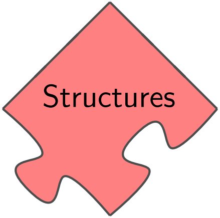
COMP9020

Foundations of Computer Science

Lecture 3: Sets and Formal Languages

Topic 1: Structures



		[LLM]	[RW]
Week 2	Sets and Formal Languages; Set Theory	4.1, 4.2	Ch. 1
Week 3	Relations	4.4	Ch. 3
Week 4	Functions	4.3, 13.7	Ch. 3
Week 5	Graph Theory	Ch. 11, 12	Ch. 6

Structures in Computer Science

Sets:

- Sets are the building blocks of nearly all mathematical structures
- Data structures based around sets can be a space-efficient storage system
- Set theory is a good introduction to formal reasoning (logic)

Formal languages:

- Formal languages are essential for compilers and programming language design
- Formal languages provide a good introduction to recursive structures (recursion and induction)

Structures in Computer Science

Relations:

- Relations are the building blocks of nearly all structures used in Computer Science
- Databases are collections of relations
- Any ordering is a relation
- Common data structures (e.g. graphs) are relations
- Functions/procedures/programs compute relations between their input and output

Functions:

- Functions, methods, procedures in programming
- Computer programs "are" functions
- Graphical transformations
- Algorithmic analysis

Structures in Computer Science

Graphs:

- Route planning in navigation systems, robotics
- Optimisation, e.g. timetables, utilisation of network structures, bandwidth allocation
- Compilers using "graph colouring" to assign registers to program variables
- Circuit layout (Untangle game)
- Determining the significance of a web page (Google's pagerank algorithm)
- Modelling the spread of a virus in a computer network or news in social network

Outline

Outline

Sets

Definition

A **set** is a collection of objects (**elements**). If x is an element of A we write $x \in A$.

NB

- *Elements are taken from a* **universe**, $\mathcal{U}$, *– but this can be quite complex. e.g. numbers, and sets of numbers, and sets of sets of numbers, etc.*
- *Not all "well-defined" universes are possible. e.g.*
  - *No "set of all sets" (Cantor's paradox)*
  - *No "sets which do not contain themselves" (Russell's paradox)*

Sets

- A set is defined by the collection of its elements. Order and multiplicity of elements is not considered.
- We distinguish between an element and the set comprising this single element. Thus always $a \neq \{a\}$.
- Set $\emptyset = \{\}$ is empty (no elements);
- Set $\{\{\}\}$ is nonempty — it has one element.

Subsets

Definition

For sets S and T, we say S is a **subset** of T, written $S \subseteq T$, if every element of S is an element of T.

NB

- *$S \subseteq T$ includes the case of $S = T$*
- *$S \subset T$ — a* **proper** *subset: $S \subseteq T$ and $S \neq T$*
- *$\emptyset \subseteq S$ for all sets S*
- *$S \subseteq \mathcal{U}$ for all sets S*
- $\mathbb{N}_{>0} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- *An element of a set; and a subset of that set are two different concepts*

$$a \in \{a, b\}, \quad a \not\subseteq \{a, b\}; \qquad \{a\} \subseteq \{a, b\}, \quad \{a\} \notin \{a, b\}$$

Outline

Defining sets

Sets are typically described by:
(a) Explicit enumeration of their elements

$$
\begin{aligned}
S_1 &= \{a, b, c\} = \{a, a, b, b, b, c\} \\
&= \{b, c, a\} = \dots \quad \text{three elements} \\
S_2 &= \{a, \{a\}\} \quad \text{two elements} \\
S_3 &= \{a, b, \{a, b\}\} \quad \text{three elements} \\
S_4 &= \{\} \quad \text{zero elements} \\
S_5 &= \{\{\{\}\}\} \quad \text{one element} \\
S_6 &= \{\, \{\}, \{\{\}\}\, \} \quad \text{two elements}
\end{aligned}
$$

Defining sets

(b) Defining a subset of the universal set $\mathcal{U}$. Including:

- Specifying the properties their elements must satisfy. A typical description involves a **logical** property $P(x)$. For example, with $\mathcal{U} = \mathbb{N}$ and $P(x) =$ "x is even":

$$\{x : x \in \mathbb{N} \text{ and } x \text{ is even}\} = \{0, 2, 4, \ldots\}$$

- Derived sets of integers

$$2\mathbb{Z} = \{\ 2x : x \in \mathbb{Z}\ \} \qquad \text{the even numbers}$$
$$3\mathbb{Z} + 1 = \{\ 3x + 1 : x \in \mathbb{Z}\ \}$$

- Using interval notation.

Intervals

Intervals of numbers (applies to any type)

$$[a, b] = \{x : a \leq x \leq b\}; \qquad (a, b) = \{x : a < x < b\}$$

$$[a, b) = \{x : a \leq x < b\}; \qquad (a, b] = \{x : a < x \leq b\}$$

$$(-\infty, b] = \{x : x \leq b\}; \qquad (-\infty, b) = \{x : x < b\}$$

$$[a, \infty) = \{x : a \leq x\}; \qquad (a, \infty) = \{x : a < x\}$$

NB

$(a, a) = (a, a] = [a, a) = \emptyset$; *however* $[a, a] = \{a\}$.

Intervals of $\mathbb{N}, \mathbb{Z}$ are finite: if $m \leq n$

$$[m, n] = \{m, m + 1, \ldots, n\}$$

Examples

Examples

- $[1,5] = \{1,2,3,4,5\}$ (when $\mathcal{U} = \mathbb{Z}$)
- $[1,5] = \{1, 1.1, 1.01, 1.001, \ldots, 2, \ldots, \pi, e, \ldots\}$ (when $\mathcal{U} = \mathbb{R}$)
- Number of multiples of k between n and m (inclusive)in $[n, m]$:
$$\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$$
- $0 \leq (m \% n) < n(m \% n) \in [0, n)$

Defining sets

(c) Constructions from other, already defined, sets

- Union ($\cup$), intersection ($\cap$), complement ($\cdot^c$), set difference ($\setminus$), symmetric difference ($\oplus$)
- Power set $\mathrm{Pow}(X) = \{\, A : A \subseteq X \,\}$
- Cartesian product ($\times$)

Outline

Basic Set Operations

Definition

$A \cup B$ – **union** (a or b):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (a and b):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set $\mathcal{U}$):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Basic Set Operations

Other set operations

Definition

$A \setminus B$ – **set difference**, relative complement (a but not b):

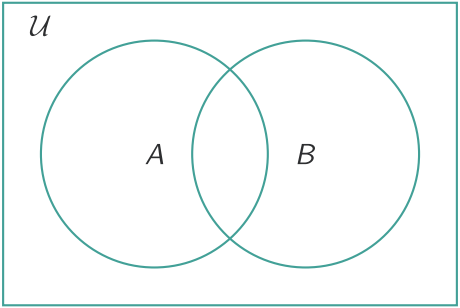
$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (a and not b or b and not a; also known as a or b exclusively; a xor b):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Venn Diagrams

A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.



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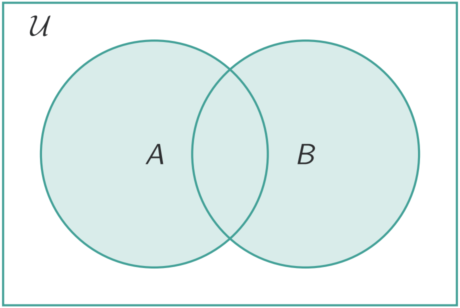

$A \cup B$

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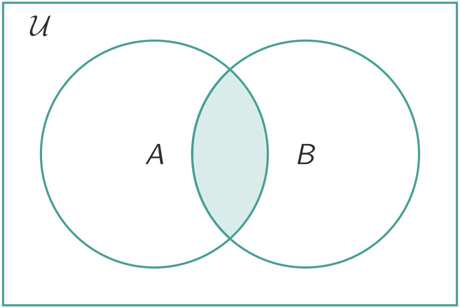

$A \cap B$

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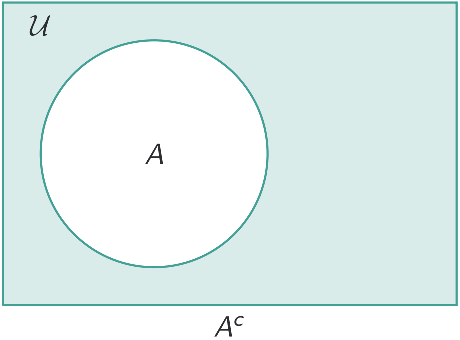

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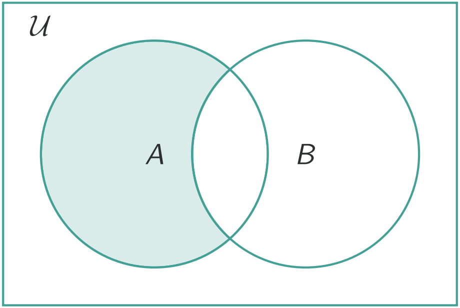

$A \setminus B$

Venn Diagrams

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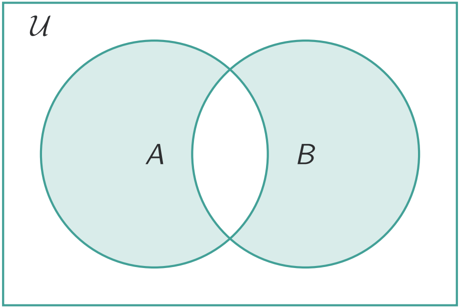

$A \oplus B$

Set Operations and Subset

Fact

$$A \cup B = B \quad \textit{iff} \quad A \cap B = A \quad \textit{iff} \quad A \subseteq B$$

There is a correspondence between set operations and logical operators (to be discussed in Week 7).

Exercises

Exercises

$\boxed{\text{RW: 1.4.7 (a)}}$ $A \oplus A =$

$\boxed{\text{RW: 1.4.7 (b)}}$ $A \oplus \emptyset =$

Power set

Definition

The **power set** of a set X, $\mathrm{Pow}(X)$, is the set of all subsets of X

Example

$$\mathrm{Pow}(\{a, b\}) = \Big\{\emptyset, \{a\}, \{b\}, \{a, b\}\Big\}$$

Cardinality

Definition

The **cardinality** of a set X (various notation) is the number of elements in that set.

$$|X| = \#(X) = \text{card}(X)$$

Fact

Always $|Pow(X)| = 2^{|X|}$

Exercises

Exercises

- $|\emptyset| \stackrel{?}{=}$
- $\mathrm{Pow}(\emptyset) \stackrel{?}{=}$
- $|\mathrm{Pow}(\emptyset)| \stackrel{?}{=}$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) \stackrel{?}{=}$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| \stackrel{?}{=}$
- $|\{a\}| \stackrel{?}{=}$
- $\mathrm{Pow}(\{a\}) \stackrel{?}{=}$
- $|\mathrm{Pow}(\{a\})| \stackrel{?}{=}$
- $|[m, n]| \stackrel{?}{=}$

Exercises

$\boxed{\text{RW: 1.3.2}}$ Find the cardinalities of sets

(a) $|\{\ \frac{1}{n} : n \in [1,4]\ \}| \stackrel{?}{=}$

(b) $|\{\ n^2 - n : n \in [0,4]\ \}| \stackrel{?}{=}$

(c) $|\{\ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11\ \}| \stackrel{?}{=}$

(d) $|\{\ 2 + (-1)^n : n \in \mathbb{N}\ \}| \stackrel{?}{=}$

Exercises

$\boxed{\text{RW: 1.4.8}}$ Relate the cardinalities to $|A \cap B|$, $|A|$, $|B|$

- $|A \cup B|$
- $|A \setminus B|$
- $|A \oplus B|$

Cartesian Product

Definition

The **Cartesian product** of two sets S and T is the set of **ordered pairs**:

$$S \times T \stackrel{\text{def}}{=} \{\, (s,t) : s \in S,\ t \in T \,\}$$

The **Cartesian product** of a collection of n sets $S_1, S_2, \ldots, S_n$ is the set of **ordered n-tuples**:

$$\bigtimes_{i=1}^{n} S_i \stackrel{\text{def}}{=} \{\, (s_1, \ldots, s_n) : s_k \in S_k, \text{ for } 1 \leq k \leq n \,\}$$

When all the S_i are equal:

$$S^2 = S \times S, \quad S^3 = S \times S \times S, \ldots, \quad S^n = \times_1^n S, \ldots$$

Cartesian product

Fact

- $\emptyset \times S = \emptyset$, *for every* S
- $|S \times T| = |S| \cdot |T|$
- $|\times_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i|$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

$$\begin{aligned} A^3 = \{&(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), \\ &(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \end{aligned}$$

Exercise

Exercise

Let A, B, C be sets.

Is $A \times (B \times C) = (A \times B) \times C$?

Outline

Formal Languages: Symbols

Σ — **alphabet**, a finite, nonempty set

Examples (of various alphabets and their intended uses)

$\Sigma = \{a, b, \ldots, z\}$ for single words (in lower case)
$\Sigma = \{\text{true-}, -, a, b, \ldots, z\}$ for composite terms
$\Sigma = \{0, 1\}$ for binary integers
$\Sigma = \{0, 1, \ldots, 9\}$ for decimal integers

The above cases all have a natural ordering; this is not required in general, thus the set of all Chinese characters forms a (formal) alphabet.

Formal Languages: Words

Definition

word — any finite string of symbols from Σ
empty word — λ

Example

$w = aba$, $w = 01101\ldots1$, etc.

$\text{length}(w)$ — $\#$ of symbols in w
$\text{length}(aaa) = 3, \text{length}(\lambda) = 0$
The only operation on words (discussed here) is **concatenation**, written as juxtaposition $vw, wvw, abw, wbv, \ldots$

NB

$\lambda w = w = w\lambda$
$\text{length}(vw) = \text{length}(v) + \text{length}(w)$

Examples

Examples

Let $w = abb$, $v = ab$, $u = ba$

- $vw = ababb$
- $ww = abbabb = vubb$
- $w\lambda v = abbab$
- $\text{length}(vw) = \text{length}(ababb) = 5$

Formal Languages: Sets of words

Notation:

Definition

- Σ^k or $\Sigma^{=k}$: The set of all words of length k
- $\Sigma^{\leq k}$: The set of all words of length at most k
- Σ^*: The set of all finite words
- Σ^+: The set of all nonempty words

We often identify $\Sigma^1 = \Sigma$

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \dots; \quad \Sigma^{\leq n} = \bigcup_{i=0}^{n} \Sigma^i$$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \dots = \Sigma^* \setminus \{\lambda\}$$

Formal Languages: Languages

Definition

A **language** is a subset of Σ^*.

Typically, only the subsets that can be formed (or described) according to certain rules are of interest. Such a collection of 'descriptive/formative' rules is called a **grammar**.

Example (Decimal numbers)

The "language" of all numbers written in decimal to at most two decimal places can be described as follows:

- $\Sigma = \{-, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Consider all words $w \in \Sigma^*$ which satisfy the following:
  - w contains at most one instance of $-$, and if it contains an instance then it is the first symbol.
  - w contains at most one instance of ., and if it contains an instance then it is preceeded by a symbol in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and followed by either one or two symbols in that set.
  - w contains at least one symbol from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

NB

According to these rules 123, 123.0 *and* 123.00 *are all (distinct) words in this language.*

Example (HTML documents)

Take $\Sigma = \{$"<html>", "</html>", "<head>", "</head>", "<body>", $\ldots\}$.

The (language of) **valid HTML documents** is loosely described as follows:

- Starts with "<html>"
- Next symbol is "<head>"
- Followed by zero or more symbols from the set of HeadItems (defined elsewhere)
- Followed by "</head>"
- Followed by "<body>"
- Followed by zero or more symbols from the set of BodyItems (defined elsewhere)
- Followed by "</body>"
- Followed by "</html>"

Exercises

$\boxed{\text{RW: 1.3.10}}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$?

(f) $\{\ w \in \Sigma^* : \text{length}(w) \leq 4\ \}$ where $\Sigma = \{a, b, c\}$?

Set Operations for Languages

Languages are sets, so the standard set operations ($\cap$, $\cup$, $\setminus$, $\oplus$, etc) can be used to build new languages.

Two set operations that apply to languages uniquely:

- Concatenation (written as juxtaposition):
  $XY = \{xy \ : \ x \in X \text{ and } y \in Y\}$
- Kleene star: X^* is the set of words that are made up by concatenating 0 or more words in X
  - $X^0 = \{\lambda\}$; $X^{i+1} = XX^i$
  - $X^* = X^0 \cup X^1 \cup X^2 \cup \ldots$

NB

The set of all finite words over Σ is the Kleene star of Σ (hence notation).

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* = \{\lambda\}$