Homotopy Theory

Cloudifold

August 8, 2021

1 Notations

Category of sets	: **Set**
Category of topological spaces	: **Top**
Category of (one-point-)based topological spaces	: $\mathbf{Top}_*$
Category of pairs (X, A) of space X and subspace A	: $\mathbf{Top}(2)$
Topological space X with topology $\mathcal{T}$	: $X_{\mathcal{T}}$
Euclidean space of dimension n	: $\mathbb{R}^n$
Unit cube of dimension n	: I^n
Boundary of I^n	: ∂I^n
Unit interval I	: $I = I^1$
Unit cell of dimension n	: $\mathring{\mathbb{D}}^n$
Unit disk of dimension n	: $\mathbb{D}^n$
Unit sphere of dimension $n-1$	: $\mathbb{S}^{n-1}$
Inclusion or Embedding	: $\hookrightarrow$
Monomorphsim	: $\rightarrowtail$
Epimorphsim	: $\twoheadrightarrow$
Hom functor of category $\mathcal{C}$	: $\mathrm{Hom}_{\mathcal{C}}(-,-)$
Limit (inverse limit) (projective limit)	: $\varprojlim$
Colimit (direct limit) (inductive limit)	: $\varinjlim$

2 Preliminary Definitions

Definition 2.1. A object A is called a **retract** of B if there are morphisms $s : A \to B, r : B \to A$ such that $r \circ s = \mathrm{id}_A$. In this case, r is called a **retraction** of s and s is called a *section* of r.

$$\mathrm{id}_A : A \xrightarrow[\textit{section}]{s} B \xrightarrow[\textit{retraction}]{r} A$$

Proposition 2.1. *For any category $\mathcal{C}$, $\mathrm{Iso}(\mathcal{C})$ is closed under forming retracts in $\mathrm{Arr}(\mathcal{C})$.*

Proof. For $f \in \mathrm{Iso}(\mathcal{C})$, g is a retract of f in $\mathrm{Arr}(\mathcal{C})$, $g^{-1} = b \circ f^{-1} \circ a$

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[r, "b"] \arrow[d, bend right, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "a"'] & \bullet \arrow[r] \arrow[u, bend right, "f^{-1}"'] & \bullet
\end{tikzcd}$$

□

3 Abstract Homotopy Theory

3.1 Introduction

In generality, homotopy theory is the study of mathematical contexts in which morphisms are equipped with a concept of homotopy between them, hence with a concept of "equivalent deformations" of morphisms, and then iteratively with homotopies of homotopies between those, and so forth.

A fundamental insight due to (Quillen 67) is that in fact all constructions in homotopy theory are elegantly expressible via just the abstract interplay of these classes of morphisms. This was distilled in (Quillen 67) into a small set of axioms called a **model category** structure (because it serves to make all objects be models for homotopy types.)

This abstract homotopy theory is the royal road for handling any flavor of homotopy theory, in particular the stable homotopy theory. Here we discuss the basic constructions and facts in abstract homotopy theory, then below we conclude this Introduction to Homotopy Theory by showing that topological spaces equipped with the above system of classes continuous functions is indeed an example of abstract homotopy theory in this sense.

3.2 Factorization Systems

Definition 3.1. A **category with weak equivalences** is

1. A category $\mathcal{C}$;
2. A sub-class $W \subseteq \mathrm{Mor}(\mathcal{C})$;

such that

1. W contains all isomorphisms in $\mathcal{C}$
2. W is closed under **two-out-of-three** :

   In every commutative diagram in $\mathcal{C}$ of the form
   $$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr] & & Z \end{tikzcd}$$
   if two of the three morphisms are in W, then so is the third.

Note. The further axioms of a **model category** serve the sole purpose of making the universal homotopy theory induced by a category with weak equivalences be tractable.

Definition 3.2. A **model category** is

1. A category $\mathcal{C}$ with limits and colimits;
2. Three sub-classes W, Fib, Cof $\subseteq \mathrm{Mor}(\mathcal{C})$ of morphisms in $\mathcal{C}$;

such that

1. $(\mathcal{C}, W)$ is a **category with weak equivalences**;
2. Pairs $(W \cap \mathrm{Cof}, \mathrm{Fib})$ and $(\mathrm{Cof}, W \cap \mathrm{Fib})$ are both **weak factorization systems**.

Note. We say:

- elements in W are **weak equivalences**;
- elements in Fib are **fibrations**;
- elements in Cof are **cofibrations**;
- elements in $W \cap \mathrm{Fib}$ are **acyclic(trivial) fibrations**;
- elements in $W \cap \mathrm{Cof}$ are **acyclic(trivial) cofibrations**;

Definition 3.3. extension, lift and lifting :

We assert that diagrams, morphisms and objects below are in a category $\mathcal{C}$.

Given diagram of the form $\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ Z & \end{tikzcd}$, the **extension** of f along p is a morphism $\overline{f} : Z \to Y$

such that the diagram
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \\ Z \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$
commutes.

Dually, given a diagram of the form
$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
, the **lift** of f through p is a morphism $\overline{f} : X \to Z$ such that the diagram
$$\begin{tikzcd} & Z \arrow[d, "p"] \\ X \arrow[ur, "\overline{f}"] \arrow[r, "f"] & Y \end{tikzcd}$$
commutes.

Combining the two cases, given a commutative square $\mathscr{C}$:
$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "p_l"'] & Z \arrow[d, "p_r"] \\ Y \arrow[r, "f_b"] & W \end{tikzcd}$$
the **lifting** in $\mathscr{C}$ is a morphism $\overline{f} : Y \to Z$ such that the diagram
$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "p_l"'] & Z \arrow[d, "p_r"] \\ Y \arrow[ur, "\overline{f}"] \arrow[r, "f_b"] & W \end{tikzcd}$$
commutes.

Definition 3.4. right and left lifting property

Diagrams, morphisms and objects below are in a category $\mathcal{C}$.
With a given sub-class of morphisms $K \subseteq \mathrm{Mor}(\mathcal{C})$:

A morphism p_r is said to have the **right lifting property** against K or to be a K**-injective morphism** if in all commutative diagrams
$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "p_l"'] & Z \arrow[d, "p_r"] \\ Y \arrow[r, "f_b"] & W \end{tikzcd}$$
with p_r on the right and any $p_l \in K$ on the left, the **lifting** in the diagram exists.

Dually, A morphism p_l is said to have the **left lifting property** against K or to be a K**-projective morphism** if in all commutative diagrams
$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "p_l"'] & Z \arrow[d, "p_r"] \\ Y \arrow[r, "f_b"] & W \end{tikzcd}$$
with p_l on the left and any $p_r \in K$ on the right, the **lifting** in the diagram exists.

Definition 3.5. A **weak factorization system** (WFS) on a category $\mathcal{C}$ is a pair $(\mathrm{Proj}, \mathrm{Inj})$ of two classes of morphisms of $\mathcal{C}$ such that

1. Every morphism $f : X \to Y$ in $\mathcal{C}$ can be factored as $X \xrightarrow{p \in \mathrm{Proj}} Z \xrightarrow{i \in \mathrm{Inj}} Y$
2. The classes are closed under having the **lifting property** against each other. That is:
   (a) Proj is *precisely* the class of morphisms having the **left lifting property** against Inj.
   (b) Inj is *precisely* the class of morphisms having the **right lifting property** against Proj.

Remark. The factorization just ensured *existence*.

Definition 3.6. For $\mathcal{C}$ a category, a **functorial factorization** of the morphisms in $\mathcal{C}$ is a functor $fact : \mathcal{C}^{\Delta[1]} \to \mathcal{C}^{\Delta[2]}$, which is the right inverse of the composition functor $d_1 : \mathcal{C}^{\Delta[2]} \to \mathcal{C}^{\Delta[1]}$. Equalently, $d_1 \circ fact = \mathrm{id}_{\mathcal{C}^{\Delta[1]}}$

Remark. The notation above is defined at **simplex category** and at **nerve** of a category. The **arrow category** of a category $\mathcal{C}$ is $\mathrm{Arr}(\mathcal{C}) := \mathcal{C}^{\Delta[1]} := \mathrm{Funct}(\Delta[1], \mathcal{C})$ whose objects are morphisms in $\mathcal{C}$. $\mathcal{C}^{\Delta[2]} := \mathrm{Funct}(\Delta[2], \mathcal{C})$, its objects are pairs of two composable morphisms in $\mathcal{C}$.

Definition 3.7. A $weak\ factorization\ system$ is a **functorial weak factorization system** if the factorization of morphisms can be *chosen* to be a $functorial\ factorization\ fact : \mathcal{C}^{\Delta[1]} \to \mathcal{C}^{\Delta[2]}$, such that $(d_2 \circ fact)(\mathrm{Obj}(\mathcal{C}^{\Delta[1]})) \subseteq \mathrm{Proj}$ and $(d_0 \circ fact)(\mathrm{Obj}(\mathcal{C}^{\Delta[1]})) \subseteq \mathrm{Inj}$. Where

$$\begin{aligned}
d_2, d_0 : &\quad \mathcal{C}^{\Delta[2]} \longrightarrow \mathcal{C}^{\Delta[1]} \\
d_2 : &\quad * \xrightarrow{f} * \xrightarrow{s} * \longmapsto * \xrightarrow{f} * \\
d_0 : &\quad * \xrightarrow{f} * \xrightarrow{s} * \longmapsto * \xrightarrow{s} *
\end{aligned}$$

Note. Not all $weak\ factorization\ system$ is $functorial$, although most (including those produced by the **small object argument**) are.

Proposition 3.1. *Let $\mathcal{C}$ be a category, $K \subseteq \mathrm{Mor}(\mathcal{C})$ a class of morphisms in $\mathcal{C}$. Write $K\,\mathrm{Proj}$, $K\,\mathrm{Inj}$ respectively for the sub-classes of K-projective morphisms and of K-injective morphisms. We have properties of $K\,\mathrm{Proj}$ and $K\,\mathrm{Inj}$ below:*

1. *Both contain $\mathrm{Iso}(\mathcal{C})$.*
2. *Both are closed under composition in $\mathcal{C}$.*
   And $K\,\mathrm{Proj}$ is closed under **transfinite composition**.
3. *Both are closed under forming retracts in the arrow category $\mathrm{Arr}(\mathcal{C})$.*
4. *$K\,\mathrm{Proj}$ is closed under pushouts in $\mathcal{C}$ ("cobase change").*
   $K\,\mathrm{Inj}$ is closed under pullbacks in $\mathcal{C}$ ("base change").
5. *$K\,\mathrm{Proj}$ is closed under coproducts in $\mathrm{Arr}(\mathcal{C})$.*
   $K\,\mathrm{Inj}$ is closed under products in $\mathrm{Arr}(\mathcal{C})$.

Proof.

1. Containing isomorphisms

Trivial for $K\,\mathrm{Inj}$:
$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "i\in\mathrm{Iso}"'] & \bullet \arrow[d, "p\in K"] \\
\bullet \arrow[r, "g"'] \arrow[ur, dotted, "f\circ i^{-1}"] & \bullet
\end{tikzcd}$$
And dual is also trivial.

2. Clousure under composition

For $K\,\mathrm{Inj}$:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dd, "k\in K"'] & X \arrow[d, "p_1\in K\,\mathrm{Inj}"] \\
 & Z \arrow[d, "p_2\in K\,\mathrm{Inj}"] \\
\bullet \arrow[r, "g"'] & Y
\end{tikzcd}$$

Compose f, p_1:

$$\begin{tikzcd}
\bullet \arrow[r, "p_1\circ f"] \arrow[d, "k\in K"'] & Z \arrow[d, "p_2\in K\,\mathrm{Inj}"] \\
\bullet \arrow[r, "g"'] \arrow[ur, dotted, "\exists l"] & Y
\end{tikzcd}$$

Then we have:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "k\in K"'] & X \arrow[d, "p_1\in K\,\mathrm{Inj}"] \\
\bullet \arrow[r, "l_1"'] \arrow[ur, dotted, "\exists\, l_2"] & Z
\end{tikzcd}$$

The l_2 is the expected lifting of the diagram:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[dd, "k\in K"'] & X \arrow[d, "p_1\in K\,\mathrm{Inj}"] \\
 & Z \arrow[d, "p_2\in K\,\mathrm{Inj}"] \\
\bullet \arrow[r, "g"'] \arrow[uur, dotted, "\exists\, l_2"] \arrow[ur, dotted, "\exists\, l_1"] & Y
\end{tikzcd}$$

For finite composition of morphisms in $K\,\mathrm{Proj}$, we have the dual forms of above. The clousure under $transfinite$ *composition* follows since it is given by *colimits* of *sequential* composition. And *successive* liftings as below constitutes a cocone. The extension of those liftings to the colimit is just the expected lifting.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[ddddd] \\
\bullet \arrow[d] \arrow[ur, dotted] \arrow[ddddr, dashed] & \\
\bullet \arrow[d] \arrow[uur, dotted] \arrow[dddr, dashed] & \\
\bullet \arrow[uuur, dotted] \arrow[ddr, dashed] & \\
\vdots & \\
\bullet \arrow[r] \arrow[uuuuur, dotted, "\varinjlim"] & \bullet
\end{tikzcd}$$

3. Closure under retracts

Let j be the retract in $\mathrm{Arr}(\mathcal{C})$ of $i \in K\,\mathrm{Proj}$

$$\begin{tikzcd}
\mathrm{id}_A : A \arrow[r] \arrow[d, "j"] & X \arrow[r] \arrow[d, "i\in K\,\mathrm{Proj}"] & A \arrow[d, "j"] \\
\mathrm{id}_B : B \arrow[r] & Y \arrow[r] & B
\end{tikzcd}$$

Then for a commutative square (named $\mathscr{C}$):

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] & \bullet \arrow[d, "k\in K"] \\
B \arrow[r] & \bullet
\end{tikzcd}$$

It is equalent to its *pasting composite* with the retract diagram above:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] & X \arrow[r] \arrow[d, "i\in K\,\mathrm{Proj}"] & A \arrow[r] \arrow[d, "j"] & \bullet \arrow[d, "k\in K"] \\
B \arrow[r, "s_2"] & Y \arrow[r] & B \arrow[r] & \bullet
\end{tikzcd}$$

The expected lifting of $\mathscr{C}$ is just $l_1 \circ s_2$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "j"] & X \arrow[r] \arrow[d, "i"] & A \arrow[r] & \bullet \arrow[d, "k \in K"] \\
B \arrow[r, "s_2"] & Y \arrow[r] \arrow[urr, dotted, "\exists l_1"] & B \arrow[r] & \bullet
\end{tikzcd}$$

4. Closure under pullbacks and pushouts

For all $p \in K\,\mathrm{Inj}$, and f^*p the *base change* of p along some f we have the pullback diagram (1):

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Our goal is to show that f^*p has the *right lifting property* against K. So, we consider (2):

$$\begin{tikzcd}
A \arrow[r, "w"] \arrow[d, "i \in K"'] & P \arrow[r, "p^*f"] \arrow[d, "f^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p \in K\,\mathrm{Inj}"] \\
B \arrow[r, "g"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Because of p has the *right lifting property* against K (3):

$$\begin{tikzcd}
A \arrow[r, "w"] \arrow[d, "i \in K"'] & P \arrow[r, "p^*f"] & X \arrow[d, "p \in K Inj"] \\
B \arrow[r, "g"'] \arrow[urr, dotted, "\exists l_{fg}"] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

By the *universal property* of pullback (4):

$$\begin{tikzcd}
& P \arrow[r, "p^*f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, dotted, "\exists l_g"] \arrow[urr, "l_{fg}"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

It remains to show the commutativity of that diagram:

$$\begin{tikzcd}
A \arrow[r, "w"] \arrow[d, "i"'] & P \\
B \arrow[ur, "l_g"'] &
\end{tikzcd}$$

By the universal property of limits : $\mathrm{Hom}(-, \varprojlim D) \overset{c}{\cong} \mathrm{Cone}(-, D)$. To prove $l_g \circ i = w$, is equalent to prove $c(l_g \circ i) = c(w)$.

$$\begin{tikzcd}
A \arrow[r, "w"] \arrow[d, "i"'] & P \arrow[r, "p^*f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, "l_g"] \arrow[urr, "l_{fg}"'] & Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

Following the diagram above (we *have not* said that the diagrams above is commutative before):

$$
\begin{aligned}
c(w) &= (p^*f \circ w, f^*p \circ w) && \textit{by def}\\
c(l_g \circ i) &= (p^*f \circ l_g \circ i,\ f^*p \circ l_g \circ i) && \textit{by def}\\
&= (l_{fg} \circ i,\ g \circ i) && \textit{by diagram } (4)\\
&= (p^*f \circ w,\ f^*p \circ w) = c(w) && \textit{by diagram } (3) \textit{ and } (2)
\end{aligned}
$$

The other case is formally dual.

5. Closure under products and coproducts

It is easy to prove that the product $\prod_{s\in S} p_s$ of a family of arrows(objects) $\{p_s : A_s \to B_s\}_{s\in S}$ in the arrow category $\mathrm{Arr}(\mathcal{C})$ is just the unique morphism determined by the morphisms $\{p_s \circ pr_{A_s}\}_{s\in S}$ out of $\prod_{s\in S} A_s$ and the *universal property* of products $\prod_{s\in S} B_s$. The s-th projection of $\prod_{s\in S} p_s$ in $\mathrm{Arr}(\mathcal{C})$ is just (pr_{A_s}, pr_{B_s}):

$$
\begin{tikzcd}
\prod_{s\in S} A_s \arrow[r, "pr_{A_s}"] \arrow[d, "\prod_{s\in S} p_s"'] & A_s \arrow[d, "p_s"] \\
\prod_{s\in S} B_s \arrow[r, "pr_{B_s}"'] & B_s
\end{tikzcd}
$$

With all $p_s \in K\,\mathrm{Inj}$, Let the diagram below commute:

$$
\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f\in K"'] & \prod_{s\in S} A_s \arrow[d, "\prod_{s\in S} p_s"] \\
Y \arrow[r, "y"'] & \prod_{s\in S} B_s
\end{tikzcd}
$$

Our goal is to get a *lifting* of the diagram above. By the universal property of the product. We have a family of commutative diagrams:

$$
\left\{
\begin{tikzcd}
X \arrow[r, "pr_{A_s}\circ x"] \arrow[d, "f\in K"'] & A_s \arrow[d, "p_s"] \\
Y \arrow[r, "pr_{B_s}\circ y"'] & B_s
\end{tikzcd}
\right\}_{s\in S}
$$

By all $p_s \in K\,\mathrm{Inj}$, we have a family of liftings:

$$
\left\{
\begin{tikzcd}
X \arrow[r, "pr_{A_s}\circ x"] \arrow[d, "f\in K"'] & A_s \arrow[d, "p_s"] \\
Y \arrow[r, "pr_{B_s}\circ y"'] \arrow[ur, dotted, "\exists l_s"] & B_s
\end{tikzcd}
\right\}_{s\in S}
$$

Therefore, there is a unique morphism $\prod_{s\in S} l_s : Y \to \prod_{s\in S} A_s$ determined by the universal property of product $\prod_{s\in S} A_s$

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, "f"'] & \prod\limits_{s\in S} A_s \arrow[r, "pr_{A_s}"] \arrow[d, "\prod\limits_{s\in S} p_s"] & A_s \arrow[d, "p_s"] \\
Y \arrow[r, "y"'] \arrow[ur, "\prod\limits_{s\in S} l_s"] \arrow[urr, "l_s"] & \prod\limits_{s\in S} B_s \arrow[r, "pr_{B_s}"'] & B_s
\end{tikzcd}$$

By the universal property of product, The diagram above (named $\mathscr{C}$) commutes. To see this, we just need to chase two sub-diagrams: $\mathscr{C}$ without vertice X and $\mathscr{C}$ without vertices B_s and $\prod\limits_{s\in S} B_s$.

Because the diagram drawed above commutes, $\prod\limits_{s\in S} l_s$ is the expected lifting.

$\square$

Proposition 3.2. *Given a model category* $(\mathcal{C}, W, \mathrm{Fib}, \mathrm{Cof})$*, then its class of weak equivalences* W *is closed under forming retracts in* $\mathrm{Arr}(\mathcal{C})$*.*

Proof. For every $w \in W$ and f is a retract of w in $\mathrm{Arr}(\mathcal{C})$:

First we consider the case $f \in \mathrm{Fib}$:

$$\begin{tikzcd}
\mathrm{id}_A : A \arrow[r, "s_1"] \arrow[d, "f"] & X \arrow[r, "r_1"] \arrow[d, "w"] & A \arrow[d, "f"] \\
\mathrm{id}_B : B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B
\end{tikzcd}$$

Factor w through $(\mathrm{Proj}, \mathrm{Inj}) := (\mathrm{Cof}, W \cap \mathrm{Fib})$:

$$\begin{tikzcd}
\mathrm{id}_A : A \arrow[r, "s_1"] \arrow[d, "\mathrm{id}_A"'] & X \arrow[r, "r_1"] \arrow[d, "w_1 \in \mathrm{Proj}"'] & A \arrow[d, "\mathrm{id}_A"'] \\
\mathrm{id}_A : A \arrow[r, "s"] \arrow[d, "f \in \mathrm{Inj}"'] & Z \arrow[r, "r"] \arrow[d, "w_2 \in \mathrm{Inj}"'] & A \arrow[d, "f \in \mathrm{Inj}"'] \\
\mathrm{id}_B : B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B
\end{tikzcd}$$

Where $s := w_1 \circ s_1$ and r is a lifting of $\begin{tikzcd} X \arrow[r, "\mathrm{id}_A \circ r_1"] \arrow[d, "w_1 \in W \cap \mathrm{Cof}"'] & A \arrow[d, "f \in \mathrm{Fib}"] \\ Z \arrow[r, "r_2 \circ w_2"'] & B \end{tikzcd}$ (by 2-out-of-3, $w_1 \in W$).
By Inj is closed under forming retracts in $\mathrm{Arr}(\mathcal{C})$, $f \in \mathrm{Inj} = W \cap \mathrm{Fib} \subseteq W$. Now we consider the general case:

$$\begin{tikzcd}
\mathrm{id}_A : A \arrow[r, "s_1"] \arrow[d, "f"] & X \arrow[r, "r_1"] \arrow[d, "w"] & A \arrow[d, "f"] \\
\mathrm{id}_B : B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B
\end{tikzcd}$$

Factor f through $(\mathrm{Proj}, \mathrm{Inj}) := (W \cap \mathrm{Cof}, \mathrm{Fib})$:

$$
\begin{tikzcd}
\mathrm{id}_A : A \arrow[r, "s_1"] \arrow[d, "f_1\in\mathrm{Proj}"'] \arrow[dd, bend left, "f"] & X \arrow[r, "r_1"] \arrow[dd, "w"'] & A \arrow[d, "f_1\in\mathrm{Proj}"'] \arrow[dd, bend left, "f"] \\
C \arrow[d, "f_2\in\mathrm{Inj}"'] & & C \arrow[d, "f_2\in\mathrm{Inj}"'] \\
\mathrm{id}_B : B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B
\end{tikzcd}
$$

Take the pushout P of f_1 and s_1, by the universal property of pushout (apply it on $(s_2 \circ f_2, w)$ and $(\mathrm{id}_C, f_1 \circ r_1)$), we have a commutative diagram:

$$
\begin{tikzcd}
\mathrm{id}_A : A \arrow[r, "s_1"] \arrow[d, "f_1"'] & X \arrow[r, "r_1"] \arrow[d, "w_1"'] \arrow[dd, bend left, "w"] & A \arrow[d, "f_1"'] \\
C \arrow[r] \arrow[rr, bend right, "\mathrm{id}_C"'] \arrow[d, "f_2"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, dotted, "\exists!\, r_3"] \arrow[d, dotted, "\exists!\, w_2"'] & C \arrow[d, "f_2"'] \\
\mathrm{id}_B : B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B
\end{tikzcd}
$$

By Proj is closed under forming pushouts and 2-*out*-*of*-3:

$$
\begin{tikzcd}
\mathrm{id}_A : A \arrow[r, "s_1"] \arrow[d, "f_1\in\mathrm{Proj}"'] & X \arrow[r, "r_1"] \arrow[d, "w_1\in\mathrm{Proj}"'] & A \arrow[d, "f_1\in\mathrm{Proj}"'] \\
\mathrm{id}_C : C \arrow[r, "s_3"] \arrow[d, "f_2\in\mathrm{Inj}"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[r, "r_3"] \arrow[d, "w_2\in W"'] & C \arrow[d, "f_2\in\mathrm{Inj}"'] \\
\mathrm{id}_B : B \arrow[r, "s_2"] & Y \arrow[r, "r_2"] & B
\end{tikzcd}
$$

Lemma 3.3. (**retract argument**) *For a pair of composable morphisms:* $f : X \xrightarrow{i} A \xrightarrow{p} Y$ *We have :*

1. *If f has the* **left lifting property** *against p, then f is a* **retract** *of i in* $\mathrm{Arr}(\mathcal{C})$.
2. *If f has the* **right lifting property** *against i, then f is a* **retract** *of p in* $\mathrm{Arr}(\mathcal{C})$.

Proof. Consider the first case, by f has the **left lifting property** against p, we have:

$$
\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "f"'] & A \arrow[d, "p"] \\
Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dotted, "\exists g"] & Y
\end{tikzcd}
$$

Rearranging the diagram above, we get:

$$
\begin{tikzcd}
\mathrm{id}_X : X \arrow[r, "\mathrm{id}_X"] \arrow[d, "f"] & X \arrow[r, "\mathrm{id}_X"] \arrow[d, "i"] & X \arrow[d, "f"] \\
\mathrm{id}_Y : Y \arrow[r, "g"'] & A \arrow[r, "p"'] & Y
\end{tikzcd}
$$

That is just what we need. And the second case follows by formally dual.

Introduction 3.8. Motivation of **small object argument**:

Given a **Set** of morphisms $C \in \mathrm{Mor}(\mathcal{C})$ in some category $\mathcal{C}$, a *natual* question is how to factor a morphism $f : X \to Y$ through a relative C-cell complex followed by a C-injective morphism.

$$
f : X \xrightarrow{\in C\text{-cell}} \hat{X} \xrightarrow{\in C\,\mathrm{Inj}} Y
$$

A first approximation is attaching all **possible** C-cells to X, where possible C-cells is just an object in $C/f := (C \cap \mathrm{Arr}(\mathcal{C}))/f$:

$$\begin{tikzcd}
\mathrm{dom}(c) \arrow[r] \arrow[d, "c"'] & X \arrow[d, "f"] \\
\mathrm{cod}(c) \arrow[r] & Y
\end{tikzcd}$$

The attaching pushout is:

$$\begin{tikzcd}
\coprod\limits_{c \in C/f} \mathrm{dom}(c) \arrow[r] \arrow[d, "\coprod\limits_{c \in C/f} c"'] & X \arrow[d, "f_1"] \\
\coprod\limits_{c \in C/f} \mathrm{cod}(c) \arrow[r, "p_1"'] & X_1 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By the fact that the coproduct is over all commuting squres to f, we have commutative diagram:

$$\begin{tikzcd}
\coprod\limits_{c \in C/f} \mathrm{dom}(c) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\coprod\limits_{c \in C/f} \mathrm{cod}(c) \arrow[r] & Y
\end{tikzcd}$$

By the universal property of colimit we have that f is factored through a relative C-cell complex f_1, the commutative diagram (1):

$$\begin{tikzcd}
\coprod\limits_{c \in C/f} \mathrm{dom}(c) \arrow[r] \arrow[d, "\mathrm{id}"'] & X \arrow[d, "f_1"] \\
\coprod\limits_{c \in C/f} \mathrm{dom}(c) \arrow[d] & X_1 \arrow[d, "i_1"] \arrow[ul, phantom, "\ulcorner", very near start] \\
\coprod\limits_{c \in C/f} \mathrm{cod}(c) \arrow[r] \arrow[ur, "p_1"'] & Y
\end{tikzcd}$$

The map p_1 **almost** exhibits the expecting right lifting of i_1 against C, the failure of that to hold on is only the fact that the morphism $\coprod\limits_{c \in C/f} \mathrm{dom}(c) \to X_1$ is missing, which means that **not all** $c \in C/i_1$ could have a lifting.

If we have $c \in C/i_1$ with $s : \mathrm{cod}(c) \to Y$ and $z : \mathrm{dom}(c) \to X_1$ be the morphism $c \to i_1$, we **don't always** have a $\hat{z} : \mathrm{dom}(c) \to X$ makes the diagram below commutes (equalently, z factors through the $X \to X_1$). Only for those $c \in C/i_1$ such that z factors through the $X \to X_1$, the lifting could be found.

$$\begin{tikzcd}
 & X \arrow[d] \\
\mathrm{dom}(c) \arrow[r, "z"'] \arrow[d, "c"'] \arrow[ur, dotted, "\hat{z}"] & X_1 \arrow[d, "i_1"] \\
\mathrm{cod}(c) \arrow[r, "s"'] & Y
\end{tikzcd}$$

The i_1 is **almost** a C-injective morphism, the idea of the **small object argument** now is to fix it, make a **real** C-injective morphism. The way it works is iterating the construction:

Next factor $X_1 \to Y$ in the same way into $X_1 \to X_2 \to Y$. Since relative cell complex is closed under transfinite composition, for α an ordinal, at stage X_α, the $X \to X_\alpha$ is still a relative C-cell complex.

Intuitively, the failure of the $X_\alpha \to Y$ to be a C-injective morphism becomes smaller and smaller, for the condition of the existence of the factorization

$$\mathrm{dom}(c) \to X_\alpha \longmapsto \mathrm{dom}(X) \to X_\beta \to X_\alpha \qquad \text{(where } \beta < \alpha\text{)}$$

is less and less (intuitively) as the X_β grow larger and larger.

The concept of **small object** is just what makes this intuition precise and finishes the small object argument. But now we just need the following simple version:

Definition 3.9. For $\mathcal{C}$ a category and $C \subseteq \mathrm{Mor}(\mathcal{C})$ a **subset** of morphisms in $\mathcal{C}$, say that C have **small domains** if there exists an ordinal α such that for any $c \in C$ and for any relative C-cell complex given by **transfinite composition of length** α

$$f : X \to X_1 \to X_2 \to \cdots \to X_\beta \to \cdots \to X_\alpha$$

any morphism $\mathrm{dom}(c) \to \hat{X}$ factors through a stage $X_\beta \to X_\alpha$ of some ordinal $\beta < \alpha$:

$$\begin{tikzcd} & X_\beta \arrow[d] \\ \mathrm{dom}(c) \arrow[ur] \arrow[r] & \hat{X} \end{tikzcd}$$

Proposition 3.4. *Let $\mathcal{C}$ be a locally small category with all small colimits. If a set $C \subseteq \mathrm{Mor}(\mathcal{C})$ of morphisms has **small domains**, then every morphism $f : X \to Y$ in $\mathcal{C}$ factors through a relative C-cell complex followed by a C-injective morphism:*

$$f : X \xrightarrow{\in C\text{-}cell} \hat{X} \xrightarrow{\in C\,\mathrm{Inj}} Y$$

Note. The locally smallness above is to restrict the size of $C/f := (C \cap \mathrm{Arr}(\mathcal{C}))/f$.

3.3 Homotopy

We discuss how the concept of homotopy is abstractly realized in model categories

Definition 3.10. Let $\mathcal{C}$ be a model category, $X \in \mathrm{Obj}(C)$.

1. A **path space object** $\mathrm{Path}(X)$ for X is a factorization of the diagonal $\Delta_X : X \to X \times X$ as

$$\Delta_X : X \xrightarrow[\in W]{i} \mathrm{Path}(X) \xrightarrow[\in \mathrm{Fib}]{(p_0,p_1)} X \times X$$

2. A **cylinder object** $\mathrm{Cyl}(X)$ for X is a factorization of the codiagonal $\nabla_X : X \sqcup X \to X$ as

$$\nabla_X : X \sqcup X \xrightarrow[\in \mathrm{Cof}]{(i_0,i_1)} \mathrm{Cyl}(X) \xrightarrow[\in W]{p} X$$