Due: Sunday, 3rd November, 23:59

Submission is through WebCMS/give and should be a single pdf file, maximum size 2Mb. Prose should be typed, not handwritten. Use of LaTeX is encouraged, but not required.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (20 marks)

Recall the relation composition operator ; defined as:

$$R_1; R_2 = \{(a, c) : \text{there is a } b \text{ with } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

For any set S, and any binary relations $R_1, R_2, R_3 \subseteq S \times S$, prove or give a counterexample to disprove the following:

(a) $(R_1; R_2); R_3 = R_1; (R_2; R_3)$ (4 marks)

(b) $I; R_1 = R_1; I = R_1$ where $I = \{(x, x) : x \in S\}$ (4 marks)

(c) $(R_1; R_2)^{\leftarrow} = R_1^{\leftarrow}; R_2^{\leftarrow}$ (4 marks)

(d) $(R_1 \cup R_2); R_3 = (R_1; R_3) \cup (R_2; R_3)$ (4 marks)

(e) $R_1; (R_2 \cap R_3) = (R_1; R_2) \cap (R_1; R_3)$ (4 marks)

Solution

(a) We have:

$$\begin{array}{rll} (a,d) \in (R_1;R_2);R_3 & \text{iff} & \text{there exists } c \in S \text{ such that } (a,c) \in R_1;R_2 \text{ and } (c,d) \in R_3 \\ & \text{iff} & \text{there exists } b,c \in S \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_2 \text{ and } (c,d) \in R_3 \\ & \text{iff} & \text{there exists } b \in S \text{ such that } (a,b) \in R_1 \text{ and } (b,d) \in R_2;R_3 \\ & \text{iff} & (a,d) \in R_1;(R_2;R_3) \end{array}$$

(b) Suppose $(a,b) \in R$. Then, because $(a,a) \in I$ we have $(a,b) \in I;R$. Also, because $(b,b) \in I$ we have $(a,b) \in R;I$.

Now suppose $(a,b) \in I;R$. Then there exists $c \in S$ such that $(a,c) \in I$ and $(c,b) \in R$. But from the definition of I, the only such c is $c = a$, so $(a,b) \in R$.

Finally suppose $(a,b) \in R;I$. Then there exists $c \in S$ such that $(a,c) \in R$ and $(c,b) \in I$. Again, from the definition of I, the only such c is $c = b$, so $(a,b) \in R$.

(c) This is not correct. Consider $S = \{a,b,c\}$, $R_1 = \{(a,b)\}$ and $R_2 = \{(b,c)\}$. Then,

$$\begin{array}{ccc} R_1;R_2 = \{(a,c)\} & & (R_1;R_2)^{\leftarrow} = \{(c,b)\} \\ R_1^{\leftarrow} = \{(b,a)\} & R_2^{\leftarrow} = \{(b,c)\} & R_1^{\leftarrow};R_2^{\leftarrow} = \emptyset \end{array}$$

It is in fact possible to show that $(R_1;R_2)^{\leftarrow} = R_2^{\leftarrow};R_1^{\leftarrow}$.

(d) We have:

$$\begin{array}{rll} (a,c) \in (R_1 \cup R_2);R_3 & \text{iff} & \text{there exists } b \in S \text{ such that } (a,b) \in R_1 \cup R_2 \text{ and } (b,c) \in R_3 \\ & \text{iff} & \text{there exists } b \in S \text{ such that } (a,b) \in R_1 \text{ and } (b,c) \in R_3, \text{ or} \\ & & \quad \text{there exists } b \in S \text{ such that } (a,b) \in R_2 \text{ and } (b,c) \in R_3 \\ & \text{iff} & (a,c) \in R_1;R_3 \text{ or } (a,c) \in R_2;R_3 \\ & \text{iff} & (a,c) \in R_1;R_3 \cup R_2;R_3 \end{array}$$

(e) This is not correct. Consider $S = \{a,b,c,d\}$ with $R_1 = \{(a,b),(a,c)\}$, $R_2 = \{(b,d)\}$, and $R_3 = \{(c,d)\}$. Then

$$\begin{array}{ccc} R_2 \cap R_3 = \emptyset & & R_1;(R_2 \cap R_3) = \emptyset \\ R_1;R_2 = \{(a,d)\} & R_1;R_3 = \{(a,d)\} & R_1;R_2 \cap R_1;R_3 = \{(a,d)\} \end{array}$$

Discussion

For each question:

- Minor errors for small logical omissions (e.g. not showing that the counterexamples work)
- Major errors include only showing one "direction" of the equality (but correctly stating whether the statement is true/false); not giving a concrete counterexample (i.e. justification for false has ambiguity)
- Shows progress includes identifying if the statement is true/false without justification.

Problem 2 (30 marks)

Let $R \subseteq S \times S$ be any binary relation on a set S. Consider the sequence of relations $R^0, R^1, R^2, \ldots$, defined as follows:

$$\begin{aligned} R^0 &:= I = \{(x,x) : x \in S\}, \text{ and} \\ R^{i+1} &:= R^i \cup (R; R^i) \text{ for } i \geq 0 \end{aligned}$$

(a) Prove that if there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$. (4 marks)

(b) Prove that if there is an i such that $R^i = R^{i+1}$, then $R^k \subseteq R^i$ for all $k \geq 0$. (4 marks)

(c) Let $P(n)$ be the proposition that for all $m \in \mathbb{N}$: $R^n; R^m = R^{n+m}$. Prove that $P(n)$ holds for all $n \in \mathbb{N}$. (8 marks)

(d) If $|S| = k$, explain why $R^k = R^{k+1}$. (*Hint: Show that if $(a,b) \in R^{k+1}$ then $(a,b) \in R^i$ for some $i < k+1$.*) (4 marks)

(e) If $|S| = k$, show that R^k is transitive. (4 marks)

(f) If $|S| = k$, show that $(R \cup R^{\leftarrow})^k$ is an equivalence relation. (6 marks)

Solution

(a) Suppose $R^i = R^{i+1}$. Let $P(j)$ be the proposition that $R^j = R^i$. We will prove that $P(j)$ holds for all $j \geq i$.

Base case $j = i$: Clearly $P(i)$ holds as $R^i = R^i$.

Inductive case. Suppose $P(j)$ holds, that is, $R^j = R^i$ for some $j \geq i$. We will show that $P(j+1)$ holds. We have:

$$\begin{aligned} R^{j+1} &= R^j \cup (R; R^j) && \text{(Definition)} \\ &= R^i \cup (R; R^i) && \text{(IH)} \\ &= R^{i+1} && \text{(Definition)} \\ &= R^i && \text{(Given)} \end{aligned}$$

So $P(j)$ implies $P(j+1)$. So by the principle of mathematical induction, $P(j)$ holds for all $j \geq i$.

(b) We first show that if $k \leq j$ then $R^k \subseteq R^j$. We prove this (for any k) by induction on j:

Base case $j = k$: Clearly $R^k \subseteq R^k$.

Inductive case: Suppose $j \geq k$ and $R^k \subseteq R^j$. Then

$$R^k \subseteq R^j \subseteq R^j \cup (R; R^j) = R^{j+1}.$$

Therefore, by the principle of induction, for all k, j, if $k \leq j$ then $R^k \subseteq R^j$.

It follows that if $R^i = R^{i+1}$ then for $k \leq i$ we have $R^k \subseteq R^i$, and for $k \geq i$ we have (from (a)) that $R^k = R^i \subseteq R^i$.

Solution (ctd)

(c) We will prove $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base case $n = 0$: For all m,

$$\begin{aligned} R^{0+m} &= R^m \\ &= I; R^m \quad \text{(From Q1(b))} \\ &= R^0; R^m \quad \text{(Def. of } R^0\text{)} \end{aligned}$$

Inductive case. Suppose $P(n)$ holds, that is, for all m, $R^n; R^m = R^{n+m}$. We have, for all m:

$$\begin{aligned} R^{n+1}; R^m &= ((R^n \cup (R; R^n)); R^m & \text{(Definition)} \\ &= (R^n; R^m) \cup ((R; R^n); R^m) & \text{(From Q1(d))} \\ &= (R^n; R^m) \cup (R; (R^n; R^m)) & \text{(From Q1(a))} \\ &= R^{n+m} \cup (R; R^{n+m}) & \text{(IH)} \\ &= R^{n+m+1} & \text{(Definition)} \\ &= R^{(n+1)+m} \end{aligned}$$

So $P(n+1)$ holds. Therefore, by the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{N}$.

(d) From (b) we have that $R^k \subseteq R^{k+1}$. We will show that $R^{k+1} \subseteq R^k$.

We observe that $(a, b) \in R^j$ if and only if there exists $c_0, \ldots, c_p \in S$, with $p \leq j$, $a = c_0$, $b = c_p$ and $(c_i, c_{i+1}) \in R$ for all $i \in [0, p]$.

Therefore, if $(a, b) \in R^{k+1}$ then either $p \leq k$, in which case $(a, b) \in R^p \subseteq R^k$; or $p = k+1$. In the latter case, if $|S| = k$ then there must exist $q, r \in [0, p]$ with $q < r$ and $c_q = c_r$. But then the we have $c_0, c_1, \ldots c_q, c_{r+1}, c_{r+2}, \ldots c_p$ as a sequence of $p - (r - q) < k + 1$ elements of S meeting the above observation. Therefore $(a, b) \in R^{p-(r-q)} \subseteq R^k$.

(e) From (d) we have that if $|S| = k$ then $R^k = R^{k+1}$. Now suppose $(a, b) \in R^k$ and $(b, c) \in R^k$. Then

$$\begin{aligned} (a, c) &\in R^k; R^k & \text{(Definition of ;)} \\ &= R^{2k} & \text{(From (c))} \\ &= R^k & \text{(From (a))} \end{aligned}$$

So R^k is transitive.

(f) We need to show that $(R \cup R^{\leftarrow})^k$ is reflexive, symmetric and transitive.

- From (b), we have that $I = (R \cup R^{\leftarrow})^0 \subseteq (R \cup R^{\leftarrow})^k$, so for all $a \in S$ we have that $(a, a) \in (R \cup R^{\leftarrow})^k$. Therefore $(R \cup R^{\leftarrow})^k$ is reflexive.
- From (e) we have that $(R \cup R^{\leftarrow})^k$ is transitive.
- Suppose $(a, b) \in (R \cup R^{\leftarrow})^k$. Following the observation in (d) we have that there exists $c_0, \ldots, c_p \in S$, with $p \leq k$, $a = c_0$, $b = c_p$ and $(c_i, c_{i+1}) \in R \cup R^{\leftarrow}$ for all $i \in [0, p]$. But if $(c_i, c_{i+1}) \in R \cup R^{\leftarrow}$ then $(c_{i+1}, c_i) \in R \cup R^{\leftarrow}$. Therefore, there exists $c_p, c_{p-1}, \ldots, c_0$ with $p \leq k$, $a = c_0$, $b = c_p$ and $(c_{i+1}, c_i) \in R \cup R^{\leftarrow}$ for all $i \in [0, p]$, so $(b, a) \in (R \cup R^{\leftarrow})^k$. Therefore $(R \cup R^{\leftarrow})^k$ is symmetric.

Discussion

- (a) and (c) should use induction (or some other sound method). Induction is not necessary for (b) or (d).
- Each property's proof in (f) is out of 2 marks
- Minor errors include: not properly referencing earlier results (applied once per part); missing a base case in induction proofs
- Major errors include: missing inductive case; reasoning with "..." in (a) or (c)
- Shows progress includes: Stating the necessary properties for an equivalence relation; correctly defining transitivity

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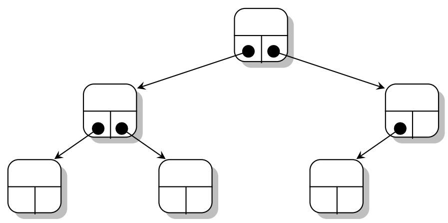
Problem 3 (26 marks)

A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we allow empty binary trees (trees with no nodes), then we can simplify the description by saying a node has *exactly two children* which are binary trees.

(a) Give a recursive definition of the binary tree data structure. *Hint: review the recursive definition of a Linked List* (6 marks)

A *leaf* in a binary tree is a node that has no successors (i.e. it has two empty trees as children). A *fully-internal node* in a binary tree is a node that has two successors. The example above has 3 leaves and 2 fully-internal nodes.

(b) Based on your recursive definition above, define the function $\mathsf{count}(\mathsf{T})$ that counts the number of nodes in a binary tree T. (4 marks)

(c) Based on your recursive definition above, define the function $\mathsf{leaves}(\mathsf{T})$ that counts the number of leaves in a binary tree T. (4 marks)

(d) Based on your recursive definition above, define the function $\mathsf{internal}(\mathsf{T})$ that counts the number of fully-internal nodes in a binary tree T. *Hint: it is acceptable to define an empty tree as having -1 fully-internal nodes.* (4 marks)

(e) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = 1 + \mathsf{internal}(\mathsf{T})$. Prove that $P(\mathsf{T})$ holds for all binary trees T. (8 marks)

Solution

(a) A binary tree is either:

- Empty: τ
- A node with two subtrees (and no data): $[T_1, T_2]$

(b) We define count recursively on trees as follows:

- $\mathsf{count}(\tau) = 0$
- $\mathsf{count}([T_1, T_2]) = 1 + \mathsf{count}(T_1) + \mathsf{count}(T_2)$

(c) We define leaves recursively on trees as follows:

- $\mathsf{leaves}(\tau) = 0$
- $\mathsf{leaves}([\tau, \tau]) = 1$ (without this condition $\mathsf{leaves}(T) = 0$ for all trees)
- $\mathsf{leaves}([T_1, T_2]) = \mathsf{leaves}(T_1) + \mathsf{leaves}(T_2)$ if T_1 and T_2 are not both empty.

(d) We define internal recursively on trees as follows:

- $\mathsf{internal}(\tau) = -1$
- $\mathsf{internal}([\tau, \tau]) = 0$ (without this condition $\mathsf{internal}(T) = -1$ for all trees)
- $\mathsf{internal}([T_1, T_2]) = 1 + \mathsf{internal}(T_1) + \mathsf{internal}(T_2)$ if T_1 and T_2 are not both empty. (As $\mathsf{internal}(\tau) = -1$, we don't have to consider separately the case where one child is empty)

(e) We prove $P(\mathsf{T})$ by structural induction on T.

Base case: ($\mathsf{T} = \tau$) From the definitions of leaves and internal we have:

$$\mathsf{leaves}(\tau) = 0 = 1 + (-1) = 1 + \mathsf{internal}(\tau).$$

So $P(\tau)$ holds.

Base case: ($\mathsf{T} = [\tau, \tau]$) (NB: this can also be considered a subcase of the inductive case) From the definitions of leaves and internal we have:

$$\mathsf{leaves}([\tau, \tau]) = 1 = 1 + 0 = 1 + \mathsf{internal}([\tau, \tau]).$$

So $P([\tau, \tau])$ holds.

Inductive case: ($\mathsf{T} = [T_1, T_2]$) Assume $P(T_1)$ and $P(T_2)$ hold and T_1 and T_2 are not both empty. That is,

$$\mathsf{leaves}(T_1) = 1 + \mathsf{internal}(T_1) \quad \text{and} \quad \mathsf{leaves}(T_2) = 1 + \mathsf{internal}(T_2).$$

We will show that $P([T_1, T_2])$ holds. We have:

$$\begin{aligned}
\mathsf{leaves}([T_1, T_2]) &= \mathsf{leaves}(T_1) + \mathsf{leaves}(T_2) && \text{(Def. of leaves)} \\
&= (1 + \mathsf{internal}(T_1)) + (1 + \mathsf{internal}(T_2)) && \text{(Induction Hypothesis)} \\
&= 1 + (1 + \mathsf{internal}(T_1) + \mathsf{internal}(T_2)) && \\
&= 1 + \mathsf{internal}([T_1, T_2]) && \text{(Def. of internal)}
\end{aligned}$$

So $P[T_1, T_2]$ holds when T_1 and T_2 are not both empty.

By the Principle of Structural Induction, we have that $P(T)$ holds for all trees T.

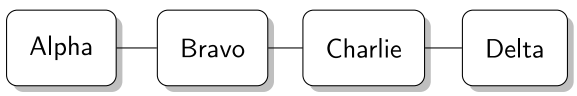
Discussion

Variants where data is included in nodes, or definitions for (a)–(d) are written in code should not be penalized other than in the cases outlined below. For full marks, the answer for (e) requires some form of abstraction [probably].

- Minor errors include missing a base case in (b), (c), (d), or (e). Each part should be assessed independently, so the penalty can be applied multiple times for multiple infractions.
- Major errors include: missing the base case in (a); missing an inductive case in any part; or defining the functions count, and/or leaves and/or internal in some way independent of the definition in (a). This includes answering (e) "by example".
- Good progress includes: only giving a base case for (b), (c), (d), or (e) (i.e. missing a base and an inductive case).

Problem 4 (24 marks)

Four wifi networks, Alpha, Bravo, Charlie and Delta, all exist within close proximity to one another as shown below.



Networks connected with an edge in the diagram above can interfere with each other. To avoid interference networks can operate on one of two channels, hi and lo. Networks operating on different channels will not interfere; and neither will networks that are not connected with an edge.

Our goal is to determine (algorithmically) whether there is an **assignment of channels to networks** so that there is no interference. To do this we will transform the problem into a problem of determining if a propositional formula can be satisfied.

(a) Carefully defining the propositional variables you are using, write **propositional formulas** for each of the following requirements: (4 marks)

  (i) φ_1: Alpha uses channel hi or channel lo; and so does Bravo, Charlie and Delta. (4 marks)

  (ii) φ_2: Alpha does not use both channel hi and lo; and the same for Bravo, Charlie and Delta. (4 marks)

  (iii) φ_3: Alpha and Bravo do not use the same channel; and the same applies for all other pairs of networks connected with an edge. (4 marks)

(b) (i) Show that $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$ is satisfiable; so the requirements can all be met. Note that it is sufficient to give a satisfying truth assignment, you do not have to list all possible combinations. (4 marks)

  (ii) Based on your answer to the previous question, which channels should each network use in order to avoid interference? (4 marks)

Solution

Let

Variable	Represent the proposition that:	Variable	Represent the proposition that:
A_h	Alpha uses channel hi	A_l	Alpha uses channel lo
B_h	Bravo uses channel hi	B_l	Bravo uses channel lo
C_h	Charlie uses channel hi	C_l	Charlie uses channel lo
D_h	Delta uses channel hi	D_l	Delta uses channel lo

(a) Then we can define the requirements as follows:

(i) $\varphi_1 = (A_h \vee A_l) \wedge (B_h \vee B_l) \wedge (C_h \vee C_l) \wedge (D_h \vee D_l)$.

(ii) $\varphi_2 = \neg(A_h \wedge A_l) \wedge \neg(B_h \wedge B_l) \wedge \neg(C_h \wedge C_l) \wedge \neg(D_h \wedge D_l)$.

(iii) $\varphi_3 = \neg((A_h \wedge B_h) \vee (A_l \wedge B_l)) \wedge \neg((B_h \wedge C_h) \vee (B_l \wedge C_l)) \wedge \neg((C_h \wedge D_h) \vee (C_l \wedge D_l))$.

(b) (i) One truth assignment could be defined as:

$$v(A_h) = v(B_l) = v(C_h) = v(D_l) = \texttt{true}$$
$$v(A_l) = v(B_h) = v(C_l) = v(D_h) = \texttt{false}$$

Under this assignment we see that $v(\varphi_1) = v(\varphi_2) = v(\varphi_3) = \texttt{true}$, so $v(\varphi_1 \wedge \varphi_2 \wedge \varphi_3) = \texttt{true}$ and hence the requirements can all be met.

(ii) In general, if the truth assignment sets A_h to `true` then the proposed solution is that Alpha uses channel hi and if the truth assignment sets A_l to `true` then the proposed solution is that Alpha uses channel lo (and similarly for Bravo, Charlie, and Delta). Note that φ_2 ensures that in any satisfying assignment at most one of A_h or A_l will be set to `true` (and likewise for the other variables), so Alpha will never be assigned to two channels; and φ_1 ensures that in any satisfying assignment at least one of A_h or A_l will be set to `true`, so Alpha will be assigned at least one channel.

In our particular example, the proposed solution is: Alpha and Charlie use channel hi; Bravo and Delta use channel lo.

Discussion

To get full marks for (b)(ii) students should state how any truth assignment translates to a solution. Failing to do so will only incur a minor penalty.
Formulas do not have to be strict wffs, but should be unambiguous.
Errors in the definition incurs a minor error for each part (e.g. Defining A to be "Alpha uses hi" and then assuming that $\neg A$ represents "Alpha uses lo" instead of "Alpha does not use hi")
A truth table is sufficient for (b)(i) as long as it includes a column for all the variables, a column for each of $\varphi_1, \varphi_2, \varphi_3$ and a column for $\varphi_1 \wedge \varphi_2 \wedge \varphi_3$.

- Minor errors include: omitting up to three columns from the truth table in (b)(i); omitting a general solution for (b)(ii) (i.e. just giving a solution based on the answer for (b)(i))
- Major errors include: incomplete specification of the truth assignment in (b)(i); incomplete, but not wholly incorrect specifications in (a)
- Good progress includes: any valid propositional formula in (a)
- No marks for propositional variables that do not represent propositions

Advice on how to do the assignment

All submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted via WebCMS (or give) as a single pdf file.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of the lecture notes or book). You can use the material presented in the lecture or book (without proving it). You do not need to write more than necessary (see comment above).
- When giving answers to questions, we always would like you to prove/explain/motivate your answers.
- If you use further resources (books, scientific papers, the internet,...) to formulate your answers, then add references to your sources.