Notes of Introducion to Commutative Algebra

Cloudifold

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0 Notations

Category of commutative rings	: **CRing**
Coordinate ring of k-variety X	: $k[X]$
Element in polynomial ring $A[x]$	: $c \in A[x] := c = \sum_i c_i x^i \ (\forall i \ . \ c_i \in A)$
Jacobson radical of ring R	: $\mathrm{Jac}(R)$
Polynomial ring with n-variables over ring R	: $R[x]_n$ or $R[x_1, \ldots, x_n]$
Radical ideal of ideal I	: $\sqrt{I}$

1 Exercises of Chapter 1 : Rings and Ideals

Exercise 1.1. $A \in \mathbf{CRing}$, $x \in \sqrt{\langle 0 \rangle} \to u \in A^\times \to u + x \in A^\times$

Solution 1.1.

$$\dfrac{\dfrac{\dfrac{\dfrac{x \in \sqrt{\langle 0 \rangle}}{\exists d \in \mathbb{Z} \ . \ x^d = 0}}{(1+x) \cdot (\sum_{i=0}^{d-1} (-x)^i) = 1}\ \text{w1}}{1 + x \in A^\times} \qquad \dfrac{u \in A^\times}{1 + x \in A^\times \leftrightarrow u + ux \in A^\times} \qquad \dfrac{x \in \sqrt{\langle 0 \rangle}}{\forall r \in A \ . \ rx \in \sqrt{\langle 0 \rangle}}}{u + x \in A^\times}\ \text{w2}$$

$$\begin{aligned} &\text{w1} : 1 - x^n = (1 + x) \cdot \left(\sum_{i=0}^{n-1} x^i \right) \\ &\text{w2} : u(1 + u^{-1}x) = u + x, u^{-1}x \in A^\times \end{aligned}$$

$\square$

Exercise 1.4. $A \in \mathbf{CRing} \to \mathrm{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}$

Solution 1.4.

$$\dfrac{\dfrac{\dfrac{\text{lemma 1} \qquad \text{lemma 2}}{c \in \sqrt{\langle 0 \rangle}_{A[x]} \leftrightarrow \forall i \ . \ c_i \in \sqrt{\langle 0 \rangle}_A}}{q \in \mathrm{Jac}(A[x]) \to q \in \sqrt{\langle 0 \rangle}_{A[x]}}\ \text{w1} \qquad \dfrac{\text{def 1} \qquad \text{def 2}}{\mathrm{Jac}(A[x]) \supseteq \sqrt{\langle 0 \rangle}_{A[x]}}\ \text{w2}}{\mathrm{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}}$$

def 1 : $\operatorname{Jac}(A[x]) = \bigcap_{m \in \operatorname{Spec}_{\max}(A[x])} m$

def 2 : $\sqrt{\langle 0 \rangle}_{A[x]} = \bigcap_{P \in \operatorname{Spec}(A[x])} P$

lemma 1 : $q \in \operatorname{Jac}(A[x]) \leftrightarrow \forall y \in A[x]\ .\ 1 - qy \in A[x]^{\times}$ (Textbook Ch1 Prop 1.9)

lemma 2 : $c \in A[x]^{\times} \rightarrow, c_0 \in A^{\times}$ and $\forall \mathrm{i} \neq 0\ .\ \mathrm{c_i} \in \sqrt{\langle 0 \rangle}_{\mathrm{A}}$ (Textbook Ch1 Ex 2.i)

w1 : $q \in \operatorname{Jac}(A[x]) \rightarrow 1 - q \in A[x]^{\times} \rightarrow \forall i, q_i \in \sqrt{\langle 0 \rangle}_{A[x]} \rightarrow q \in \sqrt{\langle 0 \rangle}_{A[x]}$

w2 : $I \in \operatorname{Spec}_{\max} A \rightarrow I \in \operatorname{Spec} A$

□

Exercise 1.6. $A \in \mathbf{CRing} \rightarrow \left(I \nsubseteq \sqrt{\langle 0 \rangle}_A \rightarrow \exists\ e \in I, e^2 = e \neq 0\right) \rightarrow \operatorname{Jac} A = \sqrt{\langle 0 \rangle}_A$

Solution 1.6.

$$\dfrac{\dfrac{\dfrac{\dfrac{I \nsubseteq \sqrt{\langle 0 \rangle}_A \rightarrow \exists\ e \in I, e^2 = e \neq 0 \qquad \operatorname{Jac} A \nsubseteq \sqrt{\langle 0 \rangle}_A}{\exists e \in \operatorname{Jac} A, e^2 = e \neq 0}}{1 - e \in A^{\times} \text{ and } e(1-e) = 0}\ \text{lemma 1}}{e = e(1-e)(1-e)^{-1} = 0(1-e)^{-1} = 0}}{\perp} \Rightarrow \operatorname{Jac} A \subseteq \sqrt{\langle 0 \rangle}_A$$

$$\dfrac{\operatorname{Jac} A \subseteq \sqrt{\langle 0 \rangle}_A \qquad \operatorname{Jac}(A) \supseteq \sqrt{\langle 0 \rangle}_A \text{ (lemma 2)}}{\operatorname{Jac} A = \sqrt{\langle 0 \rangle}_A}$$

lemma 1 : $q \in \operatorname{Jac}(A[x]) \leftrightarrow \forall y \in A[x]\ .\ 1 - qy \in A[x]^{\times}$ (Textbook Ch1 Prop 1.9)

lemma 2 : Sol 1.4. w2

□