Homework 0

Q1

(a) $f(x,y) = a_1x^2y^2 + a_4xy + a_5x + a_7$

$\frac{\partial f(x,y)}{\partial x} = 2a_1xy^2 + a_4y + a_5 \quad \frac{\partial f(x,y)}{\partial y} = 2a_1x^2y + a_4x$

$\frac{\partial f(x,y)}{\partial x\,\partial y} = 4a_1xy + a_4 \quad \frac{\partial f(x,y)}{\partial^2 x} = 2a_1y^2 \quad \frac{\partial f(x,y)}{\partial^2 y} = 2a_1x^2$

(b) $f(x,y) = a_1x^2y^2 + a_2x^2y + a_3xy^2 + a_4xy + a_5x + a_6y + a_7$

$$\frac{\partial f(x,y)}{\partial x} = 2a_1xy^2 + 2a_2xy + a_3y^2 + a_4y + a_5$$

$$\frac{\partial f(x,y)}{\partial y} = 2a_1x^y + a_2x^2 + 2a_3xy + a_4x + a_6$$

$$\frac{\partial f(x,y)}{\partial^2 x} = 2a_1y^2 + 2a_2y$$

$$\frac{\partial f(x,y)}{\partial^2 y} = 2a_1x^2 + 2a_3x$$

$$\frac{\partial f(x,y)}{\partial x\,\partial y} = 4a_1xy + 2a_2x + 2a_3y + a_4$$

(c) $\sigma'(x) = \frac{\partial \sigma}{\partial x} = \left(\frac{1}{1+e^{-x}}\right)' = \frac{e^{-x}}{(1+e^{-x})^2}$

$$\sigma(x)\big(1-\sigma(x)\big) = \frac{1}{1+e^{-x}}\left(1 - \frac{1}{1+e^{-x}}\right) = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$\therefore\ \sigma'(x) = \frac{\partial \sigma}{\partial x} = \sigma(x)\big(1-\sigma(x)\big)$$

(d) (i) $y_1' = 8x - 3$ when $y_1' = 0$ and $x = \frac{3}{8}$

therefore, when $x = \frac{3}{8}$, y has minimum points $\frac{39}{16}$

(ii) $y_2' = 12x^3 - 6x^2$ when $y_2'=0$, $x = 0$ or $x = \frac{1}{2}$

therefore, when $x = 0$, y_2 has maximum point 0, when $x = \frac{1}{2}$, y_2 has minimum $\frac{1}{2}$.

(iii) $y_3 = 4x + \sqrt{1-x}$ then $y_3' = 4 - \frac{1}{2\sqrt{1-x}}$

(iv) $y_4 = x + x^{-1}$ $y_4' = 1 - \frac{1}{x^2}$

when $y_4' = 0$, x=1 or x=-1.

therefore, when x=-1, y_4 has minimum -2, when x=1. y_4 has maximum 2.

Q2

(a) Because 20% buy products from Outlet I, 10% buy from both I and II, 40% buy from neither.
then $P(A) = 20\%$ $P(B) = 100\% - 20\% - 10\% - 40\% = 30\%$
$P(A \cup B) = 10\%$ $P(\overline{A\bar{B}}) = 40\%$

(b) (i) $r = 1 - \frac{1}{6} - \frac{1}{6} - \frac{1}{12} - \frac{1}{12} - \frac{1}{6} = \frac{1}{3}$

(ii) $P(X = 2, Y = 3) = \frac{1}{6}$

(iii) $P(X = 3) = P(X = 3, Y = 1) + P(X = 3, Y = 2) + P(X = 3, Y = 3) = 0 + \frac{1}{3} + 0 = \frac{1}{3}$

$P(Y = 2) = P(X = 1, Y = 2) + P(X = 2, Y = 2) + P(X = 3, Y = 2) = \frac{5}{12}$

$\therefore P(X = 3|Y = 2) = \frac{P(X=3,Y=2)}{P(Y=2)} = \frac{4}{5}$

(iv) $P(X = 1) = \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$ $P(X = 2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ $P(X = 3) = \frac{1}{3}$

$\therefore E(X) = P(X = 1) + 2P(X = 2) + 3P(X = 3) = \frac{1}{3} + \frac{2}{3} + 1 = 2$

$P(Y = 1) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ $P(Y = 2) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$ $P(Y = 3) = \frac{1}{12} + \frac{1}{6} = \frac{1}{4}$

$E(Y) = P(Y = 1) + 2P(Y = 2) + 3P(Y = 3) = \frac{1}{3} + \frac{10}{12} + \frac{3}{4} = \frac{23}{12}$

Because (x, y) is a discrete random variable, then XY is also a discrete random variable.

$P(XY = 1) = \frac{1}{3} * \frac{1}{3} = \frac{1}{9}$ $P(XY = 2) = \frac{1}{3} * \frac{5}{12} + \frac{1}{3} * \frac{1}{3} = \frac{1}{4}$ $P(XY = 3) = \frac{1}{3} * \frac{1}{4} + \frac{1}{3}\backslash$

$\mathbf{ast}\frac{1}{3} = \frac{7}{36}$

$\mathbf{P}(\boldsymbol{XY} = \mathbf{4}) = \frac{1}{3} * \frac{5}{12} = \frac{5}{36}$ $P(XY = 6) = \frac{1}{3} * \frac{1}{4} + \frac{1}{3} * \frac{5}{12} = \frac{8}{36}$ $P(XY = 9) = \frac{1}{3} * \frac{1}{4} = \frac{1}{12}$

$$\mathbf{E}(\boldsymbol{XY}) = \mathbf{P}(\boldsymbol{XY} = \mathbf{1}) + \mathbf{2}\boldsymbol{P}(\boldsymbol{XY} = \mathbf{2}) + \mathbf{3}\boldsymbol{P}(\boldsymbol{XY} = \mathbf{3}) + \mathbf{4}\boldsymbol{P}(\boldsymbol{XY} = \mathbf{4}) + \mathbf{6}\boldsymbol{P}(\boldsymbol{XY} = \mathbf{6}) + \mathbf{9}\boldsymbol{P}(\boldsymbol{XY} = \mathbf{9}) = \frac{23}{6}$$

(v) $P(X^2 = 1) = \frac{1}{3}$ $P(X^2 = 4) = \frac{1}{3}$ $P(X^2 = 9) = \frac{1}{3}$

$\therefore E(\mathrm{X}^2) = P(\mathrm{X}^2 = 1) + 4P(\mathrm{X}^2 = 4) + 9P(\mathrm{X}^2 = 9) = \frac{14}{13}$

$P(Y^2 = 1) = \frac{1}{3}$ $P(Y^2 = 4) = \frac{5}{12}$ $P(Y^2 = 9) = \frac{1}{4}$

$\therefore E(\mathrm{Y}^2) = P(\mathrm{Y}^2 = 1) + 4P(\mathrm{Y}^2 = 4) + 9P(\mathrm{Y}^2 = 9) = \frac{15}{4}$

(vi) $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{23}{6} - 2 * \frac{23}{12} = 0$

(vii) $Var(X) = Cov(X,X) = E(X^2) - \left(E(X)\right)^2 = \frac{14}{3} - 4 = \frac{1}{3}$

$Var(Y) = Cov(Y,Y) = E(Y^2) - \left(E(Y)\right)^2 = \frac{15}{4} - \left(\frac{23}{12}\right)^2 = \frac{1}{3}$

(viii) $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = 0$

(ix) $E(X+Y) = E(X) + E(Y) = 2 + \frac{23}{12} = \frac{47}{12}$

$E(X+Y^2) = E(X) + E(Y^2) = 2 + \frac{15}{4} = \frac{23}{4}$

$Var(X+Y) = \mathrm{Var}(X) + Var(Y) + 2Cov(X,Y) = \frac{2}{3}$

$Var(X+Y^2) = Var(X) + Var(Y^2) - 2Cov(X,Y^2) =$

Q3

(a) $A \in R^{3\times 5}$, $b \in R^{6\times 1}$, $A^T \in R^{5\times 3}$

(b) (i) AB and BA are not computable, because A is a 3x3 matrix and B is a 2x2 matrix.

(ii) $AC = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 21 & 14 & 14 \\ 20 & 10 & 10 \\ 56 & 28 & 28 \end{bmatrix}$

$CA = \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 31 & 39 & 40 \\ 10 & 12 & 12 \\ 18 & 18 & 16 \end{bmatrix}$

(iii) $AD = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 4 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 32 \\ 17 & 19 \\ 43 & 45 \end{bmatrix}$

DA is not computable, because D is a 3x2 matrix and A is a 3x3 matrix.

(iv) DC is not computable, because D is a 3x2 matrix and C is a 3x3 matrix.

$CD = \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 4 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 41 \\ 13 & 13 \\ 18 & 22 \end{bmatrix}$

$D^T C = \begin{bmatrix} 4 & 4 & 1 \\ 2 & 6 & 3 \end{bmatrix} \times \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 38 & 18 & 18 \\ 32 & 18 & 18 \end{bmatrix}$

(v) $Bu = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$

uB is not computable, because A is a 3x3 matrix and u is a 2x1 matrix.

(vi) Au is not computable, because A is a 3x3 matrix and u is a 2x1 matrix/

(vii) $Av = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix}$

vA is not computable, because v is a 3x1 matrix and A is a 3x3 matrix.

(viii) $Av + Cv = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 3 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix} + \begin{bmatrix} 29 \\ 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 47 \\ 22 \\ 45 \end{bmatrix}$

(c) (i) $||u||_1 = 1 + 3 = 4 \quad ||u||_2 = (\sum_{i=1}^{n} |u_i|^2)^{\frac{1}{2}} = \sqrt{10} \quad ||u||_2^2 = 10$

$||u||_\infty = max|u_i| = 3$

(ii) $||v||_1 = 2 + 4 + 1 = 7 \quad ||v||_2 = (\sum_{i=1}^{n} |v_i|^2)^{\frac{1}{2}} = \sqrt{21} \quad ||v||_2^2 = 21$

$||v||_\infty = max|v_i| = 4$

(iii) $v + w = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \quad \therefore ||v + w||_1 = 3 + 2 + 3 = 8$

$||v + w||_2 = (\sum_{i=1}^{n} |v_i + w_i|^2)^{\frac{1}{2}} = \sqrt{22} \quad ||v + w||_2^2 = 22$

$||v + w||_\infty = max|v_i + w_i| = 3$

(iv) $Av = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 31 \end{bmatrix} \quad A(v - w) = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 2 & 1 \\ 6 & 4 & 3 \end{bmatrix} \times \left(\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 15 \\ 13 \\ 27 \end{bmatrix}$

$||Av||_2 = (\sum_{i=1}^{n} |Av_i|^2)^{\frac{1}{2}} = \sqrt{1454} \quad ||A(v - w)||_\infty = max|A(v - w)_i| = 27$

(d) $< u, v > = u \cdot v = u^T v = [1 \quad 2] \times \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \quad < u, w > = u \cdot w = u^T w = [1 \quad 2] \times \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} =$
0

$< v, w > = u \cdot w = u^T w = [1 \quad 1] \times \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} = -\frac{1}{2}$

$cos < u, v > = \frac{u \cdot v}{||u||_2 ||v||_2} = \frac{3}{\sqrt{10}} \quad cos < u, w > = \frac{u \cdot w}{||u||_2 ||w||_2} = 0$

$cos < v, w > = \frac{v \cdot w}{||v||_2 ||w||_2} = -\sqrt{\frac{1}{10}}$

(e) The dot product equals 0 means that two vectors are perpendicular.
A negative dot product means that the angle between the two vectors is obtuse.
A positive dot product means that the angle between the two vectors is acute.

(f) $\because \begin{vmatrix} 1 & 3 \\ 4 & 1 \end{vmatrix} = -11 \neq 0$ Therefore, the matrix is invertible.

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 4 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -11 & 0 & 1 & -3 \\ 4 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -11 & 0 & 1 & -3 \\ 0 & 1 & \frac{4}{11} & -\frac{1}{11} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{11} & \frac{3}{11} \\ 0 & 1 & \frac{4}{11} & -\frac{1}{11} \end{bmatrix}$$

Therefore, A^{-1} is $\begin{bmatrix} -\frac{1}{11} & \frac{3}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix}$

(g) $\because \begin{vmatrix} 3 & 3 \\ 4 & 4 \end{vmatrix} = 0$ Therefore, the matrix is not invertible.

(h) $(X^T X)^T = X^T X$ Therefore, $X^T X$ is always symmetric.