Manifolds

Cloudifold

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0 Manifolds and Maps

Definition 0.1. A map $f : U \to \mathbb{R}^m$ (where U is an open subset of $\mathbb{R}^n$) is called C^k if the k-th derivative of f exist and continuous.
f is called C^∞ or **smooth** if f is C^k for any $k \in \mathbb{N}$.
$f : U \to V$ is called C^ω or **analytic** if f is C^∞ and forall $x \in U$, there exists open neighborhood U_x such that the Taylor series expansion of f at x pointwise converges to f on U_x

Definition 0.2. For a C^r ($r \geq 1$) map $f : U \to R^m$ (where U is an open subset of $\mathbb{R}^n$), $y \in f(U)$ is called a **regular value** of f if for any $x \in f^{-1}(y)$, the rank of the Jacobian matrix of f at x is m.

Note. By **Weierstrass M-test**, the "pointwise converges" condition in the definition of C^ω function can be replaced equivalently by "uniform converges".

Definition 0.3. An n**-dimensional** C^r **manifold** (we assume $r \geq 1$) (M, A) is datas:

- A T_2 space M whose topology have a countable basis
- $A = \{(U_j, \varphi_j)\}_{j \in J}$, which consists of open cover $\{U_j\}_{j \in J}$ of M and homeomorphisms $\{\varphi_j : U_j \to O_j\}_{j \in J}$. (where O_j is an open subset of $\mathbb{R}^n$)

with condition:

- Forall $i, j \in J$, the homeomorphism $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is C^r. (these maps are called **transition maps**)

A single (U_j, φ_j) is called a **chart**, A is called (C^r) **atlas** or **differential structure**.

Note. Manifolds defined above are locally compact, hence paracompact and compactly generated Hausdorff.

Definition 0.4. Suppose M, N are C^r manifolds with dimensions n, k and atlas $A = \{(U_j, \varphi_j)\}_{j \in J}$, $B = \{(V_i, \phi_i)\}_{i \in I}$. The **product of two manifolds** is a $(n + k)$-dimensional manifold with underlying space $M \times N$ and atlas $A \times B := \{(U_j \times V_i, \varphi_j \times \phi_i)\}_{(j,i) \in J \times I}$.

Definition 0.5. Suppose N is C^r manifolds with dimension n and atlas $A = \{(U_j, \varphi_j)\}_{j \in J}$, M is another manifold. $h : M \hookrightarrow N$ is topological embedding. The **induced atlas** on M is atlas $h^*A := \{(h^{-1}(U_j), \varphi_j \circ h)\}_{j \in J}$.

Definition 0.6. Suppose N is C^r manifolds with dimension n and atlas $A = \{(U_j, \varphi_j)\}_{j \in J}$, M is subspace of N. M is a C^r **submanifold of dimension** k of (N, A) if for any $x \in M$ there exists a chart (U_j, φ_j) such that $x \in U_j$ and

$$\varphi_j(M \cap U_j) = \phi_j(U_j) \cap \mathbb{R}^k$$

Such charts combine together give a atlas of M.
We also say that M is a submanifold of **codimension** $n - k$.

Note. Open subset U of a n-dimensional manifold M is obviously a submanifold of dimension n.

Example 0.1. The **real general linear group** $\mathrm{GL}_n(\mathbb{R})$ is an open analytic submanifold of $(n \times n)$-dimensional manifold $M_n(\mathbb{R})$ since $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ and the manifold structure (and topology) of $\mathrm{Mat}_{n\times n}(\mathbb{R})$ is given by bijection in (**Set**) $\mathrm{Mat}_{n\times n}(\mathbb{R}) \leftrightarrows \mathbb{R}^{n\times n}$.

Example 0.2. The (non-compact) **real Stiefel manifold** $\mathrm{St}_k(\mathbb{R}^n)$ is the set of $\mathbb{R}$-linear monomorphisms $\mathbb{R}^k \to \mathbb{R}^n$, which is equivalent (in **Set**) to the set of linearly independent elements

$$(a_1, \ldots, a_k) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n = \mathbb{R}^{n\times k}$$

(also called set of k-frames in $\mathbb{R}^n$)
$\mathrm{St}_k(\mathbb{R}^n)$ is an open analytic submanifold of $\mathbb{R}^{n\times k}$, since $\mathrm{St}_k(\mathbb{R}^n) = \bigcap\limits_{1\leq r\leq n-k+1} \det_r^{-1}(\mathbb{R} - \{0\})$.
Where $\det_r$ is defined by:

$$\det_r\left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ a_{2,1} & \cdots & a_{2,k} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k} \\ a_{n,1} & \cdots & a_{n,k} \end{bmatrix}\right) := \det\left(\begin{bmatrix} a_{r,1} & \cdots & a_{r,k} \\ a_{r+1,1} & \cdots & a_{r+1,k} \\ \vdots & & \vdots \\ a_{r+k-1,1} & \cdots & a_{r+k-1,k} \end{bmatrix}\right)$$

Example 0.3. The **Grassmann manifold** $\mathrm{Gr}_k(\mathbb{R}^n) = \mathrm{St}_k(\mathbb{R}^n)/\,\mathrm{GL}_k(\mathbb{R})$ is the set k-dimensional subspaces of $\mathbb{R}^n$. Define $U_V := \{W \in \mathrm{Gr}_k(\mathbb{R}^n) \mid \mathrm{rank}(p_V|_W : W \to V) = k\}$, where $p_V : \mathbb{R}^n \to V$ is the orthogonal projection. Let $i_W : V \to W$ be inverse of $p_V|_W$, define bijection:

$$\begin{aligned} U_V &\xrightarrow{\varphi_V} \mathrm{Hom}_{\mathbb{R}}(V, V^\perp) \approx \mathbb{R}^{k\times(n-k)} \\ W &\longmapsto p_{V^\perp} \circ i_W \\ \mathrm{Im}(\mathrm{id}_V \times M) &\longleftarrow M \end{aligned}$$

Where $p_{V^\perp} : \mathbb{R}^n = V \oplus V^\perp \to V^\perp$.
Atlas on $\mathrm{Gr}_k(\mathbb{R}^n)$ is given by $\{(U_V, \varphi_V)\}_{V\in \mathrm{Gr}_k(\mathbb{R}^n)}$.
We verify that the transition maps are analytic. Suppose E is spaned by $\mathbf{e}_1, \cdots, \mathbf{e}_k$, V have orthonormal basis $[\mathbf{v}_1, \cdots, \mathbf{v}_k]$ and $V^\perp$ have orthonormal basis $[\mathbf{v}_{k+1}, \cdots, \mathbf{v}_n]$. Let $\mathrm{GL}_n(\mathbb{R}) \ni T : E \oplus E^\perp \to V \oplus V^\perp$ be the transformation sends each $\mathbf{e}_i$ to $\mathbf{v}_i$. Let $[T]$ be its matrix respect to $[\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n]$, and $[T^{-1}]$ is matrix of its inverse. then $p_{V^\perp}$ is given by

$$a = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mapsto [\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n] \left[T^{-1}\right]_{i>k} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Suppose $M \in \mathrm{Hom}_{\mathbb{R}}(E, E^\perp)$ satisfy $W := \mathrm{Im}(\mathrm{id}_E \times M) \in U_V$, (that is $M \in \varphi_E^{-1}(U_E \cap U_V)$) let $[m_{i,j}]_{k\le i\le n, 1\le j\le k}$ be its matrix respect to $[\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n]$. Then $p_V|_W$ is given by:

$$a = [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n] \begin{bmatrix} I_k \\ M \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \longmapsto [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k] \left[T^{-1}\right]_{i\le k} \begin{bmatrix} I_k \\ M \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

And i_W is given by:

$$b = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \longmapsto [\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n] \begin{bmatrix} I_k \\ M \end{bmatrix} \left(\left[T^{-1}\right]_{i\le k} \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$$

Finally, we see that $p_{V^\perp} \circ i_W$ is:

$$b = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \longmapsto [\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \cdots, \mathbf{v}_n] \left[T^{-1}\right]_{i>k} \begin{bmatrix} I_k \\ M \end{bmatrix} \left(\left[T^{-1}\right]_{i\le k} \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$$

and it is an analytic function on M.

Definition 0.7. Let (M, A), (N, B) be C^r-manifolds, a continuous map $f : M \to N$ is said to be C^k $(k \le r)$ if for any charts $(U, \varphi_U) \in A$, $(V, \phi_V) \in B$, map

$$\phi_V \circ f \circ \varphi_U^{-1} : \varphi_U(U \cap f^{-1}(V)) \to \phi_V(V)$$

is C^k.
A C^r**-diffeomorphism** between two C^r-manifolds M, N is a C^r-map $f : M \to N$ with its C^r-inverse $f^{-1} : N \to M$.
If C^r-diffeomorphism between C^r-manifolds M, N exists, then M, N are said to be C^r**-diffeomorphic**.
Set of C^k maps from M to N is noted $\mathrm{Hom}_{C^r}(M, N)$.

Note. A C^r map which is a C^1-diffeomorphism is a C^r-diffeomorphism.

Example 0.4. $(-)^\perp : \mathrm{Gr}_k(\mathbb{R}^n) \to \mathrm{Gr}_{n-k}(\mathbb{R}^n)$ is a C^ω-diffeomorphism.

$$\varphi_{V^\perp} \circ (-)^\perp \circ \varphi_V^{-1} : \mathrm{Mat}_{k\times(n-k)}(\mathbb{R}) \ni M \mapsto M^T \in \mathrm{Mat}_{(n-k)\times k}(\mathbb{R})$$

Definition 0.8. Let (M, A) be a C^r-manifold, $x \in M$ the **tangency relation** defined on $\mathrm{Hom}_{C^1}(\mathbb{R}|_{\sim 0}, M)_x := \{\gamma \in \mathrm{Hom}_{C^1}(O, M) \mid 0 \in O \in \tau_{\mathbb{R}},\ \gamma(0) = x\}$ (τ_X is set of all open sets of X) is equivalence relation which have two equivalent definitions:

1. $\gamma_1(t) \sim_T \gamma_2(t)$ if there exists chart $(U, \varphi_U) \in A$ such that $x \in U$ and
$$\frac{\mathrm{d}(\varphi_U \circ \gamma_1)}{\mathrm{d}\,t}\Big|_{t=0} = \frac{\mathrm{d}(\varphi_U \circ \gamma_2)}{\mathrm{d}\,t}\Big|_{t=0}$$
2. $\gamma_1(t) \sim_T \gamma_2(t)$ if for any chart $(U, \varphi_U) \in A$ such that $x \in U$, we have
$$\frac{\mathrm{d}(\varphi_U \circ \gamma_1)}{\mathrm{d}\,t}\Big|_{t=0} = \frac{\mathrm{d}(\varphi_U \circ \gamma_2)}{\mathrm{d}\,t}\Big|_{t=0}$$

Note. 2. $\implies$ 1. is obvious.
1. $\implies$ 2. because transition maps are C^r-diffeomorphisms, whose derivatives are linear isomorphisms.

Definition 0.9. Let (M, A) be a n-dimensional C^r-manifold, $x \in M$ the **space of tangent vectors** on M at x is defined by: $T_x(M) := \mathrm{Hom}_{C^1}(\mathbb{R}|_{\sim 0}, M)_x / \sim_T$.

Note. If (U, φ_U) is a chart with $x \in U$, then φ_U defines a bijection $\mathrm{d}(\varphi_U \circ -) : T_x(M) \to \mathbb{R}^n$ by:

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & T_x(M) \\ \uplus & & \uplus \\ \mathbf{v} & \longmapsto & [\gamma_{\mathbf{v}}]_{\sim_T} \end{array} \qquad \begin{array}{ccc} f_{\mathbf{v}} : \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ \uplus & & \uplus \\ t & \longmapsto & \varphi_U(x) + t\mathbf{v} \end{array}$$

$$\frac{\mathrm{d}(\varphi_U \circ \gamma)}{\mathrm{d}\,t} \longleftarrow [\gamma]_{\sim_T} \qquad \gamma_{\mathbf{v}} : f_{\mathbf{v}}^{-1}(\varphi_U(U)) \xrightarrow{f_{\mathbf{v}}} \varphi_U(U) \xrightarrow{\varphi_U^{-1}} U$$

If (V, φ_V) is another chart with $x \in V$, then we have commutative diagram:

$$\begin{tikzcd} \mathbb{R}^n \arrow[rr, "\mathrm{d}(\varphi_V \circ \varphi_U^{-1})|_y"] \arrow[dr, "T\varphi_U^{-1}"'] & & \mathbb{R}^n \arrow[dl, "T\varphi_U^{-1}"] \\ & T_x(M) & \end{tikzcd}$$

where $y = \varphi_U^{-1}(x)$.

Definition 0.10. Let (M, A) be an n-dimensional C^{r+1}-manifold, the **space of all tangent vectors** of M is a $2n$-dimensional C^r-manifold defined by $T(M) := \bigsqcup_{x \in M} T_x(M)$ with atlas $\{(T(U), \phi_U)\}_{U \in A}$ (called **natural atlas**) defined by

$$\begin{array}{ccc} \bigsqcup_{x \in U} T_x(M) & \xrightarrow{\phi_U} & \varphi_U(U) \times \mathbb{R}^n \\ \uplus & & \uplus \\ (x, [\gamma]_{\sim_T}) & \longmapsto & (\varphi_U(x), \mathrm{d}(\varphi_U \circ \gamma)) \end{array}$$

The **tangent bundle** on M is the bundle

$$\begin{aligned} p_M : T(M) &\to M \\ (x, [\gamma]_{\sim_T}) &\mapsto x \end{aligned}$$

Definition 0.11. Let $(M, A), (N, B)$ be two C^{r+1}-manifolds, and $f : M \to N$ is an C^{r+1} map. Define an C^r map between bundles $Tf : TM \to TN$ by

$$\begin{array}{ccc} Tf : TM & \longrightarrow & TN \\ \uplus & & \uplus \\ (x, [\gamma]_{\sim_T}) & \longmapsto & (f(x), [f \circ \gamma]_{\sim_T}) \\ \downarrow\uparrow & & \downarrow\uparrow \\ (\varphi_U(x), \mathrm{d}(\varphi_U \circ \gamma)) & \longmapsto & (\phi_V(f(x)), \mathrm{d}(\phi_V \circ f \circ \varphi_U^{-1}) \circ \mathrm{d}(\varphi_U \circ \gamma)) \end{array}$$

Note. In fact, T is a functor from category of C^{r+1} manifolds to C^r manifolds. And there is natural diffeomorphism $T(M \times N) \approx_{\mathrm{d}} TM \times TN$.

Example 0.5. Use atlas of $S^n := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sqrt{\sum_i x_i^2} = 1\}$ given by stereographic projections:

$$\begin{aligned} U_0 := S^n - (1, 0, \cdots, 0) &\xrightarrow{\varphi_0} \mathbb{R}^n \\ (x_0, x_1, \cdots, x_n) &\longmapsto (\frac{x_1}{1 - x_0}, \cdots, \frac{x_n}{1 - x_0}) \\ (\frac{s^2 - 1}{s^2 + 1}, \frac{2x_1'}{s^2 + 1}, \cdots, \frac{2x_n'}{s^2 + 1}) &\longleftarrow (x_1', \cdots, x_n') \\ \text{where } s^2 := \sum_{1 \leq i \leq n} {x_i'}^2 &= \frac{1 + x_0}{1 - x_0} \\ U_\infty := S^n - (-1, 0, \cdots, 0) &\xrightarrow{\varphi_\infty} \mathbb{R}^n \\ (x_0, x_1, \cdots, x_n) &\longmapsto (\frac{x_1}{1 + x_0}, \cdots, \frac{x_n}{1 + x_0}) \\ (\frac{1 - s'^2}{s'^2 + 1}, \frac{2x_1'}{s'^2 + 1}, \cdots, \frac{2x_n'}{s'^2 + 1}) &\longleftarrow (x_1', \cdots, x_n') \\ \text{where } s'^2 := \sum_{1 \leq i \leq n} {x_i'}^2 &= \frac{1 - x_0}{1 + x_0} \end{aligned}$$

There is a diffeomorphism $TS^n \times \mathbb{R} \approx_{\mathrm{d}} S^n \times \mathbb{R}^{n+1}$ defined as:

$$\begin{aligned} TU_\lambda \times \mathbb{R} \approx \varphi_\lambda(U_\lambda) \times \mathbb{R}^n \times \mathbb{R} &\to U_\lambda \times \mathbb{R}^{n+1} \\ (\mathbf{x}', \mathbf{v}, a) &\mapsto (\varphi_\lambda^{-1}(\mathbf{x}'), \mathrm{d}(\varphi_\lambda^{-1})\mathbf{v} + a \cdot \mathbf{e}_K) \\ (\varphi_\lambda(\mathbf{x}), d(\varphi_\lambda)\mathbf{v}', p_K \mathbf{v}') &\leftarrow (\mathbf{x}, \mathbf{v}') \end{aligned}$$

where $\lambda = 0$ or ∞, $K = \ker \mathrm{d}(\varphi_0) = \ker \mathrm{d}(\varphi_\infty)$ is a 1-dimensional linear subspace, $\mathbf{e}_K$ is the unit vector in it, and p_K is the orthogonal projection onto it.
It is easy to check it is compatible with transition maps.

Note. In fact, embedding $i : M \hookrightarrow \mathbb{R}^k$ produces injective linear maps $T_x(M) \to T_{i(x)}\mathbb{R}^k$