Notes of Introducion to Commutative Algebra

Cloudifold

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0 Notations

Category of commutative rings	: **CRing**
Coordinate ring of k-variety X	: $k[X]$
Element in polynomial ring $A[x]$	: $c \in A[x] := c = \sum_i c_i x^i$ ($\forall i \ . \ c_i \in A$)
Jacobson radical of ring R	: $\mathrm{Jac}(R)$
Polynomial ring with n-variables over ring R	: $R[x]_n$ or $R[x_1, \ldots, x_n]$
Nilradical of ring A	: $\sqrt{\langle 0 \rangle}_A$
Radical ideal of ideal I	: $\sqrt{I}$

1 Exercises of Chapter 1 : Rings and Ideals

Exercise 1.1. $A \in$ **CRing**, $x \in \sqrt{\langle 0 \rangle} \to u \in A^\times \to u + x \in A^\times$

Solution 1.1.

$$\cfrac{\cfrac{\cfrac{\cfrac{x \in \sqrt{\langle 0 \rangle}}{\exists d \in \mathbb{Z} \ . \ x^d = 0}}{(1+x) \cdot \left(\sum_{i=0}^{d-1}(-x)^i\right) = 1} \ \text{w1}}{1 + x \in A^\times} \quad \cfrac{u \in A^\times}{1 + x \in A^\times \leftrightarrow u + ux \in A^\times} \quad \cfrac{x \in \sqrt{\langle 0 \rangle}}{\forall r \in A \ . \ rx \in \sqrt{\langle 0 \rangle}} \ \text{w2}}{u + x \in A^\times}$$

$$\text{w1} : 1 - x^n = (1+x) \cdot \left(\sum_{i=0}^{n-1} x^i\right)$$

$$\text{w2} : u(1 + u^{-1}x) = u + x, u^{-1}x \in A^\times$$

$\square$

Exercise 1.4. $A \in$ **CRing** $\to \mathrm{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}$

Solution 1.4.

$$\cfrac{\cfrac{\cfrac{\text{lemma 1} \quad \text{lemma 2}}{c \in \sqrt{\langle 0 \rangle}_{A[x]} \leftrightarrow \forall i \ . \ c_i \in \sqrt{\langle 0 \rangle}_A}}{q \in \mathrm{Jac}(A[x]) \to q \in \sqrt{\langle 0 \rangle}_{A[x]}} \ \text{w1} \quad \cfrac{\text{def 1} \quad \text{def 2}}{\mathrm{Jac}(A[x]) \supseteq \sqrt{\langle 0 \rangle}_{A[x]}} \ \text{w2}}{\mathrm{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}}$$

def 1 : $\mathrm{Jac}(A[x]) = \bigcap\limits_{m \in \mathrm{Spec}_{\max}(A[x])} m$

def 2 : $\sqrt{\langle 0 \rangle}_{A[x]} = \bigcap\limits_{P \in \mathrm{Spec}(A[x])} P$

lemma 1 : $q \in \mathrm{Jac}(A[x]) \leftrightarrow \forall y \in A[x] \ . \ 1 - qy \in A[x]^{\times}$ $\hspace{2cm}$ (Textbook Ch1 Prop 1.9)

lemma 2 : $c \in A[x]^{\times} \to, c_0 \in A^{\times}$ and $\forall i \neq 0 \ . \ c_i \in \sqrt{\langle 0 \rangle}_A$ $\hspace{2cm}$ (Textbook Ch1 Ex 2.i)

w1 : $q \in \mathrm{Jac}(A[x]) \to 1 - q \in A[x]^{\times} \to \forall i, q_i \in \sqrt{\langle 0 \rangle}_{A[x]} \to q \in \sqrt{\langle 0 \rangle}_{A[x]}$

w2 : $I \in \mathrm{Spec}_{\max} A \to I \in \mathrm{Spec}\, A$

$\hspace{13cm}\square$

Exercise 1.6. $A \in \mathbf{CRing} \to \left(I \nsubseteq \sqrt{\langle 0 \rangle}_A \to \exists\, e \in I, e^2 = e \neq 0 \right) \to \mathrm{Jac}\, A = \sqrt{\langle 0 \rangle}_A$

Solution 1.6.

$$\dfrac{\dfrac{\dfrac{\dfrac{I \nsubseteq \sqrt{\langle 0 \rangle}_A \to \exists\, e \in I, e^2 = e \neq 0 \quad \mathrm{Jac}\, A \nsubseteq \sqrt{\langle 0 \rangle}_A}{\exists e \in \mathrm{Jac}\, A, e^2 = e \neq 0}}{1 - e \in A^{\times} \text{ and } e(1-e) = 0}\ \text{lemma 1}}{e = e(1-e)(1-e)^{-1} = 0(1-e)^{-1} = 0}}{\bot} \quad \Rightarrow \mathrm{Jac}\, A \subseteq \sqrt{\langle 0 \rangle}_A$$

$$\dfrac{\mathrm{Jac}\, A \subseteq \sqrt{\langle 0 \rangle}_A \quad \mathrm{Jac}(A) \supseteq \sqrt{\langle 0 \rangle}_A \ (\text{lemma 2})}{\mathrm{Jac}\, A = \sqrt{\langle 0 \rangle}_A}$$

lemma 1 : $q \in \mathrm{Jac}(A) \leftrightarrow \forall y \in A \ . \ 1 - qy \in A^{\times}$ $\hspace{2cm}$ (Textbook Ch1 Prop 1.9)

lemma 2 : Sol 1.4. w2

$\hspace{13cm}\square$