CW complexes

Cloudifold

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0 Basic Definitions and Lemmas

Definition 0.1. A CW-complex is a space constructed by successively attaching cells:
For $n \in \mathbb{N}, n \geq 1$, there are maps $\{\varphi_i : S^{n-1} \to X^{n-1}\}_{i \in I_n}$ (called characteristic maps). The way to construct X^n (called n-skeleton of X) is :
(starting from $X^0 = \coprod_{I_0} *$)

$$\begin{tikzcd} \coprod_{i\in I_n} S^{n-1} \arrow[r, "\coprod_{i\in I_n}\varphi_i"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \coprod_{i\in I_n} D^n \arrow[r] & X^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad (pushout)$$

and the resulting CW-complex X is $\mathrm{Colim}\{X^0 \to \cdots \to X^n \to X^{n+1} \to \cdots\}$. The images of $\overset{\circ}{D}{}^n_i$ in X is called open cell e^n_i of X.

Definition 0.2. A is a subcomplex of CW-complex X iff for any open cell e^n_i of X, A satisfy: $A \cap e^n_i \neq \emptyset \implies \bar{e^n_i} \subseteq A$.
Pair of X and subcomplex A : (X, A) is called a CW-pair.

Definition 0.3. The Infinite Symmetric Product of a pointed space (X, x_0) is colimit of its n-th Symmetric Products ($\mathrm{SP}^n X := (\prod_{\{0,1,\ldots,n-1\}} X)/S_n$) :

$$\begin{gathered} \mathrm{Colim}\{\cdots \hookrightarrow \mathrm{SP}^n X \hookrightarrow \mathrm{SP}^{n+1} X \hookrightarrow \cdots\} \\ \{x_1, \ldots, x_n\} \mapsto \{x_0, x_1, \ldots, x_n\} \end{gathered}$$

Definition 0.4. For $n \geq 1$, a map between pairs $f : (X, A) \to (Y, B)$ is an n-equivalence if:

- $f_*^{-1}(\mathrm{Im}(\pi_0 B \to \pi_0 Y)) = \mathrm{Im}(\pi_0 A \to \pi_0 X)$
- For all choices of basepoint a in A,

  $$f_* : \pi_q(X, A, a) \to \pi_q(Y, B, f(a))$$

  is isomorphism for $1 \leq q \leq n-1$ and epimorphism for $q = n$.

Definition 0.5. A pair (X, A) of topological spaces is n-**connected** if $\pi_0(A) \to \pi_0(X)$ is surjection and $\pi_q(X, A) = 0$ for $1 \leq q \leq n$.

Definition 0.6. For topological spaces $A \hookrightarrow X$, A is a **strong deformation retract** of a neighborhood V in X if:
$\exists h : V \times I \to X$ such that
$\forall x \in V,\ h(x, 0) = x$
$h(V, 1) \subseteq A$
$\forall (a, t) \in A \times I,\ h(a, t) = a$

Definition 0.7. For topological spaces $i : A \hookrightarrow X$, A is a **deformation retract** of X if:
$\exists h : X \times I \to X$ such that
$\forall x \in X,\ h(x, 0) = x$
$h(X, 1) = A$
$\forall (a, t) \in A \times I,\ h(a, t) = a$
(That is, there are retraction $r : X \to A$ and homotopy $h : \mathrm{id}_X \simeq i \circ r \,\mathrm{rel}\, A$)
And $r := h(-, 1)$ is called a **deformation retraction**.

Definition 0.8. For topological spaces $A \hookrightarrow X$, a neighborhood V of A is **deformable** to A if:
$\exists h : X \times I \to X$ such that
$\forall x \in X,\ h(x, 0) = x$
$h(A \times I) \subseteq A$, $h(V \times I) \subseteq V$.
$h(V, 1) \subseteq A$

A criterion of weak homotopy equivalence:

Lemma 0.1. *The following on a map $e : Y \to Z$ and any fixed $n \in \mathbb{N}$ are equivalent:*

1. *For any $y \in Y$, $e_* : \pi_q(Y, y) \to \pi_q(Z, e(y))$ is monomorphism for $q = n$ and is epimorphism for $q = n + 1$.*
2. *(HELP of (D^{n+1}, S^n)) Given maps $f : D^{n+1} \to Z$, $g : S^n \to Y$ and homotopy $h : f \circ i \simeq e \circ g$:*

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, phantom, "\Leftarrow h"] \\ Y \arrow[r, "e"'] & Z \end{tikzcd}$$

   then we have extension $g^+ : D^{n+1} \to Y$ of g and $h^+ : f \simeq e \circ g^+$:

$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, "g^+"'] \\ Y \arrow[r, "e"'] & Z \arrow[ul, phantom, "\Leftarrow h^+"] \end{tikzcd}$$

3. *Conclusion above holds when the given h is $id_{f \circ i}$.*

Proof. Trivially *2.* implies *3.*

Our first goal : *3.* implies *1.*

Fix $n \in \mathbb{N}$. $\pi_n(e)$ is monomorphism:
For $n = 0$, *3.* says if we have path $e(y) \simeq e(y')$ then we have path $y \simeq y'$.That is to say e can not map two path-connected component to one.
For $n > 0$, *3.* says if $e \circ g$ is nullhomotopic, then $g : S^n \to Y$ could be extend to $g^+ : D^{n+1} \to Y$, which can be used to construct nullhomotopy of g.

Fix $n \in \mathbb{N}$. $\pi_{n+1}(e)$ is epimorphism:
For $[f] \in \pi_{n+1}(Z, e(y)) \cong [D^{n+1}, S^n; Z, e(y)]$, let $g := s \mapsto y$, the extension g^+ satisfy $e_*([g^+]) = [f]$, that proves e_* is epimorphism.

Second goal : *1.* implies *2.*

Fix g, f, h in the condition of *2.* first. And observe that $\pi_n(Y, y) = [S^n, *; Y, y]$, $\pi_{n+1}(Y, y) = [D^{n+1}, S^n; Y, y]$.
There is a map $f' : (D^{n+1}, S^n) \to Z$ homotopic to f defined by $f' = f \circ b(-, 1)$ where

$$b : CS^n \times I \to CS^n$$
$$(\overline{(x,t)}, s) \mapsto \begin{cases} \overline{(x, 1-2t)} & t \leq \frac{s}{2} \\ \overline{(x, \frac{t-s/2}{1-s/2})} & t \geq \frac{s}{2} \end{cases}$$

(recall that $D^{n+1} \simeq CS^n$) Therefore we can replace f with f'. Using the epimorphism leads to $h' : e \circ g^{+'} \simeq f'$, using the monomorphism leads to $r : g^{+'} \circ i \simeq g$. Construct $g^+ := a(-, 1)$ using

$$a : CS^n \times I \to Z$$
$$(\overline{(x,t)}, s) \mapsto \begin{cases} r(x, s-2t) & t \leq \frac{s}{2} \\ g^{+'}(x, \frac{t-s/2}{1-s/2}) & t \geq \frac{s}{2} \end{cases}$$

And that is the end of the proof:

$$
\begin{tikzcd}
S^n \arrow[rr] \arrow[dd] & & D^{n+1} \arrow[ddll, bend right] \arrow[ddll, bend left] \arrow[dd, bend right, "f'"'] \arrow[dd, bend left, "f"] \\
& \Uparrow b \quad \Leftarrow h' & \Leftarrow \\
Y \arrow[rr] & & Z
\end{tikzcd}
$$

$\square$

1 Properties and Examples

Theorem 1.1. *Homotopy Extension and Lifting property:*
A : a topological space
X : result of successively attaching cells on A of dimensions $0, 1, \dots, k$ ($k \leq n$)
$e : Y \to Z$: n-equivalence
$g : A \to Y$, $f : X \to Z$
$h : f|_A \simeq e \circ g$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\swArrow h"] & X \arrow[d, "f"] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Then there exists $g^+ : X \to Y$ extends g ($g^+|_A = g$)
and $h^+ : X \times I \to Z$ extends h, $h^+ : f \simeq e \circ g^+$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] \arrow[ur, phantom, near start, "\Leftarrow h^+"] & Z
\end{tikzcd}
$$

Proof. It suffices to prove the case $A = S^{k-1}, X = D^k$, e is inclusion. (replace Z by M_e) Apply HEP of (D^k, S^{k-1}):

$$
\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d] & D^k \arrow[d] \arrow[ddr, bend left, "f"] & \\
S^{k-1} \times I \arrow[r] \arrow[drr, bend right, "h"'] & D^k \times I \arrow[dr, dashed, "\hat{h}"] & \\
& & Z
\end{tikzcd}
$$

$f' := \hat{h}(-, 1)$, replace f with f' the diagram would be strictly commute. Therefore, f' is map of pairs $(D^k, S^{k-1}) \to (Z, Y)$, $k \leq n$ implies f' is nullhomotopic, suppose $h^+ : D^k \times I \to Z$ is the nullhomotopy, then $g^+ := h^+(-, 1)$ satisfy $g^+(D^k) \subseteq Y$.

$\square$

Note. In HELP, at condition $Y = Z$ and $e = \mathrm{id}$, HELP says (X, A) have HEP

Corollary. *If*
A : a topological space
X : result of successively attaching cells on A of any dimensions
Then, (X, A) have HEP.

Theorem 1.2. *If X is an CW-complex, $e : Y \to Z$ is an n-equivalence, Then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$, and a surjection if $\dim X = n$.(Also valid for pointed case)*

Proof. Surjectivity:
Apply HELP of $(X, \emptyset)$ ((X, x_0) for pointed case) to obtain $e_*[g^+] \simeq [f]$:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dl, "g^+"'] \\ Y \arrow[r, "e"'] & Z \end{tikzcd}$$

Injectivity ($\dim X < n$):
Suppose $[g_0], [g_1] \in [X, Y]$, $e_*[g_0] = e_*[g_1]$.
Let $f : e \circ g_0 \simeq e \circ g_1$ Apply HELP to $(X \times I, X \times \partial I)$:

$$\begin{tikzcd} X \times \partial I \arrow[r] \arrow[d, "g"'] & X \times I \arrow[d, "f"] \arrow[dl, "g^+"'] \\ Y \arrow[r, "e"'] & Z \end{tikzcd}$$

□

Corollary. *If X is a CW-complex, $e : Y \to Z$ is weak homotopy equivalence, then $e_* : [X, Y] \to [X, Z]$ is bijection.*

1.1 CW-approximation

Definition 1.1. A **CW-approximation** of $(X, A) \in \mathbf{Top(2)}$ is a CW-pair $(\widetilde{X}, \widetilde{A})$ and a weak homotopy equivalence of pairs $\varphi : (\widetilde{X}, \widetilde{A}) \to (X, A)$.

Lemma 1.3. *φ, ψ are CW-approximations of X, Y, $f : X \to Y$, then*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varphi"] \arrow[d, dashed, "\exists \widetilde{f}"'] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutes up to homotopy, and $\widetilde{f}$ is unique up to homotopy.

Proof. Directly from $\psi_* : [\widetilde{X}, \widetilde{Y}] \to [\widetilde{X}, Y]$ is bijection. □

Theorem 1.4. *φ, ψ are CW-approximations of $(X, A), (Y, B)$, $f : (X, A) \to (Y, B)$, then*

$$\begin{tikzcd} (\widetilde{X}, \widetilde{A}) \arrow[r, "\varphi"] \arrow[d, dashed, "\exists \widetilde{f}"'] & (X, A) \arrow[d, "f"] \\ (\widetilde{Y}, \widetilde{B}) \arrow[r, "\psi"'] & (Y, B) \end{tikzcd}$$

commutes up to homotopy, and $\widetilde{f}$ is unique up to homotopy.

Proof. Apply Lemma 1.3 to obtain map $\widetilde{f}_A : \widetilde{A} \to \widetilde{B}$ and homotopy $h : \psi|_{\widetilde{B}} \circ \widetilde{f}_A \simeq f \circ \varphi|_{\widetilde{A}}$ Use

HELP of $(\widetilde{X}, \widetilde{A})$ to extend it:

$$\begin{tikzcd}
\widetilde{A} \arrow[r, hook] \arrow[d, "\widetilde{f}_A"'] & \widetilde{X} \arrow[d] \arrow[ddl, "\widetilde{f}"'] \\
\widetilde{B} \arrow[d, hook] & X \arrow[d] \\
\widetilde{Y} \arrow[r] & Y
\end{tikzcd}$$

ψ_* is bijection implies the uniqueness up to homotopy of $\widetilde{f}$. $\square$

Theorem 1.5. *(Whitehead's Theorem)*
Every n-equivalence between CW-complexes whose dimension is lower than n, is homotopy equivalence. Every weak homotopy equivalence between CW-complexes is homotopy equivalence.

Proof. $e : Y \to Z$ induce bijections $[Y, Y] \to [Y, Z]$ and $[Z, Y] \to [Z, Z]$, $[f] = e_*^{-1}[\mathrm{id}_Z]$ implies $[e \circ f] = [\mathrm{id}_Z]$ and $[e \circ f \circ e] = [e]$ ($[f \circ e] = e_*^{-1}[e] = [\mathrm{id}_Y]$).

$\square$

Corollary. *CW-approximation is unique up to homotopy.*

Example 1.1. Polish circle (Warsaw circle) : closed topologist's sine curve. It is n-connected forall n but not contractible.

Definition 1.2. A cellular map between CW-pairs is $g : (X, A) \to (Y, B)$ such that $g(A \cup X^n) \subseteq B \cup Y^n$.

Theorem 1.6. *For any map between CW-pairs $f : (X, A) \to (Y, B)$ there exists a cellular map g such that $g \simeq f \operatorname{rel} A$*

Proof. Construct g inductively:
Start from $A \cup X^0$:
take paths $\gamma_i : f(x_i) \simeq y_i$, where y_i is any point in Y^0 and $x_i \in X^0 - A$.
Construct $h_0 : (X^0 \cup A) \times I \to Y$: $h_0|_A(a, t) := f(a)$, $h_0|_{X^0 - A}(x_i, t) := \gamma_i(t)$. This is a homotopy from f to $g_0 := h_0(-, 1) : A \cup X^0 \to B \cup Y^0$
Inductive step:
Assume $g_n : A \cup X^n \to B \cup Y^n$ and homotopy $h_n : f|_{A \cup X^n} \simeq g_n$ is given, try to construct g_{n+1}:
For each characteristic map $\varphi_i : S^n \to X^n$, take the resulting cell map $\varphi_i^+ : D^{n+1} \to X^{n+1}$ and use HELP of (D^{n+1}, S^n):

$$\begin{tikzcd}
S^n \arrow[r, hook] \arrow[d, "\varphi_i"'] & D^{n+1} \arrow[d, "\varphi_i^+"] \arrow[ddl, "g_{n+1,i}"'] \\
X^n \arrow[d, "g_n"'] & X^{n+1} \arrow[d, "f"] \arrow[dl, Rightarrow, "h_{n+1,i}"] \\
B \cup Y^{n+1} \arrow[r, hook] & Y
\end{tikzcd}$$

Glue all $g_{n+1,i}$ and $h_{n+1,i}$ to produce g_{n+1} and $h_{n+1} : f|_{A \cup X^{n+1}} \simeq g_{n+1}$.
Final stage:
Maps g_n determine a cellular map $g : X \to Y$ since X has the final topology determined by skeletons. $\square$

1.2 Operation of CW-complexes

Product of CW-complexes:

Example 1.2. Product topology of two CW-complexes does not coincide with the final topology (union topology):
X (star of countably many edges) : $X = X^1 = \bigvee_{n\in\omega} I_n$
Y (star of ω^ω many edges) : $Y = Y^1 = \bigvee_{f\in\omega^\omega} I_f$ ($(I_n, 0) \cong (I_f, 0) \cong (I, 0)$)
Consider subset H of $X \times Y$: $H := \{(\frac{1}{f(n)+1}, \frac{1}{f(n)+1}) \in I_n \times I_f \mid n \in \omega, f \in \omega^\omega\}$.
H is closed under the final topology since every cell of $X \times Y$ contains at most one point of H. But closure of H contains $(0, 0)$ at product topology:
Let $U \times V$ be an open neighborhood (at product topology) of $(0, 0)$, let $g : \omega \to \omega - 0$ be an increasing function such that forall $n \in \omega, [0, \frac{1}{g(n)}) \subseteq U \cap I_n$, let $k \in omega$ be sufficiently large that $\frac{1}{g(k)+1} \subseteq V \cap I_g$, then $(\frac{1}{g(k)+1}, \frac{1}{g(k)+1}) \in U \times V \cap H$.

Note. Another way to realize $X \times Y$ as CW-complex is to change its topology to the compactly generated topology $k(X \times Y)$.

Proposition 1.7. *X and Y are CW-complexes, $X \times Y$ is CW-complex if*
X or Y is locally compact
or
both X and Y have countably many cells.

Quotient of CW-pair:

Proposition 1.8. *For CW-complex X and subcomplex A, the Quotient space X/A have a CW-complex structure induced by X and A.*

Proof. Suppose the characteristic maps of X are indexed by $\{I_n\}_{n\in\mathbb{N}}$ and of A are indexed by $\{I'_n\}_{n\in\mathbb{N}}$ ($I'_n \subseteq I_n$). Then the characteristic maps of X/A are indexed by $\{K_n\}_{n\in\mathbb{N}}$, which defined below:
$K_0 := (I_0 - I'_0) \cup \{i_0\}$ where i_0 is an arbitrary element in I'_0
$K_n := I_n - I'_n$ for $n > 0$.
Verify the maps determine the CW-complex structure:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d] & X^{n-1}/A^{n-1} \arrow[d] \arrow[ddd, bend left] & \\
D^n \arrow[r] \arrow[ddrr, bend right] & X^n \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & X^n/A^{n-1} \arrow[d] \arrow[dd, dashed, bend right] \arrow[ul, phantom, "\ulcorner", very near start] & A^n \arrow[l] \arrow[d] \\
& & X^n/A^n \arrow[d, dashed] \arrow[ur, phantom, "\urcorner", very near start] & * \arrow[l] \arrow[dl, bend left] \\
& & Z &
\end{tikzcd}$$

□

Smash product of CW-complexes:

Proposition 1.9. *If (X, x_0) , (Y, y_0) are pointed CW-complexes with both countably many cell, and $X^{r-1} = \{x_0\}$, $Y^{s-1} = \{y_0\}$, then $X \wedge Y := X \times Y/X \vee Y$ is an $(r + s - 1)$-connected CW-complex.*

Proof. $X \times Y$ is CW-complex with cells of the form $e^n_{i,X} \times \{y_0\}$, $\{x_0\} \times e^m_{j,Y}$ or $e^n_{i,X} \times e^m_{j,Y}$ for $n \geq r$, $m \geq s$. Cells of the first two forms are contianed in $X \vee Y$, therefore $(X \wedge Y)^{r+s-1} = *$. □

Corollary. *If X is a pointed CW-complex, then $\Sigma^n X$ is an $(n - 1)$-connected CW-complex.*

1.3 Properties of Infinite Symmetric Product

Functoriality:

Pointed map $f : X \to Y$ induces

$$\begin{aligned} f_n : \mathrm{SP}^n X &\to \mathrm{SP}^n Y \\ \{x_1, \dots, x_n\} &\mapsto \{f(x_1), \dots, f(x_n)\} \end{aligned}$$

$$\begin{tikzcd} {} \arrow[r] & \mathrm{SP}^n X \arrow[r] \arrow[d, "f_n"] & \mathrm{SP}^{n+1} X \arrow[r] \arrow[d, "f_{n+1}"] & {} \\ {} \arrow[r] & \mathrm{SP}^n Y \arrow[r] & \mathrm{SP}^{n+1} Y \arrow[r] & {} \end{tikzcd}$$

Which induces map $\mathrm{SP}\, f : \mathrm{SP}\, X \to \mathrm{SP}\, Y$. And Functorial properties are directly from the constructions above.

Commute with directed colimit:
Suppose P is a directed poset (that is $\forall x, y \in P,\ \exists z \in P,\ x \leq z, y \leq z$) and X_i are pointed spaces indexed by P satisfying $i \leq j \implies X_i \subseteq X_j$.
Then $\mathrm{SP}^n(\mathrm{Colim}_i\, X_i) \approx \mathrm{Colim}_i(\mathrm{SP}^n X_i)$
(Proof is obtained by showing that $\mathrm{SP}^n f$ is continuous iff f is, which implies final topology on $\mathrm{Colim}_i(\mathrm{SP}^n X_i)$ agree on $\mathrm{SP}^n(\mathrm{Colim}_i\, X_i)$)

Suppose $i : A \hookrightarrow X$ is an pointed inclusion, then $\mathrm{SP}\, i : \mathrm{SP}\, A \hookrightarrow \mathrm{SP}\, X$ is also inclusion. Furthermore, if A is open (or closed) in X, then $\mathrm{SP}\, A$ is open (or closed) in $\mathrm{SP}\, X$.

Pointed homotopy $h : X \times I \to Y$ induces

$$\begin{aligned} h_n : \mathrm{SP}^n X \times I &\to \mathrm{SP}^n Y \\ (\{x_1, \dots, x_n\}, t) &\mapsto \{h(x_1, t), \dots, h(x_n, t)\} \end{aligned}$$

which induces $\mathrm{SP}\, h : \mathrm{SP}\, X \times I \to \mathrm{SP}\, Y$.

Then we observe:
$f \simeq g$ implies $\mathrm{SP}\, f \simeq \mathrm{SP}\, g$,
$e : X \to Y$ is homotopy equivalence implies $\mathrm{SP}\, e : \mathrm{SP}\, X \to \mathrm{SP}\, Y$ is,
X is contractible implies $\mathrm{SP}^n X$ and $\mathrm{SP}\, X$ is.

Theorem 1.10. *(Dold-Thom Theorem)*
If X is T_2 space and A is closed path-connected subspace of X, and there is neighborhood V deformable to A in X.
Then the quotient map $q : X \to X/A$ induces quasi-fibration $\mathrm{SP}\, q : \mathrm{SP}\, X \to \mathrm{SP}(X/A)$, which satisfy $\forall x \in \mathrm{SP}(X/A),\ (\mathrm{SP}\, q)^{-1}\{x\} \simeq \mathrm{SP}\, A$.

Corollary. *If X , Y are T_2 spaces and Y is connected, $f : X \to Y$. Then consider $X \to Y \to C_f \to \Sigma X$, the map $p : C_f \to \Sigma X$ induces quasi-fibration $\mathrm{SP}\, p : \mathrm{SP}\, C_f \to \mathrm{SP}(\Sigma X)$ with fiber $\mathrm{SP}\, Y$.*

Corollary. *If X is T_2 and path-connected, then for any $q \geq 0$, there is $\pi_{q+1}(\mathrm{SP}(\Sigma X)) \cong \pi_q(\mathrm{SP}\, X)$.*

Proof. CX is contractible implies $\mathrm{SP}\, CX$ is contractible, use the exat homotopy sequence of quasi-fibration to see:

$$\longrightarrow \pi_{q+1}(\mathrm{SP}\, CX) \longrightarrow \pi_{q+1}(\mathrm{SP}\, \Sigma X) \xrightarrow[\partial]{\cong} \pi_q(\mathrm{SP}\, X) \longrightarrow \pi_q(\mathrm{SP}\, CX) \longrightarrow$$

$\square$

Note. The inverse of the isomorphism ∂ above is given by

$$[S^q, \mathrm{SP}\, X] \ni [g] \mapsto [\Sigma g] \in [S^{q+1}, \Sigma\, \mathrm{SP}\, X]$$

$(\Sigma\,\mathrm{SP}\,X \cong \mathrm{SP}\,\Sigma X)$. Because ∂ is given by:

$$\begin{tikzcd}
\pi_q(\mathrm{SP}\,\Sigma X) \arrow[r] & \pi_q(\mathrm{SP}\,CX, \mathrm{SP}\,X) \arrow[r] & \pi_{q-1}(\mathrm{SP}\,X) \\
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
{[f]} \arrow[r, mapsto] & {[\hat{f}]} \arrow[r, mapsto] & {[\hat{f}|_{S^{q-1}}]} \\
{[p \circ Cg] = [\Sigma g]} & {[Cg]} \arrow[l, mapsto] & {[g]} \arrow[l, mapsto]
\end{tikzcd}$$

. $\square$

Corollary. *If X is T_2 space and A is path-connected subspace of X, then the canonical map $\mathrm{SP}(X \cup (A \times I)) \to \mathrm{SP}(X \cup CA)$ is a quasi-fibration with fiber $\mathrm{SP}\,A$.*

Theorem 1.11. *If X is T_2 space and A is path-connected subspace of X, and $A \hookrightarrow X$ is a cofibration.*
Then the quotient map $q : X \to X/A$ induces quasi-fibration $\mathrm{SP}\,q : \mathrm{SP}\,X \to \mathrm{SP}(X/A)$, which satisfy $\forall x \in \mathrm{SP}(X/A)$, $(\mathrm{SP}\,q)^{-1}\{x\} \simeq \mathrm{SP}\,A$.

Proof. If $A \hookrightarrow X$ is cofibration, then $X \cup CA \simeq X/A$ and $X \cup (A \times I) \simeq X$. $\square$

Proposition 1.12. *The inclusion $S^1 \to \mathrm{SP}\,S^1$ is homotopy equivalence, therefore $\pi_q(S^1) \cong \pi_q(\mathrm{SP}\,S^1)$.*

Proof. $S^1 \simeq S^2 - \{0, \infty\}$
$\mathrm{SP}^n\,S^2 = \{\{a_1, \dots, a_n\} \mid a_i \in \mathbb{C} \cup \{\infty\}\} = \{\prod_{\{a_1,\dots,a_n\}}(z - a_i) \mid a_i \in \mathbb{C} \cup \{\infty\}\}$ where $(z - \infty) := 1$
$\mathrm{SP}^n\,S^2 = \{f \in \mathbb{C}[z] - \{0\} \mid \deg(f) \le n\} = \mathbb{CP}^n$

$\mathrm{SP}^n(S^2 - \{0, \infty\}) = \{f \in \mathbb{C}[z] - \{0\} \mid \deg(f) \le n, f_n \ne 0, f_0 \ne 0\} = \mathbb{C}^n - \mathbb{C}^{n-1} \times 0 = \mathbb{C}^{n-1} \times (\mathbb{C} - 0)$

it have the same homotopy type of S^1

Corollary. *$\pi_q(\mathrm{SP}\,S^n) = \mathbb{Z}$ if $q = n$, otherwise $\pi_q(\mathrm{SP}\,S^n) = 0$. (use corollary of 1.10 to see $\pi_{q+1}(\mathrm{SP}\,\Sigma X) \cong \pi_q(\mathrm{SP}\,X)$)*

2 Homology Groups

2.1 Reduced Homology Groups

Definition 2.1. For a path-connected pointed CW-complex X, define its **n-th reduced homology group** for $n \ge 0$:

$$\tilde{H}_n(X) := \pi_n(\mathrm{SP}\,X)$$

Note. All reduced homology groups are abelian since $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$. Thus, we can extend the definition above to those X which does not necessarily be path-connected.

As SP, $\tilde{H}_n$ also satisfy functoriality. Furthermore, $\tilde{H}_n$ maps homotopic maps $f \simeq g$ to identical maps $f_* = g_*$. (SP maps homotopic maps to homotopic maps)

Exact Property:

Proposition 2.1. *For any pointed map between CW-complexes $f : X \to Y$, we have an exact sequence:*

$$\tilde{H}_n(X) \xrightarrow{f_*} \tilde{H}_n(Y) \xrightarrow{i_*} \tilde{H}_n(C_f)$$

where C_f is the mapping cone of f, $i : Y \hookrightarrow C_f$.

Proof. $Z_f := Y \cup_f (X \times I)/\{x_0\} \times I$ is the **reduced mapping cylinder** of f.
$q : Z_f \to C_f$ is defined by

$$y \mapsto y$$
$$\overline{(x,t)}^{Z_f} \mapsto \overline{(x,t)}^{C_f}$$

By Dold-Thom theorem, the induced map $\mathrm{SP}\, q$ is quasi-fibration $\mathrm{SP}\, Z_f \to \mathrm{SP}\, C_f$ with fiber $\mathrm{SP}\, X$. By definition of quasi-fibration, we have

$$\pi_n(\mathrm{SP}\, X) \cong \tilde{H}_n(X) \xrightarrow{f_*} \pi_n(\mathrm{SP}\, Z_f) \cong \tilde{H}_n(Y) \xrightarrow{i_*} \pi_n(\mathrm{SP}\, C_f) = \tilde{H}_n(C_f)$$

□

Proposition 2.2. *There does not exist retraction $r : \mathbb{D}^n \to S^{n-1}$.*

Proof. $id = r \circ i : \mathbb{S}^{n-1} \to \mathbb{D}^n \to \mathbb{S}^{n-1}$ induces

$$id_* = r_* \circ i_* : \mathbb{Z} \cong \tilde{H}_{n-1}\mathbb{S}^{n-1} \to \tilde{H}_{n-1}\mathbb{D}^n \cong 0 \to \tilde{H}_{n-1}\mathbb{S}^{n-1} \cong \mathbb{Z}$$

which lead to contradiction. □

Theorem 2.3. *Fix-point theorem:*
If $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then exist $x_0 \in \mathbb{D}^n$ such that $x_0 = f(x_0)$.

Proof. (non-constructive) No such x_0 implies $\forall x \in \mathbb{D}^n, f(x) \neq x$ therefore, we can construct continuous retraction $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ by
$r(x)$:= the intersection of "ray starting from $f(x)$ to x" and $\mathbb{S}^{n-1}$. Contradict to 2.2.

Definition 2.2. Let (X, A) be an CW-pair, define the **n-th homology group** for $n \in \mathbb{N}$ of (X, A) be:

$$H_n(X, A) := \tilde{H}_n(X \cup CA)$$

And for single space:

$$H_n(X) := H_n(X, \emptyset) = \tilde{H}(X + 1)$$

where $X + 1 := X \sqcup *$.

Note. Map between CW-pair $f : (X, A) \to (Y, B)$, induces map $\bar{f} : X \cup CA \to Y \cup CB$ defined by $(x, t) \mapsto (f(x), t)$, which induces $f_* : \tilde{H}_n(X \cup CA) \to \tilde{H}_n(Y \cup CB)$ for any $n \in \mathbb{N}$.

2.2 Axioms for Homology

Definition 2.3. A (Ordinary) Homology Theory (on **TOP** with coefficient $G \in \mathbf{Ab}$) is functors $\{H_n(-,-;G) : \mathbf{TOP(2)} \to \mathbf{Ab}\}_{n\in\mathbb{N}}$, with natural transformations $\partial_{n,(X,A)} : H_n(X, A; G) \to H_n(A, \emptyset; G)$ (called connecting homomorphism)
satisfying following axioms:

- **Dimension**:
  $H_0(*, \emptyset; G) = G$, for any $n > 0$, $H_n(*, \emptyset; G) = 0$.

- **Weak Equivalence**:
  Weak equivalence $f : (X, A) \to (Y, B)$ induces isomorphism

  $$f_* : H_*(X, A; G) \to H_*(Y, B; G)$$

- **Long Exact Sequence**:
  For any $(X, A) \in \mathbf{TOP(2)}$, maps $A \hookrightarrow X$ and $(X, \emptyset) \to (X, A)$ induce a long exact sequence together with ∂:

  $$\cdots \to H_{q+1}(A; G) \to H_{q+1}(X; G) \to H_{q+1}(X, A; G) \to H_q(A; G) \to \cdots$$

  where $H_n(X; G) := H_n(X, \emptyset; G)$.

- **Additivity**:
  If $(X,A)=\coprod_\lambda (X_\lambda,A_\lambda)$ in $\mathbf{TOP(2)}$, then inclusions $i_\lambda:(X_\lambda,A_\lambda)\to(X,A)$ induces isomorphism
  $$(\bigoplus i_{*,\lambda}):\bigoplus_\lambda H_*(X_\lambda,A_\lambda;G)\cong H_*(X,A;G)$$
- **Excision**:
  If $(X;A,B)$ is an **excisive triad** (that is, $X=\mathring{A}\cup\mathring{B}$), then inclusion $(A,A\cap B)\hookrightarrow(X,B)$ induces isomorphism
  $$H_*(A,A\cap B;G)\cong H_*(X,B;G)$$

Note. An equivalent form of Excision Axiom:

If $(X,A)\in\mathbf{TOP(2)}$, U is subspace of A and $\overline{U}\subseteq\mathring{A}$, then inclusion $i:(X-U,A-U)\hookrightarrow(X,A)$ induces isomorphism
$$i_*:H_*(X-U,A-U;G)\to H_*(X,A;G)$$

There is a critical criterion about weak homotopy equivalence between excisive triads, we prove lemmas first:

Lemma 2.4. *For*
$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i_*"] \\
X \arrow[r, "f_*"'] & X\cup_Z Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$
if D is deformation retract of X and $Z\subseteq D\subseteq X$, then $D\cup_Z Y$ is deformation retract of $X\cup_Z Y$.

Proof. Let $h:\mathrm{id}_X\simeq r\circ i$ where r is the deformation retraction $X\to D$. Define $h_*:\mathrm{id}_{X\cup_Z Y}\simeq(i\cup_Z\mathrm{id}_Y)\circ(r\cup_Z\mathrm{id}_Y)$
$$\begin{aligned}h_*:(X\cup_Z Y)\times I&\to X\cup_Z Y\\(x,t)&\mapsto f_*(h(x,t))\\(y,t)&\mapsto i_*(y)\end{aligned}$$

Observe that $(X\cup_Z Y)\times I=(X\times I)\cup_{Z\times I}(Y\times I)$, check that h^* is continuous:
$$\begin{tikzcd}
Z\times I \arrow[r] \arrow[d] & Y\times I \arrow[d] \arrow[dr, "\mathrm{id}_{\mathrm{id}_Y}"] & \\
X\times I \arrow[r] \arrow[dr, "h"'] & (X\cup_Z Y)\times I \arrow[dr, "h_*"] \arrow[ul, phantom, "\ulcorner", very near start] & Y \arrow[d, "i_*"'] \\
 & X \arrow[r, "f_*"'] & X\cup_Z Y
\end{tikzcd}$$

∎

Lemma 2.5. *For maps $i:C\to A$, $j:C\to B$ define the double mapping cylinder $M(i,j):=A\cup_{C\times\{0\}}C\times I\cup_{C\times\{1\}}B$. If i is cofibration, then the quotient map*
$$\begin{aligned}q:M(i,j)&\to A\cup_C B\\a&\mapsto a\\b&\mapsto b\\(c,t)&\mapsto c\end{aligned}$$

is a homotopy equivalence.

Proof.

$$\begin{tikzcd} C \arrow[r] \arrow[d, hook, "i"'] & B \arrow[d] \\ A \arrow[r, "i_A"'] & A \cup_C B \end{tikzcd}$$

The canonical quotient $r : M_{i_A} \to A \cup_C B$ is a deformation retraction with homotopy:

$$\begin{aligned} h : (B \cup_{C\times 0} (A \times I)) \times I &\to B \cup_{C\times 0} (A \times I) = M_{i_A} \\ (a, t, s) &\mapsto (a, (1-s)t) \\ (b, s) &\mapsto b \end{aligned}$$

Observe that $C \times I \cup_C A \times \{1\}$ is a deformation retract of $A \times I$, since $i : C \to A$ is cofibration. Then we have $M(i,j) = B \cup_{C\times\{0\}} (C \times I \cup_{C\times\{1\}} A \times \{1\})$ is a deformation retract of $B \cup_{C\times\{0\}} A \times I = M_{i_A}$. (use lemma 2.4)
Finally, an easy check shows that $M(i,j) \to M_{i_A} \xrightarrow{r} A \cup_C B$ is identical to q.

□

Theorem 2.6. *For excisive triads $(X; X_1, X_2)$, $(X'; X_1', X_2')$ and map $e : X \to X'$, if*

$$\begin{aligned} e|_{X_1} &: X_1 \to X_1' \\ e|_{X_2} &: X_2 \to X_2' \\ e|_{X_3} &: X_3 \to X_3' \end{aligned}$$

are weak equivalences, (where $X_3 := X_1 \cap X_2,\ X_3' := X_1' \cap X_2'$) then e is.

Proof. Use an important criterion of weak homotopy equivalence, it suffices to show forall $n \in \mathbb{N}$, any commutative diagram below:

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \\ X \arrow[r, "e"'] & X' \end{tikzcd}$$

can be filled like:

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, "g^+"'] \\ X \arrow[r, "e"'] \arrow[ur, phantom, "\Leftarrow h"] & X' \end{tikzcd}$$

whose upper triangle commutes.

Let

$$A_1 := g^{-1}(X - \overset{\circ}{X_1}) \cup f^{-1}(X' - \overset{\circ}{X_1'})$$
$$A_2 := g^{-1}(X - \overset{\circ}{X_2}) \cup f^{-1}(X' - \overset{\circ}{X_2'})$$

which are disjoint closed subsets of D^{n+1}. Choose CW-complex structure on D^{n+1} such that for each n-cell σ_i, $\overline{\sigma_i} \cap (A_1 \cup A_2) = \overline{\sigma_i} \cap A_1$ or $\overline{\sigma_i} \cap A_2$. Now define

$$K_1 := \bigcup \{\overline{\sigma_i} \mid g(\overline{\sigma_i} \cap S^n) \subseteq \overset{\circ}{X_1} \text{ and } f(\overline{\sigma_i}) \subseteq \overset{\circ}{X_1'}\} = \bigcup \{\overline{\sigma_i} \mid \overline{\sigma_i} \cap A_1 = \emptyset\}$$
$$K_2 := \bigcup \{\overline{\sigma_i} \mid g(\overline{\sigma_i} \cap S^n) \subseteq \overset{\circ}{X_2} \text{ and } f(\overline{\sigma_i}) \subseteq \overset{\circ}{X_2'}\} = \bigcup \{\overline{\sigma_i} \mid \overline{\sigma_i} \cap A_2 = \emptyset\}$$

which are subcomplexes of D^{n+1} and satisfy $K_1 \cup K_2 = D^{n+1}$. By HELP, we have:

$$\begin{tikzcd}
S^n \cap K_1 \cap K_2 \arrow[r, "i"] \arrow[d, "g|_{K_1 \cap K_2}"'] & K_1 \cap K_2 \arrow[d, "f|_{K_1 \cap K_2}"] \arrow[dl, "g_0"'] \\
X_1 \cap X_2 \arrow[r, "e|_{X_1 \cap X_2}"'] & X_1' \cap X_2' \arrow[ul, phantom, "\Leftarrow h_0"]
\end{tikzcd}$$

such that h_0 is $f|_{K_1 \cap K_2} \simeq e \circ g_0 \operatorname{rel} (S^n \cap K_1 \cap K_2)$. Apply HELP to:

$$\begin{tikzcd}
(S^n \cup K_1) \cap K_2 \arrow[r, "i_2"] \arrow[d, "g_{K_2}"'] & K_2 \arrow[d, "f|_{K_2}"] \\
X_2 \arrow[r] & X_2' \arrow[ul, phantom, "\Leftarrow h_{K_2}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^n \cup K_2) \cap K_1 \arrow[r, "i_1"] \arrow[d, "g_{K_1}"'] & K_1 \arrow[d, "f|_{K_1}"] \\
X_1 \arrow[r] & X_1' \arrow[ul, phantom, "\Leftarrow h_{K_1}"]
\end{tikzcd}$$

where
g_{K_i} are defined by $g_{K_i}|_{S^n \cap K_i} := g|_{S^n \cap K_i}$ and $g_{K_i}|_{K_1 \cap K_2} := g_0$,
h_{K_2} are defined by (h_{K_1} is similar):

$$h_{K_2} : ((S^n \cup K_1) \cap K_2) \times I \to X_2'$$
$$(x,t) \mapsto \begin{cases} e(g(x)) & x \in S^n \cap K_2 \\ h_0(x,t) & x \in K_1 \cap K_2 \end{cases}$$

We get:

$$\begin{tikzcd}
(S^n \cup K_1) \cap K_2 \arrow[r] \arrow[d, "g_{K_2}"'] & K_2 \arrow[d, "f|_{K_2}"] \arrow[dl, "g_2"'] \\
X_2 \arrow[r, "e|_{X_2}"'] & X_2' \arrow[ul, phantom, "\Leftarrow h_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^n \cup K_2) \cap K_1 \arrow[r] \arrow[d, "g_{K_1}"'] & K_1 \arrow[d] \arrow[dl, "g_1"'] \\
X_1 \arrow[r, "e|_{X_1}"'] & X_1' \arrow[ul, phantom, "\Leftarrow h_1"]
\end{tikzcd}$$

Define g^+ and $h : f \simeq g \operatorname{rel} S^n$ by $g^+|_{K_i} := g_i$ and $h|_{K_i \times I} := h_i$.
$h|_{S^n \times I} = (e \circ g) \times \mathrm{id}_I$ (h is $\operatorname{rel} S^n$) since $h_i(-,t)|_{S^n \cap K_i} = h_{K_i}(-,t)|_{S^n \cap K_i} = e \circ g|_{S^n \cap K_i}$.

□

Note. The proof above can be easily modified to case each weak equivalence appear in the statement is an n-equivalence.

Following theorem allow us to use CW-triads to approximate excisive triads:

Theorem 2.7. *For any excisive triad $(X; A, B)$, there is a CW-triad $(\widetilde{X}; \widetilde{A}, \widetilde{B})$ (A CW-triad $(X; A, B)$ is X and its subcomplex A, B such that $A \cup B = X$) and a map $r : \widetilde{X} \to X$ such that*

$$r|_{\widetilde{A}} : \widetilde{A} \to A$$
$$r|_{\widetilde{B}} : \widetilde{B} \to B$$
$$r|_{\widetilde{C}} : \widetilde{C} \to C$$
$$r : \widetilde{X} \to X$$

are all weak homotopy equivalences (where $\widetilde{C} := \widetilde{A} \cap \widetilde{B}$, $C := A \cap B$). Furthermore, such r is natural up to homotopy.

Proof. Choose a CW-approximation $r_C : \widetilde{C} \to C$ and extend it to $r_A : \widetilde{A} \to A$, $r_B : \widetilde{B} \to B$. $\widetilde{X} := \widetilde{A} \cup_{\widetilde{C}} \widetilde{B}$. $i : \widetilde{C} \to \widetilde{A}$ and $j : \widetilde{C} \to \widetilde{B}$ are cofibrations, by lemma 2.5 we have homotopy equivalence $q : M(i,j) \to \widetilde{X}$, which induces homotopy equivalence of triads:

$$q : M(i,j) \to \widetilde{X}$$
$$q| : \widetilde{A} \cup (\widetilde{C} \times [0, \frac{2}{3})) \to \widetilde{A}$$
$$q| : \widetilde{B} \cup (\widetilde{C} \times (\frac{1}{3}, 1]) \to \widetilde{B}$$

then we can deduce that $r \circ q$ is a weak homotopy equivalence by theorem 2.6. Consquently, r is weak homotopy equivalence. r is natural up to homotopy since each CW-approximation r_C, r_A, r_B is.

□

Then we have:

Definition 2.4. A (Ordinary) Homology Theory on CW-complexes with coefficient $G \in \mathbf{Ab}$ is functors $\{H_n(-,-;G) : \mathbf{CW\text{-}pairs} \to \mathbf{Ab}\}_{n\in\mathbb{N}}$, with natural transformations $\partial_{n,(X,A)} : H_n(X,A;G) \to H_n(A,\emptyset;G)$ (called connecting homomorphism)
satisfying axioms with the excision axiom changed to:

- **Excision**:
  If (X;A,B) is an **CW-triad** (that is $X = A \cup B$ for subcomplexes A and B) then the inclusion $(A, A\cap B) \hookrightarrow (X,B)$ induces isomorphism
  $$H_*(A, A\cap B; G) \cong H_*(X,B;G)$$

Proposition 2.8. *The homology groups defined in definition 2.2 is a ordinary homology theory on CW-complexes with coefficient $\mathbb{Z}$.*

Proof.

- Dimension: by a corollary, $H_q(*,\emptyset) = \pi_q(\mathrm{SP}\, S^0) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \geq 1 \end{cases}$.
- Weak Equivalence: SP preserves weak equivalence.
- Long Exact Sequence: use a corollary of Dold-Thom theorem.
- Additivity: For index set Λ, $P := \{S \mid S \subseteq \Lambda\}$.
  Then define $Y_S := \bigvee_{\lambda\in S} X_\lambda \cup CA_\lambda = (\coprod_{\lambda\in S} X_\lambda) \cup C(\coprod_{\lambda\in S} A_\lambda)$,
  and use fact that SP commutes with directed colimit, we have
  $\bigvee_{\lambda\in\Lambda} \mathrm{SP}(X_\lambda \cup CA_\lambda) = \mathrm{Colim}_{S\in P}\, \mathrm{SP}\, Y_S \approx \mathrm{SP}(\mathrm{Colim}_{S\in P}\, Y_S) = \mathrm{SP}((\coprod_{\lambda\in\Lambda} X_\lambda) \cup C(\coprod_{\lambda\in\Lambda} A_\lambda)) = \mathrm{SP}(X \cup CA)$.
  Which induces $\bigoplus_{\lambda\in\Lambda} \tilde{H}_n(X_\lambda \cup CA_\lambda) \cong \pi_n(\bigvee_{\lambda\in\Lambda} \mathrm{SP}(X_\lambda \cup CA_\lambda)) \cong \pi_n(\mathrm{SP}(X \cup CA)) = \tilde{H}_n(X \cup CA)$.
- Excision: For CW-triad $(X;A,B)$, $A/(A\cap B) \approx X/B$. Apply theorem 1.11 to $(Y \cup CZ, CZ)$ to show that $H_n(Y,Z) \cong \tilde{H}_n(Y/Z)$.

□

3 Homotopy and Eilenberg-Mac Lane Spaces

Theorem 3.1. *(Blakers–Massey) Homotopy Excision Theorem:*
For pointed CW-triad $(X;A,B)$ such that $C := A\cap B \neq \emptyset$, if (A,C) is $(m-1)$-connected and (B,C) is $(n-1)$-connected where $m \geq 2$, $n \geq 1$. Then $i : (A,C) \to (X,B)$ is an $(m+n-2)$-equivalence for pairs.

Note. We can replace the "CW-triad" with "excisive triad" in condition by theorem 2.7.

Proof. Define (pointed) the triad homotopy group for $q \geq 2$:

$$\pi_q(X; A, B) := \pi_{q-1}(P_{i_{B,X}}, P_{i_{C,A}})$$

where $i_{B,X} : B \hookrightarrow X$, $i_{C,A} : C \hookrightarrow A$ and P_f is the homotopy fiber

$$\{(y, \gamma) \in Y \times M(I, Z)_* \mid \gamma(1) = f(y)\}$$

of pointed map $f : Y \to Z$. Use long exact sequence of pairs:

$$\cdots \to \pi_q(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_{q-1}(P_{i_{C,A}}) \to \pi_{q-1}(P_{i_{B,X}}) \to \pi_{q-1}(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_{q-2}(P_{i_{C,A}}) \to \cdots$$
$$\cdots \to \pi_1(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_0(P_{i_{C,A}}) \to \pi_0(P_{i_{B,X}})$$

and observe that $\pi_q(P_{i_{X,B}}) \cong \pi_{q+1}(X, B)$ since for any $f : S^q \to P_{i_{X,B}}$ we have:

$$\begin{tikzcd}
S^q \arrow[dr, "f"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g"'] & & \\
& P_{i_{B,X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M(I,X)_* \arrow[d] \\
& B \arrow[r] & X
\end{tikzcd}$$

use the fact $f' \in M(S^q, M(I, X)_*)_* \cong M(S^q \wedge I, X)_* \ni f''$ and $S^q \wedge I \approx D^{q+1}$ with

$$\begin{aligned} S^q &\hookrightarrow S^q \wedge I \approx D^{q+1} \\ s &\mapsto (s, 1) \end{aligned}$$

the condition $f'(s)(1) = g(s)$ is equivalent to $f''((s, 1)) = g(s)$, that is have a map f is equivalent to have a map $f'' : (D^{q+1}, S^q) \to (X, B)$. With the analogue statement also valid for homotopies $S^q \times I \to P_{i_{X,B}}$, we have $\pi_q(P_{i_{B,X}}) = [S^q, *; P_{i_{B,X}}, *] \cong [D^{q+1}, S^q; X, B] = \pi_{q+1}(X, B)$.
Rewrites the long exact sequence of pairs above to:

$$\cdots \to \pi_{q+1}(X; A, B) \to \pi_q(A, C) \to \pi_q(X, B) \to \pi_q(X; A, B) \to \pi_{q-1}(A, C) \to \cdots$$
$$\cdots \to \pi_2(X; A, B) \to \pi_1(A, C) \to \pi_1(X, B)$$

Conditions $m \geq 1, n \geq 1$ guarantees $\pi_0(C) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(B)$ are surjections.
$m \geq 2$ is equivalent to $\pi_1(A, C) = 0$, which implies $\pi_0(C) \to \pi_0(A)$ is bijection.
For $x \in \pi_0(A \cap_C B)$, we can always find $b \in \pi_0(B), i_{B,X\,*}(b) = x$ or $a \in \pi_0(A), i_{A,X\,*}(a) = x$ which becomes $b \in \pi_0(B), i_{B,X\,*}(b) = x$ or $c \in \pi_0(C), i_{C,X\,*}(c) = x$ when $\pi_0(C) \to \pi_0(A)$ is bijection. That is equivalent to $\pi_0(B) \to \pi_0(X)$ is bijection, which means $\pi_1(X, B) = 0$.

We only need to show that for $2 \leq q \leq m + n - 2$, $\pi_q(X; A, B) = 0$.

With $J^{q-1} := (\partial I^{q-1} \times I) \cup (I^{q-1} \times \{0\})$, we have:

$$\begin{aligned}
\pi_q(P_{i_{B,X}}, P_{i_{C,A}}) &= [I^q, \partial I^q, J^{q-1}; P_{i_{B,X}}, P_{i_{C,A}}, *] \\
&= [I^q \wedge I;\ I^q,\ \partial I^q \wedge I,\ J^{q-1} \wedge I \to X; B, A, *]
\end{aligned}$$

$$:= \text{relative homotopy classes of pointed maps } f : I^q \wedge I \to X \text{ who satisfy: } \begin{cases} f(I^q) & \subseteq B \\ f(\partial I^q \wedge I) & \subseteq A \\ f(\partial I^q) & \subseteq C \\ f(J^{q-1} \wedge I) & = * \end{cases}$$

$$\text{“relative” means the homotopy } h \text{ determine the classes satisfy: } \begin{cases} h(I^q \times I) & \subseteq B \\ h((\partial I^q \wedge I) \times I) & \subseteq A \\ h(\partial I^q \times I) & \subseteq C \\ h((J^{q-1} \wedge I) \times I) & = * \end{cases}$$

(notice that $\partial I^q \wedge I \cap I^q = \partial I^q$, therefore $f(\partial I^q) \subseteq A \cap B = C$)

(this is called (relative) homotopy class of maps of tetrads)

$$\begin{aligned}
&= [(I^q \times I)/K;\ I^q \times \{1\},\ (\partial I^q \times I)/K,\ (J^{q-1} \times I)/K \to X; B, A, *] \\
&\quad (K := I^q \times \{0\} \cup \{i_0\} \times I) \\
&= [I^{q+1};\ (I^q \times \{1\}) \cup K,\ (\partial I^q \times I) \cup K,\ J^{q-1} \times I \cup K \to X; B, A, *] \\
&= [I^{q+1};\ I^q \times \{1\},\ I^{q-1} \times \{1\} \times I,\ J^{q-1} \times I \cup I^q \times \{0\} \to X; B, A, *] \\
&\quad (\text{notice that } \partial I^q = \partial I^{q-1} \times I \cup I^{q-1} \times \{0, 1\})
\end{aligned}$$

We can assume that (A, C) have no relative $q < m$-cells and (B, C) have no relative $q < n$-cells. And we can assume that X has finite many cells since I^q is compact.
For subcomplexes $C \subseteq A' \subseteq A$, where $A = e^m \cup A'$ (attaching one cell from A').
Let $X' := A' \cup_C B$, if the results hold for $(X'; A', B)$ and $(X; A, X')$, then it hold for $(X; A, B)$ since we have map between exact homotopy sequences of triples (A, A', C) and (X, X', B):

$$\begin{tikzcd}
\pi_{q+1}(A, A') \arrow[r] \arrow[d, "i_{2,q+1}"'] & \pi_q(A', C) \arrow[r] \arrow[d, "i_{1,q}"'] & \pi_q(A, C) \arrow[r] \arrow[d, "i_{3,q}"'] & \pi_q(A, A') \arrow[r] \arrow[d, "i_{2,q}"'] & \pi_{q-1}(A', C) \arrow[d, "i_{1,q-1}"'] \\
\pi_{q+1}(X, X') \arrow[r] & \pi_q(X', B) \arrow[r] & \pi_q(X, B) \arrow[r] & \pi_q(X, X') \arrow[r] & \pi_{q-1}(X', B)
\end{tikzcd}$$

induced by inclusion $(A, A', C) \hookrightarrow (X, X', B)$. If the result hold for $(X'; A', B)$ and $(X; A, X')$, maps $i_{1,q},\ i_{2,q}$ are isomorphisms when $1 \geq q \geq m + n - 3$, are epimorphisms when $q = m + n - 2$.
Notice the 5-lemma says that
if $i_{1,q}$ and $i_{2,q}$ are epimorphisms, $i_{1,q-1}$ are monomorphism, then $i_{3,q}$ is epimorphism.
if $i_{1,q}$ and $i_{2,q}$ are monomorphisms, $i_{2,q+1}$ are epimorphism, then $i_{3,q}$ is monomorphism.
We also have if $C \subseteq B' \subseteq B$ with $B = B' \cup e^n$, the result hold for CW-triads $(X'; A, B')$ and $(X; X', B)$ where $X' = A \cup_C B'$, since $(A, C) \hookrightarrow (X, B)$ factors as $(A, C) \hookrightarrow (X', B') \hookrightarrow (X, B)$.

Now we can assume that $A = C \cup D^m$ and $B = C \cup D^n$.

The current goal of proof is to prove any

$$f : (I^{q+1};\ I^q \times \{1\},\ I^{q-1} \times \{1\} \times I,\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X; B, A, *)$$

is nullhomotopic for any $q + 1$ with $2 \leq q + 1 \leq m + n - 2$.
For $a \in \mathring{D}^m$, $b \in \mathring{D}^n$ We have inclusions of based triads:

$$(A; A, A - \{a\}) \hookrightarrow (X - \{b\}; X - \{b\}, X - \{a, b\}) \hookrightarrow (X; X - \{b\}, X - \{a\}) \hookleftarrow (X; A, B)$$

The first and the third induces isomorphisms on homotopy groups of triads since B is a strong deformation retract of $X - \{a\}$ in X and A is a strong deformation retract of $X - \{b\}$ in X. $\pi_*(A; A, A - \{a\}) = 0$ since $\pi_*(A, A - \{a\}) \to \pi_*(A, A \cap (A - \{a\}))$ are isomorphisms.

Current goal : choose good a, b to show f regarded as a pointed traid map to $(X; X - \{b\}, X - \{a\})$ is homotopic to a map

$$f' : (I^{q+1};\ I^{q-1} \times \{1\} \times I,\ I^q \times \{1\},\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X - \{b\}; X - \{b\}, X - \{a, b\}, *)$$

if $2 \le q + 1 \le m + n - 2$.

Note. We want to homotopically remove some point $f^{-1}(b)$, first we may want to construct some Urysohn function u separating $f^{-1}(a) \cup J^{q-1} \times I \cup I^q \times \{0\}$ and $f^{-1}(b)$ and construct homotopy of cube $h^+ : (r, s, t) \mapsto (r, (1 - u(r, s)t)s)$ wishing that $f(h^+(r, s, 1))$ would miss b. The problem in this method is that points $f^{-1}(b)$ in the cube would be homotopically replaced by other points. Since our desire homotopy does not change the first q coordinates of the cube, we want to separate $p^{-1}(p(f^{-1}(a))) \cup J^{q-1} \times I$ and $p^{-1}(p(f^{-1}(b)))$ (where $p : I^q \times I \to I^q$). Our problem is to find suitable a, b such that $p(f^{-1}(a)) \cap p(f^{-1}(b)) = \emptyset$.

We use manifold structure on D^m and D^n to achieve it, now we homotopically approximate f by a map g which smooth on $f^{-1}(D^m_{<1/2} \cup D^n_{<1/2})$.

Let $U_{<r} := f^{-1}(D^m_{<r} \cup D^n_{<r})$, Use smooth deformation theorem to construct smooth map (for any $0 < \epsilon$) $g' : U_{<3/4} \to D^m_{<3/4} \cup D^n_{<3/4}$ with homotopy $h_1 : g' \simeq f|_{U_{<3/4}}$ (and bound $|g'(x) - f(x)| < \epsilon$ for any $x \in U_{<1}$) and take partition of unity $\{\rho, \rho'\}$ with subcoordinates $\{I^{q+1} - \overline{U_{<1/2}},\ U_{<3/4}\}$, we have:

$$\begin{aligned} g &:= \rho f + \rho' g' \\ h_2 &: g \simeq f \operatorname{rel} (I^{q+1} - U_{<3/4}) \\ h_2 &: I^{q+1} \times I \to X \\ &\quad (x, t) \mapsto \rho(x) f(x) + \rho'(x) h_1(x, t) \end{aligned}$$

with scalar multiplication and addition is already defined on smooth structure on $D^m_{<3/4} \cup D^n_{<3/4}$. We could assmue that $g(I^{q-1} \times \{1\} \times I) \cap D^n_{<1/2} = \emptyset$ (which implies g is a map of tetrads to $(X; X - \{b\}, X - \{a\}, *)$) and $g(I^q \times \{1\}) \cap D^m_{<1/2} = \emptyset$ since $f(I^{q-1} \times \{1\} \times I) \subseteq A$ and $f(I^q \times \{1\}) \subseteq B$ and we can always tighten the bound ϵ, (Similar argument also hold for h_2, then we have $h_2 : g \simeq f$ as homotopy between maps of tetrads.)

Use the manifold structure to find good (a, b):
$V := g^{-1}(D^m_{<1/2}) \times g^{-1}(D^n_{<1/2})$ is a sub-manifold of $I^{2(q+1)}$. Consider $W := \{(v, v') \in V \mid p(v) = p(v')\}$, which is the zero set of smooth submersion $(v, v') \mapsto p(v) - p(v')$. W is smooth manifold with codimension q. Therefore the map $(g, g) : W \to D^m_{<1/2} \times D^n_{<1/2}$ is smooth map between manifolds of dimension $q + 2$ and $m + n$. The map is not surjection since $q + 2 < m + n$. Then we have $(a, b) \notin (g, g)(W)$ (that is, $p(g^{-1}(a)) \cap p(g^{-1}(b))$).

Since $g(I^{q-1} \times \{1\} \times I) \cap D^n_{<1/2} = \emptyset$ and $g(J^{q-1} \times I) \cap D^n_{<1/2} = \emptyset$, we have $g(\partial I^q \times I) \cap D^n_{<1/2} = \emptyset$. By Urysohn's lemma, we have $u : I^q \to I$ separating $p(g^{-1}(a)) \cup \partial I^q$ and $p(g^{-1}(b))$. Finally we have:

$$\begin{aligned} h' &: I^q \times I \times I \to I^q \times I \\ &\quad (r, s, t) \mapsto (r, (1 - u(r)t)s) \end{aligned}$$

and $h := g \circ h'$, $f' := h(-, 1)$. $f'(I^{q+1}) \cap \{b\} = \emptyset$ since if $\exists (r, s) \in I^q \times I$, $f'(r, s) = b$, then $b = g(r, (1 - u(r))s) = g(r, 0) = *$ leads to contradiction.
Last step is to check that h is a homotopy between maps

$$(I^{q+1};\ I^{q-1} \times \{1\} \times I,\ I^q \times \{1\},\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X; X - \{b\}, X - \{a\}, *)$$

Since g is, $g \circ h'$ is too.

$\square$

Corollary. *Suppose that $Y_0 \hookrightarrow Y$ is cofibration, (Y, Y_0) is $(r-1)$-connected and Y_0 is $(s-1)$-connected, then $(Y, Y_0) \to (Y/Y_0, *)$ is $(r+s-1)$-equivalence. $(r \geq 2,\ s \geq 1)$*

Proof. $Y_0 \hookrightarrow CY_0$ is cofibration and (CY_0, Y_0) is s-connected. Use homotopy excision theorem (with $X = Y \cup CY_0,\ A = Y,\ B = CY_0,\ C = Y_0$) to see $(Y, Y_0) \to (Y \cup CY_0, CY_0)$ is $(r+s-1)$-equivalence. And $(Y \cup CY_0, CY_0) \to (Y/Y_0, *)$ is homotopy equivalence since $Y_0 \hookrightarrow Y$ is cofibration.

$\square$

Corollary. *For $n \geq 2$, $f : X \to Y$ is $(n-1)$-equivalence between $(s-1)$-connected spaces, then $(M_f, X) \to (C_f^+, *)$ is $(n+s-1)$-equivalence. Where $C_f^+ := Y \cup_f C^+X$, $C^+X := (X \times I)/(X \times \{1\})$. is the unreduced mapping cone and the unreduced cone.*

Proof. f is $(n-1)$-equivalence implies (M_f, X) is $(n-1)$-connected. Use corollary above.

$\square$

Corollary. *For $n \geq 2$, if $f : X \to Y$ is pointed map between $(n-1)$-connected well-pointed spaces (that is, pointed space whose inclusion of the base point is cofibration). Then C_f is $(n-1)$-connected and $\pi_n(M_f, X) \to \pi_n(C_f, *)$ is isomorphism.*

Proof. Use homotopy extension property to extend to unreduced case. f is map between $(n-1)$-connected space implies f is at least a $(n-1)$-equivalence. Therefore $(M_f, X) \to (C_f, *)$ is $(2n-1)$-equivalence, Since we have $n < 2n-1$ for any $n \geq 2$, $\pi_n(M_f, X) \to \pi_n(C_f, *)$ is isomorphism.

$\square$

Theorem 3.2. *(Freudenthal Suspension Theorem) If X is well-pointed and $(n-1)$-connected $(n \geq 1)$, then the map:*

$$\begin{aligned} \sigma : \pi_q(X) &\to \pi_{q+1}(\Sigma X) \\ f &\mapsto \Sigma f \end{aligned}$$

is isomorphism if $q < 2n-1$ and epimorphism if $q = 2n-1$.

Proof. If we have $f : (I^q, \partial I^q) \to (X, *)$ then $f \times \mathrm{id}_I : I^{q+1} \to X \times I$ will give a map $\overline{f \times \mathrm{id}_I}$: $(I^{q+1},\ \partial I^{q+1},\ \partial I^q \times I \cup \partial I \times \{1\}) \to (CX, X, *)$ since $J^q = \partial I^q \times I \cup \partial I \times \{0\}$, it does not give a map in $\pi_{q+1}(CX, X)$. we should change $\overline{f \times \mathrm{id}_I}$ into $\overline{f \times -\mathrm{id}_I}$. we have commutative diagram:

$$\begin{tikzcd} \pi_{q+1}(CX, X) \arrow[r, "p_*"] \arrow[d, bend right, "\partial"'] & \pi_{q+1}(CX/X, *) \arrow[d, equal] \\ \pi_q(X) \arrow[r, "-\sigma"'] \arrow[u, bend right, "i"'] & \pi_{q+1}(\Sigma X) \end{tikzcd} \qquad \begin{tikzcd} {[\overline{f \times -\mathrm{id}_I}]} \arrow[r, mapsto] \arrow[d, bend right] & {[p \circ (\overline{f \times -\mathrm{id}_I})]} \arrow[d, equal] \\ {[f]} \arrow[r, mapsto] \arrow[u, bend right] & {[-\Sigma f]} \end{tikzcd}$$

Where $p : (CX, X) \to (CX/X, *)$ is the canonical quotient map and $i : [f] \to [\overline{f \times -\mathrm{id}_I}]$ makes $\pi_{q+1}(CX) \to \pi_{q+1}(CX, X) \to \pi_q(X) \to \pi_q(CX)$ split in middle (that is, i is inverse of the connecting homomorphism ∂). We verify the commutativity:

$$\begin{aligned} -\Sigma f : (I^{q+1}, \partial I^{q+1}) &\to (CX/X, *) \\ (s, t) &\mapsto f(s) \wedge (1-t) \\ p \circ (\overline{f \times -\mathrm{id}_I}) : (I^{q+1}, \partial I^{q+1}) &\to (CX/X, *) \\ (s, t) &\mapsto f(s) \wedge (1-t) \end{aligned}$$

Since $X \hookrightarrow CX$ is cofibration and n-equivalence between $(n-1)$-connected spaces, p is an $2n$-equivalence. Therefore, $q+1 < 2n$ implies $-\sigma$ is isomorphism, $q+1 = 2n$ implies $-\sigma$ is epimorphism, and we have $-\sigma$ is iff σ is.

$\square$

Definition 3.1. We now define the q-th stable homotopy group:

$$\pi_k^s(X) := \mathrm{Colim}_r\, \pi_{k+r}(\Sigma^r X) \cong \pi_{2k+2}(\Sigma^{k+2} X) \cong \pi_{k+n}(\Sigma^n X) \qquad (n-1 > k)$$

The relation right side is directly from $\Sigma^n X$ is $(n-1)$-connected.

Note. We'll see later that $\{\pi_n^s\}_{n \in \mathbb{N}}$ defines a generalized homology theory.