CW complexes

Cloudifold

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0 Basic Definitions and Lemmas

Definition 0.1. A **CW-complex** is a space constructed by successively attaching cells:
For $n \in \mathbb{N}, n \geq 0$, there are maps $\{\varphi_i : S^{n-1} \to X^{n-1}\}_{i \in I_n}$ (called characteristic maps). The way to construct X^n (called n-skeleton of X) is :
(starting from $X^{-1} = \emptyset$, if we start from $X^{-1} = A$, we say (X, A) is a **relative CW-complex**)

$$\begin{tikzcd} \coprod_{i \in I_n} S^{n-1} \arrow[r, "\coprod_{i \in I_n} \varphi_i"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\ \coprod_{i \in I_n} D^n \arrow[r] & X^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \quad (pushout)$$

and the resulting CW-complex X is $\varinjlim\{X^0 \to \cdots \to X^n \to X^{n+1} \to \cdots\}$. The images of $\overset{\circ}{D}{}^n_i$ in X is called open cell e^n_i of X.

Definition 0.2. A is a subcomplex of CW-complex X iff for any open cell e^n_i of X, A satisfy: $A \cap e^n_i \neq \emptyset \implies \bar{e^n_i} \subseteq A$.
Pair of X and subcomplex A : (X, A) is called a CW-pair.

Definition 0.3. The Infinite Symmetric Product of a pointed space (X, x_0) is colimit of its n-th Symmetric Products ($\mathrm{SP}^n X := (\prod_{\{0,1,\ldots,n-1\}} X)/S_n$) :

$$\varinjlim\{\cdots \hookrightarrow \mathrm{SP}^n X \hookrightarrow \mathrm{SP}^{n+1} X \hookrightarrow \cdots\}$$
$$\{x_1, \ldots, x_n\} \mapsto \{x_0, x_1, \ldots, x_n\}$$

Definition 0.4. For $n \geq 1$, a map between pairs $f : (X, A) \to (Y, B)$ is an n-equivalence if:

- $f_*^{-1}(\mathrm{Im}(\pi_0 B \to \pi_0 Y)) = \mathrm{Im}(\pi_0 A \to \pi_0 X)$
- For all choices of basepoint a in A,

  $$f_* : \pi_q(X, A, a) \to \pi_q(Y, B, f(a))$$

  is isomorphism for $1 \leq q \leq n-1$ and epimorphism for $q = n$.

Definition 0.5. A pair (X, A) of topological spaces is **n-connected** if $\pi_0(A) \to \pi_0(X)$ is surjection and $\pi_q(X, A) = 0$ for $1 \leq q \leq n$.

Definition 0.6. For topological spaces $A \hookrightarrow X$, A is a **strong deformation retract** of a neighborhood V in X if:
$\exists h : V \times I \to X$ such that
$\forall x \in V,\ h(x, 0) = x$
$h(V, 1) \subseteq A$
$\forall (a, t) \in A \times I,\ h(a, t) = a$

Definition 0.7. For topological spaces $i : A \hookrightarrow X$, A is a **deformation retract** of X if:
$\exists h : X \times I \to X$ such that
$\forall x \in X,\ h(x, 0) = x$
$h(X, 1) = A$
$\forall (a, t) \in A \times I,\ h(a, t) = a$
(That is, there are retraction $r : X \to A$ and homotopy $h : \mathrm{id}_X \simeq i \circ r \,\mathrm{rel}\, A$)
And $r := h(-, 1)$ is called a **deformation retraction**.

Definition 0.8. For topological spaces $A \hookrightarrow X$, a neighborhood V of A is **deformable** to A if:
$\exists h : X \times I \to X$ such that
$\forall x \in X,\ h(x, 0) = x$
$h(A \times I) \subseteq A$, $h(V \times I) \subseteq V$.
$h(V, 1) \subseteq A$

Definition 0.9. For a topological group G, a **relative G-(equivariant) CW-complex** (X, A) is a space constructed by successively attaching **G-equivariant cells** $G/H \times D^n$ on a G-space A: For $n \in \mathbb{N}, n \geq 0$, there are maps $\{\varphi_i : G/H_i \times S^{n-1} \to X^{n-1}\}_{i \in I_n}$ (called characteristic maps) where each H_i is closed subgroup of G and G acts trivially on D^n, S^{n-1}. The way to construct X^n (called n-skeleton of X) is:
(starting from $X^{-1} = A$ where A is an G-space)

$$\begin{tikzcd}
\coprod_{i\in I_n}(G/H_i \times S^{n-1}) \arrow[r, "\coprod_{i\in I_n}\varphi_i"] \arrow[d, hook] & X^{n-1} \arrow[d, hook] \\
\coprod_{i\in I_n}(G/H_i \times D^{n}) \arrow[r, "\coprod_{i\in I_n}\phi_i"] & X^{n}
\end{tikzcd} \quad \text{(pushout in category of } G\text{-spaces)}$$

The resulting X is $\varinjlim\{X^{-1} \to X^0 \to \cdots \to X^n \to X^{n+1} \to \cdots\}$. The images of $G/H_i \times \overset{\circ}{D}{}^n_i$ in X is called open n-cell of type G/H_i. ϕ_i is called the attaching map and $\varphi_i(G/H_i \times S^{n-1})$ is called the boundary of $\phi_i(G/H_i \times D^n)$. If $A = \emptyset$, then X is called a **G-(equivariant) CW-complex**.

A criterion of weak homotopy equivalence:

Lemma 0.1. *The following on a map $e : Y \to Z$ and any fixed $n \in \mathbb{N}$ are equivalent:*

1. *For any $y \in Y$, $e_* : \pi_q(Y, y) \to \pi_q(Z, e(y))$ is monomorphism for $q = n$ and is epimorphism for $q = n + 1$.*
2. *(HELP of (D^{n+1}, S^n)) Given maps $f : D^{n+1} \to Z$, $g : S^n \to Y$ and homotopy $h : f \circ i \simeq e \circ g$:*

$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, phantom, "\swarrow h"] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}$$

then we have extension $g^+ : D^{n+1} \to Y$ of g and $h^+ : f \simeq e \circ g^+$:

$$\begin{tikzcd}
S^n \arrow[r, hook] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] & Z \arrow[l, phantom, bend right, "\Leftarrow h^+"']
\end{tikzcd}$$

3. *Conclusion above holds when the given h is $id_{f \circ i}$.*

Proof. Trivially *2.* implies *3.*

Our first goal : *3.* implies *1.*

Fix $n \in \mathbb{N}$. $\pi_n(e)$ is monomorphism:
For $n = 0$, *3.* says if we have path $e(y) \simeq e(y')$ then we have path $y \simeq y'$.That is to say e can not map two path-connected component to one.
For $n > 0$, *3.* says if $e \circ g$ is nullhomotopic, then $g : S^n \to Y$ could be extend to $g^+ : D^{n+1} \to Y$, which can be used to construct nullhomotopy of g.

Fix $n \in \mathbb{N}$. $\pi_{n+1}(e)$ is epimorphism:
For $[f] \in \pi_{n+1}(Z, e(y)) \cong [D^{n+1}, S^n; Z, e(y)]$, let $g := s \mapsto y$, the extension g^+ satisfy $e_*([g^+]) = [f]$, that proves e_* is epimorphism.

Second goal : *1.* implies *2.*

Fix g, f, h in the condition of *2.* first. And observe that $\pi_n(Y, y) = [S^n, *; Y, y]$, $\pi_{n+1}(Y, y) = [D^{n+1}, S^n; Y, y]$.
There is a map $f' : (D^{n+1}, S^n) \to Z$ homotopic to f defined by $f' = f \circ b(-, 1)$ where

$$
\begin{aligned}
b : CS^n \times I &\to CS^n \\
(\overline{(x,t)}, s) &\mapsto \begin{cases} \overline{(x, 1-2t)} & t \leq \frac{s}{2} \\ \overline{(x, \frac{t-s/2}{1-s/2})} & t \geq \frac{s}{2} \end{cases}
\end{aligned}
$$

(recall that $D^{n+1} \simeq CS^n$) Therefore we can replace f with f'. Using the epimorphism leads to $h' : e \circ {g^+}' \simeq f'$, using the monomorphism leads to $r : {g^+}' \circ i \simeq g$. Construct $g^+ := a(-, 1)$ using

$$
\begin{aligned}
a : CS^n \times I &\to Z \\
(\overline{(x,t)}, s) &\mapsto \begin{cases} r(x, s-2t) & t \leq \frac{s}{2} \\ {g^+}'(x, \frac{t-s/2}{1-s/2}) & t \geq \frac{s}{2} \end{cases}
\end{aligned}
$$

And that is the end of the proof:

$$
\begin{tikzcd}
S^n \arrow[r] \arrow[d] & D^{n+1} \arrow[dl, bend right] \arrow[dl, bend left] \arrow[dl, phantom, "b\Uparrow"] \arrow[d, bend right, "f'"'] \arrow[d, bend left, "f"] \arrow[d, phantom, "\Leftarrow"] \arrow[dl, phantom, shift right=3ex, "\Leftarrow h'"'] \\
Y \arrow[r] & Z
\end{tikzcd}
$$

$\square$

1 Right Notion For Spaces

Theorem 1.1. *Homotopy Extension and Lifting property:*
A : a topological space
X : result of successively attaching cells on A of dimensions $0, 1, \ldots, k$ ($k \leq n$)
$e : Y \to Z$: n-equivalence
$g : A \to Y$, $f : X \to Z$
$h : f|_A \simeq e \circ g$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow h"] & X \arrow[d, "f"] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Then there exists $g^+ : X \to Y$ extends g ($g^+|_A = g$)
and $h^+ : X \times I \to Z$ extends h, $h^+ : f \simeq e \circ g^+$

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \arrow[dl, "g^+"'] \arrow[dl, phantom, shift left=2ex, "\swarrow h^+"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Proof. It suffices to prove the case $A = S^{k-1}, X = D^k$, e is inclusion. (replace Z by M_e) Apply HEP of (D^k, S^{k-1}):

$$
\begin{tikzcd}
S^{k-1} \arrow[r] \arrow[d] & D^k \arrow[d] \arrow[ddr, bend left, "f"] & \\
S^{k-1} \times I \arrow[r] \arrow[drr, bend right, "h"'] & D^k \times I \arrow[dr, dashed, "\hat{h}"] & \\
& & Z
\end{tikzcd}
$$

$f' := \hat{h}(-,1)$, replace f with f' the diagram would be strictly commute. Therefore, f' is map of pairs $(D^k, S^{k-1}) \to (Z, Y)$, $k \leq n$ implies f' is nullhomotopic, suppose $h^+ : D^k \times I \to Z$ is the nullhomotopy, then $g^+ := h^+(-,1)$ satisfy $g^+(D^k) \subseteq Y$.

□

Note. In HELP, at condition $Y = Z$ and $e = \mathrm{id}$, HELP says (X, A) have HEP

Corollary 1.2. *If*
A : a topological space
X : result of successively attaching cells on A of any dimensions
Then, (X, A) have HEP.

Theorem 1.3. *If X is an CW-complex, $e : Y \to Z$ is an n-equivalence, Then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$, and a surjection if $\dim X = n$.(Also valid for pointed case)*

Proof. Surjectivity:
Apply HELP of $(X, \emptyset)$ ((X, x_0) for pointed case) to obtain $e_*[g^+] \simeq [f]$:

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

Injectivity ($\dim X < n$):
Suppose $[g_0], [g_1] \in [X, Y]$, $e_*[g_0] = e_*[g_1]$.
Let $f : e \circ g_0 \simeq e \circ g_1$ Apply HELP to $(X \times I, X \times \partial I)$:

$$
\begin{tikzcd}
X \times \partial I \arrow[r] \arrow[d, "g"'] & X \times I \arrow[d, "f"] \arrow[dl, "g^+"'] \\
Y \arrow[r, "e"'] & Z
\end{tikzcd}
$$

□

Corollary 1.4. *If X is a CW-complex, $e : Y \to Z$ is weak homotopy equivalence, then $e_* : [X, Y] \to [X, Z]$ is bijection.*

1.1 CW-approximation

This subsection shows that CW-complexes encode all weak-homotopy types of **TOP**.

Definition 1.1. A **CW-approximation** of $(X, A) \in \mathbf{Top(2)}$ is a CW-pair $(\widetilde{X}, \widetilde{A})$ and a weak homotopy equivalence of pairs $\varphi : (\widetilde{X}, \widetilde{A}) \to (X, A)$.

Theorem 1.5. *(Existence of CW-approximation) If X is path-connected pointed space (0-connected), then there is a CW-approximation $(\widetilde{X}, *) \xrightarrow{\phi} (X, *)$. If X is n-connected then $\widetilde{X}$ could be chosen to satisfy $\widetilde{X}^n = *$. (Moreover, each characteristic map of X is pointed)*

Proof. If X is n-connected, then $\phi_n : Y^n := * \to X$ is n-equivariance. Assmue inductively that we already have m-equivalence $Y^m \xrightarrow{\phi_m} X$ $(m \geq n)$, Our goal is construct Y^{m+1} and $\phi_{m+1} : Y^{m+1} \to X$.
Let

$$f_m^+ : \bigoplus_{a \in A} \mathbb{Z}_a \twoheadrightarrow \ker(\phi_{m*}) \subseteq \pi_m(Y^m)$$

be a free resolution of $\ker(\phi_{m*})$ ($\coprod_{a\in A} \mathbb{Z}_a$ if $m = 1$), and obtain a (unique up to homotopy) map $f_m : \bigvee_{a\in A} S_a^m \to Y^m$ defined by $f_m|_{S_a^m} := k_a$ where $[k_a] = f_m^+(1_a) \in [S^m, Y^m]_*$. We have: (since $[\phi_m \circ f_m] = 0$)

$$\begin{tikzcd} \bigvee_{a\in A} S_a^m \arrow[r, "f_m"] & Y^m \arrow[r] \arrow[dr, "\phi_m"'] & C_{f_m} \arrow[d, dashed, "\varphi_{m+1}"] \\ & & X \end{tikzcd}$$

C_{f_m} is a CW-complex with dim $= n+1$ with m-skeleton Y^m. $\varphi_{m+1*} : \pi_m(C_{f_m}) \to \pi_m(X)$ is isomorphism, but $\varphi_{m+1*} : \pi_{m+1}(C_{f_m}) \to \pi_{m+1}(X)$ is not necessarily an epimorphism.
Define the set $B := \pi_{m+1}(X) - \varphi_{m+1*}(\pi_{m+1}(C_{f_m}))$ and $Y^{m+1} := C_{f_m} \vee (\bigvee_{b\in B} S_b^{m+1})$.
Define ϕ^{m+1} by $\phi^{m+1}|_{C_{f_m}} := \varphi_{m+1}$ and $\phi^{m+1}|_{S_b^{m+1}} := r_b$ where $[r_b] = b \in [S^{m+1}, X]_*$.
$\widetilde{X} := \varinjlim_m \{Y^0 \hookrightarrow \cdots \hookrightarrow Y^m \hookrightarrow Y^{m+1} \hookrightarrow \cdots\}$, and $\phi = \varinjlim_m \phi_m$
If X is not path-connected, construct CW-approximation for each path-connected component. $\square$

Note. The proof of existence of CW-approximation uses homotopy excision theorem (CW-triad version). Proof of CW-traid version does not need CW-approximation. There is no circular argument.

Proposition 1.6. *For any pair (X, A), there exists CW-approximation $\phi : (\widetilde{X}, \widetilde{A}) \to (X, A)$.*

Proof. Construct $\phi_A : \widetilde{A} \to A$ first and use analogue method in proof of theorem 1.5 with $Y^0 := \widetilde{A}$. $\square$

Lemma 1.7. *φ, ψ are CW-approximations of X, Y, $f : X \to Y$, then*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\varphi"] \arrow[d, dashed, "\exists \widetilde{f}"'] & X \arrow[d, "f"] \\ \widetilde{Y} \arrow[r, "\psi"'] & Y \end{tikzcd}$$

commutes up to homotopy, and $\widetilde{f}$ is unique up to homotopy.

Proof. Directly from $\psi_* : [\widetilde{X}, \widetilde{Y}] \to [\widetilde{X}, Y]$ is bijection. $\square$

Theorem 1.8. *φ, ψ are CW-approximations of $(X, A), (Y, B)$, $f : (X, A) \to (Y, B)$, then*

$$\begin{tikzcd} (\widetilde{X}, \widetilde{A}) \arrow[r, "\varphi"] \arrow[d, dashed, "\exists \widetilde{f}"'] & (X, A) \arrow[d, "f"] \\ (\widetilde{Y}, \widetilde{B}) \arrow[r, "\psi"'] & (Y, B) \end{tikzcd}$$

commutes up to homotopy, and $\widetilde{f}$ is unique up to homotopy.

Proof. Apply Lemma 1.7 to obtain map $\widetilde{f}_A : \widetilde{A} \to \widetilde{B}$ and homotopy $h : \psi|_{\widetilde{B}} \circ \widetilde{f}_A \simeq f \circ \varphi|_{\widetilde{A}}$ Use HELP of $(\widetilde{X}, \widetilde{A})$ to extend it:

$$\begin{tikzcd}
\widetilde{A} \arrow[r, hook] \arrow[d, "\widetilde{f}_A"'] & \widetilde{X} \arrow[d] \arrow[ddl, "\widetilde{f}"'] \\
\widetilde{B} \arrow[d, hook] & X \arrow[d] \\
\widetilde{Y} \arrow[r] & Y
\end{tikzcd}$$

ψ_* is bijection implies the uniqueness up to homotopy of $\widetilde{f}$. ☐

Theorem 1.9. *(Whitehead's Theorem)*
Every n-equivalence between CW-complexes whose dimension is lower than n, is homotopy equivalence. Every weak homotopy equivalence between CW-complexes is homotopy equivalence.

Proof. $e : Y \to Z$ induce bijections $[Y, Y] \to [Y, Z]$ and $[Z, Y] \to [Z, Z]$, $[f] = e_*^{-1}[\mathrm{id}_Z]$ implies $[e \circ f] = [\mathrm{id}_Z]$ and $[e \circ f \circ e] = [e]$ ($[f \circ e] = e_*^{-1}[e] = [\mathrm{id}_Y]$). ☐

Corollary 1.10. *CW-approximation is unique up to homotopy.*

Example 1.1. Polish circle (Warsaw circle) : closed topologist's sine curve. It is n-connected forall n but not contractible.

Definition 1.2. A cellular map between CW-pairs is $g : (X, A) \to (Y, B)$ such that $g(A \cup X^n) \subseteq B \cup Y^n$.

Theorem 1.11. *For any map between CW-pairs $f : (X, A) \to (Y, B)$ there exists a cellular map g such that $g \simeq f \operatorname{rel} A$*

Proof. Construct g inductively:
Start from $A \cup X^0$:
take paths $\gamma_i : f(x_i) \simeq y_i$, where y_i is any point in Y^0 and $x_i \in X^0 - A$.
Construct $h_0 : (X^0 \cup A) \times I \to Y$: $h_0|_A(a, t) := f(a)$, $h_0|_{X^0 - A}(x_i, t) := \gamma_i(t)$. This is a homotopy from f to $g_0 := h_0(-, 1) : A \cup X^0 \to B \cup Y^0$
Inductive step:
Assume $g_n : A \cup X^n \to B \cup Y^n$ and homotopy $h_n : f|_{A \cup X^n} \simeq g_n$ is given, try to construct g_{n+1}:
For each characteristic map $\varphi_i : S^n \to X^n$, take the resulting cell map $\varphi_i^+ : D^{n+1} \to X^{n+1}$ and use HELP of (D^{n+1}, S^n):

$$\begin{tikzcd}
S^n \arrow[r, hook] \arrow[d, "\varphi_i"'] & D^{n+1} \arrow[d, "\varphi_i^+"] \arrow[ddl, "g_{n+1,i}"'] \\
X^n \arrow[d, "g_n"'] & X^{n+1} \arrow[d, "f"] \arrow[dl, Rightarrow, "h_{n+1,i}"] \\
B \cup Y^{n+1} \arrow[r, hook] & Y
\end{tikzcd}$$

Glue all $g_{n+1,i}$ and $h_{n+1,i}$ to produce g_{n+1} and $h_{n+1} : f|_{A \cup X^{n+1}} \simeq g_{n+1}$.
Final stage:
Maps g_n determine a cellular map $g : X \to Y$ since X has the final topology determined by skeletons. ☐

Corollary 1.12. *If X is a pointed CW-complex, then the inclusions $X^{n+1} \hookrightarrow X^{n+2} \hookrightarrow \cdots \hookrightarrow X$ induce $\pi_n(X^{n+1}) \cong \pi_n(X^{n+2}) \cong \cdots \cong \pi_n(X)$.*

Proof. For $k \geq 1$, $X^{n+k} \hookrightarrow X^{n+k+1}$ induces epimorphism $\pi_n(X^{n+k}) \twoheadrightarrow \pi_n(X^{n+k+1})$ since every $f : (S^n, *) \to (X^{n+k+1}, *)$ is homotopic (rel $*$) to an $g : (S^n, *) \to (X^n, *) \hookrightarrow (X^{n+k}, *)$.
Now we want to prove it is monomorphism, that is, $i_*[f] = 0 \implies [f] = 0$
If $h : (S^n, *) \times I \to X^{n+k+1}$ is a nullhomotopy in X^{n+k+1}
of a map $f : (S^n, *) \to (X^{n+k}, *) \hookrightarrow (X^{n+k+1}, *)$,
then $h : (CS^n, S^n) \to (X^{n+k+1}, X^{n+k})$ is homotopic (rel S^n) to an $h' : (CS^n, S^n) \to (X^{n+k}, X^{n+k})$, which is equivalent to $h' : S^n \times I \to X^{n+k}$ with $h(S^n, 1) = *$, $h(*, t) = *$, $h|_{S^n \times \{0\}} = f$.

□

Lemma 1.13. *If (X, A) is CW-pair and all cells of $X - A$ have* $\dim > n$*, then (X, A) is n-connected.*

Proof. For each $q \leq n$, and each $[f] \in \pi_q(X, A)$, $f \simeq g \operatorname{rel} S^{q-1}$ where g is an cellular map. (use theorem 1.11) $\pi_q(X, A) \ni [g] = 0$ since $g(S^{n-1} \cup e^n) = g(D^n) \subseteq A \cup X^n = A$.

□

1.2 Operation of CW-complexes

We show that Product, Smash Product of CW-complexes and Quotient of CW-pairs (with compact-open topology) are CW-complexes. (Compact-open topology is the right topology on CW-complexes)

Product of CW-complexes:

Example 1.2. Product topology of two CW-complexes does not coincide with the final topology (union topology):
X (star of countably many edges) : $X = X^1 = \bigvee_{n \in \omega} I_n$
Y (star of ω^ω many edges) : $Y = Y^1 = \bigvee_{f \in \omega^\omega} I_f$ ($(I_n, 0) \cong (I_f, 0) \cong (I, 0)$)
Consider subset H of $X \times Y$: $H := \{(\frac{1}{f(n)+1}, \frac{1}{f(n)+1}) \in I_n \times I_f \mid n \in \omega, f \in \omega^\omega\}$.
H is closed under the final topology since every cell of $X \times Y$ contains at most one point of H.
But closure of H contains $(0, 0)$ at product topology:
Let $U \times V$ be an open neighborhood (at product topology) of $(0, 0)$, let $g : \omega \to \omega - 0$ be an increasing function such that forall $n \in \omega, [0, \frac{1}{g(n)}) \subseteq U \cap I_n$, let $k \in omega$ be sufficiently large that $\frac{1}{g(k)+1} \subseteq V \cap I_g$, then $(\frac{1}{g(k)+1}, \frac{1}{g(k)+1}) \in U \times V \cap H$.

Proposition 1.14. *X and Y are CW-complexes, $X \times Y$ is CW-complex if*
X or Y is locally compact
or
both X and Y have countably many cells.

Another way to realize $X \times Y$ as CW-complex is to change its topology to the compactly generated topology $k(X \times Y)$:

Definition 1.3. For subspace A of X, A is compactly closed if

$$\forall \text{ compact space } K$$
$$\forall \text{ continuous } g : K \to X$$
$$g^{-1}(A) \text{ is closed in } K$$

Definition 1.4. X is k-space if any compactly closed subset is closed.

Definition 1.5. X is weak Hausdorff if

$$\forall \text{ compact space } K$$
$$\forall \text{ continuous } g : K \to X$$
$$g(K) \text{ is closed in } K$$

Definition 1.6. The k-ification of a space X is defined by: $k(X) := (X, \tau)$
where $\tau = \{X - A \mid A \text{ is compactly closed set}\}$

Definition 1.7. X is compactly generated space if it is k-space and weak Hausdorff.

Note. If X is weak Hausdorff, then $A \subseteq X$ is compactly closed iff

$$\forall \text{ compact subspace } K \subseteq X$$
$$A \cap K \text{ is closed in } X$$

If X is a CW-complex, then the topology defined on $k(X)$ automatically coincide with the final topology induced by its CW-complex structure. We have CW-complex structure of $k(X \times Y)$ is given by:

$$\begin{tikzcd}
\partial I^n \times I^m \cup I^n \times \partial I^m \arrow[r] \arrow[d] & X^{n-1} \times Y^m \cup X^n \times Y^{m-1} \arrow[d] \\
I^n \times I^m \arrow[r] & X^n \times Y^m
\end{tikzcd}$$

Furthermore, the k-ification is right adjoint of the inclusion functor i:

$$\begin{tikzcd}
\mathbf{TOP_{CptGen}} \arrow[r, hook, bend left, "i"] \arrow[r, phantom, "\perp"] & \mathbf{TOP_{weakHaus}} \arrow[l, bend left, "k(-)"]
\end{tikzcd}$$

This allows us to define the CW-complex structure on any limit of CW-complexes: $\varprojlim_i X_i \approx \varprojlim_i k(X_i) \approx k(\varprojlim_i X_i)$ ($X \approx k(X)$ and right adjoint preserve limits).
Note. Category of CW-complexes is not cartesian closed, but category of compactly generated spaces $\mathbf{TOP_{CG}}$ is. And its pointed version $\mathbf{TOP}^{*/}_{\mathbf{CG}}$ have based exponential law: $\mathrm{Hom}(X \wedge Y, Z) \approx \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$.

Quotient of CW-pair:

Proposition 1.15. *For CW-complex X and subcomplex A, the Quotient space X/A have a CW-complex structure induced by X and A.*

Proof. Suppose the characteristic maps of X are indexed by $\{I_n\}_{n\in\mathbb{N}}$ and of A are indexed by $\{I'_n\}_{n\in\mathbb{N}}$ ($I'_n \subseteq I_n$). Then the characteristic maps of X/A are indexed by $\{K_n\}_{n\in\mathbb{N}}$, which defined below:
$K_0 := (I_0 - I'_0) \cup \{i_0\}$ where i_0 is an arbitrary element in I'_0
$K_n := I_n - I'_n$ for $n > 0$.
Verify the maps determine the CW-complex structure:

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d] & X^{n-1}/A^{n-1} \arrow[d] \arrow[ddd, bend left] & \\
D^n \arrow[r] \arrow[ddrr, bend right] & X^n \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & X^n/A^{n-1} \arrow[d] \arrow[dd, dashed, bend right] \arrow[ul, phantom, "\ulcorner", very near start] & A^n \arrow[l] \arrow[d] \\
 & & X^n/A^n \arrow[d, dashed] \arrow[ur, phantom, "\urcorner", very near start] & * \arrow[l] \arrow[dl] \\
 & & Z &
\end{tikzcd}$$

$\square$

Smash product of CW-complexes:

Proposition 1.16. *If (X, x_0) , (Y, y_0) are pointed CW-complexes with both countably many cell, and $X^{r-1} = \{x_0\}$, $Y^{s-1} = \{y_0\}$, then $X \wedge Y := X \times Y / X \vee Y$ is an $(r + s - 1)$-connected CW-complex.*

Proof. $X \times Y$ is CW-complex with cells of the form $e^n_{i,X} \times \{y_0\}$, $\{x_0\} \times e^m_{j,Y}$ or $e^n_{i,X} \times e^m_{j,Y}$ for $n \geq r$, $m \geq s$. Cells of the first two forms are contianed in $X \vee Y$, therefore $(X \wedge Y)^{r+s-1} = *$. $\square$

Corollary 1.17. *If X is a pointed CW-complex, then $\Sigma^n X$ is an $(n-1)$-connected CW-complex.*

1.3 Properties of Infinite Symmetric Product

Functoriality:

Pointed map $f : X \to Y$ induces

$$f_n : \mathrm{SP}^n X \to \mathrm{SP}^n Y$$
$$\{x_1, \dots, x_n\} \mapsto \{f(x_1), \dots, f(x_n)\}$$

$$\begin{tikzcd}
{} \arrow[r] & \mathrm{SP}^n X \arrow[r] \arrow[d, "f_n"] & \mathrm{SP}^{n+1} X \arrow[r] \arrow[d, "f_{n+1}"] & {} \\
{} \arrow[r] & \mathrm{SP}^n Y \arrow[r] & \mathrm{SP}^{n+1} Y \arrow[r] & {}
\end{tikzcd}$$

Which induces map $\mathrm{SP}\, f : \mathrm{SP}\, X \to \mathrm{SP}\, Y$. And Functorial properties are directly from the constructions above.

$\mathrm{SP}(X_1 \vee X_2) \approx \mathrm{SP}(X_1) \times \mathrm{SP}(X_2)$, the homeomorphism is given by:

$$\mathrm{SP}(X_1) \times \mathrm{SP}(X_2) \leftrightarrows \mathrm{SP}(X_1 \vee X_2)$$
$$(\{a_1, a_2, \cdots, a_k\}, \{b_1, b_2, \cdots, b_m\}) \mapsto \{a_1, a_2, \cdots, a_k, b_1, b_2, \cdots, b_m\}$$

Commute with directed colimit:
Suppose P is a directed poset (that is $\forall x, y \in P,\ \exists z \in P,\ x \le z, y \le z$) and X_i are pointed spaces indexed by P satisfying $i \le j \implies X_i \subseteq X_j$.
Then $\mathrm{SP}^n(\varinjlim_i X_i) \approx \varinjlim_i(\mathrm{SP}^n X_i)$
(Proof is obtained by showing that $\mathrm{SP}^n f$ is continuous iff f is, which implies final topology on $\varinjlim_i(\mathrm{SP}^n X_i)$ agree on $\mathrm{SP}^n(\varinjlim_i X_i)$)

Suppose $i : A \hookrightarrow X$ is an pointed inclusion, then $\mathrm{SP}\, i : \mathrm{SP}\, A \hookrightarrow \mathrm{SP}\, X$ is also inclusion. Furthermore, if A is open (or closed) in X, then $\mathrm{SP}\, A$ is open (or closed) in $\mathrm{SP}\, X$.

CW-complex structure of SP:

We can have natural CW-complex structure on $\prod_n X$ by applying $k(-)$. following theorems allows us to prove that $\mathrm{SP}^n X = \prod_n X / S_n$ have a CW-complex structure.

Definition 1.8. G **acts cellularly** on a CW-complex X if:

$$\forall g \in G, e_i^n \text{ is open } n\text{-cell (of } X)$$
$$g(e_i^n) = e_j^n \text{ is open } n\text{-cell (of } X)$$

and $g(e_i^n) = e_i^n$ implies $g|_{e_i^n} = \mathrm{id}_{e_i^n}$.

Lemma 1.18. *If G is a discrete group, X is CW-complex with G cellularly act on X. Then X is a G-CW-complex with n-skeleton X^n.*

Proof. The goal is to show X^n is obtained from X^{n-1} by attaching G-equivariant cells. Since $\coprod_{i \in I_n} Y = I_n \times Y$ (I_n with discrete topology). We have:

$$\begin{tikzcd}
I_n \times S^{n-1} \arrow[r, "\varphi"] \arrow[d, hook] & X^{n-1} \arrow[d] \\
I_n \times D^n \arrow[r, "\phi"'] \arrow[ur, phantom, "\ulcorner", very near end] & X^n
\end{tikzcd}$$

G acts cellularly on open n-cells implies G acts on I_n. Decomposite I_n into disjoint unions of obrits $\coprod_{\alpha \in A} I_\alpha$ choose G-isomorphisms

$$G/H_\alpha \cong I_\alpha$$
$$gH_\alpha \mapsto gi_\alpha$$

And we have a well-defined G-map.

$$\begin{aligned}\phi_\alpha|_{e^n} : G/H_\alpha \times e^n \cong I_\alpha \times e^n &\to X^n \\ (gH_\alpha, x) \mapsto (gi_\alpha, x) &\mapsto \phi_{gi_\alpha}(x) = g\phi_{i_\alpha}(x)\end{aligned}$$

Since we have $e^n = \overset{\circ}{D}{}^n$, we obtain the following (by continuity):

$$\begin{aligned}\phi_\alpha : G/H_\alpha \times D^n &\to X^n \\ (gH_\alpha, x) &\mapsto g\phi_{i_\alpha}(x) \\ \phi_\alpha|_{S^{n-1}} = \varphi_\alpha : G/H_\alpha \times S^{n-1} &\to X^{n-1} \\ (gH_\alpha, s) &\mapsto g\varphi_{i_\alpha}(s)\end{aligned}$$

Let $\varphi' := \coprod_{\alpha \in A} \varphi_\alpha$ and $\phi' := \coprod_{\alpha \in A} \phi_\alpha$ we have:

$$\begin{tikzcd}
\coprod_{\alpha\in A}(G/H_\alpha \times S^{n-1}) \arrow[r, "\varphi'"] \arrow[d, hook] & X^{n-1} \arrow[d] \arrow[ddr, bend left, "f"] & \\
\coprod_{\alpha\in A}(G/H_\alpha \times D^n) \arrow[r, "\phi'"'] \arrow[drr, bend right, "\coprod f_\alpha"'] & X^n \arrow[dr, dashed, "\exists! f^+"] & \\
& & Z
\end{tikzcd}$$

Verify it is indeed a pushout of G-spaces: f^+ (is already determined uniquely as map between G-sets) is map between G-spaces.
Since X have compactly generated topology, f^+ is continuous on each compact subspace of X implies f^+ is continuous on each compactly closed subspace of X^n, which implies f^+ is continuous on total X^n.
f^+ is continuous on each closed n-cell $\{gH_\alpha\} \times D^n$ and f^+ is continuous on X^{n-1} implies f^+ is continuous on each compact subspace. (since each compact subspace intersect finitely with n-cells and X^{n-1} (We use X^n is T_2 to construct open cover))

□

Theorem 1.19. *For any topological group morphism $\phi : H \to G$ we have induced functors: pullback action:*

$$\begin{aligned}G{-}\mathbf{TOP} &\xrightarrow{\phi^*} H{-}\mathbf{TOP} \\ (\alpha(-,-) : G \times X \to X) &\longmapsto (\alpha(\phi(-),-) : H \times X \to X) \\ (f : X \to Y) &\longmapsto (f : X \to Y)\end{aligned}$$

induced action:

$$\begin{aligned}H{-}\mathbf{TOP} &\xrightarrow{G\times_H -} G{-}\mathbf{TOP} \\ X &\longmapsto G \times_H X := (G \times X)/[\ (g\phi(h), x) \sim (g, hx) \mid h \in H] \\ (f : X \to Y) &\longmapsto (\mathrm{id}_G \times_H f : G \times_H X \to G \times_H Y)\end{aligned}$$

Which are adjunctions:

$$\begin{tikzcd}
H{-}\mathbf{TOP} \arrow[r, bend left, "G\times_H -"] \arrow[r, phantom, "\perp"] & G{-}\mathbf{TOP} \arrow[l, bend left, "\phi^*"]
\end{tikzcd}$$

Proof. By G-equivariance, f is determined uniquely by its restriction $f|_{\phi(H)\times_H X}$. And $\tilde{f} : X \to$

$\phi^*(Y)$ uniquely determine a map $\phi(H) \times_H X \to Y$.

$$\begin{tikzcd}
(G \times_H X \arrow[r, "f"] & Y) \arrow[r, mapsto] & (X = \phi(H) \times_H X \arrow[r, "f|_{H \times_H X}"] & Y) \\
\Cup & \Cup & \Cup & \Cup \\
(g\phi(h), x) \arrow[r, mapsto] & g\phi(h)f(x) & (\phi(h), x) = (e, hx) \arrow[r, mapsto] & \phi(h)x
\end{tikzcd}$$

$$X \xrightarrow{\tilde{f}} \phi^*(Y)$$

Naturality:

$$\begin{tikzcd}
(G \times_H X' \arrow[r, "\mathrm{id}_G \times_H f'"] & G \times_H X \arrow[r, "f"] & Y \arrow[r, "f''"] & Y') \\
\Cup & \Cup & \Cup & \Cup \\
(g, hx') \arrow[r, mapsto] & (g, hf'(x)) \arrow[r, mapsto] & g\phi(h)f(f'(x)) \arrow[r, mapsto] & g\phi(h)f''(f(f'(x)))
\end{tikzcd}$$

$$\leftrightarrows$$

$$\begin{tikzcd}
(X' \arrow[r, "f'"] & X = \phi(H) \times_H X \arrow[r, "f|_{H \times_H X}"] & \phi^*(Y) \arrow[r, "\phi^*(f'')"] & \phi^*(Y')) \\
\Cup & \Cup & \Cup & \Cup \\
(hx') \arrow[r, mapsto] & (\phi(h), f'(x')) = (e, hf'(x')) \arrow[r, mapsto] & \phi(h)f(f'(x)) \arrow[r] & \phi(h)f''(f(f'(x)))
\end{tikzcd}$$

□

Proposition 1.20. *If (X, A) is relative G-equivariant CW-complex, then $(X/G, A/G)$ is relative CW-complex with n-skeleton X^n/G.*

Proof.

$$\begin{tikzcd}
\coprod_{i \in I_n} S^{n-1} \arrow[r] \arrow[d, hook] & X^{n-1}/G \arrow[d] \\
\coprod_{i \in I_n} D^n \arrow[r] & X^n/G
\end{tikzcd}$$

Is still pushout since $-/G = 1 \times_G -$, and left adjoint preserves colimits.

□

Since $k(\prod_n X)$ have CW-complex structure, and S_n (as a discrete group) acts cellularly on it, $k(\prod_n X)$ is an S_n-equivariant CW-complex. Therefore $\mathrm{SP}^n X = k(\prod_n X)/S^n$ is CW-complex. Since $\mathrm{SP}\, X = \varinjlim\{\mathrm{SP}^1 X \hookrightarrow \cdots \hookrightarrow \mathrm{SP}^n X \hookrightarrow \mathrm{SP}^{n+1} X \hookrightarrow \cdots\}$, $\mathrm{SP}\, X$ is also a CW-complex.

Pointed homotopy $h : X \times I \to Y$ induces

$$\begin{aligned} h_n &: \mathrm{SP}^n X \times I \to \mathrm{SP}^n Y \\ (\{x_1, \ldots, x_n\}, t) &\mapsto \{h(x_1, t), \ldots, h(x_n, t)\} \end{aligned}$$

which induces $\mathrm{SP}\,h : \mathrm{SP}\,X \times I \to \mathrm{SP}\,Y$.

Then we observe:
$f \simeq g$ implies $\mathrm{SP}\,f \simeq \mathrm{SP}\,g$,
$e : X \to Y$ is homotopy equivalence implies $\mathrm{SP}\,e : \mathrm{SP}\,X \to \mathrm{SP}\,Y$ is,
X is contractible implies $\mathrm{SP}^n\,X$ and then $\mathrm{SP}\,X$ is.

Theorem 1.21. *(Dold-Thom Theorem)*
If X is T_2 space and A is closed path-connected subspace of X, and there is neighborhood V deformable to A in X.
Then the quotient map $q : X \to X/A$ induces quasi-fibration $\mathrm{SP}\,q : \mathrm{SP}\,X \to \mathrm{SP}(X/A)$, which satisfy $\forall x \in \mathrm{SP}(X/A)$, $(\mathrm{SP}\,q)^{-1}\{x\} \simeq \mathrm{SP}\,A$.

Proof. See here.

Corollary 1.22. *If X , Y are T_2 spaces and Y is connected, $f : X \to Y$. Then consider $X \to Y \to C_f \to \Sigma X$, the map $p : C_f \to \Sigma X$ induces quasi-fibration $\mathrm{SP}\,p : \mathrm{SP}\,C_f \to \mathrm{SP}(\Sigma X)$ with fiber $\mathrm{SP}\,Y$.*

Corollary 1.23. *If X is T_2 and path-connected, then for any $q \geq 0$, there is $\pi_{q+1}(\mathrm{SP}(\Sigma X)) \cong \pi_q(\mathrm{SP}\,X)$.*

Proof. CX is contractible implies $\mathrm{SP}\,CX$ is contractible, use the exat homotopy sequence of quasi-fibration to see:

$$\longrightarrow \pi_{q+1}(\mathrm{SP}\,CX) \longrightarrow \pi_{q+1}(\mathrm{SP}\,\Sigma X) \xrightarrow[\partial]{\cong} \pi_q(\mathrm{SP}\,X) \longrightarrow \pi_q(\mathrm{SP}\,CX) \longrightarrow$$

□

Note. The inverse of the isomorphism ∂ above is given by

$$[S^q, \mathrm{SP}\,X] \ni [g] \mapsto [\Sigma g] \in [S^{q+1}, \Sigma\,\mathrm{SP}\,X]$$

$(\Sigma\,\mathrm{SP}\,X \cong \mathrm{SP}\,\Sigma X)$. Because ∂ is given by:

$$\begin{array}{ccccc}
\pi_q(\mathrm{SP}\,\Sigma X) & \longrightarrow & \pi_q(\mathrm{SP}\,CX, \mathrm{SP}\,X) & \longrightarrow & \pi_{q-1}(\mathrm{SP}\,X) \\
\Uparrow\!\!\!\!\cup & & \Uparrow\!\!\!\!\cup & & \Uparrow\!\!\!\!\cup \\
[f] & \longmapsto & [\hat{f}] & \longmapsto & [\hat{f}|_{S^{q-1}}] \\
 & & & & \\
[p \circ Cg] = [\Sigma g] & \longleftarrow\!\shortmid & [Cg] & \longleftarrow\!\shortmid & [g]
\end{array}$$

. □

Corollary 1.24. *If X is T_2 space and A is path-connected subspace of X, then the canonical map $\mathrm{SP}(X \cup (A \times I)) \to \mathrm{SP}(X \cup CA)$ is a quasi-fibration with fiber $\mathrm{SP}\,A$.*

Theorem 1.25. *If X is T_2 space and A is path-connected subspace of X, and $A \hookrightarrow X$ is a cofibration.*
Then the quotient map $q : X \to X/A$ induces quasi-fibration $\mathrm{SP}\,q : \mathrm{SP}\,X \to \mathrm{SP}(X/A)$, which satisfy $\forall x \in \mathrm{SP}(X/A)$, $(\mathrm{SP}\,q)^{-1}\{x\} \simeq \mathrm{SP}\,A$.

Proof. If $A \hookrightarrow X$ is cofibration, then $X \cup CA \simeq X/A$ and $X \cup (A \times I) \simeq X$. □

Proposition 1.26. *The inclusion $S^1 \to \mathrm{SP}\,S^1$ is homotopy equivalence, therefore $\pi_q(S^1) \cong \pi_q(\mathrm{SP}\,S^1)$.*

Proof. $S^1 \simeq S^2 - \{0, \infty\}$
$\mathrm{SP}^n\, S^2 = \{\{a_1, \ldots, a_n\} \mid a_i \in \mathbb{C} \cup \{\infty\}\} = \{\prod_{\{a_1,\ldots,a_n\}}(z - a_i) \mid a_i \in \mathbb{C} \cup \{\infty\}\}$ where $(z - \infty) := 1$
$\mathrm{SP}^n\, S^2 = \{f \in \mathbb{C}[z] - \{0\} \mid \deg(f) \leq n\} = \mathbb{CP}^n$

$\mathrm{SP}^n(S^2 - \{0, \infty\}) = \{f \in \mathbb{C}[z] - \{0\} \mid \deg(f) \leq n, f_n \neq 0, f_0 \neq 0\} = \mathbb{C}^n - \mathbb{C}^{n-1} \times 0 = \mathbb{C}^{n-1} \times (\mathbb{C} - 0)$
it have the same homotopy type of S^1

Corollary 1.27. *$\pi_q(\mathrm{SP}\, S^n) = \mathbb{Z}$ if $q = n$, otherwise $\pi_q(\mathrm{SP}\, S^n) = 0$. (use corollary of 1.21 to see $\pi_{q+1}(\mathrm{SP}\, \Sigma X) \cong \pi_q(\mathrm{SP}\, X)$)*

2 Homology Groups

2.1 Reduced Homology Groups

Definition 2.1. For a path-connected pointed CW-complex X, define its **n-th reduced homology group** for $n \geq 0$:

$$\tilde{H}_n(X) := \pi_n(\mathrm{SP}\, X)$$

Note. All reduced homology groups are abelian since $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$. Thus, we can extend the definition above to those X which does not necessarily be path-connected.

As SP, $\tilde{H}_n$ also satisfy functoriality. Furthermore, $\tilde{H}_n$ maps homotopic maps $f \simeq g$ to identical maps $f_* = g_*$. (SP maps homotopic maps to homotopic maps)

Exact Property:

Proposition 2.1. *For any pointed map between CW-complexes $f : X \to Y$, we have an exact sequence:*

$$\tilde{H}_n(X) \xrightarrow{f_*} \tilde{H}_n(Y) \xrightarrow{i_*} \tilde{H}_n(C_f)$$

where C_f is the mapping cone of f, $i : Y \hookrightarrow C_f$.

Proof. $Z_f := Y \cup_f (X \times I)/\{x_0\} \times I$ is the **reduced mapping cylinder** of f.
$q : Z_f \to C_f$ is defined by

$$y \mapsto y$$
$$\overline{(x,t)}^{Z_f} \mapsto \overline{(x,t)}^{C_f}$$

By Dold-Thom theorem, the induced map $\mathrm{SP}\, q$ is quasi-fibration $\mathrm{SP}\, Z_f \to \mathrm{SP}\, C_f$ with fiber $\mathrm{SP}\, X$.
By definition of quasi-fibration, we have

$$\pi_n(\mathrm{SP}\, X) \cong \tilde{H}_n(X) \xrightarrow{f_*} \pi_n(\mathrm{SP}\, Z_f) \cong \tilde{H}_n(Y) \xrightarrow{i_*} \pi_n(\mathrm{SP}\, C_f) = \tilde{H}_n(C_f)$$

□

Proposition 2.2. *There does not exist retraction $r : \mathbb{D}^n \to S^{n-1}$.*

Proof. $id = r \circ i : \mathbb{S}^{n-1} \to \mathbb{D}^n \to \mathbb{S}^{n-1}$ induces

$$id_* = r_* \circ i_* : \mathbb{Z} \cong \tilde{H}_{n-1}\mathbb{S}^{n-1} \to \tilde{H}_{n-1}\mathbb{D}^n \cong 0 \to \tilde{H}_{n-1}\mathbb{S}^{n-1} \cong \mathbb{Z}$$

which lead to contradiction. □

Theorem 2.3. *Fix-point theorem:*
If $f : \mathbb{D}^n \to \mathbb{D}^n$ is continuous, then exist $x_0 \in \mathbb{D}^n$ such that $x_0 = f(x_0)$.

Proof. (non-constructive) No such x_0 implies $\forall x \in \mathbb{D}^n, f(x) \neq x$ therefore, we can construct continuous retraction $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ by
$r(x)$:= the intersection of "ray starting from $f(x)$ to x" and $\mathbb{S}^{n-1}$. Contradict to 2.2.

Definition 2.2. Let (X, A) be an CW-pair, define the **n-th homology group** for $n \in \mathbb{N}$ of (X, A) be:

$$H_n(X, A) := \tilde{H}_n(X \cup CA)$$

And for single space:

$$H_n(X) := H_n(X, \emptyset) = \tilde{H}(X + 1)$$

where $X + 1 := X \sqcup *$.

Note. Map between CW-pair $f : (X, A) \to (Y, B)$, induces map $\bar{f} : X \cup CA \to Y \cup CB$ defined by $(x, t) \mapsto (f(x), t)$, which induces $f_* : \tilde{H}_n(X \cup CA) \to \tilde{H}_n(Y \cup CB)$ for any $n \in \mathbb{N}$.

2.2 Axioms for Homology

Definition 2.3. A (Ordinary) Homology Theory (on **TOP** with coefficient $G \in \mathbf{Ab}$) is functors $\{H_n(-, -; G) : \mathbf{TOP(2)} \to \mathbf{Ab}\}_{n \in \mathbb{Z}}$,
with natural transformations $\partial_{n,(X,A)} : H_n(X, A; G) \to H_{n-1}(A, \emptyset; G)$ (called connecting homomorphism)
satisfying following axioms:

- **Dimension**:
  $H_0(*, \emptyset; G) = G$, for any $n \neq 0$, $H_n(*, \emptyset; G) = 0$.

- **Weak Equivalence**:
  Weak equivalence $f : (X, A) \to (Y, B)$ induces isomorphism

  $$f_* : H_*(X, A; G) \to H_*(Y, B; G)$$

- **Long Exact Sequence**:
  For any $(X, A) \in \mathbf{TOP(2)}$, maps $A \hookrightarrow X$ and $(X, \emptyset) \to (X, A)$ induce a long exact sequence together with ∂:

  $$\cdots \to H_{q+1}(A; G) \to H_{q+1}(X; G) \to H_{q+1}(X, A; G) \to H_q(A; G) \to \cdots$$

  where $H_n(X; G) := H_n(X, \emptyset; G)$.

- **Additivity**:
  If $(X, A) = \coprod_\lambda (X_\lambda, A_\lambda)$ in $\mathbf{TOP(2)}$, then inclusions $i_\lambda : (X_\lambda, A_\lambda) \to (X, A)$ induces isomorphism

  $$(\bigoplus i_{*,\lambda}) : \bigoplus_\lambda H_*(X_\lambda, A_\lambda; G) \cong H_*(X, A; G)$$

- **Excision**:
  If $(X; A, B)$ is an **excisive triad** (that is, $X = \mathring{A} \cup \mathring{B}$), then inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphism

  $$H_*(A, A \cap B; G) \cong H_*(X, B; G)$$

Note. An equivalent form of Excision Axiom:

If $(X, A) \in \mathbf{TOP(2)}$, U is subspace of A and $\overline{U} \subseteq \mathring{A}$, then inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphism

$$i_* : H_*(X - U, A - U; G) \to H_*(X, A; G)$$

There is a critical criterion about weak homotopy equivalence between excisive triads, we prove lemmas first:

Lemma 2.4. *For*

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "i_*"] \\
X \arrow[r, "f_*"'] \arrow[ur, phantom, "\ulcorner", very near end] & X \cup_Z Y
\end{tikzcd}$$

if D is deformation retract of X and $Z \subseteq D \subseteq X$, then $D \cup_Z Y$ is deformation retract of $X \cup_Z Y$.

Proof. Let $h : \mathrm{id}_X \simeq r \circ i$ where r is the deformation retraction $X \to D$. Define $h_* : \mathrm{id}_{X \cup_Z Y} \simeq (i \cup_Z \mathrm{id}_Y) \circ (r \cup_Z \mathrm{id}_Y)$

$$\begin{aligned}
h_* : (X \cup_Z Y) \times I &\to X \cup_Z Y \\
(x, t) &\mapsto f_*(h(x, t)) \\
(y, t) &\mapsto i_*(y)
\end{aligned}$$

Observe that $(X \cup_Z Y) \times I = (X \times I) \cup_{Z \times I} (Y \times I)$, check that h^* is continuous:

$$\begin{tikzcd}
Z \times I \arrow[r] \arrow[d] & Y \times I \arrow[d] \arrow[dr, "\mathrm{id}_{\mathrm{id}_Y}"] & \\
X \times I \arrow[r] \arrow[dr, "h"'] \arrow[ur, phantom, "\ulcorner", very near end] & (X \cup_Z Y) \times I \arrow[dr, "h_*"] & Y \arrow[d, "i_*"] \\
& X \arrow[r, "f_*"'] & X \cup_Z Y
\end{tikzcd}$$

□

Lemma 2.5. *For maps $i : C \to A$, $j : C \to B$ define the double mapping cylinder $M(i, j) := A \cup_{C \times \{0\}} C \times I \cup_{C \times \{1\}} B$. If i is closed cofibration, then the quotient map*

$$\begin{aligned}
q : M(i, j) &\to A \cup_C B \\
a &\mapsto a \\
b &\mapsto b \\
(c, t) &\mapsto c
\end{aligned}$$

is a homotopy equivalence.

Proof.

$$\begin{tikzcd}
C \arrow[r] \arrow[d, hook, "i"'] & B \arrow[d] \\
A \arrow[r, "i_A"'] & A \cup_C B
\end{tikzcd}$$

The canonical quotient $r : M_{i_A} \to A \cup_C B$ is a deformation retraction with homotopy:

$$\begin{aligned}
h : (B \cup_{C \times 0} (A \times I)) \times I &\to B \cup_{C \times 0} (A \times I) = M_{i_A} \\
(a, t, s) &\mapsto (a, (1 - s)t) \\
(b, s) &\mapsto b
\end{aligned}$$

Observe that $C \times I \cup_C A \times \{1\}$ is a deformation retract of $A \times I$, since $i : C \to A$ is closed cofibration.
Then we have $M(i, j) = B \cup_{C \times \{0\}} (C \times I \cup_{C \times \{1\}} A \times \{1\})$ is a deformation retract of $B \cup_{C \times \{0\}} A \times I = M_{i_A}$. (use lemma 2.4)
Finally, an easy check shows that $M(i, j) \to M_{i_A} \xrightarrow{r} A \cup_C B$ is identical to q.

□

Theorem 2.6. *For excisive triads $(X; X_1, X_2)$, $(X'; X_1', X_2')$ and map $e : X \to X'$, if*

$$\begin{aligned} e|_{X_1} &: X_1 \to X_1' \\ e|_{X_2} &: X_2 \to X_2' \\ e|_{X_3} &: X_3 \to X_3' \end{aligned}$$

are weak equivalences, (where $X_3 := X_1 \cap X_2$, $X_3' := X_1' \cap X_2'$) then e is.

Proof. Use an important criterion of weak homotopy equivalence, it suffices to show forall $n \in \mathbb{N}$, any commutative diagram below:

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \\ X \arrow[r, "e"'] & X' \end{tikzcd}$$

can be filled like:

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "g"'] & D^{n+1} \arrow[d, "f"] \arrow[dl, "g^+"'] \arrow[dl, phantom, shift right=3, "\Leftarrow h"] \\ X \arrow[r, "e"'] & X' \end{tikzcd}$$

whose upper triangle commutes.

Let

$$\begin{aligned} A_1 &:= g^{-1}(X - \mathring{X}_1) \cup f^{-1}(X' - \mathring{X}_1') \\ A_2 &:= g^{-1}(X - \mathring{X}_2) \cup f^{-1}(X' - \mathring{X}_2') \end{aligned}$$

which are disjoint closed subsets of D^{n+1}. Choose CW-complex structure on D^{n+1} such that for each n-cell σ_i, $\overline{\sigma_i} \cap (A_1 \cup A_2) = \overline{\sigma_i} \cap A_1$ or $\overline{\sigma_i} \cap A_2$. Now define

$$\begin{aligned} K_1 &:= \bigcup \{\overline{\sigma_i} \mid g(\overline{\sigma_i} \cap S^n) \subseteq \mathring{X}_1 \text{ and } f(\overline{\sigma_i}) \subseteq \mathring{X}_1'\} = \bigcup \{\overline{\sigma_i} \mid \overline{\sigma_i} \cap A_1 = \emptyset\} \\ K_2 &:= \bigcup \{\overline{\sigma_i} \mid g(\overline{\sigma_i} \cap S^n) \subseteq \mathring{X}_2 \text{ and } f(\overline{\sigma_i}) \subseteq \mathring{X}_2'\} = \bigcup \{\overline{\sigma_i} \mid \overline{\sigma_i} \cap A_2 = \emptyset\} \end{aligned}$$

which are subcomplexes of D^{n+1} and satisfy $K_1 \cup K_2 = D^{n+1}$. By HELP, we have:

$$\begin{tikzcd} S^n \cap K_1 \cap K_2 \arrow[r, "i"] \arrow[d, "g|_{K_1 \cap K_2}"'] & K_1 \cap K_2 \arrow[d, "f|_{K_1 \cap K_2}"] \arrow[dl, "g_0"'] \arrow[dl, phantom, shift right=3, "\Leftarrow h_0"] \\ X_1 \cap X_2 \arrow[r, "e|_{X_1 \cap X_2}"'] & X_1' \cap X_2' \end{tikzcd}$$

such that h_0 is $f|_{K_1 \cap K_2} \simeq e \circ g_0 \operatorname{rel} (S^n \cap K_1 \cap K_2)$. Apply HELP to:

$$\begin{tikzcd} (S^n \cup K_1) \cap K_2 \arrow[r, "i_2"] \arrow[d, "g_{K_2}"'] & K_2 \arrow[d, "f|_{K_2}"] \arrow[dl, Rightarrow, "h_{K_2}"] \\ X_2 \arrow[r] & X_2' \end{tikzcd} \qquad \begin{tikzcd} (S^n \cup K_2) \cap K_1 \arrow[r, "i_1"] \arrow[d, "g_{K_1}"'] & K_1 \arrow[d, "f|_{K_1}"] \arrow[dl, Rightarrow, "h_{K_1}"] \\ X_1 \arrow[r] & X_1' \end{tikzcd}$$

where
g_{K_i} are defined by $g_{K_i}|_{S^n \cap K_i} := g|_{S^n \cap K_i}$ and $g_{K_i}|_{K_1 \cap K_2} := g_0$,
h_{K_2} are defined by (h_{K_1} is similar):

$$\begin{aligned} h_{K_2} : ((S^n \cup K_1) \cap K_2) \times I &\to X_2' \\ (x, t) &\mapsto \begin{cases} e(g(x)) & x \in S^n \cap K_2 \\ h_0(x, t) & x \in K_1 \cap K_2 \end{cases} \end{aligned}$$

We get:

$$
\begin{tikzcd}
(S^n \cup K_1) \cap K_2 \arrow[r] \arrow[d, "g_{K_2}"'] & K_2 \arrow[d, "f|_{K_2}"] \arrow[dl, "g_2"'] \\
X_2 \arrow[r, "e|_{X_2}"'] \arrow[ur, phantom, "\stackrel{h_2}{\Leftarrow}"] & X_2'
\end{tikzcd}
\qquad
\begin{tikzcd}
(S^n \cup K_2) \cap K_1 \arrow[r] \arrow[d, "g_{K_1}"'] & K_1 \arrow[d] \arrow[dl, "g_1"'] \\
X_1 \arrow[r, "e|_{X_1}"'] \arrow[ur, phantom, "\stackrel{h_1}{\Leftarrow}"] & X_1'
\end{tikzcd}
$$

Define g^+ and $h : f \simeq g \operatorname{rel} S^n$ by $g^+|_{K_i} := g_i$ and $h|_{K_i \times I} := h_i$.
$h|_{S^n \times I} = (e \circ g) \times \mathrm{id}_I$ (h is $\operatorname{rel} S^n$) since $h_i(-,t)|_{S^n \cap K_i} = h_{K_i}(-,t)|_{S^n \cap K_i} = e \circ g|_{S^n \cap K_i}$.

□

Note. The proof above can be easily modified to case each weak equivalence appear in the statement is an n-equivalence.

Following theorem allow us to use CW-triads to approximate excisive triads:

Theorem 2.7. *For any excisive triad $(X; A, B)$, there is a CW-triad $(\widetilde{X}; \widetilde{A}, \widetilde{B})$ (A CW-triad $(X; A, B)$ is X and its subcomplex A, B such that $A \cup B = X$) and a map $r : \widetilde{X} \to X$ such that*

$$
\begin{aligned}
r|_{\widetilde{A}} &: \widetilde{A} \to A \\
r|_{\widetilde{B}} &: \widetilde{B} \to B \\
r|_{\widetilde{C}} &: \widetilde{C} \to C \\
r &: \widetilde{X} \to X
\end{aligned}
$$

are all weak homotopy equivalences (where $\widetilde{C} := \widetilde{A} \cap \widetilde{B}$, $C := A \cap B$). Furthermore, such r is natural up to homotopy.

Proof. Choose a CW-approximation $r_C : \widetilde{C} \to C$ and extend it to $r_A : \widetilde{A} \to A$, $r_B : \widetilde{B} \to B$. $\widetilde{X} := \widetilde{A} \cup_{\widetilde{C}} \widetilde{B}$. $i : \widetilde{C} \to \widetilde{A}$ and $j : \widetilde{C} \to \widetilde{B}$ are closed cofibrations, by lemma 2.5 we have homotopy equivalence $q : M(i,j) \to \widetilde{X}$, which induces homotopy equivalence of triads:

$$
\begin{aligned}
q &: M(i,j) \to \widetilde{X} \\
q| &: \widetilde{A} \cup (\widetilde{C} \times [0, \frac{2}{3})) \to \widetilde{A} \\
q| &: \widetilde{B} \cup (\widetilde{C} \times (\frac{1}{3}, 1]) \to \widetilde{B}
\end{aligned}
$$

then we can deduce that $r \circ q$ is a weak homotopy equivalence by theorem 2.6. Consquently, r is weak homotopy equivalence. r is natural up to homotopy since each CW-approximation r_C, r_A, r_B is.

□

Then we have:

Definition 2.4. A (Ordinary) Homology Theory on CW-complexes with coefficient $G \in \mathbf{Ab}$ is functors $\{H_n(-,-;G) : \textbf{CW-pairs} \to \mathbf{Ab}\}_{n \in \mathbb{Z}}$,
with natural transformations $\partial_{n,(X,A)} : H_n(X, A; G) \to H_n(A, \emptyset; G)$ (called connecting homomorphism)
satisfying axioms with the excision axiom changed to:

- **Excision**:
  If (X;A,B) is an **CW-triad** (that is $X = A \cup B$ for subcomplexes A and B) then the inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphism

$$H_*(A, A \cap B; G) \cong H_*(X, B; G)$$

Proposition 2.8. *The homology groups defined in definition 2.2 with $H_{-n}(X) := 0$ is a ordinary homology theory on CW-complexes with coefficient $\mathbb{Z}$.*

Proof.

- Dimension: by a corollary, $H_q(*, \emptyset) = \pi_q(\mathrm{SP}\, S^0) = \begin{cases} \mathbb{Z} & q = 0 \\ 0 & q \geq 1 \end{cases}$.
- Weak Equivalence: SP preserves weak equivalence.
- Long Exact Sequence: use a corollary of Dold-Thom theorem.
- Additivity: For index set Λ, $P := \{S \mid S \subseteq \Lambda\}$.
  Then define $Y_S := \bigvee_{\lambda \in S} X_\lambda \cup CA_\lambda = (\coprod_{\lambda \in S} X_\lambda) \cup C(\coprod_{\lambda \in S} A_\lambda)$,
  and use fact that SP commutes with directed colimit, we have
  $\bigvee_{\lambda \in \Lambda} \mathrm{SP}(X_\lambda \cup CA_\lambda) = \varinjlim_{S \in P} \mathrm{SP}\, Y_S \approx \mathrm{SP}(\varinjlim_{S \in P} Y_S) = \mathrm{SP}((\coprod_{\lambda \in \Lambda} X_\lambda) \cup C(\coprod_{\lambda \in \Lambda} A_\lambda)) = \mathrm{SP}(X \cup CA)$.
  Which induces $\bigoplus_{\lambda \in \Lambda} \tilde{H}_n(X_\lambda \cup CA_\lambda) \cong \pi_n(\bigvee_{\lambda \in \Lambda} \mathrm{SP}(X_\lambda \cup CA_\lambda)) \cong \pi_n(\mathrm{SP}(X \cup CA)) = \tilde{H}_n(X \cup CA)$.
- Excision: For CW-triad $(X; A, B)$, $A/(A \cap B) \approx X/B$. Apply theorem 1.25 to $(Y \cup CZ, CZ)$ to show that $H_n(Y, Z) \cong \tilde{H}_n(Y/Z)$.

$\square$

2.3 Cellular Homology

Lemma 2.9. *For an ordinary homology theory $H_*(-,-;G)$, if X is a CW-complex, then for any $n \in \mathbb{Z}$ $H_n(X) \cong \tilde{H}_{n+1}(\Sigma X)$.*

Proof. Apply long exact sequence axiom on (CX, X): ($H_*(CX) = 0$ due to weak equivalence axiom):

$$0 \cong H_{n+1}(CX) \to H_{n+1}(CX, X) \xrightarrow{\cong} H_n(X) \to H_n(CX) \cong 0$$

Use excision axiom and weak equivalence axiom, we have:

$$H_*(CX, X) \cong H_*(CX \cup CX, CX) \cong H_*(\Sigma X, *)$$

$\square$

Proposition 2.10. *For an ordinary homology theory $H_*(-,-;G)$, if X is a pointed CW-complex with $X^{-1} := *$, then for any $n \geq 0$*

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \begin{cases} \bigoplus_{i \in I_n} G & q = n \\ 0 & q \neq n \end{cases}$$

where I_n is set of all q-cells.

Proof. Use additivity axiom and lemma 2.9 to see that $H_n(\bigvee S^n) \cong \bigoplus G$ and $H_q(\bigvee S^n) = 0$ for $q \neq n$. Use excision axiom and weak equivalence axiom to see

$$H_q(X^n, X^{n-1}) \cong H_q(X^n \cup CX^{n-1}, CX^{n-1}) \cong H_q(X^n/X^{n-1}, *) \cong \tilde{H}_q(\bigvee_{i \in I_n} S^n)$$

$\square$

Corollary 2.11. *If $H_*(-,-)$ is an ordinary homology theory, then for a pointed CW-complex X with $X^{-1} := *$, we have:*

$$\begin{aligned} &\tilde{H}_q(X^n) = 0 && \textit{for } q > n \\ &H_q(X^n) \cong H_q(X^{n+1}) \cong H_q(X) && \textit{for } q < n \\ &H_n(X^n) \xrightarrow{i_*} H_n(X^{n+1}) && \textit{is epimorphism} \end{aligned}$$

for any $n \geq -1$.

Proof. Use long exact sequence of (X^{n+1}, X^n):

$$\cdots \to H_{q+1}(X^{n+1}, X^n) \xrightarrow{\partial_{q+1}} H_q(X^n) \xrightarrow{i_*} H_q(X^{n+1}) \to H_q(X^{n+1}, X^n) \xrightarrow{\partial_q} H_{q-1}(X^n) \to \cdots$$
$$\cdots \to H_1(X^{n+1}, X^n) \xrightarrow{\partial_1} H_0(X^n) \xrightarrow{i_*} H_0(X^{n+1}) \to H_0(X^{n+1}, X^n)$$

For $q < n$, $H_q(X^n) \cong H_q(X^{n+1}) \cong \cdots \cong \varinjlim_{i \in \mathbb{N}} H_q(X^i)$.
For $q > n$, if $n > -1$, $H_q(X^n) \cong H_q(X^{n-1}) \cong \cdots \cong H_q(X^{-1}) \cong 0$,
if $n = -1$, $\tilde{H}_0(X^{-1}) \cong 0 \cong \tilde{H}_q(X^{-1})$.
For $q = n$, we have following exact:

$$\to H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n) \xrightarrow{i_*} H_n(X^{n+1}) \to H_n(X^{n+1}, X^n) \cong 0$$

$\square$

Definition 2.5. For pointed CW-complex X with $X^{-1} := *$ and a ordinary homology theory $H_*(-,-)$ the (reduced) **cellular chain complex** $\{\tilde{C}_n(X), d_n\}$ of X is defined by:

$$\tilde{C}_n(X) := H_n(X^n, X^{n-1})$$
$$d_n : H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{i_*} H_{n-1}(X^{n-1}, X^{n-2})$$

Note. Use cellular approximation, we can see that the construction $\tilde{C}_*(-)$ is a functor.

Theorem 2.12. *For any ordinary homology theory $H_*(-,-)$ and any pointed CW-complex X, (with $X^{-1} := *$) the n-th homology of cellular chain complex is isomorphic to $\tilde{H}_n(X)$:*

$$H_n(\tilde{C}_*(X)) \cong H_n(X, *)$$

if we set $X^{-1} := \emptyset$ in our $\tilde{C}_*(X)$, then $H_n(\tilde{C}_*(X)) \cong H_n(X, \emptyset)$.

Proof. Notice that we have commutative diagram with each straight line exact: (use long exact sequence of pairs, $n > 0$)

$$\begin{tikzcd}
 & & & H_{n-1}(X^{n-2}) \cong 0 \arrow[dr] & & \\
 & & & & H_{n-1}(X^{n-1}) \arrow[dr, tail] & \\
H_{n+1}(X^{n+1}, X^n) \arrow[rrr, "d_{n+1}"] \arrow[dr, "\partial_{n+1}"'] & & & H_n(X^n, X^{n-1}) \arrow[ur, "\partial_n"] \arrow[rr, "d_n"] & & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & H_n(X^n) = \ker(\partial_n) = \ker(d_n) \arrow[urr, tail] \arrow[drr, two heads] & & & & \\
H_n(X^{n-1}) \cong 0 \arrow[ur] & & & H_n(X^{n+1}) = \mathrm{coker}(\partial_{n+1}) \arrow[dr] & & \\
 & & & & H_{n-1}(X^{n+1}, X^n) \cong 0 &
\end{tikzcd}$$

For $n = 0$:

$$H_1(X^1, X^0) \xrightarrow{d_1} H_0(X^0, X^{-1}) \twoheadrightarrow \mathrm{coker}(d_1) \cong H_0(X^1, X^{-1}) \longrightarrow H_0(X^1, X^0) \cong 0$$

$\square$

Note. If the ordinary homology theory has coefficient $\mathbb{Z}$, then the $d_n : \tilde{C}_n(X) \to \tilde{C}_{n-1}(X)$ is given by:

$$\mathbb{Z}_i \ni 1_i = e_i^n \mapsto \sum_{j \in I_{n-1}} \alpha_i^j e_j^{n-1}$$

where α_i^j is degree of map

$$\beta_i^j : S^n \approx \partial e_i^n \xrightarrow{\varphi_i} X^{n-1} \to X^{n-1}/X^{n-2} \to \bigvee_{j' \in I_{n-1}} S^{n-1} \xrightarrow{p_j} S^{n-1}$$

where φ_i is the characteristic map, p_j maps every point not in S_j^{n-1} to $*$.

Corollary 2.13. *For any ordinary homology theory $H_*(-,-)$ and any relative CW-complex (X, A), the cellular chain of X with is $X^{-1} := A$ noted $C_*(X, A)$, we have:*

$$H_n(C_*(X, A)) \cong H_n(X/A, *) \cong H_n(X, A)$$

Proposition 2.14. *If (X, A) is a (pointed) CW-pair, (with $X^{-1} := * =: A^{-1}$) use the natural relative CW-complex (X, A) to obtain $C_*(X, A)$, then $\tilde{C}_*(X)/\tilde{C}_*(A) \cong C_*(X, A)$ naturally.*

Proof. $H_n(X^n, X^{n-1})/H_n(A^n, A^{n-1}) \cong H_n((X/A)^n, (X/A)^{n-1})$
and $H_0(X^0, X^{-1})/H_n(A^0, A^{-1}) \cong H_n((X/A)^0, (X/A)^{-1})$. Naturality:

$$\begin{tikzcd}
\dfrac{\bigoplus_{I_X^n} \mathbb{Z}}{\bigoplus_{I_A^n} \mathbb{Z}} \arrow[r, "\cong"] \arrow[d, "f_*"'] & \bigoplus_{I_X^n - I_A^n} \mathbb{Z} \arrow[d, "f_*"] \\
\dfrac{\bigoplus_{I_Y^n} \mathbb{Z}}{\bigoplus_{I_B^n} \mathbb{Z}} \arrow[r, "\cong"'] & \bigoplus_{I_Y^n - I_B^n} \mathbb{Z}
\end{tikzcd}$$

where I_Z^n is the index set of n-cells of Z, $f : (X, A) \to (Y, B)$ is a cellular map.

□

3 Homotopy and Eilenberg-Mac Lane Spaces

3.1 Homotopy Excision Theorem and its Corollary

Theorem 3.1. *(Blakers–Massey) Homotopy Excision Theorem:*
For pointed CW-triad $(X; A, B)$ such that $C := A \cap B \neq \emptyset$, if (A, C) is $(m-1)$-connected and (B, C) is $(n-1)$-connected where $m \geq 2$, $n \geq 1$. Then $i : (A, C) \to (X, B)$ is an $(m+n-2)$-equivalence for pairs.

Note. We can replace the "CW-triad" with "excisive triad" in condition by theorem 2.7.

Proof. See here.

Corollary 3.2. *Suppose that $Y_0 \hookrightarrow Y$ is cofibration, (Y, Y_0) is $(r-1)$-connected and Y_0 is $(s-1)$-connected, then $(Y, Y_0) \to (Y/Y_0, *)$ is $(r+s-1)$-equivalence. $(r \geq 2,\ s \geq 1)$*

Proof. $Y_0 \hookrightarrow CY_0$ is cofibration and (CY_0, Y_0) is s-connected. Use homotopy excision theorem (with $X = Y \cup CY_0,\ A = Y,\ B = CY_0,\ C = Y_0$) to see $(Y, Y_0) \to (Y \cup CY_0, CY_0)$ is $(r+s-1)$-equivalence. And $(Y \cup CY_0, CY_0) \to (Y/Y_0, *)$ is homotopy equivalence since $Y_0 \hookrightarrow Y$ is cofibration.
□

Corollary 3.3. *For $n \geq 2$, $f : X \to Y$ is $(n-1)$-equivalence between $(s-1)$-connected spaces, then $(M_f, X) \to (C_f^+, *)$ is $(n+s-1)$-equivalence. Where $C_f^+ := Y \cup_f C^+X$, $C^+X := (X \times I)/(X \times \{1\})$. is the unreduced mapping cone and the unreduced cone.*

Proof. f is $(n-1)$-equivalence implies (M_f, X) is $(n-1)$-connected. Use corollary above.

□

Corollary 3.4. *For $n \geq 2$, if $f : X \to Y$ is pointed map between $(n-1)$-connected well-pointed spaces (that is, pointed space whose inclusion of the base point is (closed) cofibration). Then C_f is $(n-1)$-connected and $\pi_n(M_f, X) \to \pi_n(C_f, *)$ is isomorphism.*

Proof. Use homotopy extension property to extend to unreduced case. f is map between $(n-1)$-connected space implies f is at least a $(n-1)$-equivalence. Therefore $(M_f, X) \to (C_f, *)$ is $(2n-1)$-equivalence, Since we have $n < 2n-1$ for any $n \geq 2$, $\pi_n(M_f, X) \to \pi_n(C_f, *)$ is isomorphism.

□

Theorem 3.5. *(Freudenthal Suspension Theorem) If X is well-pointed and $(n-1)$-connected $(n \geq 1)$, then the map:*

$$\begin{aligned} \sigma : \pi_q(X) &\to \pi_{q+1}(\Sigma X) \cong \pi_q(\Omega\Sigma X) \\ f &\mapsto \Sigma f \end{aligned}$$

is isomorphism if $q < 2n-1$ and epimorphism if $q = 2n-1$.

Proof. If we have $f : (I^q, \partial I^q) \to (X, *)$ then $f \times \mathrm{id}_I : I^{q+1} \to X \times I$ will give a map $\overline{f \times \mathrm{id}_I}$: $(I^{q+1},\ \partial I^{q+1},\ \partial I^q \times I \cup \partial I \times \{1\}) \to (CX, X, *)$ since $J^q = \partial I^q \times I \cup \partial I \times \{0\}$, it does not give a map in $\pi_{q+1}(CX, X)$. we should change $\overline{f \times \mathrm{id}_I}$ into $\overline{f \times -\mathrm{id}_I}$. we have commutative diagram:

$$\begin{tikzcd} \pi_{q+1}(CX, X) \arrow[r, "p_*"] \arrow[d, bend right, "\partial"'] & \pi_{q+1}(CX/X, *) \arrow[d, equal] \\ \pi_q(X) \arrow[r, "-\sigma"'] \arrow[u, bend right, "i"'] & \pi_{q+1}(\Sigma X) \end{tikzcd} \qquad \begin{tikzcd} {[\overline{f \times -\mathrm{id}_I}]} \arrow[r, mapsto] \arrow[d, bend right] & {[p \circ (\overline{f \times -\mathrm{id}_I})]} \arrow[d, equal] \\ {[f]} \arrow[r, mapsto] \arrow[u, bend right] & {[-\Sigma f]} \end{tikzcd}$$

Where $p : (CX, X) \to (CX/X, *)$ is the canonical quotient map and $i : [f] \to [\overline{f \times -\mathrm{id}_I}]$ makes $\pi_{q+1}(CX) \to \pi_{q+1}(CX, X) \to \pi_q(X) \to \pi_q(CX)$ split in middle (that is, i is inverse of the connecting homomorphism ∂). We verify the commutativity:

$$\begin{aligned} -\Sigma f : (I^{q+1}, \partial I^{q+1}) &\to (CX/X, *) \\ (s, t) &\mapsto f(s) \wedge (1-t) \\ p \circ (\overline{f \times -\mathrm{id}_I}) : (I^{q+1}, \partial I^{q+1}) &\to (CX/X, *) \\ (s, t) &\mapsto f(s) \wedge (1-t) \end{aligned}$$

Since $X \hookrightarrow CX$ is cofibration and n-equivalence between $(n-1)$-connected spaces, p is an $2n$-equivalence. Therefore, $q+1 < 2n$ implies $-\sigma$ is isomorphism, $q+1 = 2n$ implies $-\sigma$ is epimorphism, and we have $-\sigma$ is iff σ is.

□

Corollary 3.6. *If Y is well pointed $(n-1)$-connected space then $Y \to \Omega\Sigma Y$ is $(2n-1)$-equivalence. By theorem 1.3, for any CW-complex X with $\dim X < 2n-1$, $\Sigma : [X, Y]_* \to [\Sigma X, \Sigma Y]_* \cong [X, \Omega\Sigma Y]_*$ is bijection.*

Definition 3.1. We now define the q-th stable homotopy group:

$$\pi_k^s(X) := \varinjlim_r \pi_{k+r}(\Sigma^r X) \cong \pi_{2k+2}(\Sigma^{k+2} X) \cong \pi_{k+n}(\Sigma^n X) \qquad (n-1 > k)$$

(Since $\Sigma^n X$ is $(n-1)$-connected)
And stable homotopy class:

$$[X, Y]_*^s := \varinjlim_r [\Sigma^r X, \Sigma^r Y]$$

Note. We'll see later that $\{\pi_n^s\}_{n \in \mathbb{N}}$ defines a generalized homology theory.

3.2 Hurewicz Theorem

First, we use homotopy excision theorem to prove following lemmas:

Lemma 3.7. *(every $S^n_a \approx S^n$) We have canonical $i_a : S^n_a \hookrightarrow \bigvee_{a\in A} S^n_a$ and for $n > 1$:*

$$\pi_n(\bigvee_{a\in A} S^n_a) \cong \bigoplus_{a\in A} \mathbb{Z}_a$$

where $[i_a] = 1 \in \mathbb{Z}_a \subseteq \bigoplus_{a\in A} \mathbb{Z}_a$ and every $\mathbb{Z}_a \cong \mathbb{Z}$.
For $n = 1$:

$$\pi_n(\bigvee_{a\in A} S^1_a) \cong \coprod_{a\in A} \mathbb{Z}_a$$

where $\coprod$ is taken in category **Grp***, $[i_a] = 1 \in \mathbb{Z}_a \subseteq \coprod_{a\in A} \mathbb{Z}_a$ and every $\mathbb{Z}_a \cong \mathbb{Z}$.*

Proof.
Case $n = 1$:
Apply the Seifert-van Kampen theorem.
Case $n > 1$:
Suppose each S^n_a have CW-complex structure with one 0-cell and one n-cell. Consider finite product $\prod_{1\le i\le k} S^n_i$ and its subcomplex, finite wedge product $\bigvee_{1\le i\le k} S^n_i$.
The inclusion

$$\bigvee_{1\le i\le k} S^n_i \hookrightarrow \prod_{1\le i\le k} S^n_i$$

is $(2n-1)$-equivalence since $\prod_{1\le i\le k} S^n_i - \bigvee_{1\le i\le k} S^n_i$ only have cells of dim $\ge 2n$. (use lemma 1.13)
Use exact homotopy sequence of pair, we deduce that $\pi_q(\bigvee_{1\le i\le k} S^n_i) \to \pi_q(\prod_{1\le i\le k} S^n_i) \cong \bigoplus_{1\le i\le k} \mathbb{Z}$ is an isomorphism for $q \le 2n-2$. And $S^n_i \hookrightarrow \bigvee_{1\le i\le k} S^n_i \hookrightarrow \prod_{1\le i\le k} S^n_i$ is just the i-th inclusion $S^n_i \hookrightarrow \prod_{1\le i\le k} S^n_i$ which represents $1 \in \mathbb{Z}_i \hookrightarrow \bigoplus_{1\le i\le k} \mathbb{Z}_i$. Infinite wedge case:

$$\begin{tikzcd}
\bigoplus_{1\le i\le k} \pi_q(S^n_i) \arrow[r, "\cong"] \arrow[d, tail] & \pi_q(\bigvee_{1\le i\le k} S^n_i) \arrow[d] \\
\bigoplus_{a\in A} \pi_q(S^n_a) \arrow[r, tail, two heads, "\bigoplus_{a\in A} i_{a*}"'] & \pi_q(\bigvee_{a\in A} S^n_a)
\end{tikzcd}$$

$\bigoplus_{a\in A} i_{a*}$ is monomorphism since every homotopy $S^n \times I \to \bigvee_{a\in A} S^n_a$ has a compact image, and $\bigoplus_{a\in A} i_{a*}$ is epimorphism since every map $S^n \times I \to \bigvee_{a\in A} S^n_a$ has a compat image.

□

Lemma 3.8. *For $n \ge 1$, if we have a map $f : \coprod_{a\in A} \mathbb{Z}_a \to \coprod_{b\in B} \mathbb{Z}_b$ (case $n = 1$) or a map $f : \bigoplus_{a\in A} \mathbb{Z}_a \to \bigoplus_{b\in B} \mathbb{Z}_b$ (case $n > 1$).*
Then there exists a map $\phi : \bigvee_{a\in A} S^n_a \to \bigvee_{b\in B} S^n_b$ unique up to homotopy and satisfy $\pi_n(\phi) = f$.

Proof. Suppose $f(1_a) = [\phi_a] \in [S^n, \bigvee_{b\in B} S^n_b]_*$, then ϕ_a is indeed a map $S^n_a \to \bigvee_{b\in B} S^n_b$. Now we define $\phi := \bigvee_a \phi_a : \bigvee_{a\in A} S^n_a \to \bigvee_{b\in B} S^n_b$. For any $a \in A$, $\phi|_{S^n_a} = \phi_a$, we have

$$\pi_n(\phi)(1_a) = [\phi|_{S^n_a} \circ \mathrm{id}_{S^n_a}] = [\phi_a] = f(1_a)$$

which implies $\pi_n(\phi) = f$ since they are group homomorphisms.

Uniqueness up to homotopy: $\pi_n(\phi)[1_a] = \pi_n(\phi')[1_a]$ implies $\phi|_{S^n_a} \simeq \phi'|_{S^n_a} \operatorname{rel} *$. Therefore $\phi \simeq \phi' \operatorname{rel} *$.

□

Definition 3.2. If H_n is a ordinary homology theory with coefficient $\mathbb{Z}$,
then the map

$$h_X : \pi_n(X) \to \tilde{H}_n(X) := H_n(X, *)$$
$$[f] \mapsto f_*(1) \qquad\qquad (\ f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(X)\ )$$

is called **Hurewicz Homomorphism**.

Note. $h_{(-)}$ is natural transformation since we have

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "h_X"] \arrow[d, "g_*"'] & \tilde{H}_n(X) \arrow[d, "g_*"] \\
\pi_n(Y) \arrow[r, "h_Y"'] & \tilde{H}_n(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[f]} \arrow[r, mapsto] \arrow[d, mapsto] & f_*(1) \arrow[d, mapsto] \\
{[g \circ f]} \arrow[r, mapsto] & (g \circ f)_*(1) = (g_* \circ f_*)(1)
\end{tikzcd}$$

commutes. Moreover, it commutes with connecting homomorphism.

Lemma 3.9. *If $X = \bigvee_{a \in A} S^n$, $h_X : \pi_n(X) \to \tilde{H}_n(X)$ is abelianization if $n = 1$, isomorphism if $n \geq 2$.*

Proof. Directly from lemma 3.8. (we used homotopic properties of spheres only in proving is lemma) □

Theorem 3.10. *(Hurewicz) If X is $(n-1)$-connected, then $h_X : \pi_n(X) \to \tilde{H}_n(X)$ is abelianization if $n = 1$, isomorphism if $n \geq 2$.*

Proof. We can assume X is CW-complex with $X^{n-1} = *$ and each characteristic map is pointed. (since we have theorem 1.5)
For CW-complex X, $\pi_n(X^{n+1}) \cong \pi_n(X)$ and $H_n(X^{n+1}) \cong H_n(X)$, Since we have cellularity of homotopy group and cellularity of homology.
Then we have $X^n = \bigvee_{b \in B} S_b^n$, $X^{n+1} = C_\phi$ where $\phi : \bigvee_{a \in A} S_a^n \to X^n$ are the characteristic maps.
Use naturality of $h_{(-)}$, we have maps between exact sequence:

$$\begin{tikzcd}
\pi_n(\bigvee_{a \in A} S_a^n) \arrow[r, "\phi_*"] \arrow[d, "h_{\bigvee_{a \in A} S_a^n}"] & \pi_n(X^n) \arrow[r] \arrow[d, "h_{X^n}"] & \pi_n(C_\phi) \arrow[r] \arrow[d, "h_{C_\phi}"] & 0 \\
\tilde{H}_n(\bigvee_{a \in A} S^n) \arrow[r, "\phi_*"'] & \tilde{H}_n(X^n) \arrow[r] & \tilde{H}_n(C_\phi) \arrow[r] & 0
\end{tikzcd}$$

If $n > 1$, exactness of top row is directly from lemma 3.2. ($(M_\phi, \bigvee_{a \in A} S_a^n)$ is $(n-1)$-connected since we have lemma 1.13) 5-lemma shows that h_{C_ϕ} is isomorphism.
If $n = 1$, Seifert-van Kampen theorem shows that $\pi_1(C_\phi) = \pi_1(X^n)/\langle \operatorname{Im} \phi_* \rangle_{nor}$. (where for $A \subseteq$ a group G, $\langle A \rangle_{nor} := \{gAg^{-1} \mid g \in G\}$). The top row is not exact, but top row's abelianization is exact since $\langle \operatorname{Im} f \rangle_{nor}/[B, B] = \operatorname{Im} f/[B, B]$ for any group morphism $f : A \to B$. Therefore we have diagram below with the middle row and the bottom row exact:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, "q"] \arrow[d] & G = B/\langle \operatorname{Im} f \rangle_{nor} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
A^{\mathrm{ab}} \arrow[r, "f^{\mathrm{ab}}"] \arrow[d, "\cong"'] & B^{\mathrm{ab}} \arrow[r, "q^{\mathrm{ab}}"] \arrow[d, "\cong"'] & G^{\mathrm{ab}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
A' \arrow[r] & B' \arrow[r] & G' \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

Finally apply 5-lemma on the middle row and the bottom row.

□

Corollary 3.11. *(Relative version of Hurewicz theorem) If (X, A) is $(n-1)$-connected CW-pair, A is 1-connected subcomplex and $n \geq 2$, then the Hurewicz morphism $h_{(X,A)} : \pi_n(X, A) \to H_n(X, A)$ (defined analogue to h_X) is isomorphism.*

Proof. Use theorem 3.2 and Hurewicz theorem of $h_{X/A}$.

□

Uniqueness of Ordinary Homology Theory:

Theorem 3.12. *If $H_*(-,-)$ is ordinary homology theory with coefficient $\mathbb{Z}$ on CW-complexes, then $H_*(-,-)$ is unique up to natural isomorphism.*

Proof. Since $H_n(C_*(X)) \cong H_n(X)$ naturally (in X), our goal is to prove the complex defined by

$$\begin{aligned} C'_n(X) &:= \pi_n(X^n, X^{n-1})_{ab} \\ d'_n &:= \pi_n(X^n, X^{n-1})_{ab} \xrightarrow{\partial} \pi_{n-1}(X^{n-1})_{ab} \to \pi_{n-1}(X^{n-1}, X^{n-2})_{ab} \end{aligned}$$

is isomorphic to $C_*(X)$ naturally. Isomorphic:

$$\begin{tikzcd} \pi_n(X^n, X^{n-1})_{ab} \arrow[r, "\partial"] \arrow[d, "\cong"'] & \pi_{n-1}(X^{n-1})_{ab} \arrow[r] \arrow[dd] \arrow[dr] & \pi_{n-1}(X^{n-1}, X^{n-2})_{ab} \arrow[d, "\cong"] \\ \pi_n(X^n/X^{n-1})_{ab} \arrow[ur] \arrow[d, "\cong"'] & & \pi_{n-1}(X^{n-1}/X^{n-2})_{ab} \arrow[d, "\cong"] \\ H_n(X^n/X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}) \arrow[r] & H_{n-1}(X^{n-1}/X^{n-2}) \end{tikzcd}$$

Naturality directly follows from naturality of Hurewicz morphism.

□

Note. Similarly uniqueness pf ordinary homology theory with coefficient G.

3.3 Moore Spaces

Definition 3.3. A space X is **Eilenberg-Mac Lane space** of **type** $K(G, n)$ (where G is group and is abelian for $n \geq 2$) if

$$\pi_q(X) \cong \begin{cases} G & n = q \\ 0 & n \neq q \end{cases}$$

We see that $\mathrm{SP}\, S^n$ is a $K(\mathbb{Z}, n)$. Now we use this to construct other $K(G, n)$.

Note. In order to construct $K(G, n)$, we construct a space $M(G, n)$ which have $\pi_n(M(G, n)) = G$, $\pi_q(M(G, n)) = 0$ for $q < n$ and we can apply SP on it to kill all dim $> n$ homotopy group.

Proposition 3.13. *For any $k \in \mathbb{Z}$, there is a map $a_k : S^1 \to S^1$ with a_k, and $C_{a_k} = S^1 \cup_{a_k} e^2$ is the desired $M(\mathbb{Z}/k\mathbb{Z}, 1)$ (that is $\mathrm{SP}(S^1 \cup_{a_k} e^2)$ is a $K(\mathbb{Z}/k\mathbb{Z}, 1)$).*

Proof. Consider sequence $S^1 \xrightarrow{a_k} S^1 \hookrightarrow C_{a_k} \twoheadrightarrow \Sigma S^1 = C_{a_k}/S^1$, we apply an usual form of Dold-Thom Theorem to see that $\mathrm{SP}(C_{a_k}) \to \mathrm{SP}(S^2)$ is a quasi-fibration with fiber $\mathrm{SP}(S^1)$. Then we have exact sequence:

$$\begin{aligned} \cdots &\to \pi_q(\mathrm{SP}\, S^1) \to \pi_q(\mathrm{SP}\, C_{a_k}) \to \pi_q(\mathrm{SP}\, S^2) \to \pi_{q-1}(\mathrm{SP}\, S^1) \to \\ \cdots &\to \pi_2(\mathrm{SP}\, S^1) \to \pi_2(\mathrm{SP}\, C_{a_k}) \to \pi_2(\mathrm{SP}\, S^2) \\ &\to \pi_1(\mathrm{SP}\, S^1) \to \pi_1(\mathrm{SP}\, C_{a_k}) \to \pi_1(\mathrm{SP}\, S^2) \end{aligned}$$

We can conclude that $\pi_q(\mathrm{SP}\, C_{a_k}) = 0$ for any $q \neq 0, 1$ and:

$$0 \to \pi_2(\mathrm{SP}\, C_{a_k}) \to \pi_2(\mathrm{SP}\, S^2) = \mathbb{Z} \xrightarrow{\partial} \pi_1(\mathrm{SP}\, S^1) = \mathbb{Z} \to \pi_1(\mathrm{SP}\, C_{a_k}) \to 0$$

exact. Where ∂ is defined by:

$$\pi_2(\mathrm{SP}\, S^2) \cong [D^2, S^1, *; \mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1, *] \ni f \mapsto f|_{S^1} \in [S^1, S^1]_*$$

(Now we want to show that ∂ is multiplication by k)
The $1 \in \mathbb{Z} \cong \pi_2(\mathrm{SP}\, S^2)$ is represented by $[i_2 : S^2 \hookrightarrow \mathrm{SP}\, S^2]$.
Since $[D^2, S^1, *; \mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1, *] \xrightarrow{p_*} [D^2, S^1; \mathrm{SP}\, S^2, *]$ is isomorphism,

and the map $\varphi : (D^2, S^1) \xrightarrow{\mathrm{id}_{e^2} \cup a_k} (C_{a_k}, S^1) \hookrightarrow (\mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1)$ satisfy $p \circ \varphi = i_2$, the $1 \in \mathbb{Z} \cong \pi_2(\mathrm{SP}\, C_{a_k}, \mathrm{SP}\, S^1)$ is represented by φ. Then we have $\partial(1)$ is represented by $\varphi|_{S^1} = i_1 \circ a_k$ where $i_1 : S^1 \hookrightarrow \mathrm{SP}\, S^1$.
The map ∂ is $\mathbb{Z} \ni n \mapsto kn \in \mathbb{Z}$ since $[i_1 \circ a_k] = k$.
Therefore $\pi_2(\mathrm{SP}\, C_{a_k}) = 0$ and $\pi_1(\mathrm{SP}\, C_{a_k}) = \mathbb{Z}/k\mathbb{Z}$.

$\square$

Proposition 3.14. *For each $n \geq 1$, $k \in Z$, $\mathrm{SP}(S^n \cup_{\Sigma^{n-1} a_k} e^{n+1})$ is a $K(\mathbb{Z}/k\mathbb{Z}, n)$.*

Proof. For $q \geq 1$, $\Sigma(S^q \cup_{\Sigma^{q-1} a_k} e^{q+1}) \approx \Sigma S^q \cup_{\Sigma^q a_k} \Sigma e^{q+1} = S^{q+1} \cup_{\Sigma^q a_k} e^{q+2}$ since Σ is left adjoint of Ω in $\mathbf{TOP}_*$ and the pushout is took in $\mathbf{TOP}_*$. Observe that $\pi_q(\mathrm{SP}\, X) \cong \pi_{q+1}(\mathrm{SP}\, \Sigma X)$, now we have done.

$\square$

Since $\tilde{H}_n(X) \cong \tilde{H}_n(X \cup C*) \cong H_n(X, *)$, we have

$$\pi_n(\mathrm{SP}(\bigvee_{i\in I} X_i)) = \tilde{H}_n(\bigvee_{i\in I} X_i) \cong H_n(\bigvee_{i\in I} X_i, *) \cong H_n(\coprod_{i\in I} X_i, \coprod_{i\in I} *) \cong \bigoplus_{i\in I} H_n(X_i, *) \cong \bigoplus_{i\in I} \pi_n(\mathrm{SP}\, X_i)$$

We can deduce the following proposition immediately:

Proposition 3.15. *For finitely generated abelian group $G \cong (\bigoplus_r \mathbb{Z}) \oplus (\bigoplus_{1\leq i\leq k} \mathbb{Z}/d_i\mathbb{Z})$, (where $r \in \mathbb{N}$, each $d_i \in \mathbb{Z}$) we have $\mathrm{SP}((\bigvee_r S^n) \vee (\bigvee_{1\leq i\leq k}(S^n \cup_{a_{d_i}} e^{n+1})))$ is a $K(G, n)$.*

Since every abelian group G have a free resolution sequence:

$$0 \to \bigoplus_{a\in A} \mathbb{Z} \xrightarrow{f} \bigoplus_{b\in B} \mathbb{Z} \twoheadrightarrow G \to 0$$

exact. And for every group $G = F(X)/\langle Y\rangle_{nor}$ (where $F(X) := \coprod_{x\in X} \mathbb{Z}_x$ is the free group functor and $\langle Y\rangle_{nor}$ is the normal subgroup generated by Y), we have:

$$1 \to \coprod_{y\in\langle Y\rangle_{nor}} \mathbb{Z}_y \xrightarrow{f:1_y\mapsto y} \coprod_{x\in X} \mathbb{Z}_x \twoheadrightarrow G \to 1$$

exact.

Next proposition allows to construct spaces $M(\bigoplus_{a\in A} \mathbb{Z}, n)$ and $M(\coprod_{a\in A} \mathbb{Z}, 1)$:

Definition 3.4. For $n > 1$, G an abelian group, we have exact sequence

$$0 \to \bigoplus_{a\in A} \mathbb{Z} \xrightarrow{f} \bigoplus_{b\in B} \mathbb{Z} \twoheadrightarrow G \to 0$$

Then we have: (with ϕ is the map obtained using lemma 3.8)

$$\bigvee_{a\in A} S^n_a \xrightarrow{\phi} \bigvee_{b\in B} S^n_b \to C_\phi$$

the **Moore space** of type (G, n) is defined as $M(G, n) := C_\phi$.

For $n = 1$, G a group, we have exact sequence:

$$1 \to \coprod_{y\in\langle Y\rangle_{nor}} \mathbb{Z}_y \xrightarrow{f:1_y\mapsto y} \coprod_{x\in X} \mathbb{Z}_x \twoheadrightarrow G \to 1$$

Then we have: (with ϕ is the map obtained using lemma 3.8)

$$\bigvee_{y\in\langle Y\rangle_{nor}} S^1_y \xrightarrow{\phi} \bigvee_{x\in X} S^1_x \to C_\phi$$

the **Moore space** of type $(G, 1)$ is defined as $M(G, 1) := C_\phi$.

Proposition 3.16. $\pi_n(M(G,n)) = G$

Proof. For $n > 1$, use diagram:

$$\begin{tikzcd}
 & M_\phi \arrow[d, bend left, "p"] \arrow[d, phantom, "\simeq"] \\
X \arrow[ur, "i"] \arrow[r, "\phi"'] & Y \arrow[u, bend left, "j"]
\end{tikzcd}$$

To see:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(\bigvee_{a\in A} S^n_a) \arrow[r, "i_*"] \arrow[dr, "\phi_*"'] & \pi_n(M_\phi) \arrow[r] & \pi_n(M_\phi, \bigvee_{a\in A} S^n_a) \arrow[r] \arrow[dd, "q_*"] & \pi_{n-1}(\bigvee_{a\in A} S^n_a) \arrow[r] & \cdots \\
 & & \pi_n(\bigvee_{b\in B} S^n_b) \arrow[u, "j_*\cong"'] & & & \\
 & \bigoplus_{a\in A} \mathbb{Z}_a \arrow[uu, "\cong"] \arrow[r, "f"'] & \bigoplus_{b\in A} \mathbb{Z}_a \arrow[u, "\cong"'] & \pi_n(M(G,n)) & &
\end{tikzcd}$$

Where q_* is induced by $q : (M_\phi, \bigvee_{a\in A} S^n_a) \to (C_\phi, *)$. $\bigvee_{a\in A} S^n_a$ is $(n-1)$-connected, implies $\pi_{n-1}(\bigvee_{a\in A} S^n_a) = 0$. $(M_\phi, \bigvee_{a\in A} S^n_a)$ is $(n-1)$-connected due to lemma 1.13. Therefore we have q_* is isomorphism using lemma 3.2. Diagram above reduces to:

$$0 \to \bigoplus_{a\in A} \mathbb{Z}_a \xrightarrow{f} \bigoplus_{b\in B} \mathbb{Z}_b \twoheadrightarrow \pi_n(M(G,n)) \to 0$$

For $n = 1$, use Seifert-van Kampen theorem.

$\square$

Proposition 3.17. *For any $n \geq 1$ and any group morphism $f : G \to G'$ there exist morphism $f_M : M(G,n) \to M(G',n)$ such that $f_{M*} = f$.*

Proof. We have following for $n > 1$: (since free $\mathbb{Z}$-module is projective)

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{a\in A} \mathbb{Z}_a \arrow[r, tail, "i"] \arrow[d, dashed, "r_1"] & \bigoplus_{b\in B} \mathbb{Z}_b \arrow[r, two heads, "q"] \arrow[d, dashed, "r_0"] & G \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \bigoplus_{a'\in A'} \mathbb{Z}_{a'} \arrow[r, tail, "i'"'] & \bigoplus_{b'\in B'} \mathbb{Z}_{b'} \arrow[r, two heads, "q'"'] & G' \arrow[r] & 0
\end{tikzcd}$$

And we have following for $n = 1$: (where $i(1_{1_a 1_b (1_{a\cdot b})^{\text{-}1}}) := 1_a 1_b (1_{a\cdot b})^{\text{-}1}$)

$$\begin{tikzcd}
1 \arrow[r] & \coprod_{(a,b)\in(G,G)} \mathbb{Z}_{1_a 1_b (1_{a\cdot b})^{\text{-}1}} \arrow[r, tail, "i"] \arrow[d, dashed, "r_1"] & \coprod_{g\in G} \mathbb{Z}_g \arrow[r, two heads, "q"] \arrow[d, dashed, "r_0"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & \coprod_{(a',b')\in(G',G')} \mathbb{Z}_{1'_a 1'_b (1_{a'\cdot b'})^{-1}} \arrow[r, tail, "i'"'] & \coprod_{g'\in G'} \mathbb{Z}_{g'} \arrow[r, two heads, "q'"'] & G' \arrow[r] & 1
\end{tikzcd}$$

We could obtain: (use lemma 3.8)

$$\begin{tikzcd}
\bigvee_{a\in A} S^n_a \arrow[r, "\phi"] \arrow[d, "\chi_1"'] \arrow[dr, phantom, "\simeq"] & \bigvee_{b\in B} S^n_b \arrow[r] \arrow[d, "\chi_0"] & C_\phi \arrow[d, dashed, "f_M"] \\
\bigvee_{a'\in A'} S^n_{a'} \arrow[r, "\phi'"'] & \bigvee_{b'\in B'} S^n_{b'} \arrow[r] & C_{\phi'}
\end{tikzcd}$$

Finally we have: (use universal property of cokernel)

$$\begin{tikzcd}
0 \arrow[r] & \pi_n(\bigvee_{a\in A} S^n_a) \arrow[r, "\phi_*=i"] \arrow[d, "\chi_{1*}=r_1"] & \pi_n(\bigvee_{b\in B} S^n_b) \arrow[r] \arrow[d, "\chi_{0*}=r_0"] & \pi_n(C_\phi) \arrow[r] \arrow[d, "f_{M*}=f"] & 0 \\
0 \arrow[r] & \pi_n(\bigvee_{a'\in A'} S^n_{a'}) \arrow[r, "\phi'_*=i'"'] & \pi_n(\bigvee_{b'\in B'} S^n_{b'}) \arrow[r] & \pi_n(C_{\phi'}) \arrow[r] & 0
\end{tikzcd}$$

□

Theorem 3.18. *$\mathrm{SP}(M(G,n))$ is a $K(G,n)$ if G is abelian.*

Proof. In the construction of Moore spaces, we have: (use notations in the construction)

$$\begin{tikzcd}
\bigvee_{a\in A} S^n_a \arrow[r, "\phi"] \arrow[dr] & \bigvee_{b\in B} S^n_b \arrow[d, "\simeq"] \arrow[dr] & \\
& M_\phi \arrow[r] & C_\phi
\end{tikzcd}$$

which induces quasi-fibration $\mathrm{SP}\, M_\phi \to \mathrm{SP}\, C_\phi$ with fiber $\mathrm{SP} \bigvee_{a\in A} S^n_a$. Then we have long exact sequence:

$$\cdots \to \pi_q(\mathrm{SP} \bigvee_{a\in A} S^n_a) \xrightarrow{\phi_*} \pi_q(\mathrm{SP}\, M_\phi) \to \pi_q(\mathrm{SP}\, C_\phi) \to \pi_{q-1}(\mathrm{SP} \bigvee_{a\in A} S^n_a) \to \cdots$$

Sequence above says if $q \neq n$ and $q \neq n+1$, then $\pi_q(\mathrm{SP}\, C_\phi) = 0$. If $q = n+1$, we have:

$$\begin{tikzcd}
0 \arrow[r] & \pi_{n+1}(C_\phi) \arrow[r] & \pi_n(\mathrm{SP} \bigvee_{a\in A} S^n_a) \arrow[r, tail, "\phi_*"] \arrow[d, phantom, "\wr\|"] & \pi_n(\mathrm{SP} \bigvee_{b\in B} S^n_b) \arrow[r, two heads] \arrow[d, phantom, "\wr\|"] & \pi_n(C_\phi) \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
& 0 \arrow[r] & \bigoplus_{a\in A} \mathbb{Z}_a \arrow[r, tail, "f"'] & \bigoplus_{b\in B} \mathbb{Z}_b \arrow[r, two heads] & G \arrow[r] & 0
\end{tikzcd}$$

We have $\pi_{n+1}(C_\phi) = 0$ since ϕ_* is monomorphism.

□

Note. We have two equivalent ways to construct ordinary homology theory with coefficient $G \in \mathbf{Ab}$ from $H_n(-,-;\mathbb{Z})$:

1. Tensor cellular chain complex with G: $C_*(X) \otimes_{\mathbb{Z}} G$ (differentials are $d_n \otimes \mathrm{id}_G$)
2. $H_n(X, A; G) := \tilde{H}_n((X \cup CA) \wedge M(G,n))$

Note. Construction of Eilenberg Mac-Lane space using Moore spaces is limited, there is another construction of $K(G,n)$ allows non-abelian group G for $n = 1$. (use geometric realization)

Definition 3.5. The **weak product** of pointed $\{Z_i\}_{i\in Z}$ spaces is

$$\mathop{\overset{\circ}{\prod}}_{i\in\mathbb{N}} Z_i := \varinjlim_{S\in \mathrm{Fin}(\mathbb{N})} (\prod_{i\in S} Z_i)$$

whose underlying set is:

$$\{(a_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} Z_i \mid \text{only finite } a_i \text{ is not } * \}$$

Theorem following shows why $K(G,n)$ is important:

Theorem 3.19. *If Y is a path-connected commutative associative H-space with strict identity ($1 \cdot y = y$), then there is a weak equivalence*

$$\mathop{\overset{\circ}{\prod}}_{n\geq 1} K(\pi_n(Y), n) \to Y$$

Moreover, we have weak equivalence

$$\prod_{n\geq 1} K(\pi_n(Y), n) \to Y$$

Proof. Take free resolution of $\pi_n(Y)$:

$$0 \to \bigoplus_{a\in A} \mathbb{Z}_a \xrightarrow{\gamma} \bigoplus_{b\in B} \mathbb{Z}_b \xrightarrow{q} \pi_n(Y) \to 0$$

(for $n = 1$, replace $\bigoplus$ with $\coprod$). and obtain:

$$\begin{tikzcd}
\bigvee_{a\in A} S^n_a \arrow[r, "\phi"] \arrow[d] \arrow[dr, phantom, "\simeq"] & \bigvee_{b\in B} S^n_b \arrow[r] \arrow[d, "\bigvee_{b\in B} g'_b"] & C_\phi \cong M(\pi_n(Y, n)) \arrow[d, dashed, "f'_n"] \\
* \arrow[r, hook, "i"'] & Y \arrow[r] & C_i \simeq Y
\end{tikzcd}$$

where $[g'_b] = q(1_b)$. We have $f'_{n*} : \pi_n(M(\pi_n(Y), n)) \to \pi_n(Y)$ is an isomorphism.
Construct $f'^k_n : \prod_k M(\pi_n(Y), n) \to Y$ by:

$$\begin{aligned} f'^k_n : \prod_k M(\pi_n(Y), n) &\to Y \\ (a_1, a_2, \ldots, a_k) &\mapsto f(a_1) \cdot f(a_2) \cdots f(a_k) \end{aligned}$$

where $- \cdot - : Y \times Y \to Y$ is the H-multiplication on Y.
Strict identity, commutativity and associativity says it is homotopically unique $\operatorname{rel} *$.
Therefore we have a well-defined map $f^k_n : \mathrm{SP}^k M(\pi_n(Y), n) \to Y$ (for each k) which commutes with inclusion $\mathrm{SP}^k \hookrightarrow \mathrm{SP}^{k+1}$.
Directly from above, we have $f_n : \mathrm{SP}\, M(\pi_n(Y), n) \to Y$ induces isomorphism on $\pi_n(-)$. (in case $n = 1$, $\pi_1(Y)$ is abelian since Y is a commutative H-space)
Similarly we have $f : \mathrm{SP}(\bigvee_n M(\pi_n(Y), n)) \to Y$ obtained from $\bigvee_n f'_n : \bigvee_n M(\pi_n(Y), n) \to Y$.
$\mathrm{SP}(\bigvee_n M(\pi_n(Y), n)) \approx \overset{\circ}{\prod}_n \mathrm{SP}\, M(\pi_n(Y), n)$ since we have $\mathrm{SP}(X_1 \vee X_2) \approx \mathrm{SP}\, X_1 \times \mathrm{SP}\, X_2$ and SP commute with directed colimit. We can deduce that $f|_{\mathrm{SP}\, M(\pi_n(Y), n)} = f_n$ from construction of the homeomorphism.
Last, $\overset{\circ}{\prod}_{n\geq 1} K(\pi_n(Y), n) \hookrightarrow \prod_{n\geq 1} K(\pi_n(Y), n)$ is weak homotopy equivalence since S^n have compact image. (is homotopy equivalence since they are CW-complexes)

□

Corollary 3.20. *If Y is a space, then there is a weak equivalence*

$$\overset{\circ}{\prod}_{n\geq 1} K(H_n(Y), n) \to \mathrm{SP}\, Y$$

Moreover, we have weak equivalence

$$\prod_{n\geq 1} K(H_n(Y), n) \to \mathrm{SP}\, Y$$

4 Cohomology and Spectra

4.1 Axiom for Cohomology and reduced Cohomology

Definition 4.1. An Unreduced **Generalized Cohomology Theory** (E^*, δ) is a functor to the category of $\mathbb{Z}$-graded abelian groups:
$E^*(-, -) : \mathbf{TOP_{CW}(2)}^{\mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}$,
with a natural transformation of degree $+1$:
$\delta_{n,(X,A)} : E^n(A, \emptyset) \to E^{n+1}(X, A)$ (called connecting homomorphism)
satisfying following 3 axioms:

- **Homotopy Invariance**:
  Homotopy equivalence of pairs $f : (X, A) \to (Y, B)$ induces isomorphism
$$E^*(f) : E^*(Y, B) \to E^*(X, A)$$
- **Long Exact Sequence**:
  Map $A \hookrightarrow X$ induces a long exact sequence together with δ:
$$\cdots E^n(X, A) \to E^n(X) \to E^n(A) \xrightarrow{\delta} E^{n+1}(X, A) \to \cdots$$
  where $E^n(X) := E^n(X, \emptyset)$.
- **Excision**:
  If $(X; A, B)$ is an **excisive triad** (that is, $X = \mathring{A} \cup \mathring{B}$), then inclusion $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphism
$$E^*(A, A \cap B) \cong E^*(X, B)$$

We say (E^*, δ) is **additive** if in addition:
- **Additivity**:
  If $(X, A) = \coprod_\lambda (X_\lambda, A_\lambda)$ in $\mathbf{TOP_{CW}(2)}$, then inclusions $i_\lambda : (X_\lambda, A_\lambda) \to (X, A)$ induces isomorphism
$$(\prod i_{*,\lambda}) : E^*(X, A) \cong \prod_\lambda E^*(X_\lambda, A_\lambda)$$

We say (E^*, δ) is **ordinary** if (E^*, δ) satisfy all axioms above and:
- **Dimension**:
$$E^{*\neq 0}(*, \emptyset) = 0$$

An unreduced ordinary cohomology theory is called with coefficient G if $E^0(*, \emptyset) = G$.

Definition 4.2. An **Reduced Generalized Cohomology Theory** $(\tilde{E}^*, \sigma)$ is a functor from opposite of category of pointed CW-complexes to the category of $\mathbb{Z}$-graded abelian groups:
$\tilde{E}^*(-) : \mathbf{TOP}_{\mathbf{CW}}^{*/\ \mathrm{op}} \to \mathbf{Ab}^{\mathbb{Z}}$,
with a natural isomorphism of degree $+1$:
$\sigma : \tilde{E}^*(-) \cong \tilde{E}^{*+1}(\Sigma(-))$ (called suspension isomorphism)
satisfying following 2 axioms:

- **Homotopy Invariance**:
  Homotopic pointed maps $f, g : X \to Y$ induces same map:
$$\tilde{E}^*(f) = \tilde{E}^*(g) : \tilde{E}^*(Y) \to \tilde{E}^*(X)$$
- **Exactness**:
  Pointed map $i : A \hookrightarrow X$ and $j : X \hookrightarrow C_i$ gives a exact sequence in $\mathbf{Ab}^{\mathbb{Z}}$
$$\tilde{E}(C_i) \xrightarrow{\tilde{E}^*(j)} \tilde{E}^*(X) \xrightarrow{\tilde{E}^*(j)} \tilde{E}^*(A)$$

We say $(\tilde{E}^*, \sigma)$ is **additive** if in addition:
- **Wedge Axiom**:
  The canonical comparison morphism (induced by morphisms $X_i \hookrightarrow \bigvee_i X_i$)
$$\tilde{E}^*(\bigvee_i X_i) \to \prod_i \tilde{E}^*(X_i)$$
  is isomorphism.

We say $(\tilde{E}^*, \sigma)$ is **ordinary** if $(\tilde{E}^*, \sigma)$ satisfy all axioms above and:
- **Dimension**:
$$\tilde{E}^{*\neq 0}(S^0) = 0$$

A reduced ordinary cohomology theory is called with coefficient G if $\tilde{E}^0(S^0) = G$.

Note. They are related to each other by $E^*(X, A) := \tilde{E}^*(X \cup CA)$ and $\tilde{E}^* := E^*(X, *)$. (proof is omitted)

4.2 Brown Representability Theorem

We will prove that any additive reduced cohomology theory is naturally isomorphic to some $[-, Y]_*$.

Definition 4.3. $C_0 := \mathrm{Ho}(C)$, where C is category of path-connected pointed CW-complexes.

Definition 4.4. A weak limit/colimit is just ordinary limit/colimit without the uniqueness its in universal property.

Lemma 4.1. *C_0 have weak coequalizers*

Proof. If we have map $f, g : A \to X$ in C_0 then define $Z := X_1 \cup_f (A \times I) \cup_g X_2/(x, 0) \sim (x, 1)$ where $X_1 = X \times \{0\}, X_2 = X \times \{1\}$. $j : X \hookrightarrow Z$ is the weak coequalizer map. $i : A \times I \hookrightarrow Z$ is the homotopy $j \circ f \simeq j \circ g$.
For $s : X \to Y$ such that there is $h : s \circ f \simeq s \circ g$, we have $s \cup h \cup s : X_1 \cup_f (A \times I) \cup_g X_2 \to Y$, and it defines a map $s' : Z \to Y$ such that $s' \circ j = s$.

□

Lemma 4.2. *Suppose $\{Y_n\}_{n \in \mathbb{N}}$ is a sequence of objects in C_0 with for all $n \in \mathbb{N}$, $i_n : Y_n \hookrightarrow Y_{n+1}$ is cofibration.*
Let $Y := \varinjlim_n Y_n$, then there is coequalizer diagram:

$$\bigvee_n Y_n \underset{\bigvee_n \mathrm{id}_{Y_n}}{\overset{\bigvee_n i_n}{\rightrightarrows}} \bigvee_n Y_n \xrightarrow{\bigvee_n j_n} Y$$

where $j_n : Y_n \hookrightarrow Y_{n+1} \hookrightarrow Y$.

Proof. $j_{n+1} \circ i_n = j_n \circ \mathrm{id}_{Y_n}$, and if we have $g : \bigvee_n Y_n \to Z$ such that $g \circ \bigvee_n i_n \simeq g \circ \bigvee_n \mathrm{id}_{Y_n}$. Define $g_n := g|_{Y_n}$, use induction on n and HEP of cofibration, we have $g'_n \simeq g_n$ such that $g'_{n+1} \circ i_n = g'_n$, there data together defines a $g' : Y \to Z$ satisfy desired properties.

□

Definition 4.5. A **Brown functor** is a functor $H : C_0^{\mathrm{op}} \to \mathbf{Set}^{*/}$ send coproducts to products, weak coequalizers to weak equalizers:

$$H(\bigvee_i X_i) \cong \prod_i H(X_i)$$

If $j : X \to Z$ is coequalizer of $f, g : A \to X$,
then $H(j) : H(Z) \to H(X)$ is equalizer of $H(f), H(g) : H(X) \to H(A)$.

Note. Every additive reduced cohomology theory $\tilde{E}^n(-) : \mathbf{TOP}_{\mathbf{CW}}^{*/\,\mathrm{op}} \to \mathbf{Ab} \to \mathbf{Set}^{*/}$ is equivalent to a Brown functor.

Definition 4.6. Any $u \in H(Y)$ determine a natural transformation $T_u : [-, Y]_* \to H(-)$ by

$$\begin{array}{ccc} [X, Y] \ni f & \longmapsto & H(f)(u) \in H(X) \\ \downarrow & & \downarrow \\ [X', Y] \ni f \circ a & \longmapsto & H(f \circ a)(u) \in H(X') \end{array}$$

where $a \in [X', X]$.
$u \in H(Y)$ is **n-universal** $(n \geq 1)$ if $T_{u, S^q} : [S^q, Y]_* \to H(S^q)$ is isomorphism for $1 \leq q \leq n-1$ and epimorphism for $q = n$.
$u \in H(Y)$ is **universal** if u is n-universal forall $n \geq 1$.
Y is called an **classifying space** for H if there exists $u \in H(Y)$ that is universal.

Lemma 4.3. *If H is a Brown functor, $Y, Y' \in C_0$, $u \in H(Y)$, $u' \in H(Y')$ are universal, and there is a map $f : Y \to Y'$ such that $H(f)(u') = u$, then f is a weak equivalence.*

Proof. Directly from T_{u,S^q}, T_{u',S^q} are isomorphisms:

$$\begin{tikzcd}
\pi_q(Y) \arrow[r, "f_*"] \arrow[dr, "T_{u,S^q}"'] & \pi_q(Y') \arrow[d, "T_{u',S^q}"] \\
 & H(S^q)
\end{tikzcd}$$

□

Lemma 4.4. *If H is a Brown functor, $Y \in C_0$ and $u \in H(Y)$, then there exists $Y' \in C_0$ obtained from Y by attaching 1-cells, and a 1-universal element $u' \in H(Y')$ such that $H(i)(u') = u \in H(Y)$. (where $i : Y \hookrightarrow Y'$)*

Proof. Let $Y' := Y \vee (\bigvee_{a \in H(Y)} S^1_a)$, $H(i)$ is just projection:
$H(Y') \cong (H(Y) \times \prod_{a \in H(S^1)} H(S^1_a)) \to H(Y)$.
Let $g_a := S^1 \approx S^1_a \hookrightarrow Y'$,
$u' := (u, \prod_a a) \in H(Y) \times H(\bigvee_{a \in H(S^1)} S^1_a)$.
$T_{u',S^1} : [S^1, Y']_* \to H(S^1)$ is epimorphism since $H(g_a)(u') = a \in H(S^1)$.

□

Lemma 4.5. *If H is a Brown functor, $Y \in C_0$ and $u \in H(Y)$ is n-universal ($n \geq 1$), then there exists $Y' \in C_0$ obtained from Y by attaching $(n+1)$-cells, and a $(n+1)$-universal element $u' \in H(Y')$ such that $H(i)(u') = u \in H(Y)$. (where $i : Y \hookrightarrow Y'$)*

Proof. Let $K := \ker(T_{u,S^n})$, we have:

$$* \to K \hookrightarrow [S^n, Y]_* \xrightarrow{T_{u,S^n}} H(S^n) \to *$$

Let $Y_1 := Y \vee (\bigvee_{i \in H(S^{n+1})} S^{n+1}_i)$. We notice a cofib sequence:

$$\bigvee_{k \in K} S^n_k \xrightarrow{f} Y_1 \to C_f$$

where $f := \bigvee_{k \in K} k$. Let $Y' := C_f$.
$u_1 := (u, \prod_{a \in H(S^{n+1})} a) \in H(Y_1)$ where $g_a := S^{n+1} \approx S^{n+1}_a \hookrightarrow Y_1$. The cofib sequence is just a weak coequalizer diagram in C_0:

$$\begin{tikzcd}
\bigvee_{k \in K} S^n_k \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & Y_1 \arrow[r] & Y'
\end{tikzcd}$$

Apply H on it:

$$\begin{tikzcd}
H(Y') \arrow[r] & H(Y_1) \arrow[r, shift left] \arrow[r, shift right] & H(\bigvee_{k \in K} S^n_k)
\end{tikzcd}$$

We have $H(f)(u_1) = \prod_{k \in K} H(k)(u_1) = \prod_{k \in K} H(k)(u) = \prod_{k \in K} T_{u,S^n}(k) = 0 = H(0)(u_1)$.
By definition of weak equalizer, there exists $u' \in H(Y')$ such that $H(i)(u') = u \in H(Y)$. ($i : Y \hookrightarrow Y'$)
Verify that u' is $(n+1)$-universal:
$T_{u',S^{n+1}}$ is epimorphism since $T_{u',S^{n+1}}(i_1 \circ g_a) = T_{u_1,S^{n+1}}(g_a) = a \in H(S^{n+1})$.
Current goal is to prove T_{u',S^q}, $q \leq n$ are isomorphisms.
We have commutative diagram:

$$\begin{tikzcd}
\pi_{q+1}(Y', Y) \arrow[r] & \pi_q(Y) \arrow[r, "i_*"] \arrow[d, "T_{u,S^q}"'] & \pi_q(Y') \arrow[r] \arrow[dl, "T_{u',S^q}"] & \pi_q(Y', Y) \\
 & H(S^q) & &
\end{tikzcd}$$

And we notice that $\pi_q(Y', Y) = 0$ for $q \leq n$. Then we have
T_{u,S^q} is isomorphism for $q < n$ and epimorphism for $q = n$ implies that

T_{u',S^q} is isomorphism for $q < n$ and epimorphism for $q = n$.
For any $k \in K \hookrightarrow \pi_n(Y)$, $i \circ k = 0 \in \pi_n(Y')$. That is, $K \subseteq \ker(i_*)$.
And we also have $\ker(i_*) \subseteq K$, since $T_{u',S^n} \circ i_* = T_{u,S^n}$.
$\ker(i_*) = K := \ker(T_{u,S^n})$ implies that T_{u',S^n} is isomorphism.

□

Theorem 4.6. *H is a Brown functor, $Y \in C_0$ and $u \in H(Y)$ then there is a classifying space Y' for H such that (Y', Y) is a relative CW-complex and the universal element $u' \in H(Y')$ satisfying $H(i)(u') = u$. ($i : Y \hookrightarrow Y'$)*

Proof. Construct spaces $\{Y_n\}_{n\in\mathbb{N}}$ and $u_n \in H(Y_n)$ as following:
1. $Y_0 := Y$, $u_0 := u$
2. Y_1, u_1 is obtained from lemma 4.4.
3. Use lemma 4.5 to construct Y_{n+1}, u_{n+1} from Y_n, u_n.

Let $Y' := \varinjlim\{Y_0 \hookrightarrow \cdots \hookrightarrow Y_n \hookrightarrow Y_{n+1} \hookrightarrow \cdots\}$ then we have weak equalizer diagram:

$$H(Y') \longrightarrow \prod_n H(Y_n) \underset{\prod_{n\in\mathbb{N}} H(\mathrm{id}_{Y_n})}{\overset{\prod_{n\in\mathbb{N}} H(i_n)}{\rightrightarrows}} \prod_n H(Y_n)$$

and

$$(\prod_{n\in\mathbb{N}} H(i_n))(\prod_{n\in\mathbb{N}} u_n) = \prod_{n\in\mathbb{N}} u_n = \prod_{n\in\mathbb{N}} H(\mathrm{id}_{Y_n})(\prod_{n\in\mathbb{N}} u_n)$$

(by $H(i_n)(u_{n+1}) = u_n$) Then there exists $u' \in H(Y')$ satisfying $\forall n \in \mathbb{N},\ H(j_n) = u_n$. (where $j_n : Y_n \hookrightarrow Y'$)
Verify that u' is universal:

$$\begin{tikzcd}
\pi_q(Y_{q+1}) \arrow[r] \arrow[drrr, "\cong"] & \pi_q(Y_{q+2}) \arrow[r] \arrow[drr, "\cong"] & \cdots \arrow[r] & \pi_q(Y') \arrow[d, "\cong"] \\
 & & & H(S^q)
\end{tikzcd}$$

(The isomorphisms in diagram are $T_{u_{q+1},S^q},\ T_{u_{q+2},S^q},\ T_{u',S^q}$).

□

Corollary 4.7. *For any Brown functor H, there exist classifying spaces for H which are CW complexes.*

Proof. Use theorem 4.6 with $Y = *$.

□

Lemma 4.8. *H is a Brown functor, $u \in H(Y)$ is a universal element, $i : A \hookrightarrow X$ is a relative CW-complex. Given map $g : A \to Y$ and $v \in H(X)$ satisfy:*

$$\begin{tikzcd}
 & H(X) \ni v \arrow[d] \\
H(Y) \ni u \arrow[r] & H(A) \ni H(g)(u) = H(i)(v)
\end{tikzcd}$$

Then exists map $g' : X \to Y$ such that $g'|_A = g$ and diagram:

$$\begin{tikzcd}
 & H(X) \ni v = H(g')(u) \arrow[d] \\
H(Y) \ni u \arrow[r] \arrow[ur, "H(g')"] & H(A)
\end{tikzcd}$$

commutes.

Proof. Let (Z, j) be weak coequalizer of the diagram:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "g"'] & X \arrow[d, hook, "i_1"] \\ Y \arrow[r, hook, "i_2"'] & X \vee Y \end{tikzcd}$$

then we have weak equalizer diagram:

$$H(Z) \longrightarrow H(X) \times H(Y) \overset{H(i_1 \circ i)}{\underset{H(i_2 \circ g)}{\rightrightarrows}} H(A)$$

We also have

$$\begin{tikzcd} H(A) & H(X) \arrow[l, "H(i)"'] \\ H(Y) \arrow[u, "H(g)"] & H(X) \times H(Y) \arrow[l, "p_2 = H(i_2)"] \arrow[u, "p_1 = H(i_1)"'] \end{tikzcd}$$

which implies $H(i) \circ H(i_1)(v, u) = H(i)(v) = H(g)(u) = H(g) \circ H(i_2)(v, u)$.
Then there is a element $u^+ \in H(Z)$ such that $H(j)(u^+) = (v, u)$. Use theorem 4.6 to obtain relative CW-complex (Z', Z) and universal element $u' \in H(Z')$ such that $H(i_Z)(u') = u^+$. $(i_Z : Z \hookrightarrow Z')$
By lemma 4.3, $j' := i_Z \circ j \circ i_2 : Y \hookrightarrow X \vee Y \hookrightarrow Z \hookrightarrow Z'$ is a weak equivalence. We also have diagram in $\mathbf{TOP}^{*/}_{\mathbf{CW}}$: (since (Z, j) is weak coequalizer in C_0)

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "g"'] \arrow[dr, phantom, "\simeq"] & X \arrow[d, "i_Z \circ j \circ i_1"] \\ Y \arrow[r, "j'"'] & Z' \end{tikzcd}$$

Apply HELP:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "g"'] & X \arrow[d, "i_Z \circ j \circ i_1"] \arrow[dl, "g'"'] \\ Y \arrow[r, "j'"'] & Z' \end{tikzcd}$$

and verify that $H(g')(u) = H(g') \circ H(j')(u') = H(i_Z \circ j \circ i_1)u' = H(i_1) \circ H(j)(u^+) = H(i_1)(v, u) = v$.

□

Theorem 4.9. *If Y is a classifying space for a Brown functor H and $u \in H(Y)$ is a universal element, then $T_u : [-, Y] \to H(-)$ is a natural isomorphism.*

Proof. $T_{u,X}$ is epimorphism:
For $v \in H(X)$, use lemma 4.8 with $(X, A) := (X, *)$ to obtain a map $g' : X \to Y$ such that $T_{u,X}(g') = H(g')(u) = v$.
$T_{u,X}$ is monomorphism:
Let $f_0, f_1 : X \to Y$ such that $T_{u,X}(f_1) = T_{u,X}(f_2)$.
Define CW-complex $X' := X \times I/\{*\} \times I$ with CW-structure $X'^q = (X^q \times \partial I \cup X^{q-1} \times I)/\{*\} \times I$ for $q \geq 0$.
Define $h : X' \to X$ by $\overline{(x, t)} \mapsto x$ and define $v \in H(X')$ by $v = H(f_0 \circ h)(u)$.
Let $A' := X \vee X = X \times \partial I/\{*\} \times \partial I$, $i : A' \hookrightarrow X'$ and define $f : A' \to Y$ by $(a, 0) \mapsto f_0(a)$, $(a, 1) \mapsto f_1(a)$. Then we have $H(f)(u) = (H(f_0)(u), H(f_1)(u)) = (H(f_0)(u), H(f_0)(u)) = H(f_0 \circ h \circ i)(u) = H(i)(v)$. Use lemma 4.8 with $(X, A) = (X', A')$ to obtain a $f' : X' \to Y$ such that $f'|_{A'} = f$ and $H(f')(u) = v$.
$h : X \times I \to X' \xrightarrow{f'} Y$ is the desired homotopy $g_0 \simeq g_1$.

□

Corollary 4.10. *If Y, Y' are classifying spaces of a Brown functor H, and $u \in H(Y), u' \in H(Y')$ are their universal elements, then there is a homotopy equivalence $f : Y \to Y$ which is unique up to homotopy and satisfy $H(f)(u') = u$.*

Proof. By theorem 4.9, $T_{u',Y} : [Y, Y'] \to H(Y)$ is isomorphism. Then there is unique $f : [Y, Y']$ such that $T_{u',Y}(f) = u$. (notice that $T_{u',Y}(f) = H(f)(u')$) By lemma 4.3 and theorem 1.9, f is homotopy equivalence.

□

Definition 4.7. A **sequential pre-spectrum** in topological spaces is:

- A $\mathbb{N}$-graded compactly generated space : $X_* := \{X_n \in \mathbf{TOP}^{*/}_{\mathbf{CG}}\}_{n\in\mathbb{N}}$.
- Structure maps : $\{\sigma_n : \Sigma X_n \to X_{n+1}\}_{n\in\mathbb{N}}$.

Map between sequential pre-spectra is map between $\mathbb{N}$-graded spaces $f_n : X_n \to Y_n$ such that

$$\begin{tikzcd} \Sigma X_n \arrow[r, "\Sigma f_n"] \arrow[d, "\sigma_n"'] & \Sigma Y_n \arrow[d, "\sigma'_n"] \\ X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1} \end{tikzcd}$$

commutes.
An **Ω-prespectrum** is a sequential spectrum X_* with adjoints of structure maps $\sigma_n : X_n \to \Omega X_{n+1}$ are weak equivalences.
For an Ω-prespectrum X_*, we can extend it into a $\mathbb{Z}$-graded space by setting $X_{-n} := \Omega^n X_0$.

Theorem 4.11. *If $(\tilde{E}^*, \sigma)$ is a reduced additive cohomology theory, then there exist homotopically unique Ω-prespectrum Y_* (each Y_n is a CW-complex) such that $E^n(-) \cong [-, Y_n]_*$ naturally. (naturality implies diagram below commutes)*

$$\begin{tikzcd} \tilde{E}^n(-) \arrow[rr] \arrow[d] & & {[-, Y_n]_*} \arrow[d] \\ \tilde{E}^{n+1}(\Sigma(-)) \arrow[r] & {[\Sigma(-), Y_{n+1}]_*} \cong & {[-, \Omega Y_{n+1}]_*} \end{tikzcd}$$

If Y_* is an Ω-prespectrum, then $\tilde{E}^n := [-, Y_n]_*$, $\sigma_n : [-, Y_n]_* \to [-, \Omega Y_{n+1}] \cong [\Sigma(-), Y_{n+1}]$ is a reduced additive cohomology theory.

Definition 4.8. For an abelian group A, the **Eilenberg-Mac Lane prespectrum** KA_* is defined by $KA_n := K(A, n)$. Structure maps is $KA_n \to M \to \Omega KA_{n+1}$ where M is a CW-approximation of ΩKA_{n+1}, and homotopy equivalence $KA_n \to M$ is obtained from corollary 4.10.

Proposition 4.12. *If a reduced additive cohomology thoery $\tilde{H}^*, \sigma$ is ordinary, then $\tilde{H}^n(-) \cong [-, KA_n]_*$ naturally.*

Proof.

5 Towers And Homotopy Limits

5.1 Pointed and Unpointed Homotopy Classes

Proposition 5.1. *There are pointed spaces $X, Y \in \mathbf{TOP}^{*/}$, if X is well-pointed, then there is a right action of $\pi_1(Y, y_0)$ on $[X, Y]_*$.*

Proof. The right action is given by: $[f] \cdot [a] := [\hat{f}_{a,1}]$ where $\hat{f}_{a,1} := \hat{f}_a(-, 1)$

$$\begin{tikzcd} * \arrow[r, "x_0"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "f"] & \\ * \times I \arrow[r] \arrow[drr, bend right, "a\in\pi_1(Y,y_0)"'] & X \times I \arrow[dr, dashed, "\hat{f}_a"] & \\ & & Y \end{tikzcd}$$

Verify it is well-defined:
By the property of closed cofibration, $\hat{f}_a$ is unique up to homotopy, hence independent from choice of $a \in [a]$ and $f \in [f]$.
Verify it is an group action:
If e is the constant loop in $\pi_1(Y, y_0)$, then $\hat{f}_e(x,t) = f(x)$.
If $[a],[b] \in \pi_1(Y, y_0)$ then $[f]\cdot[a] = [\hat{f}_{a,1}]$, $([f]\cdot[a])\cdot[b] = [\widehat{(\hat{f}_{a,1})}_{b,1}]$. define

$$\begin{aligned} h : X \times I &\to Y \\ (x,t) &\mapsto \begin{cases} \hat{f}_a(x,2t) & t \leq 1/2 \\ \widehat{(\hat{f}_{a,1})}_b(x,2t-1) & t \geq 1/2 \end{cases} \end{aligned}$$

Since $h(-,0) = f$, $h(x_0,-) = a\cdot b$ $h \simeq \hat{f}_{a\cdot b}$. $[f]\cdot([a]\cdot[b]) = [\hat{f}_{a\cdot b}(-,1)] = [h(-,1)] = ([f]\cdot[a])\cdot[b]$.
□

Theorem 5.2. *If $X, Y \in \mathbf{TOP}^{*/}$ there is a forgetful map $\phi : [X,Y]_* \to [X,Y]$ where $[X,Y]$ is the **free** homotopy class of (not necessarily pointed) maps $X \to Y$. If (X, x_0) is well-pointed and Y is path-connected, then ϕ induces bijection $\overline{\phi} : [X,Y]_*/\pi_1(Y,y_0) \cong [X,Y]$.*

Proof. $\overline{\phi}$ is well-defined:
For any $a \in \pi_1(Y, y_0)$, f is freely homotopic to $\hat{f}_a(-,1)$.
$\overline{\phi}$ is injective:
If we have $\phi([f]) = \phi([g])$ which means there is free homotopy $h : f \simeq g$, let $a := h(x_0,-)$, then $h \simeq \hat{f}_a$, $[f]\cdot[a] = [h(-,1)] = [g]$.
$\overline{\phi}$ is surjective:
Suppose $g \in \mathrm{Hom}_{\mathbf{TOP}}(X,Y)$ is an unpointed map, choose a path $a : g(x_0) \simeq y_0$, extend a, g:

$$\begin{tikzcd} * \arrow[r, "x_0"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left, "g"] & \\ * \times I \arrow[r] \arrow[drr, bend right, "a"'] & X \times I \arrow[dr, dashed, "h"] & \\ & & Y \end{tikzcd}$$

$\phi([h(-,1)]) = [g]$.
□

Theorem 5.3. *If (W,e) is a well-pointed H-space, $\mu : W \times W \to W$ is its H-multiplication. Then μ is homotopic to another H-multiplication μ' such that $\mu'(-,e) = \mathrm{id}_W = \mu'(e,-)$ is strict identity.*

Proof. Let $l := \mu \circ (e, \mathrm{id}_W) \simeq \mathrm{id}_W$, $r := \mu \circ (\mathrm{id}_W, e) \simeq \mathrm{id}_W$,

$$\begin{aligned} h : W \vee W \times I &\to W \\ (w,e,t) &\mapsto r(w,t) \\ (e,w,t) &\mapsto l(w,t) \end{aligned}$$

Then we have diagram: (since $W \vee W \to W \times W$ is cofib)

$$\begin{tikzcd} W \vee W \arrow[r] \arrow[d] & W \times W \arrow[d] \arrow[ddr, bend left, "\mu"] & \\ W \vee W \times I \arrow[r] \arrow[drr, bend right, "h"'] & W \times W \times I \arrow[dr, dashed, "\hat{h}"] & \\ & & W \end{tikzcd}$$

$\mu' := \hat{h}(-,-,1)$.
□

Proposition 5.4. *If (X, x_0) is well-pointed space, (W, e) is a well-pointed H-space, then $\pi_1(W, e)$ acts trivially on $[X, W]_*$*

Proof. For $[f] \in [X, W]_*$, $a \in \pi_1(W, e)$ define

$$\begin{aligned} h : X \times I &\to W \\ (x, t) &\mapsto \mu'(f(x), a(t)) \end{aligned}$$

$[f] \cdot [a] = [h(-, 1)] = [f]$ since $h \simeq \hat{f}_a$.

□

Corollary 5.5. *If (X, x_0) is well-pointed space, (Y, e) is a well-pointed path-connected H-space, then $\phi : [X, Y]_* \to [X, Y]$ is a bijection.*

Theorem 5.6. *$X \in$ **TOP**, $\pi_{\leq 1}(X)$ is the fundamental groupoid of X there are functors*

$$\begin{aligned} \Psi_n : \pi_{\leq 1}(X) &\to \mathbf{Grp} \\ x_0 &\mapsto \pi_n(X, x_0) \\ \mathrm{Hom}_{\pi_{\leq 1}(X)}(x_0, x_1) \ni [a] &\mapsto ([S^n, X]_* \ni [f] \mapsto [\hat{f}_a(-, 1)]) \end{aligned}$$

with property : for every $f : X \to Y, [a] \in \mathrm{Hom}_{\pi_{\leq 1}(X)}(x_0, x_1)$ diagram

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[r, "\Psi_n(a)"] \arrow[d, "f_*"'] & \pi_n(X, x_1) \arrow[d, "f_*"] \\ \pi_n(Y, f(x_0)) \arrow[r, "\Psi_n(f \circ a)"'] & \pi_n(Y, f(x_1)) \end{tikzcd}$$

commutes.

Lemma 5.7. *Assume X, Y, Z are path-connected and well-pointed. Consider functor $[-, Z]_*$ apply on* ***Barratt-Puppe sequence***

$$\cdots \to [\Sigma Y, Z]_* \xrightarrow{\Sigma f^*} [\Sigma X, Z]_* \xrightarrow{q^*} [C_f, Z]_* \xrightarrow{i^*} [Y, Z]_* \xrightarrow{f^*} [X, Z]_*$$

The sequence is exact (in category $\mathbf{Set}^{*/}$), and we have following by exactness:

1. *$[\Sigma X, Z]_*$ acts from right on $[C_f, Z]_*$.*
2. *$q^* : [\Sigma X, Z]_* \to [C_f, Z]_*$ is a map between right $[\Sigma X, Z]_*$-sets.*
3. *$q^*([x]) = q^*([x'])$ iff exists some $[y] \in [\Sigma Y, Z]_*$ such that $[x] = \Sigma f^*([y]) \cdot [x']$.*
4. *$i^*([z]) = i^*([z'])$ iff exists some $[x] \in [\Sigma X, Z]_*$ such that $[z] = [z'] \cdot [x]$.*
5. *$\mathrm{Im}(\Sigma q^* : [\Sigma^2 X, Z]_* \to [\Sigma C_f, Z]_*)$ is central subgroup of $[\Sigma C_f, Z]_*$.*

Proof.

1. Define h-coaction map:

$$\begin{aligned} u_f : C_f &\to C_f \vee \Sigma X \\ (y, 0) &\mapsto (y, 0) \\ \overline{(x, t)} &\mapsto \begin{cases} \overline{(x, 2t)} \in C_f & t \leq 1/2 \\ \overline{(x, 2t - 1)} \in \Sigma X & t \geq 1/2 \end{cases} \end{aligned}$$

2.

$$\begin{tikzcd} C_f \arrow[r, "q"] \arrow[d, "u_f"'] & \Sigma X \arrow[d, "u_{X \to *}"] \\ C_f \vee \Sigma X \arrow[r, "q \vee \mathrm{id}_{\Sigma X}"'] & \Sigma X \vee \Sigma X \end{tikzcd}$$

3. $q^*([x]) = q^*([x']) \Leftrightarrow q^*([x] \cdot [x']^{-1}) = * \Leftrightarrow$ there exists some $[y] \in [\Sigma Y, Z]_*$ such that $[x] \cdot [x']^{-1} = \Sigma f^*(y)$.

4. If $i^*([z]) = i^*([z'])$, then there are maps $c \simeq z$, $c' \simeq z'$ such that $c|_Y = c'|_Y$. (use HEP) Define

$$\begin{aligned} x : \Sigma X &\to Z \\ \overline{(x,t)} &\mapsto \begin{cases} c'(x, 1-2t) & t \leq 1/2 \\ c(x, 2t-1) & t \geq 1/2 \end{cases} \end{aligned}$$

we have $[c] = [c'] \cdot [x]$.

5. Let $G := [\Sigma C_f, Z]$, $H := \mathrm{Im}(\Sigma q^*)$. Σu_f gives the right action $\star$ of H on G. It is different from the usual product $\cdot$ (given by $v_{\Sigma C_f}$) on G:

$$\begin{aligned} v_{\Sigma C_f} : \Sigma C_f &\to \Sigma C_{f0} \vee \Sigma C_{f1} \\ (c,t) &\mapsto \begin{cases} (c, 2t)_0 & t \leq 1/2 \\ (c, 2t-1)_1 & t \geq 1/2 \end{cases} \\ \Sigma u_f : \Sigma C_f &\to \Sigma C_f \vee \Sigma^2 X \\ \overline{(y,0,t)} &\mapsto \overline{(y,0,t)} \\ \overline{(x,s,t)} &\mapsto \begin{cases} \overline{(x,2s,t)} \in \Sigma C_f & s \leq 1/2 \\ \overline{(x,2s-1,t)} \in \Sigma^2 X & s \geq 1/2 \end{cases} \end{aligned}$$

And we have $(g \star h) \cdot (g' \star h') = (g \cdot g') \star (h \cdot h')$, which is equivalent to commutativity of diagram below.

$$\begin{tikzcd} & \Sigma C_f \arrow[dl, "\Sigma u_f"'] \arrow[dr, "v_{\Sigma C_f}"] & \\ \Sigma C_f \vee \Sigma^2 X \arrow[dr, "v_{\Sigma C_f} \vee v_{\Sigma^2 X}"'] & & \Sigma C_{f0} \vee \Sigma C_{f1} \arrow[dl, "\Sigma u_f \vee \Sigma u_f"] \\ & \Sigma C_{f0} \vee \Sigma C_{f1} \vee \Sigma^2 X_0 \vee \Sigma^2 X_1 & \end{tikzcd}$$

Verify the commutativity:

$$\begin{aligned} C_f &\to \Sigma C_{f0} \vee \Sigma C_{f1} \vee \Sigma^2 X_0 \vee \Sigma^2 X_1 \\ \overline{(y,0,t)} &\mapsto \begin{cases} \overline{(y,0,2t)}_0 & t \leq 1/2 \\ \overline{(y,0,2t-1)}_1 & t \geq 1/2 \end{cases} \\ \overline{(x,s,t)} &\mapsto \begin{cases} \overline{(x,2s,2t)}_0 \in \Sigma C_{f0} & s \leq 1/2, t \leq 1/2 \\ \overline{(x,2s-1,2t)}_0 \in \Sigma^2 X_0 & s \geq 1/2, t \leq 1/2 \\ \overline{(x,2s,2t-1)}_1 \in \Sigma C_{f1} & s \leq 1/2, t \geq 1/2 \\ \overline{(x,2s-1,2t-1)}_1 \in \Sigma^2 X_1 & s \geq 1/2, t \geq 1/2 \end{cases} \end{aligned}$$

Final step:

$$\begin{aligned} g \cdot h &= (g \star 1) \cdot (1 \star h) = (g \cdot 1) \star (1 \cdot h) \\ &= (1 \cdot g) \star (h \cdot 1) = (1 \star h) \cdot (g \star 1) = h \cdot g \end{aligned}$$

□

Lemma 5.8. *Assume X, Y, Z are path-connected and well-pointed. (dual version of lemma 5.7.)*

$$\cdots \to [Z, \Omega X]_* \xrightarrow{\Omega f_*} [Z, \Omega Y]_* \xrightarrow{j_*} [Z, P_f]_* \xrightarrow{p_*} [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

The sequence is exact (in category $\mathbf{Set}^{*/}$), and we have following by exactness:

1. *$[Z, \Omega Y]_*$ acts from right on $[Z, P_f]_*$.*
2. *$j_* : [Z, \Omega Y]_* \to [Z, P_f]_*$ is a map between right $[Z, \Omega Y]_*$-sets.*
3. *$j_*([y]) = j_*([y'])$ iff exists some $[x] \in [Z, \Omega X]_*$ such that $[y] = \Omega f_*([x]) \cdot [y']$.*
4. *$p_*([z]) = p_*([z'])$ iff exists some $[y] \in [Z, \Omega Y]_*$ such that $[z] = [z'] \cdot [y]$.*
5. *$\mathrm{Im}(\Omega j_* : [Z, \Omega^2 Y]_* \to [Z, \Omega P_f]_*)$ is central subgroup of $[Z, \Omega P_f]_*$.*

A Long Proofs

A.1 Proof of Dold-Thom Theorem

A.2 Proof of Homotopy Excision Theorem

Proof. Follow notations in the statement of the theorem. Define (pointed) the triad homotopy group for $q \geq 2$:

$$\pi_q(X; A, B) := \pi_{q-1}(P_{i_{B,X}}, P_{i_{C,A}})$$

where $i_{B,X} : B \hookrightarrow X$, $i_{C,A} : C \hookrightarrow A$ and P_f is the homotopy fiber

$$\{(y, \gamma) \in Y \times M(I, Z)_* \mid \gamma(1) = f(y)\}$$

of pointed map $f : Y \to Z$. Use long exact sequence of pairs:

$$\cdots \to \pi_q(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_{q-1}(P_{i_{C,A}}) \to \pi_{q-1}(P_{i_{B,X}}) \to \pi_{q-1}(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_{q-2}(P_{i_{C,A}}) \to \cdots$$
$$\cdots \to \pi_1(P_{i_{B,X}}, P_{i_{C,A}}) \to \pi_0(P_{i_{C,A}}) \to \pi_0(P_{i_{B,X}})$$

and observe that $\pi_q(P_{i_{X,B}}) \cong \pi_{q+1}(X, B)$ since for any $f : S^q \to P_{i_{X,B}}$ we have:

$$\begin{tikzcd}
S^q \arrow[dr, "f"] \arrow[drr, bend left, "f'"] \arrow[ddr, bend right, "g"'] & & \\
& P_{i_{B,X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M(I, X)_* \arrow[d] \\
& B \arrow[r] & X
\end{tikzcd}$$

use the fact $f' \in M(S^q, M(I, X)_*)_* \cong M(S^q \wedge I, X)_* \ni f''$ and $S^q \wedge I \approx D^{q+1}$ with

$$S^q \hookrightarrow S^q \wedge I \approx D^{q+1}$$
$$s \mapsto (s, 1)$$

the condition $f'(s)(1) = g(s)$ is equivalent to $f''((s, 1)) = g(s)$, that is have a map f is equivalent to have a map $f'' : (D^{q+1}, S^q) \to (X, B)$. With the analogue statement also valid for homotopies $S^q \times I \to P_{i_{X,B}}$, we have $\pi_q(P_{i_{B,X}}) = [S^q, *; P_{i_{B,X}}, *] \cong [D^{q+1}, S^q; X, B] = \pi_{q+1}(X, B)$.
Rewrites the long exact sequence of pairs above to:

$$\cdots \to \pi_{q+1}(X; A, B) \to \pi_q(A, C) \to \pi_q(X, B) \to \pi_q(X; A, B) \to \pi_{q-1}(A, C) \to \cdots$$
$$\cdots \to \pi_2(X; A, B) \to \pi_1(A, C) \to \pi_1(X, B)$$

Conditions $m \geq 1, n \geq 1$ guarantees $\pi_0(C) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(B)$ are surjections.
$m \geq 2$ is equivalent to $\pi_1(A, C) = 0$, which implies $\pi_0(C) \to \pi_0(A)$ is bijection.
For $x \in \pi_0(A \cap_C B)$, we can always find $b \in \pi_0(B), i_{B,X\ *}(b) = x$ or $a \in \pi_0(A), i_{A,X\ *}(a) = x$ which becomes $b \in \pi_0(B), i_{B,X\ *}(b) = x$ or $c \in \pi_0(C), i_{C,X\ *}(c) = x$ when $\pi_0(C) \to \pi_0(A)$ is bijection. That is equivalent to $\pi_0(B) \to \pi_0(X)$ is bijection, which means $\pi_1(X, B) = 0$.

We only need to show that for $2 \leq q \leq m + n - 2$, $\pi_q(X; A, B) = 0$.

With $J^{q-1} := (\partial I^{q-1} \times I) \cup (I^{q-1} \times \{0\})$, we have:

$$\begin{aligned}
\pi_q(P_{i_{B,X}}, P_{i_{C,A}}) &= [I^q, \partial I^q, J^{q-1}; P_{i_{B,X}}, P_{i_{C,A}}, *] \\
&= [I^q \wedge I;\ I^q,\ \partial I^q \wedge I,\ J^{q-1} \wedge I \to X; B, A, *] \\
&:= \text{relative homotopy classes of pointed maps} \\
&\qquad f : I^q \wedge I \to X \text{ satisfying: } \begin{cases} f(I^q) & \subseteq B \\ f(\partial I^q \wedge I) & \subseteq A \\ f(\partial I^q) & \subseteq C \\ f(J^{q-1} \wedge I) & = * \end{cases} \\
&\qquad \text{“relative” means the homotopy } h \text{ determine the classes} \\
&\qquad \text{satisfy: } \begin{cases} h(I^q \times I) & \subseteq B \\ h((\partial I^q \wedge I) \times I) & \subseteq A \\ h(\partial I^q \times I) & \subseteq C \\ h((J^{q-1} \wedge I) \times I) & = * \end{cases} \\
&\qquad (\text{notice that } \partial I^q \wedge I \cap I^q = \partial I^q, \text{ therefore } f(\partial I^q) \subseteq A \cap B = C) \\
&\qquad\quad (\text{this is called (relative) homotopy class of maps of tetrads}) \\
&= [(I^q \times I)/K;\ I^q \times \{1\},\ (\partial I^q \times I)/K,\ (J^{q-1} \times I)/K \to X; B, A, *] \\
&\qquad (K := I^q \times \{0\} \cup \{i_0\} \times I) \\
&= [I^{q+1};\ (I^q \times \{1\}) \cup K,\ (\partial I^q \times I) \cup K,\ J^{q-1} \times I \cup K \to X; B, A, *] \\
&= [I^{q+1};\ I^q \times \{1\},\ I^{q-1} \times \{1\} \times I,\ J^{q-1} \times I \cup I^q \times \{0\} \to X; B, A, *] \\
&\qquad (\text{notice that } \partial I^q = \partial I^{q-1} \times I \cup I^{q-1} \times \{0, 1\})
\end{aligned}$$

We can assume that (A, C) have no relative $q < m$-cells and (B, C) have no relative $q < n$-cells. And we can assume that X has finite many cells since I^q is compact.
For subcomplexes $C \subseteq A' \subseteq A$, where $A = e^m \cup A'$ (attaching one cell from A').
Let $X' := A' \cup_C B$, if the results hold for $(X'; A', B)$ and $(X; A, X')$, then it hold for $(X; A, B)$ since we have map between exact homotopy sequences of triples (A, A', C) and (X, X', B):

$$\begin{tikzcd}
\pi_{q+1}(A, A') \arrow[r] \arrow[d, "i_{2,q+1}"'] & \pi_q(A', C) \arrow[r] \arrow[d, "i_{1,q}"'] & \pi_q(A, C) \arrow[r] \arrow[d, "i_{3,q}"'] & \pi_q(A, A') \arrow[r] \arrow[d, "i_{2,q}"'] & \pi_{q-1}(A', C) \arrow[d, "i_{1,q-1}"'] \\
\pi_{q+1}(X, X') \arrow[r] & \pi_q(X', B) \arrow[r] & \pi_q(X, B) \arrow[r] & \pi_q(X, X') \arrow[r] & \pi_{q-1}(X', B)
\end{tikzcd}$$

induced by inclusion $(A, A', C) \hookrightarrow (X, X', B)$. If the result hold for $(X'; A', B)$ and $(X; A, X')$, maps $i_{1,q},\ i_{2,q}$ are isomorphisms when $1 \geq q \geq m + n - 3$, are epimorphisms when $q = m + n - 2$.
Notice the 5-lemma says that
if $i_{1,q}$ and $i_{2,q}$ are epimorphisms, $i_{1,q-1}$ are monomorphism, then $i_{3,q}$ is epimorphism.
if $i_{1,q}$ and $i_{2,q}$ are monomorphisms, $i_{2,q+1}$ are epimorphism, then $i_{3,q}$ is monomorphism.
We also have if $C \subseteq B' \subseteq B$ with $B = B' \cup e^n$, the result hold for CW-triads $(X'; A, B')$ and $(X; X', B)$ where $X' = A \cup_C B'$, since $(A, C) \hookrightarrow (X, B)$ factors as $(A, C) \hookrightarrow (X', B') \hookrightarrow (X, B)$.

Now we can assume that $A = C \cup D^m$ and $B = C \cup D^n$.

The current goal of proof is to prove any

$$f : (I^{q+1};\ I^q \times \{1\},\ I^{q-1} \times \{1\} \times I,\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X; B, A, *)$$

is nullhomotopic for any $q + 1$ with $2 \leq q + 1 \leq m + n - 2$.

For $a \in \overset{\circ}{D^m}$, $b \in \overset{\circ}{D^n}$ We have inclusions of based triads:

$$(A; A, A - \{a\}) \hookrightarrow (X - \{b\}; X - \{b\}, X - \{a, b\}) \hookrightarrow (X; X - \{b\}, X - \{a\}) \hookleftarrow (X; A, B)$$

The first and the third induces isomorphisms on homotopy groups of triads since B is a strong deformation retract of $X - \{a\}$ in X and A is a strong deformation retract of $X - \{b\}$ in X. $\pi_*(A; A, A - \{a\}) = 0$ since $\pi_*(A, A - \{a\}) \to \pi_*(A, A \cap (A - \{a\}))$ are isomorphisms.

Current goal : choose good a, b to show f regarded as a pointed traid map to $(X; X - \{b\}, X - \{a\})$ is homotopic to a map

$$f' : (I^{q+1};\ I^{q-1} \times \{1\} \times I,\ I^q \times \{1\},\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X - \{b\}; X - \{b\}, X - \{a, b\}, *)$$

if $2 \le q + 1 \le m + n - 2$.

Note. We want to homotopically remove some point $f^{-1}(b)$, first we may want to construct some Urysson function u separating $f^{-1}(a) \cup J^{q-1} \times I \cup I^q \times \{0\}$ and $f^{-1}(b)$ and construct homotopy of cube $h^+ : (r, s, t) \mapsto (r, (1 - u(r, s)t)s)$ wishing that $f(h^+(r, s, 1))$ would miss b. The problem in this method is that points $f^{-1}(b)$ in the cube would be homotopically replaced by other points. Since our desire homotopy does not change the first q coordinates of the cube, we want to separate $p^{-1}(p(f^{-1}(a))) \cup J^{q-1} \times I$ and $p^{-1}(p(f^{-1}(b)))$ (where $p : I^q \times I \to I^q$). Our problem is to find suitable a, b such that $p(f^{-1}(a)) \cap p(f^{-1}(b)) = \emptyset$.

We use manifold structure on D^m and D^n to achieve it, now we homotopically approximate f by a map g which smooth on $f^{-1}(D^m_{<1/2} \cup D^n_{<1/2})$.

Let $U_{<r} := f^{-1}(D^m_{<r} \cup D^n_{<r})$, Use smooth deformation theorem to construct smooth map (for any $0 < \epsilon$) $g' : U_{<3/4} \to D^m_{<3/4} \cup D^n_{<3/4}$ with homotopy $h_1 : g' \simeq f|_{U_{<3/4}}$ (and bound $|g'(x) - f(x)| < \epsilon$ for any $x \in U_{<1}$) and take partition of unity $\{\rho, \rho'\}$ with subcoordinates $\{I^{q+1} - \overline{U_{<1/2}},\ U_{<3/4}\}$, we have:

$$\begin{aligned} g &:= \rho f + \rho' g' \\ h_2 &: g \simeq f \operatorname{rel} (I^{q+1} - U_{<3/4}) \\ h_2 &: I^{q+1} \times I \to X \\ &(x, t) \mapsto \rho(x) f(x) + \rho'(x) h_1(x, t) \end{aligned}$$

with scalar multiplication and addition is already defined on smooth structure on $D^m_{<3/4} \cup D^n_{<3/4}$. We could assmue that $g(I^{q-1} \times \{1\} \times I) \cap D^n_{<1/2} = \emptyset$ (which implies g is a map of tetrads to $(X; X - \{b\}, X - \{a\}, *)$) and $g(I^q \times \{1\}) \cap D^m_{<1/2} = \emptyset$ since $f(I^{q-1} \times \{1\} \times I) \subseteq A$ and $f(I^q \times \{1\}) \subseteq B$ and we can always tighten the bound ϵ, (Similar argument also hold for h_2, then we have $h_2 : g \simeq f$ as homotopy between maps of tetrads.)

Use the manifold structure to find good (a, b):
$V := g^{-1}(D^m_{<1/2}) \times g^{-1}(D^n_{<1/2})$ is a sub-manifold of $I^{2(q+1)}$. Consider $W := \{(v, v') \in V \mid p(v) = p(v')\}$, which is the zero set of smooth submersion $(v, v') \mapsto p(v) - p(v')$. W is smooth manifold with codimension q. Therefore the map $(g, g) : W \to D^m_{<1/2} \times D^n_{<1/2}$ is smooth map between manifolds of dimension $q + 2$ and $m + n$. The map is not surjection since $q + 2 < m + n$. Then we have $(a, b) \notin (g, g)(W)$ (that is, $p(g^{-1}(a)) \cap p(g^{-1}(b))$).

Since $g(I^{q-1} \times \{1\} \times I) \cap D^n_{<1/2} = \emptyset$ and $g(J^{q-1} \times I) \cap D^n_{<1/2} = \emptyset$, we have $g(\partial I^q \times I) \cap D^n_{<1/2} = \emptyset$. By Uryssohn's lemma, we have $u : I^q \to I$ separating $p(g^{-1}(a)) \cup \partial I^q$ and $p(g^{-1}(b))$. Finally we have:

$$\begin{aligned} h' &: I^q \times I \times I \to I^q \times I \\ &(r, s, t) \mapsto (r, (1 - u(r)t)s) \end{aligned}$$

and $h := g \circ h'$, $f' := h(-, 1)$. $f'(I^{q+1}) \cap \{b\} = \emptyset$ since if $\exists (r, s) \in I^q \times I$, $f'(r, s) = b$, then $b = g(r, (1 - u(r))s) = g(r, 0) = *$ leads to contradiction.
Last step is to check that h is a homotopy between maps

$$(I^{q+1};\ I^{q-1} \times \{1\} \times I,\ I^q \times \{1\},\ J^{q-1} \times I \cup I^q \times \{0\}) \to (X; X - \{b\}, X - \{a\}, *)$$

Since g is, $g \circ h'$ is too.

$\square$