Funtion Spaces

Cloudifold

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0 Notations

Category of sets	: **Set**
Category of topological spaces	: **Top**
Category of (one-point-)based topological spaces	: $\mathbf{Top}_*$
Topological space X with topology $\mathcal{T}$	: $X_{\mathcal{T}}$
Euclidean space of dimension n	: $\mathbb{R}^n$
Unit cube of dimension n	: I^n
Boundary of I^n	: ∂I^n
Unit interval I	: $I = I^1$
Unit cell of dimension n	: $\mathring{\mathbb{D}}^n$
Unit disk of dimension n	: $\mathbb{D}^n$
Unit sphere of dimension $n-1$	: $\mathbb{S}^{n-1}$
Inclusion or Embedding	: $\hookrightarrow$
Monomorphsim	: $\rightarrowtail$
Epimorphsim	: $\twoheadrightarrow$
Hom functor of category $\mathcal{C}$	: $\mathrm{Hom}_{\mathcal{C}}(-,-)$
Limit (inverse limit) (projective limit)	: $\varprojlim$
Colimit (direct limit) (inductive limit)	: $\varinjlim$

1 Funtion Spaces

1.0 Introduction

Function spaces, are origins of many important constructions such as Loop spaces, Path spaces and so on. The duality between funtion spaces and product spaces will [todo]

1.1 Admissible Topology

Definition 1.1. A topology on $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$ is **admissible** if the evaluation funtion ev is **continuous**. Where ev is defined by :

$$\begin{aligned} ev : \mathrm{Hom}_{\mathbf{Top}}(X,Y) \times X &\to Y \\ (f,x) &\mapsto f(x) \end{aligned}$$

Note. It is possible that $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$ have **no** admissible topology.

1.2 Compact-Open Topology

Definition 1.2. The **compact-open** topology on $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$ is generated by subbase $\{O^K\}$ where K varies on all compact subsets of X, O varies on all open subsets of Y. The definition of O^K is :

$$O^K := \{f \in \mathrm{Hom}_{\mathbf{Top}}(X,Y) \mid f(K) \subseteq O\}$$

We note the compact-open topology by $\mathcal{T}_{co}$

Proposition 1.1. *Property of* **compact-open** *topology : The compact-open topology is coarser than any admissible topology. (That is, for any admissible topology $\mathcal{T}$, $\mathcal{T}_{co} \subseteq \mathcal{T}$)*

Proof. We have to show that any open set in $\mathcal{T}_{co}$ is open in $\mathcal{T}$ if $\mathcal{T}$ is admissible. It suffices to show that every $O^K \in \mathcal{T}$. By definition, we have:

$$ev : \mathrm{Hom}_{\mathbf{Top}}(X,Y)_{\mathcal{T}} \times X \to Y$$

is continuous. Take $k \in K$ and $f \in O^K$ (That is, $f(K) \in O$).

By ev is continuous, $ev(f,k) = f(k) \in O$ and the property of the base of finite product topology, we have

$$\exists V_{f,k}, W_k \ .\ f \in V_{f,k} \in \mathcal{T} \text{ and } k \in W_k \in \mathcal{T}_Y \text{ and } ev(V_{f,k} \times W_k) \subseteq O$$

The family $\{W_k\}_{k\in K}$ is an open cover of K. By compactness of K, There exists a finite subcover $\{W_{k_i}\}_{i=1,\dots,n\ (k_i \in K)}$. Put $V_f := \bigcap\{V_{f,k_i}\}_{i=1,\dots,n}$ (with $ev(V_{f,k_i} \times W_{k_i}) \subseteq O$), we have $f \in V_f$ and V_f is open in $\mathrm{Hom}_{\mathbf{Top}}(X,Y)_{\mathcal{T}}$.

Then we have $V_f \subseteq O^K$, since

$$\dfrac{\dfrac{k \in K}{\exists k_i \in K \ .\ k \in W_{k_i}} \qquad g \in V_f}{g(k) \in O}\ \mathrm{r1} \ \Rightarrow \ \dfrac{g \in V_f}{g(K) \subseteq O}$$

$$\mathrm{r1}:\ g(k) = ev(g,k) \in ev(V_f \times W_{k_i}) \subseteq ev(V_{f,k_i} \times W_{k_i}) \subseteq O$$

So, $O^K = \bigcup\{V_f\}_{f\in O^K}$, which is a union of open sets in $\mathrm{Hom}_{\mathbf{Top}}(X,Y)_{\mathcal{T}}$. That is, $O^K \in \mathcal{T}$. $\square$

Note. Now we denote $\mathrm{Hom}_{\mathbf{Top}}(X,Y)_{\mathcal{T}_{co}}$ simply by $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$. That is, the default topology on $\mathrm{Hom}_{\mathbf{Top}}(X,Y)$ is the **compact-open** topology.

Proposition 1.2. *If X is locally compact and Hausdorff, then the* **compact-open** *topology is admissible.*

Proof. We have to show $ev : \mathrm{Hom}_{\mathbf{Top}}(X,Y) \times X \to Y$ is continuous. That is $V \in \mathcal{T}_Y \to ev^{-1}(V) \in \mathcal{T}_{\mathrm{Hom}_{\mathbf{Top}}(X,Y)\times X}$. By definition, $ev^{-1}(V) = \{(f,x) \mid f(x) \in V\}$, We take $(f,x) \in ev^{-1}(V)$ for the next step.

By continity of f, we have $f^{-1}(V)$ is open in X. By locally compactness of X and X is Hausdorff, there exist $O_{(f,x)} \in \mathcal{T}_X$ such that $x \in O_{(f,x)} \subseteq \overline{O_{(f,x)}} \subseteq V$ and $\overline{O_{(f,x)}}$ is compact. Put $K_{(f,x)} := \overline{O_{(f,x)}}$

Now we have $(f,x) \in V^{K_{(f,x)}} \times O_{(f,x)} \subseteq ev^{-1}(V)$, that means

$$ev^{-1}(V) = \bigcup \left\{V^{K_{(f,x)}} \times O_{(f,x)}\right\}_{(f,x)\in ev^{-1}(V)}$$

is a union of open sets. That is, $ev^{-1}(V)$ is open. $\square$