



UNSW
SYDNEY

COMP9020

Foundations of Computer Science

Lecture 14: Combinatorics

Topic 4: Probability



		[LLM]	[RW]	[Rosen]
Week 9	Combinatorics	Ch. 14	Ch. 5	Ch. 6, 8
Week 9	Probability	Ch 16, 17	Ch. 9	Ch. 7
Week 10	Statistics	Ch. 18	Ch. 9	Ch. 7

Combinatorics in Computer Science

Combinatorics:

- Computing cost functions in algorithmic analysis
- Identifying (in-)efficiencies in data management
- Developing effective techniques for enumerating objects
- Probability calculations

Probability in Computer Science

Probability and Statistics:

- Artificial Intelligence
 - Machine Learning
 - Decision theory
 - Image processing
 - Speech recognition
- Algorithms
 - Algorithm analysis
 - Big Data sampling and analysis
- Security
 - Cryptography
 - Quantum computing
- Networks
 - Network traffic modelling
 - Reliability modelling

Outline

Counting Principles

Basic Counting Rules: Union

Basic Counting Rules: Product

Combinations and Permutations

Alternative Techniques

Difficult Counting Problems

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Counting Techniques

General idea: find methods, algorithms or precise formulae to count the number of elements in various sets or collections derived, in a structured way, from some basic sets.

Examples

Single base set $S = \{s_1, \dots, s_n\}$, $|S| = n$; find the number of

- all subsets of S
- ordered selections of r different elements of S
- unordered selections of r different elements of S
- selections of r elements from S such that ...
- functions $S \rightarrow S$ (onto, 1-1)
- partitions of S into k equivalence classes
- graphs/trees with elements of S as labelled vertices/leaves

Example

Example

A restaurant has the following menu:

Starter	Main Course	Dessert
Soup	Fish	Ice-cream
Bread	Beef	Fruit
	Pork	Cheese
	Chicken	

How many:

- 3 course meals (Starter-Main-Dessert) are possible?
- 3 course meals (Any item for each course) are possible?
- 3 course meals (Any item, no duplicates) are possible?
- Meals consisting of 3 items (order is unimportant)?

Example

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A restaurant has the following menu:

Starter	Main Course	Dessert
Soup	Fish	Ice-cream
Bread	Beef	Fruit
	Pork	Cheese
	Chicken	

How many:

- Starter-Main-Dessert?
- Any item for 3 courses?
- Any item, no duplicates, for 3 courses?
- Meals of 3 items?

Basic Counting Rules: Principles

Two simple rules:

- **Union rule** (“or”): If S and T are disjoint $|S \cup T| = |S| + |T|$
- **Product rule** (“followed by”): $|S \times T| = |S| \cdot |T|$

These cover many examples, though the rule application is not always obvious.

Common strategies:

- Direct application of the rule
- Relate unknown quantities to known quantities (e.g.
 $|S| + |T| = |S \cup T| + |S \cap T|$)
- Find a bijection to a set that can be counted

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The Union Rule

Union rule — S and T *disjoint*

$$|S \cup T| = |S| + |T|$$

S_1, S_2, \dots, S_n pairwise disjoint ($S_i \cap S_j = \emptyset$ for $i \neq j$)

$$|S_1 \cup \dots \cup S_n| = \sum |S_i|$$

Example

How many numbers in $A = [1, 2, \dots, 999]$ are divisible by 31 or 41?

$\lfloor 999/31 \rfloor = 32$ divisible by 31

$\lfloor 999/41 \rfloor = 24$ divisible by 41

No number in A divisible by both

Hence, $32 + 24 = 56$ divisible by 31 or 41

Consequences of the Union Rule

Fact

For any sets X, Y, Z :

$$|Y \setminus X| = |Y| - |X \cap Y|$$

$$|X \cup Y| = |X| + |Y| - |X \cap Y|$$

$$\begin{aligned} |X \cup Y \cup Z| = & |X| + |Y| + |Z| \\ & - |X \cap Y| - |Y \cap Z| - |Z \cap X| \\ & + |X \cap Y \cap Z| \end{aligned}$$

Fact

- If $|S \cup T| = |S| + |T|$ then S and T are disjoint
- If $|\bigcup_{i=1}^n S_i| = \sum_{i=1}^n |S_i|$ then S_i are pairwise disjoint
- If $|T \setminus S| = |T| - |S|$ then $S \subseteq T$

These properties can serve to identify cases when sets are disjoint (resp. one is contained in the other).

Proof.

$|S| + |T| = |S \cup T|$ means $|S \cap T| = |S| + |T| - |S \cup T| = 0$

$|T \setminus S| = |T| - |S|$ means $|S \cap T| = |S|$ means $S \subseteq T$ □

Exercises

Exercises

RW: 5.3.1 200 people. 150 swim or jog, 85 swim and 60 do both.
How many jog?

RW: 5.6.38 (Supp) There are 100 problems, 75 of which are 'easy' and 40 'important'.
What's the smallest number of easy *and* important problems?

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The Product Rule

Product rule

$$|S_1 \times \dots \times S_k| = |S_1| \cdot |S_2| \cdots |S_k| = \prod_{i=1}^k |S_i|$$

If all $S_i = S$ (the same set) and $|S| = m$ then $|S^k| = m^k$

NB

This counts the number of sequences where the first item is from S_1 , the second is from S_2 , and so on.

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.

How many 5-letter words?

$$|\Sigma^5| = |\Sigma|^5 = 7^5 = 16,807$$

How many with no letter repeated?

Product rule: Sequences of selections

Question

How can we count sequences when the underlying set changes?

To count sequences *without replacement*:

- Define an order on the whole underlying set
- Select from $[1, n]$, where n is the size of the “remaining” set, and a selection of i represents choosing the i -th element in that set

Example

Let $\Sigma = \{a, b, c, d, e, f, g\}$.

How many 5-letter words with no letter repeated?

$$\prod_{i=0}^4 (|\Sigma| - i) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2,520$$

Exercises

Exercises

S, T finite. How many functions $S \rightarrow T$ are there?

RW: 5.1.19 Consider a *complete* graph on n vertices.

(a) No. of paths of length 3

(b) paths of length 3 with all vertices distinct

(c) paths of length 3 with all edges distinct

Exercise

Exercise

RW: 5.3.2 $S = [100 \dots 999]$, thus $|S| = 900$.

(a) How many numbers have at least one digit that is a 3 or 7?

(b) How many numbers have a 3 *and* a 7?

Combinatorial Symmetry

A **symmetry** of a mathematical object is a bijective mapping from the object to itself which preserves “structure”.

A (combinatorial) symmetry defines an equivalence relation where the equivalence classes all have the same size.

We are often interested in counting a set “up to symmetry”. That is, counting the number of equivalence classes.

This can also be stated as a constraint that identifies a specific item in each equivalence class (**symmetric constraint**).

Definition

A *k-to-1 function* is a function that maps exactly k inputs to an output.

NB

A k-to-1 function defines the equivalence relation of a combinatorial symmetry and vice-versa.

Product rule: Symmetries and duplications

Question

- *How can we count sequences when we have symmetric constraints?*
- *How can we count sequences when we have duplicates?*

Example

Let $\Sigma = \{a, b, c, d, e\}$.

- How many 5-letter words with no letter repeated and a before b before c ?
- How many 5-letter words can be made from a, a, a, d, e ?

NB

The answer will be the same.

Product rule: Symmetries and duplications

- S_1 = sequences accounting for symmetry,
- S_2 = symmetries,
- S = sequences without symmetry

$$S = S_1 \times S_2,$$

so

$$|S_1| = |S|/|S_2|$$

Alternatively, $\frac{1}{|S_2|}$ of the $|S|$ sequences meet the symmetric constraint.

Product rule: Symmetries and duplications

Example

Let $\Sigma = \{a, b, c, d, e\}$.

How many 5-letter words with no letter repeated and a before b before c ?

Let $\Sigma' = \{a, b, c\}$.

$$S = \prod_{i=0}^4 (|\Sigma| - i) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$S_2 = \prod_{i=0}^2 (|\Sigma'| - i) = 3 \cdot 2 \cdot 1 = 6$$

$$\text{So } S_1 = 120/6 = 20$$

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Combinatorial Objects: How Many?

permutations

Ordering of all objects from a set S ; equivalently: Selecting all objects while *recognising* the order of selection.

The number of permutations of n elements is

$$n! = n \cdot (n - 1) \cdots 1, \quad 0! = 1! = 1$$

r -permutations (sequences without repetition)

Selecting any r objects from a set S of size n without repetition while *recognising* the order of selection.

Their number is

$$(n)_r = {}^n P_r = n \cdot (n - 1) \cdots (n - r + 1) = \frac{n!}{(n - r)!}$$

Permutations with duplicates

Example

How many anagrams of ASSESS?

Label S's: $AS_1S_2ES_3S_4$: $6!$

In each anagram we can label the S's in $4!$ ways.

Suppose there are m anagrams. So $m \cdot 4! = 6!$, i.e. $m = \frac{6!}{4!}$

Example

Number of anagrams of MISSISSIPPI? $\frac{11!}{3!4!2!}$

***r*-selections** (or: ***r*-combinations**)

Collecting any r distinct objects without repetition;
equivalently: selecting r objects from a set S of size n and *not* recognising the order of selection.

Their number is

$$\binom{n}{r} = \frac{\Pi(n, r)}{r!} = \frac{n!}{(n-r)!r!} = \frac{n \cdot (n-1) \cdots (n-r+1)}{1 \cdot 2 \cdots r}$$

NB

These numbers are usually called binomial coefficients due to

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + b^n = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i$$

Also defined for any $\alpha \in \mathbb{R}$ as $\binom{\alpha}{r} = \frac{\alpha(\alpha-1) \cdots (\alpha-r+1)}{r!}$

Simple Counting Problems

Example

RW: 5.1.2 Give an example of a counting problem whose answer is

(a) $(26)_{10}$

(b) $\binom{26}{10}$

Draw 10 cards from a half deck (eg. black cards only)

(a) the cards are recorded in the order of appearance

(b) only the complete draw is recorded

Examples

- Number of edges in a complete graph K_n
- Number of diagonals in a convex polygon
- Number of poker hands
- Decisions in games, lotteries etc.

Exercises

Exercises

RW: 5.1.6 From a group of 12 men and 16 women, how many committees can be chosen consisting of

- (a) 7 members?
- (b) 3 men and 4 women?
- (c) 7 women or 7 men?

RW: 5.1.7 As above, but any 4 people (male or female) out of 9 and two, Alice and Bob, unwilling to serve on the same committee.

Counting Poker Hands

Exercises

RW: 5.1.15 A poker hand consists of 5 cards drawn without replacement from a standard deck of 52 cards

$$\{A, 2-10, J, Q, K\} \times \{\text{club, spade, heart, diamond}\}$$

- (a) Number of “4 of a kind” hands (e.g. 4 Jacks)

- (b) Number of non-straight flushes, i.e. all cards of same suit but *not* consecutive (e.g. 8,9,10,J,K)

Selecting items summary

Selecting k items from a set of n items:

With replacement	Order matters	Examples	Formula
Yes	Yes	Words of length k (sequences of length k)	n^k
No	Yes	k -permutations	$(n)_k$
No	No	Subsets of size k	$\binom{n}{k}$
Yes	No	Multisets of size k	$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$

“Balls in boxes”

Have n “distinguishable” boxes.

Have k balls which are either:

- ① Indistinguishable
- ② Distinguishable

How many ways to place balls in boxes with

- A At most one
- B Any number of

balls per box?

NB

Suppose K is a set with $|K| = k$ and N is a set with $|N| = n$:

- *2A counts the number of injective functions from K to N*
- *2B counts the number of functions from K to N*

“Balls in boxes”

Case	Balls	Balls per box	Number
1A	Indist.	At most 1	$\binom{n}{k}$
1B	Indist.	Any number	$\binom{n+k-1}{k}$
2A	Dist.	At most 1	$(n)_k$
2B	Dist.	Any number	n^k

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Alternative techniques

What if the current techniques are unwieldy?

Other techniques for obtaining an exact count:

- Find a different approach for counting
- Make use of symmetries
- Make use of recursion
- Write a program (running time?)

Example

Example

How many sequences of 15 coin flips have an even number of heads?

- Using “balls in boxes”: $\binom{15}{0} + \binom{15}{2} + \dots + \binom{15}{14}$
- Use symmetry: $\frac{1}{2} \times 2^{15}$
- Use recursion: $\text{Even}(n) = \text{Odd}(n-1) + \text{Even}(n-1);$
 $\text{Odd}(n) = \text{Even}(n-1) + \text{Odd}(n-1)$

Example

Example

How many sequences of n coin flips do not contain HH ?

$$C(0) = 0$$

$$C(1) = 0$$

$$C(n) = C(n-1) + C(n-2) + 2^{n-2}$$

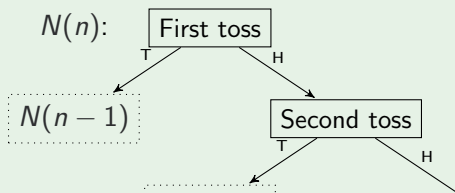
$$N(0) = 1$$

$$N(1) = 2$$

$$N(2) = 3$$

$$N(n) = N(n-1) + N(n-2)$$

We can summarise all possible outcomes in a **recursive tree**



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Example (Ramsay numbers)

An example of a *Ramsay number* is $R(3, 3) = 6$, meaning that

" K_6 is the smallest complete graph such that if all edges are painted using two colours, then there must be at least one monochromatic triangle"

This serves as the basis of a game called S-I-M (invented by Simmons), where two adversaries connect six dots, respectively using blue and red lines. The objective is to *avoid* closing a triangle of one's own colour. The second player has a winning strategy, but the full analysis requires a computer program.

Using Programs to Count

Two dice, a red die and a black die, are rolled.

(Note: one *die*, two or more *dice*)

Write a program to list all the pairs $\{(R, B) : R > B\}$

Similarly, for three dice, list all triples $R > B > G$

Generally, for n dice, all of which are m -sided ($n \leq m$), list all *decreasing* n -tuples

NB

In order to just find the number of such n -tuples, it is not necessary to list them all. One can write a recurrence relation for these numbers and compute (or try to solve) it.

Approximate Counting

NB

A Count may be a precise value or an **estimate**.

The latter should be *asymptotically correct* or at least give a good *asymptotic bound*, whether upper or lower. If S is the base set, $|S| = n$ its size, and we denote by $c(S)$ some collection of objects from S we are interested in, then we seek constants a, b such that

$$a \leq \lim_{n \rightarrow \infty} \frac{\text{est}(|c(S)|)}{|c(S)|} \leq b$$

In other words $\text{est}(|c(S)|) \in \Theta(|c(S)|)$.