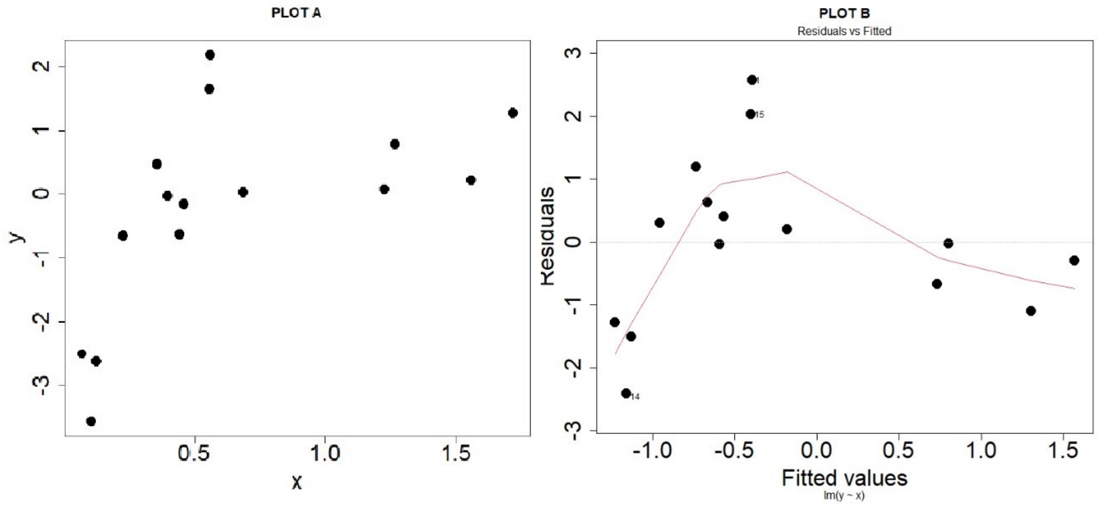
SECTION C

5. Consider two variables **X** and **Y**, for which we have observed a sample of 15 observations:

$$\mathbf{x} = (0.56, 0.23, 1.56, 0.07, 0.13, 1.72, 0.46, 1.27, 0.69, 0.45, 1.22, 0.36, 0.40, 0.11, 0.56)$$

$$\mathbf{y} = (2.18, -0.66, 0.21, -2.51, -2.63, 1.27, -0.17, 0.78, 0.02, -0.63, 0.07, 0.46, -0.04, -3.57, 1.63)$$



(a) Plot A represents the observed values of **x** and **y**. What type of plot is this? Briefly explain.

(b) Compute and report the value of the correlation coefficient for **x** and **y**. Briefly comment the obtained value. Does this value indicate that there is a causal relation between **X** and **Y**?

(c) Consider the simple linear regression of **y** over **x**, $\mathbf{y} = \alpha + \beta\mathbf{x} + \epsilon$, where α is the intercept term, β is the coefficient for **x**, and ϵ is the error term. Compute and report the estimates for α and β, given the observed sample. Interpret the estimates' values and their meaning.

(d) Are the values of α and β estimates in line with what you could have expected from plot A?

(e) Plot B represents the residual plot of the simple linear regression model $\mathbf{y} = \alpha + \beta\mathbf{x} + \epsilon$. Briefly define the equation of the residuals for the linear regression model fitted in part (c) . Are all the assumptions of the simple linear regression model fulfilled in the above situation? List all of the assumptions and discuss briefly.

[25 marks]

6. (a) Suppose that an ordinary deck of 52 cards is shuffled and the cards are then turned over one at a time until the first ace appears. If the first ace is the 20th card to appear, what is the conditional probability that the card following it is the

   (i) ace of spades?

   (ii) two of clubs?

   (b) (i) A gambler has a fair coin and a two-headed coin in their pocket. They select one of the coins at random; when they flip it, it shows heads. What is the probability that it is the fair coin?

   (ii) Suppose the same coin is flipped a second time, and, again, it shows heads. Now what is the probability that it is the fair coin?

   (iii) Suppose the same coin is flipped a third time and it shows tails. Now what is the probability that it is the fair coin?

   (c) There are two machines (M1 and M2), one that requires a single AA battery and another that uses two AAA batteries. There are two AA batteries and four AAA batteries available. Suppose the probability that any particular battery works is p and that batteries work or fail independently of one another. Suppose we have to pick one of the machines. For what values of p should we pick the machine M1?

   (d) Two different professors have submitted final exams for duplication. Let X denote the number of typographical errors on the first professor's exam and Y denote the number of such errors on the second exam. Suppose X has a Poisson distribution with parameter μ_1, and Y has a Poisson distribution with parameter μ_2, and X and Y are independent.

   (i) What is the joint probability mass function of X and Y?

   (ii) What is the probability that at most one error is made on both exams combined?

   (iii) Obtain a general expression for the probability that the total number of errors in the two exams is m (where m is a nonnegative integer).

[25 marks]

Appendix

Linear Algebra

- The cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

  where $\mathbf{x}$ and $\mathbf{y}$ are elements in $\mathbb{R}^3$.

- Given a point P (with coordinates (p_1, p_2, p_3)) in a plane and a vector $\mathbf{n}$ in $\mathbb{R}^3$ that is normal to the plane, the implicit equation of the plane is

$$n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0.$$

- Given a point P with coordinates (p_1, p_2, p_3) in a plane and two vectors $\mathbf{u}$ and $\mathbf{v}$ that lie in the plane, the parametric equation of the plane is

$$\begin{aligned} x &= p_1 + u_1 s + v_1 t \\ y &= p_2 + u_2 s + v_2 t \\ z &= p_3 + u_3 s + v_3 t \end{aligned}$$

  where $-\infty < s < \infty$ and $-\infty < t < \infty$.

- Given a point P with coordinates (p_1, p_2, p_3) on a line and a vector $\mathbf{u}$ that is parallel to the line, the implicit equation of the line is given by the symmetric equations

$$\frac{x - p_1}{u_1} = \frac{y - p_2}{u_2} = \frac{z - p_3}{u_3}$$

- Given a point P with coordinates (p_1, p_2, p_3) on a line and a vector $\mathbf{u}$ that is parallel to the line, the parametric equation of the line is

$$\begin{aligned} x &= p_1 + u_1 t \\ y &= p_2 + u_2 t \\ z &= p_3 + u_3 t \end{aligned}$$

  where $-\infty < t < \infty$.

- Orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$:

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}.\mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}$$

  Vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$:

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}.\mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}$$

Calculus

Differentiation

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2 - x^2}}$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1} \frac{x}{a}$	$\frac{a}{a^2 + x^2}$

Integration

Constants of integration omitted.

$f(x)$	$\int f'(x)$
x^n, $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
a^x $(a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln \lvert \sec x \rvert$
$\cos^2 x$	$\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$
$\sin^2 x$	$\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$
$\frac{1}{\sqrt{a^2 - x^2}}$ $(a > 0)$	$\sin^{-1} \frac{x}{a}$
$\frac{1}{x^2 + a^2}$ $(a > 0)$	$\frac{1}{a}\tan^{-1} \frac{x}{a}$
$\frac{1}{x\sqrt{x^2 - a^2}}$	$\frac{1}{a}\sec^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\ln \left\lvert \frac{x + \sqrt{x^2 + a^2}}{a} \right\rvert$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left\lvert \frac{a + x}{a - x} \right\rvert$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\ln \left\lvert \frac{x + \sqrt{x^2 - a^2}}{a} \right\rvert$

Product Rule for Differentiation

$$y = uv, \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule for Differentiation

$$y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule for Differentiation

$$y = u(v(x)), \quad \frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dx}$$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Tautologies

1. $P \vee \neg P$ Law of the excluded middle
2. $\neg(P \wedge \neg P)$ Law of non-contradiction
3. $\neg\neg P \leftrightarrow P$ Law of double negation
4. $(P \wedge Q) \rightarrow P$ Basis for simplification
5. $(P \wedge Q) \rightarrow Q$ Basis for simplification
6. $P \rightarrow P \vee Q$ Basis for addition
7. $Q \rightarrow P \vee Q$ Basis for addition
8. $Q \rightarrow (P \rightarrow Q)$
9. $\neg P \rightarrow (P \rightarrow Q)$
10. $[P \wedge (P \rightarrow Q)] \rightarrow Q$ Modus ponens
11. $[\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$ Modus tollens
12. $\neg P \wedge (P \vee Q) \rightarrow Q$
13. $P \rightarrow [Q \rightarrow (P \wedge Q)]$
14. $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$ Transitivity of implications
15. $(P \rightarrow Q) \rightarrow [(P \vee R) \rightarrow (Q \vee R)]$
16. $(P \rightarrow Q) \rightarrow [(P \wedge R) \rightarrow (Q \wedge R)]$
17. $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (P \leftrightarrow R)$ Transitivity of equivalences
18. $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ De Morgan's law
19. $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ De Morgan's law
20. $\neg(P \rightarrow Q) \leftrightarrow P \wedge \neg Q$
21. $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
22. $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$
23. $(P \leftrightarrow Q) \leftrightarrow [(P \wedge Q) \vee (\neg Q \wedge \neg P)]$
24. $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$ Law of contraposition (The contrapositive)
25. $[(P \rightarrow Q) \wedge (P \rightarrow R)] \leftrightarrow [P \rightarrow (Q \wedge R)]$
26. $[(P \rightarrow R) \wedge (Q \rightarrow R)] \leftrightarrow [(P \vee Q) \rightarrow R]$ Basis for proof by cases
27. $[P \rightarrow (Q \rightarrow R)] \leftrightarrow [(P \wedge Q) \rightarrow R]$
28. $(P \rightarrow Q \wedge \neg Q) \leftrightarrow \neg P$ Basis for indirect proofs
29. $[(P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$ Law of distributivity
30. $[(P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$ Law of distributivity
31. $(P \wedge Q) \leftrightarrow (Q \wedge P)$ Law of commutativity
32. $(P \vee Q) \leftrightarrow (Q \vee P)$ Law of commutativity
33. $[(P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$ Law of associativity
34. $[(P \wedge (Q \wedge R)] \leftrightarrow [(P \wedge Q) \wedge R]$ Law of associativity

Inference Rules

1 Rules of inference involving no quantifiers

1) **Propositional Consequence (PC):** In a proof, any statement that is a propositional consequence of previous steps in the proof can be asserted.

2) **Modus Ponens:** In a proof containing both P and $P \to Q$, the statement Q can be asserted.

$$\begin{array}{l} P \\ \underline{P \to Q} \\ \therefore Q \end{array}$$

3) **Conditional Proof/Direct Proof:** Assume P. If Q can be proven from the assumption P, then the implication $P \to Q$ can be asserted.

$$\begin{array}{l} \text{Assume } P \\ \text{Intermediate steps} \\ Q \\ \hline \therefore P \to Q \end{array}$$

4) **Indirect Proof/Proof by Contradiction/Reductio ad Absurdum:** Assume $\neg P$ and prove ANY contradiction, then P holds.

$$\begin{array}{l} \text{Assume } \neg P \\ \text{Intermediate steps} \\ \text{Any contradiction} \\ \hline \therefore P \end{array}$$

5) **Proof by Cases:** $Q \vee R$ is one of the steps of the proof. $Q \to P$ and $R \to P$ are also steps in the proof. Then P can be asserted.

$$\begin{array}{l} Q \vee R \\ \text{Assume } Q \\ \text{Intermediate steps} \\ P \text{ (end of case 1)} \\ \text{Assume } R \\ \text{Intermediate steps} \\ P \text{ (end of case 2)} \\ \hline \therefore P \end{array}$$

6) **Biconditional Rule:** If the implications $P \to Q$ and $Q \to P$ appear in the course of the proof, then $P \leftrightarrow Q$ can be asserted.

$$\begin{array}{l} P \to Q \\ Q \to P \\ \hline \therefore P \leftrightarrow Q \end{array}$$

7) **Substitution:** Let $S(P)$ be a statement containing P as a sub-statement. Let $S(P/Q)$ denote a statement that results from $S(P)$ by replacing one or more occurrences of the statement P by the statement Q. From $P \to Q$ and $S(P)$, $S(P/Q)$ can be asserted provided no free variables of P or Q become quantified in $S(P)$ or $S(P/Q)$.

$$\begin{array}{l} P \leftrightarrow Q \\ S(P) \\ \hline \therefore S(P/Q) \end{array}$$

8) **Conjunction:** If statements P and Q appear as steps in the proof, the compound statement $P \wedge Q$ can be asserted.

$$\begin{array}{l} P \\ Q \\ \hline \therefore P \wedge Q \end{array}$$

9) **Modus Tollens:** If $P \to Q$ and $\neg Q$ are both steps in the proof, then $\neg P$ can be asserted.

$$\begin{array}{l} P \to Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$$

10) **Contrapositive Conditional Proof:** If the assumption $\neg Q$ leads to the conclusion $\neg P$, then the implication $P \to Q$ can be asserted.

$$\begin{array}{l} \text{Assume} \neg Q \\ \text{Intermediate steps} \\ \neg P \\ \hline \therefore P \to Q \end{array}$$

2 Rules of inference involving quantifiers

11) **De Morgan's Laws for Quantifiers (Axioms):**

   (a) $\neg(\forall x\, P(x)) \leftrightarrow \exists x\, \neg P(x)$;

   (b) $\neg(\exists x\, P(x)) \leftrightarrow \forall x\, \neg P(x)$;

12) **Universal Specification Axiom (US):** If the domain of the variable is not empty, $\forall x\, P(x) \to P(t)$ can be asserted, where t is an object in the domain and P is a statement depending upon one free variable.

13) **Universal Generalization Rule of Inference (UG):** If $P(x)$ can be proven, where x is a free variable representing an arbitrary element of a certain domain, $\forall x\, P(x)$ can be asserted.

14) **Existential Specification Rule of Inference (ES):** If a step of the form $\exists x\, P(x)$ appears in the proof, $P(c)$ can be asserted, where c is a constant symbol.

15) **Existential Generalization Axiom (EG):** $P(t) \to \exists x\, P(x)$ can be asserted, if t is an object in the domain of x (assumed not empty).

Statistical Formulas

- Correlation coefficient: $r = r(z,h) = \dfrac{cov(z,h)}{sd(z)sd(h)}$
- Covariance: $Cov(z,h) = \dfrac{1}{n}\sum_{i=1}^{n}(z-\bar{z})\left(h-\bar{h}\right)$
- Variance: $var(z) = \dfrac{1}{n}\sum_{i=1}^{n}(z-\bar{z})^2$
- Standard deviation: $sd(z) = \sqrt{var(z)}$
- Slope coefficient (linear regression of z over h): $\hat{\beta} = r\dfrac{sd(z)}{sd(h)}$
- Intercept (linear regression of z over h): $\hat{\alpha} = \bar{z} - \hat{\beta}\bar{h}$
- Sample mean of z: $\bar{z}$

Probability Distributions

Distribution	pdf/cdf	Mean	Variance
Bernoulli	$p^x(1-p)^{1-x}$	p	$p(1-p)$
Binomial	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$
Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}$	λ	λ
Exponential	$\lambda e^{-\lambda x}$ for $(x \geq 0)$, 0 for $(x < 0)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal	$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$	μ	σ^2