

# Preamble to Discriminant Analysis

Predictive Modeling & Statistical Learning

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# Introduction

# Introduction

In these slides I'll talk about the concept of Variance decomposition taking into account a group structure.

The idea is to layout a couple of foundational principles that should allow you to understand discriminant methods in a more comprehensive way.

BTW: this material is not in the textbooks *ISL* and *APM*.

# Iris Data



# Dataset iris in R

$n = 150$  Observations, i.e. iris flowers

$p = 4$  predictors

- ▶ Sepal.Length
- ▶ Sepal.Width
- ▶ Petal.Length
- ▶ Petal.Width

One response (categorical)

- ▶ Species (3 classes: setosa, versicolor, virginica)

Famous data set collected by Edgar Anderson (1935), and used by Ronald Fisher (1936) in his paper about Discriminant Analysis.

# Dataset iris in R

```
head(iris)
```

	Sepal.Length	Sepal.Width	Petal.Length	Petal.Width	Species
1	5.1	3.5	1.4	0.2	setosa
2	4.9	3.0	1.4	0.2	setosa
3	4.7	3.2	1.3	0.2	setosa
4	4.6	3.1	1.5	0.2	setosa
5	5.0	3.6	1.4	0.2	setosa
6	5.4	3.9	1.7	0.4	setosa

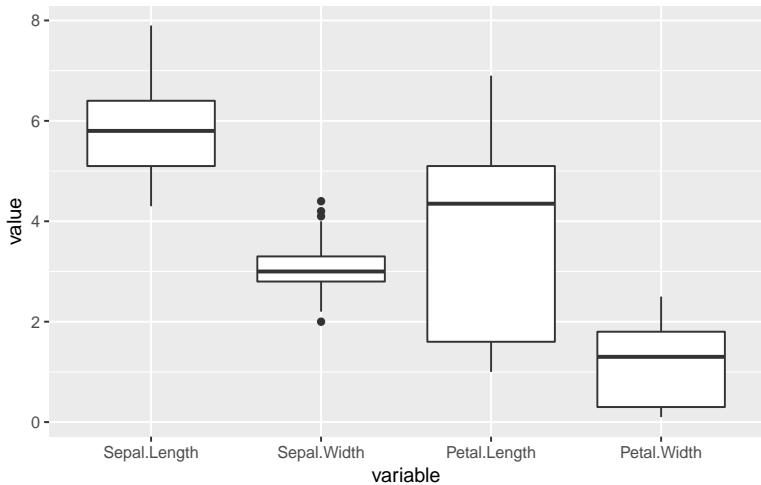
# Dataset iris in R

```
summary(iris)
```

Sepal.Length	Sepal.Width	Petal.Length	Petal.Width	Species
Min. :4.300	Min. :2.000	Min. :1.000	Min. :0.100	setosa :50
1st Qu.:5.100	1st Qu.:2.800	1st Qu.:1.600	1st Qu.:0.300	versicolor:50
Median :5.800	Median :3.000	Median :4.350	Median :1.300	virginica :50
Mean :5.843	Mean :3.057	Mean :3.758	Mean :1.199	
3rd Qu.:6.400	3rd Qu.:3.300	3rd Qu.:5.100	3rd Qu.:1.800	
Max. :7.900	Max. :4.400	Max. :6.900	Max. :2.500	

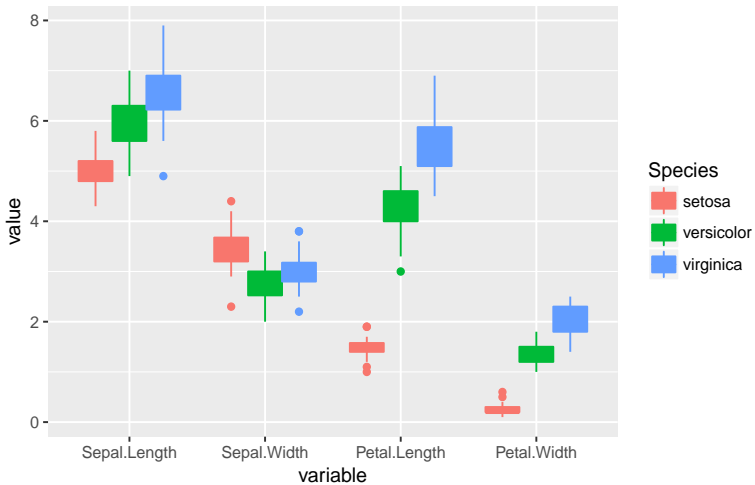


Boxplot of predictors (iris data)

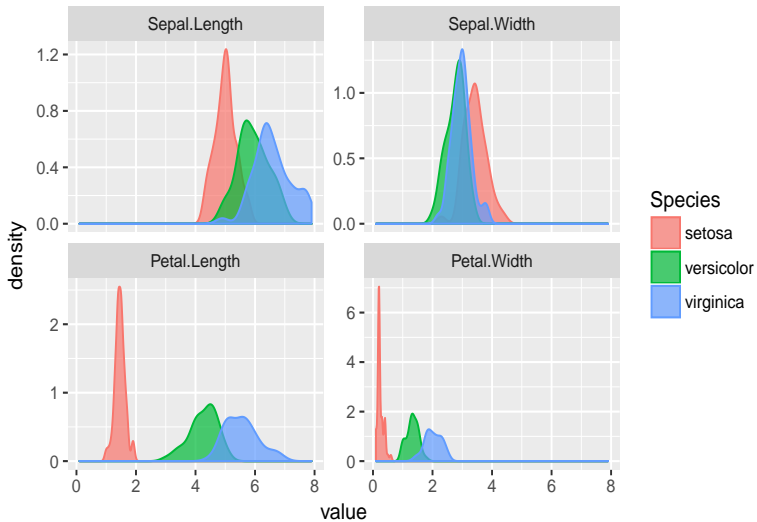


Let's take into account  
the group structure

Boxplot of predictors (iris data)



## Kernel densities of predictors (iris data)



```
library(reshape2)
library(ggplot2)

iris_melt <- melt(iris, id = "Species")

ggplot(data = iris_melt, aes(x = variable, y = value)) +
  geom_boxplot() +
  ggtitle("Boxplot of predictors (iris data)")

ggplot(data = iris_melt, aes(x = variable, y = value)) +
  geom_boxplot(aes(fill = Species, color = Species)) +
  ggtitle("Boxplot of predictors (iris data)")

ggplot(data = iris_melt, aes(x = value)) +
  geom_density(aes(fill = Species, color = Species),
              alpha = 0.7) +
  facet_wrap(~ variable, scales = 'free_y') +
  ggtitle("Kernel densities of predictors (iris data)")
```

Which predictor provides the  
“best” distinction between Species?

# Caveat: messy notation

In regression problems we've been using two indices  $i$  and  $j$

- ▶  $i$  for objects,  $i = 1, \dots, n$
- ▶  $j$  for predictors,  $j = 1, \dots, p$

## New index $k$

Now we have a new index  $k$  for groups or classes,  
 $k = 1, \dots, K$ .

# Sum of Squares

Consider a single predictor  $X$  and a categorical response  $Y$

Ignoring the response, we can obtain the mean  $\bar{x}$  and the total sum of squares (TSS) of  $X$  as:

$$\bar{x} = \sum_{i=1}^n x_i$$

$$\text{TSS} = \sum_{i=1}^n (x_i - \bar{x})^2$$



# Group (or class) structure

Let's take into account the group structure conveyed by  $Y$

- ▶ Let  $G_k$  represent the  $k$ -th group in  $Y$
- ▶ Let  $n_k$  be the number of observations in group  $G_k$ ,
- ▶ Then:

$$n = n_1 + n_2 + \cdots + n_K = \sum_{k=1}^K n_k$$

# Between Sum of Squares

Each group  $k$  will have its mean  $\bar{x}_k$ :

$$\bar{x}_k = \sum_{i \in G_k} x_{ik}$$

We can obtain the *Between-Groups Sum of Squares* (BSS)

$$\text{BSS} = \sum_{k=1}^K n_k (x_k - \bar{x})^2$$

# Within Sum of Squares

Each group  $k$  will also have its own sum-of-squares  $SS_k$

$$SS_k = \sum_{i \in G_k} (x_{ik} - \bar{x}_k)^2$$

We can obtain the *Within-Groups Sum of Squares* (WSS)

$$WSS = \sum_{k=1}^K \sum_{i \in G_k} (x_{ik} - \bar{x}_k)^2$$

# Decomposition of sums-of-squares

An important aspect has to do with looking at the squared deviations:  $(x_i - \bar{x})^2$  in terms of the group structure.

A useful trick is to rewrite the deviation terms  $x_i - \bar{x}$  as:

$$\begin{aligned}x_i - \bar{x} &= x_i - (\bar{x}_k - \bar{x}_k) - \bar{x} \\ &= (x_i - \bar{x}_k) + (\bar{x}_k - \bar{x})\end{aligned}$$

# Sum of Squares Decomposition

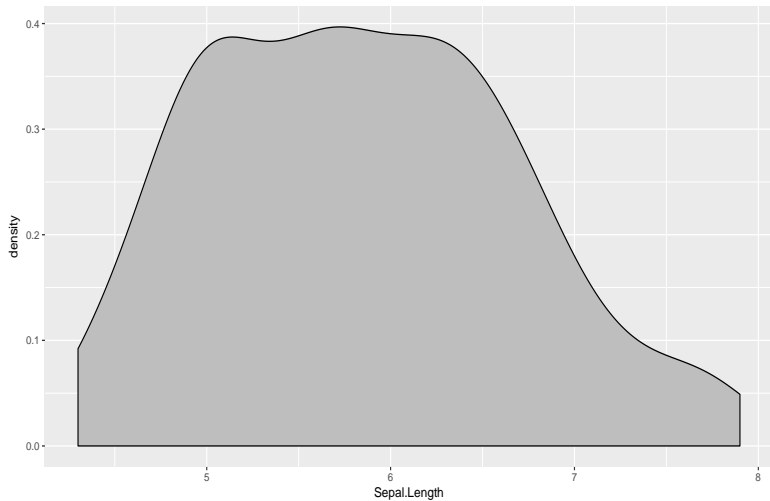
We can decompose TSS in terms of BSS and WSS:

$$\underbrace{\sum_{k=1}^K \sum_{i \in G_k} (x_{ik} - \bar{x})^2}_{\text{TSS}} = \underbrace{\sum_{k=1}^K n_k (\bar{x}_k - \bar{x})^2}_{\text{BSS}} + \underbrace{\sum_{i \in G_k} (x_{ik} - \bar{x}_k)^2}_{\text{WSS}}$$

In summary:

$$\text{TSS} = \text{BSS} + \text{WSS}$$

Density for Sepal Length



```
ggplot(data = iris, aes(x = Sepal.Length)) +  
  geom_density(fill = 'gray') +  
  ggtitle('Density for Sepal Length')
```

# TSS for Sepal.Length

```
x <- iris$Sepal.Length

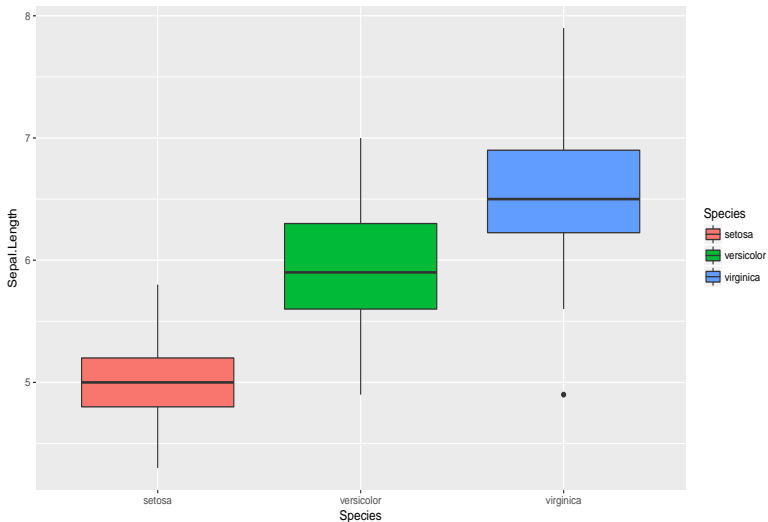
# overall mean
x_bar <- mean(x)
x_bar

## [1] 5.843333

# total sums-of-squares
tss <- sum((x - x_bar)^2)
tss

## [1] 102.1683
```

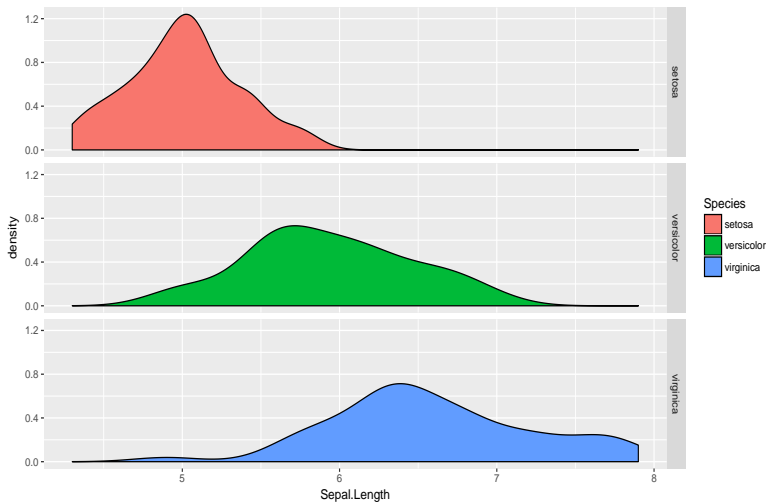
# Let's consider the group structure



```
ggplot(data = iris, aes(x = Species, y = Sepal.Length)) +  
  geom_boxplot(aes(fill = Species))
```



Sepal Length by Species



```
ggplot(data = iris, aes(x = Sepal.Length, group = Species)) +  
  geom_density(aes(fill = Species)) +  
  facet_grid(Species ~ .) +  
  ggtitle('Sepal Length by Species')
```

# BSS for Sepal.Length

```
# Sepal Length group means
x_means <- tapply(x, iris$Species, mean)

# between sums-of-squares
bss <- sum(50 * (x_means - x_bar)^2)
bss

## [1] 63.21213
```

# WSS for Sepal.Length

```
# Sepal Length group sum of squares
w1 <- sum((x[1:50] - x_means[1])^2)
w2 <- sum((x[51:100] - x_means[2])^2)
w3 <- sum((x[101:150] - x_means[3])^2)

# within sums-of-squares
wss <- w1 + w2 + w3
wss

## [1] 38.9562
```

# TSS Decomposition

Let's check that we have:

$$\text{TSS} = \text{BSS} + \text{WSS}$$

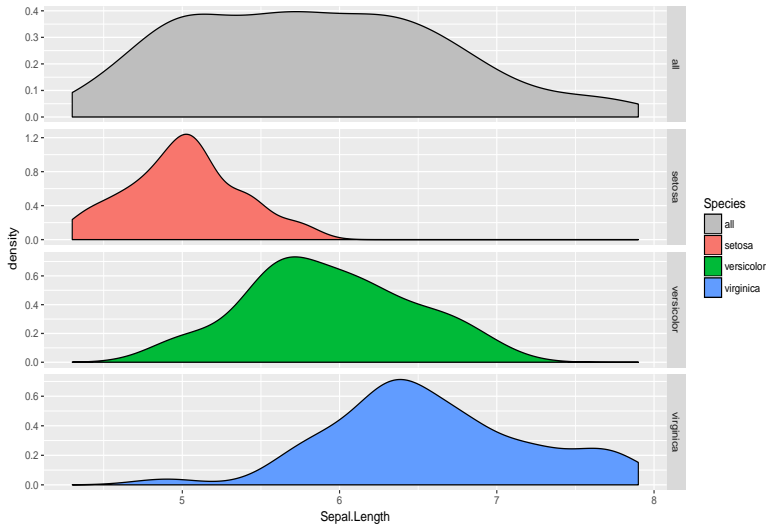
```
# tss
tss

## [1] 102.1683

# bss + wss
bss + wss

## [1] 102.1683
```

# Dispersion in Sepal.Length



Derived Ratios from  
 $TSS = BSS + WSS$

# Correlation Ratio

Correlation ratio  $\eta^2$  (proposed by K. Pearson):

$$\eta^2(X, Y) = \frac{\text{BSS}}{\text{TSS}}$$

- ▶  $\eta^2$  takes values between 0 and 1
- ▶  $\eta^2 = 0$  represents the special case of no dispersion among the means of the different categories
- ▶  $\eta^2 = 1$  refers to no dispersion within the respective categories.

The correlation ratio is a measure of the relationship between the dispersion within categories and the dispersion across all individuals.

# F Ratio

With  $TSS = BSS + WSS$ , we can also calculate:

$F$  ratio (proposed by R.A. Fisher):

$$F = \frac{BSS/(k - 1)}{WSS/(n - k)}$$

The larger the value of both ratios, the more variability is there between groups than within groups.



# Ratios for Sepal.Length

```
# correlation ratio
eta_sqr <- bss / tss
eta_sqr

## [1] 0.6187057

# F ratio
F_ratio <- (bss / (3 - 1)) / (wss / (150 - 3))
F_ratio

## [1] 119.2645
```

# Ratios for all Variables

Let's compute the decompositions for all predictors, and obtain the correlation ratios and F ratios

```
etas
```

```
## Sepal.Length Sepal.Width Petal.Length Petal.Width
##      0.6187057      0.4007828      0.9413717      0.9288829
```

```
Fs
```

```
## Sepal.Length Sepal.Width Petal.Length Petal.Width
##      119.26450      49.16004     1180.16118      960.00715
```

More Notation: generalization  
for more than 1 predictor

# Predictors and Response

- ▶  $p$  predictors  $X_1, X_2, \dots, X_p$
- ▶ One categorical response  $Y$  with  $K$  categories
- ▶  $Y$  introduces a group or class structure
- ▶ Observations divided in  $K$  groups or classes

## Caveat: messy notation

Here's some notation that I'll be using while covering classification methods:

Let  $n_k$  be the number of observations in the  $k$ -th group

Let  $x_{ijk}$  represent the  $i$ -th observation, of the  $j$ -th variable, in the  $k$ -th group.

Let  $x_{ik}$  represent  $i$ -th observation in group  $k$

Let  $x_{jk}$  represent  $j$ -th variable in group  $k$

I hope this doesn't create a lot of confusion

## Caveat: messy notation

Let  $n_k$  be the number of observations in the  $k$ -th group  $G_k$ , then:

$$n = n_1 + n_2 + \cdots + n_K = \sum_{k=1}^K n_k$$

## Caveat: messy notation

For a given variable  $X_j$ , represented with vector  $\mathbf{x}_j$ , we have:

Total or global mean:

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^n x_{ij}$$

Local mean of observations in group  $k$ :

$$\bar{x}_{jk} = \frac{1}{n_k} \sum_{i \in G_k} x_{ij}$$

where  $G_k$  represents the set of observations in group  $k$

## Caveat: messy notation

For a given variable  $X_j$ , represented with vector  $\mathbf{x}_j$ , we have:  
Total Sum of Squared deviations

$$\text{TSS}_j = \sum_{i=1}^n (x_{ij} - \bar{x}_j)^2$$

Assuming centered variables (mean = 0)

$$\text{TSS}_j = \mathbf{x}_j^\top \mathbf{x}_j$$



# Decomposition of sums-of-squares

An important aspect has to do with looking at the squared deviations:  $(x_{ij} - \bar{x}_j)^2$  in terms of the group structure.

A useful trick is to rewrite the deviation terms  $x_{ij} - \bar{x}_j$ , as:

$$\begin{aligned}x_{ij} - \bar{x}_j &= x_{ij} - (\bar{x}_{jk} - \bar{x}_{jk}) - \bar{x}_j \\&= (x_{ij} - \bar{x}_{jk}) + (\bar{x}_{jk} - \bar{x}_j)\end{aligned}$$

# Decomposition of Sums-of-Squares

Using the previous format of deviations, the sum of squared deviations can be decomposed as:

$$\sum_{i=1}^n (x_{ij} - \bar{x}_j)^2 = \sum_{k=1}^K n_k (\bar{x}_{jk} - \bar{x}_k)^2 + \sum_{k=1}^K \sum_{i \in G_k} (x_{ijk} - \bar{x}_{jk})^2$$

What's this?

# Decomposition of Sums-of-Squares

Using the previous format of deviations, the sum of squared deviations can be decomposed as:

$$\underbrace{\sum_{i=1}^n (x_{ij} - \bar{x}_j)^2}_{\text{Total SS}} = \underbrace{\sum_{k=1}^K n_k (\bar{x}_{jk} - \bar{x}_k)^2}_{\text{Between-groups SS}} + \underbrace{\sum_{k=1}^K \sum_{i \in G_k} (x_{ijk} - \bar{x}_{jk})^2}_{\text{Within-groups SS}}$$

# Decomposition of Variance

The sums-of-squares decompositions can be put in terms of **population** variances:

$$\underbrace{\frac{1}{n} \sum_{i=1}^n (x_{ij} - \bar{x}_j)^2}_{\text{Total variance}} = \underbrace{\sum_{k=1}^K \frac{n_k}{n} (\bar{x}_{jk} - \bar{x}_k)^2}_{\text{Between-groups variance}} + \underbrace{\frac{1}{n} \sum_{k=1}^K \sum_{i \in G_k} n_k (x_{ijk} - \bar{x}_{jk})^2}_{\text{Within-groups variance}}$$

Formula from one-way analysis of variance (anova)

# Decomposition of Variance

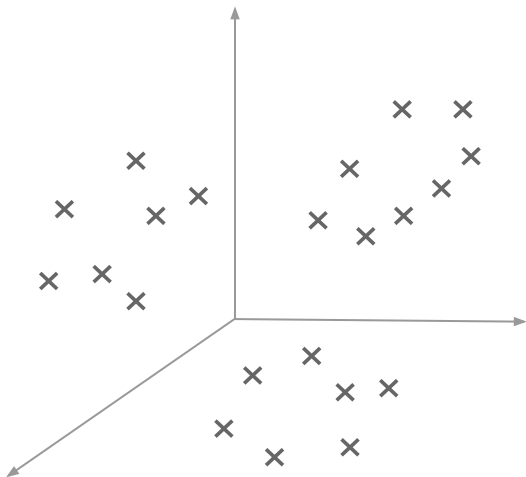
Alternatively, the sums-of-squares decompositions can also be put in terms of **sample** variances:

$$\text{TSS} = \underbrace{\frac{1}{n-1} \sum_{i=1}^n (x_{ij} - \bar{x}_j)^2}_{\text{Total variance}} =$$

$$\underbrace{\sum_{k=1}^K \frac{n_k}{n-1} (\bar{x}_{jk} - \bar{x}_k)^2}_{\text{Between-groups variance}} + \underbrace{\frac{1}{n-1} \sum_{k=1}^K \sum_{i \in G_k} (n_k - 1) (x_{ijk} - \bar{x}_{jk})^2}_{\text{Within-groups variance}}$$

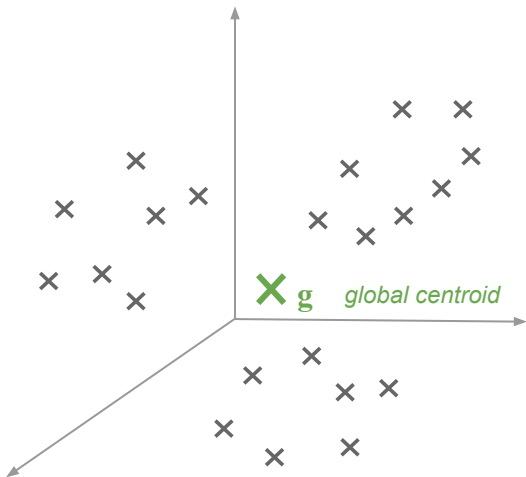
# Geometric Perspective

# Data as a cloud of points in $p$ -dim space



Cloud of  $n$  points in  $p$ -dimensional space

# Global centroid (center of gravity)



The *centroid*  $g$  is the point of averages



# Global Centroid

The global centroid  $\mathbf{g}$  is the point of averages which consists of the point formed with all the variable means:

$$\mathbf{g} = [\bar{x}_1, \bar{x}_2, \dots, \bar{x}_p]$$

where:

$$\bar{x}_j = \frac{1}{n} \sum_{i=1}^n x_{ij}$$

If all variables are mean-centered, the centroid is the origin

$$\mathbf{g} = \underbrace{[0, 0, \dots, 0]}_{p \text{ times}}$$

# Total Dispersion

Taking the global centroid as a point of reference, we can look at the amount of spread or dispersion in the data.

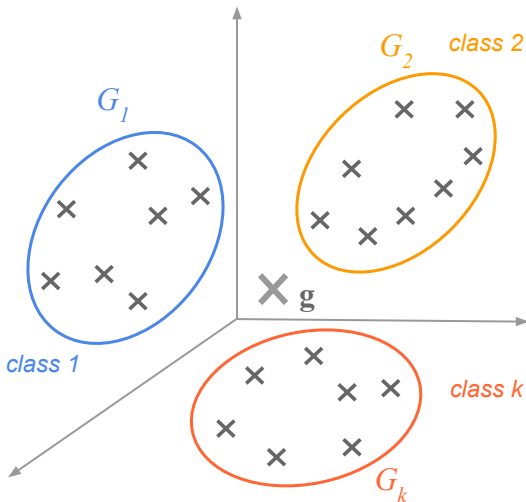
Assuming centered variables, a matrix of total dispersion is given by the *Total Sums of Squares* (TSS):

$$\text{TSS} = \mathbf{X}^T \mathbf{X}$$

Alternatively, we can get the variance-covariance matrix  $\mathbf{V}$ :

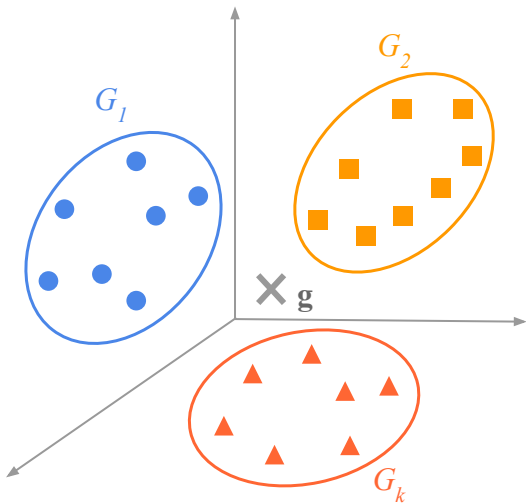
$$\mathbf{V} = \frac{1}{n-1} \mathbf{X}^T \mathbf{X}$$

# Class (group) structure



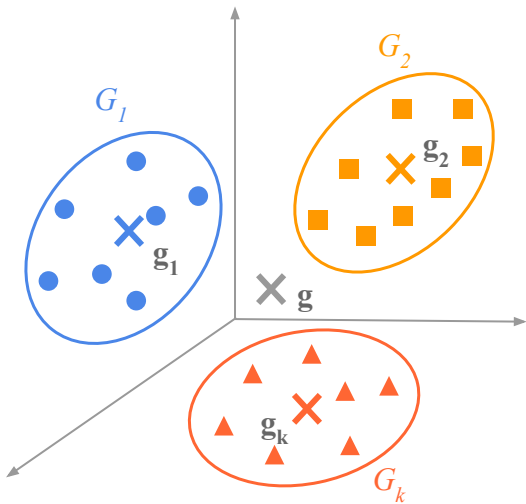
The objects are divided into classes or groups

# Sub-cloud of points for each group



Each group  $G_k$  forms its own sub-cloud

## Local or group centroids (one per class)



Each group  $G_k$  has its own centroid  $g_k$

# Group Centroids

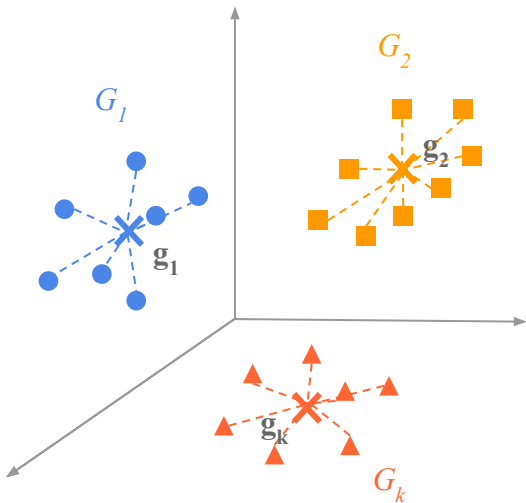
The group centroid  $\mathbf{g}_k$  is the point of averages for those observations in group  $k$ :

$$\mathbf{g}_k = [\bar{x}_{1k}, \bar{x}_{2k}, \dots, \bar{x}_{pk}]$$

where:

$$\bar{x}_{jk} = \frac{1}{n_k} \sum_{i \in G_k} x_{ij}$$

# Within-groups dispersion



We can focus on the dispersion within the clouds

# Dispersion inside a group

Each group will have an associated spread or dispersion matrix given by a *Group Sums of Squares* (GSS):

$$\text{GSS}_k = \mathbf{X}_k^T \mathbf{X}_k$$

Equivalently, there is an associated variance matrix  $\mathbf{W}_k$  for each group

$$\mathbf{W}_k = \frac{1}{n_k - 1} \mathbf{X}_k^T \mathbf{X}_k$$

where  $\mathbf{X}_k$  is the data matrix of the  $k$ -th group



# Within-groups dispersion

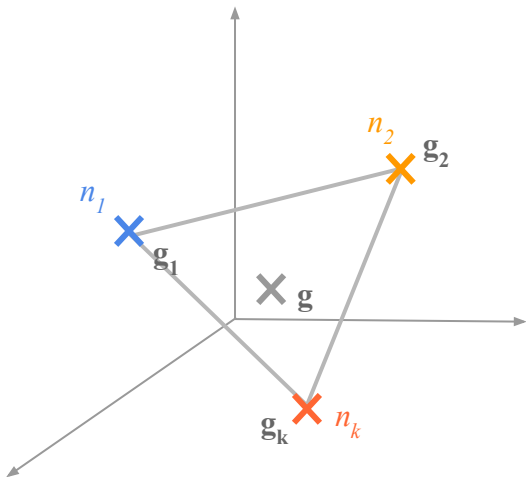
We can combine the groups dispersion to obtain a Within-groups Sums of Squares (WSS) matrix:

$$\text{WSS} = \sum_{k=1}^K \mathbf{X}_k^{\top} \mathbf{X}_k$$

Likewise, we can combine the group variances  $\mathbf{W}_k$  as a weighted average to get the **Within-groups** variance matrix  $\mathbf{W}$ :

$$\mathbf{W} = \sum_{k=1}^K \frac{n_k - 1}{n - 1} \mathbf{W}_k$$

# Global and Group Centroids



What if we focus on just the centroids?

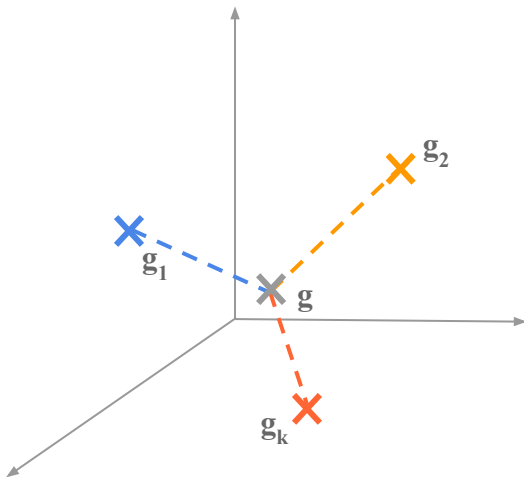
# Global and Group Centroids

Note that the global centroid  $\mathbf{g}$  can be expressed as a weighted average of the group centroids:

$$\mathbf{g} = \frac{n_1}{n} \mathbf{g}_1 + \frac{n_2}{n} \mathbf{g}_2 + \cdots + \frac{n_K}{n} \mathbf{g}_K$$

$$\mathbf{g} = \sum_{k=1}^K \left( \frac{n_k}{n} \right) \mathbf{g}_k$$

# Between-groups dispersion



We can focus on the dispersion between the centroids

# Dispersion between groups

Focusing on just the centroids, we can get its corresponding matrix of dispersion given by the *Between Sums of Squares* (BSS):

$$\text{BSS} = \sum_{k=1}^K n_k (\mathbf{g}_k - \mathbf{g})(\mathbf{g}_k - \mathbf{g})^\top$$

Equivalently, there is an associated **Between-groups** variance matrix  $\mathbf{B}$

$$\mathbf{B} = \sum_{k=1}^K \frac{n_k}{n-1} (\mathbf{g}_k - \mathbf{g})(\mathbf{g}_k - \mathbf{g})^\top$$

# Three types of Dispersions

Let's recap. We have three types of sums-of-squares matrices:

- ▶ TSS: Total Sums fo Squares
- ▶ WSS: Within-groups Sums fo Squares
- ▶ BSS: Between-groups Sums fo Squares

Alternatively, we also have three types of variance matrices:

- ▶ **V**: Total variance
- ▶ **W**: Within-groups variance
- ▶ **B**: Between-groups variance

# Dispersion Decomposition

It can be shown (Huygens theorem) for both, sums-of-squares and variances, that the total dispersion (TSS or  $V$ ) can be decomposed as:

- ▶  $TSS = BSS + WSS$
- ▶  $V = B + W$

# Dispersion Decomposition

Let  $\mathbf{X}$  be the  $n \times p$  mean-centered matrix of predictors, and  $\mathbf{Y}$  be the  $n \times K$  dummy matrix of groups

- ▶  $\text{TSS} = \mathbf{X}^T \mathbf{X}$
- ▶  $\text{BSS} = \mathbf{X}^T \mathbf{Y} (\mathbf{Y}^T \mathbf{Y})^{-1} \mathbf{Y}^T \mathbf{X}$
- ▶  $\text{WSS} = \mathbf{X}^T (\mathbf{I} - \mathbf{Y} (\mathbf{Y}^T \mathbf{Y})^{-1} \mathbf{Y}^T) \mathbf{X}$



# Bibliography

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- ▶ **Multivariate Analysis** by Maurice Tatsuoka (1988). *Chapter 7: Discriminant Analysis and Canonical Correlation.*

# French Literature

- ▶ **Statistique Exploratoire Multidimensionnelle** by Lebart et al (2004). *Chapter 3, section 3: Analyse factorielle discriminante.* Dunod, Paris.
- ▶ **Probabilites, analyse des donnees et statistique** by Gilbert Saporta (2011). *Chapter 18: Analyse discriminante et regression logistique.* Editions Technip, Paris.
- ▶ **Statistique explicative appliquee** by Nakache and Confais (2003). *Chapter 1: Analyse discriminante sur variables quantitatives.* Editions Technip, Paris.
- ▶ **Statistique: Methodes pour decrire, expliquer et prevoir** by Michel Tenenhaus (2008). *Chapter 10: L'analyse discriminante.* Dunod, Paris.