 pts)` Show that for all $t \in \mathbb{R}$,

$$e^{(A+E)t} = e^{At} + \int_0^t e^{A(t-s)} E e^{(A+E)s} \, \mathrm{d}s$$

(ii) `(2 pts)` Use (i) to show that

$$\mathbf{D} \exp(A) \cdot E = \int_0^1 e^{A(1-s)} E e^{As} \, \mathrm{d}s$$

Exercise 6.5 (4 pts) Given $A, B \in M_n(\mathbb{C})$ such that $AB = BA$, show that $e^{(A+B)t} = e^{At}e^{Bt}$ for all t.

(i) `(2 pts)` by expanding the exp function as power series.

(ii) `(2 pts)` by showing that both sides satisfy the same IVP.

Exercise 6.6 (3 pts) Given $A \in M_n(\mathbb{C})$, show that $(e^{At})^{-1} = e^{-At}$

(i) `(2 pts)` by verifying that both sides satisfy the same IVP.

(ii) `(1 pt)` by applying **Exercise 6.5**.

Exercise 6.7 (2 pts) Given a symmetric matrix $A \in M_n(\mathbb{R})$, show that A is positive definite iff all eigenvalues of A are positive.

Exercise 6.8 (2 pts) Given $f : V \to V$, the second derivative of f at x in the direction $(v, w) \in V \times V$ is given by

$$\mathbf{D}^2 f(x) \cdot (v, w) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} f(x + tv + sw)$$

Let $\mathbf{S}^n_{++}$ denote the set of $n \times n$ real symmetric positive definite matrices, and $g : \mathbf{S}^n_{++} \to \mathbb{R}$, $g(X) = -\log\det X$. Show that

$$\mathbf{D}^2 g(X) \cdot (H, H) > 0$$

for all symmetric matrix $H \in M_n(\mathbb{R}) \setminus \{0\}$. *hint:* $\|A\|^2_{\text{frob}} = \operatorname{tr}(A^\top A) = 0 \Rightarrow A = 0$

Exercise 6.9 (6 pts) Let χ be the characteristic polynomial for the n-the order ODE with constant coefficients

$$y^{(n)} = a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y$$

Suppose $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are distinct roots for χ, consider the following $n \times n$ matrix

$$A = \left[\begin{array}{c|c} 0 & \\ \vdots & I_{n-1} \\ 0 & \\ \hline a_0 & a_1 \cdots\cdots\cdots a_{n-1} \end{array}\right]$$

(i) `(2 pts)` Find eigenvalues and eigenvectors of A.

(ii) `(2 pts)` Find an invertible matrix P and diagonal matrix Λ such that $A = P\Lambda P^{-1}$.

(iii) `(2 pts)` Find the general solution to the ODE $\dot{x} = Ax$.

Exercise 6.10 (2 pts) For symmetric matrix $A \in M_n(\mathbb{R})$, consider the function over the $(n-1)$-sphere S^{n-1}

$$\begin{aligned} f_A : S^{n-1} \subset \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto x^\top A x \end{aligned}$$

where $S^{n-1} := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$. Show that x is an eigenvector of A iff $\mathbf{D}f_A(x) \cdot v = 0$ for all $v \in \ker(x^\top)$. *hint:* $\dim \ker(x^\top) = n - 1$

Exercise 6.11 (5 pts) Let $\Phi(t, t_0) \in M_n(\mathbb{R})$ satisfy the IVP over $I \subseteq \mathbb{R}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t, t_0) &= A(t)\Phi(t, t_0) \text{ for all } t, t_0 \in I \\ \Phi(t_0, t_0) &= I_n \text{ for all } t_0 \in I \end{aligned}$$

where $A(t) \in M_n(\mathbb{R})$ for all $t \in I \subseteq \mathbb{R}$. Assume all calculations are possible,

(i) `(2 pts)` Show that $\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$ for all $t_0, t_1, t_2 \in I$ by verifying both sides satisfy the same IVP.

(ii) `(1 pt)` Use (i) to conclude that $\Phi(t_0, t) = \Phi(t, t_0)^{-1}$ for all $t, t_0 \in I$

(iii) `(2 pts)` Show that

$$\det \Phi(t, t_0) = \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s)\,\mathrm{d}s\right)$$

Exercise 6.12 (4 pts) Find the general solution to the ODE $\dot{x} = Ax$ for the matrices A as follows. Express the final results in real functions.

(a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{bmatrix}$

Exercise 6.13 (2 pts) Given $A \in M_n(\mathbb{R})$, consider the IVP

$$\dot{x} = Ax + g(t), \qquad x(0) = x_0$$

where $g : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function. Show that

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} g(s)\,\mathrm{d}s$$

Exercise 6.14 (6 pts) Solve the following IVP with initial condition $x(0) = \begin{bmatrix} -1 & 2 & -31 \end{bmatrix}^\top$

(a) $\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{bmatrix} x$ (b) $\dot{x} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{bmatrix} x$ (c) $\dot{x} = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} x$

hint: one does not need matrix theory to solve (a).

Exercise 6.15 (4 pts) Given

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

(i) `(2 pts)` Use induction to show that for $n \in \mathbb{N}$,

$$J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

(ii) `(2 pts)` Determine $\exp(tJ)$

Exercise 6.16 (0 pts) Consider the *circulant matrix*

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & & c_2 \\ \vdots & c_1 & c_0 & \ddots & \vdots \\ c_{n-2} & & \ddots & \ddots & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}$$

where $c_0, \ldots, c_{n-1} \in \mathbb{C}$. Find the eigenvalues and associated eigenvectors of C_n.

Exercise 6.17 (0 pts) Given $A, B \in M_n(\mathbb{C})$.

(i) `(0 pts)` If $V \leq \mathbb{C}^n$ is an A-invariant subspace, i.e., $AV \subset V$ (or $Av \in V$ for all $v \in V$), show that V contains an eigenvector of A.

hint: consider a basis of V.

(ii) `(0 pts)` If $AB = BA$, show that A and B have a common eigenvector.

hint: find a common invariant subspace of A and B.

Exercise 6.18 (0 pts) Given a real symmetric positive definite matrix $A \in \mathbf{S}^n_{++}$, $v \in \mathbb{R}^n$,

(i) `(0 pts)` Show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^\top Ax} \,\mathrm{d}x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

hint: diagonalize A first.

(ii) `(0 pts)` Show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^\top Ax + v^\top x} \,\mathrm{d}x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}v^\top A^{-1}v}$$

(iii) `(0 pts)` Given another symmetric matrix $D \in M_n(\mathbb{R})$, show that

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}x^\top Ax + v^\top x}(x^\top Dx) \,\mathrm{d}x = \left[v^\top A^{-1}DA^{-1}v + \operatorname{tr}(DA^{-1})\right] \frac{(2\pi)^{n/2}}{\sqrt{\det A}} e^{\frac{1}{2}v^\top A^{-1}v}$$

hint: calculate the directional derivative $\mathbf{D}I(A) \cdot D$, where $I(A)$ is the integral in (ii).

(iv) `(0 pts)` Visit `https://en.wikipedia.org/wiki/Gaussian_function#Multi-dimensional_Gaussian_function` and try other identities.

(v) `(0 pts)` Given $f, g : \mathbb{R}^n \to \mathbb{R}$, $p, q \in [1, \infty)$, the Hölder's inequality states that

$$\int_{\mathbb{R}^n} |f(x)g(x)| \,\mathrm{d}x \leq \left(\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |f(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}}$$

provided the integrals converge. Use (i) and Hölder's inequality to show that for all matrices $X, Y \in \mathbf{S}^n_{++}$, $t \in [0, 1]$,

$$\det(tX + (1-t)Y) \geq (\det X)^t (\det Y)^{1-t}$$

(vi) `(0 pts)` Use (v) to conclude that the function $g : \mathbf{S}^n_{++} \to \mathbb{R}$, $g(X) = -\log\det X$ in **Exercise 6.8** is convex, i.e., $\forall X, Y \in \mathbf{S}^n_{++}$, $\forall t \in [0, 1]$, $g(tX + (1-t)Y) \leq tg(X) + (1-t)g(Y)$.