Notes of Introducion to Commutative Algebra

Cloudifold

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0 Notations

Coordinate ring of k-variety X : $k[X]$
Category of commutative rings : **CRing**
Jacobson radical of ring R : $\mathrm{Jac}(R)$
Polynomial ring with n-variables over ring R : $R[x]_n$ or $R[x_1, \ldots, x_n]$
Radical ideal of ideal I : $\sqrt{I}$

1 Exercises of Chapter 1 : Rings and Ideals

Exercise 1.1. $A \in \mathbf{CRing}$, $x \in \sqrt{\langle 0 \rangle} \to u \in A^\times \to u + x \in A^\times$

Solution 1.1.

$$\dfrac{\dfrac{\dfrac{\dfrac{x \in \sqrt{\langle 0\rangle}}{\exists d \in \mathbb{Z}\ .\ x^d = 0}}{(1+x)\cdot(\sum_{i=0}^{d-1}(-x)^i) = 1}\,w1}{1+x\in A^\times} \qquad \dfrac{u\in A^\times}{1+x \in A^\times \leftrightarrow u+ux\in A^\times} \qquad \dfrac{x\in\sqrt{\langle 0\rangle}}{\forall r \in A\ .\ rx \in \sqrt{\langle 0\rangle}}}{u+x\in A^\times}\,w2$$

$$w1 : 1 - x^n = (1+x) \cdot \left(\sum_{i=0}^{n-1} x^i\right)$$
$$w2 : u(1 + u^{-1}x) = u + x, u^{-1}x \in A^\times$$

□

Exercise 1.4. $A \in \mathbf{CRing} \to \mathrm{Jac}(A[x]) = \sqrt{\langle 0 \rangle}_{A[x]}$

Solution 1.4.

$$\mathrm{Jac}(A[x]) = \bigcap_{m \in \mathrm{Spec}_{\max}(A[x])} m$$

$$\sqrt{\langle 0 \rangle}_{A[x]} = \bigcap_{P \in \mathrm{Spec}(A[x])} P$$

$$\mathrm{Jac}(A[x]) \supseteq \sqrt{\langle 0 \rangle}_{A[x]}$$

$$q \in \mathrm{Jac}(A[x]) \leftrightarrow \forall y \in A[x]\ .\ 1 - qy \in A[x]^\times$$

$c \in A[x]^\times \to, c_0 \in A^\times$ and $\forall \mathrm{i} \neq 0\ .\ \mathrm{c_i} \in \sqrt{\langle 0 \rangle}_{\mathrm{A}}$
$c \in \sqrt{\langle 0 \rangle}_{A[x]} \leftrightarrow \forall i\ .\ c_i \in \sqrt{\langle 0 \rangle}_A$
$c = \sum_i c_i x^i\ (\forall i\ .\ c_i \in A)$
$q \in \mathrm{Jac}(A[x]) \to 1 - q \in A[x]^\times \to \forall i, q_i \in \sqrt{\langle 0 \rangle}_{A[x]} \to q \in \sqrt{\langle 0 \rangle}_{A[x]}$

□