COMP9020

Foundations of Computer Science

Lecture 10: Induction

Outline

Outline

Recursive datatypes

Describe arbitrarily large objects in a finite way

Recursive functions

Define behaviour for these objects in a finite way

Induction

Reason about these objects in a finite way

Example

Recall the recursive program:

Example

Summing the first n natural numbers:

$$
\begin{array}{l}
\texttt{sum}(n): \\
\quad \texttt{if}(n = 0)\text{: } 0 \\
\quad \texttt{else}\text{: } n + \texttt{sum}(n - 1)
\end{array}
$$

Another attempt:

Example

$$
\begin{array}{l}
\texttt{sum2}(n): \\
\quad \texttt{return}\ n * (n + 1)/2
\end{array}
$$

Induction proof **guarantees** that these programs will behave the same.

Inductive Reasoning

Suppose we would like to reach a conclusion of the form

$$P(x) \textit{ for all } x \text{ (of some type)}$$

Inductive reasoning (as understood in philosophy) proceeds from examples.
E.g. From "This swan is white, that swan is white, in fact every swan I have seen so far is white"
Conclude: "Every Swan is white"

NB

This may be a good way to discover hypotheses.
But it is not a valid principle of reasoning!

Mathematical induction *is a variant that is valid.*

Mathematical Induction

Mathematical Induction is based not just on a set of examples, but also a rule for deriving new cases of $P(x)$ from cases for which P is known to hold.

General structure of reasoning by mathematical induction:

Base Case [B]: $P(a_1), P(a_2), \ldots, P(a_n)$ for some small set of examples $a_1 \ldots a_n$ (often $n = 1$)

Inductive Step [I]: A general rule showing that if $P(x)$ holds for some cases $x = x_1, \ldots, x_k$ then $P(y)$ holds for some new case y, constructed in some way from $x_1, \ldots, x_k$.

Conclusion: Starting with $a_1 \ldots a_n$ and repeatedly applying the construction of y from existing values, we can eventually construct all values in the domain of interest.

Induction proof structure

Let $P(x)$ be the proposition that ...
We will show that $P(x)$ holds for all x by induction on x.
Base case: $x = ...$**:**

- $P(x)$: ...
- ....
- so $P(x)$ holds.

[Repeat for all base cases]

Inductive case:

- Assume $P(x)$ holds. That is, ...
- We will show $P(y)$ holds.
- ...
- So $P(x)$ implies $P(y)$.

[Repeat for all inductive cases]

Therefore, by induction, $P(x)$ holds for all x.

Outline

Motivation

Basic Induction

Variations on Basic Induction

Structural Induction

Basic induction

Basic induction is the general principle applied to the natural numbers.

Goal: Show $P(n)$ holds for all $n \in \mathbb{N}$.

Approach: Show that:

Base case (B): $P(0)$ holds; and

Inductive case (I): If $P(k)$ holds then $P(k+1)$ holds.

Example

Recall the recursive program:

Example

Summing the first n natural numbers:

$$\begin{array}{l} \texttt{sum}(n): \\ \quad \texttt{if}(n = 0): 0 \\ \quad \texttt{else}: n + \texttt{sum}(n-1) \end{array}$$

Another attempt:

Example

$$\begin{array}{l} \texttt{sum2}(n): \\ \quad \texttt{return}\ n * (n+1)/2 \end{array}$$

Induction proof **guarantees** that these programs will behave the same.

Example

Let $P(n)$ be the proposition that:

$$P(n): \qquad \sum_{i=0}^{n} i \;=\; \frac{n(n+1)}{2}.$$

We will show that $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Proof.

[B] $P(0)$, i.e.

$$\sum_{i=0}^{0} i \;=\; \frac{0(0+1)}{2}$$

[I] $\forall k \geq 0\,(P(k) \rightarrow P(k+1))$, i.e.

$$\sum_{i=0}^{k} i \;=\; \frac{k(k+1)}{2} \;\;\rightarrow\;\; \sum_{i=0}^{k+1} i \;=\; \frac{(k+1)(k+2)}{2}$$

(proof?) $\square$

Example (cont'd)

Proof.

Inductive step [I]:

$$
\begin{aligned}
\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i \right) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad \text{(by the inductive hypothesis)} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

$\square$

Outline

Motivation

Basic Induction

Variations on Basic Induction

Structural Induction

Variations

1. Induction from m upwards
2. Induction steps > 1
3. Strong induction
4. Backward induction
5. Forward-backward induction
6. Structural induction

Induction From m Upwards

If
[B] $P(m)$
[I] $\forall k \geq m\,(P(k) \to P(k+1))$
then
[C] $\forall n \geq m\,(P(n))$

Example

Theorem. For all $n \geq 1$, the number $8^n - 2^n$ is divisible by 6.

[B] $8^1 - 2^1$ is divisible by 6
[I] if $8^k - 2^k$ is divisible by 6, then so is $8^{k+1} - 2^{k+1}$, for all $k \geq 1$

Prove [I] using the "trick" to rewrite 8^{k+1} as $8 \cdot (8^k - 2^k + 2^k)$ which allows you to apply the IH on $8^k - 2^k$

Induction Steps $\ell > 1$

If

[B] $P(m)$

[I] $P(k) \rightarrow P(k+\ell)$ for all $k \geq m$

then

[C] $P(n)$ for every ℓ'th $n \geq m$

Example

Every 4th Fibonacci number is divisible by 3.

[B] $F_4 = 3$ is divisible by 3
[I] if $3 \mid F_k$, then $3 \mid F_{k+4}$, for all $k \geq 4$

Prove [I] by rewriting F_{k+4} in such a way that you can apply the IH on F_k

Strong Induction

This is a version in which the inductive hypothesis is stronger. Rather than using the fact that $P(k)$ holds for a single value, we use *all* values up to k.

If
[B] $P(m)$
[I] $[P(m) \wedge P(m+1) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ for all $k \geq m$
then
[C] $P(n)$, for all $n \geq m$

Example

Claim: All integers ≥ 2 can be written as a product of primes.

[B] 2 is a product of primes

[I] If all x with $2 \leq x \leq k$ can be written as a product of primes, then $k+1$ can be written as a product of primes, for all $k \geq 2$

Proof for [I]?

Negative Integers, Backward Induction

NB

Induction can be conducted over any subset of $\mathbb{Z}$ with least element. Thus m can be negative; eg. base case $m = -10^6$.

NB

One can apply induction in the 'opposite' direction $p(m) \to p(m-1)$. It means considering the integers with the opposite ordering where the next number after n is $n-1$. Such induction would be used to prove some $p(n)$ for all $n \leq m$.

NB

Sometimes one needs to reason about all integers $\mathbb{Z}$. This requires two separate simple induction proofs: one for $\mathbb{N}$, another for $-\mathbb{N}$. They both would start form some initial values, which could be the same, e.g. zero. Then the first proof would proceed through positive integers; the second proof through negative integers.

Forward-Backward Induction

Idea

To prove $P(n)$ *for all* $n \geq k_0$

- *verify* $P(k_0)$
- *prove* $P(k_i)$ *for infinitely many* $k_0 < k_1 < k_2 < k_3 < \dots$
- *fill the gaps*
  $P(k_1) \to P(k_1 - 1) \to P(k_1 - 2) \to \dots \to P(k_0 + 1)$
  $P(k_2) \to P(k_2 - 1) \to P(k_2 - 2) \to \dots \to P(k_1 + 1)$
  $\dots\dots\dots$

NB

This form of induction is extremely important for the analysis of algorithms.

Outline

Structural Induction

Basic induction allows us to assert properties over **all natural numbers**. The induction scheme (layout) uses the recursive definition of $\mathbb{N}$.

The induction schemes can be applied not only to natural numbers (and integers) but to any partially ordered set in general – especially those defined recursively.

The basic approach is always the same — we need to verify that

- **[B]** the property holds for all minimal objects — objects that have no predecessors; they are usually very simple objects allowing immediate verification
- **[I]** for any given object, if the property in question holds for all its predecessors (‘smaller’ objects) then it holds for the object itself

Example: Induction on Σ^*

Recall definition of Σ^*:

$$\lambda \in \Sigma^*$$

$$\text{If } w \in \Sigma^* \text{ then } aw \in \Sigma^* \text{ for all } a \in \Sigma$$

Structural induction on Σ^*:

Goal: Show $P(w)$ holds for all $w \in \Sigma^*$.

Approach: Show that:

Base case (B): $P(\lambda)$ holds; and

Inductive case (I): If $P(w)$ holds then $P(aw)$ holds for all $a \in \Sigma$.

Example: Induction on Σ^*

Recall:

Formal definition of Σ^*:

$$\lambda \in \Sigma^*$$

$$\text{If } w \in \Sigma^* \text{ then } aw \in \Sigma^* \text{ for all } a \in \Sigma$$

Formal definition of concatenation:

(concat.B) $\lambda v = v$

(concat.I) $(aw)v = a(wv)$

Formal definition of length:

(length.B) $\text{length}(\lambda) = 0$

(length.I) $\text{length}(aw) = 1 + \text{length}(w)$

Prove:

$$\text{length}(wv) = \text{length}(w) + \text{length}(v)$$

Example: Induction on Σ^*

Let $P(w)$ be the proposition that, for all $v \in \Sigma^*$:

$$\text{length}(wv) = \text{length}(w) + \text{length}(v).$$

We will show that $P(w)$ holds for all $w \in \Sigma^*$ by **structural induction on** w**.**

Proof:
Base case ($w = \lambda$**):**

$$\begin{aligned}
\text{length}(\lambda v) &= \text{length}(v) && \text{(concat.B)} \\
&= 0 + \text{length}(v) \\
&= \text{length}(w) + \text{length}(v) && \text{(length.B)}
\end{aligned}$$

Example: Induction on Σ^*

Proof cont'd:

Inductive case ($w = aw'$): Assume that $P(w')$ holds. That is, for all $v \in \Sigma^*$:

$$\text{(IH)}: \qquad \text{length}(w'v) = \text{length}(w') + \text{length}(v).$$

Then, for all $a \in \Sigma$, we have:

$$\begin{aligned}
\text{length}((aw')v) &= \text{length}(a(w'v)) && \text{(concat.I)} \\
&= 1 + \text{length}(w'v) && \text{(length.I)} \\
&= 1 + \text{length}(w') + \text{length}(v) && \text{(IH)} \\
&= \text{length}(aw') + \text{length}(v) && \text{(length.I)}
\end{aligned}$$

So $P(aw')$ holds.

We have $P(\lambda)$ and for all $w' \in \Sigma^*$ and $a \in \Sigma$: $P(w') \to P(aw')$. Hence $P(w)$ holds for all $w \in \Sigma^*$.

Example 2: Induction on Σ^*

Recall $\text{append} : \Sigma^* \times \Sigma \to \Sigma^*$ defined as:

- $\text{append}(\lambda, x) = x$
- $\text{append}(aw, x) = a\,(\text{append}(w, x))$

Prove:

For all $w, v \in \Sigma^*$ and $x \in \Sigma$:

$$\text{append}(wv, x) = w(\text{append}(v, x))$$

Example 2: Induction on Σ^*

Theorem

For all $w, v \in \Sigma^*$ *and* $x \in \Sigma$*:* $\mathit{append}(wv, x) = w(\mathit{append}(v, x))$.

Proof: By induction on w...

$$
\begin{array}{llrl}
\textbf{[B]} & \text{append}(\lambda v, x) & = \text{append}(v, x) & \text{(concat.B)} \\
\textbf{[I]} & \text{append}((aw)v, x) & = \text{append}(a(wv), x) & \text{(concat.I)} \\
 & & = a\,\text{append}(wv, x) & \text{(append.I)} \\
 & & = a\,(w\,\text{append}(v, x)) & \text{(IH)} \\
 & & = (aw)\,\text{append}(v, x) & \text{(concat.I)}
\end{array}
$$

Example 3: Induction on Σ^*

Define $\mathsf{rev} : \Sigma^* \to \Sigma^*$:

(rev.B) $\mathsf{rev}(\lambda) = \lambda$,

(rev.I) $\mathsf{rev}(a \cdot w) = \mathsf{append}(\mathsf{reverse}(w), a)$

Example 3: Induction on Σ^*

Theorem

For all $w, v \in \Sigma^*$, $\mathit{reverse}(wv) = \mathit{reverse}(v) \cdot \mathit{reverse}(w)$.

Proof: By induction on w...

$$
\begin{array}{llll}
\textbf{[B]} & \text{rev}(\lambda v) & = \text{rev}(v) & \text{(concat.B)} \\
& & = \text{rev}(v)\lambda & (*) \\
& & = \text{rev}(v)\text{rev}(\lambda) & \text{(rev.B)} \\
\\
\textbf{[I]} & \text{rev}((aw')v) & = \text{rev}(a(w'v)) & \text{(concat.I)} \\
& & = \text{append}(\text{rev}(w'v), a) & \text{(rev.I)} \\
& & = \text{append}(\text{rev}(v)\text{rev}(w'), a) & \text{(IH)} \\
& & = \text{rev}(v)\text{append}(\text{rev}(w'), a) & \text{(Example 2)} \\
& & = \text{rev}(v)\text{rev}(aw') & \text{(rev.I)}
\end{array}
$$

Example 4: Induction on more complex structures

Recall expressions in the Proof assistant:

- (B) $A, B, \ldots, Z, a, b, \ldots z$ are expressions
- (B) $\emptyset$ and $\mathcal{U}$ are expressions
- (R) If E is an expression then so is (E) and E^c
- (R) If E_1 and E_2 are expressions then:
  - $(E_1 \cup E_2)$,
  - $(E_1 \cap E_2)$,
  - $(E_1 \setminus E_2)$, and
  - $(E_1 \oplus E_2)$ are expressions.

Theorem

In any valid expression, the number of (equals the number of)

Proof: By induction on the structure of E...

Exercise

Exercise

$\boxed{\text{RW: 4.4.2}}$ Define $s_1 = 1$ and $s_{n+1} = \frac{1}{1+s_n}$ for $n \geq 1$

Then $s_1 = 1,\ s_2 = \frac{1}{2},\ s_3 = \frac{2}{3},\ s_4 = \frac{3}{5},\ s_5 = \frac{5}{8}, \ldots$

The numbers in numerator and denominator remind one of the Fibonacci sequence.

Prove by induction that

$$s_n = \frac{\mathrm{FIB}(n)}{\mathrm{FIB}(n+1)}$$