CW complexes

Cloudifold

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0 Basic Definitions and Lemmas

Definition 0.1. A **CW-complex** is a space constructed by successively attaching cells:
For $n \in \mathbb{N}, n \geq 0$, there are maps $\{\varphi_i : S^{n-1} \to X^{n-1}\}_{i\in I_n}$ (called characteristic maps). The way to construct X^n (called n-skeleton of X) is :
(starting from $X^{-1} = \emptyset$, if we start from $X^{-1} = A$, we say (X, A) is a **relative CW-complex**)

$$\begin{tikzcd}
\coprod_{i\in I_n} S^{n-1} \arrow[r, "\coprod_{i\in I_n}\varphi_i"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\
\coprod_{i\in I_n} D^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X^n
\end{tikzcd} \quad (pushout)$$

and the resulting CW-complex X is $\varinjlim\{X^0 \to \cdots \to X^n \to X^{n+1} \to \cdots\}$. The images of $\overset{\circ}{D}{}^n_i$ in X is called open cell e^n_i of X.

Definition 0.2. A is a subcomplex of CW-complex X iff for any open cell e^n_i of X, A satisfy: $A \cap e^n_i \neq \emptyset \implies \bar{e^n_i} \subseteq A$.
Pair of X and subcomplex A : (X, A) is called a CW-pair.

Definition 0.3. The Infinite Symmetric Product of a pointed space (X, x_0) is colimit of its n-th Symmetric Products ($\mathrm{SP}^n X := (\prod_{\{0,1,\ldots,n-1\}} X)/S_n$) :

$$\varinjlim\{\cdots \hookrightarrow \mathrm{SP}^n X \hookrightarrow \mathrm{SP}^{n+1} X \hookrightarrow \cdots\}$$
$$\{x_1, \ldots, x_n\} \mapsto \{x_0, x_1, \ldots, x_n\}$$

Definition 0.4. For $n \geq 1$, a map between pairs $f : (X, A) \to (Y, B)$ is an n-equivalence if:

- $f_*^{-1}(\mathrm{Im}(\pi_0 B \to \pi_0 Y)) = \mathrm{Im}(\pi_0 A \to \pi_0 X)$
- For all choices of basepoint a in A,

$$f_* : \pi_q(X, A, a) \to \pi_q(Y, B, f(a))$$

  is isomorphism for $1 \leq q \leq n-1$ and epimorphism for $q = n$.

Definition 0.5. A pair (X, A) of topological spaces is n-**connected** if $\pi_0(A) \to \pi_0(X)$ is surjection and $\pi_q(X, A) = 0$ for $1 \leq q \leq n$.

Definition 0.6. For topological spaces $A \hookrightarrow X$, A is a **strong deformation retract** of a neighborhood V in X if:
$\exists h : V \times I \to X$ such that
$\forall x \in V,\ h(x, 0) = x$
$h(V, 1) \subseteq A$
$\forall (a, t) \in A \times I,\ h(a, t) = a$

Definition 0.7. For topological spaces $i : A \hookrightarrow X$, A is a **deformation retract** of X if:
$\exists h : X \times I \to X$ such that
$\forall x \in X,\ h(x, 0) = x$
$h(X, 1) = A$
$\forall (a, t) \in A \times I,\ h(a, t) = a$
(That is, there are retraction $r : X \to A$ and homotopy $h : \mathrm{id}_X \simeq i \circ r \,\mathrm{rel}\, A$)
And $r := h(-, 1)$ is called a **deformation retraction**.

Definition 0.8. For topological spaces $A \hookrightarrow X$, a neighborhood V of A is **deformable** to A if:
$\exists h : X \times I \to X$ such that
$\forall x \in X,\ h(x, 0) = x$
$h(A \times I) \subseteq A$, $h(V \times I) \subseteq V$.
$h(V, 1) \subseteq A$

Definition 0.9. For a topological group G, a **relative G-(equivariant) CW-complex** (X, A) is a space constructed by successively attaching **G-equivariant cells** $G/H \times D^n$ on a G-space A: For $n \in \mathbb{N}, n \geq 0$, there are maps $\{\varphi_i : G/H_i \times S^{n-1} \to X^{n-1}\}_{i \in I_n}$ (called characteristic maps) where each H_i is closed subgroup of G and G acts trivially on D^n, S^{n-1}. The way to construct X^n (called n-skeleton of X) is:
(starting from $X^{-1} = A$ where A is an G-space)

$$\begin{tikzcd}
\coprod_{i \in I_n} (G/H_i \times S^{n-1}) \arrow[r, "\coprod_{i \in I_n} \varphi_i"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\
\coprod_{i \in I_n} (G/H_i \times D^n) \arrow[r, "\coprod_{i \in I_n} \bar{\phi}_i"] & X^n
\end{tikzcd} \quad \text{(pushout in category of } G\text{-spaces)}$$

The resulting X is $\varinjlim\{X^{-1} \to X^0 \to \cdots \to X^n \to X^{n+1} \to \cdots\}$. The images of $G/H_i \times \overset{\circ}{D}{}^n_i$ in X is called open n-cell of type G/H_i. ϕ_i is called the attaching map and $\varphi_i(G/H_i \times S^{n-1})$ is called the boundary of $\phi_i(G/H_i \times D^n)$. If $A = \emptyset$, then X is called a **G-(equivariant) CW-complex**.

A criterion of weak homotopy equivalence:

Lemma 0.1. *The following on a map $e : Y \to Z$ and any fixed $n \in \mathbb{N}$ are equivalent:*

1. *For any $y \in Y$, $e_* : \pi_q(Y, y) \to \pi_q(Z, e(y))$ is monomorphism for $q = n$ and is epimorphism for $q = n + 1$.*
2. *(HELP of (D^{n+1}, S^n)) Given maps $f : D^{n+1} \to Z$, $g : S^n \to Y$ and homotopy $h : f \circ i \simeq e \circ g$:*

$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow h"] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}$$

then we have extension $g^+ : D^{n+1} \to Y$ of g and $h^+ : f \simeq e \circ g^+$:

$$\begin{tikzcd}
S^n \arrow[r, hook] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] & Z \arrow[l, phantom, "h^+"]
\end{tikzcd}$$

3. *Conclusion above holds when the given h is $id_{f \circ i}$.*

Proof. Trivially *2.* implies *3.*

Our first goal : *3.* implies *1.*

Fix $n \in \mathbb{N}$. $\pi_n(e)$ is monomorphism:
For $n = 0$, *3.* says if we have path $e(y) \simeq e(y')$ then we have path $y \simeq y'$.That is to say e can not map two path-connected component to one.
For $n > 0$, *3.* says if $e \circ g$ is nullhomotopic, then $g : S^n \to Y$ could be extend to $g^+ : D^{n+1} \to Y$, which can be used to construct nullhomotopy of g.

Fix $n \in \mathbb{N}$. $\pi_{n+1}(e)$ is epimorphism:
For $[f] \in \pi_{n+1}(Z, e(y)) \cong [D^{n+1}, S^n; Z, e(y)]$, let $g := s \mapsto y$, the extension g^+ satisfy $e_*([g^+]) = [f]$, that proves e_* is epimorphism.

Second goal : *1.* implies *2.*

Fix g, f, h in the condition of *2.* first. And observe that $\pi_n(Y, y) = [S^n, *; Y, y]$, $\pi_{n+1}(Y, y) = [D^{n+1}, S^n; Y, y]$.
There is a map $f' : (D^{n+1}, S^n) \to Z$ homotopic to f defined by $f' = f \circ b(-, 1)$ where

$$
\begin{aligned}
b : CS^n \times I &\to CS^n \\
(\overline{(x,t)}, s) &\mapsto \begin{cases} \overline{(x, 1-2t)} & t \leq \frac{s}{2} \\ \overline{(x, \frac{t-s/2}{1-s/2})} & t \geq \frac{s}{2} \end{cases}
\end{aligned}
$$

(recall that $D^{n+1} \simeq CS^n$) Therefore we can replace f with f'. Using the epimorphism leads to $h' : e \circ {g^+}' \simeq f'$, using the monomorphism leads to $r : {g^+}' \circ i \simeq g$. Construct $g^+ := a(-, 1)$ using

$$
\begin{aligned}
a : CS^n \times I &\to Z \\
(\overline{(x,t)}, s) &\mapsto \begin{cases} r(x, s-2t) & t \leq \frac{s}{2} \\ {g^+}'(x, \frac{t-s/2}{1-s/2}) & t \geq \frac{s}{2} \end{cases}
\end{aligned}
$$

And that is the end of the proof:

$$
\begin{tikzcd}
S^n \arrow[r] \arrow[d] & D^{n+1} \arrow[dl, bend right] \arrow[dl, bend left] \arrow[dl, phantom, "\Uparrow b"] \arrow[d, bend right, "f'"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\Leftarrow"] \\
Y \arrow[r] \arrow[ur, phantom, "\Leftarrow h'"] & Z
\end{tikzcd}
$$

$\square$

1 Properties and Examples

Theorem 1.1. *Homotopy Extension and Lifting property:*
A : a topological space
X : result of successively attaching cells on A of dimensions $0, 1, \ldots, k$ ($k \leq n$)
$e : Y \to Z$: n-equivalence
$g : A \to Y$, $f : X \to Z$
$h : f|_A \simeq e \circ g$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow h"] & X \arrow[d, "f"] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Then there exists $g^+ : X \to Y$ extends g ($g^+|_A = g$)
and $h^+ : X \times I \to Z$ extends h, $h^+ : f \simeq e \circ g^+$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] \arrow[ur, phantom, "\swarrow h^+"] & Z
\end{tikzcd}
$$

Proof. It suffices to prove the case $A = S^{k-1}, X = D^k$, e is inclusion. (replace Z by M_e) Apply HEP of (D^k, S^{k-1}):

$$
\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d] & D^k \arrow[d] \arrow[ddr, bend left, "f"] & \\
S^{k-1} \times I \arrow[r] \arrow[drr, bend right, "h"'] & D^k \times I \arrow[dr, dashed, "\hat{h}"] & \\
 & & Z
\end{tikzcd}
$$

$f' := \hat{h}(-,1)$, replace f with f' the diagram would be strictly commute. Therefore, f' is map of pairs $(D^k, S^{k-1}) \to (Z, Y)$, $k \leq n$ implies f' is nullhomotopic, suppose $h^+ : D^k \times I \to Z$ is the nullhomotopy, then $g^+ := h^+(-,1)$ satisfy $g^+(D^k) \subseteq Y$.

$\square$

Note. In HELP, at condition $Y = Z$ and $e = \text{id}$, HELP says (X, A) have HEP

Corollary 1.2. *If*
A : a topological space
X : result of successively attaching cells on A of any dimensions
Then, (X, A) have HEP.

Theorem 1.3. *If X is an CW-complex, $e : Y \to Z$ is an n-equivalence, Then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$, and a surjection if $\dim X = n$.(Also valid for pointed case)*

Proof. Surjectivity:
Apply HELP of $(X, \emptyset)$ ((X, x_0) for pointed case) to obtain $e_*[g^+] \simeq [f]$:

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Injectivity ($\dim X < n$):
Suppose $[g_0], [g_1] \in [X, Y]$, $e_*[g_0] = e_*[g_1]$.
Let $f : e \circ g_0 \simeq e \circ g_1$ Apply HELP to $(X \times I, X \times \partial I)$:

$$
\begin{tikzcd}
X \times \partial I \arrow[r] \arrow[d, "g"'] & X \times I \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

$\square$

Corollary 1.4. *If X is a CW-complex, $e : Y \to Z$ is weak homotopy equivalence, then $e_* : [X, Y] \to [X, Z]$ is bijection.*

1.1 CW-approximation

Definition 1.1. A **CW-approximation** of $(X, A) \in \mathbf{Top(2)}$ is a CW-pair $(\widetilde{X}, \widetilde{A})$ and a weak homotopy equivalence of pairs $\varphi : (\widetilde{X}, \widetilde{A}) \to (X, A)$.

Theorem 1.5. *(Existence of CW-approximation) If X is path-connected pointed space (0-connected), then there is a CW-approximation $(\widetilde{X}, *) \xrightarrow{\phi} (X, *)$. If X is n-connected then $\widetilde{X}$ could be chosen to satisfy $\widetilde{X}^n = *$. (Moreover, each characteristic map of X is pointed)*

Proof. If X is n-connected, then $\phi_n : Y^n := * \to X$ is n-equivariance. Assmue inductively that we already have m-equivalence $Y^m \xrightarrow{\phi_m} X$ $(m \geq n)$, Our goal is construct Y^{m+1} and $\phi_{m+1} : Y^{m+1} \to X$.
Let

$$f_m^+ : \bigoplus_{a \in A} \mathbb{Z}_a \twoheadrightarrow \ker(\phi_{m*}) \subseteq \pi_m(Y^m)$$

be a free resolution of $\ker(\phi_{m*})$ ($\coprod_{a\in A} \mathbb{Z}_a$ if $m = 1$), and obtain a (unique up to homotopy) map $f_m : \bigvee_{a\in A} S_a^m \to Y^m$ defined by $f_m|_{S_a^m} := k_a$ where $[k_a] = f_m^+(1_a) \in [S^m, Y^m]_*$. We have: (since $[\phi_m \circ f_m] = 0$)

$$\begin{tikzcd} \bigvee_{a\in A} S_a^m \arrow[r, "f_m"] & Y^m \arrow[r] \arrow[dr, "\phi_m"'] & C_{f_m} \arrow[d, dashed, "\varphi_{m+1}"] \\ & & X \end{tikzcd}$$

C_{f_m} is a CW-complex with dim $= n+1$ with m-skeleton Y^m. $\varphi_{m+1*} : \pi_m(C_{f_m}) \to \pi_m(X)$ is isomorphism, but $\varphi_{m+1*} : \pi_{m+1}(C_{f_m}) \to \pi_{m+1}(X)$ is not necessarily an epimorphism.
Define the set $B := \pi_{m+1}(X) - \varphi_{m+1*}(\pi_{m+1}(C_{f_m}))$ and $Y^{m+1} := C_{f_m} \vee (\bigvee_{b\in B} S_b^{m+1})$.
Define ϕ^{m+1} by $\phi^{m+1}|_{C_{f_m}} := \varphi_{m+1}$ and $\phi^{m+1}|_{S_b^{m+1}} := r_b$ where $[r_b] = b \in [S^{m+1}, X]_*$.
$\widetilde{X} := \varinjlim_m \{Y^0 \hookrightarrow \cdots \hookrightarrow Y^m \hookrightarrow Y^{m+1} \hookrightarrow \cdots\}$, and $\phi = \varinjlim_m \phi_m$
If X is not path-connected, construct CW-approximation for each path-connected component. □

Note. The proof of existence of CW-approximation uses homotopy excision theorem (CW-triad version). Proof of CW-traid version does not need CW-approximation. There is no circular argument.

Proposition 1.6. *For any pair (X, A), there exists CW-approximation $\phi : (\widetilde{X}, \widetilde{A}) \to (X, A)$.*

Proof. Construct $\phi_A : \widetilde{A} \to A$ first and use analogue method in proof of theorem 1.5 with $Y^0 := \widetilde{A}$. □

Lemma 1.7. *φ, ψ are CW-approximations of X, Y, $f : X \to Y$, then*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varphi"] \arrow[d, dashed, "\exists \widetilde{f}"'] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutes up to homotopy, and $\widetilde{f}$ is unique up to homotopy.

Proof. Directly from $\psi_* : [\widetilde{X}, \widetilde{Y}] \to [\widetilde{X}, Y]$ is bijection. □

Theorem 1.8. *φ, ψ are CW-approximations of $(X, A), (Y, B)$, $f : (X, A) \to (Y, B)$, then*

$$\begin{tikzcd} (\widetilde{X}, \widetilde{A}) \arrow[r, "\varphi"] \arrow[d, dashed, "\exists \widetilde{f}"'] & (X, A) \arrow[d, "f"] \\ (\widetilde{Y}, \widetilde{B}) \arrow[r, "\psi"'] & (Y, B) \end{tikzcd}$$

commutes up to homotopy, and $\widetilde{f}$ is unique up to homotopy.

Proof. Apply Lemma 1.7 to obtain map $\widetilde{f}_A : \widetilde{A} \to \widetilde{B}$ and homotopy $h : \psi|_{\widetilde{B}} \circ \widetilde{f}_A \simeq f \circ \varphi|_{\widetilde{A}}$ Use HELP of $(\widetilde{X}, \widetilde{A})$ to extend it:

$$\begin{tikzcd}
\widetilde{A} \arrow[r, hook] \arrow[d, "\widetilde{f}_A"'] & \widetilde{X} \arrow[d] \arrow[ddl, "\widetilde{f}"'] \\
\widetilde{B} \arrow[d, hook] & X \arrow[d] \\
\widetilde{Y} \arrow[r] & Y
\end{tikzcd}$$

ψ_* is bijection implies the uniqueness up to homotopy of $\widetilde{f}$. $\square$

Theorem 1.9. *(Whitehead's Theorem)*
Every n-equivalence between CW-complexes whose dimension is lower than n, is homotopy equivalence. Every weak homotopy equivalence between CW-complexes is homotopy equivalence.

Proof. $e : Y \to Z$ induce bijections $[Y, Y] \to [Y, Z]$ and $[Z, Y] \to [Z, Z]$, $[f] = e_*^{-1}[\mathrm{id}_Z]$ implies $[e \circ f] = [\mathrm{id}_Z]$ and $[e \circ f \circ e] = [e]$ $([f \circ e] = e_*^{-1}[e] = [\mathrm{id}_Y])$.

$\square$

Corollary 1.10. *CW-approximation is unique up to homotopy.*

Example 1.1. Polish circle (Warsaw circle) : closed topologist's sine curve. It is n-connected forall n but not contractible.

Definition 1.2. A cellular map between CW-pairs is $g : (X, A) \to (Y, B)$ such that $g(A \cup X^n) \subseteq B \cup Y^n$.

Theorem 1.11. *For any map between CW-pairs $f : (X, A) \to (Y, B)$ there exists a cellular map g such that $g \simeq f \operatorname{rel} A$*

Proof. Construct g inductively:
Start from $A \cup X^0$:
take paths $\gamma_i : f(x_i) \simeq y_i$, where y_i is any point in Y^0 and $x_i \in X^0 - A$.
Construct $h_0 : (X^0 \cup A) \times I \to Y$: $h_0|_A(a, t) := f(a)$, $h_0|_{X^0 - A}(x_i, t) := \gamma_i(t)$. This is a homotopy from f to $g_0 := h_0(-, 1) : A \cup X^0 \to B \cup Y^0$
Inductive step:
Assume $g_n : A \cup X^n \to B \cup Y^n$ and homotopy $h_n : f|_{A \cup X^n} \simeq g_n$ is given, try to construct g_{n+1}:
For each characteristic map $\varphi_i : S^n \to X^n$, take the resulting cell map $\varphi_i^+ : D^{n+1} \to X^{n+1}$ and use HELP of (D^{n+1}, S^n):

$$\begin{tikzcd}
S^n \arrow[r, hook] \arrow[d, "\varphi_i"'] & D^{n+1} \arrow[d, "\varphi_i^+"] \arrow[ddl, "g_{n+1,i}"'] \\
X^n \arrow[d, "g_n"'] & X^{n+1} \arrow[d, "f"] \arrow[dl, Rightarrow, "h_{n+1,i}"] \\
B \cup Y^{n+1} \arrow[r, hook] & Y
\end{tikzcd}$$

Glue all $g_{n+1,i}$ and $h_{n+1,i}$ to produce g_{n+1} and $h_{n+1} : f|_{A \cup X^{n+1}} \simeq g_{n+1}$.
Final stage:
Maps g_n determine a cellular map $g : X \to Y$ since X has the final topology determined by skeletons. $\square$

Corollary 1.12. *If X is a pointed CW-complex, then the inclusions $X^{n+1} \hookrightarrow X^{n+2} \hookrightarrow \cdots \hookrightarrow X$ induce $\pi_n(X^{n+1}) \cong \pi_n(X^{n+2}) \cong \cdots \cong \pi_n(X)$.*

Proof. For $k \geq 1$, $X^{n+k} \hookrightarrow X^{n+k+1}$ induces epimorphism $\pi_n(X^{n+k}) \twoheadrightarrow \pi_n(X^{n+k+1})$
since every $f : (S^n, *) \to (X^{n+k+1}, *)$ is homotopic (rel $*$) to an $g : (S^n, *) \to (X^n, *) \hookrightarrow (X^{n+k}, *)$.
Now we want to prove it is monomorphism, that is, $i_*[f] = 0 \implies [f] = 0$
If $h : (S^n, *) \times I \to X^{n+k+1}$ is a nullhomotopy in X^{n+k+1}
of a map $f : (S^n, *) \to (X^{n+k}, *) \hookrightarrow (X^{n+k+1}, *)$,
then $h : (CS^n, S^n) \to (X^{n+k+1}, X^{n+k})$ is homotopic (rel S^n) to an $h' : (CS^n, S^n) \to (X^{n+k}, X^{n+k})$,
which is equivalent to $h' : S^n \times I \to X^{n+k}$ with $h(S^n, 1) = *$, $h(*, t) = *$, $h|_{S^n \times \{0\}} = f$.

□

Lemma 1.13. *If (X, A) is CW-pair and all cells of $X - A$ have* $\dim > n$*, then (X, A) is n-connected.*

Proof. For each $q \leq n$, and each $[f] \in \pi_q(X, A)$, $f \simeq g \operatorname{rel} S^{q-1}$ where g is an cellular map. (use theorem 1.11) $\pi_q(X, A) \ni [g] = 0$ since $g(S^{n-1} \cup e^n) = g(D^n) \subseteq A \cup X^n = A$.

□

1.2 Operation of CW-complexes

Product of CW-complexes:

Example 1.2. Product topology of two CW-complexes does not coincide with the final topology (union topology):
X (star of countably many edges) : $X = X^1 = \bigvee_{n \in \omega} I_n$
Y (star of ω^ω many edges) : $Y = Y^1 = \bigvee_{f \in \omega^\omega} I_f$ ($(I_n, 0) \cong (I_f, 0) \cong (I, 0)$)
Consider subset H of $X \times Y$: $H := \{(\frac{1}{f(n)+1}, \frac{1}{f(n)+1}) \in I_n \times I_f \mid n \in \omega, f \in \omega^\omega\}$.
H is closed under the final topology since every cell of $X \times Y$ contains at most one point of H.
But closure of H contains $(0, 0)$ at product topology:
Let $U \times V$ be an open neighborhood (at product topology) of $(0, 0)$, let $g : \omega \to \omega - 0$ be an increasing function such that forall $n \in \omega, [0, \frac{1}{g(n)}) \subseteq U \cap I_n$, let $k \in omega$ be sufficiently large that $\frac{1}{g(k)+1} \subseteq V \cap I_g$, then $(\frac{1}{g(k)+1}, \frac{1}{g(k)+1}) \in U \times V \cap H$.

Proposition 1.14. *X and Y are CW-complexes, $X \times Y$ is CW-complex if*
X or Y is locally compact
or
both X and Y have countably many cells.

Another way to realize $X \times Y$ as CW-complex is to change its topology to the compactly generated topology $k(X \times Y)$:

Definition 1.3. For subspace A of X, A is compactly closed if

$$\forall \text{ compact space } K \quad \forall \text{ continuous } g : K \to X \quad g^{-1}(A) \text{ is closed in } K$$

Definition 1.4. X is k-space if any compactly closed subset is closed.

Definition 1.5. X is weak Hausdorff if

$$\forall \text{ compact space } K \quad \forall \text{ continuous } g : K \to X \quad g(K) \text{ is closed in } K$$

Definition 1.6. The k-ification of a space X is defined by: $k(X) := (X, \tau)$
where $\tau = \{X - A \mid A \text{ is compactly closed set}\}$

Definition 1.7. X is compactly generated space if it is k-space and weak Hausdorff.

Note. If X is weak Hausdorff, then $A \subseteq X$ is compactly closed iff

$$\forall \text{ compact subspace } K \subseteq X$$
$$A \cap K \text{ is closed in } X$$

If X is a CW-complex, then the topology defined on $k(X)$ automatically coincide with the final topology induced by its CW-complex structure. We have CW-complex structure of $k(X \times Y)$ is given by:

$$\begin{tikzcd}
\partial I^n \times I^m \cup I^n \times \partial I^m \arrow[r] \arrow[d] & X^{n-1} \times Y^m \cup X^n \times Y^{m-1} \arrow[d] \\
I^n \times I^m \arrow[r] & X^n \times Y^m
\end{tikzcd}$$

Furthermore, the k-ification is right adjoint of the inclusion functor i:

$$\begin{tikzcd}
\mathbf{TOP}_{\mathbf{CptGen}} \arrow[r, hook, bend left, "i"] \arrow[r, phantom, "\perp"] & \mathbf{TOP}_{\mathbf{weakHaus}} \arrow[l, bend left, "k(-)"]
\end{tikzcd}$$

This allows us to define the CW-complex structure on any limit of CW-complexes: $\varprojlim_i X_i \approx \varprojlim_i k(X_i) \approx k(\varprojlim_i X_i)$ ($X \approx k(X)$ and right adjoint preserve limits).

Quotient of CW-pair:

Proposition 1.15. *For CW-complex X and subcomplex A, the Quotient space X/A have a CW-complex structure induced by X and A.*

Proof. Suppose the characteristic maps of X are indexed by $\{I_n\}_{n\in\mathbb{N}}$ and of A are indexed by $\{I'_n\}_{n\in\mathbb{N}}$ ($I'_n \subseteq I_n$). Then the characteristic maps of X/A are indexed by $\{K_n\}_{n\in\mathbb{N}}$, which defined below:
$K_0 := (I_0 - I'_0) \cup \{i_0\}$ where i_0 is an arbitrary element in I'_0
$K_n := I_n - I'_n$ for $n > 0$.
Verify the maps determine the CW-complex structure:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d] & X^{n-1}/A^{n-1} \arrow[d] \arrow[ddd, bend left=60] & \\
D^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right=20, start anchor=south, to=Z] & X^n \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X^n/A^{n-1} \arrow[d] \arrow[dd, dashed, bend right=40] & A^n \arrow[l] \arrow[d] \arrow[dl, phantom, "\urcorner", very near end] \\
& & X^n/A^n \arrow[d, dashed] & * \arrow[l] \arrow[dl] \\
& & |[alias=Z]| Z &
\end{tikzcd}$$

$\square$

Smash product of CW-complexes:

Proposition 1.16. *If (X, x_0) , (Y, y_0) are pointed CW-complexes with both countably many cell, and $X^{r-1} = \{x_0\}$, $Y^{s-1} = \{y_0\}$, then $X \wedge Y := X \times Y / X \vee Y$ is an $(r + s - 1)$-connected CW-complex.*

Proof. $X \times Y$ is CW-complex with cells of the form $e^n_{i,X} \times \{y_0\}$, $\{x_0\} \times e^m_{j,Y}$ or $e^n_{i,X} \times e^m_{j,Y}$ for $n \geq r$, $m \geq s$. Cells of the first two forms are contianed in $X \vee Y$, therefore $(X \wedge Y)^{r+s-1} = *$. $\square$

Corollary 1.17. *If X is a pointed CW-complex, then $\Sigma^n X$ is an $(n-1)$-connected CW-complex.*

1.3 Properties of Infinite Symmetric Product

Functoriality:

Pointed map $f : X \to Y$ induces

$$\begin{aligned} f_n : \mathrm{SP}^n X &\to \mathrm{SP}^n Y \\ \{x_1, \dots, x_n\} &\mapsto \{f(x_1), \dots, f(x_n)\} \end{aligned}$$

$$\begin{tikzcd} {} \arrow[r] & \mathrm{SP}^n X \arrow[r] \arrow[d, "f_n"] & \mathrm{SP}^{n+1} X \arrow[r] \arrow[d, "f_{n+1}"] & {} \\ {} \arrow[r] & \mathrm{SP}^n Y \arrow[r] & \mathrm{SP}^{n+1} Y \arrow[r] & {} \end{tikzcd}$$

Which induces map $\mathrm{SP}\, f : \mathrm{SP}\, X \to \mathrm{SP}\, Y$. And Functorial properties are directly from the constructions above.

$\mathrm{SP}(X_1 \vee X_2) \approx \mathrm{SP}(X_1) \times \mathrm{SP}(X_2)$, the homeomorphism is given by:

$$\begin{aligned} \mathrm{SP}(X_1) \times \mathrm{SP}(X_2) &\leftrightarrows \mathrm{SP}(X_1 \vee X_2) \\ (\{a_1, a_2, \cdots, a_k\}, \{b_1, b_2, \cdots, b_m\}) &\mapsto \{a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_m\} \end{aligned}$$

Commute with directed colimit:
Suppose P is a directed poset (that is $\forall x, y \in P,\ \exists z \in P,\ x \le z, y \le z$) and X_i are pointed spaces indexed by P satisfying $i \le j \implies X_i \subseteq X_j$.
Then $\mathrm{SP}^n(\varinjlim_i X_i) \approx \varinjlim_i(\mathrm{SP}^n X_i)$
(Proof is obtained by showing that $\mathrm{SP}^n f$ is continuous iff f is, which implies final topology on $\varinjlim_i(\mathrm{SP}^n X_i)$ agree on $\mathrm{SP}^n(\varinjlim_i X_i)$)

Suppose $i : A \hookrightarrow X$ is an pointed inclusion, then $\mathrm{SP}\, i : \mathrm{SP}\, A \hookrightarrow \mathrm{SP}\, X$ is also inclusion. Furthermore, if A is open (or closed) in X, then $\mathrm{SP}\, A$ is open (or closed) in $\mathrm{SP}\, X$.

CW-complex structure:

We can have natural CW-complex structure on $\prod_n X$ by applying $k(-)$. following theorems allows us to prove that $\mathrm{SP}^n X = \prod_n X / S_n$ have a CW-complex structure.

Definition 1.8. G **acts cellularly** on a CW-complex X if:

$$\begin{aligned} \forall g \in G, e_i^n &\text{ is open } n\text{-cell (of } X) \\ g(e_i^n) = e_j^n &\text{ is open } n\text{-cell (of } X) \end{aligned}$$

and $g(e_i^n) = e_i^n$ implies $g|_{e_i^n} = \mathrm{id}_{e_i^n}$.

Lemma 1.18. *If G is a discrete group, X is CW-complex with G cellularly act on X. Then X is a G-CW-complex with n-skeleton X^n.*

Proof. The goal is to show X^n is obtained from X^{n-1} by attaching G-equivariant cells. Since $\coprod_{i \in I_n} Y = I_n \times Y$ (I_n with discrete topology). We have:

$$\begin{tikzcd} I_n \times S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & X^{n-1} \arrow[d] \\ I_n \times D^n \arrow[r, "\phi"'] \arrow[ur, phantom, "\ulcorner"] & X^n \end{tikzcd}$$

G acts cellularly on open n-cells implies G acts on I_n. Decomposite I_n into disjoint unions of obrits $\coprod_{\alpha \in A} I_\alpha$ choose G-isomorphisms

$$\begin{aligned} G/H_\alpha &\cong I_\alpha \\ gH_\alpha &\mapsto g i_\alpha \end{aligned}$$

And we have a well-defined G-map.

$$\begin{aligned}\phi_\alpha|_{e^n} : G/H_\alpha \times e^n \cong I_\alpha \times e^n &\to X^n \\ (gH_\alpha, x) \mapsto (gi_\alpha, x) &\mapsto \phi_{gi_\alpha}(x) = g\phi_{i_\alpha}(x)\end{aligned}$$

Since we have $e^n = \overset{\circ}{D}{}^n$, we obtain the following (by continuity):

$$\begin{aligned}\phi_\alpha : G/H_\alpha \times D^n &\to X^n \\ (gH_\alpha, x) &\mapsto g\phi_{i_\alpha}(x) \\ \phi_\alpha|_{S^{n-1}} = \varphi_\alpha : G/H_\alpha \times S^{n-1} &\to X^{n-1} \\ (gH_\alpha, s) &\mapsto g\varphi_{i_\alpha}(s)\end{aligned}$$

Let $\varphi' := \coprod_{\alpha\in A} \varphi_\alpha$ and $\phi' := \coprod_{\alpha\in A} \phi_\alpha$ we have:

$$\begin{tikzcd}
\coprod_{\alpha\in A}(G/H_\alpha \times S^{n-1}) \arrow[r, "\varphi'"] \arrow[d, hook] & X^{n-1} \arrow[d] \arrow[ddr, bend left, "f"] & \\
\coprod_{\alpha\in A}(G/H_\alpha \times D^n) \arrow[r, "\phi'"'] \arrow[drr, bend right, "\coprod f_\alpha"'] & X^n \arrow[dr, dashed, "\exists! f^+"] & \\
 & & Z
\end{tikzcd}$$

Verify it is indeed a pushout of G-spaces: f^+ (is already determined uniquely as map between G-sets) is map between G-spaces.
Since X have compactly generated topology, f^+ is continuous on each compact subspace of X implies f^+ is continuous on each compactly closed subspace of X^n, which implies f^+ is continuous on total X^n.
f^+ is continuous on each closed n-cell $\{gH_\alpha\} \times D^n$ and f^+ is continuous on X^{n-1} implies f^+ is continuous on each compact subspace. (since each compact subspace intersect finitely with n-cells and X^{n-1} (We use X^n is T_2 to construct open cover))

□

Theorem 1.19. *For any topological group morphism $\phi : H \to G$ we have induced functors: pullback action:*

$$\begin{aligned}G\text{−}\mathbf{TOP} &\xrightarrow{\phi^*} H\text{−}\mathbf{TOP} \\ (\alpha(-,-) : G \times X \to X) &\longmapsto (\alpha(\phi(-),-) : H \times X \to X) \\ (f : X \to Y) &\longmapsto (f : X \to Y)\end{aligned}$$

induced action:

$$\begin{aligned}H\text{−}\mathbf{TOP} &\xrightarrow{G\times_H -} G\text{−}\mathbf{TOP} \\ X &\longmapsto G \times_H X := (G \times X)/[\ (g\phi(h), x) \sim (g, hx) \mid h \in H] \\ (f : X \to Y) &\longmapsto (\mathrm{id}_G \times_H f : G \times_H X \to G \times_H Y)\end{aligned}$$

Which are adjunctions:

$$\begin{tikzcd}
H\text{−}\mathbf{TOP} \arrow[r, bend left, "G\times_H -"] \arrow[r, phantom, "\perp"] & G\text{−}\mathbf{TOP} \arrow[l, bend left, "\phi^*"]
\end{tikzcd}$$

Proof. By G-equivariance, f is determined uniquely by its restriction $f|_{\phi(H)\times_H X}$. And $\tilde{f} : X \to$

$\phi^*(Y)$ uniquely determine a map $\phi(H) \times_H X \to Y$.

$$
\begin{array}{ccccccc}
(G \times_H X & \xrightarrow{\quad f \quad} & Y) & \longmapsto & (X = \phi(H) \times_H X & \xrightarrow{f|_{H \times_H X}} & Y) \\
\Cup & & \Cup & & \Cup & & \Cup \\
(g\phi(h), x) & \longmapsto & g\phi(h)f(x) & & (\phi(h), x) = (e, hx) & \longmapsto & \phi(h)x \\
& & & & X & \xrightarrow{\quad \tilde{f} \quad} & \phi^*(Y)
\end{array}
$$

Naturality:

$$
\begin{array}{ccccccc}
(G \times_H X' & \xrightarrow{\mathrm{id}_G \times_H f'} & G \times_H X & \xrightarrow{\quad f \quad} & Y & \xrightarrow{\quad f'' \quad} & Y') \\
\Cup & & \Cup & & \Cup & & \Cup \\
(g, hx') & \longmapsto & (g, hf'(x)) & \longmapsto & g\phi(h)f(f'(x)) & \longmapsto & g\phi(h)f''(f(f'(x)))
\end{array}
$$

$$\leftrightarrows$$

$$
\begin{array}{ccccccc}
(X' & \xrightarrow{\quad f' \quad} & X = \phi(H) \times_H X & \xrightarrow{f|_{H \times_H X}} & \phi^*(Y) & \xrightarrow{\phi^*(f'')} & \phi^*(Y')) \\
\Cup & & \Cup & & \Cup & & \Cup \\
(hx') & \longmapsto & (\phi(h), f'(x')) = (e, hf'(x')) & \longmapsto & \phi(h)f(f'(x)) & \longrightarrow & \phi(h)f''(f(f'(x)))
\end{array}
$$

$\square$

Proposition 1.20. *If (X, A) is relative G-equivariant CW-complex, then $(X/G, A/G)$ is relative CW-complex with n-skeleton X^n/G.*

Proof.

$$
\begin{array}{ccc}
\coprod_{i \in I_n} S^{n-1} & \longrightarrow & X^{n-1}/G \\
\big\downarrow & & \big\downarrow \\
\coprod_{i \in I_n} D^n & \longrightarrow & X^n/G
\end{array}
$$

Is still pushout since $-/G = 1 \times_G -$, and left adjoint preserves colimits.

$\square$

Since $k(\prod_n X)$ have CW-complex structure, and S_n (as a discrete group) acts cellularly on it, $k(\prod_n X)$ is an S_n-equivariant CW-complex. Therefore $\mathrm{SP}^n X = k(\prod_n X)/S^n$ is CW-complex. Since $\mathrm{SP}\, X = \varinjlim\{\mathrm{SP}^1 X \hookrightarrow \cdots \hookrightarrow \mathrm{SP}^n X \hookrightarrow \mathrm{SP}^{n+1} X \hookrightarrow \cdots\}$, $\mathrm{SP}\, X$ is also a CW-complex.

Pointed homotopy $h : X \times I \to Y$ induces

$$
\begin{aligned}
h_n &: \mathrm{SP}^n X \times I \to \mathrm{SP}^n Y \\
(\{x_1, \ldots, x_n\}, t) &\mapsto \{h(x_1, t), \ldots, h(x_n, t)\}
\end{aligned}
$$

which induces $\operatorname{SP} h : \operatorname{SP} X \times I \to \operatorname{SP} Y$.

Then we observe:
$f \simeq g$ implies $\operatorname{SP} f \simeq \operatorname{SP} g$,
$e : X \to Y$ is homotopy equivalence implies $\operatorname{SP} e : \operatorname{SP} X \to \operatorname{SP} Y$ is,
X is contractible implies $\operatorname{SP}^n X$ and then $\operatorname{SP} X$ is.

Theorem 1.21. *(Dold-Thom Theorem)*
If X is T_2 space and A is closed path-connected subspace of X, and there is neighborhood V deformable to A in X.
Then the quotient map $q : X \to X/A$ induces quasi-fibration $\operatorname{SP} q : \operatorname{SP} X \to \operatorname{SP}(X/A)$, which satisfy $\forall x \in \operatorname{SP}(X/A)$, $(\operatorname{SP} q)^{-1}\{x\} \simeq \operatorname{SP} A$.

Corollary 1.22. *If X , Y are T_2 spaces and Y is connected, $f : X \to Y$. Then consider $X \to Y \to C_f \to \Sigma X$, the map $p : C_f \to \Sigma X$ induces quasi-fibration $\operatorname{SP} p : \operatorname{SP} C_f \to \operatorname{SP}(\Sigma X)$ with fiber $\operatorname{SP} Y$.*

Corollary 1.23. *If X is T_2 and path-connected, then for any $q \geq 0$, there is $\pi_{q+1}(\operatorname{SP}(\Sigma X)) \cong \pi_q(\operatorname{SP} X)$.*

Proof. CX is contractible implies $\operatorname{SP} CX$ is contractible, use the exat homotopy sequence of quasi-fibration to see:

$$\longrightarrow \pi_{q+1}(\operatorname{SP} CX) \longrightarrow \pi_{q+1}(\operatorname{SP} \Sigma X) \xrightarrow[\partial]{\cong} \pi_q(\operatorname{SP} X) \longrightarrow \pi_q(\operatorname{SP} CX) \longrightarrow$$

□

Note. The inverse of the isomorphism ∂ above is given by

$$[S^q, \operatorname{SP} X] \ni [g] \mapsto [\Sigma g] \in [S^{q+1}, \Sigma \operatorname{SP} X]$$

$(\Sigma \operatorname{SP} X \cong \operatorname{SP} \Sigma X)$. Because ∂ is given by:

$$\begin{array}{ccccc}
\pi_q(\operatorname{SP} \Sigma X) & \longrightarrow & \pi_q(\operatorname{SP} CX, \operatorname{SP} X) & \longrightarrow & \pi_{q-1}(\operatorname{SP} X) \\
\Cup & & \Cup & & \Cup \\
[f] & \longmapsto & [\hat{f}] & \longmapsto & [\hat{f}|_{S^{q-1}}] \\
 & & & & \\
[p \circ Cg] = [\Sigma g] & \longleftarrow & [Cg] & \longleftarrow & [g]
\end{array}$$

. □

Corollary 1.24. *If X is T_2 space and A is path-connected subspace of X, then the canonical map $\operatorname{SP}(X \cup (A \times I)) \to \operatorname{SP}(X \cup CA)$ is a quasi-fibration with fiber $\operatorname{SP} A$.*

Theorem 1.25. *If X is T_2 space and A is path-connected subspace of X, and $A \hookrightarrow X$ is a cofibration.*
Then the quotient map $q : X \to X/A$ induces quasi-fibration $\operatorname{SP} q : \operatorname{SP} X \to \operatorname{SP}(X/A)$, which satisfy $\forall x \in \operatorname{SP}(X/A)$, $(\operatorname{SP} q)^{-1}\{x\} \simeq \operatorname{SP} A$.

Proof. If $A \hookrightarrow X$ is cofibration, then $X \cup CA \simeq X/A$ and $X \cup (A \times I) \simeq X$. □

Proposition 1.26. *The inclusion $S^1 \to \operatorname{SP} S^1$ is homotopy equivalence, therefore $\pi_q(S^1) \cong \pi_q(\operatorname{SP} S^1)$.*

Proof. $S^1 \simeq S^2 - \{0, \infty\}$
$\operatorname{SP}^n S^2 = \{\{a_1, \ldots, a_n\} \mid a_i \in \mathbb{C} \cup \{\infty\}\} = \{\prod_{\{a_1,\ldots,a_n\}}(z - a_i) \mid a_i \in \mathbb{C} \cup \{\infty\}\}$ where $(z - \infty) := 1$
$\operatorname{SP}^n S^2 = \{f \in \mathbb{C}[z] - \{0\} \mid \deg(f) \leq n\} = \mathbb{CP}^n$

$\operatorname{SP}^n(S^2 - \{0, \infty\}) = \{f \in \mathbb{C}[z] - \{0\} \mid \deg(f) \leq n, f_n \neq 0, f_0 \neq 0\} = \mathbb{C}^n - \mathbb{C}^{n-1} \times 0 = \mathbb{C}^{n-1} \times (\mathbb{C} - 0)$
it have the same homotopy type of S^1

Corollary 1.27. $\pi_q(\operatorname{SP} S^n) = \mathbb{Z}$ *if $q = n$, otherwise* $\pi_q(\operatorname{SP} S^n) = 0$. *(use corollary of 1.21 to see* $\pi_{q+1}(\operatorname{SP} \Sigma X) \cong \pi_q(\operatorname{SP} X)$*)*

2 Homology Groups

2.1 Reduced Homology Groups

Definition 2.1. For a path-connected pointed CW-complex X, define its **n-th reduced homology group** for $n \geq 0$:

$$\tilde{H}_n(X) := \pi_n(\mathrm{SP}\, X)$$

Note. All reduced homology groups are abelian since $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$. Thus, we can extend the definition above to those X which does not necessarily be path-connected.

As SP, $\tilde{H}_n$ also satisfy functoriality. Furthermore, $\tilde{H}_n$ maps homotopic maps $f \simeq g$ to identical maps $f_* = g_*$. (SP maps homotopic maps to homotopic maps)

Exact Property:

Proposition 2.1. *For any pointed map between CW-complexes $f : X \to Y$, we have an exact sequence:*

$$\tilde{H}_n(X) \xrightarrow{f_*} \tilde{H}_n(Y) \xrightarrow{i_*} \tilde{H}_n(C_f)$$

where C_f is the mapping cone of f, $i : Y \hookrightarrow C_f$.

Proof. $Z_f := Y \cup_f (X \times I)/\{x_0\} \times I$ is the **reduced mapping cylinder** of f.
$q : Z_f \to C_f$ is defined by

$$y \mapsto y$$
$$\overline{(x,t)}^{Z_f} \mapsto \overline{(x,t)}^{C_f}$$

By Dold-Thom theorem, the induced map $\mathrm{SP}\, q$ is quasi-fibration $\mathrm{SP}\, Z_f \to \mathrm{SP}\, C_f$ with fiber $\mathrm{SP}\, X$.
By definition of quasi-fibration, we have

$$\pi_n(\mathrm{SP}\, X) \cong \tilde{H}_n(X) \xrightarrow{f_*} \pi_n(\mathrm{SP}\, Z_f) \cong \tilde{H}_n(Y) \xrightarrow{i_*} \pi_n(\mathrm{SP}\, C_f) = \tilde{H}_n(C_f)$$

□

Proposition 2.2. *There does not exist retraction $r : \mathbb{D}^n \to S^{n-1}$.*

Proof. $id = r \circ i : \mathbb{S}^{n-1} \to \mathbb{D}^n \to \mathbb{S}^{n-1}$ induces

$$id_* = r_* \circ i_* : \mathbb{Z} \cong \tilde{H}_{n-1}\mathbb{S}^{n-1} \to \tilde{H}_{n-1}\mathbb{D}^n \cong 0 \to \tilde{H}_{n-1}\mathbb{S}^{n-1} \cong \mathbb{Z}$$

which lead to contradiction. □

Theorem 2.3. *Fix-point theorem:*
If $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then exist $x_0 \in \mathbb{D}^n$ such that $x_0 = f(x_0)$.

Proof. (non-constructive) No such x_0 implies $\forall x \in \mathbb{D}^n, f(x) \neq x$ therefore, we can construct continuous retraction $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ by
$r(x)$:= the intersection of "ray starting from $f(x)$ to x" and $\mathbb{S}^{n-1}$. Contradict to 2.2.

Definition 2.2. Let (X, A) be an CW-pair, define the **n-th homology group** for $n \in \mathbb{N}$ of (X, A) be:

$$H_n(X, A) := \tilde{H}_n(X \cup CA)$$

And for single space:

$$H_n(X) := H_n(X, \emptyset) = \tilde{H}(X + 1)$$

where $X + 1 := X \sqcup *$.

Note. Map between CW-pair $f : (X, A) \to (Y, B)$, induces map $\bar{f} : X \cup CA \to Y \cup CB$ defined by $(x, t) \mapsto (f(x), t)$, which induces $f_* : \tilde{H}_n(X \cup CA) \to \tilde{H}_n(Y \cup CB)$ for any $n \in \mathbb{N}$.

2.2 Axioms for Homology

Definition 2.3. A (Ordinary) Homology Theory (on **TOP** with coefficient $G \in \mathbf{Ab}$) is functors $\{H_n(-,-;G) : \mathbf{TOP(2)} \to \mathbf{Ab}\}_{n\in\mathbb{Z}}$,
with natural transformations $\partial_{n,(X,A)} : H_n(X,A;G) \to H_n(A,\emptyset;G)$ (called connecting homomorphism)
satisfying following axioms:

- **Dimension**:
  $H_0(*,\emptyset;G) = G$, for any $n \neq 0$, $H_n(*,\emptyset;G) = 0$.

- **Weak Equivalence**:
  Weak equivalence $f : (X,A) \to (Y,B)$ induces isomorphism
  $$f_* : H_*(X,A;G) \to H_*(Y,B;G)$$

- **Long Exact Sequence**:
  For any $(X,A) \in \mathbf{TOP(2)}$, maps $A \hookrightarrow X$ and $(X,\emptyset) \to (X,A)$ induce a long exact sequence together with ∂:
  $$\cdots \to H_{q+1}(A;G) \to H_{q+1}(X;G) \to H_{q+1}(X,A;G) \to H_q(A;G) \to \cdots$$
  where $H_n(X;G) := H_n(X,\emptyset;G)$.

- **Additivity**:
  If $(X,A) = \coprod_\lambda (X_\lambda, A_\lambda)$ in $\mathbf{TOP(2)}$, then inclusions $i_\lambda : (X_\lambda, A_\lambda) \to (X,A)$ induces isomorphism
  $$(\bigoplus i_{*,\lambda}) : \bigoplus_\lambda H_*(X_\lambda, A_\lambda; G) \cong H_*(X,A;G)$$

- **Excision**:
  If $(X;A,B)$ is an **excisive triad** (that is, $X = \overset{\circ}{A} \cup \overset{\circ}{B}$), then inclusion $(A, A\cap B) \hookrightarrow (X,B)$ induces isomorphism
  $$H_*(A, A\cap B; G) \cong H_*(X,B;G)$$

Note. An equivalent form of Excision Axiom:
If $(X,A) \in \mathbf{TOP(2)}$, U is subspace of A and $\overline{U} \subseteq \overset{\circ}{A}$, then inclusion $i : (X-U, A-U) \hookrightarrow (X,A)$ induces isomorphism
$$i_* : H_*(X-U, A-U; G) \to H_*(X,A;G)$$

There is a critical criterion about weak homotopy equivalence between excisive triads, we prove lemmas first:

Lemma 2.4. *For*

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i_*"] \\ X \arrow[r, "f_*"'] & X \cup_Z Y \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

if D is deformation retract of X and $Z \subseteq D \subseteq X$, then $D \cup_Z Y$ is deformation retract of $X \cup_Z Y$.

Proof. Let $h : \mathrm{id}_X \simeq r \circ i$ where r is the deformation retraction $X \to D$. Define $h_* : \mathrm{id}_{X\cup_Z Y} \simeq (i \cup_Z \mathrm{id}_Y) \circ (r \cup_Z \mathrm{id}_Y)$

$$\begin{aligned} h_* : (X \cup_Z Y) \times I &\to X \cup_Z Y \\ (x,t) &\mapsto f_*(h(x,t)) \\ (y,t) &\mapsto i_*(y) \end{aligned}$$

Observe that $(X \cup_Z Y) \times I = (X \times I) \cup_{Z\times I} (Y \times I)$, check that h^* is continuous:

$$\begin{tikzcd}
Z \times I \arrow[r] \arrow[d] & Y \times I \arrow[d] \arrow[dr, "\mathrm{id}_{\mathrm{id}_Y}"] & \\
X \times I \arrow[r] \arrow[dr, "h"'] & (X \cup_Z Y) \times I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, "h_*"'] & Y \arrow[d, "i_*"'] \\
& X \arrow[r, "f_*"'] & X \cup_Z Y
\end{tikzcd}$$

$\square$

Lemma 2.5. *For maps $i : C \to A$, $j : C \to B$ define the double mapping cylinder $M(i, j) := A \cup_{C\times\{0\}} C \times I \cup_{C\times\{1\}} B$. If i is closed cofibration, then the quotient map*

$$\begin{aligned}
q : M(i, j) &\to A \cup_C B \\
a &\mapsto a \\
b &\mapsto b \\
(c, t) &\mapsto c
\end{aligned}$$

is a homotopy equivalence.

Proof.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, hook, "i"'] & B \arrow[d] \\
A \arrow[r, "i_A"'] & A \cup_C B
\end{tikzcd}$$

The canonical quotient $r : M_{i_A} \to A \cup_C B$ is a deformation retraction with homotopy:

$$\begin{aligned}
h : (B \cup_{C\times 0} (A \times I)) \times I &\to B \cup_{C\times 0} (A \times I) = M_{i_A} \\
(a, t, s) &\mapsto (a, (1 - s)t) \\
(b, s) &\mapsto b
\end{aligned}$$

Observe that $C \times I \cup_C A \times \{1\}$ is a deformation retract of $A \times I$, since $i : C \to A$ is closed cofibration.
Then we have $M(i, j) = B \cup_{C\times\{0\}} (C \times I \cup_{C\times\{1\}} A \times \{1\})$ is a deformation retract of $B \cup_{C\times\{0\}} A \times I = M_{i_A}$. (use lemma 2.4)
Finally, an easy check shows that $M(i, j) \to M_{i_A} \xrightarrow{r} A \cup_C B$ is identical to q.

$\square$

Theorem 2.6. *For excisive triads $(X; X_1, X_2)$, $(X'; X'_1, X'_2)$ and map $e : X \to X'$, if*

$$\begin{aligned}
e|_{X_1} &: X_1 \to X'_1 \\
e|_{X_2} &: X_2 \to X'_2 \\
e|_{X_3} &: X_3 \to X'_3
\end{aligned}$$

are weak equivalences, (where $X_3 := X_1 \cap X_2$, $X'_3 := X'_1 \cap X'_2$) then e is.

Proof. Use an important criterion of weak homotopy equivalence, it suffices to show forall $n \in \mathbb{N}$, any commutative diagram below:

$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \\
X \arrow[r, "e"'] & X'
\end{tikzcd}$$

can be filled like:

$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, "g^+"'] \\
X \arrow[r, "e"'] & X' \arrow[ul, phantom, "\Leftarrow h"]
\end{tikzcd}$$

whose upper triangle commutes.

Let

$$A_1 := g^{-1}(X - \mathring{X}_1) \cup f^{-1}(X' - \mathring{X}'_1)$$
$$A_2 := g^{-1}(X - \mathring{X}_2) \cup f^{-1}(X' - \mathring{X}'_2)$$

which are disjoint closed subsets of D^{n+1}. Choose CW-complex structure on D^{n+1} such that for each n-cell σ_i, $\overline{\sigma_i} \cap (A_1 \cup A_2) = \overline{\sigma_i} \cap A_1$ or $\overline{\sigma_i} \cap A_2$. Now define

$$K_1 := \bigcup \{\overline{\sigma_i} \mid g(\overline{\sigma_i} \cap S^n) \subseteq \mathring{X}_1 \text{ and } f(\overline{\sigma_i}) \subseteq \mathring{X}'_1\} = \bigcup \{\overline{\sigma_i} \mid \overline{\sigma_i} \cap A_1 = \emptyset\}$$
$$K_2 := \bigcup \{\overline{\sigma_i} \mid g(\overline{\sigma_i} \cap S^n) \subseteq \mathring{X}_2 \text{ and } f(\overline{\sigma_i}) \subseteq \mathring{X}'_2\} = \bigcup \{\overline{\sigma_i} \mid \overline{\sigma_i} \cap A_2 = \emptyset\}$$

which are subcomplexes of D^{n+1} and satisfy $K_1 \cup K_2 = D^{n+1}$. By HELP, we have:

$$\begin{tikzcd}
S^n \cap K_1 \cap K_2 \arrow[r, "i"] \arrow[d, "g|_{K_1 \cap K_2}"'] & K_1 \cap K_2 \arrow[d, "f|_{K_1 \cap K_2}"] \arrow[dl, "g_0"'] \\
X_1 \cap X_2 \arrow[r, "e|_{X_1 \cap X_2}"'] & X'_1 \cap X'_2 \arrow[ul, phantom, "\Leftarrow h_0"]
\end{tikzcd}$$

such that h_0 is $f|_{K_1 \cap K_2} \simeq e \circ g_0 \operatorname{rel} (S^n \cap K_1 \cap K_2)$. Apply HELP to:

$$\begin{tikzcd}
(S^n \cup K_1) \cap K_2 \arrow[r, "i_2"] \arrow[d, "g_{K_2}"'] & K_2 \arrow[d, "f|_{K_2}"] \\
X_2 \arrow[r] & X'_2 \arrow[ul, phantom, "\Leftarrow h_{K_2}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^n \cup K_2) \cap K_1 \arrow[r, "i_1"] \arrow[d, "g_{K_1}"'] & K_1 \arrow[d, "f|_{K_1}"] \\
X_1 \arrow[r] & X'_1 \arrow[ul, phantom, "\Leftarrow h_{K_1}"]
\end{tikzcd}$$

where
g_{K_i} are defined by $g_{K_i}|_{S^n \cap K_i} := g|_{S^n \cap K_i}$ and $g_{K_i}|_{K_1 \cap K_2} := g_0$,
h_{K_2} are defined by (h_{K_1} is similar):

$$\begin{aligned}
h_{K_2} : ((S^n \cup K_1) \cap K_2) \times I &\to X'_2 \\
(x, t) &\mapsto \begin{cases} e(g(x)) & x \in S^n \cap K_2 \\ h_0(x, t) & x \in K_1 \cap K_2 \end{cases}
\end{aligned}$$

We get:

$$\begin{tikzcd}
(S^n \cup K_1) \cap K_2 \arrow[r] \arrow[d, "g_{K_2}"'] & K_2 \arrow[d, "f|_{K_2}"] \arrow[dl, "g_2"'] \\
X_2 \arrow[r, "e|_{X_2}"'] & X'_2 \arrow[ul, phantom, "\Leftarrow h_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^n \cup K_2) \cap K_1 \arrow[r] \arrow[d, "g_{K_1}"'] & K_1 \arrow[d] \arrow[dl, "g_1"'] \\
X_1 \arrow[r, "e|_{X_1}"'] & X'_1 \arrow[ul, phantom, "\Leftarrow h_1"]
\end{tikzcd}$$

Define g^+ and $h : f \simeq g \operatorname{rel} S^n$ by $g^+|_{K_i} := g_i$ and $h|_{K_i \times I} := h_i$.
$h|_{S^n \times I} = (e \circ g) \times \mathrm{id}_I$ (h is $\operatorname{rel} S^n$) since $h_i(-,t)|_{S^n \cap K_i} = h_{K_i}(-,t)|_{S^n \cap K_i} = e \circ g|_{S^n \cap K_i}$.

□

Note. The proof above can be easily modified to case each weak equivalence appear in the statement is an n-equivalence.

Following theorem allow us to use CW-triads to approximate excisive triads:

Theorem 2.7. *For any excisive triad $(X; A, B)$, there is a CW-triad $(\widetilde{X}; \widetilde{A}, \widetilde{B})$ (A CW-triad $(X; A, B)$ is X and its subcomplex A, B such that $A \cup B = X$) and a map $r : \widetilde{X} \to X$ such that*

$$\begin{aligned} r|_{\widetilde{A}} &: \widetilde{A} \to A \\ r|_{\widetilde{B}} &: \widetilde{B} \to B \\ r|_{\widetilde{C}} &: \widetilde{C} \to C \\ r &: \widetilde{X} \to X \end{aligned}$$

are all weak homotopy equivalences (where $\widetilde{C} := \widetilde{A} \cap \widetilde{B}$, $C := A \cap B$). Furthermore, such r is natural up to homotopy.

Proof. Choose a CW-approximation $r_C : \widetilde{C} \to C$ and extend it to $r_A : \widetilde{A} \to A$, $r_B : \widetilde{B} \to B$. $\widetilde{X} := \widetilde{A} \cup_{\widetilde{C}} \widetilde{B}$. $i : \widetilde{C} \to \widetilde{A}$ and $j : \widetilde{C} \to \widetilde{B}$ are closed cofibrations, by lemma 2.5 we have homotopy equivalence $q : M(i, j) \to \widetilde{X}$, which induces homotopy equivalence of triads:

$$\begin{aligned} q &: M(i,j) \to \widetilde{X} \\ q| &: \widetilde{A} \cup (\widetilde{C} \times [0, \frac{2}{3})) \to \widetilde{A} \\ q| &: \widetilde{B} \cup (\widetilde{C} \times (\frac{1}{3}, 1]) \to \widetilde{B} \end{aligned}$$

then we can deduce that $r \circ q$ is a weak homotopy equivalence by theorem 2.6. Consquently, r is weak homotopy equivalence. r is natural up to homotopy since each CW-approximation r_C, r_A, r_B is.

□

Then we have:

Definition 2.4. A (Ordinary) Homology Theory on CW-complexes with coefficient $G \in \mathbf{Ab}$ is functors $\{H_n(-, -; G) : \textbf{CW-pairs} \to \mathbf{Ab}\}_{n \in \mathbb{Z}}$,
with natural transformations $\partial_{n,(X,A)} : H_n(X, A; G) \to H_n(A, \emptyset; G)$ (called connecting homomorphism)
satisfying axioms with the excision axiom changed to:

- **Excision**:
  If (X;A,B) is an **CW-triad** (that is $X = A \cup B$ for subcomplexes A and B) then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphism
  $$H_*(A, A \cap B; G) \cong H_*(X, B; G)$$

Proposition 2.8. *The homology groups defined in definition 2.2 with $H_{-n}(X) := 0$ is a ordinary homology theory on CW-complexes with coefficient $\mathbb{Z}$.*

Proof.

- Dimension: by a corollary, $H_q(*, \emptyset) = \pi_q(\mathrm{SP}\, S^0) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \geq 1 \end{cases}$.
- Weak Equivalence: SP preserves weak equivalence.
- Long Exact Sequence: use a corollary of Dold-Thom theorem.
- Additivity: For index set Λ, $P := \{S \mid S \subseteq \Lambda\}$.
  Then define $Y_S := \bigvee_{\lambda \in S} X_\lambda \cup CA_\lambda = (\coprod_{\lambda \in S} X_\lambda) \cup C(\coprod_{\lambda \in S} A_\lambda)$,
  and use fact that SP commutes with directed colimit, we have
  $\bigvee_{\lambda \in \Lambda} \mathrm{SP}(X_\lambda \cup CA_\lambda) = \varinjlim_{S \in P} \mathrm{SP}\, Y_S \approx \mathrm{SP}(\varinjlim_{S \in P} Y_S) = \mathrm{SP}((\coprod_{\lambda \in \Lambda} X_\lambda) \cup C(\coprod_{\lambda \in \Lambda} A_\lambda)) = \mathrm{SP}(X \cup CA)$.
  Which induces $\bigoplus_{\lambda \in \Lambda} \tilde{H}_n(X_\lambda \cup CA_\lambda) \cong \pi_n(\bigvee_{\lambda \in \Lambda} \mathrm{SP}(X_\lambda \cup CA_\lambda)) \cong \pi_n(\mathrm{SP}(X \cup CA)) = \tilde{H}_n(X \cup CA)$.

- Excision: For CW-triad $(X; A, B)$, $A/(A \cap B) \approx X/B$. Apply theorem 1.25 to $(Y \cup CZ, CZ)$ to show that $H_n(Y, Z) \cong \tilde{H}_n(Y/Z)$.

$\square$

2.3 Cellular Homology

Lemma 2.9. *For an ordinary homology theory $H_*(-,-;G)$, if X is a CW-complex, then for any $n \in \mathbb{Z}$ $H_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$.*

Proof. Apply long exact sequence axiom on (CX, X): ($H_*(CX) = 0$ due to weak equivalence axiom):

$$0 \cong H_{n+1}(CX) \to H_{n+1}(CX, X) \xrightarrow{\cong} H_n(X) \to H_n(CX) \cong 0$$

Use excision axiom and weak equivalence axiom, we have:

$$H_*(CX, X) \cong H_*(CX \cup CX, CX) \cong H_*(\Sigma X, *)$$

$\square$

Proposition 2.10. *For an ordinary homology theory $H_*(-,-;G)$, if X is a pointed CW-complex with $X^{-1} := *$, then for any $n \geq 0$*

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \begin{cases} \bigoplus_{i \in I_n} G & q = n \\ 0 & q \neq n \end{cases}$$

where I_n is set of all q-cells.

Proof. Use additivity axiom and lemma 2.9 to see that $H_n(\bigvee S^n) \cong \bigoplus G$ and $H_q(\bigvee S^n) = 0$ for $q \neq n$. Use excision axiom and weak equivalence axiom to see

$$H_q(X^n, X^{n-1}) \cong H_q(X^n \cup CX^{n-1}, CX^{n-1}) \cong H_q(X^n/X^{n-1}, *) \cong \tilde{H}_q(\bigvee_{i \in I_n} S^n)$$

$\square$

Corollary 2.11. *If $H_*(-,-)$ is an ordinary homology theory, then for a pointed CW-complex X with $X^{-1} := *$, we have:*

$$\begin{aligned} &\tilde{H}_q(X^n) = 0 && \text{for } q > n \\ &H_q(X^n) \cong H_q(X^{n+1}) \cong H_q(X) && \text{for } q < n \\ &H_n(X^n) \xrightarrow{i_*} H_n(X^{n+1}) && \text{is epimorphism} \end{aligned}$$

for any $n \geq -1$.

Proof. Use long exact sequence of (X^{n+1}, X^n):

$$\cdots \to H_{q+1}(X^{n+1}, X^n) \xrightarrow{\partial_{q+1}} H_q(X^n) \xrightarrow{i_*} H_q(X^{n+1}) \to H_q(X^{n+1}, X^n) \xrightarrow{\partial_q} H_{q-1}(X^n) \to \cdots$$
$$\cdots \to H_1(X^{n+1}, X^n) \xrightarrow{\partial_1} H_0(X^n) \xrightarrow{i_*} H_0(X^{n+1}) \to H_0(X^{n+1}, X^n)$$

For $q < n$, $H_q(X^n) \cong H_q(X^{n+1}) \cong \cdots \cong \varinjlim_{i \in \mathbb{N}} H_q(X^i)$.
For $q > n$, if $n > -1$, $H_q(X^n) \cong H_q(X^{n-1}) \cong \cdots \cong H_q(X^{-1}) \cong 0$,
if $n = -1$, $\tilde{H}_0(X^{-1}) \cong 0 \cong \tilde{H}_q(X^{-1})$.
For $q = n$, we have following exact:

$$\to H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \xrightarrow{i_*} H_n(X^{n+1}) \to H_n(X^{n+1}, X^n) \cong 0$$

$\square$

Definition 2.5. For pointed CW-complex X with $X^{-1} := *$ and a ordinary homology theory $H_*(-,-)$ the **cellular chain complex** $\{C_n(X), d_n\}$ of X is defined by:

$$C_n(X) := H_n(X^n, X^{n-1})$$
$$d_n : H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{i_*} H_{n-1}(X^{n-1}, X^{n-2})$$

Theorem 2.12. *For any ordinary homology theory $H_*(-,-)$ and any pointed CW-complex X, (with $X^{-1} := *$) the n-th homology of cellular chain complex is isomorphic to $H_n(X)$:*

$$H_n(C_*(X)) \cong H_n(X)$$

Proof. Notice that we have commutative diagram with each straight line exact: (use long exact sequence of pairs)

$$\begin{tikzcd}
 & & H_{n-1}(X^{n-2}) \cong 0 \arrow[dr] & & \\
 & & & H_{n-1}(X^{n-1}) \arrow[dr, tail] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[rr, "d_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & & H_n(X^n, X^{n-1}) \arrow[ur, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^n) = \ker(\partial_n) = \ker(d_n) \arrow[ur, tail] \arrow[dr, two heads] & & & \\
H_n(X^{n-1}) \cong 0 \arrow[ur] & & H_n(X^{n+1}) = \operatorname{coker}(\partial_{n+1}) \arrow[dr] & & \\
 & & & H_{n-1}(X^{n+1}, X^n) \cong 0 &
\end{tikzcd}$$

□

Note. If the ordinary homology theory has coefficient $\mathbb{Z}$, then the $d_n : C_n(X) \to C_{n-1}(X)$ is given by:

$$\mathbb{Z}_i \ni 1_i = e_i^n \mapsto \sum_{j \in I_{n-1}} \alpha_i^j e_j^{n-1}$$

where α_i^j is degree of map

$$\beta_i^j : S^n \approx \partial e_i^n \xrightarrow{\varphi_i} X^{n-1} \to X^{n-1}/X^{n-2} \to \bigvee_{j' \in I_{n-1}} S^{n-1} \xrightarrow{p_j} S^{n-1}$$

where φ_i is the characteristic map, p_j maps every point not in S_j^{n-1} to $*$.

3 Homotopy and Eilenberg-Mac Lane Spaces

3.1 Homotopy Excision Theorem and its Corollary

Theorem 3.1. *(Blakers–Massey) Homotopy Excision Theorem:*
For pointed CW-triad $(X; A, B)$ such that $C := A \cap B \neq \emptyset$, if (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected where $m \geq 2$, $n \geq 1$. Then $i : (A, C) \to (X, B)$ is an $(m+n-2)$-equivalence for pairs.

Note. We can replace the "CW-triad" with "excisive triad" in condition by theorem 2.7.

Proof. Define (pointed) the triad homotopy group for $q \geq 2$:

$$\pi_q(X; A, B) := \pi_{q-1}(P_{i_{B,X}}, P_{i_{C,A}})$$

where $i_{B,X} : B \hookrightarrow X$, $i_{C,A} : C \hookrightarrow A$ and P_f is the homotopy fiber

$$\{(y, \gamma) \in Y \times M(I, Z)_* \mid \gamma(1) = f(y)\}$$

of pointed map $f : Y \to Z$. Use long exact sequence of pairs:

$$\cdots \to \pi_q(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_{q-1}(P_{i_{C,A}}) \to \pi_{q-1}(P_{i_{B,X}}) \to \pi_{q-1}(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_{q-2}(P_{i_{C,A}}) \to \cdots$$
$$\cdots \to \pi_1(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_0(P_{i_{C,A}}) \to \pi_0(P_{i_{B,X}})$$

and observe that $\pi_q(P_{i_{X,B}}) \cong \pi_{q+1}(X, B)$ since for any $f : S^q \to P_{i_{X,B}}$ we have:

$$\begin{tikzcd}
S^q \arrow[dr, "f"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g"'] & & \\
& P_{i_{B,X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M(I, X)_* \arrow[d] \\
& B \arrow[r] & X
\end{tikzcd}$$

use the fact $f' \in M(S^q, M(I, X)_*)_* \cong M(S^q \wedge I, X)_* \ni f''$ and $S^q \wedge I \approx D^{q+1}$ with

$$\begin{aligned} S^q &\hookrightarrow S^q \wedge I \approx D^{q+1} \\ s &\mapsto (s, 1) \end{aligned}$$

the condition $f'(s)(1) = g(s)$ is equivalent to $f''((s, 1)) = g(s)$, that is have a map f is equivalent to have a map $f'' : (D^{q+1}, S^q) \to (X, B)$. With the analogue statement also valid for homotopies $S^q \times I \to P_{i_{X,B}}$, we have $\pi_q(P_{i_{B,X}}) = [S^q, *; P_{i_{B,X}}, *] \cong [D^{q+1}, S^q; X, B] = \pi_{q+1}(X, B)$.
Rewrites the long exact sequence of pairs above to:

$$\cdots \to \pi_{q+1}(X; A, B) \to \pi_q(A, C) \to \pi_q(X, B) \to \pi_q(X; A, B) \to \pi_{q-1}(A, C) \to \cdots$$
$$\cdots \to \pi_2(X; A, B) \to \pi_1(A, C) \to \pi_1(X, B)$$

Conditions $m \geq 1, n \geq 1$ guarantees $\pi_0(C) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(B)$ are surjections.
$m \geq 2$ is equivalent to $\pi_1(A, C) = 0$, which implies $\pi_0(C) \to \pi_0(A)$ is bijection.
For $x \in \pi_0(A \cap_C B)$, we can always find $b \in \pi_0(B), i_{B,X\ *}(b) = x$ or $a \in \pi_0(A), i_{A,X\ *}(a) = x$ which becomes $b \in \pi_0(B), i_{B,X\ *}(b) = x$ or $c \in \pi_0(C), i_{C,X\ *}(c) = x$ when $\pi_0(C) \to \pi_0(A)$ is bijection. That is equivalent to $\pi_0(B) \to \pi_0(X)$ is bijection, which means $\pi_1(X, B) = 0$.

We only need to show that for $2 \leq q \leq m + n - 2$, $\pi_q(X; A, B) = 0$.

With $J^{q-1} := (\partial I^{q-1} \times I) \cup (I^{q-1} \times \{0\})$, we have:

$$
\begin{aligned}
\pi_q(P_{i_{B,X}}, P_{i_{C,A}}) &= [I^q, \partial I^q, J^{q-1}; P_{i_{B,X}}, P_{i_{C,A}}, *] \\
&= [I^q \wedge I;\ I^q,\ \partial I^q \wedge I,\ J^{q-1} \wedge I \to X; B, A, *] \\
&:= \text{relative homotopy classes of pointed maps}
\end{aligned}
$$

$$
f : I^q \wedge I \to X \text{ satisfying: } \begin{cases} f(I^q) & \subseteq B \\ f(\partial I^q \wedge I) & \subseteq A \\ f(\partial I^q) & \subseteq C \\ f(J^{q-1} \wedge I) & = * \end{cases}
$$

"relative" means the homotopy h determine the classes

$$
\text{satisfy: } \begin{cases} h(I^q \times I) & \subseteq B \\ h((\partial I^q \wedge I) \times I) & \subseteq A \\ h(\partial I^q \times I) & \subseteq C \\ h((J^{q-1} \wedge I) \times I) & = * \end{cases}
$$

(notice that $\partial I^q \wedge I \cap I^q = \partial I^q$, therefore $f(\partial I^q) \subseteq A \cap B = C$)

(this is called (relative) homotopy class of maps of tetrads)

$$
\begin{aligned}
&= [(I^q \times I)/K;\ I^q \times \{1\},\ (\partial I^q \times I)/K,\ (J^{q-1} \times I)/K \to X; B, A, *] \\
&\quad (K := I^q \times \{0\} \cup \{i_0\} \times I) \\
&= [I^{q+1};\ (I^q \times \{1\}) \cup K,\ (\partial I^q \times I) \cup K,\ J^{q-1} \times I \cup K \to X; B, A, *] \\
&= [I^{q+1};\ I^q \times \{1\},\ I^{q-1} \times \{1\} \times I,\ J^{q-1} \times I \cup I^q \times \{0\} \to X; B, A, *] \\
&\quad (\text{notice that } \partial I^q = \partial I^{q-1} \times I \cup I^{q-1} \times \{0,1\})
\end{aligned}
$$

We can assume that (A, C) have no relative $q < m$-cells and (B, C) have no relative $q < n$-cells. And we can assume that X has finite many cells since I^q is compact.
For subcomplexes $C \subseteq A' \subseteq A$, where $A = e^m \cup A'$ (attaching one cell from A').
Let $X' := A' \cup_C B$, if the results hold for $(X'; A', B)$ and $(X; A, X')$, then it hold for $(X; A, B)$ since we have map between exact homotopy sequences of triples (A, A', C) and (X, X', B):

$$
\begin{tikzcd}
\pi_{q+1}(A, A') \arrow[r] \arrow[d, "i_{2,q+1}"'] & \pi_q(A', C) \arrow[r] \arrow[d, "i_{1,q}"'] & \pi_q(A, C) \arrow[r] \arrow[d, "i_{3,q}"'] & \pi_q(A, A') \arrow[r] \arrow[d, "i_{2,q}"'] & \pi_{q-1}(A', C) \arrow[d, "i_{1,q-1}"'] \\
\pi_{q+1}(X, X') \arrow[r] & \pi_q(X', B) \arrow[r] & \pi_q(X, B) \arrow[r] & \pi_q(X, X') \arrow[r] & \pi_{q-1}(X', B)
\end{tikzcd}
$$

induced by inclusion $(A, A', C) \hookrightarrow (X, X', B)$. If the result hold for $(X'; A', B)$ and $(X; A, X')$, maps $i_{1,q},\ i_{2,q}$ are isomorphisms when $1 \geq q \geq m + n - 3$, are epimorphisms when $q = m + n - 2$.
Notice the 5-lemma says that
if $i_{1,q}$ and $i_{2,q}$ are epimorphisms, $i_{1,q-1}$ are monomorphism, then $i_{3,q}$ is epimorphism.
if $i_{1,q}$ and $i_{2,q}$ are monomorphisms, $i_{2,q+1}$ are epimorphism, then $i_{3,q}$ is monomorphism.
We also have if $C \subseteq B' \subseteq B$ with $B = B' \cup e^n$, the result hold for CW-triads $(X'; A, B')$ and $(X; X', B)$ where $X' = A \cup_C B'$, since $(A, C) \hookrightarrow (X, B)$ factors as $(A, C) \hookrightarrow (X', B') \hookrightarrow (X, B)$.

Now we can assume that $A = C \cup D^m$ and $B = C \cup D^n$.

The current goal of proof is to prove any

$$
f : (I^{q+1};\ I^q \times \{1\},\ I^{q-1} \times \{1\} \times I,\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X; B, A, *)
$$

is nullhomotopic for any $q + 1$ with $2 \leq q + 1 \leq m + n - 2$.
For $a \in \overset{\circ}{D}{}^m$, $b \in \overset{\circ}{D}{}^n$ We have inclusions of based triads:

$$
(A; A, A - \{a\}) \hookrightarrow (X - \{b\}; X - \{b\}, X - \{a, b\}) \hookrightarrow (X; X - \{b\}, X - \{a\}) \hookleftarrow (X; A, B)
$$

The first and the third induces isomorphisms on homotopy groups of triads since B is a strong deformation retract of $X - \{a\}$ in X and A is a strong deformation retract of $X - \{b\}$ in X. $\pi_*(A; A, A - \{a\}) = 0$ since $\pi_*(A, A - \{a\}) \to \pi_*(A, A \cap (A - \{a\}))$ are isomorphisms.

Current goal : choose good a, b to show f regarded as a pointed traid map to $(X; X - \{b\}, X - \{a\})$ is homotopic to a map

$$f' : (I^{q+1};\ I^{q-1} \times \{1\} \times I,\ I^q \times \{1\},\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X - \{b\}; X - \{b\}, X - \{a, b\}, *)$$

if $2 \leq q + 1 \leq m + n - 2$.

Note. We want to homotopically remove some point $f^{-1}(b)$, first we may want to construct some Urysohn function u separating $f^{-1}(a) \cup J^{q-1} \times I \cup I^q \times \{0\}$ and $f^{-1}(b)$ and construct homotopy of cube $h^+ : (r, s, t) \mapsto (r, (1 - u(r, s)t)s)$ wishing that $f(h^+(r, s, 1))$ would miss b. The problem in this method is that points $f^{-1}(b)$ in the cube would be homotopically replaced by other points. Since our desire homotopy does not change the first q coordinates of the cube, we want to separate $p^{-1}(p(f^{-1}(a))) \cup J^{q-1} \times I$ and $p^{-1}(p(f^{-1}(b)))$ (where $p : I^q \times I \to I^q$). Our problem is to find suitable a, b such that $p(f^{-1}(a)) \cap p(f^{-1}(b)) = \emptyset$.

We use manifold structure on D^m and D^n to achieve it, now we homotopically approximate f by a map g which smooth on $f^{-1}(D^m_{<1/2} \cup D^n_{<1/2})$.

Let $U_{<r} := f^{-1}(D^m_{<r} \cup D^n_{<r})$, Use smooth deformation theorem to construct smooth map (for any $0 < \epsilon$) $g' : U_{<3/4} \to D^m_{<3/4} \cup D^n_{<3/4}$ with homotopy $h_1 : g' \simeq f|_{U_{<3/4}}$ (and bound $|g'(x) - f(x)| < \epsilon$ for any $x \in U_{<1}$) and take partition of unity $\{\rho, \rho'\}$ with subcoordinates $\{I^{q+1} - \overline{U_{<1/2}},\ U_{<3/4}\}$, we have:

$$\begin{aligned} g &:= \rho f + \rho' g' \\ h_2 &: g \simeq f \operatorname{rel}(I^{q+1} - U_{<3/4}) \\ h_2 &: I^{q+1} \times I \to X \\ &\quad (x, t) \mapsto \rho(x) f(x) + \rho'(x) h_1(x, t) \end{aligned}$$

with scalar multiplication and addition is already defined on smooth structure on $D^m_{<3/4} \cup D^n_{<3/4}$. We could assmue that $g(I^{q-1} \times \{1\} \times I) \cap D^n_{<1/2} = \emptyset$ (which implies g is a map of tetrads to $(X; X - \{b\}, X - \{a\}, *)$) and $g(I^q \times \{1\}) \cap D^m_{<1/2} = \emptyset$ since $f(I^{q-1} \times \{1\} \times I) \subseteq A$ and $f(I^q \times \{1\}) \subseteq B$ and we can always tighten the bound ϵ, (Similar argument also hold for h_2, then we have $h_2 : g \simeq f$ as homotopy between maps of tetrads.)

Use the manifold structure to find good (a, b):
$V := g^{-1}(D^m_{<1/2}) \times g^{-1}(D^n_{<1/2})$ is a sub-manifold of $I^{2(q+1)}$. Consider $W := \{(v, v') \in V \mid p(v) = p(v')\}$, which is the zero set of smooth submersion $(v, v') \mapsto p(v) - p(v')$. W is smooth manifold with codimension q. Therefore the map $(g, g) : W \to D^m_{<1/2} \times D^n_{<1/2}$ is smooth map between manifolds of dimension $q + 2$ and $m + n$. The map is not surjection since $q + 2 < m + n$. Then we have $(a, b) \notin (g, g)(W)$ (that is, $p(g^{-1}(a)) \cap p(g^{-1}(b))$).

Since $g(I^{q-1} \times \{1\} \times I) \cap D^n_{<1/2} = \emptyset$ and $g(J^{q-1} \times I) \cap D^n_{<1/2} = \emptyset$, we have $g(\partial I^q \times I) \cap D^n_{<1/2} = \emptyset$. By Urysohn's lemma, we have $u : I^q \to I$ separating $p(g^{-1}(a)) \cup \partial I^q$ and $p(g^{-1}(b))$. Finally we have:

$$\begin{aligned} h' &: I^q \times I \times I \to I^q \times I \\ &\quad (r, s, t) \mapsto (r, (1 - u(r)t)s) \end{aligned}$$

and $h := g \circ h'$, $f' := h(-, 1)$. $f'(I^{q+1}) \cap \{b\} = \emptyset$ since if $\exists (r, s) \in I^q \times I$, $f'(r, s) = b$, then $b = g(r, (1 - u(r))s) = g(r, 0) = *$ leads to contradiction.
Last step is to check that h is a homotopy between maps

$$(I^{q+1};\ I^{q-1} \times \{1\} \times I,\ I^q \times \{1\},\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X; X - \{b\}, X - \{a\}, *)$$

Since g is, $g \circ h'$ is too.

$\square$

Corollary 3.2. *Suppose that $Y_0 \hookrightarrow Y$ is cofibration, (Y, Y_0) is $(r-1)$-connected and Y_0 is $(s-1)$-connected, then $(Y, Y_0) \to (Y/Y_0, *)$ is $(r+s-1)$-equivalence. $(r \geq 2,\ s \geq 1)$*

Proof. $Y_0 \hookrightarrow CY_0$ is cofibration and (CY_0, Y_0) is s-connected. Use homotopy excision theorem (with $X = Y \cup CY_0,\ A = Y,\ B = CY_0,\ C = Y_0$) to see $(Y, Y_0) \to (Y \cup CY_0, CY_0)$ is $(r+s-1)$-equivalence. And $(Y \cup CY_0, CY_0) \to (Y/Y_0, *)$ is homotopy equivalence since $Y_0 \hookrightarrow Y$ is cofibration.

$\square$

Corollary 3.3. *For $n \geq 2$, $f : X \to Y$ is $(n-1)$-equivalence between $(s-1)$-connected spaces, then $(M_f, X) \to (C_f^+, *)$ is $(n+s-1)$-equivalence. Where $C_f^+ := Y \cup_f C^+X$, $C^+X := (X \times I)/(X \times \{1\})$. is the unreduced mapping cone and the unreduced cone.*

Proof. f is $(n-1)$-equivalence implies (M_f, X) is $(n-1)$-connected. Use corollary above.

$\square$

Corollary 3.4. *For $n \geq 2$, if $f : X \to Y$ is pointed map between $(n-1)$-connected well-pointed spaces (that is, pointed space whose inclusion of the base point is (closed) cofibration). Then C_f is $(n-1)$-connected and $\pi_n(M_f, X) \to \pi_n(C_f, *)$ is isomorphism.*

Proof. Use homotopy extension property to extend to unreduced case. f is map between $(n-1)$-connected space implies f is at least a $(n-1)$-equivalence. Therefore $(M_f, X) \to (C_f, *)$ is $(2n-1)$-equivalence, Since we have $n < 2n-1$ for any $n \geq 2$, $\pi_n(M_f, X) \to \pi_n(C_f, *)$ is isomorphism.

$\square$

Theorem 3.5. *(Freudenthal Suspension Theorem) If X is well-pointed and $(n-1)$-connected $(n \geq 1)$, then the map:*

$$\begin{aligned} \sigma : \pi_q(X) &\to \pi_{q+1}(\Sigma X) \\ f &\mapsto \Sigma f \end{aligned}$$

is isomorphism if $q < 2n-1$ and epimorphism if $q = 2n-1$.

Proof. If we have $f : (I^q, \partial I^q) \to (X, *)$ then $f \times \mathrm{id}_I : I^{q+1} \to X \times I$ will give a map $\overline{f \times \mathrm{id}_I} : (I^{q+1},\ \partial I^{q+1},\ \partial I^q \times I \cup \partial I \times \{1\}) \to (CX, X, *)$ since $J^q = \partial I^q \times I \cup \partial I \times \{0\}$, it does not give a map in $\pi_{q+1}(CX, X)$. we should change $\overline{f \times \mathrm{id}_I}$ into $\overline{f \times -\mathrm{id}_I}$. we have commutative diagram:

$$\begin{tikzcd} \pi_{q+1}(CX, X) \arrow[r, "p_*"] \arrow[d, shift right, "\partial"'] & \pi_{q+1}(CX/X, *) \arrow[d, equal] \\ \pi_q(X) \arrow[r, "-\sigma"'] \arrow[u, shift right, "i"'] & \pi_{q+1}(\Sigma X) \end{tikzcd} \qquad \begin{tikzcd} {[\overline{f \times -\mathrm{id}_I}]} \arrow[r, mapsto] \arrow[d, shift right] & {[p \circ (\overline{f \times -\mathrm{id}_I})]} \arrow[d, equal] \\ {[f]} \arrow[r, mapsto] \arrow[u, shift right] & {[-\Sigma f]} \end{tikzcd}$$

Where $p : (CX, X) \to (CX/X, *)$ is the canonical quotient map and $i : [f] \to [\overline{f \times -\mathrm{id}_I}]$ makes $\pi_{q+1}(CX) \to \pi_{q+1}(CX, X) \to \pi_q(X) \to \pi_q(CX)$ split in middle (that is, i is inverse of the connecting homomorphism ∂). We verify the commutativity:

$$\begin{aligned} -\Sigma f : (I^{q+1}, \partial I^{q+1}) &\to (CX/X, *) \\ (s, t) &\mapsto f(s) \wedge (1-t) \\ p \circ (\overline{f \times -\mathrm{id}_I}) : (I^{q+1}, \partial I^{q+1}) &\to (CX/X, *) \\ (s, t) &\mapsto f(s) \wedge (1-t) \end{aligned}$$

Since $X \hookrightarrow CX$ is cofibration and n-equivalence between $(n-1)$-connected spaces, p is an $2n$-equivalence. Therefore, $q+1 < 2n$ implies $-\sigma$ is isomorphism, $q+1 = 2n$ implies $-\sigma$ is epimorphism, and we have $-\sigma$ is iff σ is.

$\square$

Definition 3.1. We now define the q-th stable homotopy group:

$$\pi_k^s(X) := \varinjlim_r \pi_{k+r}(\Sigma^r X) \cong \pi_{2k+2}(\Sigma^{k+2} X) \cong \pi_{k+n}(\Sigma^n X) \qquad (n-1 > k)$$

The relation right side is directly from $\Sigma^n X$ is $(n-1)$-connected.

Note. We'll see later that $\{\pi_n^s\}_{n \in \mathbb{N}}$ defines a generalized homology theory.

3.2 Hurewicz Theorem

First, we use homotopy excision theorem to prove following lemmas:

Lemma 3.6. *(every $S^n_a \approx S^n$) We have canonical $i_a : S^n_a \hookrightarrow \bigvee_{a\in A} S^n_a$ and for $n > 1$:*

$$\pi_n(\bigvee_{a\in A} S^n_a) \cong \bigoplus_{a\in A} \mathbb{Z}_a$$

where $[i_a] = 1 \in \mathbb{Z}_a \subseteq \bigoplus_{a\in A} \mathbb{Z}_a$ and every $\mathbb{Z}_a \cong \mathbb{Z}$.
For $n = 1$:

$$\pi_n(\bigvee_{a\in A} S^1_a) \cong \coprod_{a\in A} \mathbb{Z}_a$$

where $\coprod$ is taken in category **Grp**, *$[i_a] = 1 \in \mathbb{Z}_a \subseteq \coprod_{a\in A} \mathbb{Z}_a$ and every $\mathbb{Z}_a \cong \mathbb{Z}$.*

Proof.
Case $n = 1$:
Apply the Seifert-van Kampen theorem.
Case $n > 1$:
Suppose each S^n_a have CW-complex structure with one 0-cell and one n-cell. Consider finite product $\prod_{1\le i\le k} S^n_i$ and its subcomplex, finite wedge product $\bigvee_{1\le i\le k} S^n_i$.
The inclusion

$$\bigvee_{1\le i\le k} S^n_i \hookrightarrow \prod_{1\le i\le k} S^n_i$$

is $(2n-1)$-equivalence since $\prod_{1\le i\le k} S^n_i - \bigvee_{1\le i\le k} S^n_i$ only have cells of dim $\ge 2n$. (use lemma 1.13) Use exact homotopy sequence of pair, we deduce that $\pi_q(\bigvee_{1\le i\le k} S^n_i) \to \pi_q(\prod_{1\le i\le k} S^n_i) \cong \bigoplus_{1\le i\le k} \mathbb{Z}$ is an isomorphism for $q \le 2n-2$. And $S^n_i \hookrightarrow \bigvee_{1\le i\le k} S^n_i \hookrightarrow \prod_{1\le i\le k} S^n_i$ is just the i-th inclusion $S^n_i \hookrightarrow \prod_{1\le i\le k} S^n_i$ which represents $1 \in \mathbb{Z}_i \hookrightarrow \bigoplus_{1\le i\le k} \mathbb{Z}_i$. Infinite wedge case:

$$\begin{tikzcd}
\bigoplus_{1\le i\le k} \pi_q(S^n_i) \arrow[r, "\cong"] \arrow[d, tail] & \pi_q(\bigvee_{1\le i\le k} S^n_i) \arrow[d] \\
\bigoplus_{a\in A} \pi_q(S^n_a) \arrow[r, tail, two heads, "\bigoplus_{a\in A} i_{a*}"'] & \pi_q(\bigvee_{a\in A} S^n_a)
\end{tikzcd}$$

$\bigoplus_{a\in A} i_{a*}$ is monomorphism since every homotopy $S^n \times I \to \bigvee_{a\in A} S^n_a$ has a compact image, and $\bigoplus_{a\in A} i_{a*}$ is epimorphism since every map $S^n \times I \to \bigvee_{a\in A} S^n_a$ has a compat image.

□

Lemma 3.7. *For $n \ge 1$, if we have a map $f : \coprod_{a\in A} \mathbb{Z}_a \to \coprod_{b\in B} \mathbb{Z}_b$ (case $n = 1$) or a map $f : \bigoplus_{a\in A} \mathbb{Z}_a \to \bigoplus_{b\in B} \mathbb{Z}_b$ (case $n > 1$).*
Then there exists a map $\phi : \bigvee_{a\in A} S^n_a \to \bigvee_{b\in B} S^n_b$ unique up to homotopy and satisfy $\pi_n(\phi) = f$.

Proof. Suppose $f(1_a) = [\phi_a] \in [S^n, \bigvee_{b\in B} S^n_b]_*$, then ϕ_a is indeed a map $S^n_a \to \bigvee_{b\in B} S^n_b$. Now we define $\phi := \bigvee_a \phi_a : \bigvee_{a\in A} S^n_a \to \bigvee_{b\in B} S^n_b$. For any $a \in A$, $\phi|_{S^n_a} = \phi_a$, we have

$$\pi_n(\phi)(1_a) = [\phi|_{S^n_a} \circ \mathrm{id}_{S^n_a}] = [\phi_a] = f(1_a)$$

which implies $\pi_n(\phi) = f$ since they are group homomorphisms.

Uniqueness up to homotopy: $\pi_n(\phi)[1_a] = \pi_n(\phi')[1_a]$ implies $\phi|_{S^n_a} \simeq \phi'|_{S^n_a} \operatorname{rel} *$. Therefore $\phi \simeq \phi' \operatorname{rel} *$.

□

Definition 3.2. If H_n is a ordinary homology theory with coefficient $\mathbb{Z}$, then the map

$$\begin{aligned} h_X : \pi_n(X) &\to \tilde{H}_n(X) := H_n(X, *) \\ [f] &\mapsto f_*(1) \qquad\qquad (\ f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(X)\ ) \end{aligned}$$

is called **Hurewicz Homomorphism**.

Note. $h_{(-)}$ is natural transformation since we have

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h_X"] \arrow[d, "g_*"'] & \tilde{H}_n(X) \arrow[d, "g_*"] \\
\pi_n(Y) \arrow[r, "h_Y"'] & \tilde{H}_n(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[f]} \arrow[r, mapsto] \arrow[d, mapsto] & f_*(1) \arrow[d, mapsto] \\
{[g \circ f]} \arrow[r, mapsto] & (g \circ f)_*(1) = (g_* \circ f_*)(1)
\end{tikzcd}$$

commutes.

Lemma 3.8. *If $X = \bigvee_{a \in A} S^n$, $h_X : \pi_n(X) \to \tilde{H}_n(X)$ is abelianization if $n = 1$, isomorphism if $n \geq 2$.*

Proof. Directly from lemma 3.7. (we used homotopic properties of spheres only in proving is lemma) □

Theorem 3.9. *(Hurewicz) If X is $(n-1)$-connected, then $h_X : \pi_n(X) \to \tilde{H}_n(X)$ is abelianization if $n = 1$, isomorphism if $n \geq 2$.*

Proof. We can assume X is CW-complex with $X^{n-1} = *$ and each characteristic map is pointed. (since we have theorem 1.5)
For CW-complex X, $\pi_n(X^{n+1}) \cong \pi_n(X)$ and $H_n(X^{n+1}) \cong H_n(X)$, Since we have cellularity of homotopy group and cellularity of homology.
Then we have $X^n = \bigvee_{b \in B} S^n_b$, $X^{n+1} = C_\phi$ where $\phi : \bigvee_{a \in A} S^n_a \to X^n$ are the characteristic maps.
Use naturality of $h_{(-)}$, we have maps between exact sequence:

$$\begin{tikzcd}
\pi_n(\bigvee_{a \in A} S^n_a) \arrow[r, "\phi_*"] \arrow[d, "h_{\bigvee_{a \in A} S^n_a}"] & \pi_n(X^n) \arrow[r] \arrow[d, "h_{X^n}"] & \pi_n(C_\phi) \arrow[r] \arrow[d, "h_{C_\phi}"] & 0 \\
\tilde{H}_n(\bigvee_{a \in A} S^n) \arrow[r, "\phi_*"'] & \tilde{H}_n(X^n) \arrow[r] & \tilde{H}_n(C_\phi) \arrow[r] & 0
\end{tikzcd}$$

If $n > 1$, exactness of top row is directly from lemma 3.2. ($(M_\phi, \bigvee_{a \in A} S^n_a)$ is $(n-1)$-connected since we have lemma 1.13) 5-lemma shows that h_{C_ϕ} is isomorphism.
If $n = 1$, Seifert-van Kampen theorem shows that $\pi_1(C_\phi) = \pi_1(X^n)/\langle \operatorname{Im} \phi_* \rangle_{nor}$. (where for $A \subseteq$ a group G, $\langle A \rangle_{nor} := \{gAg^{-1} \mid g \in G\}$). The top row is not exact, but top row's abelianization is exact since $\langle \operatorname{Im} f \rangle_{nor}/[B, B] = \operatorname{Im} f/[B, B]$ for any group morphism $f : A \to B$. Therefore we have diagram below with the middle row and the bottom row exact:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "q"] \arrow[d] & G = B/\langle \operatorname{Im} f \rangle_{nor} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
A^{\text{ab}} \arrow[r, "f^{\text{ab}}"] \arrow[d, "\cong"'] & B^{\text{ab}} \arrow[r, "q^{\text{ab}}"] \arrow[d, "\cong"'] & G^{\text{ab}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
A' \arrow[r] & B' \arrow[r] & G' \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Finally apply 5-lemma on the middle row and the bottom row.

□

3.3 Moore Spaces

Definition 3.3. A space X is **Eilenberg-Mac Lane space** of **type** $K(G, n)$ (where G is group and is abelian for $n \geq 2$) if

$$\pi_q(X) \cong \begin{cases} G & n = q \\ 0 & n \neq q \end{cases}$$

We see that $\mathrm{SP}\, S^n$ is a $K(\mathbb{Z}, n)$. Now we use this to construct other $K(G, n)$.

Note. In order to construct $K(G, n)$, we construct a space $M(G, n)$ which have $\pi_n(M(G, n)) = G$, $\pi_q(M(G, n)) = 0$ for $q < n$ and we can apply SP on it to kill all dim $> n$ homotopy group.

Proposition 3.10. *For any $k \in \mathbb{Z}$, there is a map $a_k : S^1 \to S^1$ with a_k, and $C_{a_k} = S^1 \cup_{a_k} e^2$ is the desired $M(\mathbb{Z}/k\mathbb{Z}, 1)$ (that is $\mathrm{SP}(S^1 \cup_{a_k} e^2)$ is a $K(\mathbb{Z}/k\mathbb{Z}, 1)$).*

Proof. Consider sequence $S^1 \xrightarrow{a_k} S^1 \hookrightarrow C_{a_k} \twoheadrightarrow \Sigma S^1 = C_{a_k}/S^1$, we apply an usual form of Dold-Thom Theorem to see that $\mathrm{SP}(C_{a_k}) \to \mathrm{SP}(S^2)$ is a quasi-fibration with fiber $\mathrm{SP}(S^1)$. Then we have exact sequence:

$$\begin{aligned}
&\cdots \to \pi_q(\mathrm{SP}\, S^1) \to \pi_q(\mathrm{SP}\, C_{a_k}) \to \pi_q(\mathrm{SP}\, S^2) \to \pi_{q-1}(\mathrm{SP}\, S^1) \to \\
&\cdots \to \pi_2(\mathrm{SP}\, S^1) \to \pi_2(\mathrm{SP}\, C_{a_k}) \to \pi_2(\mathrm{SP}\, S^2) \\
&\quad \to \pi_1(\mathrm{SP}\, S^1) \to \pi_1(\mathrm{SP}\, C_{a_k}) \to \pi_1(\mathrm{SP}\, S^2)
\end{aligned}$$

We can conclude that $\pi_q(\mathrm{SP}\, C_{a_k}) = 0$ for any $q \neq 0, 1$ and:

$$0 \to \pi_2(\mathrm{SP}\, C_{a_k}) \to \pi_2(\mathrm{SP}\, S^2) = \mathbb{Z} \xrightarrow{\partial} \pi_1(\mathrm{SP}\, S^1) = \mathbb{Z} \to \pi_1(\mathrm{SP}\, C_{a_k}) \to 0$$

exact. Where ∂ is defined by:

$$\pi_2(\mathrm{SP}\, S^2) \cong [D^2, S^1, *; \mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1, *] \ni f \mapsto f|_{S^1} \in [S^1, S^1]_*$$

(Now we want to show that ∂ is multiplication by k)
The $1 \in \mathbb{Z} \cong \pi_2(\mathrm{SP}\, S^2)$ is represented by $[i_2 : S^2 \hookrightarrow \mathrm{SP}\, S^2]$.
Since $[D^2, S^1, *; \mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1, *] \xrightarrow{p_*} [D^2, S^1; \mathrm{SP}\, S^2, *]$ is isomorphism,
and the map $\varphi : (D^2, S^1) \xrightarrow{\mathrm{id}_{e^2} \cup a_k} (C_{a_k}, S^1) \hookrightarrow (\mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1)$ satisfy $p \circ \varphi = i_2$,
the $1 \in \mathbb{Z} \cong \pi_2(\mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1)$ is represented by φ. Then we have $\partial(1)$ is represented by $\varphi|_{S^1} = i_1 \circ a_k$ where $i_1 : S^1 \hookrightarrow \mathrm{SP}\, S^1$.
The map ∂ is $\mathbb{Z} \ni n \mapsto kn \in \mathbb{Z}$ since $[i_1 \circ a_k] = k$.
Therefore $\pi_2(\mathrm{SP}\, C_{a_k}) = 0$ and $\pi_1(\mathrm{SP}\, C_{a_k}) = \mathbb{Z}/k\mathbb{Z}$.

□

Proposition 3.11. *For each $n \geq 1$, $k \in Z$, $\mathrm{SP}(S^n \cup_{\Sigma^{n-1} a_k} e^{n+1})$ is a $K(\mathbb{Z}/k\mathbb{Z}, n)$.*

Proof. For $q \geq 1$, $\Sigma(S^q \cup_{\Sigma^{q-1} a_k} e^{q+1}) \approx \Sigma S^q \cup_{\Sigma^q a_k} \Sigma e^{q+1} = S^{q+1} \cup_{\Sigma^q a_k} e^{q+2}$ since Σ is left adjoint of Ω in $\mathbf{TOP}_*$ and the pushout is took in $\mathbf{TOP}_*$. Observe that $\pi_q(\mathrm{SP}\, X) \cong \pi_{q+1}(\mathrm{SP}\, \Sigma X)$, now we have done.

□

Since $\tilde{H}_n(X) \cong \tilde{H}_n(X \cup C*) \cong H_n(X, *)$, we have

$$\pi_n(\mathrm{SP}(\bigvee_{i \in I} X_i)) = \tilde{H}_n(\bigvee_{i \in I} X_i) \cong H_n(\bigvee_{i \in I} X_i, *) \cong H_n(\coprod_{i \in I} X_i, \coprod_{i \in I} *) \cong \bigoplus_{i \in I} H_n(X_i, *) \cong \bigoplus_{i \in I} \pi_n(\mathrm{SP}\, X_i)$$

We can deduce the following proposition immediately:

Proposition 3.12. *For finitely generated abelian group $G \cong (\bigoplus_r \mathbb{Z}) \oplus (\bigoplus_{1 \leq i \leq k} \mathbb{Z}/d_i\mathbb{Z})$, (where $r \in \mathbb{N}$, each $d_i \in \mathbb{Z}$) we have $\mathrm{SP}((\bigvee_r S^n) \vee (\bigvee_{1 \leq i \leq k} (S^n \cup_{a_{d_i}} e^{n+1})))$ is a $K(G, n)$.*

Since every abelian group G have a free resolution sequence:

$$0 \to \bigoplus_{a \in A} \mathbb{Z} \xrightarrow{f} \bigoplus_{b \in B} \mathbb{Z} \twoheadrightarrow G \to 0$$

exact. And for every group $G = F(X)/\langle Y \rangle_{nor}$ (where $F(X) := \coprod_{x \in X} \mathbb{Z}_x$ is the free group functor and $\langle Y \rangle_{nor}$ is the normal subgroup generated by Y), we have:

$$1 \to \coprod_{y \in \langle Y \rangle_{nor}} \mathbb{Z}_y \xrightarrow{f : 1_y \mapsto y} \coprod_{x \in X} \mathbb{Z}_x \twoheadrightarrow G \to 1$$

exact.

Next proposition allows to construct spaces $M(\bigoplus_{a \in A} \mathbb{Z}, n)$ and $M(\coprod_{a \in A} \mathbb{Z}, 1)$:

Definition 3.4. For $n > 1$, G an abelian group, we have exact sequence

$$0 \to \bigoplus_{a \in A} \mathbb{Z} \xrightarrow{f} \bigoplus_{b \in B} \mathbb{Z} \twoheadrightarrow G \to 0$$

Then we have: (with ϕ is the map obtained using lemma 3.7)

$$\bigvee_{a \in A} S_a^n \xrightarrow{\phi} \bigvee_{b \in B} S_b^n \to C_\phi$$

the **Moore space** of type (G, n) is defined as $M(G, n) := C_\phi$.

For $n = 1$, G a group, we have exact sequence:

$$1 \to \coprod_{y \in \langle Y \rangle_{nor}} \mathbb{Z}_y \xrightarrow{f: 1_y \mapsto y} \coprod_{x \in X} \mathbb{Z}_x \twoheadrightarrow G \to 1$$

Then we have: (with ϕ is the map obtained using lemma 3.7)

$$\bigvee_{y \in \langle Y \rangle_{nor}} S_y^1 \xrightarrow{\phi} \bigvee_{x \in X} S_x^1 \to C_\phi$$

the **Moore space** of type $(G, 1)$ is defined as $M(G, 1) := C_\phi$.

Proposition 3.13. $\pi_n(M(G, n)) = G$

Proof. For $n > 1$, use diagram:

$$\begin{tikzcd} & M_\phi \arrow[d, bend left, "p"] \arrow[d, phantom, "\simeq"] \\ X \arrow[ur, "i"] \arrow[r, "\phi"'] & Y \arrow[u, bend left, "j"] \end{tikzcd}$$

To see:

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(\bigvee_{a\in A} S_a^n) \arrow[r, "i_*"] \arrow[dr, "\phi_*"'] & \pi_n(M_\phi) \arrow[r] & \pi_n(M_\phi, \bigvee_{a\in A} S_a^n) \arrow[r] \arrow[dd, "q_*"] & \pi_{n-1}(\bigvee_{a\in A} S_a^n) \arrow[r] & \cdots \\ & & \pi_n(\bigvee_{b\in B} S_b^n) \arrow[u, "j_* \cong"'] & & & \\ & \bigoplus_{a\in A} \mathbb{Z}_a \arrow[uu, "\cong"] \arrow[r, "f"'] & \bigoplus_{b\in A} \mathbb{Z}_a \arrow[u, "\cong"'] & \pi_n(M(G,n)) & & \end{tikzcd}$$

Where q_* is induced by $q : (M_\phi, \bigvee_{a\in A} S_a^n) \to (C_\phi, *)$. $\bigvee_{a\in A} S_a^n$ is $(n-1)$-connected, implies $\pi_{n-1}(\bigvee_{a\in A} S_a^n) = 0$. $(M_\phi, \bigvee_{a\in A} S_a^n)$ is $(n-1)$-connected due to lemma 1.13. Therefore we have q_* is isomorphism using lemma 3.2. Diagram above reduces to:

$$0 \to \bigoplus_{a \in A} \mathbb{Z}_a \xrightarrow{f} \bigoplus_{b \in B} \mathbb{Z}_b \twoheadrightarrow \pi_n(M(G, n)) \to 0$$

For $n = 1$, use Seifert-van Kampen theorem.

□

Proposition 3.14. *For any $n \geq 1$ and any group morphism $f : G \to G'$ there exist morphism $f_M : M(G, n) \to M(G', n)$ such that $f_{M*} = f$.*

Proof. We have following for $n > 1$: (since free $\mathbb{Z}$-module is projective)

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus_{a\in A}\mathbb{Z}_a \arrow[r, tail, "i"] \arrow[d, dashed, "r_1"] & \bigoplus_{b\in B}\mathbb{Z}_b \arrow[r, two heads, "q"] \arrow[d, dashed, "r_0"] & G \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \bigoplus_{a'\in A'}\mathbb{Z}_{a'} \arrow[r, tail, "i'"'] & \bigoplus_{b'\in B'}\mathbb{Z}_{b'} \arrow[r, two heads, "q'"'] & G' \arrow[r] & 0
\end{tikzcd}
$$

And we have following for $n = 1$: (where $i(1_{1_a 1_b (1_{a\cdot b})^{-1}}) := 1_a 1_b (1_{a\cdot b})^{-1}$)

$$
\begin{tikzcd}
1 \arrow[r] & \coprod_{(a,b)\in(G,G)}\mathbb{Z}_{1_a 1_b (1_{a\cdot b})^{-1}} \arrow[r, tail, "i"] \arrow[d, dashed, "r_1"] & \coprod_{g\in G}\mathbb{Z}_g \arrow[r, two heads, "q"] \arrow[d, dashed, "r_0"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & \coprod_{(a',b')\in(G',G')}\mathbb{Z}_{1'_a 1'_b (1_{a'\cdot b'})^{-1}} \arrow[r, tail, "i'"'] & \coprod_{g'\in G'}\mathbb{Z}_{g'} \arrow[r, two heads, "q'"'] & G' \arrow[r] & 1
\end{tikzcd}
$$

We could obtain: (use lemma 3.7)

$$
\begin{tikzcd}
\bigvee_{a\in A} S^n_a \arrow[r, "\phi"] \arrow[d, "\chi_1"'] \arrow[dr, phantom, "\simeq"] & \bigvee_{b\in B} S^n_b \arrow[r] \arrow[d, "\chi_0"] & C_\phi \arrow[d, dashed, "f_M"] \\
\bigvee_{a'\in A'} S^n_{a'} \arrow[r, "\phi'"'] & \bigvee_{b'\in B'} S^n_{b'} \arrow[r] & C_{\phi'}
\end{tikzcd}
$$

Finally we have: (use universal property of cokernel)

$$
\begin{tikzcd}
0 \arrow[r] & \pi_n(\bigvee_{a\in A} S^n_a) \arrow[r, "\phi_*=i"] \arrow[d, "\chi_{1*}=r_1"] & \pi_n(\bigvee_{b\in B} S^n_b) \arrow[r] \arrow[d, "\chi_{0*}=r_0"] & \pi_n(C_\phi) \arrow[r] \arrow[d, "f_{M*}=f"] & 0 \\
0 \arrow[r] & \pi_n(\bigvee_{a'\in A'} S^n_{a'}) \arrow[r, "\phi'_*=i'"'] & \pi_n(\bigvee_{b'\in B'} S^n_{b'}) \arrow[r] & \pi_n(C_{\phi'}) \arrow[r] & 0
\end{tikzcd}
$$

□

Theorem 3.15. *$\mathrm{SP}(M(G,n))$ is a $K(G,n)$ if G is abelian.*

Proof. In the construction of Moore spaces, we have: (use notations in the construction)

$$
\begin{tikzcd}
\bigvee_{a\in A} S^n_a \arrow[r, "\phi"] \arrow[dr] & \bigvee_{b\in B} S^n_b \arrow[d, "\simeq"] \arrow[dr] & \\
& M_\phi \arrow[r] & C_\phi
\end{tikzcd}
$$

which induces quasi-fibration $\mathrm{SP}\,M_\phi \to \mathrm{SP}\,C_\phi$ with fiber $\mathrm{SP}\bigvee_{a\in A} S^n_a$. Then we have long exact sequence:

$$\cdots \to \pi_q(\mathrm{SP}\bigvee_{a\in A} S^n_a) \xrightarrow{\phi_*} \pi_q(\mathrm{SP}\,M_\phi) \to \pi_q(\mathrm{SP}\,C_\phi) \to \pi_{q-1}(\mathrm{SP}\bigvee_{a\in A} S^n_a) \to \cdots$$

Sequence above says if $q \neq n$ and $q \neq n+1$, then $\pi_q(\mathrm{SP}\,C_\phi) = 0$. If $q = n+1$, we have:

$$
\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(C_\phi) \arrow[r] & \pi_n(\mathrm{SP}\bigvee_{a\in A} S^n_a) \arrow[r, tail, "\phi_*"] \arrow[d, phantom, "\cong"] & \pi_n(\mathrm{SP}\bigvee_{b\in B} S^n_b) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & \pi_n(C_\phi) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
& 0 \arrow[r] & \bigoplus_{a\in A}\mathbb{Z}_a \arrow[r, tail, "f"'] & \bigoplus_{b\in B}\mathbb{Z}_b \arrow[r, two heads] & G \arrow[r] & 0
\end{tikzcd}
$$

We have $\pi_{n+1}(C_\phi) = 0$ since ϕ_* is monomorphism.

□

Note. Construction of Eilenberg Mac-Lane space using Moore spaces is limited, there is another construction of $K(G,n)$ allows non-abelian group G for $n=1$. (use geometric realization)

Definition 3.5. The **weak product** of pointed $\{Z_i\}_{i\in Z}$ spaces is

$$\overset{\circ}{\prod}_{i\in\mathbb{N}} Z_i := \varinjlim_{S\in\mathrm{Fin}(\mathbb{N})} \Big(\prod_{i\in S} Z_i\Big)$$

whose underlying set is:

$$\{(a_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} Z_i \mid \text{only finite } a_i \text{ is not } * \}$$

Theorem following shows why $K(G,n)$ is important:

Theorem 3.16. *If Y is a path-connected commutative H-space with strict identity ($1\cdot y = y$), then there is a weak equivalence*

$$\overset{\circ}{\prod}_{n\geq 1} K(\pi_n(Y), n) \to Y$$

Moreover, we have weak equivalence

$$\prod_{n\geq 1} K(\pi_n(Y), n) \to Y$$

Proof. Take free resolution of $\pi_n(Y)$:

$$0 \to \bigoplus_{a\in A} \mathbb{Z}_a \xrightarrow{\gamma} \bigoplus_{b\in B} \mathbb{Z}_b \xrightarrow{q} \pi_n(Y) \to 0$$

(for $n=1$, replace $\bigoplus$ with $\coprod$). and obtain:

$$\begin{tikzcd}
\bigvee_{a\in A} S^n_a \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\simeq"] & \bigvee_{b\in B} S^n_b \arrow[r] \arrow[d, "\bigvee_{b\in B} g'_b"] & C_\phi \cong M(\pi_n(Y,n)) \arrow[d, dashed, "f'_n"] \\
* \arrow[r, hook, "i"'] & Y \arrow[r] & C_i \simeq Y
\end{tikzcd}$$

where $[g'_b] = q(1_b)$. We have $f'_{n*} : \pi_n(M(\pi_n(Y),n)) \to \pi_n(Y)$ is an isomorphism.
Construct $f'^k_n : \prod_k M(\pi_n(Y),n) \to Y$ by:

$$\begin{aligned} f'^k_n : \prod_k M(\pi_n(Y),n) &\to Y \\ (a_1, a_2, \ldots, a_k) &\mapsto f(a_1)\cdot f(a_2)\cdots f(a_k) \end{aligned}$$

where $-\cdot - : Y\times Y \to Y$ is the H-multiplication on Y.
Strict identity, commutativity and associativity says it is homotopically unique $\operatorname{rel} *$.
Therefore we have a well-defined map $f^k_n : \mathrm{SP}^k M(\pi_n(Y),n) \to Y$ (for each k) which commutes with inclusion $\mathrm{SP}^k \hookrightarrow \mathrm{SP}^{k+1}$.
Directly from above, we have $f_n : \mathrm{SP}\, M(\pi_n(Y),n) \to Y$ induces isomorphism on $\pi_n(-)$. (in case $n=1$, $\pi_1(Y)$ is abelian since Y is a commutative H-space)
Similarly we have $f : \mathrm{SP}(\bigvee_n M(\pi_n(Y),n)) \to Y$ obtained from $\bigvee_n f'_n : \bigvee_n M(\pi_n(Y),n) \to Y$.
$\mathrm{SP}(\bigvee_n M(\pi_n(Y),n)) \approx \overset{\circ}{\prod}_n \mathrm{SP}\, M(\pi_n(Y),n)$ since we have $\mathrm{SP}(X_1\vee X_2) \approx \mathrm{SP}\, X_1 \times \mathrm{SP}\, X_2$ and SP commute with directed colimit. We can deduce that $f|_{\mathrm{SP}\, M(\pi_n(Y),n)} = f_n$ from construction of the homeomorphism.
Last, $\overset{\circ}{\prod}_n K(\pi_n(Y),n) \hookrightarrow \prod_n K(\pi_n(Y),n)$ is weak homotopy equivalence since S^n have compact image. (is homotopy equivalence since they are CW-complexes)

□