

# Notes of Introduction to Commutative Algebra

Cloudifold

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# 0 Notations

Category of commutative rings	: <b>CRing</b>
Coordinate ring of $k$ -variety $X$	: $k[X]$
Element in polynomial ring $A[x]$	: $c \in A[x] := c = \sum_i c_i x^i \ (\forall i . c_i \in A)$
Jacobson radical of ring $R$	: $\text{Jac}(R)$
Polynomial ring with $n$ -variables over ring $R$	: $R[x]_n$ or $R[x_1, \dots, x_n]$
Nilradical of ring $A$	: $\sqrt{\langle 0 \rangle}_A$
Radical ideal of ideal $I$	: $\sqrt{I}$

# 1 Exercises of Chapter 1 : Rings and Ideals

*Exercise 1.1.*  $A \in \mathbf{CRing}, x \in \sqrt{\langle 0 \rangle} \rightarrow u \in A^\times \rightarrow u + x \in A^\times$

*Solution 1.1.*

$$\begin{array}{c}
 \frac{x \in \sqrt{\langle 0 \rangle}}{\exists d \in \mathbb{Z} . x^d = 0} \quad \text{w1} \\
 \frac{(1+x) \cdot (\sum_{i=0}^{d-1} (-x)^i) = 1}{1+x \in A^\times} \quad \frac{u \in A^\times}{1+x \in A^\times \leftrightarrow u+ux \in A^\times} \quad \frac{x \in \sqrt{\langle 0 \rangle}}{\forall r \in A . rx \in \sqrt{\langle 0 \rangle}} \quad \text{w2} \\
 \hline
 u+x \in A^\times
 \end{array}$$

$$\begin{array}{l}
 \text{w1 : } 1 - x^n = (1+x) \cdot \left( \sum_{i=0}^{n-1} x^i \right) \\
 \text{w2 : } u(1 + u^{-1}x) = u + x, u^{-1}x \in A^\times
 \end{array}$$

□

*Exercise 1.4.*  $A \in \mathbf{CRing} \rightarrow \text{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}$

*Solution 1.4.*

$$\begin{array}{c}
 \frac{\text{lemma 1} \quad \text{lemma 2}}{c \in \sqrt{\langle 0 \rangle}_{A[x]} \leftrightarrow \forall i . c_i \in \sqrt{\langle 0 \rangle}_A} \quad \text{w1} \quad \frac{\text{def 1} \quad \text{def 2}}{\text{Jac}(A[x]) \supseteq \sqrt{\langle 0 \rangle}_{A[x]}} \quad \text{w2} \\
 \hline
 \text{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}
 \end{array}$$

$$\text{def 1 : } \text{Jac}(A[x]) = \bigcap_{m \in \text{Spec}_{\max}(A[x])} m$$

$$\text{def 2 : } \sqrt{\langle 0 \rangle}_{A[x]} = \bigcap_{P \in \text{Spec}(A[x])} P$$

$$\text{lemma 1 : } q \in \text{Jac}(A[x]) \leftrightarrow \forall y \in A[x] . 1 - qy \in A[x]^\times \quad (\text{Textbook Ch1 Prop 1.9})$$

$$\text{lemma 2 : } c \in A[x]^\times \rightarrow, c_0 \in A^\times \text{ and } \forall i \neq 0 . c_i \in \sqrt{\langle 0 \rangle}_A \quad (\text{Textbook Ch1 Ex 2.i})$$

$$\text{w1 : } q \in \text{Jac}(A[x]) \rightarrow 1 - q \in A[x]^\times \rightarrow \forall i, q_i \in \sqrt{\langle 0 \rangle}_{A[x]} \rightarrow q \in \sqrt{\langle 0 \rangle}_{A[x]}$$

$$\text{w2 : } I \in \text{Spec}_{\max} A \rightarrow I \in \text{Spec} A$$

□

$$\text{Exercise 1.6. } A \in \mathbf{CRing} \rightarrow \left( I \not\subseteq \sqrt{\langle 0 \rangle}_A \rightarrow \exists e \in I, e^2 = e \neq 0 \right) \rightarrow \text{Jac } A = \sqrt{\langle 0 \rangle}_A$$

*Solution 1.6.*

$$\begin{array}{c} \frac{I \not\subseteq \sqrt{\langle 0 \rangle}_A \rightarrow \exists e \in I, e^2 = e \neq 0 \quad \text{Jac } A \not\subseteq \sqrt{\langle 0 \rangle}_A}{\frac{\frac{\exists e \in \text{Jac } A, e^2 = e \neq 0}{1 - e \in A^\times \text{ and } e(1 - e) = 0} \quad \text{lemma 1}}{e = e(1 - e)(1 - e)^{-1} = 0(1 - e)^{-1} = 0}} \Rightarrow \text{Jac } A \subseteq \sqrt{\langle 0 \rangle}_A \\ \perp \\ \frac{\text{Jac } A \subseteq \sqrt{\langle 0 \rangle}_A \quad \text{Jac}(A) \supseteq \sqrt{\langle 0 \rangle}_A \text{ (lemma 2)}}{\text{Jac } A = \sqrt{\langle 0 \rangle}_A} \end{array}$$

$$\text{lemma 1 : } q \in \text{Jac}(A) \leftrightarrow \forall y \in A . 1 - qy \in A^\times \quad (\text{Textbook Ch1 Prop 1.9})$$

$$\text{lemma 2 : Sol 1.4. w2}$$

□