COMP9020
Foundations of Computer Science

Lecture 4: Set Theory

Administrivia

- Assignment 1 released tomorrow; due **12:00 noon Monday 11 October (AEDST)**
- Quiz 2 released tomorrow; due **14:00 Monday 27 September (AEST)**

Outline

Recap of key definitions

Set equality

Laws of Set operations

Derived laws

Two useful results

Outline

Recap of key definitions

Set equality

Laws of Set operations

Derived laws

Two useful results

Defining Sets

1. Explicitly list elements
2. Take a subset of an existing set by restricting the elements
3. Build up from existing sets using Set Operations

Set Operations

Definition

$A \cup B$ – **union** (*a* or *b*):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (*a* and *b*):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set $\mathcal{U}$):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Set Operations

Other set operations

Definition

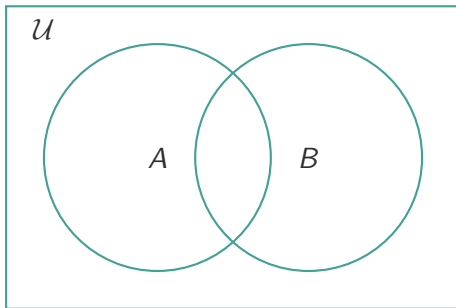$A \setminus B$ – **set difference**, relative complement (*a* but not *b*):

$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (*a* and not *b* or *b* and not *a*; also known as *a* or *b* exclusively; *a* xor *b*):

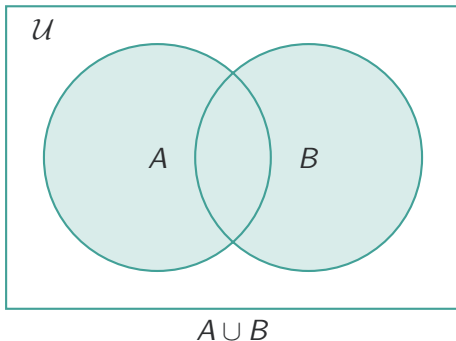$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Venn Diagrams

A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.

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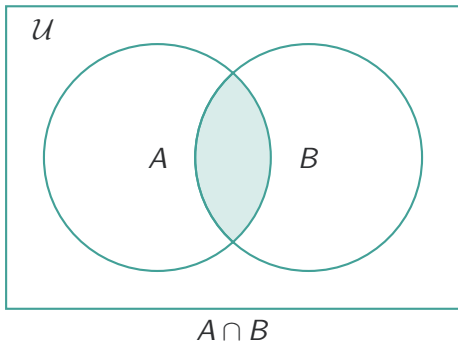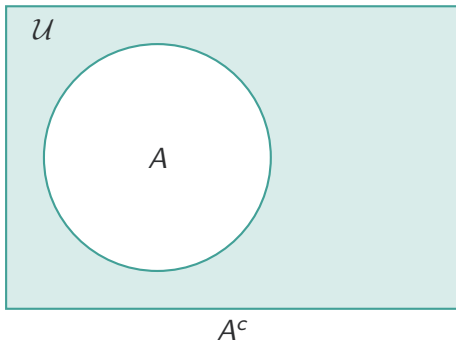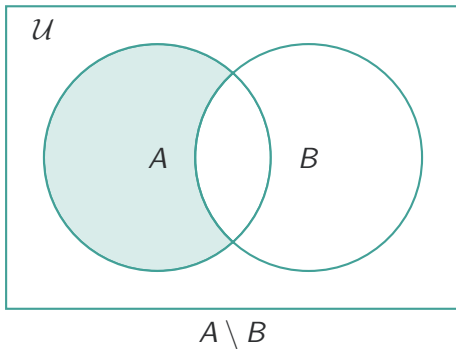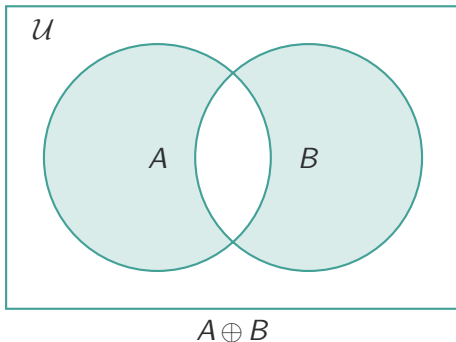

$A \cap B$

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$A \setminus B$

Venn Diagrams

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$A \oplus B$

Outline

Set Equality

Two sets are **equal** ($A = B$) if they contain the same elements

To show equality:

- Examine all the elements
- Show $A \subseteq B$ and $B \subseteq A$
- Use the Laws of Set Operations

Important!

Venn diagrams can help visualize, but are **not** rigorous.

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

$(0, 4) = \{1, 2, 3\} = \{3, 2, 1\}$.

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

$$\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{-2, -1, 0, 1, 2\}$$
$$= \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$$

Examples

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Show:

- For all $n \in \mathbb{Z}$, if $n^2 > 5$ then $|n| > 2$; and
- For all $n \in \mathbb{Z}$, if $|n| > 2$ then $n^2 > 5$.

That is, show:

$$\text{For all } n \in \mathbb{Z}: \quad n^2 > 5 \quad \text{if, and only if} \quad |n| > 2$$

Outline

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Laws of Set Operations

For all sets A, B, C:

Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distribution	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$
	$A \cap \mathcal{U} = A$
Complementation	$A \cup (A^c) = \mathcal{U}$
	$A \cap (A^c) = \emptyset$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity $\qquad\qquad A \cup B = B \cup A$

Therefore: $\qquad (C \cap D) \cup (D \oplus E) = (D \oplus E) \cup (C \cap D)$

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C && \text{[Associativity]} \\ &= C \cap (A \cap B) && \text{[Commutativity]} \\ &= C \cap (B \cap A) && \text{[Commutativity]} \end{aligned}$$

Important!

(Aim to) limit each step to a non-overlapping applications of a single rule

Outline

Other useful set laws

The following are all derivable from the previous 10 laws.

Idempotence	$A \cap A = A$
	$A \cup A = A$
Double complementation	$(A^c)^c = A$
Annihilation	$A \cap \emptyset = \emptyset$
	$A \cup \mathcal{U} = \mathcal{U}$
de Morgan's Laws	$(A \cap B)^c = A^c \cup B^c$
	$(A \cup B)^c = A^c \cap B^c$

Example (Idempotence of $\cup$)

$$
\begin{aligned}
A &= A \cup \emptyset && \text{(Identity)} \\
&= A \cup (A \cap A^c) && \text{(Complementation)} \\
&= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\
&= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\
&= (A \cup A) && \text{(Identity)}
\end{aligned}
$$

Outline

Two useful results

Definition

If A is a set defined using $\cap$, $\cup$, $\emptyset$ and $\mathcal{U}$, then $\mathrm{dual}(A)$ is the expression obtained by replacing $\cap$ with $\cup$ (and vice-versa) and $\emptyset$ with $\mathcal{U}$ (and vice-versa).

Theorem (Principle of Duality)

If you can prove $A_1 = A_2$ using the Laws of Set Operations then you can prove $\mathrm{dual}(A_1) = \mathrm{dual}(A_2)$

Example

Absorption law: $A \cup (A \cap B) = A$

Dual: $A \cap (A \cup B) = A$

Application (Idempotence of ∩)

Recall Idempotence of ∪: Invoke the dual laws!

$$
\begin{aligned}
A &= A \cup \cap \emptyset \mathcal{U} & \text{(Identity)} \\
&= A \cup \cap (A \cap \cup A^c) & \text{(Complementation)} \\
&= (A \cup \cap A) \cap \cup (A \cup \cap A^c) & \text{(Distributivity)} \\
&= (A \cup \cap A) \cap \cup \mathcal{U} \emptyset & \text{(Complementation)} \\
&= (A \cup \cap A) & \text{(Identity)}
\end{aligned}
$$

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ and $A \cup B = \mathcal{U}$ if, and only if, $B = A^c$.

Proof (Only if).

$$
\begin{aligned}
B &= B \cap \mathcal{U} & \text{(Identity)} \\
&= B \cap (A \cup A^c) & \text{(Complement)} \\
&= (B \cap A) \cup (B \cap A^c) & \text{(Distributivity)} \\
&= (A \cap B) \cup (A^c \cap B) & \text{(Commutativity)} \\
&= \emptyset \cup (A^c \cap B) & \text{(Given)} \\
&= (A \cap A^c) \cup (A^c \cap B) & \text{(Complement)} \\
&= (A^c \cap A) \cup (A^c \cap B) & \text{(Commutativity)} \\
&= A^c \cap (A \cup B) & \text{(Distributivity)} \\
&= A^c \cap \mathcal{U} & \text{(Given)} \\
&= A^c & \text{(Identity)}
\end{aligned}
$$

$\square$

Application (Double complement)

Take $A = X^c$ and $B = X$:

$$\begin{aligned} X^c \cap X &= X \cap X^c \qquad &\text{(Commutativity)} \\ &= \emptyset \qquad &\text{(Identity)} \\[1em] X^c \cup X &= \mathcal{U} \qquad &\text{(Principle of duality)} \end{aligned}$$

By the uniqueness of complement, $(X^c)^c = X$.

Exercises

Exercises

Show the following for all sets A, B, C:

- $B \cup (A \cap \emptyset) = B$
- $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
- $(A \cap B) \cup (A \cup B^c)^c = B$

Exercises

Give counterexamples to show the following do not hold for all sets:

- $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
- $(A \cup B) \setminus C = A \cup (B \setminus C)$
- $(A \setminus B) \cup B = A$