SECTION C

5. (a) A drug company is interested in assessing the research question: is the proportion of patients who are cured with drug A different from the natural recovery rate of 50%? The regulatory requirement is for a 95% confidence interval width for the estimate of population proportion to be no more than 4%. The company takes a sample of 1000 individuals treated with drug A, and records that 475 of them were cured. They also record a number of other characteristics of the patients for use in future hypothesis tests. They then perform a one-tailed t-test for the null hypothesis that the population proportion is less than or equal to 0.50. They use a significance level alpha = 0.05.

    (i) Following the approach of the drug company as described above, calculate the test statistic for this hypothesis test.

    (ii) Critically evaluate the drug companies approach to answering their research question.

   (b) A number of graphical methods exist for the assessment of a simple linear model. Choose THREE (3) of these and for each one

    (i) describe the method's purpose and draw an example of a graph that would not indicate issues with the model.

    (ii) draw an example of a graph that would indicate possible issues with the model.

    (iii) For each of your graphs in (b)(ii) explain why a simple linear model might produce a graph with such an appearance.

[33 marks]

6. (a) Suppose we model the time until the next phone call as an exponential random variable. If the chance of my getting a call during the next hour is 0.5. What is the probability that I will get a call during the next two hours?

   (b) Mike and Tom are playing a game. Suppose the score is tied and a player wins the game when they get two points ahead of the other. Suppose the probability that Mike wins a point is p, and each point is assumed to be independent. What is the probability that Mike wins the game?

   (c) Consider two events A and B with $0 < \mathbb{P}(A) < 1$ and $0 < \mathbb{P}(B) < 1$.

      (i) Suppose A and B are disjoint events. Can they be independent? Justify your answer.

      (ii) Suppose $A \subset B$. Can they be independent?

   (d) At Trinity College Dublin, 4% of men are over 200 pounds in weight and 1% of women are over 200 pounds. Women account for 60% of the total student population, and men account for 40%. If a randomly selected student is over 200 pounds in weight, what is the probability that the student is a woman?

   (e) Suppose a random integer N is picked from 1 to 3 with the three possibilities being equally likely. Another integer M is then randomly picked from N to 3 with the $4 - N$ possibilities being equally likely. What is the probability that M will be 3?

[33 marks]

Linear Algebra

- The cross product of two vectors in $\mathbb{R}^3$ is defined by

$$\mathbf{x} \times \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}.$$

  where $\mathbf{x}$ and $\mathbf{y}$ are elements in $\mathbb{R}^3$.

- Given a point P (with coordinates (p_1, p_2, p_3)) in a plane and a vector $\mathbf{n}$ in $\mathbb{R}^3$ that is normal to the plane, the implicit equation of the plane is

$$n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0.$$

- Given a point P with coordinates (p_1, p_2, p_3) in a plane and two vectors $\mathbf{u}$ and $\mathbf{v}$ that lie in the plane, the parametric equation of the plane is

$$\begin{aligned} x &= p_1 + u_1 s + v_1 t \\ y &= p_2 + u_2 s + v_2 t \\ z &= p_3 + u_3 s + v_3 t \end{aligned}$$

  where $-\infty < s < \infty$ and $-\infty < t < \infty$.

- Given a point P with coordinates (p_1, p_2, p_3) on a line and a vector $\mathbf{u}$ that is parallel to the line, the implicit equation of the line is given by the symmetric equations

$$\frac{x - p_1}{u_1} = \frac{y - p_2}{u_2} = \frac{z - p_3}{u_3}$$

- Given a point P with coordinates (p_1, p_2, p_3) on a line and a vector $\mathbf{u}$ that is parallel to the line, the parametric equation of the line is

$$\begin{aligned} x &= p_1 + u_1 t \\ y &= p_2 + u_2 t \\ z &= p_3 + u_3 t \end{aligned}$$

  where $-\infty < t < \infty$.

- Orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$:

$$\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}.\mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}$$

  Vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$:

$$\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}.\mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}$$

Calculus

Differentiation

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln x$	$\frac{1}{x}$
e^x	e^x
e^{ax}	ae^{ax}
a^x	$a^x \ln a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$\sec^2 x$
$\cos^{-1} \frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}$
$\sin^{-1} \frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}$
$\tan^{-1} \frac{x}{a}$	$\frac{a}{a^2+x^2}$

Integration

Constants of integration omitted.

$f(x)$	$\int f'(x)$
$x^n,\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x\ (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cos^2 x$	$\frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right)$
$\sin^2 x$	$\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right)$
$\frac{1}{\sqrt{a^2-x^2}}\ (a > 0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2+a^2}\ (a > 0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$
$\frac{1}{x\sqrt{x^2-a^2}}$	$\frac{1}{a}\sec^{-1}\frac{x}{a}$
$\frac{1}{\sqrt{x^2+a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2+a^2}}{a}\right\rvert$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert$

Product Rule for Differentiation

$$y = uv, \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Quotient Rule for Differentiation

$$y = \frac{u}{v}, \quad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Chain Rule for Differentiation

$$y = u(v(x)), \quad \frac{dy}{dx} = \frac{du}{dv}\frac{dv}{dx}$$

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Tautologies

1. $P \vee \neg P$ Law of the excluded middle
2. $\neg(P \wedge \neg P)$ Law of non-contradiction
3. $\neg\neg P \leftrightarrow P$ Law of double negation
4. $(P \wedge Q) \rightarrow P$ Basis for simplification
5. $(P \wedge Q) \rightarrow Q$ Basis for simplification
6. $P \rightarrow P \vee Q$ Basis for addition
7. $Q \rightarrow P \vee Q$ Basis for addition
8. $Q \rightarrow (P \rightarrow Q)$
9. $\neg P \rightarrow (P \rightarrow Q)$
10. $[P \wedge (P \rightarrow Q)] \rightarrow Q$ Modus ponens
11. $[\neg Q \wedge (P \rightarrow Q)] \rightarrow \neg P$ Modus tollens
12. $\neg P \wedge (P \vee Q) \rightarrow Q$
13. $P \rightarrow [Q \rightarrow (P \wedge Q)]$
14. $[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$ Transitivity of implications
15. $(P \rightarrow Q) \rightarrow [(P \vee R) \rightarrow (Q \vee R)]$
16. $(P \rightarrow Q) \rightarrow [(P \wedge R) \rightarrow (Q \wedge R)]$
17. $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (P \leftrightarrow R)$ Transitivity of equivalences
18. $\neg(P \wedge Q) \leftrightarrow \neg P \vee \neg Q$ De Morgan's law
19. $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ De Morgan's law
20. $\neg(P \rightarrow Q) \leftrightarrow P \wedge \neg Q$
21. $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
22. $(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$
23. $(P \leftrightarrow Q) \leftrightarrow [(P \wedge Q) \vee (\neg Q \wedge \neg P)]$
24. $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$ Law of contraposition (The contrapositive)
25. $[(P \rightarrow Q) \wedge (P \rightarrow R)] \leftrightarrow [P \rightarrow (Q \wedge R)]$
26. $[(P \rightarrow R) \wedge (Q \rightarrow R)] \leftrightarrow [(P \vee Q) \rightarrow R]$ Basis for proof by cases
27. $[P \rightarrow (Q \rightarrow R)] \leftrightarrow [(P \wedge Q) \rightarrow R]$
28. $(P \rightarrow Q \wedge \neg Q) \leftrightarrow \neg P$ Basis for indirect proofs
29. $[(P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$ Law of distributivity
30. $[(P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$ Law of distributivity
31. $(P \wedge Q) \leftrightarrow (Q \wedge P)$ Law of commutativity
32. $(P \vee Q) \leftrightarrow (Q \vee P)$ Law of commutativity
33. $[(P \vee (Q \vee R)] \leftrightarrow [(P \vee Q) \vee R]$ Law of associativity
34. $[(P \wedge (Q \wedge R)] \leftrightarrow [(P \wedge Q) \wedge R]$ Law of associativity

Inference Rules

1 Rules of inference involving no quantifiers

1) **Propositional Consequence (PC):** In a proof, any statement that is a propositional consequence of previous steps in the proof can be asserted.

2) **Modus Ponens:** In a proof containing both P and $P \to Q$, the statement Q can be asserted.

$$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \end{array}$$

3) **Conditional Proof/Direct Proof:** Assume P. If Q can be proven from the assumption P, then the implication $P \to Q$ can be asserted.

$$\begin{array}{l} \text{Assume } P \\ \text{Intermediate steps} \\ Q \\ \hline \therefore P \to Q \end{array}$$

4) **Indirect Proof/Proof by Contradiction/Reductio ad Absurdum:** Assume $\neg P$ and prove ANY contradiction, then P holds.

$$\begin{array}{l} \text{Assume } \neg P \\ \text{Intermediate steps} \\ \text{Any contradiction} \\ \hline \therefore P \end{array}$$

5) **Proof by Cases:** $Q \vee R$ is one of the steps of the proof. $Q \to P$ and $R \to P$ are also steps in the proof. Then P can be asserted.

$$\begin{array}{l} Q \vee R \\ \text{Assume } Q \\ \text{Intermediate steps} \\ P \text{ (end of case 1)} \\ \text{Assume } R \\ \text{Intermediate steps} \\ P \text{ (end of case 2)} \\ \hline \therefore P \end{array}$$

6) **Biconditional Rule:** If the implications $P \to Q$ and $Q \to P$ appear in the course of the proof, then $P \leftrightarrow Q$ can be asserted.

$$\begin{array}{l} P \to Q \\ Q \to P \\ \hline \therefore P \leftrightarrow Q \end{array}$$

7) **Substitution:** Let $S(P)$ be a statement containing P as a sub-statement. Let $S(P/Q)$ denote a statement that results from $S(P)$ by replacing one or more occurrences of the statement P by the statement Q. From $P \to Q$ and $S(P)$, $S(P/Q)$ can be asserted provided no free variables of P or Q become quantified in $S(P)$ or $S(P/Q)$.

$$\begin{array}{l} P \leftrightarrow Q \\ S(P) \\ \hline \therefore S(P/Q) \end{array}$$

8) **Conjunction:** If statements P and Q appear as steps in the proof, the compound statement $P \wedge Q$ can be asserted.

$$\begin{array}{l} P \\ Q \\ \hline \therefore P \wedge Q \end{array}$$

9) **Modus Tollens:** If $P \to Q$ and $\neg Q$ are both steps in the proof, then $\neg P$ can be asserted.

$$\begin{array}{l} P \to Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$$

10) **Contrapositive Conditional Proof:** If the assumption $\neg Q$ leads to the conclusion $\neg P$, then the implication $P \to Q$ can be asserted.

$$\begin{array}{l} \text{Assume}\neg Q \\ \text{Intermediate steps} \\ \neg P \\ \hline \therefore P \to Q \end{array}$$

2 Rules of inference involving quantifiers

11) **De Morgan's Laws for Quantifiers (Axioms):**

    (a) $\neg(\forall x\, P(x)) \leftrightarrow \exists x\, \neg P(x)$;

    (b) $\neg(\exists x\, P(x)) \leftrightarrow \forall x\, \neg P(x)$;

12) **Universal Specification Axiom (US):** If the domain of the variable is not empty, $\forall x\, P(x) \to P(t)$ can be asserted, where t is an object in the domain and P is a statement depending upon one free variable.

13) **Universal Generalization Rule of Inference (UG):** If $P(x)$ can be proven, where x is a free variable representing an arbitrary element of a certain domain, $\forall x\, P(x)$ can be asserted.

14) **Existential Specification Rule of Inference (ES):** If a step of the form $\exists x\, P(x)$ appears in the proof, $P(c)$ can be asserted, where c is a constant symbol.

15) **Existential Generalization Axiom (EG):** $P(t) \to \exists x\, P(x)$ can be asserted, if t is an object in the domain of x (assumed not empty).