

Notes of Introduction to Commutative Algebra

Cloudifold

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0 Notations

Coordinate ring of k -variety $X : k[X]$

Category of commutative rings : **CRing**

Jacobson radical of ring $R : \text{Jac}(R)$

Polynomial ring with n -variables over ring $R : R[x]_n$ or $R[x_1, \dots, x_n]$

Radical ideal of ideal $I : \sqrt{I}$

Element in polynomial ring $A[x] : c \in A[x] := c = \sum_i c_i x^i$ ($\forall i . c_i \in A$)

1 Exercises of Chapter 1 : Rings and Ideals

Exercise 1.1. $A \in \mathbf{CRing}$, $x \in \sqrt{\langle 0 \rangle} \rightarrow u \in A^\times \rightarrow u + x \in A^\times$

Solution 1.1.

$$\frac{\frac{x \in \sqrt{\langle 0 \rangle}}{\exists d \in \mathbb{Z} . x^d = 0} \quad \text{w1}}{(1+x) \cdot (\sum_{i=0}^{d-1} (-x)^i) = 1} \quad \frac{u \in A^\times}{1+x \in A^\times \leftrightarrow u+ux \in A^\times} \quad \frac{x \in \sqrt{\langle 0 \rangle}}{\forall r \in A . rx \in \sqrt{\langle 0 \rangle}} \quad \text{w2}}{u+x \in A^\times}$$

$$\text{w1 : } 1 - x^n = (1+x) \cdot \left(\sum_{i=0}^{n-1} x^i \right)$$

$$\text{w2 : } u(1 + u^{-1}x) = u + x, u^{-1}x \in A^\times$$

□

Exercise 1.4. $A \in \mathbf{CRing} \rightarrow \text{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}$

Solution 1.4.

$$\frac{\frac{\text{lemma 1} \quad \text{lemma 2}}{c \in \sqrt{\langle 0 \rangle}_{A[x]} \leftrightarrow \forall i . c_i \in \sqrt{\langle 0 \rangle}_A} \quad \text{w1}}{q \in \text{Jac}(A[x]) \rightarrow q \in \sqrt{\langle 0 \rangle}_{A[x]}} \quad \frac{\text{def 1} \quad \text{def 2}}{\text{Jac}(A[x]) \supseteq \sqrt{\langle 0 \rangle}_{A[x]}} \quad \text{w2}}{\text{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}}$$

$$\text{def 1 : } \text{Jac}(A[x]) = \bigcap_{m \in \text{Spec}_{\max}(A[x])} m$$

$$\text{def 2 : } \sqrt{\langle 0 \rangle}_{A[x]} = \bigcap_{P \in \text{Spec}(A[x])} P$$

lemma 1 : $q \in \text{Jac}(A[x]) \leftrightarrow \forall y \in A[x] . 1 - qy \in A[x]^\times$ (Textbook Ch1 Prop 1.9)

lemma 2 : $c \in A[x]^\times \rightarrow c_0 \in A^\times$ and $\forall i \neq 0 . c_i \in \sqrt{\langle 0 \rangle}_A$ (Textbook Ch1 Ex 2.i)

w1 : $q \in \text{Jac}(A[x]) \rightarrow 1 - q \in A[x]^\times \rightarrow \forall i, q_i \in \sqrt{\langle 0 \rangle}_{A[x]} \rightarrow q \in \sqrt{\langle 0 \rangle}_{A[x]}$

w2 : $I \in \text{Spec}_{\max} A \rightarrow I \in \text{Spec} A$

□

Exercise 1.6. $A \in \mathbf{CRing} \rightarrow \left(I \not\subseteq \sqrt{\langle 0 \rangle}_A \rightarrow \exists e \in I, e^2 = e \neq 0 \right) \rightarrow \text{Jac } A = \sqrt{\langle 0 \rangle}_A$

Solution 1.6.

$$\begin{array}{c}
 \frac{I \not\subseteq \sqrt{\langle 0 \rangle}_A \rightarrow \exists e \in I, e^2 = e \neq 0 \quad \text{Jac } A \not\subseteq \sqrt{\langle 0 \rangle}_A}{\frac{\frac{\exists e \in \text{Jac } A, e^2 = e \neq 0}{1 - e \in A^\times \text{ and } e(1 - e) = 0} \text{ lemma 1} \Rightarrow \text{Jac } A \subseteq \sqrt{\langle 0 \rangle}_A}{\frac{e = e(1 - e)(1 - e)^{-1} = 0(1 - e)^{-1} = 0}{\perp}}} \\
 \frac{\text{Jac } A \subseteq \sqrt{\langle 0 \rangle}_A \quad \text{Jac}(A) \supseteq \sqrt{\langle 0 \rangle}_A \text{ (lemma 2)}}{\text{Jac } A = \sqrt{\langle 0 \rangle}_A}
 \end{array}$$

lemma 1 : $q \in \text{Jac}(A[x]) \leftrightarrow \forall y \in A[x] . 1 - qy \in A[x]^\times$ (Textbook Ch1 Prop 1.9)

lemma 2 : Sol 1.4. w2

□