SECTION C

5. You may draw diagrams if desired to aid in your answers to the following questions.

   (a) Evaluate and appraise the use of p-values in the context of hypothesis testing.

   (b) Compare and contrast the following hypothesis tests: Two-Sample Independent Z-test, Two-Sample Independent t-test, paired t-test, paired z-test, chi-square test of independence.

   (c) Discuss and assess the use of simple linear regression in statistical modelling.

6. (a) Assume a system has n similar components that are connected in a series configuration, either linear or circular, which the latter means the first and the last components are adjacent and/or connected. The system is called a consecutive *2*-out-of-n:F system if failure of 2 adjacent components causes system failure.

   The lifetime of each component of the system is exponentially distributed with parameter λ.

   The reliability of a system at time t is defined as the probability the system is properly functioning at time t. Denote the reliability of linear and circular consecutive *2*-out-of-n:F systems at time t as $R_L(t, n)$ and $R_C(t, n)$.

   i. Show that when $n > 3$:

   $$R_C(t, n) \; = \; e^{-\lambda t} R_L(t, n-1) + e^{-2\lambda t}(1 - e^{-\lambda t}) R_L(t, n-3).$$

   Hint: The above definition of a consecutive *2*-out-of-n:F system implies that for a functioning system, the two components adjacent to a defective one must be working.

   ii. Hence or otherwise, show:

   $$R_C(t, 5) \; = \; e^{-3\lambda t}(5 - 5e^{-\lambda t} + e^{-2\lambda t})$$

   (b) The number of customers arriving into a service system in $(0, t)$ has Poisson distribution with the arrival rate r per time unit. Explain the equation:

   $$\sum_{i=1}^{k-1} \frac{e^{-rt}(rt)^i}{i!} = \int_t^{\infty} \frac{(rtx)^{k-1}}{(k-1)!} rte^{-rtx} dx.$$

Calculus

Differentiation

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2 - x^2}}$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2 - x^2}}$
$\tan^{-1} \frac{x}{a}$	$\frac{a}{a^2 + x^2}$

Integration

Constants of integration omitted.

$f(x)$	$\int f'(x)$
x^n, $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x$
e^x	e^x
e^{ax}	$\frac{1}{a} e^{ax}$
$a^x \ (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln \lvert \sec x \rvert$
$\cos^2 x$	$\frac{1}{2}\left(x + \frac{1}{2} \sin 2x\right)$
$\sin^2 x$	$\frac{1}{2}\left(x - \frac{1}{2} \sin 2x\right)$
$\frac{1}{\sqrt{a^2 - x^2}} \ (a > 0)$	$\sin^{-1} \frac{x}{a}$
$\frac{1}{x^2 + a^2} \ (a > 0)$	$\frac{1}{a} \tan^{-1} \frac{x}{a}$
$\frac{1}{x\sqrt{x^2 - a^2}}$	$\frac{1}{a} \sec^{-1} \frac{x}{a}$
$\frac{1}{\sqrt{x^2 + a^2}}$	$\ln \left\lvert \frac{x + \sqrt{x^2 + a^2}}{a} \right\rvert$
$\frac{1}{a^2 - x^2}$	$\frac{1}{2a} \ln \left\lvert \frac{a + x}{a - x} \right\rvert$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\ln \left\lvert \frac{x + \sqrt{x^2 - a^2}}{a} \right\rvert$

Product Rule for Differentiation

$$y = uv, \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule for Differentiation

$$y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule for Differentiation

$$y = u(v(x)), \quad \frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dx}$$

Integration by parts

$$\int u\, dv = uv - \int v\, du$$

Theorems of the Propositional Calculus

Equivalence and true

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \; \equiv \; (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of $\equiv$:** $\mathit{true} \equiv q \equiv q$

(3.4) true

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and false

(3.8) **Axiom, Definition of false:** $\mathit{false} \equiv \neg \mathit{true}$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \; \equiv \; \neg p \equiv q$

(3.10) **Axiom, Definition of $\not\equiv$:** $(p \not\equiv q) \; \equiv \; \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of false:** $\neg \mathit{false} \equiv \mathit{true}$

(3.14) $(p \not\equiv q) \; \equiv \; \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv \mathit{false}$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \; \equiv \; (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \; \equiv \; (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \; \equiv \; (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \;\; \equiv \;\; p \equiv q \not\equiv r$

Disjunction

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \; \equiv \; q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \; \equiv \; p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \; \equiv \; p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \; \equiv \; r) \; \equiv \; p \vee q \; \equiv \; p \vee r$

(3.28) **Axiom, Excluded Middle:** $p \vee \neg p$

(3.29) **Zero of $\vee$:** $p \vee \mathit{true} \; \equiv \; \mathit{true}$

(3.30) **Identity of $\vee$:** $p \vee \mathit{false} \; \equiv \; p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \; \equiv \; (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \; \equiv \; p \vee \neg q \; \equiv \; p$

Conjunction

(3.35) **Axiom, Golden rule** : $p \wedge q \; \equiv \; p \; \equiv \; q \; \equiv \; p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \; \equiv \; q \wedge p$

(3.37) **Associativity of** $\wedge$**:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of** $\wedge$**:** $p \wedge p \equiv p$

(3.39) **Identity of** $\wedge$**:** $p \wedge true \equiv p$

(3.40) **Zero of** $\wedge$**:** $p \wedge false \equiv false$

(3.41) **Distributivity of** $\wedge$ **over** $\wedge$**:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$

(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$

(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of** $\vee$ **over** $\wedge$**:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of** $\wedge$ **over** $\vee$**:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$

(3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$

(3.50) $p \wedge (q \equiv p) \equiv p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$

(3.52) **Definition of** $\equiv$**:** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

Implication

(3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$

(3.59) **Definition of implication:** $p \Rightarrow q \equiv \neg p \vee q$

(3.60) **Definition of implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$

(3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$

(3.63) **Distributivity of** $\Rightarrow$ **over** $\equiv$**:** $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$

(3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \equiv p$

(3.68) $p \vee (p \Rightarrow q) \equiv true$

(3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$

(3.71) **Reflexivity of** $\Rightarrow$**:** $p \Rightarrow p \equiv \mathit{true}$

(3.72) **Right zero of** $\Rightarrow$**:** $p \Rightarrow \mathit{true} \equiv \mathit{true}$

(3.73) **Left identity of** $\Rightarrow$**:** $\mathit{true} \Rightarrow p \equiv p$

(3.74) $p \Rightarrow \mathit{false} \equiv \neg p$

(3.75) $\mathit{false} \Rightarrow p \equiv \mathit{true}$

(3.76) **Weakening/strengthening:** (a) $p \Rightarrow p \vee q$
(b) $p \wedge q \Rightarrow p$
(c) $p \wedge q \Rightarrow p \vee q$
(d) $p \vee (q \wedge r) \Rightarrow p \vee q$
(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:** (a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

Leibniz as an axiom

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E_e^z = E_f^z$

(3.84) **Substitution:** (a) $(e = f) \wedge E_e^z \equiv (e = f) \wedge E_f^z$
(b) $(e = f) \Rightarrow E_e^z \equiv (e = f) \Rightarrow E_f^z$
(c) $q \wedge (e = f) \Rightarrow E_e^z \equiv q \wedge (e = f) \Rightarrow E_f^z$

(3.85) **Replace by** true**:** (a) $p \Rightarrow E_p^z \equiv p \Rightarrow E_{\mathit{true}}^z$
(b) $q \wedge p \Rightarrow E_p^z \equiv q \wedge p \Rightarrow E_{\mathit{true}}^z$

(3.86) **Replace by** false**:** (a) $E_p^z \Rightarrow p \equiv E_{\mathit{false}}^z \Rightarrow p$
(b) $E_p^z \Rightarrow p \vee q \equiv E_{\mathit{false}}^z \Rightarrow p \vee q$

(3.87) **Replace by** true**:** $p \wedge E_p^z \equiv p \wedge E_{\mathit{true}}^z$

(3.88) **Replace by** false**:** $p \vee E_p^z \equiv p \vee E_{\mathit{false}}^z$

(3.89) **Shannon:** $E_p^z \equiv (p \wedge E_{\mathit{true}}^z) \vee (\neg p \wedge E_{\mathit{false}}^z)$

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of** $\vee$**:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of** $\wedge$**:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$