Vv256 Honors Calculus IV (Fall 2023)

Assignment 2

Date Due: See canvas
This assignment has a total of **(48 points)**.
Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**.

Exercise 2.1 (8 pts) [BD12, p. 40] Find the solution to the given initial value problem.

(a) $y' + (2/t)y = (\cos t)/t^2 \quad y(\pi) = 0, \quad t > 0$

(b) $ty' + 2y = \sin t, \quad y(\pi/2) = 1, \quad t > 0$

(c) $t^3 y' + 4t^2 y = e^{-t}, \quad y(-1) = 0, \quad t < 0$

(d) $ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0$

Exercise 2.2 (8 pts) [BD12, p. 48] Solve the given differential equation.

(a) $y' = (\cos^2 x)(\cos^2 2y)$

(b) $xy' = (1 - y^2)^{1/2}$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x - e^{-x}}{y + e^y}$

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2}{1 + y^2}$

Exercise 2.3 (4 pts) [BD12, p. 101] Show that the given equation is not exact, but becomes exact when multiplied by the given integrating factor μ.

(a) $x^2 y^3 + x(1 + y^2)y' = 0, \qquad \mu(x, y) = 1/xy^3$

(b) $(\sin y - 2ye^{-x} \sin x) + (\cos y + 2e^{-x} \cos x)y' = 0, \qquad \mu(x, y) = e^x$

Exercise 2.4 (2 pts) [BD12, p. 101] Show that if $(N_x - M_y)/M = Q$, where Q is a function of y only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form

$$\mu(y) = \exp\left[\int Q(y)\mathrm{d}y\right]$$

Exercise 2.5 (2 pts) [BD12, p. 102] Show that if $(N_x - M_y)/(xM - yN) = R$, where R depends on the quantity xy only, then the differential equation

$$M + Ny' = 0$$

has an integrating factor of the form $\mu(xy)$. Find a general formula for this integrating factor.

Exercise 2.6 (6 pts) [BD12, p. 175] Consider the equation $ay'' + by' + cy = 0$, where $a, b, c \in \mathbb{R}$ are constants. Find the set of triples $(a, b, c) \in \mathbb{R}^3$ such that all the solutions

(i) (2 pts) tend to zero.

(ii) (2 pts) are bounded.

(iii) (2 pts) tend to a constant that depends on the initial conditions as $t \to \infty$. Determine this constant for the initial conditions $y(0) = y_0$, $y'(0) = y_0'$.

Exercise 2.7 (4 pts) [BD12, p. 192] Find the general solution to the following inhomogeneous equation. One solution to the homogeneous equation is provided.

(a) $t^2 y'' - t(t+2)y' + (t+2)y = 2t^3$, $t > 0$; $y_1(t) = t$

(b) $(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}$, $0 < t < 1$; $y_1(t) = t$

Exercise 2.8 (4 pts) [BD12, p. 239] find the solution of the given initial value problem.

(a) $y''' - 3y'' + 2y' = t + e^t$, $y(0) = 1$, $y'(0) = -\frac{1}{4}$, $y''(0) = -\frac{3}{2}$

(b) $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12 \sin t - e^{-t}$, $y(0) = 3$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 2$

Exercise 2.9 (2 pts) [BD12, p. 306] The Bessel (differential) equation of order $\alpha \in \mathbb{C}$ is given by

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - \alpha^2)y = 0$$

Show that the Bessel equation of order $\frac{1}{2}$

$$x^2 y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0, \qquad x > 0$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this, conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order $\frac{1}{2}$.

Exercise 2.10 (4 pts) [BD12, p. 307] By a suitable change of variables it is sometimes possible to transform another differential equation into a Bessel equation.

(i) `(2 pts)` Show that a solution of

$$x^2 y'' + \left(\alpha^2\beta^2 x^{2\beta} + \tfrac{1}{4} - \nu^2\beta^2\right)y = 0, \qquad x > 0$$

is given by $y = x^{1/2}f(\alpha x^\beta)$, where $f(\xi)$ is a solution of the Bessel equation of order ν.

(ii) `(2 pts)` Show that the general solution of the Airy equation

$$y'' - xy = 0, \qquad x > 0$$

is $y = x^{1/2}\left[c_1 f_1\left(\tfrac{2}{3}ix^{3/2}\right) + c_2 f_2\left(\tfrac{2}{3}ix^{3/2}\right)\right]$, where $f_1(\xi)$ and $f_2(\xi)$ are a fundamental set of solutions of the Bessel equation of order $\frac{1}{3}$.

Exercise 2.11 (4 pts) The Taylor series expansion for the solution of the differential equation

$$\dot{x} = f(t, x), \qquad x(0) = x_0$$

at times $t_n = nh$ with $x_n := x(nh)$ is given by

$$x_{n+1} = x_n + h\dot{x}_n + \frac{h^2}{2!}\ddot{x}_n + \cdots$$

with

$$\begin{aligned}\dot{x}_n &= f(t_n, x_n)\\ \ddot{x}_n &= \mathbf{D}_1 f(t_n, x_n) + \mathbf{D}_2 f(t_n, x_n)\cdot f(t_n, x_n)\end{aligned}$$

where $\mathbf{D}_1$, $\mathbf{D}_2$ stand for the derivatives wrt the first and second argument, resp. An N-th order *Runge-Kutta* method seek to approximate the first N terms of the Taylor series using not the derivatives of f, but the values of f at some intermediate values of (t, x) between (t_n, x_n) and (t_{n+1}, x_{n+1}).

(i) `(2 pts)` Show that the scheme described by

$$x_{n+1} = x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}k_1 &= f(t_n, x_n)\\ k_2 &= f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_1\right)\\ k_3 &= f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2}k_2\right)\\ k_4 &= f(t_n + h, x_n + hk_3)\end{aligned}$$

matches the first four coefficients of the Taylor series for x_{n+1}.

(ii) `(2 pts)` Show that the scheme described by

$$x_{n+1} = x_n + \frac{h}{8}(k_1 + 3k_2 + 3k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= f(t_n, x_n) \\
k_2 &= f\left(t_n + \frac{1}{3}h, x_n + \frac{h}{3}k_1\right) \\
k_3 &= f\left(t_n + \frac{2}{3}h, x_n - \frac{h}{3}k_1 + hk_2\right) \\
k_4 &= f(t_n + h, x_n + hk_1 - hk_2 + hk_3)
\end{aligned}$$

also matches the first four coefficients of the Taylor series for x_{n+1}.

References

[BD12] W.E. Boyce and R.C. DiPrima. *Elementary Differential Equations*. 10th ed. Wiley, 2012 (Cited on pages 1, 2).