STAT 151A Homework 1 Questions 1-3 Solutions

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These are rough sketches for the solutions for some of Homework 1. Some computational steps are omitted for brevity.

1

(a) In $Y = X\beta + e$ form, the model is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}.$$

So, $X = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix}$.

(b) The normal equation is always $X^\top X\beta = X^\top y$. Plugging in our X from part (a) and doing some rearranging yields

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \beta = \begin{bmatrix} \overline{y} \\ \overline{y} \end{bmatrix},$$

where $\overline{y} := \frac{1}{n}\sum_{i=1}^n y_i$.

Thus the solutions to the normal equation is the subspace of vectors (β_0, β_1) satisfying

$$\beta_0 + \beta_1 = \overline{y}.$$

One such solution is $\beta_0 = 0$, $\beta_1 = \overline{y}$.

(c) From the previous part, it is $\overline{y}$.

(d) $\beta_1 = \Lambda^\top \beta$ for $\Lambda = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By the definition of estimable, we need to check if there exists P such that $P^\top X = \Lambda^\top$. Plugging in X and Λ yields

$$P^\top \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

This is impossible because even if $P^\top$ has the right dimension $1 \times n$, the left-hand side would always be a vector of the form $\begin{bmatrix} c & c \end{bmatrix}$ for some number c, and thus can never equal the right-hand side. Thus β_1 is not estimable.

(e) In $Y = X\beta + e$ form, the model is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} e_1 \\ \vdots \\ e_n \\ e_{n+1} \end{bmatrix}.$$

The normal equation $X^\top X\beta = X^\top y$ is

$$\begin{bmatrix} n+1 & n+2 \\ n+2 & n+4 \end{bmatrix} \beta = \begin{bmatrix} y_1 + \cdots + y_n + y_{n+1} \\ y_1 + \cdots + y_n + 2y_{n+1}. \end{bmatrix}$$

Then, doing some rearranging yields

$$\beta_0 + 2\beta_1 = y_{n+1}$$
$$\beta_0 + \beta_1 = \frac{1}{n}\sum_{i=1}^{n} y_i,$$

from which we have

$$\widehat{\beta}_1 = y_{n+1} - \frac{1}{n}\sum_{i=1}^{n} y_i, \qquad \widehat{\beta}_0 = \frac{2}{n}\sum_{i=1}^{n} y_i - y_{n+1}.$$

2

(a)

$$X = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) The two columns of X form a basis for $C(X)$, since they span $C(X)$ (by definition) and are linearly independent (since one is not a scalar multiple of the other). Thus the rank is 2.

(c) The normal equation is

$$\begin{bmatrix} 3 & \\ & 3 \end{bmatrix} \beta = \begin{bmatrix} y_1 + y_2 + y_3 \\ -y_2 + y_3 + y_4 \end{bmatrix}$$

Thus

$$\widehat{\beta}_0 = \frac{1}{3}(y_1 + y_2 + y_3), \qquad \widehat{\beta}_1 = \frac{1}{3}(-y_2 + y_3 + y_4).$$

3

To show (a) and (b) are equivalent, we must show that (a) implies (b), and that (b) implies (a). For (a) implies (b), we provide two different solutions.

- **(b) $\implies$ (a).** Using the fact that projection matrices $M_0 = M_1M_2$, M_1, and M_2 are symmetric, we have

$$M_1M_2 = (M_1M_2)^\top = M_2^\top M_1^\top = M_2M_1.$$

- **(a) $\implies$ (b).** it suffices to check that M_1M_2 is idempotent and symmetric, and that its column space is $C(M_1) \cap C(M_2)$.

  (i) Idempotence:

$$(M_1M_2)(M_1M_2) = M_1M_1M_2M_2 = M_1M_2.$$

  The first equality uses the assumption (a), and the second uses the fact that M_1 and M_2 are themselves projection matrices and thus are also idempotent ($M_1M_1 = M_1$ and $M_2M_2 = M_2$).

  (ii) Symmetry:

$$(M_1M_2)^\top = M_2^\top M_1^\top = M_2M_1 = M_1M_2.$$

  Here we used the fact that M_1 and M_2 are symmetric because they are projection matrices, and we also used the assumption (a).

(iii) Finally, we need to check $C(M_1M_2) = C(M_1) \cap C(M_2)$. First, note

$$
\begin{aligned}
C(M_1M_2) &\subseteq C(M_1) \\
C(M_1M_2) = C(M_2M_1) &\subseteq C(M_2).
\end{aligned}
$$

Together, these inclusions imply

$$C(M_1M_2) \subseteq C(M_1) \cap C(M_2).$$

We now prove the reverse inclusion

$$C(M_1M_2) \supseteq C(M_1) \cap C(M_2). \tag{1}$$

Let $v \in C(M_1) \cap C(M_2)$. By the definition of M_1 and M_2, we have $M_1v = v$ and $M_2v = v$. Thus $M_1M_2v = M_1v = v$, so $v \in C(M_1M_2)$.

Another proof of (1) is as follows. Let $v \in C(M_1) \cap C(M_2)$. Since $v \in C(M_2)$, there is some vector u such that $v = M_2u$. Moreover, since $v \in C(M_1)$, we have $M_1v = v$. Therefore, $v = M_1v = M_1M_2u$, so $v \in C(M_1M_2)$. This implies $C(M_1) \cap C(M_2) \subseteq C(M_1M_2)$.

- **(a) $\implies$ (b), alternate approach.** it suffices to show the following two things.

  (i) $M_1M_2v = v$ for any $v \in C(M_1) \cap C(M_2)$

  (ii) $M_1M_2v = 0$ for any $v \in (C(M_1) \cap C(M_2))^\perp$

  For (i), note that $M_2v = v$ since $v \in C(M_2)$, and $M_1v = v$ since $v \in C(M_1)$. For (ii), the key is to note that

  $$(C(M_1) \cap C(M_2))^\perp = C(M_1)^\perp + C(M_2)^\perp.$$

  That is, v can be written as $v = w_1 + w_2$ where $w_1 \in C(M_1)^\perp$ and $w_2 \in C(M_2)^\perp$. Thus, by using the assumption (a) along with the fact that $M_1w_1 = 0$ and $M_2w_2 = 0$, we have

  $$M_1M_2v = M_1M_2(w_1 + w_2) = M_1M_2w_1 + M_1M_2w_2 = M_2M_1w_1 + M_1M_2w_2 = 0 + 0 = 0.$$