9020 Assignment 2

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November 1, 2021

Problem1

(a)

In order to proof R_f is an equivalence relation, we need to proof R(Reflexive), S(Symmetric) and T(Transive).
R(Reflexive):
for $x \in S$, $y \in T$. $\because f : S \to T$ is a function, $\therefore x$ corresponds to a unique y.
$\therefore f(x) = f(x)$, from the definition of Reflexive, for all $x \in S : (x, x) \in R_f$
Therefore, R_f is reflexive.
S(Symmetric):
for $s_1, s_2 \in S$, $t_1, t_2 \in T$ and $f(s_1) = t_1, f(s_2) = t_2$ and $(f(s_1), f(s_2)) \in R_f$
Then $t_1 = t_2$ and $f(s_1) = f(s_2)$ and $f(s_2) = f(s_1)$ and
$(s_1, s_2) \in R_f, (s2, s1) \in R_f$
From the definition of Symmetric, for all $s_1, s_2 \in S$, if $(s_1, s_2) \in R$ then
$(s_2, s_1) \in R$
Therefore, R_f is Symmetric.
T(Transitive):
For $s_1, s_2, s_3 \in S$, $t_1, t_2, t_3 \in T$ and $(s_1, s_2) \in R_f$ and $(s_2, s_3) \in R_f$
$\therefore f(s_1) = f(s_2)$ and $f(s_2) = f(s_3)$. Then $f(s_1) = f(s_3)$
$\therefore (s_1, s_2) \in R_f$. From the definition of Transitive, for all $s_1, s_2, s_3 \in S$, if
$(s_1, s_2), (s_2, s_3) \in R_f$, then $(s_1, s_3) \in R_f$
Therefore, R_f is Transitive.
Because R_f is R, S, T, then R_f is an equivalence relation.

(b)

Because $R \subseteq S \times S$ is an equivalence, then we can assume that T is an equivalence on R.
From the definition of equivalence, if $(s, s') \in R$, then $[s] = [s']$.
Because f_R is a function from S to T, then $f_R(s) = [s]$.
Therefore, $(s, s') \in R$ if and only if $f_R(s) = f_R(s')$.

Problem 2

(a)

(i)

There are 4 possibilities for a and b.

$$\begin{cases} a>0 \\ b>0 \\ a+b>0 \end{cases} \Longrightarrow \begin{cases} f(a)=1 \\ f(b)=1 \\ f(a+b)=1 \end{cases} \begin{cases} a>0 \\ b=0 \\ a+b>0 \end{cases} \Longrightarrow \begin{cases} f(a)=1 \\ f(b)=0 \\ f(a+b)=1 \end{cases}$$

$$\begin{cases} a=0 \\ b>0 \\ a+b>0 \end{cases} \Longrightarrow \begin{cases} f(a)=0 \\ f(b)=1 \\ f(a+b)=1 \end{cases} \begin{cases} a=0 \\ b=0 \\ a+b=0 \end{cases} \Longrightarrow \begin{cases} f(a)=0 \\ f(b)=0 \\ f(a+b)=0 \end{cases}$$

Therefore, $f(a+b) = max\{f(a), f(b)\}$.

(ii)

There are 4 possibilities for a and b.

$$\begin{cases} a>0 \\ b>0 \\ ab>0 \end{cases} \Longrightarrow \begin{cases} f(a)=1 \\ f(b)=1 \\ f(a+b)=1 \end{cases} \begin{cases} a>0 \\ b=0 \\ ab=0 \end{cases} \Longrightarrow \begin{cases} f(a)=1 \\ f(b)=0 \\ f(a+b)=0 \end{cases}$$

$$\begin{cases} a=0 \\ b>0 \\ ab=0 \end{cases} \Longrightarrow \begin{cases} f(a)=0 \\ f(b)=1 \\ f(a+b)=0 \end{cases} \begin{cases} a=0 \\ b=0 \\ ab=0 \end{cases} \Longrightarrow \begin{cases} f(a)=0 \\ f(b)=0 \\ f(a+b)=0 \end{cases}$$

Therefore, $f(ab) = min\{f(a), f(b)\}$.

(b)

$\because$ relation $\boxplus \subseteq \mathbb{E} \times \mathbb{E}^2$, $\therefore \boxplus$ is a relation from pairs to integers.
From problem 1, we know that $R_f \subseteq \mathbb{N} \times \mathbb{N}$, the relation given by:
$(m,n) \in R_f \Leftrightarrow f(m) = f(n)$ is an equivalence relation.

(i)

For $x_1, x_2, ..., x_n \in X,\ f(x_1) = f(x_2) = ... = f(x_n)$
For $y_1, y_2, ..., y_n \in Y,\ f(y_1) = f(y_2) = ... = f(y_n)$
Therefore, for $x \in X, y \in Y, x+y \in Z$, $\boxplus$ is a relation from $(f(x), f(y))$ to $f(x+y)$.
$\because f(x+y)$ is a function from $x+y$ to $f(x+y)$
Then for every pair $(f(x), f(y))$, there is a unique $f(x+y)$.
Therefore. relation $\boxplus$ is a function.

(ii)

For $x_1, x_2, ..., x_n \in X,\ f(x_1) = f(x_2) = ... = f(x_n)$

For $y_1, y_2, ..., y_n \in Y$, $f(y_1) = f(y_2) = ... = f(y_n)$
Therefore, for $x \in X, y \in Y, xy \in Z$, $\boxdot$ is a relation from $(f(x), f(y))$ to $f(xy)$.
$\therefore f(x+y)$ is a function from xy to $f(xy)$
Then for every pair $(f(x), f(y)$, there is a unique $f(xy)$.
Therefore, relation $\boxdot$ is a function.

(c)

(i)

Suppose that $a \in A$, then
$A \boxdot [1] = [a] \boxdot [1]$ (definition of equivalence relation)
$= [a \cdot 1] = [a]$ ((b))
$= [A]$ (definition of equivalence relation)

(ii)

Suppose that $a \in A$ and $b \in B$, then
$A \boxplus B = [a] \boxplus [b]$ (definition of equivalence relation)
$= [a + b]$ ((b))
$= [b + a]$ (commutation of $+$)
$= [b] \boxplus [a]$ ((b))
$B \boxplus A$ (definition of [ ])

(iii)

Suppose that $a \in A, b \in B$, and $c \in C$, then
$A \boxdot (B \boxplus C) = [a] \boxdot ([b] \boxplus [c])$ (definition of [ ])
$= [a] \boxdot ([b + c])$ ((b))
$= [a(b + c)]$ ((b))
$= [a \cdot b + a \cdot c]$ (multiplication left distributes over addition)
$= [a \cdot b] \boxplus [a \cdot c]$ ((b))
$= ([a] \boxdot [b]) \boxplus ([a] \boxdot [c])$ ((b))
$= (A \boxdot B) \boxplus (A \boxdot C)$ (defition of [ ])

Problem 3

(a)

(i)

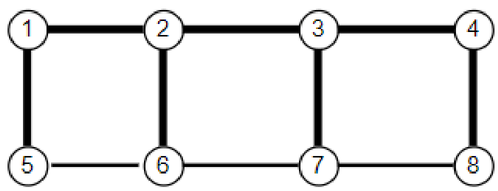
We define 8 houses as 8 vertices, and 2 houses that under the different Wifi channel become an edge.
Therefore, for each vertices, we need to connect the opposite and the left and right vertices.

(ii)

The problem of the minimum number of Wifi channels is equivalent to the problem of the minimum chromatic number.

(b)

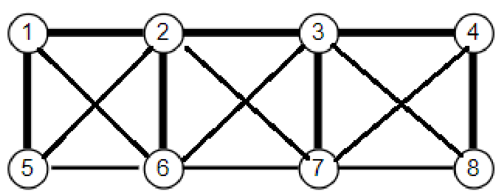
The minimum of Wifi channels is 2.
We can label the vertice as follows.



Therefore, the first channel includes: {1,3,6,8}.
The second channel includes: {2,4,5,7}.

(c)

Based on the grapg above, for each vertices, add the edge connecting the left and right of the houses over the road.
Continue to find the minimum chromatic number of the new graph.
We can label vertices as follows.



The minimum of Wifi channels is 4.
Therefore, the first channels includes : {1,3}.
The second channels includes : {2,4}.
The third channels includes : {5,7}.
The forth channels includes : {6,8}.

Problem 4

(a)

From strategy II of finding a subdivision, we know that there are 3 operations:
1. delete an edge.
2. delete a vertices and all adjacent edges.
3. Replace a vertex of degree 2 with an edge connecting its neighbours.
For Peterson graph, we need to make it to become K_5 by doing these 3 operations.
For each operation, the degree of some related vertices is reduced.
For Peterson graph, each vertices has degree of 3, and for K_5, each vertices has degree of 4.
Therefore, Peterson graph does not contain a subdivision of K_5.

(b)

From strategy II of finding a subdivision, we have 6 steps of K_5:
Step1. delete the edge $\{1,6\}$.
Step2. delete the edge $\{4,5\}$.
Step3. replace vertex 1 with edge $\{0,2\}$ connecting its neighbours.
Step4. replace vertex 4 with edge $\{0,9\}$ connecting its neighbours.
Step5. replace vertex 3 with edge $\{2,8\}$ connecting its neighbours.
Step6. replace vertex 6 with edge $\{8,9\}$ connecting its neighbours.
For our transformed grapg, there are 2 disjoint sets of vertices $\{0,7,8\}$ and $\{2,5,9\}$, all vertices from different parts are connected, vertices from the same part are disconnected, so this graph is $K_{3,3}$. Therefore, Peterson graph contains a subdivision of $K_{3,3}$.

Problem 5

(a)

1. when i=1, $R^1 = R^0 \cup (R^0; R^0) = R^0 \cup \{(x, x : x \in S)\} = R^0 \cup R^0 = R^0$
then $R^0 \subseteq R^1$. Therefore, $P_0(1)$ holds.
2. For integer $i \geq 0$, $R^{i+1} = R^i \cup (R; R^i)$
when $(R; R^i) = \emptyset$, $R^{i+1} = R^i$, then $R^i \subseteq R^{i+1}$
when $(R; R^i) \neq \emptyset$, $R^{i+1} = R^i \cup (R; R^i)$. then $R^i \subseteq R^{i+1}$
then for $i \geq 0$, $R^i \subseteq R^{i+1}$.
From we discussed above, $R^0 \subseteq R^1 \subseteq ... \subseteq R^{n+1}$.
Therefore, for all $i, j \in \mathbb{N}$, if $i \leq j$ then $R_i(j)$ holds.

(b)

$R^{n+1} = R^n \cup (R; r^n)$
$= (I; R^n) \cup (R; R^n)$ [Assignment 1 problem 8.(b)]

$= (I \cup R); R^n$ [Assignment 1 problem 8.(c)]
$= R; R^n$ [Identity]
Then $R^{n+1} = R; R^n$.
Base: When $i = 0$, $P(0) = R^0; R^m = I; R^m$.
$= R^m$ [Assignment 1 problem 8 (b)]
$= R^{0+m}$
Then $P(0)$ holds.
Inductive: When $i + 1 \in \mathbb{N}$
$\because R^{i+1} = R; R^i$, $\therefore P(i+1) = R^{i+1+m} = R; R^{i+m}$
$= (R; ...; R); R^m$ (there are i numbers of R)
$= (R; ...; R); R^2; R^m$ (there are i-1 number of R)
$= ... = R^{i+1}; R^m$
Then $P(i+1)$ holds.
Therefore, $P(n)$ holds for all $n \in \mathbb{N}$.

(c)

$\because R^{i+1} = R; R^i$ ((b))
$\therefore$ when $R^i = R^{i+1}, i \in \mathbb{N}, R^i = R; R^i$
$\therefore R = I$ [Assignment 1 problem 8 (b)]
Base: When$j = i, R^j = R^i$
Then $P(i)$ holds.
Inductive: When$j > i, R^j = R^{j-i}; R^i$ ((b))
$= (R; ...; R); R^i$ (there are j-i number of R)
$= I; R^i$ (Assignment 1 problem 8 (b))
R^i (Assignment 1 problem 8 (b))
Then P(j) holds.
Therefore, if $\exists i \in \mathbb{N}$, such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$.

(d)

(e)

From (d), we can know that $R^{k^2} = R^{k^2+1}$.
Assume that $\exists x, y$, $(x, y) \in R^{k^2}$, and $\exists y, z$, $(y, x) \in R^{k^2}$
Then $(x, z) \in R^{k^2}; R^{k^2}$ (defition of compositive relation)
$= (x, z) \in R^{2k^2}$ ((b))
$= (x, z) \in R^{k^2}$ ((a))
Therefore, if $|s| = k$, then R^{k^2} is transitive.

(f)

Problem 6

(a)

Base: if $T = \tau$, $count(T) = 0$
Inductive: if $T \neq \tau$, $count(T) = count(T_{left}) + count(T_{right})$

(b)

Base1: if $T = \tau$, $leaves(T) = 0$
Base2: if $T = (\tau, \tau)$, $leaves(T) = 1$
Inductive: if $T = (T_{left}, \tau)$ or $T = (\tau, T_{right})$ or $T = (T_{left}, T_{right})$,
$leaves(T) = leaves(T_{left}) + leaves(T_{right})$

(c)

Base1: if $T = (\tau, \tau)$, $internal(T) = 0$
Base2: if $T = \tau$, $internal(T) = -1$
Inductive: if $T = (T_{left}, \tau)$ or $T = (\tau, T_{right})$ or $T = (T_{left}, T_{right})$,
$internal(T) = internal(T_{left}) + internal(T_{right}) + 1$

(d)

Base1: when $T = \tau$, $leaves(T) = 0, internal(T) = -1$, then
$leaves(T) = internal(T) + 1$.
Base2: when $T = (\tau, \tau)$, $leaves(T) = 1, internal(T) = 0$, then
$leaves(T) = internal(T) + 1$
Inductive: when $T = (T_{left}, \tau)$ or $T = (\tau, T_{right})$ or $T = (T_{left}, T_{right})$.
If both T_{left} and T_{right} such that $leaves(T) = internal(T) + 1$
Then $leaves(T_{left}) = internal(T) + 1$ and $leaves(T_{right}) = internal(T_{right}) + 1$
Therefore, $leaves(T) = leaves(T_{left}) + leaves(T_{right})$
$= (internal(T_{left}) + 1) + (internal(T_{right}) + 1)$
$= ((internal(T_{left}) + internal(T_{right}) + 1) + 1$
$= internal(T) + 1$
Then $leaves(T) = internal(T) + 1$
Therefore, for all T, $leaves(T) = internal(T) + 1$

Problem 7

Base: if $a \in \Sigma, b \in \Sigma$, and $a < b$, then $(a, b) \in \Sigma^* \times \Sigma^*$
Inductive: if $a \in \Sigma, b \in \Sigma$, and $a < b$, $w \in \Sigma^*$, $(w_1, w_2) \in \Sigma^* \times \Sigma^*$
then $(w_1, w_2a) \in \Sigma^* \times \Sigma^*$, $(wa, wb) \in \Sigma^* \times \Sigma^*$