Math 115A - Spring 2019
Final Exam

Full Name: ____________________________________

UID: ____________________________________

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- All work including proofs should be well organized and clearly written using complete sentences.
- You may use the provided scratch paper, however this work will not be graded unless very clearly indicated there and in the exam.
- Calculators are not allowed but you may have a 3×5 inch notecard.

Page	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	

Page	Points	Score
6	10	
7	10	
8	10	
Bonus		
Total:	80	

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You may use this page for scratch work. Work found on this page will not be graded unless clearly indicated here and in the exam.

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1. (10 points) Let $V = M_{2\times 2}(\mathbb{C})$. A square root of a matrix A is a matrix B such that $B^2 = A$.

   (a) Let

   $$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

   for a fixed $\lambda \in \mathbb{C}$ with $\lambda \neq 0$. Find a square root of A.

   (b) Show that if a matrix C admits a square root, then every matrix similar to C also admits a square root.

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2. (10 points) Consider the vector space $V = \mathbb{R}^4$ with the standard inner product. Let

$$S = \{w_1 = (1,1,0,0), w_2 = (2,2,1,1), w_3 = (0,-1,1,2)\}.$$

(a) Apply the Gram-Schmidt orthogonalization algorithm to S to compute an orthogonal basis β' for $\operatorname{span}(S)$. You may use that S is linearly independent.

(b) Use your result from part (a) to compute an orthonormal basis β of $\operatorname{span}(S)$. Then let $x = (1,2,3,2) \in \operatorname{span}(S)$ and compute the coordinate vector $[x]_\beta$.

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3. (10 points) Let V be a finite-dimensional complex inner product space. Let $S = \{v_1, \ldots, v_n\}$ be an orthogonal list of nonzero vectors in V.

(a) Show that S is linearly independent.

(b) Given $\lambda_i \in \mathbb{C}$ for $1 \leq i \leq n$ show that

$$\left|\left|\sum_{i=1}^{n} \lambda_i v_i\right|\right|^2 = \sum_{i=1}^{n} |\lambda_i|^2 ||v_i||^2.$$

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4. (10 points) True or False: Prove or disprove the following statements.

   (a) The function $\det : M_{n\times n}(\mathbb{F}) \to \mathbb{F}$ sending a matrix A to its determinant $\det(A)$ is a linear map.

   (b) Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear operator. Then $V = \ker(T) \oplus \operatorname{im}(T)$.

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5. (10 points) Let $V = P_2(\mathbb{R})$ and let $\beta = \{1, x, x^2\}$ be the standard basis for V. Consider the linear map $T : V \to V$ defined by

$$T(f(x)) = f(x) + xf''(x).$$

(a) Find $[T]_\beta^\beta$.

(b) Compute the characteristic polynomial of T. Find all eigenvalues of T and their algebraic multiplicities.

(c) Is T diagonalizable? If so, find a basis γ of eigenvectors for T and write $[T]_\gamma^\gamma$.

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6. (10 points) True or False: Prove or disprove the following statements.

Let V be a finite-dimensional inner product space and let $T : V \to V$ be a linear operator.

(a) Suppose $\mathbb{F} = \mathbb{C}$. If T is self-adjoint then $\langle Tv, v \rangle$ is real for all $v \in V$.

(b) Suppose $\mathbb{F} = \mathbb{R}$. If T is normal then T is diagonalizable.

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7. (10 points) Let V, W, and Z be finite-dimensional vector spaces each of dimension n. Let $T : V \to W$ and $S : W \to Z$ be linear maps. Show that ST is an isomorphism if and only if both S and T are isomorphisms.

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8. (10 points) Let V be a finite-dimensional inner product space over the field $\mathbb{F} = \mathbb{R}$.

   (a) Let $T : V \to V$ be a self-adjoint linear operator whose only eigenvalues are zero and one. Show $T^m = T$ for all $m \geq 1$.

   (b) Suppose $V = W \oplus W^{\perp}$ for some subspace $W \subseteq V$. Let $T : V \to V$ be the orthogonal projection onto W along $W^{\perp}$. Show that T is self-adjoint.

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