

Manifolds

Cloudifold

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0 Manifolds and Maps

0.1 Manifolds

Definition 0.1. A map $f : U \rightarrow \mathbb{R}^m$ (where U is an open subset of \mathbb{R}^n) is called C^k if the k -th derivative of f exist and continuous.

f is called C^∞ or **smooth** if f is C^k for any $k \in \mathbb{N}$.

$f : U \rightarrow V$ is called C^ω or **analytic** if f is C^∞ and for all $x \in U$, there exists open neighborhood U_x such that the Taylor series expansion of f at x pointwise converges to f on U_x .

Definition 0.2. For a C^r ($r \geq 1$) map $f : U \rightarrow \mathbb{R}^m$ (where U is an open subset of \mathbb{R}^n), $y \in f(U)$ is called a **regular value** of f if for any $x \in f^{-1}(y)$, the rank of the Jacobian matrix of f at x is m .

Note. By **Weierstrass M-test**, the *pointwise converge* condition in the definition of C^ω function can be replaced equivalently by *uniform converge*.

Definition 0.3. An n -dimensional C^r manifold (we assume $r \geq 1$) (M, A) is datas:

- A T_2 space M whose topology have a countable basis
- $A = \{(U_j, \varphi_j)\}_{j \in J}$, which consists of open cover $\{U_j\}_{j \in J}$ of M and homeomorphisms $\{\varphi_j : U_j \rightarrow O_j\}_{j \in J}$. (where O_j is an open subset of \mathbb{R}^n)

with condition:

- For all $i, j \in J$, the homeomorphism $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$ is C^r . (these maps are called **transition maps**)

A single (U_j, φ_j) is called a **chart**, A is called (C^r) **atlas** or **differential structure**.

Note. In definition above, the condition $\varphi_j(U_j)$ open in \mathbb{R}^n can be replaced equivalently by $\varphi_j(U_j) = \mathbb{R}^n$ since we can use $\varphi_j^{-1}(B_r(\varphi_j(x)))$ to produce a finer cover, and \mathbb{R}^n is C^ω diffeomorphic to $B_r(y)$. (by $x \mapsto y + r \arctan(\|x\|) \cdot x$)

Note. Manifolds defined above are locally compact, hence paracompact and compactly generated Hausdorff.

Definition 0.4. Suppose M, N are C^r manifolds with dimensions n, k and atlas $A = \{(U_j, \varphi_j)\}_{j \in J}$, $B = \{(V_i, \phi_i)\}_{i \in I}$. The **product of two manifolds** is a $(n + k)$ -dimensional manifold with underlying space $M \times N$ and atlas $A \times B := \{(U_j \times V_i, \varphi_j \times \phi_i)\}_{(j,i) \in J \times I}$.

Definition 0.5. Suppose N is C^r manifolds with dimension n and atlas $A = \{(U_j, \varphi_j)\}_{j \in J}$, M is another manifold. $h : M \hookrightarrow N$ is topological embedding. The **induced atlas** on M is atlas $h^*A := \{(h^{-1}(U_j), \varphi_j \circ h)\}_{j \in J}$.

Definition 0.6. Suppose N is C^r manifolds with dimension n and atlas $A = \{(U_j, \varphi_j)\}_{j \in J}$, M is subspace of N . M is a C^r **submanifold of dimension k** of (N, A) if for any $x \in M$ there exists a chart (U_j, φ_j) such that $x \in U_j$ and

$$\varphi_j(M \cap U_j) = \phi_j(U_j) \cap \mathbb{R}^k$$

Such charts combine together give a atlas of M .

We also say that M is a submanifold of **codimension $n - k$** .

Note. Open subset U of a n -dimensional manifold M is obviously a submanifold of dimension n . Submanifold is invariant under diffeomorphisms.

Example 0.1. The **real general linear group** $GL_n(\mathbb{R})$ is an open analytic submanifold of $(n \times n)$ -dimensional manifold $M_n(\mathbb{R})$ since $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ and the manifold structure (and topology) of $Mat_{n \times n}(\mathbb{R})$ is given by bijection in **(Set)** $Mat_{n \times n}(\mathbb{R}) \xrightarrow{\sim} \mathbb{R}^{n \times n}$.

Example 0.2. The (non-compact) **real Stiefel manifold** $St_k(\mathbb{R}^n)$ is the set of \mathbb{R} -linear monomorphisms $\mathbb{R}^k \rightarrow \mathbb{R}^n$, which is equivalent (in **Set**) to the set of linearly independent elements

$$(a_1, \dots, a_k) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n = \mathbb{R}^{n \times k}$$

(also called set of k -frames in \mathbb{R}^n)

$\text{St}_k(\mathbb{R}^n)$ is an open analytic submanifold of $\mathbb{R}^{n \times k}$, since $\text{St}_k(\mathbb{R}^n) = \bigcap_{1 \leq r \leq n-k+1} \det_r^{-1}(\mathbb{R} - \{0\})$.

Where \det_r is defined by:

$$\det_r \left(\begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ a_{2,1} & \cdots & a_{2,k} \\ \vdots & & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,k} \\ a_{n,1} & \cdots & a_{n,k} \end{bmatrix} \right) := \det \left(\begin{bmatrix} a_{r,1} & \cdots & a_{r,k} \\ a_{r+1,1} & \cdots & a_{r+1,k} \\ \vdots & & \vdots \\ a_{r+k-1,1} & \cdots & a_{r+k-1,k} \end{bmatrix} \right)$$

Example 0.3. The **Grassmann manifold** $\text{Gr}_k(\mathbb{R}^n) = \text{St}_k(\mathbb{R}^n) / \text{GL}_k(\mathbb{R})$ is the set k -dimensional subspaces of \mathbb{R}^n . Define $U_V := \{W \in \text{Gr}_k(\mathbb{R}^n) \mid \text{rank}(p_V|_W : W \rightarrow V) = k\}$, where $p_V : \mathbb{R}^n \rightarrow V$ is the orthogonal projection. Let $i_W : V \rightarrow W$ be inverse of $p_V|_W$, define bijection:

$$\begin{aligned} U_V &\xrightarrow{\varphi_V} \text{Hom}_{\mathbb{R}}(V, V^\perp) \approx \mathbb{R}^{k \times (n-k)} \\ W &\longmapsto p_{V^\perp} \circ i_W \\ \text{Im}(\text{id}_V \times M) &\longleftarrow M \end{aligned}$$

Where $p_{V^\perp} : \mathbb{R}^n = V \oplus V^\perp \rightarrow V^\perp$.

Atlas on $\text{Gr}_k(\mathbb{R}^n)$ is given by $\{(U_V, \varphi_V)\}_{V \in \text{Gr}_k(\mathbb{R}^n)}$.

We verify that the transition maps are analytic. Suppose E is spanned by $\mathbf{e}_1, \dots, \mathbf{e}_k$, V have orthonormal basis $[\mathbf{v}_1, \dots, \mathbf{v}_k]$ and V^\perp have orthonormal basis $[\mathbf{v}_{k+1}, \dots, \mathbf{v}_n]$. Let $\text{GL}_n(\mathbb{R}) \ni T : E \oplus E^\perp \rightarrow V \oplus V^\perp$ be the transformation sends each \mathbf{e}_i to \mathbf{v}_i . Let $[T]$ be its matrix respect to $[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]$, and $[T^{-1}]$ is matrix of its inverse. then p_{V^\perp} is given by

$$a = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n] \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mapsto [\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \dots, \mathbf{v}_n] [T^{-1}]_{i>k} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Suppose $M \in \text{Hom}_{\mathbb{R}}(E, E^\perp)$ satisfy $W := \text{Im}(\text{id}_E \times M) \in U_V$, (that is $M \in \varphi_E^{-1}(U_E \cap U_V)$) let $[m_{i,j}]_{k \leq i \leq n, 1 \leq j \leq k}$ be its matrix respect to $[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n]$. Then $p_V|_W$ is given by:

$$a = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n] \begin{bmatrix} I_k \\ M \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix} \longmapsto [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k] [T^{-1}]_{i \leq k} \begin{bmatrix} I_k \\ M \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$$

And i_W is given by:

$$b = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \longmapsto [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n] \begin{bmatrix} I_k \\ M \end{bmatrix} \left([T^{-1}]_{i \leq k} \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$$

Finally, we see that $p_{V^\perp} \circ i_W$ is:

$$b = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \longmapsto [\mathbf{v}_{k+1}, \mathbf{v}_{k+2}, \dots, \mathbf{v}_n] [T^{-1}]_{i>k} \begin{bmatrix} I_k \\ M \end{bmatrix} \left([T^{-1}]_{i \leq k} \begin{bmatrix} I_k \\ M \end{bmatrix} \right)^{-1} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$$

and it is an analytic function on M .

0.2 Maps

Definition 0.7. Let $(M, A), (N, B)$ be C^r -manifolds, a continuous map $f : M \rightarrow N$ is said to be C^k ($k \leq r$) if for any charts $(U, \varphi_U) \in A, (V, \phi_V) \in B$, map

$$\phi_V \circ f \circ \varphi_U^{-1} : \varphi_U(U \cap f^{-1}(V)) \rightarrow \phi_V(V)$$

is C^k .

A C^r -**diffeomorphism** between two C^r -manifolds M, N is a C^r -map $f : M \rightarrow N$ with its C^r -inverse $f^{-1} : N \rightarrow M$.

If C^r -diffeomorphism between C^r -manifolds M, N exists, then M, N are said to be C^r -**diffeomorphic**. Set of C^k maps from M to N is noted $\text{Hom}_{C^k}(M, N)$.

Note. A C^r map which is a C^1 -diffeomorphism is a C^r -diffeomorphism.

Example 0.4. $(-)^{\perp} : \text{Gr}_k(\mathbb{R}^n) \rightarrow \text{Gr}_{n-k}(\mathbb{R}^n)$ is a C^{ω} -diffeomorphism.

$$\varphi_{V^{\perp}} \circ (-)^{\perp} \circ \varphi_V^{-1} : \text{Mat}_{k \times (n-k)}(\mathbb{R}) \ni M \mapsto M^T \in \text{Mat}_{(n-k) \times k}(\mathbb{R})$$

Definition 0.8. Let $(M, A), (N, B)$ be C^r -manifolds, a C^r map $f : M \rightarrow N$ is said to be a (C^r) **local diffeomorphism at** $x \in M$ if exists openset $U \ni x$, $f(U)$ is open in N and $f|_U : U \rightarrow f(U)$ is a C^r -diffeomorphism.

A C^r -map f is said to be a **local diffeomorphism** if for all $x \in M$, f is a local diffeomorphism at x .

Theorem 0.1. Inverse Function Theorem

Suppose W is open subset of \mathbb{R}^m , $f : W \rightarrow \mathbb{R}^n$ is a C^r map. If $df_p : TW_p \rightarrow T\mathbb{R}_{f(p)}^n$ is an isomorphism, then f is a local diffeomorphism at p .

Definition 0.9. Let (M, A) be a C^r -manifold, $x \in M$ the **tangency relation** defined on $\text{Hom}_{C^1}(\mathbb{R}|_{\sim 0}, M)_x := \{\gamma \in \text{Hom}_{C^1}(O, M) \mid 0 \in O \in \tau_{\mathbb{R}}, \gamma(0) = x\}$ (τ_X is set of all open sets of X) is equivalence relation which have two equivalent definitions:

1. $\gamma_1(t) \sim_T \gamma_2(t)$ if there exists chart $(U, \varphi_U) \in A$ such that $x \in U$ and

$$\frac{d(\varphi_U \circ \gamma_1)}{dt} \Big|_{t=0} = \frac{d(\varphi_U \circ \gamma_2)}{dt} \Big|_{t=0}$$

2. $\gamma_1(t) \sim_T \gamma_2(t)$ if for any chart $(U, \varphi_U) \in A$ such that $x \in U$, we have

$$\frac{d(\varphi_U \circ \gamma_1)}{dt} \Big|_{t=0} = \frac{d(\varphi_U \circ \gamma_2)}{dt} \Big|_{t=0}$$

Note. 2. \implies 1. is obvious.

1. \implies 2. because transition maps are C^r -diffeomorphisms, whose derivatives are linear isomorphisms.

Definition 0.10. Let (M, A) be a n -dimensional C^r -manifold, $x \in M$ the **space of tangent vectors** on M at x is defined by: $T_x(M) := \text{Hom}_{C^1}(\mathbb{R}|_{\sim 0}, M)_x / \sim_T$.

Note. If (U, φ_U) is a chart with $x \in U$, then φ_U defines a bijection $d(\varphi_U \circ -) : T_x(M) \rightarrow \mathbb{R}^n$ by:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\quad} & T_x(M) \\ \Downarrow & & \Downarrow \\ \mathbf{v} & \longmapsto & [\gamma_{\mathbf{v}}]_{\sim_T} \end{array} \quad \begin{array}{ccc} f_{\mathbf{v}} : \mathbb{R} & \xrightarrow{\quad} & \mathbb{R}^n \\ \Downarrow & & \Downarrow \\ t & \longmapsto & \varphi_U(x) + t\mathbf{v} \end{array}$$

$$\frac{d(\varphi_U \circ \gamma)}{dt} \longleftarrow [\gamma]_{\sim_T} \quad \gamma_{\mathbf{v}} : f_{\mathbf{v}}^{-1}(\varphi_U(x)) \xrightarrow{f_{\mathbf{v}}} \varphi_U(x) \xrightarrow{\varphi_U^{-1}} U$$

If (V, φ_V) is another chart with $x \in V$, then we have commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{d(\varphi_V \circ \varphi_U^{-1})|_y} & \mathbb{R}^n \\ & \searrow T\varphi_U^{-1} & \swarrow T\varphi_U^{-1} \\ & T_x(M) & \end{array}$$

where $y = \varphi_U^{-1}(x)$.

Definition 0.11. Let (M, A) be an n -dimensional C^{r+1} -manifold, the **space of all tangent vectors** of M is a $2n$ -dimensional C^r -manifold defined by $T(M) := \bigsqcup_{x \in M} T_x(M)$ with atlas $\{(T(U), \phi_U)\}_{U \in A}$ (called **natural atlas**) defined by

$$\begin{array}{ccc} \bigsqcup_{x \in U} T_x(M) & \xrightarrow{\phi_U} & \varphi_U(U) \times \mathbb{R}^n \\ \Psi & & \Psi \\ (x, [\gamma]_{\sim_T}) & \longmapsto & (\varphi_U(x), d(\varphi_U \circ \gamma)) \end{array}$$

The **tangent bundle** on M is the bundle

$$\begin{array}{ccc} p_M : T(M) & \rightarrow & M \\ (x, [\gamma]_{\sim_T}) & \mapsto & x \end{array}$$

Definition 0.12. Let $(M, A), (N, B)$ be two C^{r+1} -manifolds, and $f : M \rightarrow N$ is an C^{r+1} map. Define an C^r map between bundles $Tf : TM \rightarrow TN$ by

$$\begin{array}{ccc} Tf : TM & \longrightarrow & TN \\ \Psi & & \Psi \\ (x, [\gamma]_{\sim_T}) & \longmapsto & (f(x), [f \circ \gamma]_{\sim_T}) \\ \downarrow \uparrow & & \downarrow \uparrow \\ (\varphi_U(x), d(\varphi_U \circ \gamma)) & \longmapsto & (\phi_V(f(x)), d(\phi_V \circ f \circ \varphi_U^{-1}) \circ d(\varphi_U \circ \gamma)) \end{array}$$

Note. In fact, T is a functor from category of C^{r+1} manifolds to C^r manifolds. And there is natural diffeomorphism $T(M \times N) \approx_d TM \times TN$.

Example 0.5. Use atlas of $S^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sqrt{\sum_i x_i^2} = 1\}$ given by stereographic projections:

$$\begin{array}{l} U_0 := S^n - (1, 0, \dots, 0) \xrightarrow{\varphi_0} \mathbb{R}^n \\ (x_0, x_1, \dots, x_n) \mapsto \left(\frac{x_1}{1-x_0}, \dots, \frac{x_n}{1-x_0} \right) \\ \left(\frac{s^2-1}{s^2+1}, \frac{2x'_1}{s^2+1}, \dots, \frac{2x'_n}{s^2+1} \right) \longleftarrow (x'_1, \dots, x'_n) \\ \text{where } s^2 := \sum_{1 \leq i \leq n} x_i'^2 = \frac{1+x_0}{1-x_0} \\ U_\infty := S^n - (-1, 0, \dots, 0) \xrightarrow{\varphi_\infty} \mathbb{R}^n \\ (x_0, x_1, \dots, x_n) \mapsto \left(\frac{x_1}{1+x_0}, \dots, \frac{x_n}{1+x_0} \right) \\ \left(\frac{1-s'^2}{s'^2+1}, \frac{2x'_1}{s'^2+1}, \dots, \frac{2x'_n}{s'^2+1} \right) \longleftarrow (x'_1, \dots, x'_n) \\ \text{where } s'^2 := \sum_{1 \leq i \leq n} x_i'^2 = \frac{1-x_0}{1+x_0} \end{array}$$

There is a diffeomorphism $TS^n \times \mathbb{R} \approx_d S^n \times \mathbb{R}^{n+1}$ defined as:

$$\begin{array}{ccc} TU_\lambda \times \mathbb{R} \approx \varphi_\lambda(U_\lambda) \times \mathbb{R}^n \times \mathbb{R} & \rightarrow & U_\lambda \times \mathbb{R}^{n+1} \\ (\mathbf{x}', \mathbf{v}, a) & \mapsto & (\varphi_\lambda^{-1}(\mathbf{x}'), d(\varphi_\lambda^{-1})\mathbf{v} + a \cdot \mathbf{e}_K) \\ (\varphi_\lambda(\mathbf{x}), d(\varphi_\lambda)\mathbf{v}', p_K \mathbf{v}') & \longleftarrow & (\mathbf{x}, \mathbf{v}') \end{array}$$

where $\lambda = 0$ or ∞ , $K = \ker d(\varphi_0) = \ker d(\varphi_\infty)$ is a 1-dimensional linear subspace, \mathbf{e}_K is the unit vector in it, and p_K is the orthogonal projection onto it. It is easy to check it is compatible with transition maps.

Note. In fact, embedding $i : M \hookrightarrow \mathbb{R}^k$ produces injective linear maps $T_x(M) \rightarrow T_{i(x)}\mathbb{R}^k$

Definition 0.13. Let $(M, A), (N, B)$ be two C^r -manifolds, and $f : M \rightarrow N$ is an C^1 map. We call f is **immersive** at $x \in M$ if $T_x f : T_x M \rightarrow T_x N$ is monomorphism, and **submersive** if $T_x f$ is epimorphism.

We call f is an **immersion/submersion** if for all $x \in M$, f is immersive/submersive at x .

We call f is an **embedding** if f is immersion and $f : M \rightarrow f(M)$ is homeomorphism (between topological spaces).

Theorem 0.2. N is a C^r -manifold, A subspace $A \subseteq N$ is a submanifold of N iff A is the image of a C^r -embedding.

Proof. \implies : trivial

\impliedby : Suppose $A = f(M)$, $\{(N_i, \varphi_i : N_i \rightarrow \mathbb{R}^n)\}_{i \in I}$ is charts of N which covers $f(M)$. Then find $A_M := \{(M_i, \phi_i : M_i \rightarrow \mathbb{R}^k)\}_{i \in I}$, an atlas of M such that $f(M_i) = A \cap N_i$. (we could find such A_M since f is an embedding) It is enough to prove that images of C^r embeddings $f_i := \varphi_i \circ f \circ \phi_i^{-1} : \phi_i(M_i) \rightarrow \mathbb{R}^n$ are submanifolds.