

Funtion Spaces

Cloudifold

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0 Notations

Category of sets	: Set
Category of topological spaces	: Top
Category of (one-point-)based topological spaces	: Top _*
Topological space X with topology \mathcal{T}	: $X_{\mathcal{T}}$
Euclidean space of dimension n	: \mathbb{R}^n
Unit cube of dimension n	: I^n
Boundary of I^n	: ∂I^n
Unit interval I	: $I = I^1$
Unit cell of dimension n	: $\mathring{\mathbb{D}}^n$
Unit disk of dimension n	: \mathbb{D}^n
Unit sphere of dimension $n - 1$: \mathbb{S}^{n-1}
Inclusion or Embedding	: \hookrightarrow
Monomorphsim	: \rightarrowtail
Epimorphsim	: \twoheadrightarrow
Hom functor of category \mathcal{C}	: $\text{Hom}_{\mathcal{C}}(-, -)$
Limit (inverse limit) (projective limit)	: \lim_{\leftarrow}
Colimit (direct limit) (inductive limit)	: \lim_{\rightarrow}

1 Funtion Spaces

1.0 Introduction

Function spaces, are origins of many important constructions such as Loop spaces, Path spaces and so on. The duality between funtion spaces and product spaces will [todo]

1.1 Admissible Topology

Definition 1.1. A topology on $\text{Hom}_{\mathbf{Top}}(X, Y)$ is **admissible** if the evaluation funtion ev is **continuous**. Where ev is defined by :

$$\begin{aligned} ev : \text{Hom}_{\mathbf{Top}}(X, Y) \times X &\rightarrow Y \\ (f, x) &\mapsto f(x) \end{aligned}$$

Note. It is possible that $\text{Hom}_{\mathbf{Top}}(X, Y)$ have **no** admissible topology.

1.2 Compact-Open Topology

Definition 1.2. The **compact-open** topology on $\text{Hom}_{\mathbf{Top}}(X, Y)$ is generated by subbase $\{O^K\}$ where K varies on all compact subsets of X , O varies on all open subsets of Y . The definition of O^K is :

$$O^K := \{f \in \text{Hom}_{\mathbf{Top}}(X, Y) \mid f(K) \subseteq O\}$$

We note the compact-open topology by \mathcal{T}_{co}

Proposition 1.1. *Property of compact-open topology : The compact-open topology is coarser than any admissible topology. (That is, for any admissible topology \mathcal{T} , $\mathcal{T}_{co} \subseteq \mathcal{T}$)*

Proof. We have to show that any open set in \mathcal{T}_{co} is open in \mathcal{T} if \mathcal{T} is admissible. It suffices to show that every $O^K \in \mathcal{T}$. By definition, we have:

$$ev : \text{Hom}_{\mathbf{Top}}(X, Y)_{\mathcal{T}} \times X \rightarrow Y$$

is continuous. Take $k \in K$ and $f \in O^K$ (That is, $f(K) \in O$).

By ev is continuous, $ev(f, k) = f(k) \in O$ and the property of the base of finite product topology, we have

$$\exists V_{f,k}, W_k . f \in V_{f,k} \in \mathcal{T} \text{ and } k \in W_k \in \mathcal{T}_Y \text{ and } ev(V_{f,k} \times W_k) \subseteq O$$

The family $\{W_k\}_{k \in K}$ is an open cover of K . By compactness of K , There exists a finite subcover $\{W_{k_i}\}_{i=1, \dots, n}$ ($k_i \in K$). Put $V_f := \bigcap \{V_{f,k_i}\}_{i=1, \dots, n}$ (with $ev(V_{f,k_i} \times W_{k_i}) \subseteq O$), we have $f \in V_f$ and V_f is open in $\text{Hom}_{\mathbf{Top}}(X, Y)_{\mathcal{T}}$.

Then we have $V_f \subseteq O^K$, since

$$\frac{\frac{k \in K}{\exists k_i \in K . k \in W_{k_i}} \quad g \in V_f}{g(k) \in O} \text{ r1} \Rightarrow \frac{g \in V_f}{g(K) \subseteq O}$$

$$\text{r1} : g(k) = ev(g, k) \in ev(V_f \times W_{k_i}) \subseteq ev(V_{f,k_i} \times W_{k_i}) \subseteq O$$

So, $O^K = \bigcup \{V_f\}_{f \in O^K}$, which is a union of open sets in $\text{Hom}_{\mathbf{Top}}(X, Y)_{\mathcal{T}}$. □